

Proceedings of the Forty-Fifth Annual Meeting of the  
North American Chapter of the International Group for  
the Psychology of Mathematics Education

**Engaging *All* Learners**

**VOLUME 2: LEARNING**

Reno, Nevada

Oct 1-4, 2023

Editors:

Teruni Lamberg  
University of Nevada, Reno  
terunil@unr.edu

Diana Moss  
University of Nevada, Reno  
dmoss@unr.edu



**Citation:**

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol 2). University of Nevada, Reno.

ISBN: 978-1-7348057-2-7

DOI: 10.51272/pmena.45.2023

**Copyright:** Articles published in the Proceedings are copyrighted by the authors. Permission to reproduce an article or portions from an article must be obtained from the author.

# PME-NA HISTORY AND GOALS

## **PME-NA History and Goals**

PME came into existence at the Third International Congress on Mathematical Education (ICME-3) in Karlsruhe, Germany, in 1976. It is affiliated with the International Commission for Mathematical Instruction. PME-NA is the North American Chapter of PME. The first PME-NA conference was held in Evanston, Illinois in 1979. Since their origins, PME and PME-NA have expanded and continue to expand beyond their psychologically oriented foundations. The major goals of the International Group and the North American Chapter are:

1. To promote international contacts and the exchange of scientific information in the psychology of mathematics education.
2. To promote and stimulate interdisciplinary research in the aforesaid area, with the cooperation of psychologists, mathematicians, and mathematics teachers; and
3. To further a deeper and better understanding of the psychological aspects of teaching and learning mathematics and the implications thereof.

## **PME-NA Membership**

Membership is open to people who are involved in active research consistent with PME-NA's aims or who are professionally interested in the results of such research. Membership is open on an annual basis and depends on payment of dues for the current year. Membership fees for PME-NA (but not PME International) are included in the conference fee each year. If you are unable to attend the conference but want to join or renew your membership, go to the PME-NA website at <http://pmena.org>. For information about membership in PME, go to <http://www.igpme.org> and visit the "Membership" page.

# PME-NA Contributors

## PME-NA Steering Committee

### Elected Members

Doris Jeannotte (Chair) – Université du Québec à Montréal  
Ayman Aljarrah – Acadia University  
Leslie Dietiker – Boston University  
Christopher Kurz – Rochester Institute of Technology  
Zareen Rahman – James Madison University  
Angeles Dominguez – Tecnológico de Monterrey  
Xiangquan (James) Yao – Pennsylvania State University  
María S. García González – Universidad Autónoma de Guerrero  
Oyemolade (Molade) Osibodu – York University  
Casey Griffin (Graduate Student Rep.) – University of Delaware  
Tabatha Rainwater (Graduate Student Rep.) – University of Tennessee, Knoxville

### Appointed Members

Aaron Brakoniecki (Webmaster) – Boston University  
Winnie Ko (Treasurer) – Indiana State University

# Conference Organizing Committees

## Current Conference (2022-2023)

Teruni Lamberg  
Chair  
University of Nevada, Reno  
terunil@unr.edu

Shera Alberti-Annunzio  
Conference Coordinator  
University of Nevada, Reno  
shera@unr.edu

## Local Organizing Committee

Diana Moss\*  
University of Nevada, Reno  
dmoss@unr.edu

Lynda Wiest  
University of Nevada, Reno  
wiest@unr.edu

Glenn Waddell\*  
University of Nevada, Reno  
gwaddell@unr.edu

Rachael Welder\*  
University of Nevada, Reno  
rwelder@unr.edu

Alysia Goyer\*  
Stockton University  
alysia.goyer@stockton.edu

Joseph Antonides\*  
Virginia Tech  
jantonides@vt.edu

\*Served as strand leaders

## Future Conference Chair (2023-2024)

Karl W. Kosko  
Kent State University  
kkosko1@kent.edu

## Past Conference Co-Chairs (2021-2022)

Alyson Lischka  
Middle Tennessee State  
University  
alyson.lischka@mtsu.edu

Elizabeth B. Dyer  
University of Tennessee  
Knoxville  
edyer8@utk.edu

Ryan Seth Jones  
Middle Tennessee State  
University  
ryan.Jones@mtsu.edu

Jennifer N. Lovett  
Middle Tennessee State University  
jennifer.lovett@mtsu.edu

Jeremy Strayer  
Middle Tennessee State University  
jeremy.strayer@mtsu.edu

### Sponsor

University of Nevada, Reno

## Reviewers

Abbaspour Tazehkand,  
Shahabeddin  
Acevedo, Carlos Ivan  
Adeolu, Adewale  
Aguilar, Jair  
Akgul, Enisa  
Akuom, Denish Ogwen  
Aljarrah, Ayman  
Altshuler, Mari  
Alyami, Hanan  
Alzayyat, Ahmad  
Ambrose, Rebecca C  
Anthony, Monica  
Antonides, Joseph  
Aqazade, Mahtob  
Aryal, Harman Prasad  
Asempapa, Reuben  
Ataide Pinheiro, Weverton  
Austin, Christine Kathryn  
Austin, Christopher  
Azimi Asmaroud,  
Seyedekhadijeh  
Bailey, Nina Gabrielle  
Balady, Steve  
Baniahmadi, Mona  
Bennett, Amy Been  
Bermudez, Hillary  
Bertolone-Smith, Claudia  
Marie  
Bharaj, Pavneet Kaur  
Bieda, Kristen N  
Bofferding, Laura  
Boncoddo, Rebecca  
Bondurant, Liza  
Bostic, Jonathan David  
Boyce, Steven  
Bozzano, Patricia Eva  
Brakoniecki, Aaron  
Brass, Amy  
Brown, Yuriko Hoshiya  
Bui, Mai  
Butler, Rebecca  
Cabañas-Sánchez,  
Guadalupe  
Cannon, Susan  
Cardetti, Fabiana  
Castanheira, Brittney  
Cavey, Laurie Overman  
Caviness, Stephen  
Chandler, Kayla  
Chavez, Oscar  
Chen, Lizhen  
Chicalote Jiménez, Tania  
Azucena  
Choi, Kyong Mi  
Cirillo, Michelle  
Complete Name  
Cook, Candies  
Cordero Osorio, Francisco  
Corey, Douglas  
Corven, Julien  
Czocher, Jennifer A  
Daniel, Amy L  
Davis, Jon D.  
De Alba, Carlos Alejandro  
Dietiker, Leslie  
Dinapoli, Joseph  
Dobie, Tracy Elyse  
Doherty, Kristin  
Dominguez, Angeles  
Dubbs, Christopher  
Ducloux, Kanita  
Duni, Dhimitraq  
Edwards, Belinda  
Egbedeyi, Temitope  
Elizondo, Andrea Martina  
Erwin, Alicia  
Escobar Durán, Uriel  
Fajardo, Maria Del  
Carmen  
Fan, Yiyun  
Feikes, David  
Fink, Heather  
Forde, Elizabeth  
Franks, Asia  
Frazee, Leah Michelle  
Gantt, Allison L.  
García González, María S.  
Gargroetzi, Emma Carene  
Gebremichael, Andualem  
Tamiru  
Ghousseini, Hala  
Gilbertson, Nicholas J  
Gómez-Árciga, Adrián  
Goodson-Espy, Tracy  
Goyer, Alysia  
Graysay, Duane  
Greenstein, Steven  
Grewall, Tejvir Kaur  
Griffin, Camille  
Griffin, Casey  
Gualdron Pinto, Elgar  
Hackenberg, Amy J  
Haiduc, Ana-Maria  
Hall, Jennifer  
Han, Simon Byeonguk  
Hand, Victoria  
Hartmann, Christine  
He, Jia  
Hidayat, Angga  
Hohensee, Charles  
Hong, Dae S.  
Hunt, Jessica H.  
Izard, Blair  
Jarry-Shore, Michael  
Jeannotte, Doris  
Kacerja, Suela  
Kamaldar, Azar  
Kamlue, Nitchada  
Karanevich, Peter  
Kaufman-Ridout, Bailey  
Anne  
Kebreab, Lybrya  
Kent, Laura Brinker  
Kercher, Andrew  
Kerrigan, Sarah  
Kim, Hee-Jeong  
Kim, Youngjun  
Kirkland, Patrick  
Kirwan, J Vince  
Kitchen, Richard  
Ko, Winnie  
Ko, Yi-Yin  
Kochmanski, Nicholas

Kokushkin, Vladislav	Ortiz, Daniel	Simpson, Amber
Krell, Michael W	Oslund, Joy Ann	Smith, Amy
Kurz, Christopher A	Osibodu, Oyemolade	Smith, Bernard Ronald
Kwon, Oh Hoon	Otten, Samuel	Smith, Shawnda
Lamberg, Teruni	Pacheco, Mike	Smithey, Montana
Lassak, Marshall	Paliwal, Veena	Smucker, Karoline
Leatham, Keith R.	Panorkou, Nicole	Son, Kyunghoon
Lee, Boram	Paoletti, Teo	Sourwine, Jasmine
Lee, Carrie W	Park, Hyejin	Stoddard, Elyssa
Lee, Hwa Young	Park, Jungeun	Suárez, Mayra Zulay
Leonard, Helene	Partridge, Eric	Suazo-Flores, Elizabeth
Lim, Dexter	Patterson, Cody L	Sung, Yewon
Lloyd, Mary Elizabeth	Phelps-Gregory, Christine	Swart, Michael I.
Riley	M	Tague, Jenna
López-Iñesta, Emilia	Piatek-Jimenez, Katrina	Talbot, Jennifer
Love, Candice	Poling, Lisa L	Thrasher, Emily
Lovin, Louann	Polly, Drew	Towers, Jo
Macdonald, Beth L.	Provost, Amanda	Tucci, Anthony
Magiera, Marta T.	Pujiyanto, Fnu	Turner, Blake O'Neal
Mainzer, Emily	Pynes, Kristen D'Anna	Turner, Erin
Maldonado Reynoso,	Quansah, Abigail Lois	Tyburski, Brady A
Norma Patricia	Rahman, Zareen Gul	Varghese, Sijo
Males, Lorraine M	Rainwater, Tabatha	Voyias, Karley Violet
Margolis, Claudine	Raja, Waleed Ashraf	Waddell, Jr., Glenn
Marsh, Dalton Dayne	Rathouz, Margaret	Warshauer, Hiroko
Martin, Tami S.	Ricks, Thomas	Kawaguchi
Martínez Uribe, Alfredo	Roberts, Sarah A.	Watkins, Jonathan D
Martinez-Soto, Eduan	Robinson, Molly L	Webb, David C
Mask, Walker	Roman, Christopher	Webel, Corey
Matranga, Anthony	Orlando	Wei, Xinyu
Mauntel, Matthew	Roman, Kathryn E.	Weiland, Travis
Max, Brooke	Roscioli, Kate	Welder, Rachael Mae
Meagher, Michael S	Ruiz, Steven L	Wessman-Enzinger, Nicole
Méndez Huerta, Dinorah	Rupnow, Rachel	Marie
Moldavan, Alesia Mickle	Rygaard Gaspard, Brandi	Westby, Kathryn R.
Montero-Moguel, Luis E	Saldaña, Mike	Wheeler, Ann
Moss, Diana L.	Sanchez Wall, Lina	Willett, Brooklynn
Nagle, Courtney	Sankaranarayanan,	Winsor, Matthew
Naresh, Nirmala	Ananthi	Witt, Nicholas
Nitta, Kathleen	Sayavedra, Alyssa	Wynn, Lynda
Norton, Anderson	Schwartz, Gil	Yao, Xiangquan
O'Dell, Jenna R	Sepulveda, Francisco	Yavuz, Selim
Odiwuor, Brian	Sevier, John	Yilmaz, Zuhail
Orozco, Claudia	Sianturi, Iwan Andi Jonri	Zaldívar, José David
Ortiz Galarza, Mayra	Siebert, Daniel	Zhou, Lili
Lizeth	Simon, Martin	Zhuang, Yuling



Zimmerman, Stacey C.

Zolt, Holly

Zwanch, Karen

# Preface

The Forty Fifth Annual meeting of the North American chapter of the International Group for the Psychology of Mathematics Education was held PME-NA 45 in Reno, Nevada, Oct. 1-4, 2023. The conference theme is listed below:

## *Engaging All Learners*

Math learning should be a joyful experience for all students. When students are engaged and inspired, they are motivated to learn. Instruction that targets the learning needs and interests of our students makes it possible for students to excel in learning math. Participants in the conference explored how to create conditions to support learning that build on student engagement and interest in addition to other research engaged by the PME-NA community. The specific conference theme questions explored as part of the conference was:

- How can we engage all students to learn math content by building on their interest and motivation to learn?
- How do we design learning environments that take students and learning into account?
- What are the design features of tools and curricula design features considering student engagement and interest in supporting learning?
- How do we build partnerships with schools and the community to support student engagement and math learning?
- What research agendas should we pursue to ensure that all students reach their potential by paying attention to engagement and learning needs?

The acceptance rate for Research Report was 45%, the acceptance rate for brief research reports was 70 %. The acceptance rate for posters was 90%. Note: some papers were accepted in alternate format than originally proposed. The total number of participants who submitted proposals as co-authors was 1083.

# Plenary Speakers

## Motivation and Embodied Cognition

- Mitchell J. Nathan, Ph.D., University of Wisconsin at Madison
- James Middleton, Ph.D., Arizona State University

## Connecting Math to Real-world Experiences, Culture and Technology

- Lisa Lunney Borden, Ph.D., St. Francis Xavier University, Canada
- Jose Luis Cortina, Ph.D., National Pedagogical University, Mexico City
- Theodore Chao, Ph.D., Ohio State University

## Play Experiences and Math Learning Panel Presentation, "*What Do You See in Mathematical Play?*"

- Nathaniel Bryan, Ph.D., Ed.D., The University of Texas at Austin
- Melissa Gresalfi, Ph.D., Vanderbilt University
- Naomi Jessup, Ph.D., Georgia State University
- Amy Parks, Ph.D. Michigan State University
- Tran Templeton, Ed.D., Teachers College Columbia University
- Anita Wager, Ph.D. Vanderbilt University

## Preparing Teachers to Engage Students (closing the plenary sessions)

- Robert Berry III, Ph.D., University of Arizona

The local organizing committee would like to thank the steering committee for all their support and everyone who helped make this conference a success.

# Table of Contents

<b>ENGAGING ALL LEARNERS</b>	<b>I</b>
<b>PME-NA HISTORY AND GOALS</b>	<b>III</b>
<b>PME-NA CONTRIBUTORS</b>	<b>IV</b>
<b>PREFACE</b>	<b>II</b>
<b>PLENARY SPEAKERS</b>	<b>III</b>
<b>TABLE OF CONTENTS</b>	<b>4</b>
<b>STUDENT LEARNING AND RELATED FACTORS</b>	<b>5</b>
<b>TECHNOLOGY AND LEARNING ENVIRONMENT DESIGN</b>	<b>265</b>
<b>EQUITY AND JUSTICE</b>	<b>365</b>
<b>EARLY ALGEBRA, ALGEBRAIC THINKING AND FUNCTION</b>	<b>204</b>
<b>GEOMETRY AND MEASUREMENT</b>	<b>317</b>
<b>NUMBER CONCEPTS AND PROPORTIONAL REASONING</b>	<b>388</b>
<b>PRE-CALCULUS AND HIGHER-LEVEL MATH</b>	<b>454</b>
<b>STATISTICS, PROBABILITY AND DATA SCIENCE</b>	<b>559</b>
<b>WORKING GROUP PAPERS</b>	<b>590</b>

---

# STUDENT LEARNING AND RELATED FACTORS

---

## ASSESSING MATHEMATICS ENGAGEMENT IN THE MOMENT

James A. Middleton  
Arizona State University  
jimbo@asu.edu

Adi Wiezel  
Elon University  
awiezell@asu.edu

Mandy Jansen  
University of Delaware  
jansen@udel.edu

*Engagement can be described as students' tendency to work productively, think deeply, enjoy and value their learning, and to support each other in the process of learning (ZDM article). Each of these dimensions can be indexed by a variety of psychologically validated constructs such as interest and enjoyment, self-regulation, effort, emotional valence and object. The purpose of this paper is to describe a method of assessment that takes under 5 minutes to administer as students are learning mathematics concepts or shortly thereafter in the context of classroom observation and analysis of practices that may support productive engagement. A description of an online survey, its development and psychometric properties assessed in a study of over 1,000 secondary mathematics students in the US is presented, and implications for research on task-level engagement in mathematics classrooms is discussed.*

Keywords: Engagement, Emotions, Affect, Assessment

### Introduction

The engagement in, and motivation to continue, mathematics has consistently been shown to decrease as students move through compulsory education (Collie et al., 2019). Such engagement is also complex, psychologically, and socially, involving the coordination of memories of prior math experiences, with the social and cognitive characteristics of mathematics tasks and practices interpreted through affective and motivational responses (Fredricks, 2011; Middleton, Jansen, & Goldin, 2017). Moreover, the supportive features of classrooms, in the form of social relationships among learners and their teachers and peers has been shown to be related to different forms of engagement, sometimes enhancing and sometimes diminishing one feature in favor of another (Reindl et al., 2015; Strati et al., 2017). It is generally assumed that aspects of long-term engagement are impacted by the patterns of engagement students experience over time in mathematics tasks and activities. A classic example is the development of personal interest in mathematics from situational interest (Hidi & Reninger, 2016). These aspects of engagement become particularly salient in times of social transition, for example, in the transition from middle to high school mathematics, where one's peer group, level of mathematical rigor and class norms may change substantially (Middleton, Mangu & Lee, 2017).

The larger body of literature on engagement characterizes it as four related dimensions involving affective, cognitive, behavioral, and social facets (e.g., Wang, et al., 2016; Rimm-Kaufman, et al., 2015). Affective engagement consists of the immediate emotional responses to aspects of a learning environment (Fredricks, Blumenfeld, & Paris, 2004), as well as the meta-affective re-evaluations of those responses (e.g., Goldin, 2014; Goldin, 2002). Cognitive engagement can be thought of as students expending effort on coordinating prior information with current information (Middleton, Jansen, & Goldin, 2017). Behavioral engagement is the productive behavior that students engage in in a math environment (Rimm-Kaufman, Bardood, Larsen, Curby, & Abry, 2015). Finally, social engagement involves the nature of interpersonal relationships and interactions in a math classroom (Wang, Fredricks, Ye, Hofkens, and Lin, 2015). This includes the quality of the

relationships and interactions with peers as well as the teacher. We focus on the first two years of high school mathematics, in which each of these facets potentially may change as new environmental demands are placed on incoming students.

One of the difficulties in studying engagement concerns its measurement. While long-term engagement, including students' beliefs about their self-efficacy and personal interest in mathematics have historically been assessed using survey methods (Middleton, et al., 2023), standard surveys have not shown significant success when applied to engagement as experienced in the moment of learning because of the time they take, and the multidimensionality of engagement constructs. It just takes too many standard surveys, each of which takes up lots of time, to probe students' beliefs, emotions, and behaviors in the moment with much fidelity. One exception to this is Experience Sampling Methods (ESM). These typically utilize a (very) short-term survey (taking 2 to 5 minutes only) to uncover students' immediate responses to their experience (Larson & Csikszentmihaly, 1987; Shernoff; Csikszentmihaly, Schneider, and Shernoff, 2003; Shernoff, 2013). In a typical ESM, participants are signaled at random or after a specific pre-determined event (such as a particular lesson or task) to complete a series of closed or open-ended items about their experiences (Shernoff, 2013; Shiffman, Stone, and Hufford, 2008).

An advantage of using ESMs relative to retrospective assessments is that they are better able to capture in-the-moment impressions of events (Shiffman, Stone, and Hufford, 2008). This is important because prior research suggests that while in-the-moment experiences do color and direct more long-term tendencies, people's in-the-moment and after-the-fact impressions of an experience *can* diverge (Shiffman, Stone, and Hufford, 2008). Moreover, ESMs have also been implemented successfully in school settings to capture aspects of students' engagement such as level of concentration (behavioral) and interest and enjoyment (affective) (Shernoff, 2013).

In the remainder of this paper, we describe the development, administration, and psychometric analysis of an instrument designed to assess mathematics students' engagement as closely to moments of learning as possible. Following this analysis, we discuss a broader approach to studying engagement in classroom settings.

### **Method**

Here, we describe the development of an instrument designed to capture engagement in secondary mathematics in the moment. Two primary criteria guided our work: 1) Because we wanted the instrument to be administered immediately following a key activity in a mathematics class, it needed to be short (less than 5 minutes to administer) so as not to unduly interfere with students' learning; and 2) the instrument needed to be multidimensional, meaningfully capturing important features of each of the 4 dimensions of engagement.

### **Item Development**

To ensure broad conceptual coverage, a team of five content experts assembled to define and write items based on the literature across a variety of content areas, including cognitive engagement, behavioral engagement, affective engagement, social engagement, perceived instrumentality, and mathematics self-efficacy, the latter being two important concepts related to engagement that can also be captured in the moment. These items received several rounds of iterations, including conceptual pairing with a long-term survey designed as part of the same project. Chosen items were reviewed by a focus group of five 9<sup>th</sup> graders in a high school in the Southwest US and further revised to ensure students' endorsement and understanding of the language used.

Thirty-two 5-point Likert items and one checklist of emotions were retained for pilot testing. Because the instrument was designed to be an in-the-moment assessment, and time limits were a

concern, three versions were initially developed for pilot administration. Ten items (including the emotion checklist) that were determined by the team to reflect the most important items for each of the core concepts were selected to form a “core” that would be used across each of the three versions. The remaining 23 items were split across versions A, B, and C. This enabled us to eliminate poorly worded items, note discrepancies in students’ interpretations before creating a single instrument with the most consistent items.

### **ESM Instrument Creation**

Following the pilot, a comparison of common items across the three versions, revealed little temporal variation and nearly no structural variation in exploratory factor analyses performed on each of the three versions. We then performed a further comparison of the instrument to another instrument measuring longer-term engagement patterns (Zhang, et al., 2018), yielding a final, single version which included one item about what participants were doing in the moment, 16 multiple-choice items covering cognitive engagement, behavioral engagement, social engagement, perceived instrumentality, and mathematics self-efficacy, and a checklist of 16 emotions which participants would choose, indicating the object of the emotions (e.g., feeling *frustrated* (checked emotion) at the math *activity*, their *peers*, *themselves*, and/or their *teacher* (the objects of the emotion). The survey concludes with a space in which participants can provide any additional comments on their experience. The final instrument was implemented and online using Qualtrics®.

### **Sample**

The data were collected from first-year high school mathematics classrooms from fourteen teachers across the Mid-Atlantic and the Southwest US in Fall 2018. 450 students had complete responses on all 16 emotion checklist items for each of the four emotion objects as well as the 16 Likert items. 45.8% of these students were from the Southwest, and 54.2% were from the Mid-Atlantic. Student demographics for the schools in the Southwest were: 85-94% low income, 2- 5% White, 1-15% Black, 74-96% Latinx, and 0-5% Asian, Native American, or Multi-Racial; student demographics for the schools in the Mid-Atlantic were: 9-30% low income, 24-57% White, 27-46% Black, 7-24% Latinx, and 0-5% Asian, Native American, or Multi-Racial. Of the sample, 48% of students identified as male, 49% identified as female, and 1% identified as neither or both.

### **Analysis Strategy**

We collected data from students after a focal classroom activity during two semesters, Fall 2018 and Spring 2019. We conducted exploratory factor analyses on the data collected from consented students in Fall 2018 and conducted confirmatory factor analyses on the data collected from consented students in Spring 2019, to examine model fit and configural measurement invariance over time.

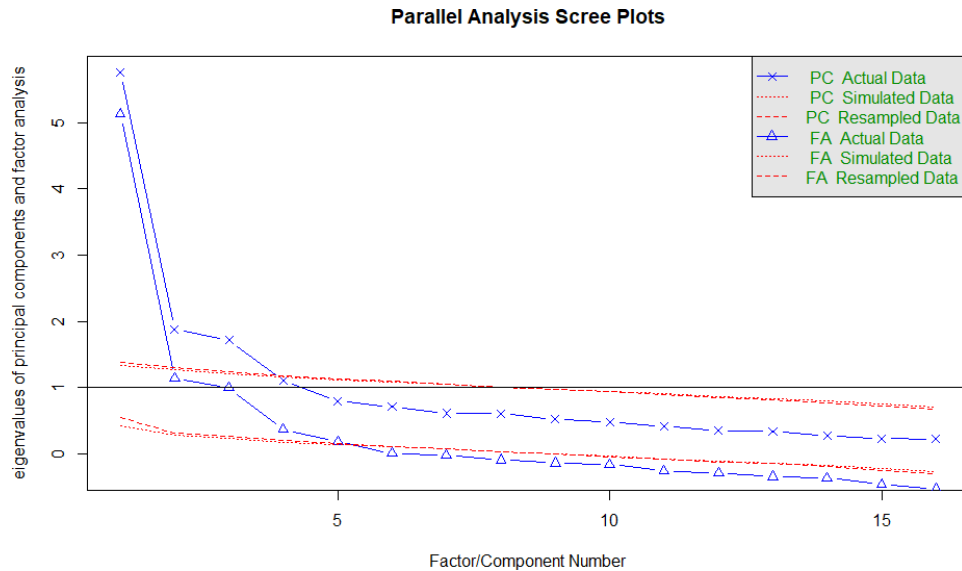
Because our items varied in response styles, from 5-point Likert items to binary responses on the checklist of emotions, we assessed the psychometrics for the Likert items and binary emotion checklist items separately. Data was analyzed using polychoric correlations as this limits potential attenuation which may occur if items with relatively few response options are treated as continuous (Byrne, 2005). Moreover, to help clarify the final sample size used in each analysis (as polychoric correlations can make it difficult to determine the final sample size with pairwise deletion), we used listwise deletion to only analyze cases that had complete data on all items, in both our Fall 2018 and Spring 2019 samples.



## Results

### Likert Items

The optimal number of factors for the 16 Likert items were assessed using a polychoric parallel analysis, which compares a scree plot generated from the actual to one generated using random simulations of the data. The analysis was conducted using the statistical software package R version 3.3.1 and suggested that a maximum of five factors represent the data better than randomness (see Figure 1).



**Figure 1. Parallel analysis for Likert items.**

Based on a visual inspection of the scree plot from the actual data, we decided to estimate two, three, four, and five factor solutions, using an unweighted least squares (ULS) EFA with Oblimin rotation to allow the factors to correlate. Based on interpretability, we felt the 5-factor solution fit best.

**Table 1. Inter-factor correlations.**

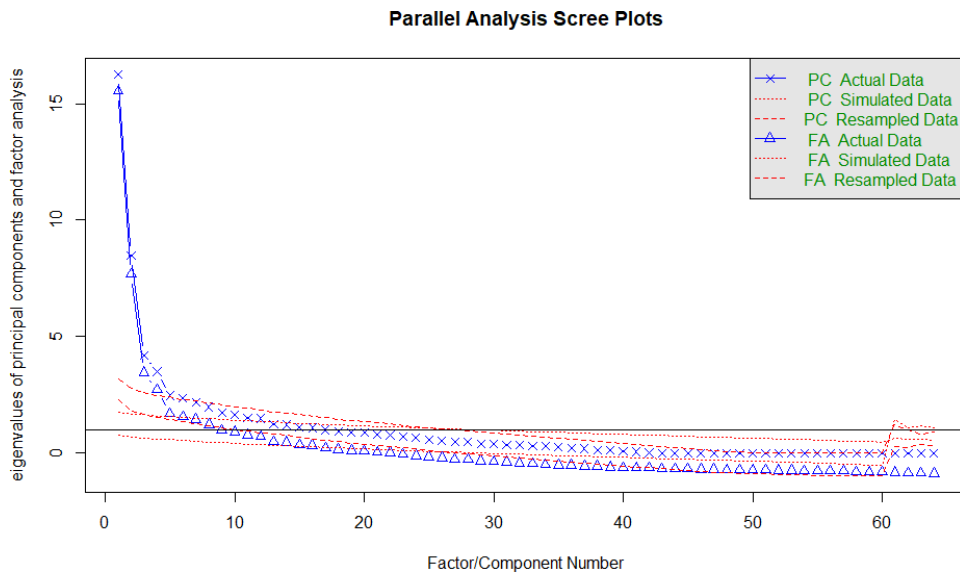
	Task-Efficacy	Effort	Social Engagement	Instrumentality	Situational Interest
Task-Efficacy	1				
Effort	0.039	1			
Social Engagement	<b>0.369</b>	<b>0.448</b>	1		
Instrumentality	0.050	<b>0.272</b>	<b>0.388</b>	1	
Situational Interest	<b>0.318</b>	<b>0.288</b>	<b>0.578</b>	<b>0.529</b>	1

*Note.* Bold text indicates correlations significant at the  $p < .05$  level.

### Emotion Checklist

The optimal number of factors for the 16 emotion checklist items (across 4 possible objects each: the math activity, the classroom, the teacher, and the self) were assessed using a polychoric

parallel analysis. The analysis suggested that a maximum of fifteen factors represent the data better than randomness (see triangles displayed in *Figure 2*).



**Figure 2. Polychoric Parallel analysis for Emotion Checklist items.**

Based on a visual inspection of the scree plot from the actual data, we decided to estimate 4, 5, and 6 factor solutions, each of which was estimated in Mplus version 8 using a mean- and variance-adjusted weighted least squares (WSMLV) EFA with Geomin rotation to address the ordinal nature of the data. Based on interpretability, we felt the six-factor solution best modeled the data. Due to the sheer number of items, to aid interpretation, we eliminated nine items that cross-loaded one more than one factor at a loading level of 0.4 or higher, as well as one item that did not load on any factor at the 0.4 level. This solution generated six factors corresponding to negative emotions about teachers/classmates, positive emotions about teachers/classmates, negative emotions about the self, positive emotions about the self, negative emotions about the math, and positive emotions about the math.

**Table 2. Inter-factor correlations of Emotion Checklist Items.**

	Negative Emotions about Teachers/Classmates	Positive Emotions about Teachers/Classmates	Negative Emotions about the Self	Positive Emotions about the Math	Positive Emotions about the Self	Negative Emotions about the Math
Negative Emotions about Teachers/Classmates	1					
Negative Emotions about Teachers/Classmates	0.24	1				
Negative Emotions about the Self	<b>0.311</b>	0.155	1			

Positive Emotions about the Math	-0.004	<b>0.293</b>	-0.082	1		
Positive Emotions about the Self	0.208	<b>0.289</b>	0.053	<b>0.322</b>	1	
Negative Emotions about the Math	<b>0.293</b>	0.137	0.245	0.095	<b>0.204</b>	1

Note. Bold text indicates correlations significant at the  $p < .05$  level.

### Confirmatory Psychometric Properties

We next aimed to examine whether these models for the Likert items and the emotion checklist showed measurement invariance by running the model extracted in the Fall, 2018 on data from Spring 2019 using categorical confirmatory factor analyses (CFA) with factor loadings as specified in each of the exploratory analyses for Likert items and the emotion checklist.

**Sample.** The data were collected from first-year high school mathematics classrooms from fifteen teachers across Southwest and Mid-Atlantic US in Spring 2019. 690 students had complete responses on all 16 emotion checklist items for each of the four emotion objects as well as the 16 Likert items. 45.1% of these students were from the Southwest, and 54.9% were from the Mid-Atlantic.

**Likert Items.** We examined the fit of the factor structure of the 16 items using the Spring 2019 dataset. Specifically, we used a categorical CFA with polychoric correlations and WLSMV estimation using Mplus version 8 (Muthén & Muthén, 2017). Factors were identified by setting the variance of each latent variable equal to unity, thus standardizing our CFAs. Fits are indicated below.

**Table 3. CFA fit statistics for 15 item Likert model**

Confirmatory Model	X <sup>2</sup>	df	RMSEA	RMSEA 90% CI	CFI	TLI	WRMR
5 Factor Model	462.16	80	0.083	[0.076, 0.091]	0.963	0.952	1.341

In general, this revised model fit the data adequately, per some sources, on the basis of RMSEA, as the value was  $< 0.1$  (Browne & Cudeck, 1993). CFI and TLI were also now very good by standard cutoffs (Tucker & Lewis, 1973; Bentler, 1990). Interitem correlations show that these facets of engagement are interrelated, yet distinct—each contributing variability to the final factor structure.

**Table 4. CFA inter-factor correlations for the Likert items**

	Task-Efficacy	Effort	Social Engagement	Instrumentality	Situational Interest
Task-Efficacy	1				
Effort	<b>0.270</b>	1			
Social Engagement	<b>0.623</b>	<b>0.520</b>	1		

Instrumentality	<b>0.390</b>	<b>0.324</b>	<b>0.588</b>	1	
Situational Interest	<b>0.586</b>	<b>0.397</b>	<b>0.682</b>	<b>0.754</b>	1

Note. Bold text indicates correlations significant at the  $p < .05$  level.

Reliability as assessed by Cronbach's Alpha for each of the Likert Subscales was good to very good: *Efficacy* ( $\alpha = 0.796$ ), *Effort* ( $\alpha = 0.765$ ), *Social Engagement* ( $\alpha = 0.755$ ), *Instrumentality* ( $\alpha = 0.798$ ), and *Situational Interest* ( $\alpha = 0.857$ ).

### Emotion Items

We then examined the fit of the factor structure of the emotion items. Specifically, we used a Categorical CFA with polychoric correlations and WLSMV estimation using Mplus version 8 (Muthén & Muthén, 2017). Fit statistics are indicated below.

**Table 5. CFA fit statistics for the emotion items**

Confirmatory Model	$\chi^2$	df	RMSEA	RMSEA 90% CI	CFI	TLI	WRMR
6 Factor Model	1778.69	1310	0.023	[0.020, 0.025]	0.883	0.877	1.194

In general, this revised model again fit data well based on the RMSEA (Steiger & Lind, 1980, CFI and TLI were also a little higher, although still a bit low by standard cutoffs (Tucker & Lewis, 1973; Bentler, 1990). Recent work using Monte-Carlo simulations of SEM analyses suggests that for smaller sample sizes (less than 500 records), the fit statistics, CFI and TLI, are negatively biased, while RMSEA tends to be positively biased (Shi et al., 2018). As the number of free parameters increases, the relative bias in these estimates becomes more pronounced. The TLI, in particular is affected by number of parameters relative to sample size. Given a  $\frac{\chi^2}{df}$  ratio

of 1.36, well under the recommended ratio of 3, and our excellent RMSEA, we judge this model to show relatively good fit despite lower estimated values of CFI and TLI (Hu & Bentler, 1999).

**Table 6. CFA inter-factor correlations for Emotion subscales, using 54 items.**

	Negative Emotions about Teachers/Classmates	Positive Emotions about Teachers/Classmates	Negative Emotions about the Self	Positive Emotions about the Math Activity	Positive Emotions about the Self	Negative Emotions about the Math Activity
Negative Emotions about Teachers/Classmates	1					
Negative Emotions about Teachers/Classmates	<b>0.324</b>	1				
Negative Emotions about the Self	<b>0.527</b>	<b>0.270</b>	1			

Positive Emotions about the Math Activity	0.180	<b>0.433</b>	0.094	1		
Positive Emotions about the Self	<b>0.329</b>	<b>0.536</b>	<b>0.329</b>	<b>0.446</b>	1	
Negative Emotions about the Math Activity	<b>0.446</b>	0.115	<b>0.606</b>	<b>0.161</b>	0.090	1

Reliability as assessed by Cronbach’s Alpha for each of the Emotion subscales was good to very good *Negative Emotions about Teacher/Classmates* ( $\alpha = 0.655$ ), *Positive Emotions about Teacher/Classmates* ( $\alpha = 0.817$ ), *Negative Emotions about the Self* ( $\alpha = 0.807$ ), *Positive Emotions about Self* ( $\alpha = 0.807$ ), *Negative Emotions about the Math* ( $\alpha=0.700$ ), and *Positive Emotions about the Math Activity* is comprised of (the sum of) eight items ( $\alpha = 0.749$ ).

### Discussion

Using experience sampling methods to create and pilot an assessment of mathematical engagement, we were able to meet our two most important research goals: 1) to design an instrument practical that could be administered in under 5 minutes, targeting tasks that have the potential to be interesting and engaging for students, and 2) to maintain multidimensionality across each of the 4 dimensions of engagement: Affective (emotion checklist, interest), behavioral (Effort), cognitive (Efficacy, Instrumentality), and social (Social Engagement) dimensions. It has long been known that Interest (personal and situational), Efficacy (task- and subject-specific Self-Efficacy), Effort and feelings of Instrumentality are highly related, and that each is implicated in the interpretation of one’s experience and the decisions one makes when engaging in challenging content like mathematics (Wiesel et al., 2019). Research subsequent to the development of this instrument has shown that these dimensions of task-level engagement are related to teacher and peer support and are predictive of longer-term motivation in mathematics as well as achievement (Middleton et al., 2023).

In particular, one innovation we have achieved in this effort is the ability to assess the impact of emotion/object interactions. Previously, emotions have been assessed using similar checklists, but object, which is critical to the interpretation of experience and the learner’s response to emotional information, has not been studied in mathematics classrooms. The emotion checklist, because it pairs basic emotions with objects that are seen as causes of the emotion (e.g., being frustrated with the mathematics, or proud of oneself) allows for hypotheses about the ways in which emotional experiences are interpreted in academic tasks, and their differential impact on task-level motivation and behavior (see Middleton, et al., 2023). Our research has shown that differential patterns among the emotional objects show that students look to different cues in interpreting their experiences, wherein positive math emotions appear to increase interest and efficacy beliefs about the tasks, while negative emotions are associated with decreased interest and efficacy. Negative emotions about the mathematics appeared to be negatively associated with social engagement and feelings of instrumentality. Study of the impact of task-level emotions on engagement is in its infancy, but the work we are doing with this instrument is proving to be fruitful in this regard.

Such a short assessment of such a complex set of behaviors and attitudes has many limitations including lack of comprehensive coverage of the constructs that make up engagement, reliability of self-reports, and others. In addition, the transient nature of in-the- moment attitudes and emotions precludes using such an instrument as a diagnostic tool. Instead, we recommend a mixed-methods approach, wherein the actions that give rise to students’ interpretations of their experiences are captured, and conjectures can be tested *over time*, such that the overall pattern of students’ engagement may be recorded and coupled with potential causal factors.

The full instrument and administration guidelines may be obtained by writing the corresponding author.

### Acknowledgments

This material is based upon work supported by the National Science Foundation under Grant No. 1661180. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

### References

- Bentler, P. M. (1990). Comparative fit indexes in structural models. *Psychological Bulletin*, 107, 238–246.
- Browne, M. W., & Cudeck, R. (1993). Alternative ways of assessing model fit. In K. A. Bollen and J. S. Long (Eds.), *Testing structural equation models* (pp. 136-162). Newbury Park, CA: Sage.
- Byrne, B. M. (2005). Factor analytic models: Viewing the structure of an assessment instrument from three perspectives. *Journal of Personality Assessment*, 85(1), 17–32. doi: 10.1207/s15327752jpa8501\_02
- Csikszentmihaly, M., & Larson, R. (1987). Validity and reliability of the experience-sampling method. *Journal of Nervous & Mental Disease*, 175, 525-536.
- Fredricks, J., Blumenfeld, P., & Paris, A. (2004). School engagement: Potential of the concept, state of the evidence. *Review of Educational Research*, 74(1), 59-109.
- Goldin, G. A. (2002). Affect, meta-affect, and mathematical belief structures. In G. Leder, E. Pehkonen, & G. Torner (Eds.), *Beliefs: A hidden variable in mathematics education?* (pp. 59–72). Dordrecht, The Netherlands: Kluwer.
- Goldin, G. A. (2014). Perspectives on emotion in mathematical engagement, learning, and problem solving. In R. Pekrun & L. Linnenbrink-Garcia (Eds.), *Handbook of emotions in education* (pp. 391–414). New York, NY: Taylor & Francis.
- Middleton, J. A., Jansen, A., & Goldin, G. A. (2017). The complexities of mathematical engagement: Motivation, affect, and social interactions. In J. Cai (Ed.), *Compendium for Research in Mathematics Education* (pp. 87- 119). Reston, VA: National Council of Teachers of Mathematics.
- Middleton, J. A., Mangu, D., & Lee, A. (2019). A longitudinal study of mathematics and science motivation patterns for stem-intending high schoolers in the us. In M. Hannula, G. C. Leder, F. Morselli, M. Vollstedt, and Q. Zhang (Eds.), *Affect And Mathematics Education*. New York: Springer.
- Middleton, J. A., Wiezel, A., Jansen, A., & Smith, E. P. (2023). Tracing mathematics engagement in the first year of high school: relationships between prior experience, observed support, and task-level emotion and motivation. *ZDM—Mathematics Education*, 55(2), 427-445.
- Muthén, L. K., & Muthén, B. O. (2017). *Mplus*. Los Angeles: Muthén & Muthén.
- Rimm-Kaufman, S., Baroody, A., Larsen, R., Curby, T., & Abry, T. (2015). To what extent do teacher–student interaction quality and student gender contribute to fifth graders’ engagement in mathematics learning? *Journal of Educational Psychology*, 107(1), 170-185.
- Sheroff, D. J. (2013). *Measuring student engagement in high school classrooms and what we have learned. Optimal Learning Environments to Promote Student Engagement*. New York, NY: Springer New York.
- Sheroff, D. J., Csikszentmihalyi, M., Schneider, B., & Sheroff, E. S. (2003). Student engagement in high school classrooms from the perspective of flow theory. *School Psychology Quarterly*, 18(2), 158-176.
- Shiffman, S., Stone, A.A., & Hufford, M.R. (2008). Ecological momentary assessment. *Annual Review of Clinical Psychology*, 4, 1-32.
- Steiger, J. H., & Lind, J. C. (1980, May). Statistically-based tests for the number of common factors. Paper presented at the annual meeting of the Psychometric Society, Iowa City, IA.
- Tucker, L. R., & Lewis, C. (1973). A reliability coefficient for maximum likelihood factor analysis. *Psychometrika*, 38, 1–10.
- Wang, M.-T., Fredricks, J. A., Ye, F., Hofkens, T. L., & Linn, J. S. (2016). The math and science engagement scales: Scale development, validation, and psychometric properties. *Learning and Instruction*, 43, 16–26.
- Wiezel, A., Middleton, J. A., Zhang, Z. V., Grimm, K., & Jansen, A. (2019). Interest and Emotion Predictors of Motivation in Secondary Mathematics Classrooms. In *Proceedings of the 43rd annual meeting of the International Group for the Psychology of Mathematics Education*. Pretoria, South Africa: University of Pretoria.

# A STUDY OF WHAT STUDENTS FOCUS ON AND NOTICE ABOUT QUADRATIC FUNCTIONS REPRESENTATIONS DURING INSTRUCTION

Charles Hohensee  
University of Delaware  
hohensee@udel.edu

Sara Gartland  
National University of Ireland Galway  
sara.gartland@nuigalway.ie

Yue Ma  
University of Delaware  
yuemajoy@udel.edu

Srujana Acharya  
University of Delaware  
srujana@udel.edu

*Student focusing and noticing, which drive reasoning, are important but under researched aspects of student learning. Quadratic functions representations are perceptually and conceptually complex and thus, offer much for students to focus on and notice. Our study compared a teacher's goals for student focusing and noticing during quadratic functions instruction with what students actually focused on and noticed. Qualitative analysis revealed some alignment but also informative ways that the teacher's goals and student outcomes for focusing and noticing were misaligned. These results will further the field's understanding of how students learn about quadratic functions and may have implications for student focusing and noticing of other mathematics topics as well.*

Keywords: Algebra and Algebraic Thinking, Cognition, Learning Theory.

This study is about what students focus on and notice during quadratic functions instruction. The topic of student focusing and noticing is a vitally important consideration for the field of mathematics education because (a) focusing and noticing are crucial processes for any and all reasoning (i.e., what remains unnoticed cannot be reasoned with or about; Bransford & Schwartz, 1999), and (b) what students focus on and notice might not always align with what their teachers want students to focus on and notice (Mason, 1999). Although teacher noticing has become a thriving area of research among teacher education researchers, student focusing and noticing has yet to garner the same kind of attention among student learning researchers. Our research sets out to help address that imbalance. Foremost, our research sets out to produce insights about how students learn about quadratic functions. However, we also view our findings as having potential relevance for the learning of other topics as well.

## **Theoretical Orientation Towards Student Noticing and the Learning of Quadratic Functions**

In this section, we first present our theoretical orientation toward student focusing and noticing. We then present our theoretical orientation toward the learning of quadratic functions.

### **Student Focusing and Noticing**

We conceive of student focusing and noticing as the last two parts of a three-part process involving *attending*, *focusing*, and *noticing*. All three are needed for student noticing to occur (i.e., *attending* and *focusing* are precursors to noticing). To conceptualize these processes, we turn to von Glasersfeld (1995). We conceptualize attending as “pick[ing] a chunk of experience, isolate[ing] it from what came before and from what follows, and treat[ing] it as a closed entity,” (p. 91). In other words, to attend to an object in a complex perceptual or conceptual context requires that the object be mentally isolated or foregrounded. While driving, for example, someone could foreground traffic signals against a background of other aspects of the

environment.

Next, we conceptualize focusing on an object as “the mind, then, ‘to posit it as object against itself’ . . . to re-present it,” (von Glasersfeld, 1995, p. 91). In other words, focusing on an object in a complex perceptual and/or conceptual field goes beyond simply attending to the object, and requires the creation of a mental record of the object, and the re-presentation of the object to oneself. For example, while driving, someone may focus on the license plate of the car in front of them by creating a mental record of the license plate and re-presenting that mental record to themselves in a way that goes beyond the way they simply attend to the traffic signals.

Finally, we conceptualize noticing as about “(‘establish[ing] regularities in the flow of experience” (von Glasersfeld, 1995, p. 144). In other words, noticing requires more than just focusing on an object in a complex perceptual and/or conceptual field; it requires mentally identifying particular regularities, features, properties, etc. of the object. For example, someone who is focusing on the licence plate in front of them while driving may notice that the plate is from out of state, that the numbers on the plate form a pattern, etc.

Although noticing can be viewed as the culmination of the three-part process, our study examined both student focusing *and* student noticing because what one focuses on does not predetermine what one notices (Goodwin, 1994). So, for instance, two students could focus on the same perceptual or conceptual object yet notice different features. We did not consider attending for this study because, of the three processes, attending is more difficult to track (e.g., can occur at a subconscious level), and in our view would not add significantly to what a study of focusing and noticing would reveal.

### **Quadratic Functions**

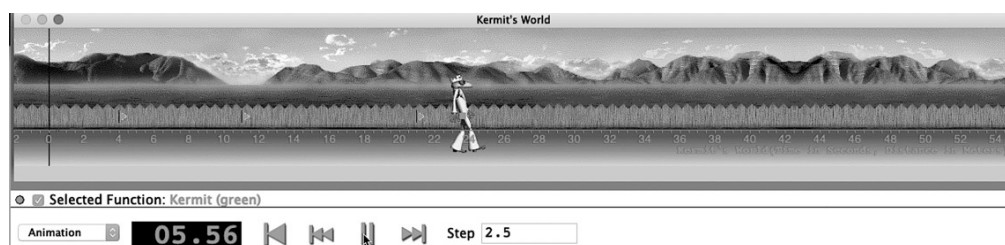
As Lobato et al. explained, quadratic functions represent a “complex mathematical domain” (2012, p. 85). We add that quadratic functions are complex, both perceptually and conceptually. Quadratic functions are a particularly important topic for which to study student focusing and noticing because there are many aspects of quadratic functions teachers could want students to focus on and notice and many aspects students could actually focus on and notice. An overarching goal teachers may have with respect to quadratic functions is to help students focus on and notice those features of quadratic functions that put students in position to understand what makes a function quadratic. This was the instructional goal for our study as well.

Although there are a number of ways to determine a function is quadratic (e.g., the distance from vertex of a parabola to a point on the parabola being equal to the shortest distance from the point on the parabola to a directrix of the parabola, second order function, etc.), the way quadratic functions were promoted in our study were as that “the rate of change of the rate of change . . . of a quadratic function is constant” (Cooney et al., 2010, p. 9). Thus, our study emphasized focusing on and noticing features of quadratic functions that put students in position to understand that functions with a constant rate of change of the rate of change are quadratic.

An instructional approach to teaching functions that aligns with the instructional goal described above is a *covariational reasoning approach*. This approach aligns with our goal because covariational reasoning is needed for understanding and reasoning about rates of change (Carlson et al., 2002), and hence also for understanding and reasoning about rates of change of rates of change. Confrey and Smith (1994) described a covariational reasoning approach as “being able to move operationally from  $y_m$  to  $y_{m+1}$  coordinating with movement from  $x_m$  to  $x_{m+1}$ ” (p. 137). Moreover, Confrey and Smith (1995) found that “creating tables of data can often serve as a point of entry for students” (p. 78) for covariational reasoning. Additionally, Carlson et al. explained that covariational reasoning can be fostered by creating dynamic situations.



For our instructional approach, we fostered covariational reasoning about quadratic functions by using data tables as the primary representation. Additionally, we created dynamic quadratic functions on SimCalc Mathworlds™ software. Specifically, we created distance-time quadratic function (DTQF) animations (see Figure 1). For those unfamiliar with SimCalc, this computer software can be programmed with a great variety of distance-time functions, including all manner of DTQFs. When played, animation characters walk across the screen according to the pre-programmed function and tools exist that enable students to collect precise measurements of the accumulating distance and time that they can then explore for patterns and relationships.



**Figure 1. SimCalc Mathworlds™ DTQF animation**

### Research Question

The research question that guided our study was the following: During quadratic functions instruction that emphasizes covariational reasoning with dynamic situations, how does what students focus on and notice about quadratic functions during instruction align with the teacher's goals for student focusing and noticing? As stated above, in addressing this question, we hoped to generate insights about the learning of quadratic functions, as well as more general insights that potentially could also apply to the learning of other mathematics topics.

## Methods

### Participants and Context

This study took place during a summer mathematics program for 9<sup>th</sup> and 10<sup>th</sup> grade students held at a university in the Mid-Atlantic region of the United States. Participants included 18 students ( $N = 18$ ), and one teacher, who was also the first author, and thus a researcher-participant. The remaining four members of the research team, consisted of the second author, another researcher, and two public high school mathematics teachers from the Mid-Atlantic region.

The summer program was designed to focus on quadratic functions and to promote students' covariational reasoning. During the two-week program, the students met each day for two 1-hour instructional sessions, separated by a short break. Each 1-hour instructional session typically involved a single instructional activity. Students manipulated and explored DTQF SimCalc animations on laptops in small groups during these instructional activities. Then, students engaged in whole-group discussions about the quantities and relationships associated with each DTQF.

### Data Collection and Analysis

Data were also collected during the summer program. In particular, the instructional sessions were recorded. These recordings were of whole-group and small-group activities, although only whole-group recordings were reported on for this paper. Specifically, recordings and the subsequent transcripts of these whole-group discussions were analyzed qualitatively.

As stated above, data analysis focused on the whole-group discussion that occurred at the end of each instructional activity (we called each discussion an episode). Analysis was used to develop inductive codes for the teacher's goals for student focusing and noticing and the student outcomes for focusing and noticing (Strauss & Corbin, 1998). Analytic memos were written to develop

themes. This report focuses on themes emerging from four early episodes, namely Episode 2B (E2B), E3A, E3B1, and E3B2.

### Results

We now present four themes that emerged during data analysis with respect to the teacher’s goals and students’ outcomes for focusing on and noticing features of DTQFs, a general theme and three quantity-specific themes. The general theme was about the association between the teacher’s goals and student outcomes for focusing and noticing and the amount of structure in the instructional activities. The three quantity-specific themes involved *distance* (3-part theme), *time* (2-part theme), and *speed* (2-part theme). Table 1 presents a comparison of teacher goals and student outcomes for focusing on DTQFs. Table 2 presents a comparison of teacher goals and student outcomes for noticing features of DTQFs. We say more about the tables below.

**Table 1: Comparison of Teacher Goals and Student Outcomes for Focusing on DTQFs**

Episode	Teacher Goals for Student Focusing	Student Focusing Outcomes
E2B	Focus on individual pairs of accum. time & distance	Focus on individual pairs of accum. time & distance, <b>speed*</b>
E3A	Focus on multiple accum. times	Focus on multiple accum. times, <b>total time*</b> , <b>speed*</b>
E3B1	Focus on multiple accum. distances	Focus on multiple accum. distances, <b>total distance*</b> , <b>displacement*</b> , <b>speed*</b>
E3B2	Focus simultaneously on multiple accum. times & distances	Focus simultaneously on multiple accum. times & distances

---

\*Aspects of teachers’ goals and student outcomes for focusing and noticing not aligned.

**Table 2: Comparison of Teacher Goals and Student Outcomes for Noticing DTQFs**

Episode	Teacher Goals for Student Noticing	Student Noticing Outcomes
E2B	Noticing features of accum. time and distance pairs that indicate which DTQF is faster	Noticing features of accum. time and distance pairs that indicate which DTQF is faster, <b>speed constant*</b> , <b>speed changing*</b>
E3A	Noticing accum. times at multiple notable points in DTQF, <b>how accum. time changing*</b>	Noticing accum. times at multiple notable points in DTQF, <b>total time*</b> , <b>speed constant*</b> , <b>speed changing*</b>
E3B1	Noticing accum. distances at multiple notable points in DTQF, <b>how accum. distance changing*</b>	Noticing accum. distances at multiple notable points in DTQF, <b>sign of the distances (pos/ neg)*</b> , <b>total distance*</b> , <b>displace. distance*</b> , <b>speed changing*</b>
E3B2	Noticing multiple associated accum. times and distances	Noticing multiple associated accum. times and distances

\*Aspects of teachers' goals and student outcomes for focusing and noticing not aligned.

### Association Between Focusing and Noticing and the Amount of Structure in the Instructional Activities

The first theme we present is about the association between how aligned the teacher's goals and students' outcomes for focusing on and noticing features of DTQFs were and the amount of structure in the instructional activities. This theme was more general than the other themes. What we specifically observed was that more structure in the DTQF instructional activities was associated with more alignment between the teacher's goals and students' outcomes for focusing and noticing. We explain the kinds of structure we observed and describe how more versus less structure was associated with alignment between the teacher's goals and student outcomes for focusing and noticing.

For the episodes reported on in this paper, there appeared to be two kinds of structure in the DTQF instructional activities that guided student focusing and noticing. One type of structure was when the instructional activity had a clearly defined goal. E2B had a clearly defined goal because it involved students in deciding which of two DTQF animations, which could not be played simultaneously, would win a race that went beyond the length of either animation (i.e., each animation ended before 150 m and students were to decide which animation would win a 150 m race). E3A, E3B1, and E3B2 did not have as clearly defined a goal.

The second type of structure was when the activity involved clear patterns. E3B2 involved focusing on and noticing patterns when students used patterns to focus on and notice associated accumulating distances and times. Some students focused on and noticed accumulating times associated with a pattern of accumulating distance (e.g., every 10 m). Other students focused on and noticed accumulating distances associated with a pattern of accumulating times (e.g., every 0.8 sec). E2B, E3A, and E3B1 did not involve clear patterns.

Comparisons between the teacher's goal column and the student outcome column for Table 1 and 2, indicate greater alignment between the teacher's goals for focusing on and noticing features of DTQFs in E2B and E3B2 than in E3A and E3B1. In E2B and E3B2, all of the teacher's goals for noticing were evident in the student outcomes but not in E3A and E3B1.

Moreover, in E2B and E3B1, fewer aspects of DTQFs not part of the teacher's goals were focused on and noticed by students (e.g., for E3A, total time and speed were aspects students focused on that were not part of the teacher's goals, whereas for E3B2 no aspects we observed students focus on were not part of the teacher's goals). Thus, our conclusion was that EB2 and E3B2, which involved activities that had more structure in the form of having a more clearly defined goal or that involved focusing on and noticing patterns, also involved greater alignment between the teacher's goals and the student outcomes for focusing on and noticing DTQFs.

### **Focusing On and Noticing Distance**

The second theme that emerged with respect to the teacher's goals and students' outcomes for focusing on and noticing features of DTQFs was quantity-specific and involved *distance* (i.e., the dependent variable of a DTQF). The theme about focusing on and noticing distance had three parts, (a) focusing on accumulating distances, (b) noticing that the accumulating distance changed direction, and (c) noticing the speed that accompanies the accumulating distance.

**Focusing on accumulating distance.** A teacher's goal in these episodes for student focusing in the context of DTQFs was for students to focus on the accumulating distance. Our use of the term accumulating is consistent with how it is used in Carlson et al. (2002), in particular that "the accumulating quantity can be imagined to be made of infinitesimal accruals in the quantities" (p. 165). Our analysis revealed that, although students' outcomes for focusing with respect to the accumulating distances sometimes matched the teacher's goals, there were two additional aspects of the distance of DTQFs not part of the teacher's goals on which students sometimes focused.

One aspect of distance of DTQFs on which students sometimes focused, in addition to the accumulating distance, was the *displacement distance*. For example, students were considering a DTQF animation in which a dog on the negative side of a distance scale, walked toward and into the positive side of the scale, and then turned around and walked toward and into the negative side. In the following excerpt, Kevin revealed that he focused on and noticed displacement distance: "the starting point for us was -9, and he ended up at -41 meters, -9 meters and he ended up at -41 meters. So, we thought the displacement was 32 meters."<sup>1</sup>

The second aspect of distance of DTQFs students sometimes focused on, in addition to the accumulating distance, was the *total distance*. For example, in the same DTQF animation referred to above, Rashana focused on the total distance as the animation went from -7 up to +44 and back down to -41: "136 [total distance] is the distance in the forest space . . . it's from -7, because that's when it started . . . all the way to 44. And then from 44 again, all the way to -41" (note that Kevin and Rashana noticed different starting distances). In sum, two aspects of distance, namely the displacement distance and the total distance, appeared to be competing with the accumulating distance for what students focused on.

**Noticing that the accumulating distance changed direction.** A teacher goal in these episodes for student noticing was for students to notice that the accumulating distance changed direction (i.e., that somewhere on a DTQF the accumulating distance grew and then shrunk or vice versa). Our analysis revealed that, although students' outcomes for noticing distances sometimes matched the teacher's goals, not all students appeared to notice that the accumulating distance changed direction and noticed another feature instead.

The aspect of the accumulating distance of DTQFs students sometimes noticed instead of noticing that the accumulating distance changed directions, was that the accumulating distance changed sign. To illustrate, consider the following exchange between the teacher and Yolanda, in which the teacher wanted students to notice the change in direction of the accumulating distance:

T: We're focusing specifically on how distance is changing. And someone from Group 3 is gonna come up and share with us their group was thinking about distance.

Y: We said the distance means amount of meters that Rover walked towards and moved away from the forest . . . So, like, you know the ruler thing? That would be like -5, then the distance from—is a negative, but then when it gets to like 1 or 2 which is positive.

T: One thing that I want to just connect with what you said. Did people notice that the distance quantity changed directions?

Y: I don't think so.

In this excerpt, a teacher goal was for students to notice the accumulating distance changed direction. However, Yolanda instead noticed that the accumulating distance changed sign. In this instance, the teacher's goal for student noticing and what they actually noticed did not align.

#### **Noticing the speed that accompanies focusing on and noticing accumulating distances.**

Another teacher goal for noticing was for students to notice aspects of distance and to ignore speed (i.e., to delay focusing on and noticing speed until future episodes). However, our analysis revealed that when students were supposed to focus on and notice accumulating distance, sometimes speed appeared to grab their attention. In other words, when they focused on and noticed distances, they often also reported noticing an aspect of speed. For example, as Yolanda explained, "we noticed that it ends at 40 meters, and it starts at 70 meters. The least amount of distance is -25 . . . it slows, or speeds up based on the negative and positive side, on the ruler." In this example, Yolanda linked what she noticed about the accumulating distances of the animation with how fast or slow the animation was going.

#### **Focusing On and Noticing Time**

The third theme that emerged with respect to the teacher's goals for focusing and noticing and students' outcomes for noticing and focusing was also quantity-specific and involved *time* (i.e., the independent variable of a DTQF). The theme about focusing and noticing time had two parts, (a) focusing on the accumulating time, and (b) noticing the speed that accompanies the accumulating time.

**Focusing on accumulating time.** A teacher goal in these episodes for focusing in the context of DTQFs was for students to focus on the *accumulating time*. Our analysis revealed that, although students' outcomes for focusing with respect to the accumulating time sometimes aligned with the teacher's goals, there was an additional aspect of the time of DTQFs not part of the teacher's goals on which students sometimes focused.

The aspect of the time of DTQFs on which students sometimes focused, in addition to focusing on accumulating time, was *total time*. For example, consider Damarcus's statement:

We noticed that time starts at a negative number . . . that it's from negative 4.5 to 8.7 . . . and we said that the total distance was 13.2, er what—not total distance, but total time is 13.2 seconds.

In this and other instances, whereas the teacher's goal for focusing was to focus on accumulating times, students sometimes also focused on total time. In other words, total time appeared to be competing with the accumulating time for what students focused on.

**Noticing the speed that accompanies accumulating times.** As stated above, a teacher goal for noticing was for students to ignore speed. However, our analysis revealed that sometimes speed appeared to grab students' attention, not only when students were to focus on and notice accumulating distance as described above, but also when they were to focus on and notice accumulating time.

In the following example, Natasha stated what her group noticed about accumulating times:

We noticed that the time continues as he turns around to go away from the forest . . . like when he turns around, he pauses, but the time still continues. He enters the forest, when it's zero . . . it seems like his speed is consistent, looking at it, we think he's like moving at a consistent pace, except when he turns around because he pauses. It starts at negative 4.5 seconds and ends at 8.7 seconds.

In sum, this excerpt further supports our claim that speed, more than time and distance, was an attention-grabbing quantity in a DTQF context.

### **Focusing On and Noticing Speed**

The fourth theme that emerged with respect to the teacher's goals for focusing and noticing and students' outcomes for noticing and focusing was also quantity-specific and involved *speed* (i.e., the rate of change of a DTQF). The theme about focusing and noticing speed had two parts, (a) not ignoring speed, and (b) noticing speed as constant.

**Not ignoring speed.** As stated above, the teacher's goal for these episodes was for students to ignore speed and instead focus on distance and time. However, as explained earlier, speed was not always ignored. As further illustrated in Tables 1 and 2, in three of the four episodes students focused on speed and noticed features of speed. In E2B, the teacher intended for students to notice features of accumulating distances and times for two characters. While presenting what his group noticed, Bryan explained, "And then ah, for the Frog, it travelled about 28 meters in 1 second, umm and that was at its top speed." In E3A, the teacher intended for students to notice features of time. During the whole-group discussion, Demarcus shared, "...so, ah we said the time was the distance times speed." In E3B1, the teacher intended for students to notice features of distance. While explaining what her group noticed, Yazmin said, "...and then, it slows or speeds up based on the negative and positive side, on the ruler."

**Noticing speed as constant.** A feature of speed that students sometimes noticed was that the speed was constant (i.e., the rate of change was constant). Of course, for all quadratic functions, the rate of change is always changing (i.e., not constant). However, that is not what students always noticed; sometimes students noticed constant speed. For example, as stated above Natasha noticed that "he's like moving at a consistent pace." Similarly, Bob noticed the following about one of the DTQF animations:

And then ah, for the Frog, it traveled about 28 meters in 1 second and that was at its top speed. So that was about a pretty constant pace. So, we just used that throughout the rest of the time when ah for the Frog went.

This was not always true because sometimes students noticed that the speed was changing, such as Halima who said, "Another thing that I noticed with the Clown is, each time it moves, it travels way greater than it did last frame." Thus, not only did speed grab students' attention, but what students noticed about speed in terms of whether it was constant or changing was not consistent. Next, we explain the significance of these findings and implications for research and practice.

### **Discussion**

We presented four themes emerging from our analysis that are specifically about students focusing on and noticing aspects of quadratic functions. Our results suggest that what students focus on and notice about DTQFs to put them in position to understand what makes DTQFs quadratic can be shaped by the amount of structure for focusing and noticing in the instructional

activities (e.g., clear goal, leveraging patterns). The results also suggest that the teacher goal that students should focus on and notice DTQFs' accumulating times, accumulating distances, or both, may also be accompanied by noticing other quantities (e.g., total time, displacement distance) or other features of quantities (e.g., constant speed). These results could be useful to practitioners who teach quadratic functions with a covariational approach and who use dynamic DTQFs and tabular representations.

In addition to these content-specific insights, we also have three hypotheses about how our themes may have applicability to focusing on and noticing of other mathematics topics. First, we hypothesize that in other content areas, some quantities attract student focusing and noticing more than others, just as speed did in our study. Second, we hypothesize that sometimes students may focus on the quantities their teachers want them to focus on, but not the variation of the quantities the teacher intended just as, for example, students in our study focused on total time instead of accumulating time. Third, we hypothesize that, in some cases, what students actually notice about aspects of mathematics content, may actually not be present just as, for example, students in our study noticed that the speed was constant when, in actuality, it always changing. Of course, these are exploratory hypotheses that will require testing in future studies. We hope this report will serve as a catalyst for new research on student focusing and noticing.

### References

- Bransford, J. D., & Schwartz, D. L. (1999). Rethinking transfer: A simple proposal with multiple implications. In A. Iran-Nejad & P. D. Pearson (Eds.), *Review of research in education* (Vol. 24, pp. 61–100). Washington, DC: American Educational Research Association.
- Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education*, 33(5), 352–378. <https://doi.org/10.2307/4149958>
- Confrey, J., & Smith, E. (1994). Exponential functions, rates of change, and the multiplicative unit. *Educational Studies in Mathematics*, 26(2/3), 135–164. <https://doi.org/10.1007/BF01273661>
- Confrey, J., & Smith, E. (1995). Splitting, covariation, and their role in the development of exponential functions. *Journal for Research in Mathematics Education*, 26(1), 66–88. <https://doi.org/10.5951/jresmetheduc.26.1.0066>
- Cooney, T. J., Beckmann, S., & Lloyd, G. M. (2010). *Developing essential understanding of functions for teaching mathematics in grades 9–12*. Reston, VA: National Council of Teachers of Mathematics.
- Goodwin, C. (1994). Professional vision. *American Anthropologist*, 96(3), 606–633. <http://www.jstor.org/stable/682303>
- Lobato, J., Hohensee, C., Rhodehamel, B., & Diamond, J. (2012). Using student reasoning to inform the development of conceptual learning goals: The case of quadratic functions. *Mathematical Thinking and Learning*, 14(2), 85–119. <http://dx.doi.org/10.1080/10986065.2012.656362>
- Mason, J. (1998). Enabling teachers to be real teachers: Necessary levels of awareness and structure of attention. *Journal of Mathematics Teacher Education*, 1(3) 243–267. <https://doi.org/10.1023/A:1009973717476>
- Strauss, A., & Corbin, J. (1998). *Basics of qualitative research: Techniques and procedures for developing grounded theory*. Thousand Oaks, CA: Sage.
- von Glasersfeld, E. (1995). *Radical constructivism: A way of knowing and learning*. Bristol, PA: Falmer Press.

# BARRIERS TO PERCEIVED USEFULNESS OF MATHEMATICS AMONG MIDDLE SCHOOL STUDENTS

Tracy E. Dobie  
University of Utah  
tracy.dobie@utah.edu

Keywords: Affect, Emotion, Beliefs, and Attitudes; Equity, Inclusion, and Diversity; Middle School Education; Utility Value

## Objectives & Perspectives

Two focal questions for this year's conference are how we can engage all students by building on their interests and motivation, and what research agendas we should pursue to achieve that goal. One promising avenue is working to enhance students' perceived *utility value*, or the relevance of mathematics tasks for their current and future goals, or other aspects of their lives (Eccles & Wigfield, 2002). Perceiving a subject as useful can have numerous positive benefits including enhanced interest in a subject and improved course performance (Hulleman et al., 2010; Hulleman & Harackiewicz, 2009). However, as Dobie (2019) highlighted, existing studies have primarily represented the perspectives of White, middle-class college students. As middle school can be a time of declining attitudes towards mathematics (Midgley et al., 1989), it is important to understand what barriers might negatively affect early adolescents' perceived usefulness of mathematics. Additionally, students from a range of cultural contexts should be represented given the important role of culture in learning and development (Lee, Spencer, & Harpalani, 2003; Rogoff, Moore, Correa-Chávez, & Dexter, 2014). Thus, needed is research that elevates the voices of a diverse group of middle school students, adding both developmental and systems-level lenses to understand the challenges they face. This study asks, *What potential barriers to perceiving mathematics as useful do early adolescents experience?* Knowledge of those barriers will then be used to suggest research agendas that have the potential to enhance adolescents' perceptions of usefulness and engagement with mathematics.

## Response to Issue

This study draws on data from interviews with 39 11-14-year-old adolescents with a range of ethnicities, gender identities, and levels of perceived competence. Interviews involved open-ended questions and card-sorting tasks (Dobie, 2019) designed to understand students' views of the usefulness of mathematics. Interviews were transcribed, and through a process of open coding (Strauss, 1987), four themes emerged that indicated potential barriers to perceived utility: viewing mathematics as useful in ways that are relevant for others but not oneself; contrived examples of math's utility; narrow definitions of mathematics; and trusting the messages of significant others without question. This poster unpacks and provides examples of those themes, drawing on research that includes but is not limited to the important influence of significant others in early adolescence (Clark-Lempers et al., 1991), authenticity and math word problems (Palm, 2008), everyday and school mathematics (Carragher & Schliemann, 2002), possible selves (Markus & Nurius, 1986), conceptions of mathematics in society (D'Ambrosio, 1990), and racial capitalism and STEM education (Morales-Doyle & Gutstein, 2019). Emphasis is placed on highlighting how existing systems within mathematics education and society contribute to the barriers faced by adolescents. I conclude by suggesting directions for future research to better understand both the potential impact of each phenomenon and ways to overcome these barriers.



## References

- Carraher, D. W., & Schliemann, A. D. (2002). Chapter 8: is everyday mathematics truly relevant to mathematics education?. *Journal for Research in Mathematics Education. Monograph*, 131-153.
- Clark-Lempers, D. S., Lempers, J. D., & Ho, C. (1991). Early, middle, and late adolescents' perceptions of their relationships with significant others. *Journal of Adolescent Research*, 6(3), 296-315.
- D'Ambrosio, U. (1990). The role of mathematics education in building a democratic and just society. *For the learning of mathematics*, 10(3), 20-23.
- Dobie, T. E. (2019). Expanding conceptions of utility: middle school students' perspectives on the usefulness of mathematics. *Mathematical Thinking and Learning*, 21(1), 28-53.
- Eccles, J. S., & Wigfield, A. (2002). Motivational beliefs, values, and goals. *Annual Review of Psychology*, 53(1), 109–132.
- Hulleman, C. S., Godes, O., Hendricks, B. L., & Harackiewicz, J. M. (2010). Enhancing interest and performance with a utility value intervention. *Journal of Educational Psychology*, 102(4), 880.
- Hulleman, C. S., & Harackiewicz, J. M. (2009). Promoting Interest and Performance in High School Science Classes. *Science*, 326(5958), 1410–1412. <https://doi.org/10.1126/science.1177067>
- Lee, C. D., Spencer, M. B., & Harpalani, V. (2003). “Every shut eye ain’t sleep”: Studying how people live culturally. *Educational Researcher*, 32(5), 6-13.
- Markus, H., & Nurius, P. (1986). Possible selves. *American Psychologist*, 41(9), 954.
- Midgley, C., Feldlaufer, H., & Eccles, J. S. (1989). Student/teacher relations and attitudes toward mathematics before and after the transition to junior high school. *Child Development*, 981–992.
- Morales-Doyle, D., & Gutstein, E. R. (2019). Racial capitalism and STEM education in Chicago Public Schools. *Race Ethnicity and Education*, 22(4), 525-544.
- Rogoff, B., Moore, L., Najafi, B., Dexter, A., Correa-Chávez, M., & Solís, J. (2007). Children's development of cultural repertoires through participation in everyday routines and practices.
- Strauss, A. L. (1987). *Qualitative Analysis for Social Scientists*. Cambridge University Press.

## BEYOND THE STATISTICS: JOY IN MATHEMATICS

Nicole M. Wessman-Enzinger  
George Fox University  
nenzinger@georgefox.edu

Laura Bofferding  
Purdue University  
lbofferd@purdue.edu

*In this paper, we share about specific tenets of joy in mathematics and conditions for supporting joy. A pair of students, Julia and Javier, young Latinx mathematicians in Grades 3 and 4 respectively, participated in a study that involved addition and subtraction with negative integers within game play. Drawing on descriptions of joy and supporting joy, we describe specific instances of joy from Julia and Javier and the conditions that supported their joy. Julia and Javier's joy with mathematics above grade level provides a story of empowerment that counters a narrative that they are in an afterschool program where students are not meeting grade level expectations. Further, this qualitative analysis provides explicit instances of joy in mathematics and the conditions supported in a structured mathematical game, which is useful to both teachers and researchers.*

Keywords: Affect, Emotion, Beliefs, and Attitudes; Equity, Inclusion, and Diversity; Number Conceptions and Operations

For young children, especially within early childhood education, there is strong support for play-based instruction (Parks, 2015) and advocating for creating joyful mathematics (Parks, 2020). Yet, the importance of play and joy are not exclusive to young children or only social encounters. Students of all ages can engage in mathematics in playful ways (Featherstone, 2000; Wessman-Enzinger, 2018). Parks (2020) advocates that strategies such as supporting play, choice, exploration, social interaction, engagement with materials, relaxing on time, and supporting caring relationships can facilitate joy in mathematics. These strategies for supporting joy in a PreK-2 classroom offered by Parks seem to naturally extend to all mathematical classrooms at all grade levels. Joyful mathematics offers a way of engaging in mathematics that counters the challenges that many students face with mathematics. For example, mathematics instruction focused on conceptual understanding and depth of thinking as opposed to correct solutions and speed moves toward joyfulness (Parks, 2020). Yet, including all of the strategies described above within instruction may prove daunting to teachers (and researcher-teachers) who feel the pressure of getting through particular material.

Joy is also a space of empowerment (Lama & Tutu, 2016). In thinking about joy, we acknowledge that mathematics instruction has historically left some students feeling joyless and excluded (e.g., Battey & Leyva, 2016; Martin, 2019; Joseph et al., 2019). Latinx students are a group traditionally underrepresented in STEM disciplines (Bensimon et al., 2019; Rodriguez & Blaney, 2021). Therefore, by highlighting Latinx students' joy, while they do mathematics far above grade level expectations, we tell a counter narrative that is asset-based rather than deficit-based. Jones and Gomez Marchant (2021) highlight the importance of de-centering Anglos and centering Latinx stories in asset-based ways. Further, this study broadens the work on joy in mathematics to older students and provides a case that illustrates how to include the strategies for incorporating joy, even in what could be interpreted as a structured learning opportunity.

### **Conceptual Framework: Defining Joy in Mathematics**

Joy is often equated with play. Play, according to Huizinga (1955), is voluntary, creates a separate world, creates order, involves tension, and has rules (Featherstone, 2000). Further

definitions even distinguish between mathematical play and play with mathematics (Ginsburg, 2006). For example, mathematical play might involve playing with counting for the sake of counting; whereas play with mathematics involves using mathematics that is taught as part of play. Play (whether mathematical or not) may be joyful, but situations that are not voluntary (or outside the scope of play) might also be joyful. But what is joy in mathematics? Parks (2020) describes joy as going beyond fun. Parks also describes joy as wonder, excitement, pride, and pleasure for practitioners. Other frameworks (see Cottrell, 2016), outside mathematics, describe joy as having temporality (i.e., it is brief and temporary), intentionality (i.e., it is spontaneous and voluntary), awareness (i.e., it involves sudden sense of insight), connection with others, and physical attributes (e.g., clapping, laughing). Parks (2020) suggests a few strategies for supporting joy, such as providing space for play, allowing for choice, social interaction, and caring, and relaxing time constraints. Exploring joy in mathematics has primarily been discussed by Parks with early education students. Understanding experiences and what joy is in mathematics for older students will be helpful for supporting joy in mathematics at all levels.

Based on definitions of play (e.g., Burghardt, 2011; Ginsburg, 2006), most elementary school math might not be considered play because the activities are generally not voluntary and are structured. In this research report, we investigated attributes of joy that emerged when doing mathematics in a structured activity during an afterschool program. In particular, we investigated the following research questions: In what ways does joy emerge (if any) in a structured mathematics game about negative integers? How can we characterize joy in mathematics? What conditions support joy?

## Methods

### Context and Data Collection

One of the researchers worked with students in an afterschool program at Redwood Elementary in the PNW. At Redwood Elementary, 91% of the students are Hispanic and 15% of the students are at grade level expectations for mathematics according to the state. Students in grades 3 and 4 worked for 20 minutes a day in their afterschool program on mathematics. The researcher, conducting a study on children’s formal and invented notation about negative integers, worked with the students individually or in pairs using mathematical games (See Figure 1 for research design). The purpose of the study began in supporting thinking and learning about negative integers. It is within that context that the researchers began thinking about joyful experiences in mathematics, inspired by the students during the structured game play.

We recruited 10 participants from an afterschool program at an elementary school in the Pacific Northwest for students in grades 3 and 4. We considered students from grades 3 and 4 as ideal participants because they have not yet been formally introduced to the concept of negative integers and the representation for negative integers on a number line. On the other hand, Grade 3 and 4 students have already been introduced to and are familiar with the addition and subtraction of *positive* integers (whole numbers), which means that they would be able to leverage and extend their prior knowledge to negative numbers.



Figure 1: Research Design

In this paper, we use case study methodology (Merriam, 1998) and highlight one of the pairs from group 1 (the mathematicians' group). The students in this group played structured mathematics games that provided negative integers with traditional notation, while the other group used invented notation for negative integers. We have intentionally selected this group for two reasons. First, after this particular session, the first researcher felt "joy" and immediately reached out to the second researcher to discuss the moments of "joy." Because of this, it seemed like an apt set of data to first start to qualitatively explore. Second, we know that mathematical play, which should be inherently joyful, is not necessarily present in the context of a game with rules, provided in a top-down way. This made us wonder how this particular context could be joyful, and it felt like that describing spaces of joy in this particular context would be most beneficial to both researchers and teachers.

### Structured Game Play

The students in the session that we selected for this study played the Draw or Discard game (Bofferding & Wessman-Enzinger, 2015; Wessman-Enzinger & Bofferding, 2014) and used traditional notation for negatives, which we called mathematicians' notation. The objective of this game is to have the most points at the end of the game using the cards in Figure 2.

0	1	2	3	4	-1	-2	-3	-4
5	6	7	8	-5	-6	-7	-8	

**Figure 2: Cards in Draw and Discard**

Each player starts with two cards and the players use a pile of cards in the center of the playing space. Players must begin by deciding how many rounds of turns they want to play. On each player's turn, the player must turn over the top card in the center deck so that everyone can see it. Then the player must decide to either draw or discard a card. If the player chooses to draw the card from the center pile, they will add it to their hand. If the player does not want to draw the face up card, they must discard a card from their hand. The next player repeats the same steps, and play continues until the previously determined number of rounds have been completed. A player can have as many or as few cards in their hand as they wish, and the sum of integers in a player's hand represents the total points.

### Data Analysis

We began qualitatively analyzing how students engaged with negative numbers and joy by first reading the transcript separately and taking notes. After individual reflections on the video and transcript, we met together and first conducted a form of qualitative analysis called thematic analysis (Braun & Clarke, 2020). After noting themes, we saw ways that these themes aligned with moments of joy we saw in Parks (2020) and also delineated by Cottrell (2016).

### Results

We structure and discuss the themes on how the students experienced joy chronologically. We specifically chose this to highlight how the students' experienced joy throughout the game, which looked different towards the end of the game compared to the beginning of the game.

#### Beginning: Explaining the Rules of the Game

The researcher first shared: "When we play our game, it will be you two on one team against me." Javier and Julia laughed and smiled about this. Both made silly faces and said, "Ok, ok, ok..." with excitement about playing against the researcher. The researcher then explained the

rules of the game. First, the researcher engaged the students in discussion around what the words “draw” and “discard” mean, modeling with the cards. Then, she modeled the rules of the game, employing pedagogical strategies for supporting joy, such as care and social interaction, by asking the students, “Have you ever made a game?” She continued by saying, “We can show each other what our cards are, because it’s a strategy game.”

**Table 1: Tenets of Joy, Joyful Experiences, and Conditions of Joy for Beginning**

Tenets of Joy	Joyful Experiences of Students	Conditions for Supporting Joy
Excitement	Joking	Students playing collaboratively <i>against</i> the teacher
Physical Attributes	Smiling, laughing, joking	Sharing the cards that were used with each other

**Learning: Modeling the Game**

The students began by drawing their first two cards: -4 and -3. The transcript excerpt below is Javier and Julia’s first interaction with negative integers and these initial cards.

Javier: (Makes a joke.) It’s a draw. It’s a draw.  
 Researcher: Why are you saying it’s a draw?  
 Javier: Because it’s seven.

In this excerpt we see Javier and Julia laughing together when they draw -3 and -4, perhaps because they are seeing negative integers for the first time. Javier makes a joke to draw new cards again. There is spontaneity in Javier’s joke and their first attempt at considering  $-3 + -4$ .

Researcher: You think it’s seven?  
 Julia: Wait... Wait...  
 Researcher: What do you think it is?  
 Julia: Three minus... Well, it would be zero.  
 Javier: One minus.  
 Julia: No it would be one.  
 Javier: One minus.  
 Julia: (Nods yes.) One minus.

Javier and Julia’s initial idea for  $-3 + -4$  is “one minus.” The researcher did not comment, instead saying, “How about you help me with my cards (shows cards 0 and -7)? What’s my score? I’ve got 0 and what do you think this is called?” Julia started saying, “Seven minus (referring to -7), and the researcher suggested they could call it negative seven. Then both Julia and Javier said her score would be “Zero!” Instead of correcting them, the researcher created space for student thinking by saying, “I’m going to make an analogy for you. If I have three plus zero what is it?”

Julia: Three plus zero... three.  
 Researcher: Ok. If I have five plus zero, what is it?  
 Javier: Five.

Julia: Oh. So that would be negative seven.

The researcher used an analogy with the students, comparing “0 + 3” and “0 + 5” to her cards, “0 + -7,” helping them make sense of her hand before returning to their hand. She asked students to think about what negative three and negative four points are.

Javier: One minus.

Julia: (Nods.) One minus.

Researcher: Ok. Let’s make an analogy here together. If I had three points and I got four points, how many would I have?

Julia/Javier: Seven.

Researcher: So if I had negative three points and negative four points, how much would I have?

Javier: Seven.

Julia: Negative seven.

Javier: (makes a joke) We draw...

Researcher: Can you explain?

Javier: It’s like, it’s like, it’s just like plus.

Researcher: So you have -4 points and -3 points, so how many points do you have?

Julia: Oh, I get it. It’s like you added.

Javier: But minus it.

Through the help of another analogy and her partner explaining, “I’d just like plus,” Julia stated “oh I get it” with increased volume, a big smile, and a sense of new insight.

After the students determined the sum of their initial cards, they encountered the first move of the game where they needed to decide if they were going to draw the -5 card that was flipped over or discard one of their cards. Although discarding the -4 would be the strategic move towards getting more positive points, Julia and Javier decided to add -5 (i.e., draw the card) to their sum. The researcher did not comment and supported the students in this choice. Julia and Javier worked collaboratively and spoke in Spanish about their new total. They verbally stated that the sum of -4 + -3 + -5 was “doce” (or +12) and wrote -12 on their recording sheet. The researcher affirmed their language and solution, by nodding at -12 and also calling it “doce.”

**Table 2: Tenets of Joy, Joyful Experiences, and Conditions of Joy for Modeling the Game**

<b>Tenets of Joy</b>	<b>Joyful Experiences of Students</b>	<b>Conditions for Supporting Joy</b>
Physical Attributes	The students laughed at the negative integers and made jokes about re-drawing new cards.	Giving students unfamiliar cards with negative integers on them.
Intentionality and Connection	The students collaborated on what -3 + -4 equals. They both adjusted their thinking.	The researcher supported brief exploration in thinking before moving onto the next card draw.

Awareness	The students made the connection that $-3 + -4 = -7$ .	The researcher modeled making analogies.
Connection	The students made choices, such as what card they wanted to draw or discard and used language to use.	The researcher supported student choices (“one negative” for -1; Spanish “doce” for -12).

### Playing: Finding Flow in the Game

Next, with the cards -3, -4, and -5, Javier and Julia flipped +6 in the center pile. They needed to decide if they would draw the +6 or discard one of their cards. As the students made a choice about what to do next they smiled and whispered to each other (truly playing “against” the researcher now). They decided to discard the -3, which would leave them with -4 and -5. The students then needed to decide how many points they had left, the sum of -4 and -5. They worked collaboratively and negotiated together that their sum, or total points, was -9.

On their next move, they flipped over a +3 in the center deck with a hand of -4 and -5. Javier spontaneously started clapping, saying, “Take it, take it, take it.” Although Javier and Julia both now agreed that they should take the positive card, +3, they had different ideas about what  $3 + -4 + -5$  are. Javier thought  $3 + -4 + -5 = -6$  and Julia thought that  $3 + -4 + -5 = -12$ . The transcript below highlights Javier’s argument within the game. Figure 3 highlights their physical smiles and laughing as they argued about the sum of  $3 + -4 + -5$ .



**Figure 3: Javier and Julia discuss different sums for  $3 + -4 + -5$**

- Javier: Minus six. Minus six.  
 Researcher: Instead of saying minus six, try explaining it to her.  
 Javier: These two bro (pointing at the negatives).  
 Researcher: Ok. How about this...  
 Javier: These two... these two ... (pointing at cards) these two bro.  
 Researcher: Ok, what two?  
 Javier: These two. If you minus these two (points at -5 and -4) to three then that’s six minus. You are plussing. You are trying to plus to get higher.

Through discussion, the researcher helped the students remember that  $-3 + -4 = -7$ . Then, she asked about  $-3 + 4$ .

- Javier: If you minus these two it’s one.  
 Researcher: Yes, four minus three is one. But, you are saying one minus? Can you explain that?  
 Javier: Because this is lower than the four (points at the 3).  
 Researcher: What’s lower than the four?

Javier: The three.

Eventually Julia stated, “Let’s put down negative six. ... I’m convinced.”

Next, Javier and Julia, flipped over a 0 in the center pile with a hand of -5, -4, and 3. They collectively agreed to discard their -5 and debated the sum of 3 and -4. Julia thought the sum of 3 and -4 was 1 because  $4 - 3 = 1$  Javier argued the sum of “one minus.” And Julia changed her mind to “one minus.” Then, Javier and Julia decided to draw a card, +3, and add it to their hand. They worked together to determine the sum of  $3 + 0 + -4 + 3$ . As they decided this, they exhibited excitement. Notably, Julia started obtaining the correct solutions for the sums of negatives now.

Researcher: (Spreads cards out on the table for the students to look at.) Let’s see here, we have three, zero, three, and negative four.

Julia: So this would be six (combines the two 3 and 3 cards together).

Javier: (moves cards 3 and -4 together)

Julia: (takes the cards back) Six minus four is two. Two points.

Javier: Yeah, two. Yeah, let’s catch up! Let’s go!

The students and the researcher continued taking turns. The students, without questions or support from the researcher, decided which cards to discard or draw. The students had emotional reactions to the researcher’s choices. When the researcher made a move to draw or discard a card, they laughed, and Javier would cover his face with the hood of his sweatshirt.

**Table 3: Tenets of Joy, Joyful Experiences, and Conditions of Joy for Playing**

Tenets of Joy	Joyful Experiences of Students	Conditions for Supporting Joy
Connection	They each stated their ideas and questioned each other (e.g., debated the sums of $3 + -4 + -5$ and then $3 + -4$ ).	The researcher supported the students’ choices in drawing or discarding, even if it veered from the most strategic move.
Awareness and Pride	The students began to solve problems on their own and showed enthusiasm about it.	The researcher encouraged them to look at their recording sheet and talk to each other.
Excitement	“Take it, take it, take it.” “Yeah, let’s catch up! Let’s go!”	The students played a game against the researcher.
Wonder	The students each had moments of curiosity about the sums of their cards.	The researcher encouraged them to share their thinking.

### Ending: Concluding the Game

As the game concluded, the researcher won the game. Javier put his shirt over his head as the researcher confirmed, “I did win.” Both students laughed and asked to play another round. The researcher said, “We do have ten more minutes,” which Javier took as a good sign, saying, “Yeah, yeah, yeah!” The researcher confirmed, “We can see who wins after 10 minutes.” The researcher



and the students began their second game, with the students playing against the researcher. On the researcher’s turn, she had a -8 in her draw pile with -4 and 5 in her hand.

Researcher: I’ve got a negative four and five. So negative four and five is one. And, I drew a negative eight. I’m going to...

Javier: Take it! (Keeps laughing hard, this is a joke.)

Researcher: I’m definitely not taking the negative eight.

Julia and Javier: (laughing)

The students made a math joke by telling the researcher to make a move that would not help her win. Adding -8 to the hand would make the total point score as low as possible. Julia and Javier continued teasing the researcher, based on their integer understandings, as they concluded their time playing the game. Javier and Julia continued teasing the researcher the rest of the game. Eventually, the researcher said: “So whatever you are saying is what I should not do...”

**Table 4: Tenets of Joy, Joyful Experiences, and Conditions of Joy for Ending**

<b>Tenets of Joy</b>	<b>Joyful Experiences of Students</b>	<b>Conditions for Supporting Joy</b>
Excitement	The students wanted to play another round of the game.	The researcher let them play again even with the brief time.
Awareness	The students made math jokes about integer values (indicating cleverness).	The students collaboratively played against the researcher.
Physical Attributes	The students cheered for a second game and laughed at their math joke.	The researcher asked for their opinion on what moves to make.

### **Implications**

Julia and Javier are young Latinx mathematicians in grades 3 and 4, respectively. They not only engaged with negative integer operations in intellectual and mathematical ways, but they also experienced joy. The most significant part of these results is highlighting how two young, Latinx mathematicians not only worked with negative integers (a topic far above grade level), but they also experienced joy in working with negative integers. Recommendations include operations with negative integers in middle school; yet, Julia and Javier cleverly did this mathematics as young mathematicians in Grades 3 and 4. Similarly, Julia and Javier, demonstrated many instances of joy within in mathematics. The context of this school matters: statistics paint these children in an afterschool program where only 15% of the students are at grade level. Yet, we meet Julia and Javier, joyfully engaging in advanced mathematics.

This joy is immersed in a structured study and a structured mathematical game, which may seem like a space where there is little opportunity for joy. Yet, the examples provided here included many different experiences with joy (e.g., wonder, excitement, connection, cleverness). The results here point to an inaugural avenue for thinking about how we can support joy in different pedagogical and research environments, such as structured mathematical game play. As we consider future implications, we think about the tenets and conditions of joy, such as collective

and collaborative. For example, the students played against the researcher, someone positioned in power. Julia and Javier shared their opinions on the game play and expressed themselves mathematically in both English and Spanish. We wonder about ways to modify games to be collaborative and about the potential of collaborative games. Relaxing, allowing choices, and encouraging the students are aspects to supporting joy that centers on caring.

### References

- Batthey, D., & Leyva, L. A. (2016). A framework for understanding whiteness in mathematics education. *Journal of Mathematics Education*, 9(2), 49–80.
- Bensimon, E. M., Dowd, A. C., Stanton-Salazaar, & Dávila, B. A. (2019). The role of institutional agents in providing institutional support to Latinx students in STEM. *The Review of Higher Education*, 42(4), 1689–1721.
- Bofferding, L., & Wessman, N. M. (2015). Solutions to the integers: Draw or discard game. *Teaching Children Mathematics*, 21(8), 460–463.
- Braun, V., & Clarke, V. (2020). Thematic analysis. In H. Cooper (Ed.), *APA Handbook of Research Methods in Psychology* (Vol. 2., pp. 57–71). American Psychological Association.
- Burghardt, G. M. (2011). Defining and recognizing play. In A. Pelligrini (Ed.), *The Oxford handbook of the development of play* (pp. 9–18). New York, NY: Oxford University Press.
- Cottrell, L. (2016). Joy and happiness: A simultaneous and evolutionary concept analysis. *Journal of Advanced Nursing*, 1506 – 1517. doi: 10.1111/jan.12980.
- Featherstone, H. (2000). “-Pat + Pat = 0”: Intellectual play in elementary mathematics. *For the Learning of Mathematics*, 20(2), 14–23.
- Ginsburg, H. P. (2006). Mathematical play and playful mathematics: A guide for early education. In D. Singer, R. Golinkoff, & K. Hirsh-Pasek (Eds.), *Play= Learning: How play motivates and enhances children’s cognitive and social-emotional growth* (pp. 145–165). New York, NY: Oxford University Press.
- Huizinga, J. (1955). *Homo ludens: A study of the play-element in cult*. Boston, MA: Beacon Press.
- Jones, S. R., & Gomez Marchant, C. N. (2021). Narratives of Raza in mathematics education research literature. In D. Olanoff, K. Johnson, & S. Spitzer (Eds.), *Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 134–142). Philadelphia, PA.
- Joseph, N., Frank, T. J., & Taqiyyah, Y. E. (2021). A call for a critical-historical framework in addressing the mathematical experiences of black teachers and students. *Journal for Research in Mathematics Education*, 52(4), 476–490.
- Lama, D., & Tutu, D. (2016). *The book of joy*. Penguin Random House.
- Martin, D. B. (2019). Equity, inclusion, and antiblackness in mathematics education. *Race Ethnicity and Education*, 22(4), 459–478. <https://doi.org/10.1080/13613324.2019.1592833>
- Merriam, S. B. (1998). *Qualitative research and case study applications in education*. San Francisco, CA: Jossey-Bass.
- Parks, A. N. (2020). Creating joy in PK-Grade 2 mathematics classrooms. *Mathematics Teacher: Learning & Teaching PK-12*, 61–64.
- Parks, A. N. (2015). *Exploring mathematics through play in the early childhood classroom*. Teachers’ College Press.
- Rodriguez, S., & Blaney, J. (2021). “We’re the unicorns of STEM”: Understanding how academic and social experiences influence sense of belonging for Latina undergraduate students. *Journal of Diversity in Higher Education*, 14(3), 441–455.
- Wessman-Enzinger, N. M., & Bofferding, L. (2014). *Integers: Draw or discard! game*. *Teaching Children Mathematics*, 20(8), 476–480.

## CLAIM MAKING AS A TOOL FOR ANALYZING STUDENT THINKING IN STEM CONTEXTS

David Slavit  
Washington State University  
Vancouver  
dslavit@wsu.edu

Amber Simpson  
Binghamton University  
asimpson@binghamton.edu

Kristin Lesseig  
Washington State University  
Vancouver  
kristin.lesseig@wsu.edu

*We articulate a framework for delineating student thinking in active, STEM-rich learning environments. Researchers have identified ways of reasoning that relate to specific content areas and practices within each of the STEM disciplines. However, attempts at characterizing student thinking in transdisciplinary STEM environments remains in its infancy, in need of theoretical and analytic frameworks to support this emerging research area (Li et al., 2019). This paper advances the field along both of these dimensions by articulating an analytic framework for student thinking in STEM contexts and operationalizing it through an analysis of two groups of students in elementary, informal learning situations. Our results suggest that student thinking, in these environments, is often grounded in personal experience and authority and consists of a rich mix of general and disciplinary-oriented reasoning. Implications are provided.*

Keywords: Integrated STEM, Cognition, Reasoning and Proof, Problem-Based Learning

### Perspectives

Is STEM a discipline in and of itself? What is integrated, multidisciplinary, or transdisciplinary STEM? While perspectives on the precise nature of STEM vary widely (Holmlund et al., 2018), we believe that conceiving of STEM as a stand-alone discipline of study can be problematic given the differences found in the epistemological orientations of the individual STEM disciplines (Kelley & Knowles, 2016; Ortiz-Revilla et al., 2020; Slavit et al., 2021). For example, mathematical truth is found by logical deduction based on a set of axioms, scientific truth is based on systematic observation and an analysis of available evidence, and engineering truth is often subjectively decided by the client or what constitutes a *best* design. Given this variety of epistemologies, conceptualizing STEM as a stand-alone discipline can be rather difficult; characterizing *STEM thinking* can be even more difficult.

However, we do believe that student thinking in STEM contexts can be analyzed in a systematic manner. Through the analysis of multiple learning situations, we have determined that a fruitful avenue of research lies in the identification and analysis of student claims made in STEM contexts. Specifically, we have developed an analytic framework designed to characterize the nature of the claims, evidence, and reasoning that surface in student comments and/or activity while engaged in active, group-based STEM learning activities. This approach allows us to draw from the plethora of existing reasoning types inside the individual STEM disciplines, but also provides an avenue to explore the kind of thinking that might be unique to STEM learning contexts. While we align ourselves most closely with transdisciplinary STEM due to its synergy with project-based learning environments (Vasquez, 2014), due to space limitations we refer broadly to “STEM environments” rather than attempt to make distinctions between integrated, multidisciplinary, and transdisciplinary perspectives on STEM.

### Disciplinary STEM Reasoning

Over the past decades, educational researchers from individual disciplines have identified specific kinds of reasoning related to thinking and problem solving in mathematics, science,

engineering, and technology. Hjelte et al. (2020) conducted a systematic review of the types of reasoning found in mathematics education research and identified spatial sense, quantitative reasoning, additive/multiplicative reasoning, proportional reasoning, and algebraic reasoning as common examples of student reasoning grounded in specific mathematical ideas and concepts. Hjelte et al. (2020) also mentioned informal inferential reasoning and transformational reasoning as commonly researched areas. In engineering education, Worsley and Bilkstein (2017) identified four types of reasoning observed in student solutions to design tasks. These were unexplained reasoning, materials-based reasoning, example-based reasoning, and principle-based reasoning. Worsley and Bilkstein (2017, p. 64) described the latter as “working forward from deep structural features.” In science education, Kind and Osborne (2017) explored six styles of reasoning, which they labeled mathematical deduction, experimental evaluation, hypothetical modeling, categorization and classification, probabilistic reasoning, and historical-based evolution. Kind and Osborne discussed the presence of these types of reasoning in science education, such as the predominance of hypothetical modeling and experimental evaluation in the traditional scientific method (Windschitl et al., 2008). However, while student thinking about specific scientific ideas has been researched, there is a general lack of content-specific frameworks for analyzing thinking in science education beyond the levels of Earth, physical, and life sciences (Roehrig et al., 2021) or key “cross-cutting concepts” (NGSS Lead States, 2013). For example, while proportional reasoning exists as a construct in mathematics education, constructs such as *velocity reasoning* or *organismic reasoning* are not found in the science education literature. Engineering education is also generally devoid of such constructs. Further, we believe that all four types of engineering-based reasoning described by Worsley and Bilkstein (2017) are compatible with the other three STEM disciplines and, in addition to their applications to the discipline of engineering, can be considered as more general types of STEM reasoning.

While not exhaustive, the above discussion of disciplinary reasoning suggests that there are multiple reasoning types inside the individual STEM disciplines which can support an analysis of STEM thinking. Further, some of these are specific to an aspect of a discipline, the overall discipline, or are broad enough in nature to generalize across the disciplines, such as example-based or discipline-based reasoning.

### **Claim Making as a Tool for Analyzing STEM Thinking**

Is STEM reasoning a combination of the above kinds of reasoning? Is it something that transcends this collection? Space does not allow an adequate response to these questions, nor for a thorough theoretical discussion of the precise nature of STEM. However, we now provide a discussion of how we approached the task of analyzing STEM thinking.

Drawing from Toulmin (1958) and a variety of perspectives on student claim making in science (McNeill & Krajcik, 2012), mathematics (Kazemi et al., 2021), engineering (Siverling et al., 2019), and technology, we approach our analysis of student thinking in STEM contexts by focusing on the nature of the student claims, evidence, and reasoning that surface during active, collaborative learning experiences. We view a claim as the onset of an argumentation process, and reasoning as a tool that can be used to support or challenge the validity of the claim (Gray & Kang, 2014; Lee et al. 2014). Because claims involve positioning, they elicit arguments that utilize evidence and reasoning in support of that position. Further, claims are often contextualized and therefore dependent on the perspectives of the claimer (Toulmin, 1958; Forman et al., 1998).

A claim can consist of a relatively short, concise statement made by an individual. However, claims can be more complex and consist of a set of statements and actions by an individual or collection of individuals. When such statements and actions are related in nature or substance, we

consider the collective a *claim sequence* and treat it as a single claim for coding purposes. Note that challenges and differing perspectives might exist inside a claim sequence.

Certain types of claims were excluded from our data sets. Claims not relevant to the current STEM-based activity were not recorded, nor were claims that were procedural in nature or did not add significantly to the disciplinary value of the student activity. For example, a statement such as “I think we need to reread the directions” was not identified as a claim. Restated or repeated claims were also not recorded. Because gestures and other non-verbal student activity can be an important part of the claim-making process, and even fully constitute a claim, we took care to incorporate descriptions of these non-verbal activities when identifying and describing claims.

### Methods

Drawing on a sociocultural perspective (Vygotsky, 1978; Cobb, 1992), this qualitative research study is grounded in naturalistic observations (Lincoln & Guba, 1985) of student activity. This paper reports on work from a larger project that analyzed over ten learning situations across Grades 1-12 in both formal and informal learning environments. In this paper we focus on two learning episodes, both of which involved groups of elementary students working a STEM-rich task in an informal learning environment.

### Settings and Participants

Two videos of small-group learning experiences were selected for analysis and comparison. The two videos had many similarities, including students at the elementary level, an informal and active learning environment, and the students working in a collaborative manner. However, key differences in the learning experiences included the nature of the STEM content emphasized and the nature of the materials provided to support the student activity. Further details of the two videos can be found in Table 1.

**Table 1: Student Learning Activity Descriptions**

Video	Brief Description	Setting	Participants	# of Claims	Length
Dash Coding	One group of students design a path and then program a robot (Dash) to follow a path created by another group.	Informal; Free time in TinkerLab in elementary school	5 <sup>th</sup> grade students; 3 boys and 1 girl	36	18 minutes
Roller Coaster Wall	One group of students used grooved rubber strips and other materials to build a “roller coaster” affixed to a wall.	Informal, after school; hosted by local museum	2 <sup>nd</sup> grade students; 3 boys	29	32 minutes

### Data Collection

Both episodes were videotaped using a GoPro camera attached to the chest of one of the student participants. This allowed for a unique view of the activity as well as high-quality sound. For both videos, two of the three authors independently identified the claims and claim sequences articulated by the student participants, and then transcribed the student comments into the first

column of an Excel spreadsheet, the top row of which contained the names of all codes (see below). As noted above, both the words and actions that constituted a claim were recorded. Therefore, our unit of analysis was a *claim* or *claim sequence*, as described above. Once the pair of authors had individually recorded all claims in a given video, they met to compare the claims they each identified. Claims were removed, added, and modified based on these discussions. The two videos and the accompanying set of transcribed claims represent the data used in this paper.

### **Data Analysis**

All claims were coded using an analytic framework we developed from the literature as well as our analysis of over ten videotaped learning activities. The framework includes a variety of individual codes related to the claims, evidence, and reasoning that surfaced during student activity. Further details of the development and nature of the codes have been provided elsewhere (Slavit et al., 2022), but the following is a summary description:

#### *Claim Codes:*

Explicit/Implicit – if the claim has been overtly communicated using words and/or symbols

Formal/Informal – if the claim uses academic language or is grounded in disciplinary practices

Tentative/Certain – if the claim is made with confidence

Novel/Challenge – if the claim introduces a new idea or perspective into the activity

Disciplinary – if the claim is grounded in math, science, engineering, or technology contexts

#### *Evidence Codes:*

Fact – use of a past-learned fact, example, or definition

Prior Experience – use of an event that occurred prior to the learning experience

Test – use of a recently-occurring event

Perception/Observation/Manipulation – use of in-the-moment event or “seeing in the moment”

None or unable to determine

#### *Reasoning Codes:*

Explicit/Implicit – if the reasoning has been overtly communicated using words and/or symbols

Experiential/Abstract – if the reasoning is based on observation/evidence or theory

Personal/External Authority – if the source of the reasoning comes from the claimer

Disciplinary – if the reasoning is grounded in math, science, engineering, or technology contexts

General - principle-based, test-based, experience-based, unexplained

Mathematically-grounded – spatial, numeric, algebraic

Scientifically-grounded – cause-effect, analogic

Engineering-grounded – constraints-based, materials-based

Technology-grounded – algorithmic

Other – as these are identified and described, new categorizations are continuously considered

The authors used both the transcriptions of the claims and the video to make coding decisions. First, one or both codes from each of the first four pairs of claim codes were applied, followed by as many disciplinary codes as were relevant to the nature of the claim. If the source of evidence was able to be determined, then one or more of the evidence codes were applied. Finally, to code the reasoning, one or both codes from each of the first three pairs of reasoning codes were applied, followed by as many disciplinary codes as were relevant to the nature of the reasoning. Then, any general (e.g., principle-based, test-based) or discipline-specific (e.g, spatial, cause-effect) reasoning codes were applied as relevant.

For both videos, the two authors who created the set of transcribed claims individually coded the set of claims for that video using the above procedures. Interrater agreement was calculated using exact coding comparisons, dividing the number of agreements by the number of coding decisions. Interrater agreement across the two videos was 85.8%. The pairs of authors met to resolve the discrepancies in the codes, producing the final results.

### Results

Tables 2 and 3 illustrate the results for both the Dash and Roller Coaster Wall videos, respectively, related to the nature of the claims, evidence, and reasoning operationalized by the student participants. Note that some coding pairs or sets do not always add to 100% due to instances of double coding. This usually occurred in an extended claim sequence involving multiple comments or actions by more than one individual. For example, one student might initiate a claim using formal language, and another might immediately provide a challenge to the claim using informal language or a contradictory action. This situation would produce double codes for both of these coding pairs.

**Table 2: CER Codes for Dash Learning Episode**

36 Claims or Claim Sequences								
Claim	Evidence			General Reasoning		Discipline-Oriented Reasoning		
Explicit	35	Fact	0	Explicit	11	Math	Spatial	22
Implicit	2	Prior Exp	2	Inferred	26	Math	Numeric	2
Formal	14	Test	11	Experiential	36	Math	Algebraic	0
Informal	27	Perc/Obs/Man	30	Abstract	0	Science	Cause-effect	3
Tentative	10	None or N/A	3	Pers Auth	35	Science	Analogic	0
Certain	31			Ext Auth	2	Engineering	Constraints	6
Novel	35			Science	0	Engineering	Materials	5
Challenge	12			Math	20	Technology	Algorithmic	2
Science	0			Engineering	24	Multiple Disciplines		19
Math	23			Technology	13			
Engineering	23			Principle	5			
Technology	17			Test	10			
				Experience	2			
				Unexplain	3			

**Table 3: CER Codes for Roller Coaster Building Learning Episode**

29 Claims or Claim Sequences								
Claim	Evidence			General Reasoning		Discipline-Oriented Reasoning		
Explicit	28	Fact	5	Explicit	8	Math	Spatial	12
Implicit	1	Prior Exp	1	Inferred	21	Math	Numeric	0
Formal	1	Test	13	Experiential	27	Math	Algebraic	0
Informal	28	Perc/Obs/Man	26	Abstract	5	Science	Cause-effect	14
Tentative	5	None or N/A	2	Pers Auth	28	Science	Analogic	1
Certain	26			Ext Auth	1	Engineering	Constraints	1
Novel	24			Science	8	Engineering	Materials	10
Challenge	6			Math	1	Technology	Algorithmic	0
Science	8			Engineering	27	Multiple Disciplines		9
Math	1			Technology	0			
Engineering	29			Principle	7			
Technology	0			Test	17			
				Experience	1			
				Unexplain	12			

Overall, several commonalities emerged in the nature of the claims, evidence, and reasoning across the two learning episodes. The nature of the claims made by the students were mostly explicit, informal, and certain. These data collectively indicate that the students felt confident in their own ideas but chose to express them using language and gestures outside of the disciplinary registers. We also see that the nature of the activity influenced the disciplinary nature of the claims. For example, all student claims in the Roller Coaster activity related to an aspect of the engineering activity of the student work, which was framed by the design and building of a structure. However, student claims in the Dash video relied more on multiple disciplinary influences, drawing heavily from mathematics, engineering, and technology.

The evidence used to support the claims came mostly from tests and observations. This suggests that student reasoning was mostly emergent, influenced by in-the-moment advances in the student activity. Finally, the reasoning used to support the claims was mostly inferred and came from personal authority as opposed to an external source. The reliance on personal authority is in line with the prior result regarding the influence of in-the-moment activity as a source of evidence. In other words, the role of personal authority in the claims likely emerged, in part, from the personally constructed nature of the evidence.

The large number of claims in the Dash video that drew on more than one discipline emerged from the synthesis of mathematics, engineering, and technology embedded in the learning experience. The students were continuously measuring angles, considering design elements of the path of a robot, and programming path directions. This led to more than 50% (19 of 36) of the claims incorporating more than one STEM discipline. A smaller percentage of claims (31.0%, 9 of 29) from the Roller Coaster video incorporated more than one STEM discipline, and these were mostly grounded in engineering and science concepts and practices. For example, at the very beginning of the Roller Coaster activity, the following claim sequence occurred:

Gary: Now we need to go up. [referring to the design of the roller coaster path]

Steve: No, we can't do it or else it won't go. [referring to the need for speed to continue an upward path]



Gary: We need to just make it straight.

Steve: That would make it straight down like a roller coaster. [added rubber strip to extend the roller coaster path down]

Gary: And then up just a little bit...Because I imagine it has a lot of energy in this.

Here we see a rich blend of design-based thinking and scientific properties related to motion in the claims being made throughout this sequence. We also see both novel and challenge claims, the use of both informal and informal language, and a reliance on past tests and current observations to support the STEM-based reasoning. Approximately five minutes later, the following claim sequence occurred that was again grounded in both science and engineering concepts and practices:

Gary: We should start it over here. [placed rubber strip to reaffirm intention] So I can put one in over here. I am just keeping it here, and it should be angled downward. It should be angled down to give it more energy.

A key difference between this claim and the initial example lies in the degree of abstract thinking and use of formal language used by the student. Rather than relying on a test to make a claim about the utility of the design, Gary drew from his own knowledge of properties of motion and force to make a claim about the increase in energy due to the proposed design. This type of claim was much less common throughout both of the learning experiences.

There was an approximate 1:3 ratio in the number of challenge claims to novel claims in the Dash video, with a ratio of 1:4 for the Roller Coaster video. The presence of challenge claims in both learning episodes is important, as it signifies moments of disagreement, opening up opportunities for argumentation and further learning. For example, the following discussion occurred early in the Dash video after a test run of the robot:

Ryan: I think the 45 degrees might be too much.

Alex: No, because that means he'll only have to do a tiny turn. [moves Dash in air to mimic this]

Ryan: Ok, then we have to do a turn left 45 degrees or turn left 30 degrees.

Alex: Move forward 10 cm.

[Alex reprograms, then Dash runs]

Alex's challenge of Ryan's initial claim led to refinement of their programming, including changes to both angle and length. Alex's spatial sense led to the use of a slightly smaller angle, a claim articulated by both actions and gestures. Ryan was convinced and further advanced the claim with a statement about the length of the robot's path. Aspects of mathematics, engineering, and technology appear in both the nature of the claims and the associated reasoning.

### **Discussion**

This paper provides a framework for analyzing student thinking in STEM environments and provides empirical evidence on the nature of student thinking during two STEM learning experiences. The two learning episodes have unique features, and we do not claim to have captured the full range of thinking that happens within STEM learning environments. However, as demonstrated here, our framework is useful for looking at the nature of thinking from the perspective of student claim making, and our results provide insight into the nature of student thinking in STEM environments.

In these two examples from elementary, informal settings, there are many consistencies. Of special importance is the use of in-the-moment activity (usually test and observation) as the

primary source of evidence, and the dominant role of explicit, certain, and informal language in the student claims. This collectively produced an experimental quality to the evidence and reasoning. Our results suggest that student thinking, in these environments, is often grounded in personal experience and authority and consists of a rich mix of general and disciplinary-oriented reasoning, as well as the use of multiple disciplines inside a single claim. In contrast to the nature of the claims, most of the reasoning was not explicitly stated and had to be inferred. Educators should note these thinking tendencies and attempt to nurture and build on these practices during student activity.

While only one claim in the Roller Coaster video was coded as mathematical in nature, 12 instances of spatial reasoning occurred throughout the activity, often tightly connected to reasoning from other disciplines. In contrast, there were 23 claims and 20 examples of reasoning that incorporated mathematics in the Dash video. Most of this reasoning also involved spatial reasoning related to the nature of the robot's path, but some instances of numeric reasoning, usually associated with measurement, also occurred. Specifically, the students in the Dash video often relied on estimation or trial-and-error, rather than careful tool-based measurement, exhibiting a strong, intuitive sense of angle and length estimations. Much of the reasoning was also done in conjunction with engineering-based reasoning about the design of the path and technology-based reasoning related to the programming parameters of the robot.

Contexts, participants, materials, and content all played roles in the emergence, nature, and evolution of the thinking observed, including thinking that drew from multiple disciplines. Although the learning experiences had a clearly defined goal, they were generally ill-structured. An initial prompt and materials were provided, but then very little subsequent direction was provided for the students to follow. This afforded a personal dimension in the student thinking, as the students had freedom to explore the context, develop their own ideas, and think collaboratively. While the inherent nature of each activity placed some degree of definition on the nature of the disciplinary content that emerged, the students were also free to draw on their formal and informal disciplinary knowledge in a manner of their own choosing. This is quite different from the more siloed and structured learning environments found in many classrooms. As a result, the student thinking incorporated claims that often involved more than one disciplinary influence and were initiated by ideas and activities generated by the students.

While the above discussion is helpful, we call for more examples of STEM thinking in a variety of contexts to support educators in building learning experiences more in line with the ways in which students think about disciplinary and STEM ideas. For example, Foster et al. (2022) drew on Walton's (1998, 2022) argumentation framework to analyze student dialogue in elementary STEM classrooms. Similar to our results, they found multiple disciplines emerging in student comments, but their use of Walton's framework allowed for an analysis that further discerned the nature of the arguments. For example, they identified arguments that sought out information or helped to determine a best course of action as most common. We urge others to extend on these existing frameworks and further explore the student aspects of claim making in STEM environments.

## References

- Cobb, P., Yackel, E., & Wood, T. (1992). A constructivist alternative to the representational view of mind in mathematics education. *Journal for Research in Mathematics Education*, 23(1), 2-33.
- Forman, E. A., Larreamendy-Joerns, J., Stein, M. K., & Brown, C. A. (1998). "You're going to want to find out which and prove it": Collective argumentation in a mathematics classroom. *Learning and Instruction*, 8(6), 527-548.

- Foster, J., Schneider, J. G., Franco, L., Zhuang, Y., Crawford, B., & Conner, A. (2022). Categorizing classroom-based argumentation in elementary STEM lessons: Applying Walton's types of argument dialogue. *Journal of Research in STEM Education*, 8(2), 79-110.
- Gray, R., & Kang, N. H. (2014). The structure of scientific arguments by secondary science teachers: Comparison of experimental and historical science topics. *International Journal of Science Education*, 36(1), 46-65.
- Hjelte, A., Schindler, M., & Nilsson, P. (2020). Kinds of mathematical reasoning addressed in empirical research in mathematics education: A systematic review. *Education Sciences*, 10(10), 289.
- Holmlund, T. H., Lesseig, K., & Slavit, D. (2018). Making sense of "STEM education" in K-12 contexts. *International Journal of STEM Education*, 5(32), 1-18.
- Kind, P. E. R., & Osborne, J. (2017). Styles of scientific reasoning: A cultural rationale for science education? *Science Education*, 101(1), 8-31.
- Kazemi, E., Ghousseini, H., Cordero-Siy, E., Prough, S., McVicar, E., & Resnick, A. F. (2021). Supporting teacher learning about argumentation through adaptive, school-based professional development. *ZDM—Mathematics Education*, 53(2), 435-448.
- Kelley, T. R., & Knowles, J. G. (2016). A conceptual framework for integrated STEM education. *International Journal of STEM education*, 3(1), 1-11.
- Kind, P. E. R., & Osborne, J. (2017). Styles of scientific reasoning: A cultural rationale for science education? *Science Education*, 101(1), 8-31.
- Lee, H. S., Liu, O. L., Pallant, A., Roohr, K. C., Pryputniewicz, S., & Buck, Z. E. (2014). Assessment of uncertainty-infused scientific argumentation. *Journal of Research in Science Teaching*, 51(5), 581-605.
- Li, Y., Schoenfeld, A. H., diSessa, A. A., Graesser, A. C., Benson, L. C., English, L. D., & Duschl, R. A. (2019). On thinking and STEM education. *Journal for STEM Education Research*, 2, 1-13.
- Lincoln, Y. S., & Guba, E. G. (1985). *Naturalistic inquiry*. Sage.
- McNeill, K.L., and J. Krajcik. 2012. *Supporting grade 5–8 students in constructing explanations in science: The claim, evidence, and reasoning framework for talk and writing*. Allyn and Bacon.
- NGSS Lead States. 2013. *Next Generation Science Standards: For States, By States*. Washington, DC: The National Academies Press.
- Ortiz-Revilla, J., Adúriz-Bravo, A., & Greca, I. M. (2020). A framework for epistemological discussion on integrated STEM education. *Science & Education*, 29, 857-880.
- Roehrig, G. H., Dare, E. A., Ring-Whalen, E., & Wieselmann, J. R. (2021). Understanding coherence and integration in integrated STEM curriculum. *International Journal of STEM Education*, 8(1), 1-21.
- Siverling, E. A., Suazo-Flores, E., Mathis, C. A., & Moore, T. J. (2019). Students' use of STEM content in design justifications during engineering design-based STEM integration. *School Science and Mathematics*, 119(8), 457-474.
- Slavit, D., Lesseig, K., & Grace, E. (2021). Student ways of thinking in STEM contexts: A focus on claim making and reasoning. *School Science and Mathematics*, 121(8), 466-480.
- Toulmin, S. E. (1958). *The uses of argument*. Cambridge University Press.
- Vasquez, J. (2014). STEM—Beyond the acronym. *Educational Leadership*, 72(4), 10-15.
- Vygotsky, L. S. (1978). *Mind in society: Development of higher psychological processes*. Harvard University Press.
- Walton, D. N. (1998). *The New Dialectic: Conversational contexts of argument*. University of Toronto Press.
- Walton, D. (2022). Formal dialogue models for argumentation in education and linguistics. *Learning, Culture and Social Interaction*, 36, 1-4.
- Windschitl, M., Thompson, J., & Braaten, M. (2008). Beyond the scientific method: Model-based inquiry as a new paradigm of preference for school science investigations. *Science Education*, 92(5), 941-967.
- Worsley, M., & Blikstein, P. (2016). Reasoning strategies in the context of engineering design with everyday materials. *Journal of Pre-College Engineering Education Research (J-PEER)*, 6(2), 4.

## CONSIDERING NEW MEASURES OF CHILDREN'S AND PARENTS' MATH ATTITUDES AND THEIR ASSOCIATIONS WITH MATH PERFORMANCE

Caroline Byrd Hornburg  
Virginia Tech  
chornburg@vt.edu

Katherine E. D'Ercole  
Virginia Tech  
katherinedercole@vt.edu3

Jisun Kim  
Virginia Tech  
jisunk@vt.edu

Christine A. Berry  
Virginia Tech  
christineb43@vt.edu

*Children's math anxiety is an important predictor of math performance; however, children's other math attitudes as well as parents' math attitudes have not often been considered in tandem with math anxiety. It may be that other attitudes beyond typical measures of anxiety for children and parents play a unique role. This study examined relations among children's math attitudes (i.e., math-specific anxiety, avoidance, competence, interest, and value), parents' math attitudes (i.e., math anxiety, math teaching attitudes) and children's math performance. Children (N = 59; 2nd-4th graders; 35 girls, 24 boys) and their parents completed surveys, and children solved word problems. Results highlighted correlations among most math attitudes and performance in the expected directions; however, it was a new construct of parents' math teaching attitudes that was significantly correlated with child performance rather than parents' math anxiety.*

Keywords: Affect, Emotion, Beliefs, and Attitudes; Elementary Education; Problem Solving

Children's math attitudes are linked to their parents' math attitudes (Gunderson et al., 2011), and children's math attitudes are associated with performance, in that math anxiety predicts lower performance (Ramirez et al., 2016) and competence, interest, and value predict higher performance (Petersen & Hyde, 2017). Parents' math anxiety is also negatively linked to children's performance (Maloney et al., 2015). Prior work examining affective factors, however, has not considered children's math avoidance behaviors, even though math avoidance has been established as a critical factor in adolescents' performance (Choe et al., 2019), and avoidant behaviors in academic situations are one dimension of math anxiety (Ashcraft, 2002; Szczygieł & Pieronkiewicz, 2022). In addition, it is unclear whether parents' attitudes surrounding helping their children with math, which we label parents' math teaching attitudes, may be uniquely related to children's math attitudes and performance, in a distinct way from parents' own math anxiety. Indeed, prior work has examined parents' own math anxiety rather than anxiety in the context of supporting their children's math learning. Thus, this study focused on two questions:

1. What are the relations among children's math attitudes and parents' math attitudes?
2. How are children's and parents' math attitudes associated with math performance?

Specifically, we hypothesized that there would be associations among each of the children's math attitudes and between the two parent math attitudes measured, in line with prior work, but that the two new variables of children's math avoidance and parents' math teaching attitudes would be distinct from math anxiety for children and parents, respectively. We also expected that children's competence, interest, and value, as well as parents' math teaching attitudes would be

positively associated with children's math performance, and that parents' and children's math anxiety and children's math avoidance would be negatively related to math performance.

## Method

### Participants and Procedure

Participants included 59 2nd (18), 3rd (23), and 4th (18) grade children ( $M_{\text{age}} = 9.00$  years, range = 6.8-10.8 years; 35 girls, 24 boys) and their parents (51 mothers, 7 fathers, 1 other caregiver). Children's racial backgrounds were as follows: 71% White, 7% Asian, 3% Hispanic, 2% Black, 12% multi-racial, 5% other. Parents' education included 15% less than a bachelor's degree, 19% Bachelor's degree, 66% Master's degree or higher. Parents completed a REDCap survey after consent. Child measures were administered on Zoom during the COVID-19 pandemic, and verbal responses were recorded in REDCap. Math performance and survey questions were asked in separate sessions, and all questions were read aloud to children.

### Measures

Children's *math performance* was measured with eight items ( $\alpha = .79$ ) drawn from the Texas Word Problems: Brief, designed for use for 3rd graders (Powell et al., 2020). Each word problem was displayed and read aloud. Analyses used proportion correct. Children's *math anxiety* was measured with nine items ( $\alpha = .83$ ) adapted from prior work (Carey et al., 2017). Children were shown five faces and needed to select the face (and label, 1 = not nervous at all to 5 = very, very nervous) that represented how they felt about certain situations (e.g., "Listening to another child explain a math problem."). Children's *math avoidance* was measured using eight items (e.g., "I wish we spent less time doing math at school.") from recent work (Conlon et al., 2022). Children responded to each item using a 4-point scale: 1 (no), 2 (not really), 3 (kind of), to 4 (yes). There was moderate item reliability ( $\alpha = .79$ ;  $\alpha = .78$  for the 7-item version used in analyses).

The remaining child math attitudes were each measured with five items adapted from prior work (Fredricks & Eccles, 2002), using the same 4-point scale as above. These attitudes included *math competence* (e.g., "I am good at math.") ( $\alpha = .79$ ), *math interest* (e.g., "Math is fun.") ( $\alpha = .86$ ), and *math value* (e.g., "What I learn in math is important.") ( $\alpha = .68$ ).

Parents' *math anxiety* was measured with one item, in which parents rated how much they agreed with the statement "Doing mathematics makes me anxious" using a 5-point scale (1 = strongly disagree, 5 = strongly agree). Parents' *math teaching attitudes* were assessed with three items (e.g., "I feel comfortable teaching my child mathematics"), adapted from prior work with pre-service teachers (Boyd et al., 2014). Parents responded using the 5-point scale ( $\alpha = .87$ ). The *parent education* measure used a 9-point scale (1 = eighth grade or less to 9 = doctoral degree), and if applicable, the education for a second caregiver was obtained (highest level was used).

## Results

Descriptive statistics are presented in Table 1. First, we examined skewness and kurtosis of individual items. This resulted in the removal of one item from math avoidance because of high kurtosis (8.50). Then, the average score was calculated across items for all variables. Child grade and gender differences across variables were then examined. Third and 4th graders showed better performance than 2nd graders ( $t = -2.66, p = .011$ ;  $t = -3.05, p = .004$ , respectively). A significant gender difference was found in child math anxiety ( $F = 4.52, p = .038$ ) and math competence ( $F = 4.42, p = .04$ ), with girls showing higher anxiety and lower competence beliefs.

**Table 1: Descriptive Statistics for Study Variables**

Variable	<i>M</i>	<i>SD</i>	Range	Skewness	Kurtosis
Grade	3.00	0.79	2-4	0.00	-0.38
Parent Education	7.63	1.74	1-9	-1.83	3.62
Parent Math Anxiety <sup>1</sup>	2.59	1.35	1-5	0.41	-0.98
Parent Math Teaching Attitudes	3.12	1.08	1-5	-0.34	-1.05
Child Math Anxiety	1.99	0.70	1-4.11	0.93	0.51
Child Math Avoidance	1.82	0.72	1-4	1.29	1.26
Child Math Competence	3.22	0.66	1-4	-1.22	1.34
Child Math Interest	3.24	0.78	1.2-4	-1.15	0.18
Child Math Value	3.25	0.60	1-4	-1.25	2.56
Child Math Performance	0.53	0.30	0-1	-0.27	-1.13

Note. Ranges for all variables presented are observed ranges. <sup>1</sup>*N* = 58

### Relations Among Children’s Math Attitudes and Parents’ Math Attitudes

Correlations are presented in Table 2. Each of the children’s attitudes were significantly correlated, in the expected directions, with the exception of anxiety and both interest and value. As expected, children’s math anxiety and math avoidance were moderately correlated, but distinct constructs ( $r = .56, p = .001$ ). Although the correlation between parent variables of math anxiety and math teaching attitudes was a meaningful size, it was not statistically significant ( $r = -.25, p = .06$ ), and each variable was differently related to child attitudes. Parents’ math anxiety was related to their child’s math competence ( $r = -.27, p = .042$ ) and math interest ( $r = -.38, p = .003$ ) only, whereas parent math teaching attitudes were not related to any child math attitudes.

**Table 2: Zero-Order Correlations Among Study Variables**

Variable	1	2	3	4	5	6	7	8	9
1. Grade <sup>1</sup>	—								
2. Parent Education <sup>1,2</sup>	.34*	—							
3. Child Math Anxiety	-.04	-.11	—						
4. Child Math Avoidance	-.11	-.24	.56***	—					
5. Child Math Competence	.21	.28*	-.49***	-.62***	—				
6. Child Math Interest	.05	.06	-.24	-.48**	.65**	—			
7. Child Math Value	-.02	-.11	.03	-.34**	.51***	.52***	—		
8. Parent Math Anxiety <sup>3</sup>	-.06	-.08	.04	.18	-.27*	-.38**	-.21	—	
9. Parent Math Teaching Attitudes	.02	.19	-.19	-.16	.18	.15	-.06	-.25	—
10. Child Math Performance	.40**	.51**	-.28*	-.47***	.54***	.36**	.22	-.22	.30*

<sup>1</sup>Correlations among grade, parent education, and each other variable are nonparametric (Spearman’s rho). <sup>2</sup>1 = eighth grade or less, 2 = some high school, 3 = GED, 4 = high school diploma, 5 = some college, 6 = associate’s degree, 7 = bachelor’s degree, 8 = master’s degree, 9 = doctoral/postgraduate degree. <sup>3</sup>*N* = 58.

### Children’s and Parents’ Math Attitudes as Related to Children’s Math Performance

In a regression analysis, control variables (grade, parent education) were included in Step 1 and all child and parent attitudes with significant zero-order correlations with performance were added in Step 2 (note no multicollinearity was found). Only child math competence remained significantly related to math performance ( $b = 0.14, p = .042$ ) when all relevant attitudes and control variables were considered in the model (see Table 3). Note that the zero-order correlation between child math avoidance and performance ( $r = -.47$ ) was larger than that of child math anxiety and performance ( $r = -.28$ ), and within the full regression model, the magnitude of the partial correlation was also larger for child math avoidance ( $r = -.25$ ) than anxiety ( $r = .08$ ).

**Table 3: Relations Among Math Attitudes and Children’s Math Performance**

Predictor	<i>b</i>	<i>t</i>	<i>p</i>	<i>r<sub>p</sub></i>	<i>R</i> <sup>2</sup> $\Delta$
Step 1					.254***
Grade	0.12*	2.51	.015	.32	
Parent Education	0.06**	2.73	.008	.34	
Step 2					.257***
Child Math Anxiety	0.03	0.61	.054	.08	
Child Math Avoidance	-0.10	-1.83	.073	-.25	
Child Math Competence	0.14*	2.08	.042	.28	
Child Math Interest	-0.02	-0.30	.763	-.04	
Parent Math Teaching Attitudes	0.05	1.91	.062	.26	

Note.  $N = 59$ . Statistics reported for Step 1 variables are from the model before the Step 2 predictors were included. \* $p < .05$ , \*\* $p < .01$ , \*\*\* $p < .001$

### Discussion

The present study examined the relations among elementary school children’s and parents’ math attitudes and children’s math performance, including two constructs not previously examined in tandem (i.e., children’s math avoidance and parents’ math teaching attitudes). Without rehashing all results from Tables 2 and 3, the key takeaways here included the finding that these new measures were indeed distinct constructs from math anxiety in both children and parents, respectively. In addition, in contrast to prior work linking parents’ math anxiety to child performance, parent math anxiety was not significantly related to performance or to most child attitudes. Instead, the newly created measure of parents’ math teaching attitudes (i.e., feelings surrounding helping children with math), was significantly correlated with child performance. This highlights the importance of the home math environment, in line with prior literature considering caregivers’ beliefs and attitudes (see Hornburg et al., 2021, for a review).

The sample size was small and most parents were highly educated. We had inadequate power to examine any possible interactions among parent gender, child gender, or grade, and many variables in the regression were important but not statistically significant. Future studies with larger samples from diverse backgrounds are needed. Also, with only one time point, causality cannot be determined, and some of the key constructs were measured using relatively few items.

In sum, this small-scale study is a first step towards a broader perspective of children’s and parents’ math attitudes, including constructs not considered previously. Given the context of the COVID-19 pandemic in which children’s learning was remote, it may be especially important to examine how parent teaching attitudes may be related to children’s success in online contexts.

## Acknowledgments

Thank you to the participating children and parents who made this study possible. Thank you also to Colleen Ganley and members of the FSU Math Thinking and Learning Lab who were collaborators in the development of the child math avoidance measure used in this study.

## References

- Ashcraft, M. H. (2002). Math anxiety: Personal, educational, and cognitive consequences. *Current Directions in Psychological Science*, 11(5), 181-185. <https://doi.org/10.1111/1467-8721.00196>
- Boyd, W., Foster, A., Smith, J., & Boyd, W.E. (2014). Feeling good about teaching mathematics: Addressing anxiety amongst pre-service teachers. *Creative Education*, 5(4), 207–217. <https://doi.org/10.4236/ce.2014.54030>
- Carey, E., Hill, F., Devine, A., & Szűcs, D. (2017). The modified abbreviated math anxiety scale: A valid and reliable instrument for use with children. *Frontiers in Psychology*, 8, Article 11. <https://doi.org/10.3389/fpsyg.2017.00011>
- Choe, K. W., Jenifer, J. B., Rozek, C. S., Berman, M. G., & Beilock, S. L. (2019). Calculated avoidance: Math anxiety predicts math avoidance in effort-based decision-making. *Science Advances*, 5(11). Article 1062. <https://doi.org/10.1126/sciadv.aay1062>
- Conlon, R. A., Krentz, V. L., Dasher, J. M., Merritt, K. E., Ganley, C. M., Hornburg, C. B., Meyer, A., & Hart, S. A. (2022, June). Measuring math avoidance in children to better understand its relation with math anxiety and achievement. In M. C. Passolunghi (Chair), *Towards math equity for all students: The link between math anxiety, math achievement, math avoidance and career interests*. Paper presented at the 5th Annual Meeting of the Mathematical Cognition and Learning Society (MCLS), Antwerp, Belgium.
- Fredricks, J. A., & Eccles, J. S. (2002). Children's competence and value beliefs from childhood through adolescence: Growth trajectories in two male-sex-typed domains. *Developmental Psychology*, 38(4), 519–533. <https://doi.org/10.1037/0012-1649.38.4.519>
- Gunderson, E. A., Ramirez, G., Levine, S. C., & Beilock, S. L. (2011). The role of parents and teachers in the development of gender-related math attitudes. *Sex Roles: A Journal of Research*, 66, 153–166. <https://doi.org/10.1007/s11199-011-9996-2>
- Hornburg, C. B., Borriello, G. A., Kung, M., Lin, J., Litkowski, E., Cosso, J., Ellis, A., King, Y. A., Zippert, E., Cabrera, N. J., Davis-Kean, P., Eason, S. H., Hart, S. A., Iruka, I. U., LeFevre, J.-A., Simms, V., Susperreguy, M. I., Cahoon, A., Chan, W. W. L., Cheung, S. K., Coppola, M., De Smedt, B., Elliott, L., Estévez-Pérez, N., Gallagher-Mitchell, T., Gardner-Neblett, N., Gilmore, C., Leyva, D., Maloney, E. A., Manolitsis, G., Melzi, G., Mutaf-Yildiz, B., Nelson, G., Niklas, F., Pan, Y., Ramani, G. B., Skwarchuk, S.-L., Sonnenschein S., & Purpura, D. J. (2021). Next directions in measurement of the home mathematics environment: An international and interdisciplinary perspective. *Journal of Numerical Cognition*, 7(2), 195-220. <https://doi.org/10.5964/jnc.6143>
- Maloney, E. A., Ramirez, G., Gunderson, E. A., Levine, S. C., & Beilock, S. L. (2015). Intergenerational effects of parents' math anxiety on children's math achievement and anxiety. *Psychological Science*, 26, 1480-1488. <https://doi.org/10.1177/0956797615592630>
- Petersen, J. L., & Hyde, J. S. (2017). Trajectories of self-perceived math ability, utility value and interest across middle school as predictors of high school math performance. *Educational Psychology*, 37(4), 438-456. <https://doi.org/10.1080/01443410.2015.1076765>
- Powell, S. R., Berry, K. A., & Barnes, M. A. (2020). The role of pre-algebraic reasoning within a word-problem intervention for third-grade students with mathematics difficulty. *ZDM Mathematics Education*, 52(1), 151–163. <https://doi.org/10.1007/s11858-019-01093-1>
- Ramirez, G., Chang, H., Maloney, E. A., Levine, S. C., & Beilock, S. L. (2016). On the relationship between math anxiety and math achievement in early elementary school: The role of problem solving strategies. *Journal of Experimental Child Psychology*, 141, 83–100. <https://doi.org/10.1016/j.jecp.2015.07.014>
- Szczygiel, M., & Pieronkiewicz, B. (2022). Exploring the nature of math anxiety in young children: Intensity, prevalence, reasons. *Mathematical Thinking and Learning*, 24(3), 248-266. <https://doi.org/10.1080/10986065.2021.1882363>

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.



# DEVELOPING A TOOL FOR MEASURING STUDENT ORIENTATIONS WITH RESPECT TO UNDERSTANDING IN MATHEMATICAL LEARNING

Siqi Huang  
University of California, Berkeley  
shuang123@berkeley.edu

*The goal of this paper is twofold. First, the paper clarifies and elaborates on an important theoretical construct called orientation with respect to understanding in mathematics, which denotes the degree to which students exhibit an inclination towards and demonstrate an earnest concern for understanding in mathematical learning. Second, the paper reports on the creation and evaluation of a methodological tool for measuring the aforementioned construct. The tool was operationalized from analyses of 38 college students' problem-solving behaviors as well as their verbal self-reflections in semi-structured task-based interviews. Results showed decent validity and reliability evidence on the proposed research tool. This study contributes to a better conceptualization of learning orientation as a fundamental shaper of how students engage with mathematics; it also holds practical potential for enhancing mathematics classrooms.*

Keywords: Affect, Emotion, Beliefs, and Attitudes; Measurement; Research Methods

Helping students understand is one of the most important goals in mathematics education (Common Core State Standards, 2010; Schoenfeld et al., 2023). Literature on understanding provides well-articulated conceptualizations about how people develop understanding (Piere & Kieren, 1989, 1994) as well as what understanding is from a psychological perspective (Hiebert & Carpenter, 1992; Sierpiska, 1994), performance-based perspective (Perkins & Blythe, 1994), sociocultural perspective (Godino, 1996; Johnson, 1987), and affective perspective (Duffin & Simpson, 2000). Yet there is an under-addressed challenge in precisely explaining students' different understanding-related behaviors and using it to inform classroom teaching and learning. For example, what accounts for some students' consistent inclination to memorize and apply formulas without necessarily making sense of them, as compared to others' tendency to scrutinize the underlying mathematical logic? In an attempt to articulate the difference more precisely and investigate its potential impact on student learning, I extend Schoenfeld's (2010) definition and define *orientation* as including people's goals, attitudes, values, dispositions, beliefs, and preferences. I further define students' *orientations with respect to understanding* to be the degree to which students exhibit an inclination towards and demonstrate an earnest concern for understanding in mathematical learning. Students with a strong orientation towards understanding care deeply about how things work, prioritize goals and actions for developing understanding, and are willing to put in much effort to understand underlying principles.

## Theoretical Background

The construct of student orientations with respect to understanding is important and is theoretically consistent with existing literature. Schoenfeld's (2010) theory on goal-oriented decision making argues that the key determinants of human behaviors are resources (including knowledge, heuristics and other perceived resources), goals (their conscious or unconscious aims or objectives), and orientations (including their beliefs, values, dispositions, and biases). This suggests a fundamental relationship between student orientations and their behaviors, providing a theoretical foundation that the more we want to understand how students are engaged in meaningful learning and problem solving, the more we need to know about their orientations with

respect to understanding. Moreover, the construct appears to significantly correlate with (if not explain) students' mathematical achievements, as Mashaal (2006) reports that lower-achieving students tend to rely on memorization strategies while higher-achieving students tend to put in more effort to understand mathematical concepts and principles. Since the present body of literature has not rigorously examined the construct of student orientations with respect to understanding (Schoenfeld, personal communication, January 24, 2023), it is imperative that we operationalize it and develop a research tool to support more sophisticated discussions about students' orientations in relation to their mathematical learning behaviors and outcomes.

There are indeed works from Dweck and colleagues that suggest a type of orientation in particular to goals and mindsets is central to student learning. According to the achievement goals framework (Dweck, 1986; Maehr, 1984; Nicholls, 1984), students' performance-oriented goals seek to outperform others and to demonstrate comparative competencies (Elliot, 2005), and mastery-oriented goals emphasize the desire to "understand a task, acquire new knowledge, and develop abilities" (Darnon, Butera & Harackiewicz, 2007, p. 61). Performance-oriented goals are often assumed to result in undesirable outcomes (such as poor achievement, avoidance of challenges, anxiety) and their benefits are limited to simple tasks (Dweck, 1986; Nicholls, 1984; Senko, 2019). Mastery-oriented goals, in contrast, have been found to improve student engagement and perseverance in challenging tasks and are believed to be conducive to high achievements, openness to collaboration, effective study strategies, and students' well-being (Senko, 2019). According to Dweck (2006, 2012), how students perceive their abilities also play an important role in their motivation and achievement. More specifically, those who believe that intelligence can be learned through effort are said to have a "growth mindset" (or an "incremental" belief) and those who believe that intelligence is fixed, and someone is either smart or not are said to have a "fixed mindset" (or an "entity" belief). Literature shows that students with a growth mindset are more likely to exert effort to overcome challenges and display greater resilience when encountering failure (Blackwell et al., 2007; Dweck & Leggett, 1988; Dweck & Yeager, 2019); students with a fixed mindset are more prone to evading challenges and conceding to setbacks (Dweck, 2007a, 2007b, 2013).

The Pirie-Kieren model for growth of mathematical understanding (Pirie & Kieren, 1989, 1994) and Huang's (2022) observation system provide useful insights for understanding potential indicators of students' orientations with respect to understanding in mathematical learning. In Pirie-Kieren's works, growth in understanding is seen as a dynamic, active, non-linear, and transcendently recursive process involving the building of and acting in the world (Pirie & Kieren, 1991; Pirie & Martin, 2000). The metaphor of recursion is fundamental to the Pirie-Kieren model, and the term *fold back* is defined specifically to describe the phenomenon of returning to an initial understanding, reflecting on and reorganizing earlier conceptions, and effectively building a thicker understanding in response to complex problem situations (Pirie & Martin, 2000, p. 131). Huang (2022) proposes an observation system that characterizes students' understanding-oriented behaviors in the context of problem solving, suggesting that students' orientation with respect to understanding indeed matters and that it is correlated with students' problem-solving performances. Huang (2022) further argues that the behavioral indicators of (a) problematizing ongoing works, (b) suggesting solving with multiple approaches, (c) examining proposed ideas, and (d) examining problem statements give rise to promising profiles of students' orientations with respect to understanding. Collectively, existing literature offers a theoretically coherent and rich foundation upon which I seek to push the conversation further by developing a more refined research tool.

### The Refined Construct: Orientation with Respect to Understanding

I develop a taxonomy of four qualitatively different levels in the continuum of students' orientations with respect to understanding in mathematical learning. At the highest level, *proactive to understanding* (PU), students proactively seek understanding regardless of the learning and problem-solving situations; they care deeply about how things work and are willing to put in much effort to understand underlying principles. At the second level, *open to understanding* (OU), students typically make an effort to understand mathematical ideas under most circumstances. Students at this level are open to developing a deeper understanding of mathematics; they may seek understanding when prompted, but they may not take initiative in doing so. At the third level, *indifferent to understanding* (IU), students do not have a strong preference for developing a deep understanding of mathematics; they may resort to memorization most of the time. Students at this level are generally comfortable with memorizing formulas, although they do make an effort to understand complex mathematical concepts on occasion. At the lowest level, *resistant to understanding* (RU), students resist understanding by ignoring their confusions, shutting down conflicting ideas, or blindly applying formulas or procedures without making sense of them. Table 1 summarizes the levels with descriptions and examples operationalized from my analyses of 38 college students' behaviors and reflections.

**Table 1: Orientation with Respect to Understanding**

Levels	Descriptions	Examples
Proactive to Understanding (PU)	Care deeply about how things work and are willing to put in much effort to understand underlying principles; proactively seek understanding in learning and problem solving.	Spend time understanding confusing ideas even after getting an answer; problematize ongoing works ("What if ...?"); proactively suggest solving a problem with multiple approaches.
Open to Understanding (OU)	Make an effort to understand mathematics under most circumstances; seek a deeper understanding when prompted but may not take initiative in doing so.	Fold back (Pirie & Kieren, 1991) to initial understanding when confronted with challenges; examine conflicting ideas; probe further clarification or justification from their peers.
Indifferent to Understanding (IU)	Do not have a strong preference for developing a deep understanding of mathematics; resort to memorization most of the time.	Watch their peers sort out confusions with no evidence of contributing; only reexamine the problem statement when prompted.
Resistant to Understanding (RU)	Not willing to put in effort for understanding; want to memorize and apply formulas or procedures without making sense of them.	Ignore or shut down conflicting ideas; ignore their own confusions; blindly apply formulas or procedures without examining their applicability.

### Methods

#### Item Design

An overarching assumption underlying my work is that what students say is not always the same as what they actually think or believe in, positioning their learning behaviors in actual problem situations as potentially more reliable measurement targets than their self-assessments or verbal reports. On the other hand, students' reflections about their own learning could offer insights about their orientations that might not be otherwise evident from purely examining their behaviors. Thus, my work seeks to operationalize the construct of student orientations with respect

to understanding using both students' behaviors and their verbal self-reflections as a triangulation (Thurmond, 2001) strategy. I designed a semi-structured task-based interview (Ginsburg, 1997; Goldin, 2000) and asked students to (a) think aloud and try to solve two challenging mathematical problems (so as to elicit behaviors that might indicate certain level of orientations), and (b) reflect on their mathematical learning experiences (so as to elicit what they actually think about understanding and how they approach it on a daily basis).

**Problem-Solving Tasks.** This item was built on the assumption that challenging problem situations afforded distinction between students at different levels of the continuum of learning orientations: If students encountered a challenging problem that they did not immediately know how to solve, some might work hard to understand relevant mathematics and build a more sophisticated understanding, while others might try to plug in (not-necessarily related) formulas they remembered without necessarily making sense of the situation or the mathematics. Below is one example task that I used to elicit college students' orientations in a problem situation.

You have 100 pennies (\$0.01) on a table. Abby comes in and replaces every second penny with a nickel (\$0.05), which means the 2<sup>nd</sup>, 4<sup>th</sup>, ..., 100<sup>th</sup> coins become nickels. Oliver then replaces every third coin with a dime (\$0.10), which means now the 3<sup>rd</sup>, 6<sup>th</sup>, ..., 99<sup>th</sup> coins become dimes. After that, Emma replaces every fourth coin with a quarter (\$0.25), which means now the 4<sup>th</sup>, 8<sup>th</sup>, ..., 100<sup>th</sup> coins become quarters. Summing all the 100 coins on the table, how much money will you have?

**Open-Ended Interview Questions.** To zoom out of the problem-solving situation, I developed an interview protocol to explore students' orientations in their general mathematical learning experiences. Two personal-reflection item bundles were particularly relevant: one investigated students' treatments to understanding versus memorizing, and the other investigated students' behaviors and their reasoning behind when they did not understand something about mathematics. The first bundle (I\_1) prompted the following: (a) "Reflecting on your mathematical experience in classrooms, when do you make an effort to understand mathematical ideas, and when would you be okay with just memorizing formulas or procedures?", and (b) "Reflecting on your experience feeling stuck in mathematical problems, when would you make an effort to go back and try to understand the underlying mathematics, and when would you be okay with recalling and applying formulas without evaluating whether they are applicable?". The second bundle (I\_2) asked students to reflect on and describe the last time they did not understand something about mathematics, with a focus on what they did in the situations.

**Accounting for the External Variable of Math Anxiety.** Among other things, students' math anxiety (Ashcraft, 2002) might affect the extent to which students were oriented towards understanding in mathematical learning. I hypothesized that if students felt comfortable learning and doing mathematics, they were more likely to try and make sense of underlying concepts and principles; if students had high levels of math anxiety, they were more likely to refrain from confusions or challenging mathematical ideas. For this reason, I drew on May's (2009, p. 75) math anxiety questionnaire and added a subsection at the end of the interview to explore students' levels of math anxiety. The questions I asked were (A\_1) "To what extent do you feel stressed in your math class?", (A\_2) "To what extent do you worry that you will not be able to understand the math you learn?", and (A\_3) "To what extent is doing math stressful?".

### **Data Collection**

I used semi-structured task-based interviews (Ginsburg, 1997; Goldin, 2000) along with think-aloud protocols (Leighton, 2017; Schoenfeld, 1985) to conduct this study. 19 self-selected pairs of

friends (defined by feeling comfortable discussing math together) from UC Berkeley volunteered to participate sequentially in Summer 2022. The participants took about 50-90 minutes to complete the problem-solving tasks as well as the subsequent semi-structured interview in pairs. I video-recorded all participation and collected all participants' written works after they completed the study. Among the 38 participants, 21 (55.3%) were female and 17 (44.7%) were male. 12 (31.6%) were STEM undergraduates, nine (23.7%) were non-STEM undergraduates, five (13.2%) were undergraduates with undeclared majors, eight (21.1%) were non-STEM graduate students, and four (10.1%) were STEM graduate students.

### **Data Analysis**

I approached data analysis in five phases: (1) holistically reviewed the video-recordings of students' problem solving as well as their interview reflections to develop an overview of the data, (2) developed analytic memos to record my critical examination of each recording along with the corresponding time and the reasons I hypothesized to be potential indicators of student orientations, (3) identified emerging themes from the analytic memos and mapped each theme to the most reasonable level, (4) developed transcripts with high precision that captured a detailed account of students' problem solving behaviors as well as their verbal reflections and coded them, (5) took another holistic review of the recordings to evaluate whether the preliminary analysis provided a full story, compared it with my initial impression of the data, and revised the coding rubric accordingly. The results informed the revision of my proposed taxonomy (Table 1).

The description column in Table 1 (which was iteratively revised in the data analysis process) provided an overarching coding rubric for the interview responses. Building on Pirie & Kieren's (1989, 1991, 1994) as well as Huang's (2022) work, I operationalized some behavioral indicators of each level for the problem-solving tasks. Student behaviors were coded as *proactive to understanding* if they showed evidence of at least one of the following: (a) spent time understanding confusing ideas even after getting a correct answer; (b) problematized their ongoing works (e.g., "What if ...?"), (c) proactively generalized problem-solving strategies, and (d) proactively suggested solving a given problem with multiple approaches. Students were coded as *open to understanding* if they (a) folded back (Pirie & Kieren, 1991) to prior knowledge and elaborated on or reconstructed incomplete understandings when confronted with challenges, (b) carefully examined conflicting ideas, or (c) probed further clarification or justification from their peers. Students were coded as *indifferent to understanding* if they (a) watched their peers sort out confusions with no evidence of contributing or (b) only reexamined (clarified, reread, or rephrased) the problem statement when prompted. Students were coded as *resistant to understanding* if they (b) ignored or shut down emergent, conflicting ideas, (c) ignored their own confusions, or (c) blindly applied formulas or procedures without examining their applicability. With the operationalized rubric, I used the BEAR Assessment System Software (Fisher & Wilson, 2019) to evaluate the item fit statistics, reliability, and validity of my proposed measure. I further created a scatterplot to investigate potential correlations between students' estimated orientations and their levels of math anxiety.

## **Results**

### **Item Fit Statistics**

I used the Rasch model (Bond & Fox, 2013) and item fit statistics to evaluate the proposed instrument and the extent to which the observed data fitted the model-implied distribution. Infit is an inlier-sensitive or information-weighted fit statistic, which measures the difference between the discrimination of an item and the average discrimination of other items in an instrument (Wu & Adams, 2013). Outfit is an outlier-sensitive fit statistic, which is a measure sensitive to unexpected

responses such as incorrect answers about easy questions by a high-performing student (Waterbury, 2020). The expected value of both Infit and Outfit is 1, with smaller values indicating an over-fit of the data to the Rasch model (i.e., responses are too predictable or Guttman-like) and greater values indicating an under-fit of the data (i.e., responses are not as predictable as expected). In applied Rasch measurement, it is common to flag items with fit-index values less than 0.75 and values greater than 1.33 (Blum et al., 2020). According to Wilson (in press), high fit-index values are often more problematic, suggesting the relevant item might measure something different from the intended construct. As shown in Table 2, all items except A\_2 showed decent Infit and Outfit values; the Infit and Outfit values of A\_2 were slightly smaller than 0.75 (meaning responses were “too predictable”), which might have been because A\_2 was a reliable item for measuring students’ levels of math anxiety (May, 2009).

**Table 2: Item Fit Statistics**

	Problem Solving Tasks	I_1	I_2	A_1	A_2	A_3
Infit	1.28	0.86	0.82	1.02	0.73	0.98
Outfit	1.16	0.87	0.88	0.88	0.71	0.99

### Reliability

I implemented two types of reliability assessment for my measure. First, I calculated the internal consistency coefficient using the BEAR Assessment System Software (Fisher & Wilson, 2019): the EAP reliability was 0.909, the WLE reliability was 0.890, and the variance/covariance was 8.711. Second, I measured the inter-rater reliability using the classical test theory approach. I was lucky to have a volunteer from my undergraduate research mentees who independently coded the data using the rubric I provided. The percentage agreement between her and my initial code was 79.6%. All code conflicts were resolved, and the resulting discussions contributed to the revised coding rubric as shown in the Data Analysis section.

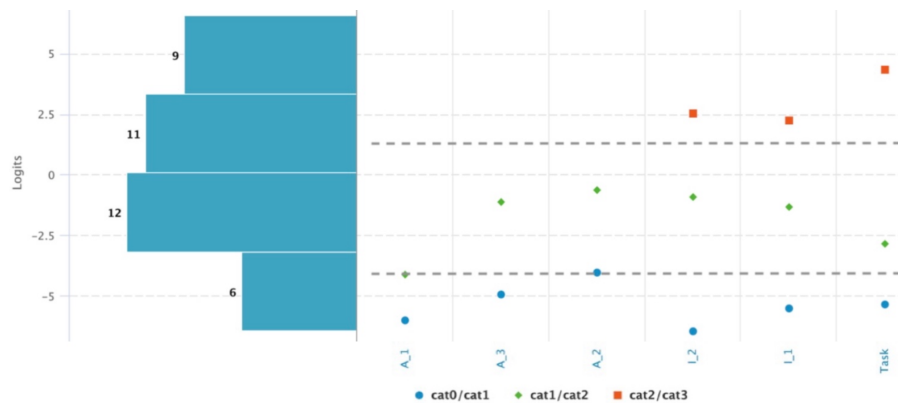
### Validity

**Content Validity.** I drew from Schoenfeld’s theory on decision making (2010) as well as Dweck’s and colleagues’ works (e.g., Dweck, 1986, 2006, 2012; Nicholls, 1984) to develop my proposed taxonomy of student orientations with respect to understanding. I provided a detailed description of the levels and a coding rubric with examples building on Pirie & Kieren’s (1989, 1991, 1994) model, Huang’s (2022) observation system, as well as careful operationalization from my data. I further calibrated my proposed measurement using the BEAR Assessment System Software (Fisher & Wilson, 2019) and the feedback I received from my informants.

**Response Process Validity.** I used think-aloud protocols (Leighton, 2017; Schoenfeld, 1985) to encourage the participants share their thinking process while completing the problem-solving tasks, so that I could triangulate their problem-solving behaviors using their spontaneous utterances, body movements, and the flow of mathematical ideas. At the end of the post-task interview, I asked the participants to evaluate which tasks or questions, if any, they had trouble understanding. The result was that all participants found the materials to be very accessible.

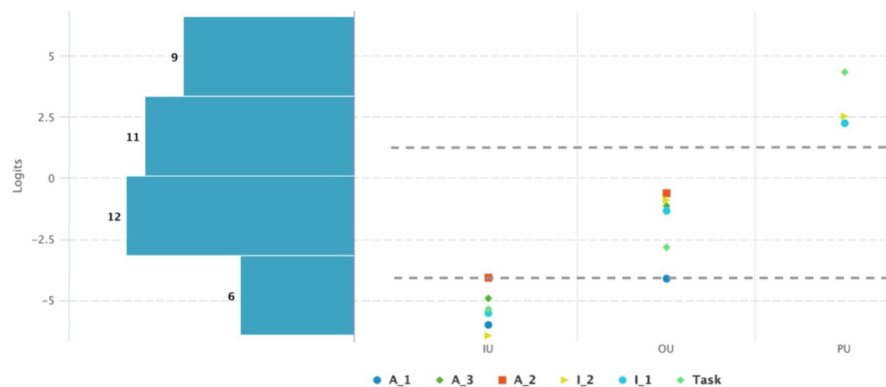
**Internal Structure Validity.** I used a WrightMap by Items (Figure 1) and a WrightMap by Levels (Figure 2) to evaluate the structural validity and consistency within my proposed measurement. According to Figure 1, the first thresholds (marked blue) of all items were generally below the second thresholds (marked green), which were all below the third threshold (marked red). This phenomenon is called banding, which implies that it was the levels (PU, OU, IU, RU) of

orientations as opposed to the items that appeared to primarily determine the threshold locations. This supports the hypothesis that the proposed levels were distinct and that the approach of mapping different levels of student orientations within different items was valid.



**Figure 1: WrightMap by Items**

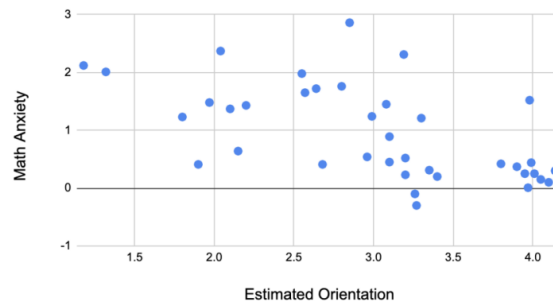
Figure 2 shows, from another angle, that the levels were almost clearly separated and that all items seemed to show an increase of mean location across levels. This reinforces the claim that the proposed levels were distinct with little overlap and that they characterized qualitatively different levels in the continuum of students’ understanding-related orientations.



**Figure 2: WrightMap by Levels**

**External Variable Validity.** Based on my learning and teaching experiences, I hypothesized that if students felt comfortable learning and doing math, they were more likely to want to understand underlying principles; if they had high levels of math anxiety, they were more likely to resist understanding when it came to learning math. To investigate potential correlations between students’ levels of math anxiety and their estimated orientations using the above coding rubric, I created a scatterplot as shown in Figure 3. Results showed that there appeared to be a weak, negative linear correlation between students’ level of math anxiety and their estimated orientations with respect to understanding. This suggested that researchers and teachers should take into account (or even better, try to address) students’ math anxiety when planning for interventions that

aim to help students prioritize goals and actions that seek to develop deeper understanding of mathematics.



**Figure 3: Correlation with Math Anxiety**  
**Concluding Remarks**

Results from the above analyses show that the proposed delineation of student orientations and the instrument for measurement had good reliability and validity evidence. This paper provides significant theoretical implications to the field. First, it foregrounds the importance of orientation with respect to understanding in student learning, pushing the field forward into articulating more precisely the hidden differences in students’ goals, values, attitudes, and beliefs about understanding and how that impacts students’ learning behaviors (and hence learning outcomes). Second, by offering a valid and reliable tool for measuring student orientations through problem solving and interview reflections, the paper paves the way for understanding students’ learning and engagement with mathematics more rigorously and comprehensively. Indeed, how intellectually meaningful could it be if students show high levels of engagement in appearance but upon careful examination, they show evidence of readily accepting and applying formulas without bothering to ask why the formulas are true or understand where a particular step comes from? Third, it contributes to a more solid foundation for addressing disparities in different groups of students’ intellectual experiences with mathematics and perhaps equivalently importantly, disparities in different groups of students’ disciplinary identities and habits of mind.

Practically, the proposed taxonomy and the accompanying coding rubric can serve as a teaching tool for teachers to diagnose students’ current states of orientations and model productive, meaningful ways of engaging with mathematics in classrooms at all grade levels. It can also serve as a learning tool for students to self-assess and bootstrap their development of productive learning orientations and mathematical practices. In the spirit of teaching for (Blythe et al., 1998; Newton, 2011; Putnam et al., 1992) and with (Hiebert, 1997) understanding, this study advocates for — besides inviting students to explore, evaluate, reflect, and take risks (as opposed to merely memorize) in expanding their understanding of mathematical and social issues — explicitly supporting students to see the value in and care more about understanding in mathematical learning. With more students orienting towards understanding in the classroom, there is reason to believe that students’ learning experiences with school mathematics as both individuals and as a collective will be more positive and productive, contributing to richer, more engaging, and more beautiful mathematical discourses in the classroom and beyond.

### References

Ashcraft, M. H. (2002). Math anxiety: Personal, educational, and cognitive consequences. *Current directions in psychological science, 11*(5), 181-185.

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.



- Blackwell, L. S., Trzeniewski, K. H., & Dweck, C.S. (2007). Implicit theories of intelligence predict achievement across an adolescent transition: A longitudinal study and an intervention. *Child Development*, 78, 246-263.
- Blum, A. M., Mason, J. M., Kim, J., & Pearson, P. D. (2020). Modeling question-answer relations: The development of the integrative inferential reasoning comic assessment. *Reading and Writing*, 33(8), 1971-2000.
- Blythe, T., Croft, A., & Strelec, N. (1998). *Teaching for understanding*. Paidós: Teacher's Guide Buenos Aires.
- Bond, T. G., & Fox, C. M. (2013). *Applying the Rasch model: Fundamental measurement in the human sciences*. Psychology Press.
- Common Core State Standards. (2010). *Mathematics standards*. <http://www.corestandards.org/the-standards/mathematics>. Accessed May 14, 2022.
- Darnon, C., Butera, F., & Harackiewicz, J. M. (2007). Achievement goals in social interactions: Learning with mastery vs. performance goals. *Motivation and Emotion*, 31(1), 61-70.
- Duffin, J. M., & Simpson, A. P. (2000). A search for understanding. *The Journal of Mathematical Behavior*, 18(4), 415-427.
- Dweck, C. S. (1986). Motivational processes affecting learning. *American psychologist*, 41(10), 1040.
- Dweck, C. S. (2006). *Mindset: The new psychology of success*. Random House.
- Dweck, C. S. (2007a). Is math a gift? Beliefs that put students at risk. *Why aren't more women in science*.
- Dweck, C. S. (2007b). The perils and promises of praise. *Ascd*, 65(2), 34-39.
- Dweck, C.S. (2012). Mindsets and human nature. *American Psychologist*, 67, 614-622.  
<http://dx.doi.org/10.1037/a0029783>
- Dweck, C. S. (2013). *Self-theories: Their role in motivation, personality, and development*. Psychology press.
- Dweck, C. S., & Leggett, E. L. (1988). A social-cognitive approach to motivation and personality. *Psychological Review*, 95, 256-273.
- Dweck, C. S., & Yeager, D. S. (2019). Mindsets: A view from two eras. *Perspectives on Psychological science*, 14(3), 481-496.
- Elliot, A. J. (2005). A conceptual history of the achievement goal construct. In A. J. Elliot, & C. S. Dweck (Eds.). *Handbook of competence and motivation* (pp. 52-72). New York, NY: Guilford Publications.
- Fisher, W. P., & Wilson, M. (2019). An online platform for sociocognitive metrology: The BEAR Assessment System Software. *Measurement Science and Technology*, 31(3), 034006.
- Ginsburg, H. P. (1997). Not a cookbook: Guidelines for conducting a clinical interview. *Entering the child's mind: The clinical interview in psychological research and practice*, 115-137.
- Godino, J. D. (1996). Mathematical concepts, their meanings and understanding. In L. Puig & A. Gutierrez (Eds.), *Proceedings of the 20th International Conference of PME*, Vol. 2, pp. 417-424. Valencia.
- Goldin, G. A. (2000). A scientific perspective on structured, task-based interviews in mathematics education research. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 517-545). Mahwah, NJ: Lawrence Erlbaum Associates.
- Hiebert, J. (1997). *Making sense: Teaching and learning mathematics with understanding*. Portsmouth, NH: Heinemann.
- Hiebert, J., & Carpenter, T. P. (1992). Learning and teaching with understanding. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning: A project of the National Council of Teachers of Mathematics* (pp. 65-97). Macmillan Publishing Co, Inc.
- Huang, S. (2022). Mathematical Problem Solving with Understanding. In A. E. Lischka, E. B. Dyer, R. S. Jones, J. Lovett, J. F. Strayer, S. Drown (Eds.), *Proceedings of the forty-fourth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, pp. 790-795. Middle Tennessee State University.
- Johnson, M. (1987). *The body in the mind. The bodily basis of meaning, imagination, and reason*. Chicago: The University of Chicago Press.
- Leighton, J. P. (2017). *Using think aloud interviews and cognitive labs in educational research*. Oxford, UK: Oxford University Press.
- Maehr, M. L. (1984). Meaning and motivation: Toward a theory of personal investment. In R. E. Ames, & C. Ames (Vol. Eds.), *Motivation in Education: Student Motivation: Vol. 1*, (pp. 115-1114). New York: Academic Press.
- Mashaal, M. (2006). *Bourbaki: A secret society of mathematics*. American Mathematical Society.
- May, D. K. (2009). *Mathematics self-efficacy and anxiety questionnaire* (Doctoral dissertation, University of Georgia).
- Newton, D. P. (2011). *Teaching for understanding: What it is and how to do it*. Routledge.
- Nicholls, J. G. (1984). Achievement motivation: Conceptions of ability, subjective experience, task choice, and performance. *Psychological review*, 91(3), 328.

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

- Perkins, D., & Blythe, T. (1994). Putting Understanding Up Front. *Educational Leadership*, 51(5), 4-7.
- Pirie, S. E. B., & Kieren, T. E. (1989). A recursive theory of mathematical understanding. *For the Learning of Mathematics*, 9(3), 7-11.
- Pirie, S. E. B., & Kieren, T. E. (1991). Folding back: Dynamics in the growth of mathematical understanding. In F. Furinghetti (Ed.), *Proceedings of the 15th annual meeting of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 169-176). Assisi, Italy: Program Committee.
- Pirie, S. E. B., & Kieren, T. E. (1994). Growth in mathematical understanding: How can we characterise it and how can we represent it? *Educational Studies in Mathematics*, 26, 165-190.
- Pirie, S. E. B., & Martin, L.C. (2000). The role of collecting in the growth of mathematical understanding. *Mathematics Education Research Journal*, 12, 127-146.
- Putnam, R. T., Heaton, R. M., Prawat, R. S., & Remillard, J. (1992). Teaching mathematics for understanding: Discussing case studies of four fifth-grade teachers. *The Elementary School Journal*, 93(2), 213-228.
- Schoenfeld, A. H. (1985). *Mathematical problem solving*. Orlando, FL: Academic Press.
- Schoenfeld, A. H. (2010). *How we think: A theory of goal-oriented decision making and its educational applications*. Routledge.
- Schoenfeld, A., Fink, H., Zun˜iga-Ruiz, S., Huang, S., Wei, X, Chirinda, B. (2023). *Helping Students Become Powerful Mathematical Thinkers: Case Studies of Teaching for Robust Understanding*. New York: Routledge.
- Senko, C. (2019). When do mastery and performance goals facilitate academic achievement?. *Contemporary Educational Psychology*, 59, 101795.
- Sierpiska, A. (1994). *Understanding in mathematics*. London: The Falmer Press.
- Thurmond, V. A. (2001). The point of triangulation. *Journal of nursing scholarship*, 33(3), 253-258.
- Waterbury, G. T. (2020). *RISE and shine: A comparison of item fit statistics for the Rasch model* (Doctoral dissertation, James Madison University).
- Wilson, M. (in press). *Constructing Measures: An Item Response Modeling Approach, 2nd edition*. New York: Routledge.
- Wu, M. L., & Adams, R. J. (2013). Properties of Rasch residual fit statistics. *Journal of Applied Measurement*, 14(4), 339–355.

# EMBRACING STUDENT LANGUAGE AS SCAFFOLDING DURING MATHEMATICAL MODELING

Abigail Quansah  
Texas State University  
alq8@txstate.edu

Jennifer A. Czoher  
Texas State University  
czoher.1@txstate.edu

Keywords: Communication, Modeling, Advanced Mathematical Thinking, Cognition

Mathematics prefers precision in language and especially at advanced levels of mathematics, concepts are often presented through definitions (Tall, 1988). Comprehension of mathematical language in the forms of terminology, phrases, definitions, and so on is vital (Adams et al., 2005). Yet students also develop rich, varied, and robust images of those concepts through repeated exposure to examples of the concept or problems that build on the concept (Goldenberg & Mason, 2008; Tall, 1988). Along with those concept images come informal ways of referring to the concepts and objects they refer to. For instance, through working on many different word problems, students may learn to assimilate situations about interest growth or radioactive decay as examples of exponential growth. They may refer to the situation as an “exponential” situation or a “ $Pe^{rt}$  formula” situation even though neither of those phrases precisely indicates what is meant by “exponential function.”

The purpose of this study is to extend this work into research on the teaching and learning of mathematical modeling, where students use mathematical language and notation to represent real-world conditions and assumptions. We wanted to know: How can student language be embraced for efficient mathematical modeling process? From a larger study that examined successful scaffolding strategies for aiding undergraduate STEM majors to learn to use advanced mathematics for mathematically modeling real-world scenarios, we discuss a successful case of an interviewer embracing one student’s meanings, from enabling the student, Navani, to engage with mathematics she neither formally nor formerly studied to model real-world scenarios. Navani worked on two tasks: the Sorting Task, based on the Wisconsin Card Sort task (Eling et al., 2008), to document participants’ awareness of linear vs. exponential growth in various scenarios, and the Cancerous mass task, which helped her develop mathematical models for absolute change and percent change in mass to reconceptualize exponential growth by coordinating constant multiplicative change in cells mass with constant additive change in time (an approach advocated by Ellis et al. (2012)) to arrive at the equation  $m' = r \cdot m$ .

Our results reveal that Navani's framework relied on language conventions. This finding supports previous research showing that building on everyday language in mathematics allows students to express their initial conceptual understandings, gives educators access to the student's thinking, and influences instructional choices (Adams et al., 2005). How students convey their mental models affects their ideas. Thus, while working on modeling activities, it is vital to have introductory documenting tasks (like the sorting task) before sophisticated tasks to build a shared mathematical vocabulary. In this study, familiarity with Navani's associations between language and concepts allowed the interviewer to enter the space of what she could do unassisted and to help her through the more complex modeling tasks, expanding her potential with guidance.

## Acknowledgments

This manuscript is based on research supported by National Science Foundation Grant No. 1750813.

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

## References

- Adams, T., Thangata, F., & King, C. (2005). "Weigh" to go! Exploring mathematical language. *Mathematics Teaching in the Middle School*, 10(9), 444-448.
- Eling, P., Derckx, K., & Maes, R. (2008). On the historical and conceptual background of the Wisconsin Card Sorting Test. *Brain and Cognition*, 67, 247-253.
- Ellis, A. B., Lockwood, E., Williams, C. C., Dogan, M. F., & Knuth, E. (2012). Middle School Students' Example Use In Conjecture Exploration And Justification. 34th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Kalamazoo, MI.
- Goldenberg, P., & Mason, J. (2008). Shedding light on and with example spaces. *Educational Studies in Mathematics*, 69, 183-194. <https://doi.org/10.1007/s10649-008-9143-3>
- Tall, D. (1988). *Concept Image and Concept Definition Senior Secondary Mathematics Education*, Utrecht.

# EMOTIONAL INTELLIGENCE AND MATHEMATICS ANXIETY'S IMPACT ON MATHEMATICS PERFORMANCE OF STUDENTS WITH AND WITHOUT MLD

Qingli Lei  
University of Illinois Chicago  
qlei@uic.edu

Di Liu  
East China Normal University  
dliu@spe.ecnu.edu.cn

Xiuhan Chen  
University of Missouri  
xcthc@missouri.edu

Megan Hirni  
University of Missouri  
mj.hirni@missouri.edu

Heba Abdelnaby  
University of Missouri  
hzadmb@mail.missouri.edu

*This study investigated the relationship between non-cognitive factors (mathematics anxiety, Emotional Intelligence, and mathematics self-concept) and mathematics performance in students with and without Mathematics Learning Disability (MLD). Participants were 340 3rd, 4th, and 5th grade students from a public elementary school. Results showed that students with MLD had significantly lower mathematics performance compared to their peers. Mathematics anxiety was found to have a negative impact on mathematics performance among students without MLD. While low Emotional Intelligence scores were significant predictors of lower math performance for students with and without MLD. Additionally, mathematics self-concept mediated the relationship between mathematics anxiety and mathematics performance. These findings have important implications for educators who work with students with and without MLD.*

Keywords: Equity, Inclusion, and Diversity, Elementary School Education, Students with Disabilities, Mathematical Knowledge for Teaching

## Introduction

### Students with Mathematics Learning Disability

Students with math learning disabilities (MLD) often encounter challenges in grasping mathematical concepts and skills, resulting in lower performance compared to their peers (Lei et al., 2018, 2020ab). Beyond struggling with mathematical content, individuals with MLD may also manifest non-cognitive characteristics that further influence their mathematical performance. These characteristics include math anxiety, a diminished math self-concept, and low emotional intelligence. While these non-cognitive factors can play a role in the academic outcomes of students with MLD, their precise impact varies. Some students may face substantial negative effects on their math performance, while others may not be as adversely affected. It is crucial to recognize that not all students with MLD will necessarily exhibit these non-cognitive characteristics, and the severity of effects on MLD can differ from one student to another, resulting in varying impacts on math performance (Lei, 2021; Lei & Xin, 2023).

### Mathematics Anxiety (MA)

Researchers have linked math anxiety with poor math performance of students across educational levels (Ashcraft, 2002). Math anxiety is defined as “a negative reaction to math and to mathematical situations” (Ashcraft & Ridley, 2005, p. 315) which negatively affects math performance. Individuals with math anxiety develop feelings of tension when introduced to academic and daily life situations involving math and solving number problems (Richardson & Suinn, 1972). Math anxiety limits efficiency in solving simple math problems and negatively

impacts performance on standardized tests, numerical reasoning courses and math-problem solving (Chang & Beilock, 2016).

### **Emotional Intelligence**

Emotional intelligence includes the ability to recognize, understand, and manage one's own emotions, as well as the emotions of others. Emotional intelligence, also known as emotional self-efficacy, examines individuals' emotional characteristics and self-perceptions. It is typically assessed using self-reporting measures (Petrides et al., 2007). Emotional intelligence impacts academic performance and affects the allocation of time and resources by educators when structuring academic interventions (Perera & DiGiacomo, 2013). However, the relationship between emotional intelligence and student achievement has been a topic of debate among scholars. Mavroveli and Sanchez-Ruiz (2011) found that young learners' mathematics performance could be improved by achieving higher emotional intelligence scores. Additionally, the study reported that students with disabilities had lower emotional intelligence scores compared to their peers.

### **Mathematics Self-concept**

A positive math self-concept, characterized by confidence, a sense of efficacy, and a belief in one's mathematical abilities, has been found to be a robust predictor of improved math performance (Wigfield et al., 2015). Ahmed et al. (2012) investigated the reciprocity between self-concept and anxiety in mathematics and found that lower mathematics self-concept leads to higher mathematics anxiety. In addition, Meece et al. (2006) established a positive association between math self-concept and math achievement, indicating that students with higher levels of self-concept tend to perform better in mathematics. This relationship suggests that individuals who possess a strong belief in their mathematical abilities are more likely to approach mathematical tasks with confidence, persistence, and a growth mindset, leading to improved performance outcomes.

We intended to answer following research questions:

1. What are the relationships between mathematics performance and student noncognitive characteristics (i.e., mathematics anxiety, mathematics self-concept, and emotional intelligence)?
2. Does mathematics self-concept mediate the relationship between mathematics anxiety and mathematics performance, and does this apply to both groups of students - those with and without MLD?

## **Research Methodology**

### **Participants**

Participants were recruited from the 3rd, 4th, and 5th grades in two ordinary public elementary schools in Shanghai, People's Republic of China. This project sampled 340 elementary students (179 girls and 153 boys) from 9 different classrooms. Among them, 123 students are third graders from three classrooms, 97 students are fourth graders from three classrooms, and 112 students are fifth graders from three classrooms. No individual within the sample received special education services or had documented brain injury or behavioral problems. Student scores in three recent mathematics mid-term and final tests were collected; each test was proctored within one hour, and three distinct mathematics scores (discussed further in the methods section) were computed from test results.

Using standardized diagnostic criteria (Cai et al., 2013), MLD group students are individuals with standard scores in the standardized mathematics test—as well as in three recent math tests—which ranked at the bottom 20% of the class. Control group students are those with standard scores

in standardized mathematics tests—as well as in three recent math tests—which ranked in the top 20% of the class. There were two exclusion criteria. Firstly, students with Motivation Adaption Assessment Test (MAAT) scores (Zhou, 1991) lower than two standard deviations were excluded; the MAAT tested student learning motivation. Secondly, intellectually impaired students were also excluded based on daily observations of mathematics teachers. No student's IQ score was below 80. There were no students excluded from MAAT and IQ tests.

We received the permission from all students' parents. The two groups had no significant difference in learning motivation but had significant difference in terms of mathematics scores. The mathematics achievements of both groups had no gender difference ( $t = 1.65, p = 0.10$ ).

## Measures

### Mathematics Achievement

Three measures of mathematics were administered: calculation fluency, numerical operations, and math reasoning (problem solving). **Calculation fluency** was adopted from WIAT-III (Wechsler, 2009) and included two subtests: addition fluency and subtraction fluency. In each subtest, children were asked to solve as many additions or subtractions as possible within a 60-second time limit. Each subtest included two pages (24 problems per page). A participant's score was the total number of addition and subtraction problems completed within the time limit (96 problems total). **Numerical operation** was also adopted from WIAT-III (Wechsler, 2009) and required children to solve mathematical operations. The task contains 38 items. **Mathematics reasoning** was assessed with the math standard achievement test (MSAT), which was based on the National Standards for Mathematics Curriculum of China. The test included 30 items: 26 items were multiple choice questions, and 4 items were fill-in questions (e.g., *Based on the map you have in front of you, how long will it take Fang to go to the bookstore, if she first passes by Hong's home?*).

### The Trait Emotional Intelligence Questionnaire-Child Form (TEI)

TEI contained 75 short statements with 5-point Likert scale response options (Mavroveli et al., 2008). The respondents were asked to rate each statement (e.g., *"I always find the words to show how I feel"*) using a 5-point scale that ranged from strongly disagree to strongly agree. The TEI comprises nine facets (i.e., adaptability, affective disposition, emotion expression, emotion perception, emotion regulation, low impulsivity, peer relations, self-esteem, and self-motivation) and it has demonstrated satisfactory reliability and validity in children between 8 and 12 years (Mavroveli et al., 2008; Mavroveli & Sanchez-Ruiz, 2011). The Chinese TEI was prepared with a user interface appropriate to the age of the respondents and pretested on a small group of subjects to assess comprehension and ease of answering. For each participant, scores on the nine facets and on global trait EI were computed. In this sample, Cronbach's alpha was 0.93.

### Mathematics Anxiety Scale for Children (MASC)

MASC contains 22 items. Children rated these items according to a 4-point scale in terms of how much anxiety they experienced. A rating of four points represents *extremely nervous*, three points *very nervous*, two points *a little nervous*, and one point represents *not nervous* (Chiu & Henry, 1990). The total score on these 22 items indicates the student's mathematics anxiety level. Moreover, MASC includes four different factors. Factor one was defined by eight of the items which were relevant to the *evaluation of mathematics learning*. Factor two was defined by six items; these items were *concerned* with the activity or process of learning mathematics. Factor three was defined by five items which related to *solving math problems* in a non-testing situation. For example, item two (*"Reading and interpreting graphs or charts"*), item three (*"Listening to another student explain a math problem"*), item nine (*"Picking up a math book to begin working on a homework assignment"*),

item ten (“Working on a mathematical problem”), and item 14 (“Being told how to interpret mathematics statements”). Factor three was labeled as mathematics problem solving. Factor four was relevant to mathematics teacher anxiety.

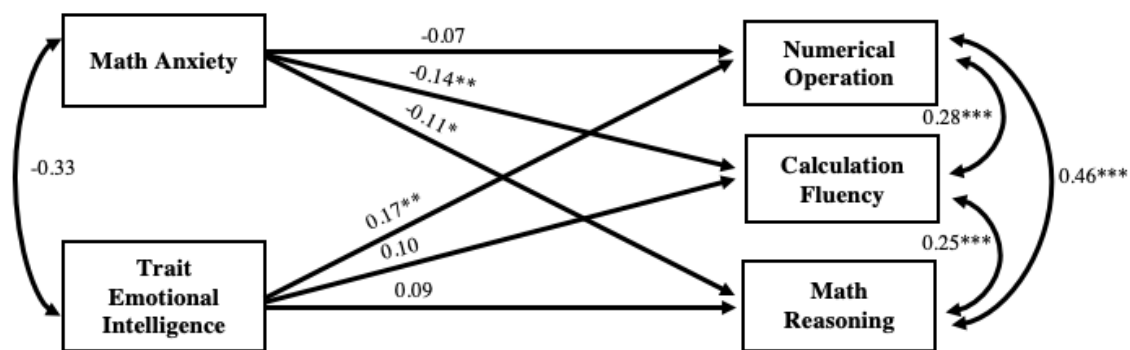
### Theoretical Framework and Procedures

A multivariate analysis was employed within the Structural Equation Modeling (SEM) framework to explore the relationship between math anxiety and math achievement and between the TEI and math achievement. A full sample ( $n = 326$ ) was used to fit the multivariate structure. In subsequent analytical steps, the model was fit by MLD ( $n = 75$ ) and non-MLD group ( $n = 251$ ). The coefficient of determination ( $R^2$ ) was used to evaluate the model fit. To further explore the data, a mediation model was adopted with the full sample ( $n = 326$ ). In the next phase of the analysis, the mediation model was implemented to fit the MLD ( $n = 75$ ) and non-MLD group ( $n = 251$ ) data. The comparative fit index (CFI; cutoff  $> .90$ ; Bentler, 1990) and the standardized root mean square residual (RMSEA; cutoff  $< .08$ ; Bentler, 1995) were used to evaluate the mediation model fit. The lavaan package (Rosseel, 2012) was utilized for data analysis using the software R (R Core Team, 2020).

### Results

The current study aims to examine the effects of two independent variables on three dependent variables within a multivariate framework using SEM. The two independent variables are MA and TEI, and the three dependent variables are numerical operation, calculation fluency, and mathematics reasoning.

A full sample of 326 was employed to explore the structural model. Bootstrapping was employed with 2000 iterations to estimate the 95% confidence interval for standardized coefficients. Path coefficients indicated that MA significantly predicted calculation fluency ( $\beta = -0.14$ , 95% CI [-0.25, -0.05]) and math reasoning ( $\beta = -0.11$ , 95% CI [-0.21, -0.01]), and TEI significantly predicted numerical operation ( $\beta = -0.17$ , 95% CI [0.07, 0.26]; see Figure 1 for path coefficients), controlling for grade and gender. The  $R^2$  values were 0.52, 0.33, and 0.40 for numerical operation, calculation fluency, and math reasoning, respectively, indicating that 52%, 33%, and 40% of the variance in the numerical operation, calculation fluency, and math reasoning was explained by MA and TEI (see Table 1).



**Figure 1. Baseline Model for Full Sample (N = 326)**

\* $p < 0.05$ , \*\* $p < 0.01$ , \*\*\* $p < 0.00$

**Table 1. Baseline Model Regression Coefficients for Full Sample (N = 326)**



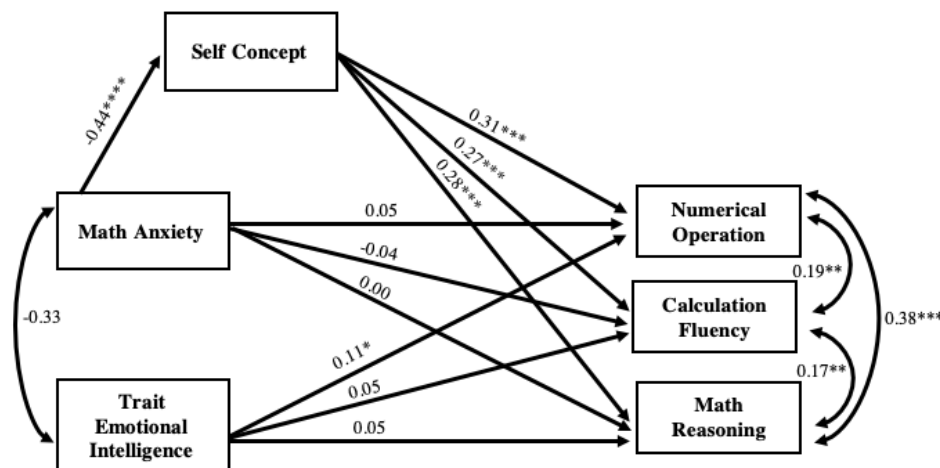
Regressions	Estimate	SE	Z	p
Numeric Operation ~ MA	<b>-0.07</b> [-0.15, -0.01]	0.04	-1.64	0.10
Calculation Fluency ~ MA	<b>-0.14</b> [-0.25, -0.05]	0.05	-2.83	< 0.01
Math Reasoning ~ MA	<b>-0.11</b> [-0.21, 0.01]	0.05	-2.24	< 0.05
Numeric Operation ~ TEI	<b>0.17</b> [0.07, 0.26]	0.05	3.46	< 0.01
Calculation Fluency ~ TEI	<b>0.10</b> [-0.02, 0.20]	0.06	1.71	0.09
Math Reasoning ~ TEI	<b>0.09</b> [-0.02, 0.19]	0.05	1.72	0.09

After further examining the model, the results from group without MLD ( $n = 251$ ) revealed that MA significantly predicted calculation fluency ( $\beta = -0.15$ , 95% CI [-0.26, -0.03]) and math reasoning ( $\beta = -0.10$ , 95% CI [-0.19, -0.02]), and TEI significantly predicted math reasoning ( $\beta = 0.10$ , 95% CI [0.01, 0.19]) and numerical operation ( $\beta = 0.13$ , 95% CI [0.03, 0.24]) while controlling for grade and gender. The  $R^2$  values for the model without MLD group were 0.48, 0.38, and 0.43 for numerical operation, calculation fluency, and math reasoning, respectively, indicating that 48%, 38%, and 43% of the variance in the numerical operation, calculation fluency, and math reasoning was explained by MA and TEI. Interestingly, controlling for grade and gender, the results from group with MLD ( $n = 75$ ) showed that TEI was a significant predictor to predict numerical operation ( $\beta = 0.29$ , 95% CI [0.09, 0.50]; see Table 2). The  $R^2$  values for the model without MLD group were 0.57, 0.34, and 0.60 for numerical operation, calculation fluency, and math reasoning, indicating that 57%, 34%, and 60% of the variance in the numerical operation, calculation fluency, and math reasoning was explained by MA and TEI. The overall result suggested that there was a group difference in terms of using MA and TEI to predict numerical operation, math reasoning, and calculation fluency.

**Table 2. Baseline Models for MLD and No MLD Groups**

Regression	Group: MLD				Group: No MLD			
	Estimate	SE	Z	p	Estimate	SE	Z	p
Numeric Operation ~ MA	<b>0.06</b> [-0.12, 0.23]	0.09	0.69	0.49	<b>-0.09</b> [-0.18, 0.00]	0.05	-1.81	0.07
Calculation Fluency ~ MA	<b>-0.08</b> [-0.31, 0.14]	0.12	-0.71	0.48	<b>-0.15</b> [-0.26, 0.03]	0.06	-2.58	< 0.05
Math Reasoning ~ MA	<b>0.03</b> [-0.19, 0.26]	0.12	0.24	0.81	<b>-0.10</b> [-0.19, -0.02]	0.04	-2.42	< 0.05
Numeric Operation ~ TEI	<b>0.22</b> [-0.01, 0.43]	0.11	1.98	0.05	<b>0.13</b> [0.03, 0.24]	0.05	2.54	< 0.05
Calculation Fluency ~ TEI	<b>0.29</b> [0.09, 0.50]	0.10	2.81	< 0.01	<b>0.03</b> [-0.08, 0.15]	0.06	0.43	0.67
Math Reasoning ~ TEI	<b>-0.10</b> [-0.32, 0.10]	0.11	-0.95	0.34	<b>0.10</b> [0.01, 0.19]	0.05	2.28	< 0.05

Subsequently, we implemented a mediation model using maximum likelihood estimation in SEM to explore the underlying mechanisms of the relationship between the two independent variables and three dependent variables ( $n = 326$ ) with self-concept selected as a mediator. Four direct paths from MA to self-concept, from self-concept to numeric operation, from self-concept to math reasoning, and from self-concept to calculation fluency were specified. Certain fit statistics, such as RMSEA, indicated a fit below the desired level of acceptability in our model ( $\chi^2(14) = 491.91, p < 0.05$ ; CFI = 0.97; RMSEA = 0.13(90% CI [0.09, 0.18]); notably, the small sample size may have led to an artificially larger RMSEA value (Kenny et al., 2015), so we include the 90% confidence interval. Figure 2 displays standardized path coefficients. The results indicated that MA had a significant direct effect on self-concept ( $\beta = -0.44, 95\% \text{ CI } [-0.55, -0.31]$ ). The direct effects from self-concept to numerical operation ( $\beta = 0.31, 95\% \text{ CI } [0.23, 0.39]$ ), to calculation fluency ( $\beta = 0.27, 95\% \text{ CI } [0.17, 0.37]$ ), and to math reasoning ( $\beta = 0.28, 95\% \text{ CI } [0.19, 0.37]$ ) were also significant (see Figure 2). Using 2000 bootstrapped samples, significant indirect effects from MA to numerical operation ( $\beta = 0.31, 95\% \text{ CI } [0.23, 0.39]$ ), calculation fluency ( $\beta = 0.27, 95\% \text{ CI } [0.17, 0.37]$ ), and math reasoning ( $\beta = 0.28, 95\% \text{ CI } [0.19, 0.37]$ ) through self-concept were observed, while controlling for grade and gender. The full mediation model structure was supported.



**Figure 2. Mediation Model for Full Sample (N = 326)**

\* $p < 0.05$ , \*\* $p < 0.01$ , \*\*\* $p < 0.001$

Furthermore, MLD group was used to further explore the mediation structure. Both models were controlled for grade and gender. In the group without MLD ( $n = 251$ ), the model demonstrated a poor fit of the data ( $\chi^2(4) = 18.465, p < 0.05$ ; CFI = 0.98; RMSEA = 0.12 (90% CI [0.07, 0.18])). The results indicated that MA has a significant direct effect on self-concept ( $\beta = -0.41, 95\% \text{ CI } [-0.55, -0.24]$ ); see Table 3). In addition, the direct effects from self-concept to numerical operation ( $\beta = 0.25, 95\% \text{ CI } [0.16, 0.34]$ ), to calculation fluency ( $\beta = 0.26, 95\% \text{ CI } [0.15, 0.36]$ ), and to math reasoning ( $\beta = 0.19, 95\% \text{ CI } [0.11, 0.26]$ ) were significant. Significant indirect effects from MA to numerical operation ( $\beta = 0.00, 95\% \text{ CI } [-0.10, 0.10]$ ), calculation fluency ( $\beta = -0.05, 95\% \text{ CI } [-0.16, 0.05]$ ), and math reasoning ( $\beta = -0.04, 95\% \text{ CI } [-0.12, 0.04]$ ) via self-concept were identified. The results were consistent with the model using the full sample size. In the group with MLD ( $n = 75$ ), the model fit the data relatively well  $\chi^2(4) = 7.974, p > 0.05$ ; CFI = 0.97; RMSEA = 0.12 (90% CI [0.00, 0.23]). Notably, only the partial mediation from MA to

numerical operation through self-concept was supported because a significant direct path ( $\beta = 0.22$ , 95% CI [0.06, 0.36]) and a significant indirect path ( $\beta = -0.18$ , 95% CI [-0.30, -0.09]) from MA to numerical operation were observed (see Table3).

In conclusion, the full mediation structure was presented in the non-MLD group, while the partial mediation structure was presented in the MLD group. There exist significant relationships between predictors and the mediator, and significant relationships between the mediator and the outcome variables (indirect effects).

**Table 3. Models for MLD and No MLD Groups with Self Concept as a Mediator**

Regression	Group: MLD				Group: No MLD			
	Estimate	SE	Z	p	Estimate	SE	Z	p
Self Concept ~ MA	<b>-0.47</b> [-0.64, -0.27]	0.09	-4.94	< 0.001	<b>-0.41</b> [-0.55, -0.24]	0.08	-5.33	< 0.001
Numeric Operation ~ Self Concept	<b>0.39</b> [0.25, 0.51]	0.07	5.64	< 0.001	<b>0.25</b> [0.16, 0.34]	0.05	5.20	< 0.001
Calculation Fluency ~ Self Concept	<b>0.19</b> [-0.02, 0.40]	0.11	1.78	0.08	<b>0.26</b> [0.15, 0.36]	0.05	4.83	< 0.001
Math Reasoning ~ Self Concept	<b>0.05</b> [-0.16, 0.24]	0.11	0.49	0.63	<b>0.19</b> [0.11, 0.26]	0.04	4.69	< 0.001
Numeric Operation ~ MA	<b>0.22</b> [0.06, 0.36]	0.07	3.01	< 0.01	<b>0.00</b> [-0.10, 0.10]	0.05	0.03	0.98
Calculation Fluency ~ MA	<b>-0.01</b> [-0.27, 0.27]	0.14	-0.03	0.98	<b>-0.05</b> [-0.16, 0.05]	0.05	-1.03	0.31
Math Reasoning ~ MA	<b>0.05</b> [-0.19, 0.28]	0.12	0.40	0.69	<b>-0.04</b> [-0.12, 0.04]	0.04	-0.91	0.37
Numeric Operation ~ TEI	<b>0.17</b> [-0.05, 0.37]	0.10	1.64	0.10	<b>0.09</b> [-0.01, 0.19]	0.05	1.82	0.07
Calculation Fluency ~ TEI	<b>0.26</b> [0.07, 0.48]	0.11	2.53	< 0.05	<b>-0.02</b> [-0.12, 0.10]	0.05	-0.30	0.76
Math Reasoning ~ TEI	<b>-0.11</b> [-0.32, 0.08]	0.11	-1.01	0.31	<b>0.07</b> [-0.02, 0.16]	0.05	1.62	0.11

### Conclusions and Implications

The study found that for students both with and without MLD, mathematics outcomes were impacted by low TEI scores. Mathematics anxiety was found to have a negative impact on math performance, and the relationship was partially mediated by mathematics self-concept. The findings emphasize the importance of addressing non-cognitive factors, such as mathematics anxiety and emotional intelligence, to improve the mathematics performance of students with and without MLD. The findings highlight the negative impact of mathematics anxiety on mathematics performance and the importance of addressing low mathematics self-concept to improve mathematics performance. Therefore, it is important for educators to address mathematics anxiety and provide support to individuals who experience it in order to help them develop positive attitudes toward mathematics and to improve mathematics performance. The findings have important implications for educators and policymakers who should consider non-cognitive factors in designing interventions to improve mathematics performance, particularly for students with MLD.

### References

Ashcraft, M. H. (2002). Math anxiety: Personal, educational, and cognitive consequences. *Current directions in psychological science*, 11(5), 181-185.

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

- Ashcraft, M. H., & Ridley, K. S. (2005). Math anxiety and its cognitive consequences. *Handbook of mathematical cognition*, 315-327.
- Bentler, P. M. (1990). Comparative fit indexes in structural models. *Psychological Bulletin*, 107(2), 238–246. <https://doi.org/10.1037/0033-2909.107.2.238>
- Bentler, P. M. (1995). *EQS structural equations program manual* (Vol. 6). Encino, CA: Multivariate software.
- Chang, H., & Beilock, S. L. (2016). The math anxiety-math performance link and its relation to individual and environmental factors: A review of current behavioral and psychophysiological research. *Current Opinion in Behavioral Sciences*, 10, 33-38.
- Chiu, L. H., & Henry, L. L. (1990). Development and validation of the Mathematics Anxiety Scale for Children. *Measurement and evaluation in counseling and development*.
- Lei, Q. (2021). *The effect of scaffolded model-based instruction on word problem-solving performance of English learners with learning difficulties in mathematics* (Doctoral dissertation, Purdue University Graduate School).
- Lei, Q., Mason, R. A., Xin, Y. P., Davis, J. L., David, M., & Lory, C. (2020a). A meta-analysis of single-case research on mathematics word problem-solving interventions for English learners with learning disabilities and mathematics difficulties. *Learning Disabilities Research & Practice*, 35(4), 201-217. <https://doi.org/10.1111/ldrp.12233>
- Lei, Q., & Xin, Y. P. (2023). A synthesis of mathematical word problem-solving instructions for English learners with learning disabilities in mathematics. *Review of Education*, 11(2), e3396.
- Lei, Q., Xin, Y. P., Morita-Mullaney, T., & Tzur, R. (2020b). Instructional Scaffolds in Mathematics Instruction for English Learners With Learning Disabilities: An Exploratory Case Study. *Learning Disabilities: A Contemporary Journal*, 18(1), 123-144.
- Lei, Q., Xin, Y. P., Morita-Mullaney, T., & Tzur, R. (2018). Analyzing a Discourse of Scaffolds for Mathematics Instruction for an ELL with Learning Disabilities. *North American Chapter of the International Group for the Psychology of Mathematics Education*.
- Mavroveli, S., Petrides, K. V., Shove, C., & Whitehead, A. (2008). Investigation of the construct of trait emotional intelligence in children. *European child & adolescent psychiatry*, 17(8), 516-526.
- Mavroveli, S., & Sánchez-Ruiz, M. J. (2011). Trait emotional intelligence influences on academic achievement and school behaviour. *British Journal of Educational Psychology*, 81(1), 112-134.
- Meece, J. L., Anderman, E. M., & Anderman, L. H. (2006). Classroom goal structure, student motivation, and academic achievement. *Annu. Rev. Psychol.*, 57, 487-503.
- Perera, H. N., & DiGiacomo, M. (2013). The relationship of trait emotional intelligence with academic performance: A meta-analytic review. *Learning and individual differences*, 28, 20-33.
- Petrides, K. V., Pita, R., & Kokkinaki, F. (2007). The location of trait emotional intelligence in personality factor space. *British journal of psychology*, 98(2), 273-289.
- Richardson, F. C., & Suinn, R. M. (1972). The mathematics anxiety rating scale: psychometric data. *Journal of counseling Psychology*, 19(6), 551.
- Kenny, D. A., Kaniskan, B., & McCoach, D. B. (2015). The Performance of RMSEA in Models With Small Degrees of Freedom. *Sociological Methods & Research*, 44(3), 486–507. <https://doi.org/10.1177/0049124114543236>
- Rosseel Y (2012). lavaan: An R Package for Structural Equation Modeling. *Journal of Statistical Software*, 48(2), 1–36.

## EPISTEMOLOGICAL OBSTACLES RELATED TO TREATING LOGICAL IMPLICATIONS AS ACTIONS: THE CASE OF MARY

Anderson Norton  
Virginia Tech  
norton3@vt.edu

Rachel Arnold  
Virginia Tech  
rlongley@vt.edu

Joseph Antonides  
Virginia Tech  
jantonides@vt.edu

Vladislav Kokushkin  
Virginia Tech  
vladkok@vt.edu

*Understanding how students reason with logical implication is essential for supporting students' construction of increasingly powerful ways of reasoning in proofs-based mathematics courses. We report on the results of an NSF-funded case study with a mathematics major enrolled in an introductory proofs course. We investigate the epistemological obstacles that she experienced and how they might relate to her treatment of logical implications as actions. Evidence shows that an action conception may pose challenges when students transform or quantify implications and may contribute to erroneous assumptions of biconditionality. Our report on available ways of operating with logical implications as actions is a first step in designing instructional tasks that leverage students' existing reasoning skills to support their continued development.*

Keywords: Cognition, Learning Theory, Reasoning and Proof, Undergraduate Education

From a constructivist perspective, engaging all learners requires teachers and researchers to understand students' available ways of operating and invite students to bring them forth to make meaning in new mathematical contexts. At the same time, teachers and researchers must recognize their role in supporting students' constructions of increasingly powerful ways of operating. For example, Dubinsky (1986) identified two distinct ways that students operate with logical implications (LIs): as actions or as objects. Instruction on LI must therefore be sensitive to students who reason in each of these ways.

As an action, an LI ( $P \rightarrow Q$ ) involves three components: a predicate,  $P$ ; a conclusion,  $Q$ ; and a transformation between them. Students who treat LI as an action can reason by *modus ponens*: the truth of  $P$  transfers, by implication, to the truth of  $Q$ . However, in treating an LI as an action across three components, rather than as a single object, students might experience persistent challenges in transforming and quantifying them. In particular, they might not reason by *modus tollens*:  $Q$  is false implies  $P$  must also be false ( $\sim Q \rightarrow \sim P$ ). They might also not attend to the quantification of an LI when determining its negation.

Here, we report on results from a case study with Mary—a mathematics major enrolled in an introductory proofs course in Fall 2022—who consistently treated LIs as actions. Data comes from three clinical interviews, as part of our larger study called The Proofs Project. Within our project, we frame epistemological obstacles (EOs) as persistent challenges experienced within mathematical interactions between the teacher and the student (cf. Brousseau, 2002). The purpose of this report is to document EOs related to LIs, their transformations, and their quantification, especially related to the treatment of LIs as actions. We address the following two research questions:

1. What specific EOs arise in mathematical interactions with Mary?

- How might these EOs relate to Mary’s treatment of LIs as actions?

### Theoretical Framework

We adopt an overarching Piagetian framework in which students’ logical-mathematical reasoning is understood in terms of their available ways of operating. In this section, we elaborate on what that means in terms of reasoning with LIs. Then, we interpret existing research on students’ logical reasoning within that framework.

#### Action-Object Theory Applied to Logical Implications

We conceptualize student reasoning about LI using an action-object perspective, grounded in Piaget’s (1970) genetic epistemology. According to this view, mathematical concepts arise as objects first through the coordination of mental actions (a process called *reflective abstraction*). APOS theory (Arnon et al., 2014; Dubinsky, 1991) is one Piagetian-based perspective that adopts an action-object view of mathematical thinking. Within APOS, an *action* is an explicit transformation that must be carried out step-by-step. Through interiorization, sequences of actions become coordinated into *processes* that may be carried out in thought. Note, however, that in our analyses, we do not distinguish between an action and a process; so for us, an “action” understanding may involve processes. Processes may be subsequently encapsulated into *objects* on which actions may be applied. When a student assimilates a task situation, they call to mind a *schema* of associated actions, processes, and objects.

Dubinsky (1991) hypothesized that for students to develop powerful ways of reasoning about logic and proof (e.g., to make sense of mathematical induction), they need to hold LI as an object. For this to occur, LI needs to be encapsulated (see Figure 1). Characteristically, a student with an action conception of LI might think about the statement “If  $P$ , then  $Q$ ” as a command to operate: verify that  $P$  is true, and if so, then  $Q$  is true. A student with an object conception of LI has interiorized this way of operating, and they can meaningfully transform an LI, such as by taking its converse, contrapositive, or negation.

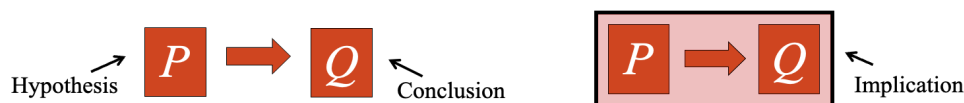


Figure 1: Logical Implication as Action or Object

#### Epistemological Obstacles Related to Logical Implications

The concept of an EO originates from the work of Sierpińska (1987) and Brousseau (1997, 2002), in which EOs were conceptualized as the necessary challenges in students’ mathematical development. We frame EOs as cognitive challenges experienced by both teachers and students during instructional interactions. EOs persist over time, even in research-based instruction.

Prior literature has identified multiple challenges related to student reasoning with LIs. For example, students tend to interpret a conditional statement (*if  $P$ , then  $Q$* ) as a biconditional one ( *$P$  if and only if  $Q$* ) (e.g., Giroto, 1990; Wilkins, 1928). Although such interpretations are not valid from the standpoint of formal logic, they are deeply rooted in everyday language and can be explained from a pragmatic point of view (Epp, 1999; Geis & Zwicky, 1971; Romain et al., 1983; Wagner-Egger, 2007). Transforming LIs presents a related challenge. Empirical studies have shown that undergraduate students often conflate an LI ( $P \rightarrow Q$ ) with its converse ( $Q \rightarrow P$ ) (Durand-Guerrier, 2003; O’Brien et al., 1971) and with its inverse ( $\sim P \rightarrow \sim Q$ ) (Goetting, 1995; Knuth, 1999, 2002). Other scholars reported on students’ struggles to understand the equivalence between an LI

and its contrapositive ( $\sim Q \rightarrow \sim P$ ) (Dawkins & Hub, 2017; Stylianides et al., 2004) or between an LI and a disjunction ( $\sim P \vee Q$ ) (Hawthorne & Rasmussen, 2015).

Quantification presents another challenge, particularly when quantifiers are hidden. Shipman (2016) observed that the LI  $P \rightarrow Q$  in fact has the form  $P(x) \rightarrow Q(x)$ . The latter contains a hidden universal quantifier and is conventionally interpreted as " $\forall x, P(x) \rightarrow Q(x)$ ." Overlooking hidden quantifiers can result in logical fallacies, which are especially evident when students attempt to transform LIs (e.g., negate them). In a similar vein, Durand-Guerrier (2003) reported on the challenge of dealing with LIs when the quantifiers are implicit.

Logical statements containing multiple quantifiers (i.e., multiply quantified statements) exacerbate students' struggles with quantification. Prior studies have documented naïve readings of such statements (Dubinsky & Yiparaki, 2000; Epp, 2003). Dawkins and Roh (2020) found that undergraduate students tend to read multiply quantified statements semantically, i.e., applying their previous knowledge to make sense of the formal statements, thereby neglecting the syntax of the statement and the order of the quantifiers in particular (Piatek-Jimenez, 2010). In our framing, such EOs should be deliberately evoked and carefully addressed over time.

### Methods

Our study took place in the context of an introductory proofs course in the Fall of 2022. Using initial data from the classroom, we applied the technique of purposeful sampling (Maxwell, 2013) to select students who we thought would provide diverse ways of thinking and who seemed to articulate their reasoning. In total, four students agreed to participate in three clinical interviews. Each interview was between 30-60 minutes and was video recorded to capture students' interactions with the interviewer. The first interview focused on students' reasoning with LIs. The second interview focused on quantification of variables, statements, and LIs. The third interview employed stimulated recall, using video clips from the first two interviews to question students about their prior responses and underlying reasoning. Tasks for the first two interviews were given separately on slips of paper, and students were provided with a Livescribe pen to record what they wrote and drew.

We used the constant comparative method (Corbin & Strauss, 2014) to build and iteratively refine a system of codes for categorizing interview data. Two researchers conducted a first round of coding of each student's first two interviews, first analyzing all students' first interviews, then all students' second interviews. This initial analysis allowed us to (a) build a draft codebook drawn from literature reviewed within our theoretical framework; and (b) select important moments in each student's first two interviews to be presented to students during the stimulated recall portion of their third interview.

In a second round of coding, the researchers applied the draft codebook to analyze video data student-by-student. During this process, we clarified the definitions and indicators of codes in the initial draft codebook, defined new and emergent codes as needed, and resolved disagreements by discussing our interpretations of the data with the research team until reaching consensus. As we coded, we also wrote analytic memos of our hypotheses and interpretations of the data, and we noted quotes that seemed to capture important moments in students' reasoning.

### Results

Within our framework, we report results in two sections: one assessing Mary's treatment of LIs as actions, versus objects; the other identifying EOs experienced during the interviews.

#### **Mary's Treatment of Logical Implications as Actions**

As indicated in Table 1, we coded 13 segments indicating how Mary operated with LIs: nine segments from the first interview and four segments from the second interview. Each segment

provided evidence that Mary treated LIs as actions. The strongest evidence came from the middle of the first interview, as Mary responded to the first question (a) shown in the following Probability Task:

Let  $P$  and  $Q$  be events that have some nonzero probability of occurring, and suppose the following two implications are true:

- If  $P$  and  $Q$  are mutually exclusive, their probabilities are not independent.
  - If the probabilities of  $P$  and  $Q$  are independent, the probability of  $P$  and  $Q$  is the product of the probability of  $P$  and the probability of  $Q$
- (a) What can you conclude if  $P$  and  $Q$  are independent?  
 (b) What can you conclude if the probability of  $P$  and  $Q$  is not the product of the probability of  $P$  and the probability of  $Q$   
 (c) What can you conclude if  $P$  and  $Q$  are mutually exclusive?

Mary: It's an if statement, so it's either, that happens, or it doesn't happen.  
 R1: Oh, I see. So, you are talking about this part happening or not [pointing to the left side of the second LI (second bullet point)].  
 Mary: So, we already know that this part happens because P and Q are independent, and it's asking if they are independent [pointing to the LI]. So, this is true; if is a true statement in this case [points thumb up]... If happens. But in this one [pointing to the first LI], we don't necessarily know if it happens because... In my understanding of how if-statements kind of work, being dependent has to depend on being mutually exclusive, but being mutually exclusive does not necessarily mean it's dependent. So, if P and Q are mutually exclusive... Actually, if they are mutually exclusive... [long pause] No, I'm going to stick with that. If P and Q are mutually exclusive, they could still be independent... So, we don't know if P and Q are mutually exclusive, but we do know what their combined probability is, but I can't say anything else because I don't have...  
 R1: Okay, so in addition to knowing that P and Q are independent, you can also tell me what other fact?  
 Mary: For "a", the only thing else we know for a fact is that the probability of P and Q is the product of the probability of P and the probability of Q.

**Table 1: Common Codes**

Code	Description	Occurrences
AO	evidence of student treating LI as an action or object	9, 4
Liff	interpreting LIs as biconditional	6, 0
Lit	transforming LIs into their converses, contrapositives and negations	5, 1
Qh	hidden quantification, especially leading to ambiguity in meaning	2, 6
Qord	reasoning about order of quantification in the meaning of a statement	0, 7

Mary's response at the start of the transcript indicates that she thought about LIs as on/off switches: when the "if" part of an LI switches on, the "then" part switches on. This way of operating fits an action conception of LI: the action of turning on the first part of the LI (the hypothesis) is followed by the action of turning on the second part of the LI (the conclusion). This way of operating worked well for Mary when the hypothesis was given, as it was for the case of



the second bulleted LI. However, when the hypothesis was not given, Mary was prone to reversing the LI (as she did for the first bulleted LI) and did not attempt to reason with *modus tollens* via its contrapositive.

Mary's reference to LIs as on/off switches persisted across all three interviews (including the SRI) and, at times, led her to think of LIs as biconditional (if and only if): either the hypothesis and conclusion of an LI were both true or both false. We will see evidence of this reasoning in the next section, specifically as we discuss the Liff code.

### **Epistemological Obstacles Experienced during Instructional Interactions with Mary**

Among the potential EOs identified in prior literature, four appeared frequently (more than three times each) in our coding of Mary's first two interviews: Liff, LIIt, Qh, and Qord (see Table 1). In this section we provide examples for each code and characterize the EOs in relation to Mary's ways of operating with LIs.

**Liff.** We have seen that Mary could apply an LI to a given hypothesis to draw the pertinent conclusion (*modus ponens*), using her on/off switch. When representing an LI abstractly (outside of directly applying it to a given hypothesis) Mary relied on a subset relationship and generated concrete examples. For instance, during the first interview, in considering  $P \rightarrow Q$ , Mary introduced a diagram showing that horses, dogs, and porcupines form subsets of the set of mammals. She explained that if we knew something were a horse, we would already know it was a mammal; but there are other subsets of mammals, so an animal being a mammal does not necessarily imply it is a horse (i.e., the converse  $Q \rightarrow P$  does not necessarily follow). Elsewhere in the first interview, she explained that if-and-only-if relations occur when there is only one possible subset: "A is the only subset of B."

After reviewing the first interview, the researchers identified a potential conflict between Mary's two ways of operating with LIs: the on/off switch and the subset relationship. We found initial evidence as Mary responded to question (c) in the Probability Task. Mary read the question aloud and proceeded as follows.

- Mary: So if they are not mutually exclusive, then they're independent, and then this is also true [pointing to second bulleted LI].
- R1: How did you know that?
- Mary: Because, if they are mutually exclusive then their probabilities are not independent [pointing to first bulleted LI], so that means they are dependent. So, if they are mutually exclusive, they are dependent. But we are saying that they are not mutually exclusive, so they can't be dependent, so they have to be independent. Because dependent and independent is just like an on and off switch [waving hands back and forth]. It's either one or the other. It's binary. And since we know that it's independent, we know... [points to second bulleted LI]

We inferred from Mary's explanation that she had applied her binary on/off switch to the first LI, taking the negation of the hypothesis to infer the negation of the conclusion: because the "if" was off, it followed that the "then" was off. Note that this way of operating equates the LI  $P \rightarrow Q$  with its inverse  $\sim P \rightarrow \sim Q$  (the converse of the contrapositive) rendering the relation biconditional. This way of operating served Mary well in reasoning with the contrapositive of an LI, but it conflicted with her subset meaning for LIs whereby she had explicitly denied that the converse

must follow. We investigated this potential cognitive conflict further during the third interview, which we discuss in the next subsection.

**LI**t. Early in the first interview, Mary drew an “umbrella” set with two subsets under it. She argued that if something did not exist in the umbrella set, it could not exist in either of the subsets under it (an argument for the contrapositive). However, shortly after, she opened up the possibility that there could be a second umbrella under which the subset could fall so that something could exist in the subset without existing in the first umbrella. Thus, it seems that Mary was not certain about the logical equivalence of an LI and its contrapositive. Further evidence for this claim comes from Mary’s responses to the questions in the Probability Task. For example, in response to question (a), Mary drew the direct conclusion using the second bulleted LI, summarizing, “that’s the only thing I can say with 100% certainty.” She did not apply the given condition to the negation of the conclusion in the first bulleted LI to infer the negation of its hypothesis.

On the other hand, when Mary treated LIs as biconditional, using her on/off switch, both the contrapositive and converse logically followed. There is no need to transform a biconditional statement because the hypothesis and conclusion are paired with the same truth value: they are both on or both off. Thus, Mary’s treatment of LIs as actions resolved any ambiguity she might have experienced in attempting to transform an LI into its contrapositive. At the same time, it equated an LI with its converse, which Mary knew, via her subset representations, did not always follow. The researcher attempted to induce this potential cognitive conflict in the third interview by showing Mary a video clip of her responses (from the first interview) to the Probability Task.

- Mary: [having just reviewed an audio-video clip of her saying, “they could still be mutually exclusive without being dependent”] I would like to amend that statement.
- R1: How would you amend it?
- Mary: If-then, um, if P and Q are mutually exclusive, their probabilities are not independent. Since we know P and Q *are* independent, they are *not* mutually exclusive.
- R1: Oh! OK. That *is* something new. So, what do you think changed from that Mary to this Mary, to where you had that realization?
- Mary: I think we went through the if-then unit [in class]. Before that I always used if-then as, like, it *could* work, but kind of like two inputs could converge and get one outcome, but now I know that’s not true, I guess.
- R1: What do you mean “two inputs could converge and give you one output.”
- Mary: Like, before I thought, with an if-then statement, the then could be true but the if not necessarily, but if the if is true, then the then also has to be true. But on its own the then could be true. I thought that before, but now I do not think that.
- R1: Oh, you don’t think the then can be true unless the if is true.

After Mary affirmed the researcher’s interpretation, the interview continued with the researcher showing Mary a video clip of her explaining (to the contrary) that “being dependent has to depend on being mutually exclusive, but being mutually exclusive does not necessarily mean it’s dependent.” Just then, Mary interjected, “yes!” and explained as follows.

- Mary: It’s like, if W implies X, just because you have X doesn’t mean W is true.

- R1: But I thought you said just a little while ago that the only way the then happens is if the if happens.
- Mary: [Long pause] Mm-hmm. I did say that.
- R1: But what you just said now...
- Mary: Contradicts that [laughing]? It does. Can I draw this out?

Mary then drew an Euler diagram in which the circle representing cases in which  $W$  is true was contained within the circle representing cases in which  $X$  was true. Then she exclaimed, “Oh dear! That just contradicts directly with what I was saying before. I take back what I said before. I’m going back to my original.”

In treating LI as an action, Mary had accommodated her on/off switch to account for the logical equivalence of the contrapositive. However, this action of switching hypotheses and conclusions on and off together introduced the logical equivalence of the converse as well. Relying on a subset representation, such as an Euler diagram, Mary recognized the contradiction, but by the end of the third interview it remained unclear how she might reliably transform LIs into their contrapositives without also equating them with their converses.

**Qh.** Mary attended to the language of quantification within mathematical statements, including LIs, but she sometimes did so in unconventional ways. For instance, at the start of the first interview, she was considering an LI whose conclusion specified that all objects  $x$  in set  $S$  satisfy a particular property,  $P$ . Then, when considering a statement that “not all” objects satisfy that property, rather than taking this statement as the negation of the conclusion (thus, negating the hypothesis as well, via the contrapositive), she indicated that “not all” meant the conclusion might or might not happen. Rather than treating the conclusion as an invariant condition (that all  $x$  in  $S$  satisfy  $P$ ), she had broken it into two cases: the case where the condition was met and the case where it was not. The quantification “not all” meant for her that both cases could exist, and so the conclusion was true for one case and not the other. In the second interview, we find stronger evidence for how Mary quantified LIs themselves.

At the start of the second interview, the researcher asked Mary to respond to the following task from Shipman (2016), which contains an LI within an LI: “If  $A$  and  $B$  are sets, then  $(x \in A \Rightarrow x \in B) \Rightarrow A \subseteq B$ .” The following exchange began as the researcher asked Mary whether she could quantify the LI shown in parentheses.

- R1: Does this mean there exists an  $x$  such that  $x$  of  $A$  implies  $x$  of  $B$ , or does it mean for all  $x$ ,  $x$  of  $A$  implies  $x$  is in  $B$ ? Yeah, how would you quantify this implication, just this part [pointing to the LI in parentheses]
- Mary: Oh, I would say there is an  $x$  within  $A$  that implies... There exists an  $x$  such that an  $x$  within  $A$  implies that  $x$  is also in  $B$ .
- R1: What if I changed the quantification and said for all  $x$ ,  $x$  is in  $A$  implies  $x$  is in  $B$ . Would that change your answer?
- Mary: Yes. For all values of  $x$ ,  $x$  of  $A$  implies  $x$  of  $B$ . Actually, it’s false, because there could be a number outside of  $x$ , like a  $y$  outside of  $x$ , that doesn’t make the implication true.
- R1: Alright, I’m going to quantify it one more time: for all  $x$  in  $A$ ,  $x$  being an element of  $A$  implies  $x$  is also an element of  $B$ . So, if this implication is true for all  $x$  in  $A$ ...
- Mary: Then it is a subset, yes.

- R1: What changed?  
 Mary: Because you told me that  $x$  is  $A$ . You basically told me that  $x$  is within  $A$ , and for any  $x$  in  $A$ , it's also going to be in  $B$

Mary seemed more confident about the truth of the conclusion (that  $A$  is a subset of  $B$ ) when the universal quantification of the LI was restricted to  $A$ . This would seem illogical because the universal quantification of the LI within the entire universe of numbers would include its quantification within  $A$ . However, Mary's reasoning makes more sense when we consider her tendency to move the quantification of an LI within the LI, applying it to the hypothesis that  $x$  is in  $A$ , instead: When  $x$  is in  $A$ , the first switch is on. when  $x$  is not in  $A$ , it's off. Thus, we conclude that Mary noticed the role of quantifications of statements within implications, but she did not seem to quantify LIs themselves.

**Qord.** The second interview included two sets of tasks designed specifically to assess whether students attended to the order of quantification when a statement included multiple quantifiers. All of the Qord codes occurred in our analysis of Mary's responses to those tasks. One such task asked Mary to evaluate the following statement: "For all positive real numbers  $k$ , there is a natural number  $M$  such that  $1/k < M$ " (Piatek-Jiminez, 2010). Mary responded as follows: " $k$  could be like 1 over a billion, and therefore this  $1/k$  would be a billion. And there technically is an  $M$ —a billion and 1—that is bigger than  $1/k$ , so I say that, yes, it's true. You could literally make  $M$  the inverse of  $k$ , plus 1."

From this response and similar responses to the other tasks, we inferred that Mary did not seem to attend to the order of quantifiers and how this order might alter the meaning and validity of statements. However, because these quantifiers occurred within statements and not across LIs, we do not currently have a hypothesis about how Mary's treatment of LIs as actions might influence her treatment of multiple quantifiers and their order. We conjecture that students might need to objectify sequences of quantifiers in the same way that they objectify LIs.

### Conclusions

Mary had developed powerful ways of reasoning with LIs (e.g., reasoning by *modus ponens*). However, in treating LIs as actions (not objects), Mary seemed to experience challenges anticipated within our action-object framework (Dubinsky, 1991; Piaget, 1970), including EOs identified in prior research (e.g., Epp, 1999; Wagner-Egger, 2007). Specifically, she experienced persistent challenges in transforming and quantifying LIs because they were not, for her, objects to be transformed or quantified. In attempting to transform LIs into their contrapositives, Mary accommodated her on-off scheme for LIs to include biconditionality, rendering the LI logically equivalent to its converse (Goetting, 1995). In attending to quantification of an LI, Mary moved the quantification into the LI, applying it to the hypothesis instead (Norton et al., 2022). Thus, we find that many EOs identified in prior research might relate to treatment of LI as an action.

Mary's reasoning relied on particular ways of operating with LIs, such as her on/off switch and her use of subset relationships. Our understanding of those ways of operating is a necessary starting point for engaging students like Mary and for supporting their continued development. It enables us to "design learning environments that take student engagement and learning into account." We are particularly interested in designing tasks that might support Mary's objectification of LIs, as a potential means of meaningfully addressing the EOs she experienced with regard to transforming and quantifying LIs. We recognize that as much as we have learned about logical ways of operating from Mary, we have more to learn from other students.

## Acknowledgments

This project is supported by the National Science Foundation under grant numbers DUE-2141626. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the NSF.

## References

- Arnon, I., Cottrill, J., Dubinsky, E., Oktaç, A., Fuentes, S. R., Trigueros, M., & Weller, K. (2014). *APOS theory: A framework for research and curriculum development in mathematics education*. Springer.
- Brousseau, G. (1997). *Theory of didactical situations in mathematics 1970–1990*. Translation from French: M. Cooper, N. Balacheff, R. Sutherland & V. Warfield. Dordrecht, The Netherlands: Kluwer Academic Publishers.
- Brousseau, G. (2002). Epistemological obstacles, problems, and didactical engineering. *Theory of Didactical Situations in Mathematics: Didactique des Mathématiques, 1970–1990*, 79–117.
- Corbin, J., & Strauss, A. (2014). *Basics of qualitative research: Techniques and procedures for developing grounded theory* (4th ed.). Sage.
- Dawkins, P. C., & Hub, A. (2017). Explicating the concept of contrapositive equivalence. In Galindo, E., & Newton, J., (Eds.). (2017). *Proceedings of the 39th Annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.
- Dawkins, P. C., & Roh, K. H. (2020). Assessing the influence of syntax, semantics, and pragmatics in student interpretation of multiply quantified statements in mathematics. *International Journal of Research in Undergraduate Mathematics Education*, 6(1), 1–22.
- Durand-Guerrier, V. (2003). Which notion of implication is the right one? From logical considerations to a didactic perspective. *Educational Studies in Mathematics*, 53(1), 5–34
- Dubinsky, E. (1986). Teaching mathematical induction: I. *The Journal of Mathematical Behavior*, 5, 305–317.
- Dubinsky, E. (1991). Constructive aspects of reflective abstraction in advanced mathematics. In L. P. Steffe (Ed.), *Epistemological foundations of mathematical experience* (pp. 160-202). Springer.
- Dubinsky, E., & Yiparaki, O. (2000). On student understanding of AE and EA quantification. *Research in Collegiate Mathematics Education*, IV, 239–289.
- Ernst, P. (1984). Mathematical induction: A pedagogical discussion. *Educational Studies in Mathematics*, 15(2), 173–189.
- Epp, S. (1999). The language of quantification in mathematics instruction. *Developing mathematical reasoning in grades K-12 (1999 Yearbook)*, 188–197.
- Epp, S. (2003). The role of logic in teaching proofs. *The American Mathematical Monthly*, 110, 886–899.
- Geis, M. L., & Zwicky, A. M. (1971). On invited inferences. *Linguistic Inquiry*, 11, 561–565.
- Giroto, V. (1990). Biases in children's conditional reasoning. In J-P. Caverni, J.-M. Fabre, & M. Gonzalez (Eds.), *Cognitive biases*. Amsterdam: Elsevier.
- Goetting, M. M. (1995). *The college student's understanding of mathematical proof*. University of Maryland, College Park.
- Hawthorne, C., & Rasmussen, C. (2015). A framework for characterizing students' thinking about logical statements and truth tables. *International Journal of Mathematical Education in Science and Technology*, 46(3), 337–353.
- Knuth, E.J. (1999). *The Nature of Secondary School Mathematics Teachers' Conceptions of Proof*, Doctoral Dissertation, University of Colorado, Boulder.
- Knuth, E.J. (2002). Secondary school mathematics teachers' conceptions of proof. *Journal for Research in Mathematics Education*, 33(5), 379–405.
- Maxwell, J. A. (2013). *Qualitative research design: An interactive approach* (3rd ed.). Sage.
- Norton, A., Arnold, R., Kokushkin, V., & Tiraphatna, M. (2022). Addressing the cognitive gap in proof by mathematical induction. *International Journal for Research in Undergraduate Mathematics Education*, 1-27.
- O'Brien, T. C., Shapiro, B. J., & Reali, N. C. (1971). Logical thinking-language and context. *Educational studies in mathematics*, 201–219.
- Piaget, J. (1970). *Genetic epistemology* (E. Duckworth, Trans.). Norton.
- Piatek-Jimenez, K. (2010). Students' interpretations of mathematical statements involving quantification. *Mathematics Education Research Journal*, 22(3), 41–56.
- Rumain, B., Connell, J., & Braine, M. D. (1983). Conversational comprehension processes are responsible for reasoning fallacies in children as well as adults: If is not the biconditional. *Developmental psychology*, 19(4), 471–481.
- Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

- Shipman, B. A. (2016). Subtleties of hidden quantifiers in implication. *Teaching Mathematics and its Applications*, 35(1), 41–49.
- Sierpińska, A. (1987). Humanities students and epistemological obstacles related to limits. *Educational Studies in Mathematics* 18, 371–397.
- Stylianides, A. J., Stylianides, G. J., & Philippou, G. N. (2004). Undergraduate students' understanding of the contraposition equivalence rule in symbolic and verbal contexts. *Educational Studies in Mathematics*, 55(1-3), 133–162.
- Wagner-Egger, P. (2007). Conditional reasoning and the Wason selection task: Biconditional interpretation instead of reasoning bias. *Thinking & Reasoning*, 13(4), 484–505.

## ESTABLISHING A MATHEMATICAL BELONGINGNESS CONSTRUCT: EXPLORATORY FACTOR ANALYSIS OF IES'S HIGH SCHOOL LONGITUDINAL STUDY 2009

Lybrya Kebreab  
Saint Louis University  
lybrya.kebreab@slu.edu

Sarah B. Bush  
University of Central Florida  
Sarah.Bush@ucf.edu

Debbie Hahs-Vaughn  
University of Central Florida  
debbie@ucf.edu

Farshid Safi  
University of Central Florida  
Farshid.Safi@ucf.edu

Janet Andreasen  
The Geneva School  
Jbandreasen@genevaschool.org

Christa Jackson  
Saint Louis University  
Christa.jackson@slu.edu

*This investigation utilized publicly available data from the High School Longitudinal Study 2009 (HSLs:09) by the National Center for Educational Statistics (NCES) to examine any latent structures among variables which may empirically support the validity and reliability of a mathematical sense of belonging (MSB) construct. Using the nationally representative survey data in the HSLs:09, the proposed study performed an exploratory factor analysis (EFA) to create a complex construct of students' MSB or mathematical belongingness. A weighted sample of over 13,000 (N=13,354) high school students was used to conduct the EFA, which was then mapped onto Mahar and colleagues' (2012) intersecting, transdisciplinary themes of belonging. Possible implications for preservice teachers, professional development for practicing teachers, as well as future research directions are discussed.*

Keywords: Affect, Emotion, Beliefs, and Attitudes; Teacher Educators; Professional Development; High School Education

### Introduction

Research is scarce on empirical, quantitative studies of variables related to secondary students' mathematical sense of belonging, especially based on data from a nationally representative sample. The field could benefit from a deeper understanding of high school students' perceptions of their mathematical belongingness, based on the five themes of belongingness that have been found across disciplines (Mahar et al., 2012). Consequently, this investigation seeks to utilize the publicly available data from the High School Longitudinal Study 2009 (HSLs:09) by the National Center for Educational Statistics (NCES) to examine any latent structures among variables which may empirically support the validity and reliability of a mathematical sense of belonging (MSB) construct.

Using nationally representative survey data in the HSLs:09, the proposed study performed exploratory factor analyses to parsimoniously explain the shared variance among a plethora of variables related to a complex construct of MSB or mathematical belongingness (hereafter used interchangeably). Experts in exploratory factor analysis (EFA) support these parsimonious solutions as they are "generally considered to have greater external validity and, as such, are more likely to replicate" (Henson & Roberts, 2006, p. 394). Importantly, this research extended and shed new light on over two decades of previous studies on belonging with respect to not only school sense of belonging, but also domain-specific belongingness. Simultaneously, the study addressed contemporary needs to elucidate factors which may be important for increasing equity and access in mathematics education as it a well-known gatekeeper for preparation and belongingness in the STEM fields (U.S. Department of Education, 2022).

## **Purpose and Research Questions**

Belongingness is basic human need which can impact several different areas of a student's school experiences as well as life and career choices (U.S. Department of Education, 2022). However, studies suggest upwards of 84% of students (Holian et al., 2020; NCTM, 2014) retain no interest (nor intention to remain) in mathematics/STEM related careers after they finish their compulsory, high school education. There is an ongoing debate regarding which current mathematics classroom norms (e.g., mathematics as a gift not a learned skill; too much focus on testing and answer-getting; lack of academic safety; little to no high-quality studies of wonder, beauty and joy) are unproductive and may impede a mathematical sense of belonging for many students (NCTM, 2018).

Despite this unfortunate state of affairs, there is still hope that we can change this situation through research which provides a deeper understanding of students' affective states, i.e., mathematical belongingness. If students' motivation and achievement in mathematics tends to wane as they enter adolescence (Wang & Pomerantz, 2009), it is important to empirically define affective variables, i.e., MSB, in secondary education to begin to understand why. Using common themes from and alignment to extant literature, students' MSB will be operationally defined, then empirically examined through a composition of variables in the HSLs:09, a large-scale, publicly available data set. Hence, the purpose of this study is to refine the literature on mathematical belongingness and its importance as an area of study in mathematics education. To this end, the following questions were researched:

1. Which factors have been found to contribute to various disciplinary theories of belongingness irrespective of the academic discipline?
2. What factor(s) can be empirically researched to establish a construct for a discipline-specific mathematical sense of belonging (MSB) within existing theoretical frameworks in the research?

## **Literature Review**

### **Extant Literature on Belonging**

Researchers have long found a sense of belonging to be a broadly established (Lerner et al. 2005) basic human need (Maslow, 1943; Schunk, 2012). Belongingness in school has been theorized by Baumeister and Leary (1995) as a prerequisite for overall school functioning. Further, multiple studies have found a sense of belonging at school to positively relate to students' motivation, self-esteem, classroom behavior and academic achievement, while negatively relating to school dropout (Korpershoek et al., 2020).

Teachers are empowered to and responsible for establishing optimal learning environments for students in which their cognitive and affective needs can be met (Graesser & D'Mello, 2011). Graesser and D'Mello (2011) contend that deep learning occurs when the learning environment responds to learners in ways that are sensitive to the learners' affective as well as their cognitive states. Therefore, a holistic approach to learning, where both academic and socioemotional skill development are addressed simultaneously, is necessary for student success in mathematics (de Royston et al., 2020). Secondary mathematics preservice and practicing teachers experience professional development related to honing flexible and effective pedagogical strategies, which can be described as pedagogical fluency (Kebreab et al., 2021), to address cognitive aspects of the classroom (e.g., content standards, curriculum, formative and summative assessments). However, it can be challenging to overcome so-called traditional beliefs and practices (Anderson et al., 2018) in order to cultivate pedagogical fluency related to the affective domain (e.g., students' MSB, intrinsic motivation) (Hannula et al., 2019). Teachers may not be cognizant of their own



unproductive beliefs and practices (Jackson & Delaney, 2017; Schoenfeld, 2015) which may also impede their students' MSB. In the affective domain, belongingness is a vital component of learning that cannot be emphasized enough nor overstated (Arends & Visser, 2019).

Although numerous studies have examined the impact of belongingness on *general* academic and social outcomes in schools, research literature is scarce on students' sense of belonging specifically in the mathematics community (Barbieri & Miller-Cotto, 2021). It is crucial for mathematics education researchers and practitioners to develop productive beliefs and effective practices (Kilpatrick et al., 2001; NCTM, 2014) that attend to the ways in which belongingness and learning are associated in mathematics (Barbieri & Booth, 2016; Barbieri & Miller-Cotto, 2021; Good et al., 2012; OECD, 2017) in order to cultivate students' genius across academic disciplines (Muhammad, 2022) and focus on their brilliance (Leonard & Martin, 2013). Additionally, the need for mathematics education has evolved from economic and national defense motives (Tate, 2013) to becoming a necessity for understanding the world and fully engaging in a democratic society (NCTM/NCSM, 2019). Furthermore, Moses and Cobb (2001) argue "[i]n today's world, economic access and full citizenship depend crucially on math and science literacy," and that "math literacy...is the key to the future of disenfranchised communities" (Moses & Cobb 2001, pp. 4-5). For these reasons, belongingness in mathematics, as a precursor for learning in which successful achievement provides access to developing mathematical literacy and prepares students as informed members of a democratic society, can also be considered a 21<sup>st</sup>-century civil rights issue (Moses & Cobb, 2001).

At the time of this writing, there is a dearth in the research regarding empirical studies specifically focused on belongingness in secondary mathematics classrooms and its impact on student learning and achievement outcomes. Importantly, there are few studies which explicitly investigate connections between students' MSB to mathematics learning and/or academic performance (Barbieri & Booth, 2016; Barbieri & Miller-Cotto, 2021; Good et al., 2012). Although limited in quantity, the high *quality* of the insights in these studies provided important frames of reference and evidence of the field's need for this study.

### **Defining Belonging**

As a foundation for the operationalized definition of belongingness used in this study, a brief summary of how researchers have previously conceptualized sense of belonging is provided. Theoretical frameworks on sense of belonging are commonly based on the belongingness hypothesis from Baumeister and Leary (1995). The belongingness hypothesis contends that people have "a pervasive drive to form and maintain...lasting, positive, and significant interpersonal relationships" (Baumeister & Leary 1995, p. 497). Additionally, many studies cite the participation-identification model (Finn, 1989) which suggests only students who identify with their school develop a sense of belonging to it. Importantly, Mahar and colleagues (2012) conducted a scoping, transdisciplinary literature review of previous research on belongingness. They used the key search term "sense of belonging", and reviewed referent lists of relevant papers. After reaching theoretical saturation at 40 papers, including 22 qualitative works, the researchers uncovered five intersecting themes: subjectivity, groundedness to an external referent; reciprocity; dynamism; and self-determination (Mahar et al., 2012). The final product of their review resulted in defining "sense of belonging" as follows:

[A] subjective feeling of value and respect derived from a reciprocal relationship to an external referent that is built on a foundation of shared experiences, beliefs, or personal characteristics. These feelings of external connectedness are grounded to the context or referent group, to

whom one chooses, wants, and feels permission to belong. This dynamic phenomenon may be either hindered or promoted by complex interactions between environmental and personal factors. (Mahar et al., 2012, p.1031)

To address the first research question, Table 1 defines the five intersecting, transdisciplinary themes and connects them with the related research on sense of belonging in secondary mathematics education research (Mahar et al., 2012).

**Table 1: Connecting 5 Themes and Mathematics Education**

5 Intersecting, Transdisciplinary Themes of Belongingness (Mahar et al., 2012)	Mathematics Education Research Connections
<u>Subjectivity</u> -a perception that is unique to each individual and centers on feelings of being valued, respected and fitting in	Students need to develop a positive mathematics identity (Aguirre et al. 2013). Students' mathematical skills in the affective domain are as important as those in the cognitive domain for deep, lasting learning (Graesser & D'Mello, 2011; Hannula et al., 2019).
<u>Groundedness to an External Referent</u> -required referent group to anchor the subjective feeling of belongingness. One must belong to something	The field of mathematics education must broaden the purposes of learning mathematics, dispel the myth of a math person, and dismantle structural inequities (Ball & Moses, 2009; NCTM 2014; NCTM, 2018; NCTM, 2020).
<u>Reciprocity</u> -a sense of relatedness or connectedness that is shared by the person and the external referent	Teachers must attend to their own mathematical identities, adopt evidence-based productive beliefs, equitably apply effective mathematics teaching practices, and be culturally relevant/responsive to students' identities (Aguirre et al., 2013; Gay, 2000; Ladson-Billings, 1995; NCTM, 2014; Thomas & Berry III, 2019).
<u>Dynamism</u> -refers to dynamic physical and social environments which may contribute to or detract from one's sense of belonging	All stakeholders in mathematics education must be committed to lifelong learning, e.g., continuous development of mathematical habits of mind in service of the 5 strands of proficiency and the standards for mathematical practice, where learning and teaching are done with social others in context (DeRoyston et al., 2020; Kilpatrick et al., 2001; National Governors Association, 2010).
<u>Self-Determination</u> -respects a person's right to decide whether to interact with referents and their perceived power in the interaction	Students need to be given multiple, high-quality options, which do not lead to dead-end pathways, regarding which courses to take after completing essential concepts in high school (NCTM, 2018).

For the purposes of this study, mathematical sense of belonging (MSB) is defined as human beings' feelings of rightful presence (Isler, et al., 2021) in the broader mathematics community and/or its specialized communities of practice (e.g., mathematical artists, researchers, data analysts, educators, etc.), in which people choose to identify, participate, and persist in engaging with any dimension of mathematics (i.e., applied, pure, art, physics, data science, modeling, learning and teaching, etc.). This definition of MSB *intentionally centers the humanity and identity of the learner*, as well as an inclusive perspective on the definition of mathematics.

**Extant Analyses of the HSLs:09**

Although there have been several studies which utilize the data from the HSLs:09, their foci varied greatly. Some studies analyzed students' self-efficacy, identity and achievement, but were

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

centered on select subgroups of students, i.e., Black girls (Joseph et al., 2020), African American males (Briggs, 2014), or students with visual impairments and the deaf (Lund, 2020). Another study focused on these variables but only compared male and female students during their 9<sup>th</sup> (2009) and 11<sup>th</sup> grade (first follow-up) years (Bonitto, 2020). For studies which took a more holistic approach to analyzing high school students' post-secondary education goals, researchers report students' college application trends (U.S. Department of Education, 2017) or their trajectories of academic achievement (Alhadabi & Li, 2020; Sharpe & Marsh, 2021) but did not use the most recent data on students' actualized career paths thus far. Finally, one finding closely related to the focus of this study centered on students' readiness and intentions to continue their education within Science, Technology, Engineering and Mathematics (STEM) fields, but did not explore their actualized STEM college and career accomplishments (Kurban & Cabrera, 2020).

### **Methodology**

Composite index creation of a theory-based construct which cannot be directly observed or measured is common practice in many research fields (i.e., social science, psychology, and political science). For instance, the NASDAQ, a composite index used to measure the "health" of the National Association of State Departments of Agriculture (Chakrabarty, 2017), was founded in 1916 on the New York Stock Exchange. Chakrabarty and colleagues (2017) also note the well-known Human Development Index (HDI) as another example of an overall index combining indicators of health, education and income. Validation of new constructs resulting from the composite indices of related measures often involves advanced statistical methods, i.e., exploratory factor analysis (EFA).

To address the second research question, an EFA was conducted to create a construct of students' perceived mathematical sense of belonging (MSB) using Likert-scale and binary affective variables. EFA affords researchers the opportunity to work with several variables and concurrently reveal patterns in their underlying data structure. Therefore, EFA is often used as evidence of construct validity (Hahs-Vaughn, 2016). As mathematics-specific sense of belonging has limited theoretical and empirical research, an EFA was used as a "data-driven approach" (Brown, 2006, p.14) to establish an MSB construct. Details regarding the design of the EFA are discussed in the next section.

### **Delimiting the Sample**

The publicly available HSLs:09 included 21,444 unweighted cases. The initial unweighted sample ( $N = 21,444$ ) was first restricted to students for whom there was existing data on the base-year weight, which resulted in the sample size  $N = 15,558$ . The independent variables included in the EFA were delimited using listwise deletion as they met the threshold of less than 10% of the sample. Missing data on the affective mathematics questions from the survey were removed individually in sequential order. These variables were used in the EFA to create an MSB construct based on the theory of the transdisciplinary, scoping review of belongingness in education (Mahar et al., 2012). The variables used to construct the MSB subscales had missing that ranged from a low of  $n = 59$ , 0.4% for *C01A (9th grader sees himself/herself as a math person)* to a high of  $n = 350$ , 2.2% for *C07C (9th grader thinks fall 2009 math course is useful for future career)*. After removing this missing data, the final analytic sample size was over 13,000 high school students ( $N = 13,534$ ).

### **Addressing EFA Assumptions**

To create a mathematical belongingness construct, the data were screened to determine the extent to which the assumptions associated with EFA were met. The assumptions were generated in SPSS with base weights applied only and may contain minimal bias. Nonetheless, assumptions

of a conventional EFA, including (a) independence, (b) linearity, (c) absence of outliers (both univariate and multivariate), and (d) lack of extreme multicollinearity and singularity (Hahs-Vaughn, 2016), were met for this analysis. Having addressed the assumptions, there will now be a discussion of the results of the EFA used to create a mathematical sense of belonging (MSB) construct.

## Results

Evidence for construct validity for the MSB construct was obtained using an EFA. The initial factorability criteria included examination of the following: (1) bivariate correlations, (2) Kaiser-Meyer-Olkin measure of sampling adequacy (overall and individual), (3) Bartlett's test of sphericity, and (4) communalities. Then, the proceeding EFA results were presented in two parts due to the complex (ordinal and binary components) sample design. These proceeding items were chosen in that they either captured the definitive letter (parallel syntax) or the spirit (approximate semantics) of the five transdisciplinary themes of belonging (Table 1). The first is based on 17 Likert-scale items mapped onto four of the five themes, namely subjectivity, groundedness to an external referent, reciprocity, and dynamism (Mahar et al., 2012). The second used 9 binary items to theoretically map onto self-determination, the fifth and final theme. As the binary items were not included in the EFA, their relationships are based on theory not statistical analysis.

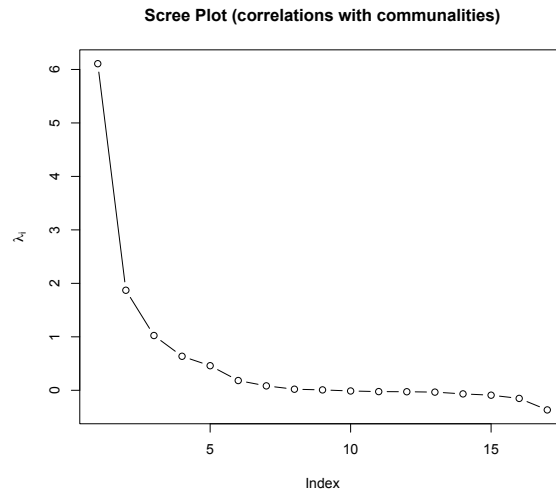
Using a Spearman's correlation coefficient because the data are ordinal, each of the items correlated at least .30 with at least one other item. The overall Kaiser-Meyer-Olkin measure of sampling adequacy was .90, larger than the recommended value of .50. In addition, the measure of sampling adequacy values for the individual items were all greater than the recommended value of .50. Next, the uniqueness (computed as  $1 - \text{communality}$ ) exported from R (R Core Team, 2020) using the complex survey design (with base and replicate weights) was assessed. Generally, the uniqueness values are below .30 and indicate a large proportion of the variance of the item being accounted for by the factor.

All Likert-scale items contributed to a simple factor structure and had a primary factor loading above the recommended .30. Therefore, each Likert-scaled item was retained in the EFA. Although three items are the widely accepted minimum number of factors to index a subscale, the fourth factor only has two items. Nonetheless, it was retained as the two items combined were already a composite scale, named mathematics identity (coefficient of reliability  $\alpha = 0.84$ ), by the survey designers (Ingels et al., 2011). Mathematical identity is an established, well-researched construct in extant mathematics literature (Aguirre et al., 2013). Similarly, the third factor also mirrored the results of the original survey in a composite scale named mathematics utility with a coefficient of reliability  $\alpha = 0.78$  (Ingels et al., 2011). Given the other criteria for determining factorability were met, it was reasonable to proceed with determining the factor structure of the four items mapped onto the first four transdisciplinary themes of belonging (Mahar et al., 2012).

Principal component analysis (PCA) was conducted with the complex survey design in R, meaning both base and replicate weights were applied. The PCA was conducted first with Promax (an oblique rotation because the data are related in the survey), then varimax (to clarify the relationships between the variables) rotations were used to extract the factors from the data in R. Parallel analysis was used to determine the number of factors to retain. The results suggested a four-factor model was appropriate (i.e., the first four raw data eigenvalues were greater than the random and permuted mean and 95th percentile eigenvalues; all other raw data eigenvalues were less in value). Although a more subjective tool for determining the number of factors, the scree plot (Figure 1) indicated the eigenvalues leveled off after four factors, again supporting a four-factor solution. The interpretation of a four-factor solution was also plausible and was a consideration in

retaining two factors. All but one component variable, *SI C11F 9th grader's fall 2009 math teacher treats some kids better than others*, were retained before testing for internal consistency and reliability. Post extraction, the four-factor solution represented about 59% of the variance explained.

Scree plots are a graphic tool that can be used to decide on the number of factors to retain when conducting an EFA. In interpreting the scree plot in Figure 1, researchers examine where the line goes from being diagonal to being horizontal. There is some subjectivity with the decision to determine the four factors to retain in this graph, but scree plots generally perform better than the eigenvalue greater than one rule (Hahs-Vaughn, 2016).



**Figure 1: EFA Scree Plot**

Table 2 provides the factor loading pattern matrix for the final solution. The names of the four sub-scaled factors for students' MSB are Reciprocity (Factor 1), Dynamism (Factor 2), Groundedness to an External Referent/Mathematics Utility (Factor 3), and Subjectivity/Mathematics Identity (Factor 4). The results of the EFA lend support to internal structure validity evidence supporting the conclusion that the scores from this instrument are a valid assessment of the sub-scales MSB (based on four of the five transdisciplinary themes), thereby justifying the retention of the theoretical names. Factor values bolded in Table 2 were chosen for the final EFA analyses of internal consistency.

**Table 2: Exploratory Factor Analysis- Factor Loadings Pattern Matrix**

Variables	Factor 1	Factor 2	Factor 3	Factor 4
S1 C01A 9th grader sees himself/herself as a math person		0.208	0.162	<b>0.837</b>
S1 C01B Others see 9th grader as a math person		0.153	0.135	<b>0.790</b>
S1 C11A 9th grader's fall 2009 math teacher values/listens to students' ideas	<b>0.757</b>	0.283		
S1 C11B 9th grader's fall 2009 math teacher treats students with respect	<b>0.869</b>	0.138	0.110	
S1 C11C 9th grader's fall 2009 math teacher treats every student fairly	<b>0.857</b>	0.161		
S1 C11D 9th grader's fall 2009 math teacher thinks all student can be successful	<b>0.720</b>	0.159	0.164	
S1 C11E 9th grader's fall 2009 math teacher thinks mistakes OK if students learn	<b>0.620</b>	0.175		
S1 C11F 9th grader's fall 2009 math teacher treats some kids better than others	<b>0.520</b>	0.170		
S1 C07A 9th grader thinks fall 2009 math course is useful for everyday life	0.111	0.214	<b>0.648</b>	
S1 C07B 9th grader thinks fall 2009 math course will be useful for college	0.134		<b>0.690</b>	0.117
S1 C07C 9th grader thinks fall 2009 math course is useful for future career		0.129	<b>0.803</b>	0.135
S1 C06A 9th grader is enjoying fall 2009 math course very much	0.266	<b>0.621</b>	0.246	0.339
S1 C06B 9th grader thinks fall 2009 math course is a waste of time	-0.212	<b>-0.466</b>	-0.321	-0.152
S1 C06C 9th grader thinks fall 2009 math course is boring	-0.232	<b>-0.629</b>	-0.203	-0.178
S1 C11G 9th grader's fall 2009 math teacher makes math interesting	0.508	<b>0.616</b>	0.102	
S1 C11I 9th grader's fall 2009 math teacher makes math easy to understand	0.510	<b>0.545</b>		0.105
S1 C11F 9th grader's fall 2009 math teacher treats some kids better than others	0.395	0.113		
SS Loadings	4.129	2.035	1.882	1.570
Proportion Variance	0.243	0.120	0.111	0.092
Cumulative Variance	0.243	0.363	0.473	0.566

The degrees of freedom for the model was 74 and the fit was 0.5888

Subscales were measured for internal consistency via Cronbach's alpha ( $\alpha$ ) of reliability. Table 3 provides the results of the final variable components retained for each factor with the corresponding Cronbach's  $\alpha$  indices. All Cronbach's  $\alpha$  measures in Table 9 were greater than the recommended .70 for acceptable internal consistency reliability (UCLA Statistical Consulting Group, 2022). These analyses were conducted in R.

**Table 3: Mathematical Sense of Belonging (MSB) Subscale Reliability**

Transdisciplinary Theme	Component Variables	Cronbach's Alpha Coefficient
Subjectivity/Mathematics Identity	C01A, C01B	$\alpha = .838$
Groundedness to an External Referent/Mathematics Utility	C07A, C07B, C07C	$\alpha = .779$
Reciprocity	C11A, C11B, C11C, C11D, C11E, C11F	$\alpha = .878$
Dynamism	C06A, C11G, C11I	$\alpha = .809$

Lastly, the binary items were theoretically mapped onto the fifth transdisciplinary theme of self-determination. This mapping was based on parallel syntax and semantics of the survey items and the intersecting, transdisciplinary self-determination theme's description (Mahar et al., 2012). As binary items need a different correlation matrix to generate, it was not yet possible to apply the complex design in R nor SPSS with the available packages. Therefore, these survey items were assigned a weighted, composite index.

### Discussion

This study used exploratory factor analysis and long-standing belongingness and mathematics educational theoretical research to establish a set of sub-scales for creating an empirically constructed mathematical sense of belonging (MSB) construct. The data from the HSLS:09 were used to analyze mathematics affective survey variables and establish external validity (using base and replicate weights) and internal consistency and reliability (alpha scores greater than .70). The large-sized ( $N = 13,534$ ) and nationally representative sample contributed to the success of this research as the results are generalizable. Results showed four of five *transdisciplinary* themes of belonging (namely Subjectivity, Groundedness to an External Referent, Reciprocity, and Dynamism) empirically unveiled statistically reliable subscales of a new mathematical sense of belonging (MSB) construct within a complex sample (Table 3).

Although the EFA was successful in this study, there are limitations. First, the participants were high school students, therefore the generalizability is limited to secondary students. Further studies could be conducted on primary/elementary students to establish an MSB for younger students. Finally, one of the five transdisciplinary themes of belonging, namely self-determination, could only be theoretically mapped to binary items from the HSLS:09 and were not a part of the EFA. The authors attempted to reach out to other R community experts, but none were able to apply a complex survey design to the EFA in order to include the binary items. Hopefully, appropriate R-packages and corresponding codes will be available in the near future. Further detailed statistical results may be requested of the corresponding author.

### Implications

The results of this study provide a foundation for identifying subscales for an empirically established, discipline specific MSB. Mathematics teacher educators are encouraged to implement and instruct preservice teachers on the importance of each of transdisciplinary themes of belongingness and the themes' relationships to existing literature on supporting students' positive MSB (Table 1). Practitioners could use this research to justify the need for professional development which focuses on a holistic approach to mathematics instruction.

### References

- Aguirre, J. M., Mayfield-Ingram, K., & Martin, D. B. (2013). *The Impact of Identity in K-8 Mathematics: Rethinking Equity-based Practices*. Reston, VA: National Council of Teachers of Mathematics.
- Alhadabi, A., & Li, J. (2020). Trajectories of academic achievement in high schools: Growth mixture model. *Journal of Educational Issues*, 6(1), 140-165.
- Anderson, R. K., Boaler, J., & Dieckmann, J. A. (2018). Achieving Elusive Teacher Change through Challenging Myths about Learning: A Blended Approach. *Education Sciences*, 8(3), 98.
- Arends, F., & Visser, M. (2019). The contribution of South African teachers to students' sense of belonging and mathematics achievement: Students' perspective from the 2015 Trends in International Mathematics and Science study. *South African Journal of Childhood Education*, 9(1), 1-11.
- Ball, D., & Moses, R. (2009, October 1). *Equity and mathematics: an interview with Deborah Ball and Bob Moses*, 54-59. (J. Richardson, Interviewer) Phi Delta Kappa.
- Barbieri, C., & Booth, J. L. (2016). Support for struggling students in algebra: Contributions of incorrect worked examples. *Learning and Individual Differences*, 48, 36-44.

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

- Barbieri, C., & Miller-Cotto, D. (2021). The importance of adolescents' sense of belonging to mathematics for algebra learning. *Learning and Individual Differences, 87*, 101993.
- Baumeister, R. F., & Leary, M. R. (1995). The need to belong: a desire for interpersonal attachments as a fundamental human motivation. *Psychological bulletin, 117*(3), 497.
- Bonitto, C. M. (2020). Comparing male and female student self-efficacy, identification, and achievement in Mathematics: A cross-lagged panel analysis of causal effects. [Doctoral dissertation]. University of Kansas.
- Briggs, C. (2014). Mathematics: Self-efficacy, identity, and achievement among African American males from the high school longitudinal study. *Publication No. 3583106*. ProQuest Dissertations and Theses Global.
- Brown, T. A. (2006). *Confirmatory factor analysis for applied research*. Guilford Press.
- Chakrabarty, S. N. (2017). Composite index: methods and properties. *Journal of Applied Quantitative Methods, 12*(2), 25-33.
- DeRoyston, M. M., Lee, C., Nasir, N. S., & Pea, R. (2020, October 26). *Rethinking schools, rethinking learning*. Retrieved from Kappanonline.org: <https://kappanonline.org/rethinking-schools-rethinking-learning-mckinney-de-royston-lee-nasir-pea/>
- Finn, J. D. (1989). Withdrawing from school. *Review of educational research, 59*(2), 117-142.
- Gay, G. (2000). *Culturally responsive teaching: Theory, research, and practice*. New York, NY: Teachers College Press.
- Good, C., Rattan, A., & Dweck, C. S. (2012). Why do women opt out? Sense of belonging and women's representation in mathematics. *Journal of personality and social psychology, 102*(4), 700-717.
- Graesser, A., & D'Mello, S. K. (2011). Theoretical perspectives on affect and deep learning. In R. A. Calvo, & S. K. D'Mello (Eds.), *New Perspectives on Affect and Learning Technologies* (pp. 11-21). New York, New York: Springer.
- Hahs-Vaughn, D. L. (2016). *Applied multivariate statistical concepts*. Chicago: Routledge.
- Hannula, M., Leder, G. C., Morselli, F., Vollstedt, M., & Zhang, Qiaoping, Z. (2019). Affect and mathematics education: Fresh perspectives on motivation, engagement, and identity. Springer.
- Henson, R. K., & Roberts, J. K. (2006). Use of exploratory factor analysis in published research. *Educational and Psychological Measurement, 66*(3), 393-416. <https://doi.org/10.1177/0013164405282485>
- Holian, L., & Kelly, E. (2020). *STEM Occupational Intentions Stability and Change through High School*. *Stats in Brief*. National Center for Education Statistics.
- Ingels, S. J., Pratt, D. J., Herget, D. R., Burns, L. J., Dever, J. A., Ottem, R., ... & Leinwand, S. (2011). High School Longitudinal Study of 2009 (HSLs: 09): Base-Year Data File Documentation. NCES 2011-328. *National Center for Education Statistics*.
- Isler, J. C., Berryman, N. V., Harriot, A., Vilfranc, C. L., Carrington, L. J., & Lee, D. N. (2021). Defining the Flow—Using an Intersectional Scientific Methodology to Construct a Vanguard STEM Hyperspace. *Genealogy, 5*(1), 8.
- Jackson, C., & Delaney, A. (2017). Mindsets and practices shifting to an equity centered paradigm. In A. Fernandes, S. Crespo, & M. Civil (Eds.), *Access and Equity Promoting High-Quality Mathematics in Grades 6-8* (pp. 143-155). Reston: National Council of Teachers of Mathematics.
- Joseph, N. M., Tyler, A. L., Howard, N. R., Akridge, S. L., & Kelsi, R. R. (2020). The role of socialization in shaping Black girls' mathematics identity: An analysis of the high school longitudinal study 2009. *Teachers College Record*.
- Kebreab, L., Bush, S. B., & Jackson, C. (2021). Leveraging student identities to develop pedagogical fluency. *Mathematics Teacher: Learning and Teaching PK-12, 114*(12), 948-955.
- Kilpatrick, J., Swafford, J., & Bradford, F. (2001). *Adding it up: Helping children learn mathematics (Vol. 2101)*. Washington, DC: National Academy Press.
- Korpershoek, H., Canrinus, E. T., Fokkens-Bruinsma, M., & de Boer, H. (2020). The relationships between school belonging and students' motivational, social-emotional, behavioural, and academic outcomes in secondary education: A meta-analytic review. *Research Papers in Education, 35*(6), 641-680.
- Kurban, E. R., & Cabrera, A. F. (2020). Building Readiness and Intention Towards STEM Fields of Study: Using HSLs:09 and SEM to Examine This Complex Process Among High School Students. *The Journal of Higher Education, 91*(4), 620-650.
- Ladson-Billings, G. (1995). But that's just good teaching! The case for culturally relevant pedagogy. *Theory Into Practice, 34*(3), 159-165.
- Lerner, R. M., Lerner, J. V., Almerigi, J. B., Theokas, C., Phelps, E., Gestsdottir, S., & ... & Smith, L. M. (2005). Positive youth development, participation in community youth development programs, and community
- Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (Vol. 2)*. University of Nevada, Reno.



- contributions of fifth-grade adolescents: Findings from the first wave of the 4-H study of positive youth development. *The Journal of Early Adolescence*, 25(1), 17-71.
- Leonard, J., & Martin, D. B. (2013). *The brilliance of Black children in mathematics*. IAP.
- Lund, E. M. (2020). Post-school employment and education outcomes of young adults reporting deafblindness on the High School Longitudinal Study of 2009. *Journal of the American Deafness & Rehabilitation Association (JADARA)*, 53(3), 11-22.
- Mahar, A., Cobigo, V., & Stuart, H. (2012). Conceptualizing belonging. *Disability and rehabilitation*, 35(12), 1026-1032.
- Maslow, A. H. (1943). A theory of human motivation. *Psychology Review*, 50(4), 370-396.
- Muhammad, G. (2022, May 18). *The Blueprint of Our Lives*. Retrieved from HILL Pedagogies: <https://hillpedagogies.com/resources/>
- Moses, R. P., & Cobb, C. E. (2001). *Radical equations: Civil rights from Mississippi to the algebra project*. Boston: Beacon Press.
- NCTM/NCSM. (2019). *Building, S. T. E. M. Education on a Sound Mathematical Foundation*. Retrieved from <https://www.nctm.org/Standards-and-Positions/Position-Statements/Building-STEM-Education-on-a-Sound-Mathematical-Foundation>
- National Council of Teachers of Mathematics. (2014). *Principles to actions: ensuring mathematical success for all*. Reston, VA: National Council of Teachers of Mathematics.
- National Council of Teachers of Mathematics. (2018). *Catalyzing change in high school mathematics: Initiating critical conversations*. Reston, VA: National Council of Teachers of Mathematics.
- National Council of Teachers of Mathematics. (2020). *Catalyzing change in middle school mathematics: initiating critical conversations*. Reston, VA: National Council of Teachers of Mathematics.
- National Governors Association. (2010). *Common core state standards*. Retrieved from <http://www.corestandards.org/Math/>
- OECD. (2017). Retrieved 10 2019, from PISA 2015 results (volume III): Students' well-being: <https://doi.org/10.1787/9789264273856-en>
- Schoenfeld, A. H. (2015). What counts, when? Reflection on beliefs, affect, attitude, orientations, habits of mind, grain size, time scale, context, theory, and method. In *From beliefs to dynamic affect systems in mathematics education* (pp. 395-404). Switzerland: Springer International Publishing.
- Schunk, D. H. (2012). *Learning theories and educational perspective sixth edition*. New York, NY: Pearson.
- Sharpe, S. T., & Marsh, D. D. (2021). A systematic review of factors associated with high schoolers' algebra achievement according to HSLS: 09 results. *Educational Studies in Mathematics*, 1-24.
- Tate, W. F. (2013). Race, retrenchment, and the reform of school mathematics. In E. Gutstein, & B. Peterson (Eds.), *Rethinking mathematics: Teaching social justice by the numbers, second edition* (pp. 42-51). Milwaukee, WI.
- Thomas, C. A., & Berry III, R. Q. (2019). A qualitative meta-synthesis of culturally relevant pedagogy & culturally responsive teaching: Unpacking mathematics teaching practices. *Journal of Mathematics Education at Teachers College*, 10(1), 21-30.
- UCLA Statistical Consulting Group. (2022). *OARC Stats. Retrieved May 9, 2022, from*. Retrieved from UCLA Advanced Research Computing: Statistical Methods and Data Analytics: <https://stats.oarc.ucla.edu/spss/faq/what-does-cronbachs-alpha-mean/>
- U.S. Department of Education. (2017). *College Applications of 2009 High School Freshman: Differences by Race/Ethnicity*. Data Point, IES-NCES.
- U.S. Department of Education. (2022). U.S. Department of Education Launches New Initiative to Enhance STEM Education for All Students: Teacher development and partnerships with external organizations are key elements. (2022, December 7). *Home-U.S. Department of Education*. Retrieved February 22, 2023, from <https://www.ed.gov/news/press-releases/us-department-education-launches-new-initiative-enhance-stem-education-all-students>
- Wang, Q., & Pomerantz, E. M. (2009). The motivational landscape of early adolescence in the United States and China: A longitudinal investigation. *Child Development*, 80(4), 1272-1287.

# EXAMINING MATHEMATICS IDENTITIES OF SUCCESSFUL UNIVERSITY DEVELOPMENTAL STUDENTS

Francis Nzuki  
Stockton University  
francis.nzuki@stockton.edu

Keywords: Developmental Math, Mathematics Identity, Equity, Narrative Inquiry, Schlossberg's Transition Theory

Nationally, about 60 percent of high school graduates are referred to one or several developmental courses (Attewell, Lavin, Domina, & Levey, 2006; Bailey, Jeong, & Cho, 2010) before they are eligible to enter college-level coursework. Furthermore, a vast majority of first-year college students taking a developmental course at four-year institutions are required to do so in mathematics (Duranczyk & Higbee, 2006). The body of research examining the participation and achievement of developmental students has usually focused on group underperformance and comparisons to students in non-remedial classes. In the recent past, however, there has been a considerable shift, with a growing number of researchers turning their analytic lenses to classroom and out-of-school mathematics experiences and toward understanding the nature of successful practices. According to Cobb (2004) it is vital to understand the factors that contribute to students' learning and "the process by which students' emerging identities in the mathematics classroom might, over time, involve changes in their more enduring sense of who they are and who they want to become," (p. 336).

This study uses a narrative inquiry to explore developmental students' experiences and construct stories for their academic success, viewed through the lens of the Schlossberg's transition theory (Schlossberg, 1981), by addressing their home and school experiences during the mathematics transition from pre-college and beyond. In addition, Martin's (2000) identity framework is utilized to examine how these experiences helped to shape their mathematical identities and how these identities in turn influenced their successful completion of developmental mathematics. To this end, the following research questions guided this study of successful developmental math students at a mid-sized, urban public university:

1. What factors contributed to the development of mathematics identities of successful developmental math students?
2. What math identities did the participants construct as they moved through developmental math?
3. What impact did taking developmental math have on the participants?

The first research question explores the factors that influenced the formation and development of math identities among successful developmental math students while the second research question investigates the specific math identities that the participants constructed as they progressed and advanced through the developmental math curriculum. The last research question aims to understand the overall impact that taking developmental math had on the participants, encompassing its effects on their academic journeys, confidence levels, and overall development.

Data was collected using semi-structured interviews, which were conducted with the guidance of an interview protocol whose development was informed by the research questions and the theoretical frameworks guiding the study. Data analysis included an inductive approach to identify

narratives regarding participants' mathematics experiences and how these experiences influenced their identities.

In addition, thematic coding and analytic memoing were used to help flesh out concepts, patterns and themes emerging in the data (Miles, Huberman, & Saldana, 2013). Also, a timeline of the participants' experiences moving in, moving through, and moving out of developmental mathematics, and perceptions of the factors behind their persistence and success, and future aspirations was created. Moreover, in interpreting the participants' narratives with regards to their mathematical-learning experiences, I drew on Martin's (2000) identity framework.

Results showed being placed into developmental math courses was an unanticipated transition for most of the participants. Also, despite moving in developmental math course with tarnished math identities, their persistence led to successful course completion. Factors like self-determination, dedication, resilience, and support from teachers, family, and peers influenced identity evolution. Beyond skill enhancement, developmental math sparked increased confidence, laid a foundation for further learning, and propelled participants towards STEM majors. It also provided vital tools and support systems, fostering a conducive learning environment for their academic growth.

### References

- Attewell, P., Lavin, D., Domina, T., & Levey, T. (2006). New evidence on college remediation. *The Journal of Higher Education*, 77(5), 886–924.
- Bailey, T., Jeong, D. W., & Cho, S. W. (2010). Referral, enrollment, and completion in developmental education sequences in community colleges. *Economics of Education Review*, 29(2), 255–270.
- Cobb, P. (2004). Mathematics, literacies, and identity. *Reading Research Quarterly*, 39(3), 333–337.
- Duranczyk, I. M., & Higbee, J. L. (2006). Developmental mathematics in 4-year institutions: Denying access. *Journal of Developmental Education*, 30(1), 22-31.
- Martin, D.B. (2000). *Mathematics success and failure among African-American youth: The roles of sociohistorical context, community forces, school influence, and individual agency*. Mahwah, NJ: Lawrence Erlbaum Associates.
- Miles, M. B., Huberman, A. M., & Saldana, J. (2013). *Qualitative data analysis: A methods sourcebook*. Thousand Oaks, CA: SAGE Publications, Incorporated.
- Schlossberg, N. K. (1981). A model for analyzing human adaptation to transition. *The Counseling Psychologist*, 9(2), 2–18.

# EXPLORING FACTORS INFLUENCING SUCCESS IN MATHEMATICAL PROBLEM SOLVING

## EXPLORANDO LOS FACTORES QUE INFLUYEN EN EL ÉXITO EN LA RESOLUCIÓN DE PROBLEMAS MATEMÁTICOS

Sam Rhodes  
Georgia Southern University  
srhodes@georgiasouthern.edu

Rick Bryck  
Landmark College  
rickbryck@landmark.edu

Antonio Gutierrez de Blume  
Georgia Southern University  
agutierrez@georgiasouthern.edu

*Supporting students in becoming effective problem solvers is a critical component of K-12 mathematics instruction. Unfortunately, little is known about the factors that are related to problem solving proficiency in middle school students. We report the results of a study that employed a hierarchical linear regression analysis to examine the factors that influenced the problem-solving scores in a sample of 213 sixth and seventh grade students. Results support theoretical predictions that metacognition, executive function, student beliefs, and content knowledge all relate to problem solving proficiency.*

Keywords: Problem solving; Metacognition; Affect, Emotion, Beliefs, and Attitudes.

Research has suggested that teaching with and through mathematical problem solving (PS) has vast potential for engaging all students in mathematical practices that support increased motivation, achievement, positive beliefs about mathematics, and conceptual understanding of mathematical concepts (Boaler, 2002; Boaler & Staples, 2008; Cai, 2003; Lester & Cai, 2013; Stein et al., 2003). Consequently, PS is often considered to be a focal point of effective mathematics instruction (NCTM, 2014; 2018). However, relatively little is known about how to effectively teach mathematical PS (Lester & Cai, 2016).

Proficiency in PS is thought to be related to interactions between multiple factors such as metacognition (MC), strategic thinking, content knowledge, and the beliefs and dispositions that students have about themselves and mathematics (Chapman, 2015; Schoenfeld, 2013). Previous studies have sought to investigate the relationship between individual factors and PS, with factors such as MC (e.g., Özsoy & Ataman, 2009; Tan & Limjap, 2018), executive function (EF; Viterbori et al., 2017), and student anxieties and beliefs (Kramarski et al., 2010; Mayer & Wittrock, 2006) being investigated. However, few studies to date have empirically explored the relationship between PS and a combination of cognitive, metacognitive, and affective factors. The current study—which was part of a larger study on EF and mathematical PS, funded by the Advanced Education Research and Development Fund (AERDF)—responds to this need by using a hierarchical linear regression to determine which factors explained unique variance in the PS scores of middle school students. Better understanding these relationships may be vital in developing future PS interventions and instruction to support students.

### Review of Related Literature

#### Mathematical Problem Solving and Executive Function

Executive function refers to goal-directed, attention regulation skills that predict a wide range of important academic outcomes, including mathematics performance (Zelazo et al., 2016). These include the ability to keep information in mind (working memory; WM), the ability to stop oneself from responding immediately (inhibition), and the ability to think in multiple ways or perspectives (cognitive flexibility). Theoretical and empirical evidence show a relationship between these three

core components of EF and mathematics: WM (Raghubar et al., 2010), inhibition (Lemaire & Lecacheur, 2011), and cognitive flexibility (Yeniad et al., 2013).

Moreover, EF skills appear to be particularly needed for multi-step PS skills, and evidence suggests that developing EF skills support the development of math PS, and vice versa (Clements et al., 2016). WM, in particular, has been shown to be an important predictor of math PS accuracy (Passolunghi & Siegel, 2001; Swanson et al., 2008; Steinberg & Roditi, 2018). Inhibition, in turn, is thought to be critical in suppressing inefficient strategies and ignoring irrelevant information during PS (Bull & Lee, 2014), and cognitive flexibility allows students to fluidly shift between mathematical representations. These three core EF components seem to fill unique needs at different stages of the PS process (Viterbori et al., 2017).

### **Mathematical Problem Solving and Metacognition**

Research has suggested that metacognitive knowledge—one's awareness of their own cognition—and regulation—the ability to control one's cognition—are critical components of PS (Favell, 1976, 1979; Lester, 2013; Schoenfeld, 1992, 2013; Stillman & Galbraith, 1998). Specifically, MC has been found to be critical throughout the PS process, impacting everything from how one engages in sense-making, to selecting and utilizing strategies, to implementing plans and reflecting on solutions (Tan and Limjap, 2018). This link has been empirically tested in numerous studies, with findings often suggesting that metacognitive training and supports result in significant improvement in students' PS (e.g., Hensberry & Jacobbe, 2012, Montague et al., 2011, Özsoy & Ataman, 2009).

Moreover, while the use of heuristics or strategic thinking has long been considered a key component of successful PS, as noted above, research has suggested that heuristics are insufficient unless they are coupled with effective metacognitive awareness and regulation. These are necessary to support problem solvers in adapting heuristics to new problems and monitoring what they are doing and evaluating their plan throughout the PS process (Lesh & Zawojewski, 2007; Jitendra et al., 2015). Indeed, students' metacognitive verbalizations have been found to increase along with problem difficulty (Rosenzweig et al., 2011), and students' metacognitive awareness has been found to be especially critical when students utilize new strategies or engage with non-routine problems (Carr & Jessup, 1995).

### **Mathematical Problem Solving and Affect**

Numerous affective factors are thought to impact PS proficiency. These include self-efficacy, beliefs, anxiety, and math identity (Chapman, 2015; Irhamna et al., 2020; Tzohar-Tozen & Kramarski, 2014). Exemplifying these links, WM is used to attend to various functions, meaning that increased anxiety may decrease cognitive performance on other tasks (Moran, 2016). Moreover, student beliefs can support or inhibit students' ability to regulate their cognition, as well as their motivation and persistence as they encounter challenges inherent in PS (Goldin et al., 2016; Tzohar-Tozen & Kramarski, 2014).

### **Theoretical Framework**

Research has suggested that self-regulated learning (SRL) is a key aspect of mathematical PS (Tzohar-Rozen & Kramarski, 2014). SRL theory refers to the ability to control one's learning environment and is posited to encompass cognition, MC, and motivation processes. Several theoretical accounts of SRL have been proposed in the literature (see Panadero, 2017, for a review). For instance, Zimmerman's Cyclical Phases Model (CPM; Zimmerman & Moylan, 2009) describes SRL as a cyclical process involving three parts: (1) forethought (e.g., goal setting, strategic planning, self-efficacy beliefs, and intrinsic motivation); (2) performance and volitional control (e.g., attention focusing, self-instruction, and self-monitoring); and (3) self-reflection (e.g.,

self-evaluation, attributions, and self-reactions). Similarly, Winne and Hadwin (1998) developed a Metacognitive Perspective Model (MPM) of SRL in which metacognitive processes play a central role. According to the tenets of this model, learners are perceived as being active and involved self-regulated individuals who control their own learning through the implementation of metacognitive monitoring and strategy use, which are central to the goals of the present study. The model was subsequently expanded to include self-regulatory actions (Winne & Hadwin, 2008). Along a similar vein, Efklides (2011) devised the Metacognitive and Affective Model of Self-Regulated Learning (MASRL) in which metacognitive and motivational processes are also key. Even though all these models vary regarding labels and what aspects to include, they all agree that learning is regulated by a variety of dynamic interacting and cyclical cognitive, metacognitive, and motivational factors (Butler & Winne, 1995; Panadero, 2017). The present investigation was centered on the MASRL (Efklides, 2011) and MPM (Winne & Hadwin, 2008), as those are the models in which metacognitive skills play a central role. Specifically, SRL informed both the factors under consideration within the present study and the method of model-building used for the hierarchical linear regression.

## Methods

### Participants

Participants were drawn from three middle schools from a large, suburban school district located on the West Coast of the United States. All grade 6-8 mathematics teachers at these schools, and their respective students, were given the opportunity to participate in the study. Within this, 115 sixth grade students and 98 seventh grade students completed all measures, yielding a total sample of 213 for the analyses reported herein. The participants identified their gender as male (40.8%), female (53.1%), non-binary (1.9%), or other (2.8%), with 1.4% of participants electing not to specify a gender. The participants also identified as Hispanic/Latinx (29.6%), Middle Eastern (28.6%), 2 or more races (16.4%), Asian (6.6%), Black/African American (4.2%), or White (2.8%), with 11.7% of participants preferring not to specify.

### Research Instruments and Scoring

**Executive Function.** Students' EF was measured using the Adaptive Cognitive Evaluation (ACE; Younger et al., 2022)—gamified versions of well-known computerized cognitive tasks of three core EFs: inhibition (Flanker), working memory (change detection), and cognitive flexibility (task switching; Miyake et al., 2000). Mean and standard deviation scores were calculated for both reaction time and accuracy measures. Difference scores were the primary variables of interest for each measure, including “Flanker effect” (incongruent minus congruent trials), “K” (estimate of WM capacity, hits minus false alarms), “WM filtering” (set size 2 with 2 distractors minus set size 2), and “switch cost” (task switching switch minus stay trials).

**Content Knowledge.** Given that the partnering district utilized the i-Ready diagnostic assessments (Curriculum Associates, 2022) to gain regular data on students' content knowledge in mathematics, these scores were utilized as a proxy for mathematics content knowledge.

**Metacognition.** Subjective metacognitive awareness was measured using a shortened version of the Junior Metacognitive Awareness Inventory (MAI, Jr.; Gutierrez de Blume et al., under review; Sperling et al., 2002). The survey was administered using a sliding scale of 1 to 100 to allow for more continuous data collection. In addition, objective MC was measured through confidence judgements in which students predicted how well they would do on the PS measure (see below) prior to taking the measure, and then postdicted their performance immediately after completing the measure. Scores for objective MC were calculated by computing the absolute value

of the difference scores between an individual's confidence judgements and their actual performance.

**Affective Instruments.** Several surveys were used to measure various affective factors. These included the modified Abbreviated Math Anxiety Scale (mAMAS; Carey et al., 2017), and belief scales 1, 5, and 6 of the Indiana Mathematics Beliefs Scales (IMBS; Kloosterman & Stage, 1992), which measured students' beliefs about whether they can solve time-consuming problems, about whether effort increases ability, and about the usefulness of mathematics in their lives, respectively. In addition, students were given five questions about their feelings about mathematics within their classrooms (e.g., "I feel encouraged to solve challenging problems in math") and a question about how close they felt to the subject of mathematics.

All surveys were administered as originally developed, with the following exceptions: 1) all surveys were administered using a sliding scale of 1 to 100; 2) the mAMAS was slightly adapted for American English (e.g., "maths" to "math"); and 3) an abbreviated version of the IMBS (Rhodes et al., forthcoming), was employed to reduce testing fatigue. For each survey, all questions within each scale were averaged to obtain a single score for each distinct construct.

**Problem Solving Measure.** The researchers developed a distinct 3-item PS measure for each grade, with one item aligning to content taught in each trimester per the pacing guide that was provided by the partnering school district. All measure items were drawn from problems developed by Illustrative Mathematics (IM) and were chosen for inclusion based on A) the degree to which they were cognitively demanding and rigorous; B) the degree to which they aligned to standards that the district identified as priorities within their respective grade levels; and C) the degree to which they required students to show or explain their thinking.

If questions met criteria A and B but not C, problems were slightly modified (e.g., adding directions requiring students to show or explain their thinking). Each problem was scored as being correct or incorrect (referred to herein as accuracy) using answer keys developed by IM. In addition, problems were scored using rubrics developed by an external researcher that assessed the degree to which a student's work demonstrated relevant mathematical understandings, regardless of the correctness of the final answer (referred to herein as understanding). Given the links between the use of heuristics and MC noted above, we intentionally disregarded scoring students' strategies, focusing instead only on the mathematically understandings they demonstrated. This choice was made to avoid scoring in ways that specifically considered MC prior to then correlating the scores with measures of MC. Interrater agreement on scoring for accuracy and understanding was measured by Fleiss' kappa. The Fleiss' kappas were .961 and .880 for accuracy and .703 and .842 for understanding for 6<sup>th</sup> and 7<sup>th</sup> grade, respectively.

### Analyses

Prior to data analyses, data were screened for requisite statistical assumptions including normality, linearity, and multicollinearity. Data met all statistical assumptions. However, two outliers were detected in the data that were determined to be two standard deviations from the line of best fit. These outliers were removed and omitted from subsequent analyses. To address the research objective, two hierarchical linear regressions were performed: one with PS accuracy as criterion, and a second with PS understanding as criterion. Guided by SRL theory, predictors for each criterion variable were entered in the following order. Cognitive measures (i.e., EF measures, iReady scores) were entered in the first block. Affective factors (e.g., the IMBS) were entered in the second block, and MC factors (e.g., objective metacognitive monitoring accuracy, MAI, Jr. scores) were entered into the third block. Given that all assumptions were met, the results were analyzed, and findings are reported below.

## Results

### Accuracy

The omnibus model for accuracy as the criterion was statistically significant,  $F(4,206) = 47.44$ ,  $p < .001$ ,  $R^2 = .480$ . EF were entered as the first block of predictors, and they contributed significant incremental variance to the prediction of accuracy,  $\Delta F(1,209) = 8.42$ ,  $\Delta p = .004$ ,  $\Delta R^2 = .039$ . Task switching was the only significant predictor of accuracy. Mathematics beliefs, as measured by the IMBS, was entered as the second block of predictors, and it also contributed significant incremental variance to the prediction of accuracy,  $\Delta F(1,208) = 13.46$ ,  $\Delta p < .001$ ,  $\Delta R^2 = .058$ . Scale 1, measuring the extent to which a student can solve time consuming math problems, was the only significant predictor. Metacognitive monitoring, as assessed by students prediction feeling of knowing judgments, was entered as the third block predictor, and it contributed significant incremental variance to the prediction of accuracy,  $\Delta F(1,207) = 141.17$ ,  $\Delta p < .001$ ,  $\Delta R^2 = .366$ . Finally, the degree to which students felt encouraged to solve challenging problem in math class, was entered as the fourth block predictor, and it also contributed significant incremental variance to the prediction of accuracy,  $\Delta F(1,206) = 6.45$ ,  $\Delta p < .012$ ,  $\Delta R^2 = .016$ . Table 1 includes the model coefficients for all four blocks.

**Table 1: Regression Coefficients Results for Math Problem Solving Accuracy**

Predictor	<i>b</i>	$\beta$	<i>t</i>	<i>p</i>	CI <sub>95%</sub>	
					LB	UB
Block 1: Executive Function						
Task Switching	.001	.197	2.90	.004	.000	.002
Block 2: IMBS						
Solving Time Consuming Problems	.007	.196	2.97	.003	.003	.011
Block 3: Objective Metacognitive Monitoring Accuracy						
Prediction Judgments	-.017	-.614	-11.88	< .001	-.020	-.014
Block 4: Student Feelings About Their Mathematics Classroom						
Solving Challenging Math Problems	.003	.142	2.54	.012	.001	.006

Key: *b* = unstandardized regression coefficient;  $\beta$  = standardized regression coefficient; 95% confidence interval for the unstandardized regression coefficients; LB = Lower bound value; UB = Upper bound value.

*N* = 213

The omnibus model for understanding as the criterion was statistically significant,  $F(6,204) = 11.46$ ,  $p < .001$ ,  $R^2 = .251$ . iReady math achievement scores for Fall 2021 was entered as a block 1 predictor, and it contributed significant incremental variance to the prediction of understanding,  $\Delta F(1,209) = 16.46$ ,  $\Delta p < .001$ ,  $\Delta R^2 = .073$ . EF were entered as the second block of predictors, and they contributed significant incremental variance to the prediction of accuracy,  $\Delta F(1,208) = 4.25$ ,  $\Delta p = .041$ ,  $\Delta R^2 = .019$ . Task switching, as a measure of cognitive flexibility, was the only significant predictor of understanding. Mathematics beliefs, as measured by the IMBS, was entered as the third block of predictors, and it also contributed significant incremental variance to the prediction of understanding,  $\Delta F(1,207) = 7.85$ ,  $\Delta p < .006$ ,  $\Delta R^2 = .033$ . Scale 1, measuring the



extent to which a student can solve time consuming math problems, was the only significant predictor. Metacognitive monitoring, as assessed by students prediction feeling of knowing judgments, was entered as the fourth block predictor, and it contributed significant incremental variance to the prediction of understanding,  $\Delta F(1,206) = 19.09$ ,  $\Delta p < .001$ ,  $\Delta R^2 = .074$ . Postdiction, as another metacognitive monitoring measure (after students have seen the task), was entered as the fifth block predictor, and it contributed significant incremental variance to the prediction of understanding,  $\Delta F(1,205) = 10.26$ ,  $\Delta p = .002$ ,  $\Delta R^2 = .038$ . Finally, the degree to which students felt encouraged to solve challenging problem in math class, was entered as the sixth block predictor, and it also contributed significant incremental variance to the prediction of understanding,  $\Delta F(1,204) = 4.05$ ,  $\Delta p = .045$ ,  $\Delta R^2 = .015$ . Table 2 includes the model coefficients for all six blocks.

**Table 2: Regression Coefficients Results for Math Problem Solving Understanding**

Predictor	<i>b</i>	$\beta$	<i>t</i>	<i>p</i>	CI <sub>95%</sub>	
	LB	UB				
Block 1: Math Achievement						
iReady Math	.004	.270	4.06	< .001	.002	.005
Block 2: Executive Function						
Task Switching	.000	.136	2.06	.041	.000	.001
Block 3: IMBS						
Solving Time Consuming Problems	.003	.186	2.80	.006	.001	.005
Block 4: Objective Metacognitive Monitoring Accuracy						
Prediction Judgments	-.005	-.281	-4.37	< .001	-.007	-.003
Block 5: Objective Metacognitive Monitoring Accuracy						
Postdiction Judgements	.004	.226	3.20	.002	.001	.006
Block 6: Student Feelings About Their Mathematics Classroom						
Solving Challenging Math Problems	.002	.136	2.01	.045	.000	.003

Key: *b* = unstandardized regression coefficient;  $\beta$  = standardized regression coefficient; 95% confidence interval for the unstandardized regression coefficients; LB = Lower bound value; UB = Upper bound value.

*N* = 213

### Discussion

The purpose of the present investigation was to determine the predictive effect of EF, students' beliefs about their math skills, students' self-report math anxiety, math achievement, and objective metacognitive monitoring on students' PS accuracy and PS understanding. The findings of the study substantiate prior research that demonstrated factors such as anxiety and self-confidence (Irhamna et al., 2020), MC (Lesh & Zawojewski, 2007; Jitendra et al., 2015), content knowledge (Chapman, 2015), and EF (Clements et al., 2016; Viterbori et al., 2017), relate to PS proficiency. This study also extends this research by confirming that cognitive, metacognitive, and affective factors each explain unique variance within the PS scores of middle school students.

Regarding PS accuracy, results revealed that task switching, students' beliefs that they can solve time-consuming math problems, prediction judgments, and students' feelings regarding teachers' encouragement to solve challenging math problems, significantly predicted PS accuracy. Regarding PS understanding, math achievement, task switching, students' beliefs that they can solve time-consuming math problems, prediction and postdiction feeling of knowing metacognitive judgments, and students' feelings regarding teachers' encouragement to solve challenging math problems, were significant.

Cognitively, content knowledge was only predictive within the model of understanding, but not accuracy. Given that the individual correlation between content knowledge and PS understanding,  $r = .058, p > .05$ , was also non-significant, this potentially suggests that correct answers alone are insufficient to consider when examining students' mathematical understandings within PS solutions. EF, however, explained unique variance within both accuracy and understanding. Although task switching, generally considered a proxy of cognitive flexibility, predicted PS accuracy and understanding, the direction was opposite of our theoretical prediction. Our measure of interest, reaction time (RT) "switch cost," is the difference in RT between trials in which the task switches compared to repeat trials. In short, a robust finding is that "switch" trials have longer RTs on average than "repeat" trials, with the longer time (or difference between switch and stay) indicative of the additional EF processes that must occur. Thus, lower switch costs are thought to represent more efficient EF processing; as such, we predicted lower switch costs to be correlated with higher PS performance, but we observed the opposite. It is possible that a complex speed-accuracy trade-off is occurring, such that students that take more time during PS to engage in reflective processing perform similarly during task-switching, that is they pause to reflect after a task switch, which would in turn inflate their observed switch costs.

Interestingly, MC has a rather nuanced relation with students' PS accuracy and understanding. Whereas prediction judgments significantly predicted PS accuracy, both prediction and postdiction judgements predicted PS understanding. This is interesting because the literature on MC has consistently demonstrated that students tend to be more accurate in their postdictions (after the task) than their predictions (e.g., Gutierrez & Schraw, 2015; Hacker et al., 2008), especially when the postdictions are delayed (Thiede et al., 2012), presumably because students' have had additional opportunities to reflect on their performance and adjust their confidence in their performance to more closely align to their actual performance (Gutierrez de Blume et al., 2021).

Further, predictions, which are made before the task, were the best predictors of PS accuracy, uniquely explaining the majority of variability. Conversely, the combined predictive effect of predictions and postdictions on PS understanding was significantly attenuated. It is plausible that PS accuracy, which utilizes similar cognitive processes to metacomprehension judgments, is more closely associated with metacognitive skills than PS understanding. Indeed, this relation between metacomprehension and PS accuracy has been found in domains other than math (e.g., driving ability, Ackerman et al., 2010; physics, Sharma & Bewes, 2011).

In the affective domain, students' self-efficacy regarding their ability to solve time consuming mathematics problems was predictive in both models, along with the question measuring the degree to which students felt encouraged to solve challenging problems in their mathematics classes. Interestingly, mathematics anxiety, student beliefs about whether effort increases ability, and students' perceptions of how close they felt to mathematics were not unique predictors within either model, even though their individual correlations were significant with both accuracy and understanding. This potentially suggests that the variance was subsumed by other factors such as MC, EF, and self-efficacy.

## Limitations and Avenues for Future Research

There are number of limitations to consider from the present study. Chiefly, all factors were measure at single time points. However, many of these factors are dynamic and can change based on unique situations, contexts, and environment. Given that correlational analyses are limited in their ability to attend to factors that are context dependent (Schoenfeld, 2002), future research should consider utilizing state-based measures along with qualitative analyses to better understand how these factors interact in-the-moment, and how best to support students in becoming proficient problem solvers.

Secondly, although the ACE offers the advantage of being short and gamified (improving engagement), these shortened versions of classic cognitive batteries, typically involving hundreds of trials per condition, likely produce more variability and noise. Moreover, these computerized EF measures assess “core” EF, but may not fully predict “real-world” EF, such as those required in a math classroom. Finally, the present study was limited in the affective factors considered and thus future research should explore additional factors (e.g., motivation) and more robust measures of students’ math identity.

## Implications and Recommendations for Research, Theory, and Practice and Conclusion

As utilizing rigorous PS within instruction continues to be a focal point of K-12 mathematics courses, the results of the present study offer critical insights into factors related to PS proficiency that are essential to consider in supporting student learning. As previously suggested in research (e.g., Chapman, 2015), PS proficiency is complex, comprised of myriad cognitive, metacognitive, and affective factors. These results suggest that future research and instruction on PS should simultaneously attend to cognitive, metacognitive, and affective factors within PS, rather than parsing them out for individual attention. Thus, strengthening student PS may best be approached via a holistic, multi-pronged instructional approach that incorporates each of these factors, rather than attempting to teach these skills independently.

Moreover, the present investigation demonstrated that an objective measure of MC (monitoring accuracy before and after a task) was the best predictor of PS accuracy, and a significant predictor of PS understanding, albeit to a lesser extent. This has relevant implications for further research targeting these factors more deeply, and for the theories of SRL, MC and EF. For educators, this supports previous findings on the importance of attending to MC within PS instruction (e.g., Montague et al., 2011), and highlights the need for continued attention to developing students’ metacognition within the context of instruction on mathematical PS.

## Acknowledgements

The research reported here was supported by the EF+Math Program of the Advanced Education Research and Development Fund (AERDF) through funds provided to CueThinkEF+. The perspectives expressed are those of the authors and do not represent views of the EF+Math Program or AERDF.

## References

- Ackerman, M., Crowe, M., Vance, D., Wadley, V., Owsley, C., & Ball, K. (2010). The impact of feedback on self-rated driving ability and driving self-regulation among older adults. *The Gerontologist*, *51*(3), 367-378. <https://doi.org/10.1093/geront/gnq082>
- Boaler, J. (2002). *Experiencing school mathematics: Traditional and reform approaches to teaching and their impact on student learning*. Routledge.
- Boaler, J., & Staples, M. (2008). Creating mathematical futures through an equitable teaching approach: The case of Railside School. *Teachers College Record*, *110*(3), 608–645. <https://doi.org/10.1177/016146810811000>
- Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

- Bull, R., & Lee, K. (2014). Executive functioning and mathematics achievement. *Child Development Perspectives*, 8(1), 36–41. <https://doi.org/10.1111/cdep.12059>
- Butler, D. L., & Winne, P. H. (1995). Feedback and self-regulated learning: A theoretical synthesis. *Review of Educational Research*, 65(3), 245–281. <https://doi.org/10.3102/00346543065003245>
- Cai, J. (2003). What research tells us about teaching mathematics through problem solving. In F. K. Lester & R. I. Charles (Eds.), *Research and issues in teaching mathematics through problem solving* (pp. 241–254). National Council of Teachers of Mathematics.
- Carey, E. (2017). The modified abbreviated math anxiety scale: A valid and reliable instrument for use with children. *Frontiers in Psychology*, 8, 13. <https://doi.org/10.3389/fpsyg.2017.00011>
- Carr, M., & Jessup, D. L. (1995). Cognitive and metacognitive predictors of mathematics strategy use. *Learning and Individual Differences*, 7(3), 235–247.
- Chapman, O. (2015). Mathematics teachers' knowledge for teaching problem solving. *LUMAT: International Journal on Math, Science and Technology Education*, 3(1), 19–36. <https://doi.org/10.31129/lumat.v3i1.1049>
- Clements, D. H., & Sarama, J. (2007). Effects of a preschool mathematics curriculum: Summative research on the Building Blocks project. *Journal for Research in Mathematics Education*, 38, 136–163. DOI: 10.2307/30034954
- Clements, D. H., Sarama, J., & Germeroth, C. (2016). Learning executive function and early mathematics: Directions of causal relations. *Early Childhood Research Quarterly*, 36, 79–90. <https://doi.org/10.1016/j.ecresq.2015.12.009>
- Curriculum Associates (2022). Research base for *i-Ready Personalized Instruction for Mathematics: Executive Summary*. [https://www.curriculumassociates.com/-/media/mainsite/files/i-ready/iready-research-foundations-mathematics-executive\\_summary-2020.pdf](https://www.curriculumassociates.com/-/media/mainsite/files/i-ready/iready-research-foundations-mathematics-executive_summary-2020.pdf)
- Efklides, A. (2011). Interactions of metacognition with motivation and affect in self-regulated learning: The MASRL model. *Educational Psychologist*, 46, 6–25. <https://doi.org/10.1080/00461520.2011.538645>
- Flavell, J. H. (1976). Metacognitive aspects of problem solving. In L. B. Resnick (Ed.), *The nature of intelligence*. Erlbaum.
- Flavell, J. H. (1979). Metacognition and cognitive monitoring: A new area of cognitive-developmental inquiry. *American Psychologist*, 34, 906–911.
- Goldin, G. A., Hannula, M. S., Heyd-Metzuyanim, E., Jansen, A., Kaasila, R., Lutovac, S., ... Zhang, Q. (2016). Attitudes, beliefs, motivation and identity in mathematics education: An overview of the field and future directions. In *ICME-13 Topical Surveys* (pp. 1–35). [https://doi.org/https://doi.org/10.1007/978-3-319-32811-9\\_1](https://doi.org/https://doi.org/10.1007/978-3-319-32811-9_1)
- Gutierrez, A. P., & Schraw, G. (2015). Effects of strategy training and incentives on students' performance, confidence, and calibration. *The Journal of Experimental Education*, 83, 386–404. <https://doi.org/10.1080/00220973.2014.907230>
- Gutierrez de Blume, A. P., Schraw, G., Kuch, F., & Richmond, A. S. (2021). General accuracy and general error factors in metacognitive monitoring and the role of time in predicting metacognitive judgments. *CES Psicología*, 14(2), 179–208. <https://doi.org/10.21615/cesp.5494>
- Gutierrez de Blume, A.P., Rhodes, S., & Bryck, R. (under review). *Metacognitive awareness amount middle school adolescents: Development and validation of a shortened version of the MAI, Jr.*
- Hacker, D. J., Bol, L., & Bahbahani, K. (2008). Explaining calibration accuracy in classroom contexts: The effects of incentives, reflection, and explanatory style. *Metacognition Learning*, 3, 101–121. <https://doi.org/10.1007/s11409-008-9021-5>
- Hensberry, K. K. R., & Jacobbe, T. (2012). The effects of Polya's heuristic and diary writing on children's problem solving. *Mathematics Education Research Journal*, 24(1), 59–85. DOI: 10.1007/s13394-012-0034-7
- Irhamna, Amry, Z., Syahputra, H. (2020). Contribution of mathematical anxiety, learning motivation and self-confidence to student's mathematical problem solving. *Budapest International Research and Critics in Linguistics and Education*, 3, 1759–1772. <https://doi.org/10.33258/birle.v3i4.1343.g1806>
- Jitendra, A. K., Petersen-Brown, S., Lein, A. E., Zaslofsky, A. F., Kunkel, A. K., Jung, P. G., & Egan, A. M. (2015). Teaching mathematical word problem solving: The quality of evidence for strategy instruction priming the problem structure. *Journal of Learning Disabilities*, 48(1), 51–72. <https://doi.org/10.1177/0022219413487408>
- Kloosterman, P., & Stage, F. K. (1992). Measuring beliefs about mathematical problem solving. *School Science and Mathematics*, 92(3), 109–115. <https://doi.org/10.1111/j.1949-8594.1992.tb12154.x>
- Kramarski, B., Weisse, I., & Kololshi-Minsker, I. (2010). How can self-regulated learning support the problem solving of third-grade students with mathematics anxiety? *ZDM - International Journal on Mathematics Education*, 42(2), 179–193. <https://doi.org/10.1007/s11858-009-0202-8>

- Lemaire, P., & Lecacheur, M. (2011). Age-related changes in children's executive functions and strategy selection: A study in computational estimation. *Cognitive Development*, 26, 282–294. <https://doi.org/10.1016/j.cogdev.2011.01.002>
- Lesh, R., & Zawojewski, J. S. (2007). Problem solving and modeling. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 763–804). Information Age.
- Lester, F. K. (2013). Thoughts About Research On Mathematical Problem- Solving Instruction. *The Mathematics Enthusiast*, 10(1), 245-278. <https://doi.org/10.54870/1551-3440.1267>
- Lester, F. K., & Cai, J. (2016). Can mathematical problem solving be taught? Preliminary answers from 30 years of research. In P. Felmer, E Pehkonen, & J. Kilpatrick (Eds.), *Posing and solving mathematical problems* (pp. 117–135). <https://doi.org/10.1007/978-3-319-28023-3>
- Mayer, R. E., & Wittrock, R. C. (2006). Problem solving. In P. A. Alexander & P. H. Winne (Eds.), *Handbook of educational psychology* (2nd ed.), (pp. 287-304). Mahwah, NJ: Erlbaum.
- Montague, M., Enders, C., & Dietz, S. (2011). Effects of cognitive strategy instruction on math problem solving of middle school students with learning disabilities. *Learning Disability Quarterly*, 34(4), 262–272. <https://doi.org/10.1177/0731948711421762>
- Moran, T. P. (2016). Anxiety and working memory capacity: A meta-analysis and narrative review. *Psychological Bulletin*, 142(8), 831–864. <https://doi.org/10.1037/bul0000051>
- National Council of Teachers of Mathematics. (2014). *Principles to actions: Ensuring mathematical success for all*. National Council of Teachers of Mathematics
- National Council of Teachers of Mathematics. (2018). *Catalyzing change in high school mathematics: Initiating critical conversations*. Author.
- Özsoy, G., & Ataman, A. (2009). The effect of metacognitive strategy training on mathematical problem solving achievement. *International Electronic Journal of Elementary Education*, 1(2), 68–82.
- Panadero, E. (2017). A review of self-regulated learning: Six models and four directions for research. *Frontiers in Psychology*, 8, 1-28. <https://doi.org/10.3389/fpsyg.2017.00422>
- Passolunghi, M. C., & Siegel, L. S. (2001). Short-term memory, working memory, and inhibitory control in children with difficulties in arithmetic problem solving. *Journal of Experimental Child Psychology*, 80(1), 44–57. <https://doi.org/10.1006/jecp.2000.2626>
- Raghubar, K. P., Barnes, M. A., & Hecht, S. A. (2010). Working memory and mathematics: A review of developmental, individual difference, and cognitive approaches. *Learning and Individual Differences*, 20(2), 110–122. <https://doi.org/10/fs4ghp>
- Rhodes, S., Gutierrez de Blume, A., & Bryck, R. (in preparation). *Development and validation of a shortened version of the Indiana Mathematics Beliefs Scales*.
- Rosenzweig, C., Krawec, J., & Montague, M. (2011). Metacognitive strategy use of eighth-grade students with and without learning disabilities during mathematical problem solving: A think-aloud analysis. *Journal of Learning Disabilities*, 44, 508– 520. <https://doi.org/10.1177/0022219410378445>
- Schoenfeld, A. H. (1992). Learning to think mathematically: Sense-making in mathematics. In D. Grouws (Ed.), *Handbook for research on mathematics teaching and learning* (pp. 334–370). MacMillan.
- Schoenfeld, A.H. (2002). How can we examine the connections between teachers' world views and their educational practices? *Issues in Education*, 8(2), 217-227.
- Schoenfeld, A. H. (2013). Reflections on problem solving theory and practice. *The Mathematics Enthusiast*, 10(2), 9-34. <https://doi.org/10.54870/1551-3440.1258>
- Sharma, M. D., & Bewes, J. (2011). Self-monitoring: Confidence, academic achievement and gender differences in physics. *Journal of Learning Design*, 4(3), 1-13. doi: <http://dx.doi.org/10.5204/jld.v4i3.76>
- Sperling, R. A., Howard, B. C., & Miller, L. A. (2002). Measures of children's knowledge and regulation of cognition. *Contemporary Educational Psychology*, 27, 51-79. <https://doi.org/10.1006/ceps.2001.1091>
- Stein, M., Boaler, J., & Silver, E. A. (2003). Teaching mathematics through problem solving: Research perspectives. In *Teaching mathematics through problem solving: Grades 6-12*. National Council of Teachers of Mathematics.
- Steinberg, J., & Roditi, B. N. (2018). The strategic math classroom: How executive function impacts math learning. In *Executive Function in Education: From Theory to Practice*, 2nd ed (pp. 300–325). Guilford Press.
- Stillman, G. A., & Galbraith, P. L. (1998). Applying mathematics with real world connections: Metacognitive characteristics of secondary students. *Educational Studies in Mathematics*, 36, 157–189.
- Swanson, H. L., Jerman, O., & Zheng, X. (2008). Growth in working memory and mathematical problem solving in children at risk and not at risk for serious math difficulties. *Journal of Educational Psychology*, 100, 343–379. <https://doi.org/10.1037/0022-0663.100.2.343>

- Tan, D. A., & Limjap, A. A. (2018). Filipino students use of metacognitive skills in mathematical problem solving: An emergent model. *International Journal of Development Research*, 8, 20430–20439.
- Thiede, K. W., Redford, J. S., Wiley, J., & Griffin, T. D. (2012). Elementary school experience with comprehension testing may influence metacomprehension accuracy among seventh and eighth graders. *Journal of Educational Psychology*, 104(3), 554–564. <https://doi.org/10.1037/a0028660>
- Tzohar-Rozen, M., & Kramarski, B. (2014). Metacognition, motivation, and emotions: Contributions on self-regulated learning to solve mathematical problems. *Global Education Review*, 1(4), 76–94.
- Viterbori, P., Traverso, L., & Usai, M. C. (2017). The role of executive function in arithmetic problem-solving processes: A study of third graders. *Journal of Cognition and Development*, 18(5), 595–616. <https://doi.org/10.1080/15248372.2017.1392307>
- Winne, P. H., & Hadwin, A. F. (1998). Studying as self-regulated engagement in learning. In D. Hacker, J. Dunlosky, & A. Graesser (Eds.), *Metacognition in educational theory and practice* (pp. 277–304). Erlbaum.
- Winne, P. H., & Hadwin, A. F. (2008). The weave of motivation and self-regulated learning. In D. H. Schunk & B. J. Zimmerman (Eds.), *Motivation and self-regulated learning: Theory, research, and applications* (pp. 297–314). Lawrence Erlbaum Associates.
- Yeniad, N., Malda, M., Mesman, J., van IJzendoorn, M. H., & Pieper, S. (2013). Shifting ability predicts math and reading performance in children: A meta-analytical study. *Learning and Individual Differences*, 23, 1–9. <https://doi.org/10.1016/j.lindif.2012.10.004>
- Younger, J., O’Laughlin, K., Anguera, J. A., Bunge, S., Ferrer, E. E., Hoeft, F., McCandliss, B., Mishra, J., Rosenberg-Lee, M., Gazzaley, A., & Uncapher, M. (2022). *Better together: Novel methods for measuring and modeling development of executive function diversity while accounting for unity*. PsyArXiv. <https://doi.org/10.31234/osf.io/ctzwx>
- Zelazo, P. D., Blair, C. B., & Willoughby, M. T. (2017). *Executive function: Implications for education (NCER 2017-2000)*. Washington, DC: National Center for Education Research, Institute of Education Sciences, U.S. Department of Education. This report is available on the Institute website at <http://ies.ed.gov/ncer/pubs/20172000/>
- Zimmerman, B. J., & Moylan, A. R. (2009). Self-regulation: Where metacognition and motivation intersect. In D. J. Hacker, J. Dunlosky, & A. C. Graesser (Eds.), *Handbook of metacognition in education* (pp. 299-315). Routledge. <https://doi.org/10.4324/9780203876428>

## EXPLORING HIGH SCHOOL STUDENTS' CARELESS OR INSUFFICIENT EFFORT SURVEY RESPONSES WHEN INVESTIGATING ATTITUDES TOWARD MATHEMATICS

Xiaohui Wang

The University of Texas Rio Grande Valley  
xiaohui.wang@utrgv.edu

Mayra Ortiz Galarza

The University of Texas Rio Grande Valley  
mayra.ortizgalarza@utrgv.edu

Sergey Grigorian

The University of Texas Rio Grande Valley  
sergey.grigorian@utrgv.edu

u

Aaron Wilson

The University of Texas Rio Grande Valley  
aaron.wilson@utrgv.edu

John Knight

The University of Texas Rio Grande Valley  
john.knight01@utrgv.edu

*A modified, bilingual Attitudes Toward Mathematics Inventory (ATMI) instrument was administered to 1,258 high school students in South Texas in an NSF-funded project on informal learning of mathematics and near peer mentoring. We explore students' survey response behaviors and examine the existence of careless and insufficient effort (CIE) responses. This is empirical research for handling the challenge of CIE responses that leads to improved survey data quality, thus eventually validating the intervention effect of the mathematical informal learning project.*

**Keywords:** high school education; informal education; affect, emotion, beliefs and attitudes; measurement

It is well-known that in educational and psychological research self-report measures (i.e., surveys, questionnaires, inventories, etc.) are commonly used by researchers and practitioners to measure non-cognitive constructs (Ulitzsch, Yildirim-Erbasli, Gorgun, & Bulut, 2022). However, in these measures, Careless Response (CR) and Insufficient Effort Response (IER) have been prevalently identified as common sources for biased estimates, and invalid inferences and response data (see for example, Zhong, Li, & Li, 2021; Ward & Meade, 2023).

As part of an NSF-funded project on informal learning of mathematics, the authors implemented an informal educational intervention in which high school students interacted with their near-peer college students in the context of novel mathematical explorations (Wilson, et al., 2023). In particular, high school students were exposed to *MathShows*, which were interactive mathematical presentations, as well as math social media, and a math summer internship. A smaller scale pilot study (Wilson & Grigorian 2019) has shown that such near peer interventions have the potential to positively affect attitudes to mathematics. To understand the impact of such interactions on high school students' attitudes to mathematics, the project team administered surveys to the high school students involved in the project before the start of the project, at specific times during the project, and at the end of the project. The experimental design involved some schools receiving the intervention, while others serving as control groups with no intervention (i.e., intervention offered after data collection). Over the course of 3 years, this project involved 1,258 students from 4 high schools in two majority-Hispanic school districts in South Texas. One of the survey instruments was the 19-item shortened Attitudes Toward Mathematics Inventory (ATMI) survey (Lim & Chapman, 2013; Majeed et al., 2013, Wilson & Grigorian 2019). This is a 5-point

Likert scale survey that measures four subscales: mathematics enjoyment, mathematics motivation, self-confidence in mathematics, and perceived value of mathematics.

Measuring beliefs and attitudes toward mathematics is important to mathematics educational research. In this empirical study, we address a necessary concern about validity of responses in any survey-based educational research project.

### **Research Context and Methods of the Study**

A qualitative component of the study involved focus group interviews with high school students. Reported data derived from some focus groups show that some participants answered the surveys carelessly. Some of the careless survey completion behavior includes answering questions without reading them, choosing answers at random, and copying answers from their classmates. This anecdotal evidence of CIE (careless or insufficient effort) responses prompted concern on how widespread this practice may be, especially since some surveys were administered remotely during the COVID-19 pandemic without project team supervision. Therefore, to accurately measure the impact of the intervention activities on student attitudes toward mathematics, it has become necessary to implement robust techniques to detect and filter CIE responses. C/IE responses can evidently invalidate research results since they represent data that is derived from a scenario where the survey respondent failed to provide genuine responses (Curran, 2015). Throughout the paper we use the acronym CIE for *careless and/or insufficient effort survey responses*: “in which a person responds to items without sufficient regard to the content of the items and/or survey instructions” (Huang, Liu, & Bowling, 2015, p. 828).

In recent years, multiple methods for detecting both CR and IER have been developed and amended to better detect a wide array of CR and IER patterns indicators (i.e., response-pattern or response-time-based, model-based approaches, etc.) (Ulitzsch, et al., 2022). For instance, Curran (2015) reviews a variety of methods that have been used to detect survey responses that were evidently provided by participants who were careless or else gave insufficient effort C/IE. For detection of such responses, Curran suggests applying certain methods in sequence, as deemed appropriate to the circumstances of the survey administration and data received (Curran, 2015). Multiple post-hoc metrics or methods for the detection of C/IE responses are highlighted by Curran, including: 1) response time, 2) long-string analysis, 3) Mahalanobis distance, 4) odd-even consistency, 5) resampled individual reliability, 6) semantic antonyms/synonyms, 7) inter-item standard deviation, 8) polytomous Guttman errors, and 9) person total correlation. Beyond these, Curran (2015) also suggests other checks that can be included when developing the survey, before data collection. For instance, of particular interest to our study is Curran’s observation that surveys involving reverse-worded items introduce additional complexity into the detection of C/IE responses. In this paper we address data from a survey that included some reverse-worded items, and thus our findings elaborate the detection of C/IE responses for this more complex case.

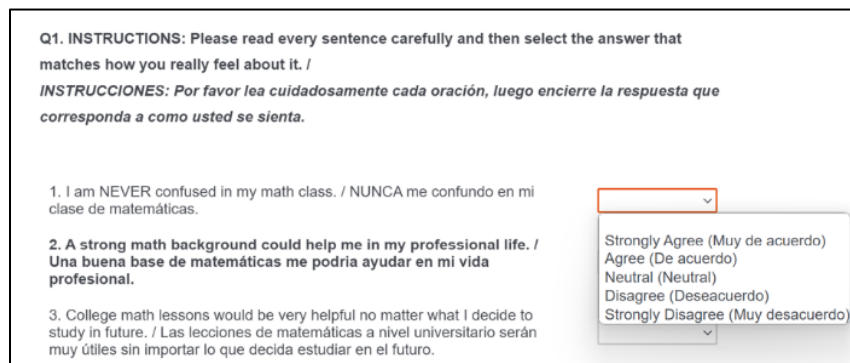
### **Research Objectives and Data**

Our research goal is to investigate to what extent high school students make CIE responses when taking the ATMI survey. Table 1 lists the 19 ATMI items used in our study, with the tone of negative or positive marked on the side. The two research questions we will address are:

- What are the CIE indicators for this study?
- To what extent did high school students make CIE responses?



We use the baseline data collected digitally via Qualtrics surveys from 2021 through 2022. Figure 1 gives a sample view of how the survey appeared to students. Students' participation was voluntary. The data consists of survey responses and time stamp information from 1,258 students from four different schools (A, B, C and D). The first two schools (A intervention, and B control) were in the remote schooling format (due to COVID pandemic) during the data collection stage, thus surveys had to be supervised by teachers. The other two schools (C and D) were in the face-to-face schooling format when data was collected. The survey was administered by the research team in the school classrooms. It is worth noting that two different administration methods were used among the latter two schools. In School C, an intervention school, students were given the survey including consent and assent forms right after the introduction of the project and research team personnel. In School D, a control school, a short motivating *mini-MathShow* (a “math-magician” show involving a number trick) was performed after those introductions and before the survey administration, in an effort to build up an initial rapport between the project team and the students. During the surveys administration in Schools C and D, the researchers persistently reminded students about the requirement and importance of reading the survey questions carefully and responding accurately and honestly.



**Figure 1: Sample View of The Online ATMI Survey**

The table below gives all 19 items of ATMI as implemented in this study, noting the working orientation (positive or negative), as well as providing the Spanish translation as used in our administration.

**Table 1: The 19 ATMI Items Used (Four constructs: SC – Self-Confidence; VAL – Sense of Value Toward Math, ENJ – Enjoyment of Math, MOV – Motivation to Do Math)**

Tone and Construct	ATMI Items
+ SC	1. I am never confused in my math class. (Nunca me confundo en mi clase de matemáticas)
+ VAL	2. A strong math background could help me in my professional life. (Una buena base de matemáticas me podría ayudar en mi vida profesional)
+ VAL	3. College math lessons would be very helpful no matter what I decide to study in future. (Las lecciones de matemáticas a nivel universitario serán muy útiles sin importar lo que decida estudiar en el futuro)
- ENJ	4. Math is NOT a very interesting subject. (Las matemáticas NO son una materia muy interesante.)
+ ENJ	5. I really like math. (Me gustan mucho las matemáticas)
- SC	6. It makes me nervous to even think about having to do a math problem. (Me pone nervioso tan solo pensar en hacer un problema de matemáticas)
- ENJ	7. I don't like to solve new problems in math. (No me gusta resolver problemas nuevos de matemáticas.)
- VAL	8. Math is one of the LEAST important subjects for people to study. (Las matemáticas son una de las materias MENOS importantes que la gente debe estudiar.)
- SC	9. I feel a sense of insecurity when attempting math. (Siento una sensación de inseguridad cuando intento hacer matemáticas.)
- VAL	10. Math is a worthless and unnecessary subject. (Matemáticas es una materia sin valor e innecesaria.)
- MOV	11. The challenge of math does not appeal to me. (El reto de hacer matemáticas no me llama la atención.)
+ ENJ	12. I am happier in a math class than in any other class. (Soy más feliz en una clase de matemáticas que en cualquier otra clase.)
- SC	13. Studying math makes me feel nervous. (Estudiar matemáticas me hace sentir nervioso.)
- MOV	14. I plan to take as little math as I can during my education. (Yo planeo tomar las menos matemáticas posibles durante mi educación.)
+ ENJ	15. I have usually enjoyed studying math in school. (Usualmente he disfrutado estudiar matemáticas en la escuela.)
- VAL	16. Math is NOT important in everyday life. (Las matemáticas NO son importantes en la vida diaria.)
+ SC	17. I am always calm and relaxed in a math class. (Siempre estoy calmado y relajado en una clase de matemáticas.)
+ MOV	18. I am willing to take more than the required amount of math. (Estoy dispuesto a tomar más matemáticas de lo requerido.)
+ MOV	19. I am confident that I could learn advanced math. (Me siento seguro de que podría aprender matemáticas avanzadas.)

### CIE Identifying Methods

We coded the 5-point Likert scales for the ATMI survey items as shown in Figure 1 and Table 1 using values of (-2, -1, 0, 1, 2). To distinguish the severeness of the CIE responses, we labeled them in four categories: non-CIE, slight CIE, moderate CIE, and severe CIE. We assigned flag value of 0 to both non-CIE and slight CIE cases, value 1 to moderate CIE, and value 2 to severe CIE. Combining the methods existing in the literature and our survey, we propose the following criteria to identify potential CIE responses:

1. *All-item-same criterion*: if a student's responses for all 19 items are the same, the responses will be classified as severe CIE. Flag values for this criterion are 0 or 1, with 1 representing CIE flag.

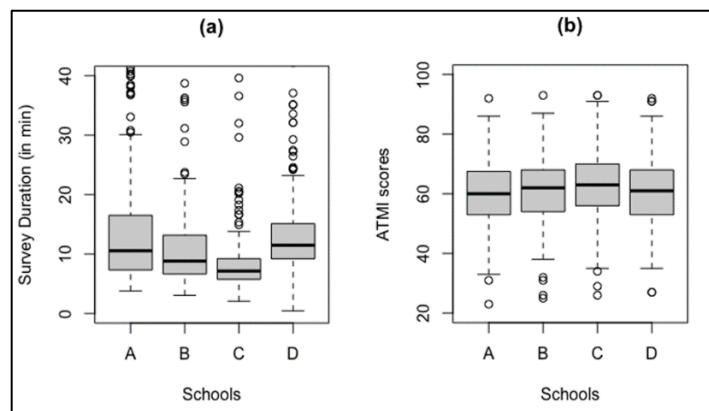
2. *Response time criterion*: Abandoned or non-submitted surveys are automatically recorded by Qualtrics after a period of 7 days, so existence of such responses leads to extreme outliers in survey duration. On the other hand, testing of the online surveys by the research team has shown that it is not feasible to complete the entire survey too quickly, while reading the questions and answering thoughtfully. If a student's survey duration (recorded by Qualtrics) is shorter than the threshold, defined by "median – median absolute deviation" that is calculated based on all survey durations in his/her school. Median and median absolute deviation are used for threshold computing because the response time data has many far outliers. This is the total response time, not just the ATMI, which was only one component of a longer survey. Flag values for this criterion is 0 or 1, with 1 representing CIE flag.
3. *ATMI score outliers*: if a student's ATMI score appeared to be an outlier compared to the ATMI scores in the corresponding school, we classify it as a potential CIE.
4. *Neighbor Opposite-Item-Pair consistency criterion*: For a pair of two neighbor items, if one is positively toned and the other negatively toned, it is a Neighbor Opposite-Item-Pair. There are three such pairs as shown in Table 1, and they are items 4&5, 11&12, and 14&15. Out of the 25 possible responses combinations of the choices for an opposite pair, if the two codes had different signs or being zero in one of the items, we consider that the student had polarized answers, which indicates that she/he reacted to the tone switch correctly. These types of combinations are classified as non-CIE. The combinations (-1)to(-1) or (1)to(1) fall outside of the class of non-CIE but since the inconsistency is minor, we classify it as slight CIE. We classify (-2)to(-2) or (2)to(2) as severe CIE for these choices pair indicating maximum possible inconsistency. Four other combinations are classified as moderate CIE.
5. *Same Construct Opposite-Item-Pair consistency criterion*: For a pair of two items in a construct, if one is positively toned and the other negatively toned and that expert opinion confirms a high level of similarity among them, it is a Same Construct Opposite-Item-Pair. There are four such pairs as shown in Table 1: three pairs of SC construct items, 6&17, 9&17, and 13&17; and one pair of ENJ construct items, 4&5. This last pair is already included in the Neighbor Opposite-Item-Pair consistency criterion. The same classification methods as in Neighbor Opposite-Item-Pair consistency criterion are used for pairs in this criterion.
6. *Same Construct Similar-Item-Pair consistency criterion*: For a pair of two items in a construct, if they are toned in the same direction and that an expert opinion confirms a high level of similarity among them, it is a Same Construct Similar-Item-Pair. There are eight such pairs as shown in Table 1: three pairs of SC construct items, 6&9, 6&13, and 9&13; one pair of ENJ construct items: 12&15; and four pairs of VAL construct items: 2&3, 8&10, 8&16, and 10&16. There are maximum of four units steps between the choices, from (-2)to(2) or (2)to(-2). We classify any response combinations having one step apart or completely agree is non-CIE. If the two choices are two steps apart, it is a slight CIE; three steps apart, moderate CIE, and then four steps apart, severe CIE.

All the aforementioned criteria are constructed in a conservative way to allow students to express their opinions and to avoid disqualifying responses that have limited accidental errors.

## Results

Of the 1,258 student participants, we found that there were 19 students (1.5%) giving the same answers to all 19 ATMI items. According to the response time criterion, there were 90 students (7.8%) flagged for responding too fast. Figure 2 (Panel a) displays the distribution of survey durations for four schools. The distributions of survey durations varied. One reason is that the total numbers of survey items varied (51, 51, 36 and 47 items for Schools A, B, C and D). School D seemed to have relatively longer survey durations than other schools. According to the ATMI score outliers criterion, there were 20 students (1.6%) flagged for their extreme ATMI scores. Of them, nine students had low ATMI scores ( $\leq 34$ ) and eleven students reported high ATMI scores ( $\geq 87$ ). Figure 2 (Panel b) displays the distribution of ATMI scores for four schools. We see that the four schools have almost identical distributions of ATMI scores at baseline. Note in Figure 2 that the y-axis for survey duration was cut-off at 40 mins, as there were a number of extremely far outliers, indicative of students that forgot to submit their form until hours and days later.

Table 2 lists the students' response pairs and corresponding counts for the Opposite-Item-Pairs. Those pairs are grouped into four levels of CIE. We found that more than 84% of students gave consistent responses to those six pairs, respectively. Students' CIE rates for the Neighbor Opposite-Item-Pairs increased for questions later in the survey, from 5.3% on the first pair of 4&5, to 12.3% on the second pair of 11&12, then to 16% on the last pair 14&15. The rates of the three CIE sub-categories also increased: slight CIE increased from 3.4% to 8.4% and then to 10.1%; moderate CIE increased from 1.4% to 3.6% and then to 4.5%; severe CIE increased from 0.6% to 1.2% then to 1.4%.



**Figure 2: The Distributions of Survey Durations (Panel a) and ATMI Scores (Panel b) for Four Schools.**

**Table 2. Students' Opposite-Item-Pair Responses in Different CIE Levels**

Pair-wise Responses	Neighbor Opposite-Item-Pair			Same Construct Opposite-Item-Pair (SC)		
	4&5	11&12	14&15	6&17	9&17	13&17
9 pairs with one or two 0s	648	825	678	676	652	629
(-2)to(1)	30	21	30	41	43	37
(-2)to(2)	55	13	20	29	32	32
(-1)to(1)	203	68	166	186	175	203
(-1)to(2)	36	7	39	34	24	30
(1)to(-2)	40	43	8	18	22	20
(1)to(-1)	83	72	70	81	101	107
(2)to(-2)	61	28	21	32	28	30
(2)to(-1)	30	13	24	34	31	19
Non-CIE	1186 (94.7%)	1090 (86.7%)	1056 (84.0%)	1131 (90.0%)	1108 (88.1%)	1107 (88.1%)
(-1)to(-1)	20	94	34	38	45	44
(1)to(1)	22	11	93	44	63	61
Slight CIE	42 (3.4%)	105 (8.4%)	127 (10.1%)	82 (6.5%)	108 (8.6%)	105 (8.4%)
(-2)to(-1)	2	8	3	3	1	4
(-1)to(-2)	10	31	9	8	11	11
(1)to(2)	3	6	30	14	9	15
(2)to(1)	3	2	15	11	7	6
Moderate CIE	18 (1.4%)	47 (3.7%)	57 (4.5%)	36 (2.9%)	28 (2.2%)	36 (2.9%)
(-2)to(-2)	4	9	5	4	5	4
(2)to(2)	3	6	12	4	8	5
Severe CIE	7 (0.6%)	15 (1.2%)	17 (1.4%)	8 (0.6%)	13 (1.0%)	9 (0.7%)
Total	1253	1257	1257	1257	1257	1257

The Similar-Item-Pairs counts for all possible response pairs and three levels of CIE are reported in Table 3. We found that almost more than 90% of students gave consistent responses on those eight pairs, respectively. The pairs involving later items in the survey had higher CIE rates than pairs involving earlier items.

**Table 3. Students' Similar-Item-Pair Responses in Different CIE Levels**

Constructs	SC (3 pairs)			ENJ (1 pair)	VAL (4 pairs)			
Pair-wise Responses	6&9	6&13	9&13	12&15	2&3	8&10	8&16	10&16
Non-CIE (15 pairs with values less than 2 units apart)	1196 (95.2%)	1209 (96.3%)	1197 (95.3%)	1136 (90.3%)	1178 (93.9%)	1193 (95.0%)	1148 (91.3%)	1150 (91.5%)
(-2)to(0)	9	3	4	35	4	4	38	57
(0)to(-2)	9	8	13	2	5	25	19	4
(0)to(2)	12	7	8	47	18	5	12	7
(2)to(0)	12	11	15	3	25	9	8	3
Slight CIE	42 (3.3%)	29 (2.3%)	40 (3.2%)	87 (6.9%)	52 (4.1%)	43 (3.4%)	77 (6.1%)	71 (5.6%)
(-2)to(1)	1	2	4	15	3	1	6	16
(-1)to(2)	6	2	5	8	7	2	7	10
(1)to(-2)	1	5	2	4	6	8	5	1
(2)to(-1)	5	6	5	2	3	4	4	3
Moderate CIE	13 (1.0%)	15 (1.2%)	16 (1.3%)	29 (2.3%)	19 (1.5%)	15 (1.2%)	22 (1.8%)	30 (2.4%)
(-2)to(2)	2	2	2	5	3	0	4	4
(2)to(-2)	3	1	1	1	2	5	6	2
Severe CIE	5 (0.4%)	3 (0.2%)	3 (0.2%)	6 (0.5%)	5 (0.4%)	5 (0.4%)	10 (0.8%)	6 (0.5%)
Total	1256	1256	1256	1258	1254	1256	1257	1257

Based on the six criteria, the maximum possible total count of flags is 17. The more flags a response has, the more likely it is a CIE response. Table 4 gives the distribution of flag counts for the 1258 student participants. There were 390 (31%) students having at least one CIE flag, 135 (10.7%) students having at least two CIE flags, and 57 (4.5%) students having at least three CIE flags.

**Table 4. Summary of Students with Various Number of Flags**

No. of Flags	7	6	5	4	3	2	1	0
Counts (%)	2(0.2%)	3 (0.2%)	5 (0.4%)	13 (1%)	34 (2.7%)	78 (6.2%)	255 (20.3%)	868 (70%)
Cumulative Counts (%)	2 (0.2%)	5 (0.4%)	10 (0.8%)	23 (1.8%)	57 (4.5%)	135 (10.7%)	390 (31%)	1258 (100%)

### Discussion

Some methods for identifying CIE responses as found in the literature (e.g., Curran, 2015). are not suitable for our digital survey situation. For example, the long-string or straight-lining criterion does not seem to be relevant for the survey administered in this study, because participants needed

to select responses from drop-down boxes, as shown in Figure 1. Moreover, long strings may be genuine responses for some items in our survey (such as items 6 through 11 could give long-string = 6). The digital survey platform, Qualtrics, allowed us to record the survey duration. We used survey questions with mixed tones consisting of ten negative tone items and nine positive tone items. This mixture gives us a total of 14 pairs to examine consistency of participants' responses. With such survey questionnaire design, we were able to propose the six criteria to identify CIE responses for the study population, leading to our answer for the first research question.

For the second research question, there were significant number of students who gave CIE responses as close to a third of participating students had at least one CIE flag. As noted in literature (Curran, 2015; Zhong et al., 2021) the rate of CIE responses increases as the survey progresses, and we have observed the same pattern with CIE rates for the Neighbor Opposite-Item-Pairs increasing for later questions in the survey.

### Conclusion

The findings in this study confirm the existence of CIE responses. In the next phases of this study, it will be essential to examine how CIE responses impact the ATMI reliability measures and the corresponding confirmatory factor analysis. Further, we will examine whether different survey administration methods (remote vs face-to-face schooling formats) affect the occurrence of CIE responses. The surveys analyzed for this paper are the baseline surveys at the beginning of the project for each school. The study reported here is ongoing and we will soon collect end-of-project (EOP) surveys. The CIE criteria calibrated from the baseline surveys will be applied to the EOP surveys. It is worth noting that calibration of CIE criteria should not be based on EOP surveys to avoid bias.

Using the methods detailed in this paper allows filtering CIE responses to measure the differences more accurately in math attitudes between control and intervention schools. The survey design strategy and CIE detection methods are generalizable to research aiming at learner's belief and attitude evaluations, especially study involving high school students.

### References

- Curran, P. G. (2015). Methods for the detection of carelessly invalid responses in survey data. *Journal of Experimental Social Psychology*, 66 (2016), 4–19. <https://doi.org/10.1016/j.jesp.2015.07.006>
- Huang, J. L., Bowling, N. A., Liu, M., & Li, Y. (2015). Detecting insufficient effort responding with an infrequency scale: Evaluating validity and participant reactions. *Journal of Business and Psychology*, 30(2), 299–311. <https://doi.org/10.1007/s10869-014-9357-6>
- Lim, S. Y., Chapman, E. (2013). Development of a short form of the attitudes toward mathematics inventory. *Educational Studies in Mathematics*, 82(1), 145–164.
- Majeed, A. A., Darmawan, I. G. N., Lynch, P. (2013). A confirmatory factor analysis of attitudes toward mathematics inventory. *The Mathematics Educator*, 15(1), 121–135.
- Ulitzsch, E., Yildirim-Erbasli, S. N., Gorgun, G., & Bulut, O. (2022). An explanatory mixture IRT model for careless and insufficient effort responding in self-report measures. *British Journal of Mathematical and Statistical Psychology*, 75(3), 668–698. <https://doi.org/10.1111/bmsp.12272>
- Ward, M. K., & Meade, A. W. (2023). Dealing with careless responding in survey data: Prevention, identification, and recommended best practices. *Annual Review of Psychology*, 74, 577–596. <https://doi.org/10.1146/annurev-psych-040422-045007>
- Wilson, A. T., & Grigorian, S. (2018). The near-peer mathematical mentoring cycle: Studying the impact of outreach on high school students' attitudes toward mathematics. *International Journal of Mathematical Education in Science and Technology*, 50(1), 46–64. <https://doi.org/10.1080/0020739X.2018.1467508>

- Wilson, A. T., Wang, X., Galarza, M. O., Knight, J., & Patino, E. (2023). Math attitudes and identity of high schoolers impacted through participating in informal, near-peer mentoring. *International Journal of Research in Education and Science (IJRES)*, 9(2), 535-545. <https://doi.org/10.46328/ijres.3093>
- Zhong, X., Li, M., & Li, L. (2021). Preventing and detecting insufficient effort survey responding. *Advances in Psychological Science*, 29(2), 225–237. <https://doi.org/10.3724/SP.J.1042.2021.00225>



# FACTORS THAT INFLUENCE COLLEGE STUDENTS' MATHEMATICS IDENTITY AND BELIEFS ABOUT THE NATURE OF MATHEMATICS

## FACTORES QUE INFLUYEN LA IDENTIDAD Y CREENCIAS SOBRE LA NATURALEZA DE LAS MATEMÁTICAS DE LOS ESTUDIANTES UNIVERSITARIOS

Parker Parrent  
University of Kentucky  
parker.parrent@uky.edu

Cindy Jong  
University of Kentucky  
cindy.jong@uky.edu

William Wiseman  
University of Kentucky  
william.wiseman@uky.edu

Pooja Sidney  
University of Kentucky  
pooja.sidney@uky.edu

Benjamin Braun  
University of Kentucky  
benjamin.braun@uky.edu

*Students' beliefs about the nature of mathematics can impact many aspects of their academic and life pursuits and be influenced by a myriad of prior educational experiences. To examine undergraduate college students' ( $n = 50$ ) past mathematics education experiences, beliefs about the nature of mathematics, and experiences in a College Algebra course, pre- and post-surveys were administered in Spring 2022. The figured worlds conceptual framework was used to understand in the multi-layered components of identity in varying mathematical learning contexts. Results of open-ended responses demonstrated four themes, including: affective factors (e.g., stress, enjoyment, confusion), achievement (e.g., success, failure, course requirements), acknowledgement (e.g., feeling seen, participation) and application (e.g., careers, daily life) of mathematics among undergraduate college students.*

Keywords: Affect, Emotion, Beliefs, and Attitudes; Undergraduate Education; Identity

### Perspectives

The nature of mathematical knowledge is loosely defined by Kean (2017) as an individual's belief of how mathematical knowledge is organized and the source, justification, and content of this knowledge. Students' beliefs about the nature of mathematics can impact many aspects of their academic and life pursuits and be influenced by a myriad of prior educational experiences. In this study, we specifically examine the beliefs and experiences of college students enrolled in college algebra courses, as we know the course is a gatekeeper that often predicts students' graduation (Bailey, et al., 2010). We use the figured worlds conceptual framework to understand the multi-layered components of identity in varying contexts related to the learning of mathematics (Holland, et al., 1998).

Boaler and Greeno (2000) explore the ideas of students' figured worlds and how they are affected by the classroom environments teachers create. According to Holland et al. (1998), "figured worlds are places where agents come together to construct joint meanings and activities. Figured worlds are socially and culturally constructed realms of interpretation in which particular characters and actors are recognized, significance is assigned to certain acts, and particular outcomes are valued over others" (p.52). Students within these figured worlds develop their own positional identities, which is the way in which a student perceives themselves within their own figured worlds (i.e., the contexts in which they participate, such as a college mathematics course or

a work environment), and the actions they take to fit into these roles they have developed (Boaler & Greeno, 2000).

## Methodology

### Context of Project

To examine undergraduate college students' past mathematics education experiences, beliefs about the nature of mathematics, and experiences in a College Algebra course, pre- and post-surveys were administered in spring 2022. While there were 92 participants who completed the pre-survey and 131 who completed the post-survey, 50 participants completed both the pre- and post-surveys. For the purposes of our study, we decided to focus on this sample of 50 to compare pre- and post-survey results. Our guiding research question was: *In what ways do college students' past and current experiences learning mathematics affect their views on the nature of mathematics?*

### Instrumentation

The College Math Beliefs and Belonging (CMBB) survey, which consisted of Likert-type items and open-ended questions, was developed by a collaborative group of researchers in the Mathematics, Psychology, and STEM Education departments at the same university where data were collected. The survey included scales intended to measure beliefs about the nature of mathematics, perceptions of mathematics proficiency, and sense of belonging in mathematics; responses were on a 6-point scale ranging from *strongly agree* to *strongly disagree*. The open-ended questions probed students about past mathematics experiences, experiences in College Algebra, and views on mathematics. In addition to examining the psychometric properties of the scales, researchers conducted cognitive interviews to collect further validity evidence (Leighton, 2017). One open-ended question posed at both timepoints asked participants: *What does mathematics mean to you?* Questions unique to the pre-survey included the following: Describe at least one *positive* mathematics experience from when you were in elementary, middle, and/or high school. (It can be something you experienced in school or out of school.) *and* Describe at least one *negative* mathematics experience from when you were in elementary, middle, and/or high school. (It can be something you experienced in school or out of school.) Questions unique to the post-survey included the following: Have any experiences this semester influenced your view of mathematics? If so, how? *and* Did your experience in this course influence your decision whether or not to take math courses in the future? If yes, how did it influence you?

### Data Analysis

For the purposes of this paper, our analysis focused on responses to the open-ended questions in the CMBB. We used open coding by taking an inductive approach to themes that directly came from the data (Strauss & Corbin, 1998). The raw data were organized into a Google spreadsheet and all responses were read. As two researchers read through the raw data, notes were taken of the types of responses being made to inform preliminary codes. This process was repeated as codes were refined and defined. Then, the data were coded by indicating a "1" when a particular code was present within a participant's response to allow for frequencies to be computed. Within each theme, we also coded one of three sub-categories to capture the nuances among the responses: positive, negative, or neither. In some cases, responses had multiple codes and included experiences or perspectives that were both positive and negative. Finally, we categorized the codes into major themes and developed a description for each theme (Creswell, 2009). After the two researchers coded the data, disagreements were discussed and negotiated to establish 100% agreement for the 300 responses (50 participants with three questions at each of the two timepoints).

## Results

Four themes emerged from the open-ended responses. Included in these themes were: affective factors (e.g., stress, confusion, enjoyment), achievement (e.g., success, failure, course requirements), acknowledgement (e.g., feeling seen, participation) and application (e.g., careers, daily life) of mathematics. The themes of affect and acknowledgement related to identity while the application related to perceptions of the nature of mathematics; achievement related to both. We also included an “other” category to capture statements that did not fit into one of the four themes. Table 1 provides examples of responses that were coded as one of the four themes. Patterns still need to be examined within and among the frequencies due to several of the pre- to post-questions being different. However, the following frequencies were present among responses in the pre-survey, which included college students’ prior learning experiences in mathematics: achievement = 79 (positive = 36, negative = 39, neither = 4); affect = 37 (positive = 19, negative = 18); acknowledgement = 14 (positive = 9, negative = 5); application = 28 (positive = 18, negative = 1, neither = 9). In response to what mathematics means, college students described it as formulas, studying numbers and shapes, ways to solve problems, being difficult, important, and applicable to real life.

**Table 1. Sample Responses among Themes based on Open-ended Questions**

Theme	Responses
Achievement	Getting all A's in my math classes in high school.  My 10th grade geometry teacher didn't really teach us, so I almost failed my first high school class.
Affect	I enjoyed math all of my years because of all of the teachers I had.  It makes me feel stressed and that is a requirement not something I should feel like I want to do.
Acknowledgement	One positive experience I've had was in high school when I was in a stats class. I felt appreciated and was able to engage in class/learn.  I was told that I have a great memory with concepts compared to other students and remembering the definitions and ideas of certain topics.
Application	Math helped me to be better in physics.  I help my grandma to solve the problems in daily life when we buy something in the market.

## Discussion

Boaler and Greeno (2000) analyzed figured worlds and mathematical identities by addressing the four ways of knowing as developed by Belenky et. al (1986), which include: received knowing, subjective knowing, separate knowing, and connected knowing. We found that two of these ways of knowing were shown in our undergraduate mathematics context: received knowing and connected knowing. Received knowledge is seen as a type of learning in which the individual looks to an authoritative source for their knowledge. Connected knowing is developed primarily through interactions with people, most of the knowledge comes from understanding other experiences, reasoning and combining that with those of the individual. We plan to further investigate these types of knowing and how they relate to the nature of mathematics and identity

among undergraduate students. Findings further supported the multi-layered and complex nature of identities in mathematics, and how they are shaped by various learning experiences.

### References

- Boaler, J. & Greeno, J. G. (2000) Identity, Agency and Knowing in Mathematical Worlds. In J. Boaler (Ed.) Multiple perspectives on mathematics teaching and learning (pp. 171-200), Westport, CT: Ablex.
- Belenky, M. F., Clinchy, B. M., Goldberger, N. R., & Tarule, J. M. (1986). Women's Ways of Knowing: The Development of Self, Voice and Mind. New York: Basic Books Inc.
- Creswell, J.W. (2009). Research design: Qualitative, quantitative, and mixed methods approaches (3rd ed.). Thousand Oaks, CA: Sage.
- Holland, D., Lachcotte, W., Skinner, D., & Cain, C. (1998). Identity and agency in cultural worlds. Cambridge, MA: Harvard University Press.
- Kean, L. (2017) Using Beliefs about the Nature of Mathematical Knowledge in Teaching College Algebra. In B. Gold, C.E. Behrens, & R.A. Simons (Eds.) Using the Philosophy of Mathematics in Teaching Undergraduate Mathematics (pp.63-74), MAA Press: Washington, DC.
- Leighton, J. P., & Ohio Library and Information Network. (2017). Using think-aloud interviews and cognitive labs in educational research. New York, NY: Oxford University Press.
- Strauss, A., & Corbin, J. (1998). Basics of qualitative research: Techniques and procedures for developing grounded theory (2nd ed.). Thousand Oaks, CA: Sage.

# FOLD TO MOLD: ORIGAMI-BASED ACTIVITIES' EFFECT ON MIDDLE SCHOOL STUDENTS' BELIEFS ABOUT MATHEMATICS

Yi-Yin Ko  
Indiana State University  
Winnie.Ko@indstate.edu

Connor Goodwin  
Indiana State University  
Connor.Goodwin@indstate.edu  
u

Erick Herrera  
Indiana State University  
eherrera2@sycamores.indstate.edu

*To better support middle school students whose mathematics learning has been affected by the COVID-19 pandemic, we designed a series of origami-based activities to improve their attitudes and beliefs towards mathematics throughout a semester-long program. The results suggest that students' beliefs about solving time-consuming mathematics problems and the importance of understanding mathematical concepts shift positively from the pre-assessment to the post-assessment. The results also show that the majority of students strongly agreed that understanding the correctness of solutions is important in the pre-assessment, mid-assessment, and post-assessment.*

Key Words: Design Experiments, Problem Solving, Middle School Education

## Introduction and Theoretical Framework

It is acknowledged that students' beliefs about mathematics affect not only their interest in mathematics but also their learning of mathematics (e.g., Kloosterman, 1996). However, students losing interest in mathematics during their middle school years has been around for a long time (e.g., Christensen, & Knezek, 2020). Despite middle school students' lack of interest in mathematics, available research has demonstrated that some students' mathematical learning was affected by the COVID-19 pandemic (e.g., Kuhfeld, Tarasawa, Johnson, Ruzek, & Lewis, 2020; Store & Store, 2022). One possible way to assist middle school students who have struggled with mathematics through this pandemic and/or have encountered more difficulty with mathematics than before is to implement origami-based activities. While using origami activities can develop middle school students' mathematical reasoning and communication skills (e.g., Georgeson, 2011; Miles, 2011), there is a need to provide insight into how engaging origami-based tasks impact middle school students' beliefs about mathematics over the semester under current post-pandemic learning environments. To address this research gap, this study is guided by the following two research questions:

1. What are similarities and differences among middle school students' beliefs towards working on time-consuming mathematics problems over the semester?
2. What are similarities and differences among middle school students' beliefs towards the importance of understanding mathematical concepts over the semester?

To answer the above research questions, this study draws upon students' beliefs about mathematics. First, students' beliefs about mathematics play a vital role in their engagement and motivation with mathematics (e.g., Kloosterman, 1996). To examine students' perceptions on mathematics, Kloosterman and Stage (1992) suggested five categories of beliefs and reworded the last belief from the Fennema-Sherman (1976) Usefulness of Mathematics scale. The six beliefs in

Kloosterman and Stage’s (1992) Indiana Mathematics Beliefs Scales are: “Belief 1: I can solve time-consuming mathematics problems; Belief 2: There are word problems that cannot be solved with simple, step-by-step procedures; Belief 3: Understanding concepts is important in mathematics; Belief 4: Word problems are important in mathematics; Belief 5: Effort can increase mathematical ability; and Belief 6: Mathematics is useful and relevant to my life” (p. 115). These items are designed to measure students’ perceptions of (a) their self-confidence in working on demanding problems; (b) mathematics problems that always follow step-by-step manner; (c) the importance of understanding mathematics conceptually; (d) the importance of word problems in mathematics; (e) the relationships between individuals’ efforts and capabilities in learning mathematics ; and (f) the relevance of mathematics in real lives.

## Methods

### Program Context and Participants

The program reported in this paper took place twice per week for a 17-week long semester in Fall 2022. Each section lasted 30 minutes, and each grade level had its own session in our partnered school. These sessions took place during the school day and did not interfere with the students’ required classes scheduled through the school. The program’s activities were developed with the school’s state standards to deepen students’ conceptual understanding of mathematical ideas, as well as to create an engaging learning environment by using hands-on activities centered around origami. We used design-research methodology which involved implementing activities and refining them after each implementation (e.g., Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003). Due to space limitation, an example of our geometric transformations and reflections activities using the ideas of one-straight cut to create a desired English can be seen by clicking the links of our [PowerPoint slides](#) and [worksheet](#). Regarding the recruitment process for students participating in our program, the first two authors were in collaboration with our partnered school’s administration and mathematics teachers. With one mathematics teacher per grade level, each teacher gave a roster of approximately 25 students that they believed would benefit from the program. An important note to make in the recruitment process is that this program was not required by any of the students selected. Therefore, parents or guardians could remove their students from the program at any time. In addition, we gave participating students a brief introduction about our program prior to giving an assent and a consent form. If either student, parent or guardian said no, then the students work would not be used as evidence in our research but remained in the program.

**Table 1: Pre-, Mid-, & Post-Semester Assessment Questions about Mathematical Beliefs  
(Adapted from Kloosterman and Stage, 1992)**

Belief Category	Belief Question
Belief 1: I can solve time-consuming mathematics problems.	Q1: I feel I can do math problems that take a long time to complete. (+) Q6: I find I can do hard math problems if I just hang in there. (+)
Belief 3: Understanding concepts is important in mathematics.	Q3: Time used to figure out why a solution to a problem works is time well spent. (+) Q8: It is important to understand why the answer is correct. (+)

## Data Collection and Analysis

To measure students' beliefs about mathematics, we drew upon Kloosterman and Stage's (1992) Indiana Mathematics Beliefs Scales to develop our questionnaire items. Based on our research questions described previously, this paper only focuses on Belief 1 and Belief 3 with minor modifications to Belief 3 (see Table 1). To examine changes in students' beliefs about mathematics throughout the semester, we distributed the same mathematical belief questionnaire to our sixth, seventh, and eighth graders at the beginning, middle, and end of the semester.

Because of absenteeism during the pre-assessment, the mid-assessment, and/or post-assessment, we only had 29 students across the grade levels who completed all three assessments in Fall 2022. As all the students were asked to use a numerical score to rate each questionnaire item, our data analysis focused on descriptive statistics. Students could choose from 1 to 4, with 1 signifying a negative attitude towards mathematics, and 4 signifying a positive attitude towards mathematics. For students who did not respond to a question or chose more than one rating, their responses were coded as NA.

## Results and Discussion

In this section, we organize our results into two different subsections based on our research questions concerning students' beliefs about solving demanding mathematics problems and their beliefs about the importance of conceptual understanding in mathematics, respectively.

**Table 2: Statistics of Students' Belief 1 (Q1 & Q6)**

	1	2	3	4	N/A	Total	Median	Mean
Pre-Assessment	$n = 5$	$n = 10$	$n = 32$	$n = 10$	$n = 1$	$n = 58$	3	2.78
Mid-Assessment	$n = 3$	$n = 6$	$n = 29$	$n = 18$	$n = 2$	$n = 58$	3	3
Post-Assessment	$n = 3$	$n = 3$	$n = 25$	$n = 27$	$n = 0$	$n = 58$	3	3.31

## Students' Beliefs about Solving Demanding Mathematics Problems

Table 2 shows students' ratings regarding Belief 1 with respect to solving demanding mathematics problems at the beginning, middle, and the end of the semester of our designed program. For the first belief, the mean score increases from the pre-assessment (mean = 2.78), the mid-assessment (mean = 3), to the post-assessment (mean = 3.31). As seen in Table 2, 8% of the students (5 cases out of 58) who strongly disagreed that they could do time-consuming mathematics problems at the beginning of our program. The percentage on the same question declines to 5% in the middle and at the end of our program (3 cases out of 29 for both assessments). For the students who strongly agreed that they can solve time-demanding mathematics problems, the percentage increases from 17% on the pre-assessment (10 cases out of 29) to 31% on the mid-assessment (18 cases out of 58). Also, the percentage on the same question inclined to 47% (27 cases out of 58) at the end of our program. In addition, our results show that our students' confidence in completing time-consuming mathematics problems or working on hard mathematics problems has increased from the beginning to the end of the semester. When students came to our program, they were encouraged to complete unfamiliar origami-based mathematical tasks independently. These experiences might help them develop their patience with complex and

time-demanding tasks. Students learn, understand, and master mathematical concepts at different times. It is quite evident in the data collected that students' confidence scores increased after they went through the semester-long program. Learning is not always a straight linear line, but there should be positive slope between the initial score and final score collected, implying that learning did occur at the student's pace. Thus, we take this as evidence pertaining to one of our program goals, which implements enjoyable origami-based activities to enhance students' critical thinking skills by working on mathematics problems with various entry points.

**Table 3: Statistics of Students' Belief 3**

	1	2	3	4	N/A	Total	Median	Mean
Pre-Assessment	$n = 5$	$n = 10$	$n = 32$	$n = 10$	$n = 1$	$n = 58$	3	2.77
Mid-Assessment	$n = 3$	$n = 6$	$n = 29$	$n = 18$	$n = 2$	$n = 58$	3	3
Post-Assessment	$n = 3$	$n = 3$	$n = 25$	$n = 27$	$n = 0$	$n = 58$	3	3.31

### Students' Beliefs about the Importance of Conceptual Understanding

Table 3 illustrates students' ratings of Belief 3 regarding the importance of conceptual understanding in mathematics at the beginning, middle, and the end of the semester of our designed program. Under this belief, the mean score increases from the pre-assessment (mean = 2.77), the mid-assessment (mean = 3), to the post-assessment (mean = 3.31). At the beginning of our program, 8% of the students (5 cases out of 58) believed that it is not worth spending time understanding why a solution to a mathematics problem works. This percentage decreases to 5% in the middle and at the end of the semester (3 cases out of 58 in both assessments). Also, the percentage of the students who believed that it is important to spend time understanding why behind solutions increases from 17% in the pre-assessment (10 cases out of 58) to 31% in the mid-assessment (18 cases out of 58) and 47% in the post-assessment (27 cases out of 58). As seen in Table 3, more students who strongly agreed the importance of understanding correct answers to mathematics problems in the mid-semester than the pre-semester and the post- semester. The fact that more of our students changed their beliefs in the mid-semester might be because of the design of our origami-based activities. More specifically, we implemented one-cut origami letters activities in the mid-program, which allowed students to explore multiple ways to make their desired figures. Because of this series of activities, our students might be interested in figuring out why a variety of approaches lead to the same correct final products. This can lead to students discovering "other" solutions to mathematical questions in their mathematics classes and improve their analyzation skills. In adding more to this program down the line, to develop students' beliefs about the importance of understanding why an answer is correct, we are currently adding more explanation questions to our worksheets. In doing so, we could gain more insight into why students' beliefs about understanding correct solutions change or not change in the future.

### Acknowledgement

This research was supported by a grant from the Indiana Department of Education. The opinions expressed in the article are those of the authors and do not necessarily reflect the position, policy, or endorsement of the Indiana Department of Education.



## References

- Christensen, R., & Knezek, G. (2020). Indicators of middle school students' mathematics enjoyment and confidence. *School Science and Mathematics*, 120(8), 491-503.
- Cobb, P., Confrey, J., diSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. *Educational Researcher*, 32(1), 5-8.
- Fennema, E., & Sherman, J. (1976). Fennema-Sherman mathematics attitudes scales: Instruments designed to Measure attitudes toward the learning of mathematics by females and males. Madison, WI: Wisconsin Center for Educational Research.
- Georgeson, J. (2011). Fold in origami and unfold math. *Mathematics Teaching in the Middle School*, 16(6), 354-61.
- Kloosterman, P. (1996). Students' beliefs about knowing and learning mathematics: Implications for motivation. In M. Carr (Ed), *Motivation in mathematics* (pp. 131-156). Cresskill, NJ: Hampton Press.
- Kloosterman, P., & Cougan, M. C. (1994). Student's beliefs about learning school mathematics. *Elementary School Journal*, 94, 375-388.
- Kloosterman, P., Raymond, A. M., & Emenaker, C. (1996). Students' beliefs about mathematics: A three-year study. *The Elementary School Journal*, 97(1), 39-56.
- Kuhfeld, M, Tarasawa, B., Johnson, A., Ruzek, I, & Lewis, K. (2020). Learning during COVID-19: Initial findings on students' reading and math achievement and growth. NWEA.
- Miles, V. (2011). Modular origami: Moving beyond cubes. *Mathematics Teaching in the Middle School*, 17(3), 180-184.
- Store, D., & Store, J. C. (2022). Mathematics learning loss due to interruptions in formal education. In Bateiha, S. and Cobbs, G. (Eds.), *Proceedings of the Forty-Ninth Annual Meeting of the Research Council on Mathematics Learning* (pp. 12-19). Grapevine, TX.

## I DON'T KNOW WHAT TO DO: MATHEMATICS ANXIETY AS PERCEIVED BY STUDENTS AND THEIR PARENTS

Dr. Andrea Martina Elizondo

Del Mar College  
aelizondo43@delmar.edu

Dr. Kathleen Lynch-Davis  
Texas A&M Corpus Christi  
kathleen.lynch-davis@tamucc.edu

Dr. Corrinne Valadez  
Texas A&M Corpus Christi  
corinne.valadez@tamucc.edu

Dr. Faye Bruun  
Texas A&M Corpus Christi  
faye.bruun@tamucc.edu

*A problem influencing mathematics performance and student's perceptions of their ability to learn mathematics is mathematics anxiety. This mixed method study examined the perceptions, correlations, and mathematical conversations of Hispanic fourth and fifth grade low-performing students and their parents. A psychological approach provided the framework necessary to understand the theories behind mathematics self-concept. Instrumentation included the Mathematics Anxiety Rating Scale for Elementary students (MARS-E), the Mathematics Anxiety Rating Scale for adults (MARS-SV), and interviews. Various mathematical situations and mathematical conversations were found to affect mathematics self-concept. The psychological implications disclosed specific characteristics to further understand how to create conditions to support mathematical learning.*

Keywords: Affect, Emotion, Beliefs, and Attitudes

The problem of concern relates to the number of under-represented elementary students continually performing low academically and specifically in mathematics. The National Association of Education Progress (NAEP) reported only 36% of fourth graders were considered proficient in mathematics according to their mathematics assessment (NAEP, 2022). Hispanic students scored 21 points below white students and the achievement gap continues to widen according to NAEP. This becomes problematic as the mastery of mathematics is critical for STEM (science, technology, engineering, and mathematics) professions which are predicted to increase (Lacey & Wright, 2009), mathematics skills are an essential life skill (Vijayan & Joshith, 2018), and mathematics is a gateway in providing access to higher education and better employment opportunities (Bryk & Treisman, 2010). More importantly, research has shown several problems connected to low mathematics achievement including mathematics anxiety (Ashcraft & Krause, 2007; Luttenberger, Wimmer, & Paechter, 2018; Ma & Xu, 2004; Papuosek, Ruggeri, & Macher, 2012; Ramirez, Gunderson, Levine, & Beilock, 2013; Wigfield & Meece, 1988), parent mathematics anxiety (Berkowitz, Schaeffer, Maloney, Peterson, Gregor, Levine, & Beilock, 2015; Elizondo, Bruun, & Pletcher, 2021a; Schaffer, Rozek, Berkowitz, Levine, & Beilock, 2018), and issues related to self-beliefs such as low self-concept (McCleod, 1992; Shavelson, Hubner, Stanton, 1976; Shavelson & Bolus, 1981). Studies have shown a parent's mathematic anxiety correlates to their child's mathematics anxiety and affects mathematics performance in positive and negative ways (Berkowitz, Schaeffer, Maloney, Peterson, Gregor, Levine, & Beilock, 2015; Schaffer, Rozek, Berkowitz, Levine, & Beilock, 2018; Elizondo, Valadez, Lynch-Davis, & Bruun, 2021b; Soni & Kumari, 2017).

The problem of low performance in mathematics persists nationally (Fleischmann, Hopstock, Pelscar, Shelley, 2010) and further research is needed to fully comprehend the psychology of mathematics related to performance and motivation. There is little empirical evidence that studies the examination of Hispanic fourth and fifth grade under-represented students considered low performing in mathematics, what contributes to their mathematics self-concept (MSC), and if Hispanic parents contribute to MSC. The following four research questions guided a larger research study:

3. What are the perceptions of mathematics anxiety of fourth and fifth grade students considered low performing in mathematics?
4. What are the perceptions of parents' mathematics anxiety?
5. Is there a relationship between a parents' mathematics anxiety and a student's mathematics anxiety?
6. How do parents and their students talk about mathematics?

This research report will predominantly focus on results related to mathematics self-concept and the fourth research question concerning mathematics talk.

### **Theoretical Framework and Literature Review**

Self-concept provides one perspective to help us understand an individual's behavior and it is a central concept relevant in psychology (Epstein, 1973). Researchers broadly defined self-concept as a person's perception of him or herself and these perceptions are created by one's experience and by their environment (Shavelson, Hubner, & Stanton, 1976; Shavelson & Bolus, 1981). Shavelson and his associates (1976) identified perceptions to be influenced by reinforcements, evaluations of significant others, and how one acknowledges their own behavior. They provided a definition of self-concept, created a theoretical framework for self-concept research, established a hierarchically model of self-concept, and further identified the construct of self-concept to be defined by seven critical features. The features more prominent for this study included: (1) self-concept was formed by participants categorizing information about themselves and then disclosing that information, (2) self-concept was hierarchical with perceptions of behavior at the core and moving to judgements about the self in subareas such as how participants felt about their self-concept related to mathematics, (3) general self-concept was more positive, however, in situation-specific experiences such as participants experienced mathematics anxiety within certain contexts, it became problematic, and (4) self-concept was descriptive and evaluative, individuals described themselves as "I enjoy mathematics" or evaluated themselves as "I do well in mathematics".

#### **Mathematics Self-Concept**

A student's mathematical self-concept (MSC) refers to their perceptions or belief in their ability to do well in mathematics or their confidence in learning mathematics (Elizondo et al., 2021b; Reyes, 1984;) and educators have realized the importance of fostering a student's self-concept as a critical component of mathematics education (McLeod, 1992; NTCM, 2000). A study by Wilkins (2004) conducted an international study consisting of 39 countries and found a correlation between MSC and mathematics performance to be positive and significant. Students with positive MSC had greater achievement and students with negative MSC had lower achievement. Other research discovered a reciprocal relationship may exist between MSC and mathematics anxiety (Ahmed, Minnaert, Kuyper, & Van Der Werf, 2012) and mathematics anxiety can foster negative self-concepts regarding mathematics abilities (Ashcraft & Kirk, 2001; Wu,

Barth, Marsh, Craven, & Yeung, 2014). In addition, research investigated high school Latino students and results revealed MSC can help minimize a decrease in mathematics interest and low levels in mathematics interest placed a student at risk due to their decline in MSC (Denner, Valdes, Dickson, & Laursen, 2019).

### **Mathematics Anxiety**

Mathematics anxiety can be defined as feelings of tension and nervousness (Richardson & Suinn, 1972), when individuals manipulate numbers and solve mathematical problems in academic situations (Carey, Hill, Devine, & Szucs, 2017; Hopko, Mahadevan, Bare, & Hunt, 2003; Paechter, Macher, Martskvishvili, Wimmer, Papousek, 2017; Richardson et. al, 1972), or have fear of making mistakes in mathematics (Chipman, Krantz, & Silver, 1992; Dutton, 1951). Studies have found students' mathematics anxiety to negatively impact one's ability to master mathematics content (Ashcraft & Krause, 2007; Hembree, 1990; Ma & Xu, 2004; Meece, Wigfield, & Eccles, 1990), mathematics anxiety affects working memory which is critical to the arithmetic and mathematics performance (Ashcraft & Krause, 2007; Luttenberger, et. al, 2018), mathematics anxiety affects mathematics motivation (Ashcraft & Krause, 2007; Hembree, 1990; Ma & Xu, 2004; Meece, Wigfield, & Eccles, 1990), and effort put into mathematics learning (Paechter et. al, 2017; Wigfield & Meece, 1998).

### **Impact of Parents Mathematics Anxiety**

Parent's mathematics anxiety is associated with lower student mathematics achievement (Maloney, Ramierez, Gunderson, Levine, & Beilock, 2015; Berowitz, Schaeffer, Maloney, Peterson, Gregor, Levine, & Beilock, 2015). First grade students of higher anxious parents learned less mathematics (Berkowitz et. al, 2015), and children learn less mathematics when their parents dislike or feel inadequate in mathematics (Maloney, et. al, 2015). Negative attitudes concerning mathematics are harmful and can result in low mathematics motivation, low mathematics self-efficacy (Hembree, 1990), and eliminating negative associations regarding parents' mathematics anxiety resulted in improved mathematics achievement (Schaffer, Rozek, Berkowitz, Levine, Beilock, 2018). Gunderson and associates (2012) discussed how parents' ideologies based on mathematics anxiety, mathematics self-concepts, and other mathematical beliefs affect their interactions with their child concerning mathematics and can influence their child's mathematics anxiety, attitudes, and achievement.

### **Parent Involvement and Mathematics Achievement**

Not all types of parent involvement are beneficial for students' academic outcomes and what is more important is the way parents help (Moroni, Dumont, Trautwein, Niggli, & Baeriswyl, 2015). Studies have shown when parents exert control in mathematical homework situations such as doing it their way, this diminishes a child's sense of competence, mathematics self-concept, task persistence (Silinskas & Kikas, 2019), and lowered mathematics performance (Dumont, Trautwein, Ludtke, Neumann, Niggli, & Schnyder, 2012; Hill & Tyson, 2009; Nunez, 2015; Silinskas & Kikas, 2019). When children perceived their parent's help as interference, intrusive, or felt negativity during parent controlling homework assistance, this had a negative effect (Pomerantz & Eaton, 2001) and decreased mathematics self-concept, performance, and task persistence (Silinskas & Kikas, 2019).

### **Funds of Knowledge and Mathematics Talk**

Everyday mathematical concepts learned at home provide the building blocks to develop mathematical concepts learned in school (Gonzalez, Andrade, Civil, & Moll, 2001) and families offer unique types of knowledge that can and do promote learning for their children (Moll, Amanti, Neff, & Gonzalez, 1992). Families contribute to a student's mathematical learning (Civil, 2007),

parents helped their children make real-world connections during mathematical homework time, held discussions related to mathematics homework, taught financial literacy (Williams, Tunks, Gonzalez-Carriedo, Faulkenberry, & Middlemiss, 2020), and families encouraged their children to learn mathematics in a positive way (Williams et al., 2020). Discussions regarding mathematics in the home (Williams et al., 2020) and children being exposed to “math talk” before starting school was positively related to early mathematical abilities a year later (Susperreguy & Davis-Keen, 2016). Research also found that the number of times parents engaged their children in number related activities at home predicted their numerical knowledge and development (Ramani, Rowe, Eason, & Leech, 2015).

### Methodology

The basis for this study resulted from mixed method research (Elizondo et al., 2021a) conducted in the summer of 2018 and showed mathematics anxiety (MA) to be an issue for both parents and students. The current study took place at three elementary schools (Title I) in south Texas. Eligibility included students who did not pass the State of Texas Assessment of Academic Readiness (STAAR) mathematics and scored low on mathematics benchmarks. Students (n=38) identified with the following demographics 36 Hispanic, 2 African American, 22 males, and 16 females. In addition, one parent from each child participated in the study (n=38) for a total of 76 participants. Parents associations included 36 Hispanic, One African American, 11 males, and 27 females. Purposeful sampling (Patton, 2002) was used to select information-rich cases based on the MARS-E (Suinn, Taylor, & Edwards, 1988) and MARS-SV (Suinn & Winston, 2002).

### Data Collection

The quantitative part of the study utilized the MARS- E (26 items) and measured the degree of anxiety experienced by students in a variety of mathematics-related situations. Students were asked to indicate the degree of anxiety or nervousness that they felt in each situation, using a five-point Likert scale from 1 (not at all nervous) to 5 (very, very nervous). Total scores were calculated by taking the sum of all 26-items, the lowest possible total score of 26 signified low anxiety and the highest score of 130 indicated extreme anxiety (Suinn et al., 1988). Parent’s MA was measured using the MARS-SV. The scale had 30-items designed to measure the anxiety experienced when completing mathematics-related activities. The MARS-SV is a five-point Likert scale that parents used to rate their anxiety using descriptors of (1) not at all, (2) a little, (3) a fair amount, (4) much, or (5) very much. The 30-items were calculated, and total scores ranged from 30 to 150 with higher scores indicating higher levels of MA.

The qualitative portion of the study was a descriptive analysis of students and parents’ perceptions of MA and the discussions between parent and students regarding mathematics. Five families were purposively selected and consented to be interviewed using semi-structured interview questions. Parents were selected according to five profiles established by the MARS. (Table 1).

**Table 1: Five Profiles Selected for Interviews**

Parent	Student (child of parent)	Family
High Anxiety	High Anxiety	La Fuente Family
Moderate Anxiety	Moderate Anxiety	Vidal Family
Low Anxiety	Low Anxiety	Hernandez Family
High Anxiety	Low Anxiety	Lucido-Schwartz Family
High Anxiety	Moderate Anxiety	Jones-Mendez Family

## Data Analysis

Descriptive statistics was used to portray the participants' mathematics anxiety in a collection of data, to depict patterns in the data, and to summarize the details of the data (Vogt, Gardner, Haeffele, & Vogt, 2014). The Pearson  $r$  correlation measured the association between students and parents' mathematics anxiety (Vogt et al., 2014). Analysis of the MARS results were broken down by specific constructs related to performance anxiety and test anxiety. Each question was analyzed according to their constructs finding the mean and standard deviation. This allowed a closer examination of the type of anxiety experienced by participants in certain mathematical situations. (Table 2)

Data was also gathered from the semi-structured interviews and the data was transcribed. MARS scale results informed the types of interview questions. For instance, participants showed more mathematics anxiety when learning new difficult mathematics material and interview questions were designed to provide a more in-depth rich description of that particular mathematics anxiety as perceived by parents and students. The researcher used descriptive coding or topic/theme coding (Wolcott, 1988) to allow any additional themes to naturally emerge from the transcriptions. Four themes emerged from data that included performance anxiety, test anxiety, positive mathematics self-concept and mathematics talk. In addition, three sub-themes materialized from data and included school, real-world, and assessment mathematics talk. The second cycle coding utilized was axial coding (Boeije, 2010; Glaser, 1978; Glaser & Strauss, 1967; Strauss, 1987; Strauss & Corbin, 1998). This linked the themes found in the first cycle with interview transcriptions (Charmaz, 2014). After the data had been arranged and the main themes were verified, themes were placed onto a concept map in a diagramming process to show connections and sequential progression recommended for axial coding (Strauss, 1987).

## Results

This mixed-method research study examined 38 students and parents' perceptions of mathematics anxiety (MA) as measured by the MARS-E and the MARS-SV. The Pearson product-moment correlation coefficient determined there was a moderate relationship between parents and students MA in all five profiles ( $r=.39$ ,  $n=38$ ,  $p>0.05$ ). The findings for this study were consistent with other mathematics anxiety (MA) research that showed a moderate relationship between parents and students with MA (Berkowitz, Schaeffer, Maloney, Peterson, Gregor, Levine, & Beilock, 2015; Gunderson, Ramirez, Levine, Beilock, 2012; Maloney, Ramirez, Gunderson, Levine, & Beilock, 2015; Schaffer, Rozek, Berkowitz, Levine, & Beilock, 2018). More importantly, data found both parents and students within the five profiles had positive mathematics self-concept (PMSC) despite exhibiting character MA. The quantitative findings from the MARS scale results produced five profiles (Table 1) and qualitative results further explained the various mathematical situations that contributed to a PMSC.

The current study found both parents and students within the five profiles had positive MSC despite experiencing MA. For example, both parent and child in the La Fuente family had high MA. However, Jacob (the child) talked about his mathematics self-concept and stated, "I don't think people will think that I am stupid in math, math is easy to learn, and it is my favorite subject." Marisol the mother explained her MSC, "My math anxiety has nothing to do with how people look at me, my everyday math skills are an eight (scale of 1-10). and math was my favorite subject in school." The Vidal family experienced moderate mathematics anxiety (MMA) and Alejandro (the child) expressed his own perceptions regarding his MSC, "I know I am doing good, and I feel fine learning math. I think others think I am good at math. Math makes me better and

whenever I don't understand the teacher, or my mom helps, and I get it." Marcus his father disclosed his beliefs about his mathematical skills and said, "I use math at work all the time and people depend on me to be good at it. It comes natural for me, and I am good at it (math)." Parents and students believed their mathematical learning and skills were good, and their perceptions of how others viewed their mathematical skills was positive. Therefore, they exhibited a PMSC.

Positive MSC was further demonstrated as students continued to discuss their perceptions of their mathematical skills and their belief that they could learn mathematics. Jacob with the La Fuente family (HMA) provided additional insight by stating "Math is easy to learn, but division is difficult. I could be better, I need to practice more. I am excited to learn math and what I don't understand my mom can help me. My parents think I can do math and I am not nervous doing math in front of them." Other students like Marcus (MMA) and Arris (MMA) talked about their mathematical skills, Marcus explained, "My math skills are like a 7 (scale 1-10). I feel fine about learning math, and I know that I am doing good in math. My parents think I am good at math, and I feel good doing math with my mom and dad." Arris (MMA) also stated, "My parents rate me at 8.5 (regarding his mathematics skills) and I can learn math, and I like it." This data further validates the theory behind MSC, and this study demonstrates that other's perceptions about mathematical skills may have contributed to students having a positive MSC.

Positive MSC was further validated as all profiles demonstrated parent support instead of parent control as previous research (Silinkskas & Kikas, 2019) found parent control contributed to a negative MSC. Parent support for this study consisted of parents assisting their child with homework only when asked, parents understood their child's learning needs, parents were actively involved, and learned new mathematical concepts for their children when needed. For example, Jacob (La Fuente Family) described mathematics homework time at home "I go to my mom first, she does not make me feel nervous and she helps me understand." Marisol his mother further explained by saying "We go over each problem, step by step, I try to keep him from getting nervous. I do old school use items in the house to explain the math. I learn how to better help him with his math homework from my husband." Other parent support was shown with the Jones-Mendez family, Azalea the mother had high MA and her son Arris had moderate MA. Arris explained that he asks his mom to help him when he needs it "I understand when she helps me and after all it is just math, I like math." Azalea described the mathematical discussions that occur and stated "I don't feel nervous helping him, I can listen to all his math lessons right now because of CO-VID and what he does not get, I help him with that. I don't project my lack of math confidence on my son, and can I learn what is needed to help him."

Students in this study felt their parents' support and involvement contributed to their success in mathematics, they were happy for the homework assistance, and the interactions between parents and students were very positive and could have contributed to their positive MSC. As illustrated when Jacob stated "It makes me feel a little happy when my mom helps me with my math homework. She gets close to me, and we look over the answers at the dinner table and she makes sure it is correct. She helps me a lot." Alejandro (Vidal Family) also commented about how he feels about his mathematics homework interactions with his mom and stated "Doing my math homework in front of my mom makes me feel kind of good because I am getting help. I am happy when my mom helps me, and I ask my mom when I don't get it (math).

The three types of mathematics talk occurring within the profiles included schoolwork, real-world, and assessment mathematics talk. Mathematics talk may have contributed to the PMSC experienced. Parents with high mathematics anxiety (MA) showed increased occurrences of mathematics talk when compared to other profiles with the exception of the parent with high MA

and the student with low MA (the Lucido-Schwartz family). Examples of mathematics talk that occurred in the La Fuente family, Marisol (HMA) explained “I try to help him as much as possible, we study for his math tests together, and he asks me for help on his homework. We talk about his math grades, and I encourage him to do better when he needs to do better. We talk about his STARR math results, and when he does not do well, I tell him he will do better next time.” Other profiles displayed similar mathematics talk such as Marcus (Vidal family MMA) “We talk about why he got a bad grade in math and what can we do to get a better grade. We talk about his STAAR math results and why he did not do well or if I need to talk to his teacher like does he need a tutor.” Parents may contribute to a PMSC through mathematical talk occurring in the home especially if discussions are supportive and positive in nature.

Williams et al., (2020) found parents supported mathematics learning and that they intentionally promoted mathematics in their household like financial literacy. This was also the case in the current study, parents intentionally taught their children financial literacy. For example, Marisol (HMA) explained “I teach him the value of money like when we buy clothes, we talk about spending \$40 on one shirt or spending \$40 on one shirt, one pair of pants, and have money left over for ice cream.” Jacob also discussed how his mom teaches him how to budget his money “She teaches me how to budget, and how to spend my money wisely.” Two parents in the study discussed how they explained their household budgets to their kids. Marcus (MMA) stated “We talk about money when we go shopping like if he wants to buy something, I will explain that funds are kind of tight and explain why. He needs to learn how to budget and stay within a budget.” Another parent Asia (HMA) said “We talk about saving money and how to budget money. I teach him the value of money and I have to pay the bills and what that means.

The following table provides a few examples of the mathematical situations that caused high MA and participants interview comments showed their positive MSC despite experiencing HMA.

**Table 2: Mathematical Situations and Positive MSC**

Profile	Mathematical Situation	Positive MSC
La Fuente Family (HMA)	Learning Math (HMA)	“My everyday math skills are an eight and I can learn math.” (student and parent)
	Homework beyond basic functions (HMA)	“I know adding, subtracting, multiplying and dividing, (student and parent) I do not know the new math. Eventually, I will learn it.” (parent)
	Getting the answer wrong when helping with homework (HMA)	“I listen and learn from my husband. Once I learn it, I can help him.” (parent)



Jones-Mendez Family  
(PHMA/SMMA)

Getting the right answer  
(HMA)

“I do it over and over until I get it. If I don’t get it, then I ask my mom. It is just a subject and I can learn math.”  
(student)

“I learn from his teacher (online classes) and I teach him what he does not understand. Like I said, if I don’t know it, I will google it and I end up getting the answers right.” (parent)

Vidal Family (MMA)

Learning New Difficult Math  
(HMA)

“I am fine learning new stuff. You just have to learn it.” (student)

“I get anxious with new stuff because people depend on me, but I can do it and I am good at math.” (parent)

---

### Conclusion

Data provided evidence to further understand the psychology of Hispanic fourth and fifth graders low performing in mathematics and their parents. What can be learned from this study is that parent support through mathematical discussions can create conditions to support positive mathematics self-concept despite experiencing mathematics anxiety. It will be critical to create conditions in the classroom that support a positive mathematics self-concept (PMSC), train pre-service and in-service teachers to create curriculum and conditions that support learning through PMSC, and build partnerships with parents to teach them about fostering the concept of parent support instead of parent control during mathematics homework. Mathematics learning should be enjoyed instead of students feeling anxious or unmotivated. Designing learning environments that support PMSC, reduce mathematics anxiety, and support the psychological development of students is needed. It will be necessary to continue research to further understand the implications of PMSC, what curriculum cultivates a PMSC, what teacher and parent interactions and discussions with students increase PMSC, what reduces mathematics anxiety, and how can we engage all learners through this process. Researchers could learn from students like Jacob who experienced high mathematics anxiety but had a PMSC, “Math is my favorite because it is kind of easy to learn. I am excited to be learning math and I don’t feel nervous when my mom helps me with my math. I could be better in math, I just need to practice.”

### References

Ahmed, W., Minnaert, A., Kuyper, H., & van der Werf, G. (2012). Reciprocal relationships between math self-concept and math anxiety. *Learning and individual differences*, 22(3), 385-389.

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

- Ashcraft, M. H., & Kirk, E. P. (2001). The relationships among working memory, math anxiety, and performance. *Journal of experimental psychology: General*, 130(2), 224. <https://doi.org/10.1037/0096-3445.130.2.224>
- Ashcraft, M. H., & Krause, J. A. (2007). Working memory, math performance, and math anxiety. *Psychonomic bulletin & review*, 14(2), 243-248. <https://doi.org/10.3758/bf03194059>
- Ashcraft, M. H., & Moore, A. M. (2009). Mathematics anxiety and the affective drop in performance. *Journal of Psychoeducational Assessment*, 27(3), 197-205. <https://doi.org/10.1177/0734282908330580>
- Baloglu, M., & Balgalmis, E. (2010). The Adaptation of the Mathematics Anxiety Rating Scale- Elementary Form into Turkish, Language Validity, and Preliminary Psychometric Investigation. *Educational Sciences: Theory and Practice*, 10(1), 101-110. <https://doi.org/10.1037/t71636-000>
- Berkowitz, T., Schaeffer, M. W., Maloney, E. A., Peterson, L., Gregor, C., Levine, S. C., & <https://doi.org/10.1126/science.aac7427>
- Beilock, S. L. (2015). Benefits of Math App.
- Boeije, H. (2010). Doing qualitative analysis. *Analysis in qualitative research*, 93-12. <https://doi.org/10.4135/9781473906006.n12>
- Bryk, A. S., & Treisman, U. (2010). Make math a gateway, not a gatekeeper. *Chronicle of Higher Education*, 56(32), B19-B20.
- Carey E, Hill F, Devine A, Szücs D. The Modified Abbreviated Math Anxiety Scale: a valid and reliable instrument for use with children. *Front Psychol*. 2017;8:11. <https://doi.org/10.3389/fpsyg.2017.00011>
- Civil, M. (2007). Building on community knowledge: An avenue to equity in mathematics education. *Improving access to mathematics: Diversity and equity in the classroom*, 105-117.
- Chipman, S. F., Krantz, D. H., & Silver, R. (1992). Mathematics anxiety and science careers among able college women. *Psychological science*, 3(5), 292-296. <https://doi.org/10.1111/j.1467-9280.1992.tb00675.x>
- Denner, J., Valdes, O., Dickson, D. J., & Laursen, B. (2019). Math interest and self-concept among latino/a students: Reciprocal influences across the transition to middleschool. *Journal of adolescence*, 75, 22-36. <https://doi.org/10.1016/j.adolescence.2019.06.015>
- Dumont, H., Trautwein, U., Lüdtke, O., Neumann, M., Niggli, A., & Schnyder, I. (2012). Does parental homework involvement mediate the relationship between family background and educational outcomes?. *Contemporary Educational Psychology*, 37(1), 55-69. <https://doi.org/10.1016/j.cedpsych.2011.09.004>
- Dutton, W. H. (1951). Attitudes of prospective teachers toward arithmetic. *The Elementary School Journal*, 52(2), 84-90. <https://doi.org/10.1086/459310>
- Elizondo, A., Bruun, C. F., Pletcher, B. C. (2021). Perceptions of students', parents', and instructors' experience in a summer math camp. *School Science and Mathematics* <http://doi.org/10.1111/ssm.12465>
- Epstein, S. (1973). The self-concept revisited: Or a theory of a theory. *American psychologist*, 28(5), 404. <https://doi.org/10.1037/h0034679>
- Fleischman, H. L., Hopstock, P. J., Pelczar, M. P., & Shelley, B. E. (2010). Highlights from PISA 2009: Performance of US 15-Year-Old Students in Reading, Mathematics, and Science Literacy in an International Context. NCES 2011-004. National Center for Education Statistics.
- Glaser, B. G. (1978). *Theoretical sensitivity*. Sociology Press.
- Glaser, B. G., & Strauss, A. L. (1967). The constant comparative method of qualitative analysis. *The discovery of grounded theory: Strategies for qualitative research*, 101, 158.
- González, N., Andrade, R., Civil, M., & Moll, L. (2001). Bridging funds of distributed knowledge: Creating zones of practices in mathematics. *Journal of Education for students placed at risk*, 6(1-2), 115- 132. [https://doi.org/10.1207/s15327671espr0601-2\\_7](https://doi.org/10.1207/s15327671espr0601-2_7)
- Guay, F., Marsh, H. W., & Boivin, M. (2003). Academic self-concept and academic achievement: Developmental perspectives on their causal ordering. *Journal of educational psychology*, 95(1), 124. <https://doi.org/10.1037/0022-0663.95.1.124>
- Gunderson, E. A., Ramirez, G., Levine, S. C., & Beilock, S. L. (2012). The role of parents and teachers in the development of gender-related math attitudes. *Sex roles*, 66(3-4), 153-166.
- Hembree, R. (1990). The nature, effects, and relief of mathematics anxiety. *Journal for research in mathematics education*, 33-46. <https://doi.org/10.1007/s11199-011-9996-2>
- Hill, N. E., & Tyson, D. F. (2009). Parental involvement in middle school: a meta-analytic assessment of the strategies that promote achievement. *Developmental psychology*, 45(3), 740. <https://doi.org/10.1037/a0015362>
- Jameson, M. M., & Fusco, B. R. (2014). Math anxiety, math self-concept, and math self-efficacy in adult learners compared to traditional undergraduate students. *Adult Education Quarterly*, 64(4), 306-322. <https://doi.org/10.1177/0741713614541461>

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

- Justicia-Galiano, M. J., Martín-Puga, M. E., Linares, R., & Pelegrina, S. (2017). Math anxiety and math performance in children: The mediating roles of working memory and math self-concept. *British Journal of Educational Psychology*, 87(4), 573-589. <https://doi.org/10.1111/bjep.12165>
- Lecky, P. (1945). Self-consistency; a theory of personality.
- Lee, J. (2009). Universals and specifics of math self-concept, math self-efficacy, and math anxiety across 41 PISA 2003 participating countries. *Learning and individual differences*, 19(3), 355-365. <https://doi.org/10.1016/j.lindif.2008.10.009>
- Levpušček, M., & Zupančič, M. (2009). Math achievement in early adolescence: The role of parental involvement, teachers' behavior, and students' motivational beliefs about math. *The Journal of Early Adolescence*, 29(4), 541-570. <https://doi.org/10.1177/0272431608324189>
- Levpušček, M. P., Zupančič, M., & Sočan, G. (2013). Predicting achievement in mathematics in adolescent students: The role of individual and social factors. *The Journal of Early Adolescence*, 33(4), 523-551. <https://doi.org/10.1177/0272431612450949>
- Luttenberger, S., Wimmer, S., & Paechter, M. (2018). Spotlight on math anxiety. *Psychology research and behavior management*, 11, 311. <https://doi.org/10.2147/prbm.s141421>
- Ma, X. (1999). A meta-analysis of the relationship between anxiety toward mathematics and achievement in mathematics. *Journal for research in mathematics education*, 520-540. <https://doi.org/10.2307/749772>
- Ma, X. (2006). Cognitive and affective changes as determinants for taking advanced mathematics courses in high school. *American Journal of Education*, 113(1), 123-149. <https://doi.org/10.1086/506496>
- Ma, X., & Xu, J. (2004). The causal ordering of mathematics anxiety and mathematics achievement: a longitudinal panel analysis. *Journal of Adolescence*, 27(2), 165-179. <https://doi.org/10.1016/j.adolescence.2003.11.003>
- Maloney, E. A., Ramirez, G., Gunderson, E. A., Levine, S. C., & Beilock, S. L. (2015). Intergenerational effects of parents' math anxiety on children's math achievement and anxiety. *Psychological Science*, 26(9), 1480-1488. <https://doi.org/10.1177/0956797615592630>
- McLeod
- Marsh, H. W. (1990). The structure of academic self-concept: The Marsh/Shavelson model. *Journal of Educational Psychology*, 82(4), 623. <https://doi.org/10.1037/0022-0663.82.4.623>
- Marsh, H. W., Byrne, B. M., & Shavelson, R. J. (1988). A multifaceted academic self-concept: Its hierarchical structure and its relation to academic achievement. *Journal of educational psychology*, 80(3), 366. <https://doi.org/10.1037/0022-0663.80.3.366>
- Marsh, H. W., & Craven, R. G. (2006). Reciprocal effects of self-concept and performance from a multidimensional perspective: Beyond seductive pleasure and unidimensional perspectives. *Perspectives on psychological science*, 1(2), 133-163. <https://doi.org/10.1111/j.1745-6916.2006.00010.x>
- Marsh, H. W., & O'Mara, A. (2008). Reciprocal effects between academic self-concept, self-esteem, achievement, and attainment over seven adolescent years: Unidimensional and multidimensional perspectives of self-concept. *Personality and Social Psychology Bulletin*, 34(4), 542-552. <https://doi.org/10.1177/0146167207312313>
- Marsh, H. W., & Shavelson, R. (1985). Self-concept: Its multifaceted, hierarchical structure. *Educational psychologist*, 20(3), 107-123. [https://doi.org/10.1207/s15326985ep2003\\_1](https://doi.org/10.1207/s15326985ep2003_1)
- Mathematics Performance, [https://nces.ed.gov/programs/coe/pdf/coe\\_cnc.pdf](https://nces.ed.gov/programs/coe/pdf/coe_cnc.pdf) (NAEP, 2018)
- D. B. (1992). Research on affect in mathematics education: A reconceptualization. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 575-596). New York: Macmillan.
- Mead, G. H. (1934). *Mind, self and society* (Vol. 111). University of Chicago Press.: Chicago.
- Meece, J. L., Wigfield, A., & Eccles, J. S. (1990). Predictors of math anxiety and its influence on young adolescents' course enrollment intentions and performance in mathematics. *Journal of educational psychology*, 82(1), 60. <https://doi.org/10.1037/0022-0663.82.1.60>
- Moll, L. C., Amanti, C., Neff, D., & Gonzalez, N. (1992). Funds of knowledge for teaching: Using a qualitative approach to connect homes and classrooms. *Theory into practice*, 31(2), 132-141. <https://doi.org/10.1080/00405849209543534>
- Moroni, S., Dumont, H., Trautwein, U., Niggli, A., & Baeriswyl, F. (2015). The need to distinguish between quantity and quality in research on parental involvement: The example of parental help with homework. *The Journal of Educational Research*, 108(5), 417-431. <https://doi.org/10.1080/00220671.2014.901283>
- National Assessment of Educational Progress [NAEP]. (2019). *Nations Report Card: Mathematics*. Washington, D.C.: National Center for Education Statistics, Institute of Education Sciences, U.S. Dept. of Education.
- National Council of Teachers of Mathematics [NCTM]. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.
- Núñez, J. C., Suárez, N., Rosário, P., Vallejo, G., Valle, A., & Epstein, J. L. (2015). Relationships between perceived parental involvement in homework, student homework behaviors, and academic achievement: differences among

- elementary, junior high, and high school students. *Metacognition and learning*, 10(3), 375-406. <https://doi.org/10.1007/s11409-015-9135-5>
- Paechter M, Macher D, Martskvishvili K, Wimmer S, Papousek I. Mathematics anxiety and statistics anxiety. Shared but also unshared components and antagonistic contributions to performance in statistics. *Front Psychol*. 2017; 8:1196. <https://doi.org/10.3389/fpsyg.2017.01196>
- Pajares, F., & Kranzler, J. (1995). Self-efficacy beliefs and general mental ability in mathematical problem-solving. *Contemporary educational psychology*, 20(4), 426- 443. <https://doi.org/10.1006/ceps.1995.1029>
- Pajares, F., & Miller, M. D. (1994). Role of self-efficacy and self-concept beliefs in mathematical problem solving: A path analysis. *Journal of educational psychology*, 86(2), 193. <https://doi.org/10.1037/0022-0663.86.2.193>
- Pajares, F., & Schunk, D. H. (2001). Self-beliefs and school success: Self-efficacy, self-concept, and school achievement. *Perception*, 11, 239-266.
- Pletzer, B., Wood, G., Scherndl, T., Kerschbaum, H. H., & Nuerk, H. C. (2016). Components of mathematics anxiety: Factor modeling of the MARS30-Brief. *Frontiers in psychology*, 7, 91. <https://doi.org/10.3389/fpsyg.2016.00091>
- Ramani, G. B., Rowe, M. L., Eason, S. H., & Leech, K. A. (2015). Math talk during informal learning activities in Head Start families. *Cognitive Development*, 35, 15-33. <https://doi.org/10.1016/j.cogdev.2014.11.002>
- Ramirez, G., Gunderson, E. A., Levine, S. C., & Beilock, S. L. (2012). Spatial anxiety relates to spatial abilities as a function of working memory in children. *The Quarterly Journal of Experimental Psychology*, 65(3), 474-487. <https://doi.org/10.1080/17470218.2011.616214>
- Reyes, L. H. (1984). Affective variables and mathematics education. *The Elementary School Journal*, 84(5), 558-581. <https://doi.org/10.1086/461384>
- Richardson, F. C., & Suinn, R. M. (1972). The mathematics anxiety rating scale: psychometric data. *Journal of counseling Psychology*, 19(6), 551. <https://doi.org/10.1037/h0033456>
- Schaeffer, M. W., Rozek, C. S., Berkowitz, T., Levine, S. C., & Beilock, S. L. (2018). Disassociating the relation between parents' math anxiety and children's math achievement: Long-term effects of a math app intervention. *Journal of Experimental Psychology: General*, 147(12), 1782. <https://doi.org/10.1037/xge0000490.supp>
- Seaton, M., Parker, P., Marsh, H. W., Craven, R. G., & Yeung, A. S. (2014). The reciprocal relations between self-concept, motivation and achievement: juxtaposing academic self-concept and achievement goal orientations for mathematics success. *Educational psychology*, 34(1), 49-72. <https://doi.org/10.1080/01443410.2013.825232>
- Seo, E., Shen, Y., & Benner, A. D. (2019). The paradox of positive self-concept and low achievement among Black and Latinx youth: A test of psychological explanations. *Contemporary Educational Psychology*, 59, 101796. <https://doi.org/10.1016/j.cedpsych.2019.101796>
- Sewasew, D., & Schroeders, U. (2019). The developmental interplay of academic self-concept and achievement within and across domains among primary school students. *Contemporary Educational Psychology*, 58, 204-212. <https://doi.org/10.1016/j.cedpsych.2019.03.009>
- Shavelson, R. J., & Bolus, R. (1982). Self-concept: The interplay of theory and methods. *Journal of educational Psychology*, 74(1), 3. <https://doi.org/10.1037/0022-0663.74.1.3>
- Shavelson, R. J., Hubner, J. J., & Stanton, G. C. (1976). Self-concept: Validation of construct interpretations. *Review of educational research*, 46(3), 407-441. <https://doi.org/10.3102/00346543046003407>
- Silinskas, G., & Kikas, E. (2019). Parental involvement in math homework: Links to children's performance and motivation. *Scandinavian Journal of Educational Research*, 63(1), 17- 37. <https://doi.org/10.1080/00313831.2017.1324901>
- Skaalvik, E. M., & Skaalvik, S. (2002). Internal and external frames of reference for academic self-concept. *Educational Psychologist*, 37(4), 233-244. [https://doi.org/10.1207/s15326985ep3704\\_3](https://doi.org/10.1207/s15326985ep3704_3)
- Soni, A., & Kumari, S. (2017). The role of parental math anxiety and math attitude in their children's math achievement. *International Journal of Science and Mathematics Education*, 15(2), 331-347. <https://doi.org/10.1007/s10763-015-9687-5>
- Strauss, A. L. (1987). *Qualitative analysis for social scientists*. Cambridge university press.
- Strauss, A., & Corbin, J. (1998). *Basics of qualitative research techniques*. Thousand Oaks, CA: Sage publications.
- Suinn, R. M., & Edwards, R. (1982). The measurement of mathematics anxiety: The mathematics anxiety rating scale for adolescents—MARS-A. *Journal of Clinical Psychology*, 38(3), 576-580. [https://doi.org/10.1002/1097-4679\(198207\)38:3<576::aid-jclp2270380317>3.0.co;2-v](https://doi.org/10.1002/1097-4679(198207)38:3<576::aid-jclp2270380317>3.0.co;2-v)
- Suinn, R. M., Taylor, S., & Edwards, R. W. (1988). Suinn mathematics anxiety rating scale for elementary school students (MARS-E): Psychometric and normative data. *Educational and Psychological Measurement*, 48(4), 979-986. <https://doi.org/10.1177/0013164488484013>

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

- Suinn, R. M., Taylor, S., & Edwards, R. W. (1989). The Suinn mathematics anxiety rating scale (MARS-E) for Hispanic elementary school students. *Hispanic Journal of Behavioral Sciences*, 11(1), 83-90. <https://doi.org/10.1177/07399863890111007>
- Suinn, R. M., & Winston, E. H. (2003). The mathematics anxiety rating scale, a brief version: psychometric data. *Psychological reports*, 92(1), 167-173. <https://doi.org/10.2466/pr0.2003.92.1.167>
- Sullivan, H. S. (1953). *The collected works* (Vol. 1). Norton.
- Susperreguy, M. I. (2013). "Math Talk" in Families of Preschool-Aged Children: Frequency and Relations to Children's Early Math Skills across Time.
- Susperreguy, M. I., & Davis-Kean, P. E. (2016). Maternal math talks in the home and math skills in preschool children. *Early Education and Development*, 27(6), 841-857. <https://doi.org/10.1080/10409289.2016.1148480>
- Vijayan, V., & Joshith, V. P. (2018). Reflection of Problem-Solving Skill in Life and Mathematics Education through Modeling and Applying. *i-Manager's Journal on Educational Psychology*, 12(2), 1. <https://doi.org/10.26634/jpsy.12.2.15141>
- Vogt, W. P., Gardner, D. C., Haeffele, L. M., & Vogt, E. R. (2014). *Selecting the right analyses for your data: Quantitative, qualitative, and mixed methods*. Guilford Publications.
- Wigfield, A., & Meece, J. L. (1988). Math anxiety in elementary and secondary school students. *Journal of Educational Psychology*, 80(2), 210. <https://doi.org/10.1037/0022-0663.80.2.210>
- Wilkins, J. L., & Ma, X. (2003). Modeling change in student attitude toward and beliefs about mathematics. *The Journal of Educational Research*, 97(1), 52-63.
- Williams, J. J., Tunks, J., Gonzalez-Carriedo, R., Faulkenberry, E., & Middlemiss, W. (2020). Supporting mathematics understanding through funds of knowledge. *Urban Education*, 55(3), 476-502. <https://doi.org/10.1177/0042085916654523>
- Wolcott, H. F. (1994). *Transforming qualitative data: Description, analysis, and interpretation*. Sage.

# INTERPRETACIONES DE ESTUDIANTES DE BACHILLERATO AL RESOLVER UNA MEA DONDE SUBYACE EL CONCEPTO DE FUNCIÓN LINEAL

Dinorah Méndez Huerta  
Universidad de Guadalajara  
Dinorah.mendez7682@alumnos.udg.mx

Verónica Vargas Alejo  
Universidad de Guadalajara  
veronica.vargas@academicos.udg.mx

Luis E. Montero Moguel  
The University of Texas at San Antonio  
luis.monteromoguel@utsa.edu

Palabras clave : Proporcionalidad, Actividad Provocadora de Modelos, Interpretación

## Introducción

Aprender matemáticas en ambientes donde se promueva la resolución de situaciones problemas cercanos a la vida real es uno de los retos actuales en la educación matemática (Brady et al., 2015). En este póster se describen los resultados de la implementación de una actividad cercana a la vida real asociada con uno de los problemas que actualmente ha cobrado mayor relevancia alrededor del mundo, la escasez del agua (Organización de las Naciones Unidas, 2017). La pregunta de investigación es: ¿Qué interpretaciones siguieron estudiantes de bachillerato para responder la MEA ¡Día Cero!?

## Marco Teórico

El marco teórico es la Perspectiva de Modelos y Modelación [MMP, por sus siglas en inglés]. Aprender matemáticas es un proceso en el cual los individuos desarrollan sistemas conceptuales o modelos (Lesh, 2010).

Los modelos son sistemas conceptuales (formados por elementos, relaciones, operaciones y reglas que gobiernan las interacciones) que se expresan utilizando notación externa y que se utilizan para construir, describir o explicar los comportamientos de otro(s) sistema(s), para poder manipular o predecir otro sistema de forma inteligente. (Lesh & Doerr, 2003, p. 10)

La MMP propone situaciones problemas cercanos a la vida real, denominadas MEAs (Lesh, 2010); resolverlas implica varios ciclos de modelación en los cuales los estudiantes revelan su forma de pensar al mismo tiempo que la modifican, extienden y refinan (Doerr & Pratt, 2008).

## Metodología

La investigación fue de carácter cualitativo. Se diseñó la MEA ¡Día Cero! con base en Lesh et al. (2000). En ella se solicitaba la escritura de una carta para explicar ¿Cómo disminuirá el agua de la presa de Valsequillo tras el abastecimiento al Distrito de Riego 030 con el paso de los años? El concepto de función lineal está implícitamente involucrado. Participaron 10 alumnos de bachillerato de la ciudad de Puebla, a quienes un profesor de otro estado de la república mexicana atendió de manera virtual en dos sesiones de Zoom de 60 min cada una. Los instrumentos de recolección de datos fueron grabaciones de video y cartas.

## Resultados y conclusiones

Los alumnos exhibieron tres diferentes interpretaciones al resolver la MEA ¡Día cero! Primero, atribuyeron el desabasto de agua de la presa a la población y dieron propuestas para cuidar y mantener el agua. Después, estimaron el tiempo que la presa podría abastecer al campo; utilizaron proporcionalidad. Finalmente, construyeron un modelo mediante la sistematización de sus procedimientos. La experiencia permitió que los estudiantes aprendieran sobre modelación y que el profesor conociera cómo evolucionaron las interpretaciones al resolver la MEA.

## Referencias

- Brady, C., Eames, C. L., & Lesh, R. (2015). Connecting real-world and in-school problem-solving experiences. *Quadrante*, 24(2), 5-38.
- Carmona, G., Lima, L., Vargas-Alejo, V., & Montero-Moguel, L. (2023, March 9-14). ¿Cuál es el papel de la modelación matemática dentro de una visión interdisciplinaria en Educación en Ciencias, Ingeniería, Tecnología y Matemáticas? *Primer Congreso de la Sociedad Mexicana de Investigación y Divulgación de la Educación Matemática (SOMIDEM)*. Ciudad de México, México.
- Doerr, H. M., & Pratt, D. (2008). The learning of mathematics and mathematical modeling. *Research on technology and the teaching and learning of mathematics: Research syntheses*, 1, 259-285.
- Lesh, R. (2010). Tools, Researchable Issues & Conjectures for Investigating What it Means to Understand Statistics (or Other Topics) Meaningfully. *Journal of Mathematical Modelling and Application*, 1(2), 16-48.
- Lesh, R. & Doerr, H. M., (2003). Foundations of a Models and Modeling Perspective on Mathematics Teaching, Learning, and Problem Solving. In R. Lesh & H. M. Doerr (Eds.), *Beyond constructivism: Models and modeling perspectives on mathematics problem solving, learning, and teaching* (pp. 3-34). Lawrence.
- Lesh, R., Hoover, M., Hole, B., Kelly, A., & Post, T. (2000). Principles for developing thought revealing activities for students and teachers. En M. Hoover y R. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 591–645). Routledge.
- Lesh, R., Kathleen, C., Doerr, H. M., Post, T. & Zawojewski, J. S. (2003). Model Development Sequences. In R. Lesh & H. M. Doerr (Eds.), *Beyond constructivism: Models and modeling perspectives on mathematics problem solving, learning, and teaching* (pp. 35-58). Lawrence.
- United Nations Educational Scientific and Cultural Organization. (2017). *Education for sustainable development goals: Learning objectives*. UNESCO.

## INTUITION AND FORMALIZATION IN THE UNDERSTANDING OF THE MATHEMATICAL INFINITE: THE CASE OF OMAR

### INTUICIÓN Y SIMBOLIZACIÓN EN LA COMPRENSIÓN DEL INFINITO MATEMÁTICO: EL CASO DE OMAR

Liliana Aurora Tabares  
Sánchez  
Cinvestav IPN  
ltabaress@cinvestav.mx

Luis Enrique Moreno Armella  
Cinvestav IPN  
lmorenoarmella@gmail.com

Isaías Miranda Viramontes  
CICATA, Legaria  
imirandav@ipn.mx

*The development of the mathematical concept of the infinite, through the reflections that arise from personal notions and perceptions and the analysis of some ideas of Galileo and Cantor, invites us to investigate the relationship between intuition and formalization for the understanding of the said concept. This paper aims to observe and describe the shift from intuition to formalization that Omar, a first-semester undergraduate student in applied mathematics, goes through when accounting for the number of objects between two equal mirrors, one in front of the other. The results indicate that the interaction between intuition and formalism is necessary for Omar to conceive of mathematical infinity.*

Keywords: advanced mathematical thinking, cognition, learning theory, mathematical knowledge for teaching.

Mathematical infinity is a concept that has had a diversity of meanings throughout the history of mathematics (Kidron and Tall, 2015). Across the times, it has been observed that the efforts of thinkers to make sense of this concept were characterized by a permanent intertwining between symbolism and intuition. The following mental exercise represents how this intertwining can occur. If you could have a bag full of 5,000,000 candies and you decide to give one of them away, you would surely feel that the bag is still full; it would be said that still the same amount of sweets remains. If we instead have a bag with 50 candies, and we decide to give one away, perhaps we could still affirm that the bag is full, but our intuition suggests that the number of candies is a bit less than 50. In number symbols, we rather doubt that 49 is 50. Finally, if we have a bag with 5 candies, and we give one away, we could no longer say that we continue to have the same number of candies. Our intuition cannot allow us to say that 4 is almost 5; neither could a symbolic justification allow it. From this mental exercise we can affirm that as a set has increasingly more elements, our intuition exhibits a certain insensitivity to the loss of a few elements, that is, a small number with respect to the original size of the set. However, if the original set is small, losing one is a sensitive loss. Reading the thought experiment, we felt a certain familiarity with its plot line. We have internalized our experiences with numbers and have transformed them into a symbolic, mental model that we can manipulate as if it were something material.

Donald (2001), in his book, *A mind so rare: The evolution of human consciousness*, explains how human cognition, shared with other species, in addition to its analogical or holistic nature, has evolutionarily acquired a capacity for symbolic representation. That duplicates, in a certain sense, the world of our experiences from the material world. Of course, this symbolic ability gradually acquires a (relative) autonomy that is reflected both in the world of art and in the world of science.



And, of course, in mathematics. The mental exercise that we previously described is related to the encounter that we have, at an early age, with the sequence of natural numbers. When counting 1,2, 3..., at some point we feel that we are not going to finish. No matter how hard we try. Here we have an encounter between the meaning of the endless and the infinite —as an action without end. Our symbolic abilities, apart from allowing us to recode the world of our experiences, enable an access to a level of reality that no other species experiences: the world of the abstract.

Our cognitive ability is hybrid: we have a direct contact to the material world around us and we can symbolize (at least partially) this world transforming symbolically those raw materials. As if the symbolic structures were an external version of a world of experiences that lives within us. There is a permanent swinging between these two worlds every moment we begin exploring a piece of mathematical knowledge previously unknown to us. In general, the material world and the symbolic world are intertwined to generate a swing between our intuitions and the corresponding mathematical formalism. The infinite is right there: in the coming and going of intuitions and symbols as we induce, generalize and deduce. Like an endless dance, so is the cognitive act: a constant intertwining between the material world and the symbolic world. In mathematics education, the concept of mathematical infinity has been taught precisely from a symbolic perspective: infinity as the distant, the unattainable, the imperceptible. And, at the same time, as the uncountable, the inexpressible.

Let us now continue with our introductory discussion of infinity. To do this, let's go with our students to read a short passage from Galileo's book, *Dialogue on two new sciences* (Galilei, 1638, pp. 31-33). There Galileo's characters (Salviati, Sagredo, and Simplicius) discuss how each natural number can be associated with its square. Then a list of the squares can be made: 1, 4, 9, 16...and so on. Galileo concludes that such a list is endless. But then you must accept that the list is never-ending, since you can order the members of that list: first, second, third, fourth,... and this requires the use of all natural numbers.

Galileo is facing a paradox: All natural numbers are needed to count the squares. However, it is clear to him that there are considerably more naturals than there are squares. The squares constitute a fragment, so to speak, of the (set of) natural numbers. Then he concludes that, under these circumstances, it does not follow that the whole is greater than a proper part, as common-sense dictates. Thus, he manages to avoid the paradoxical situation. Our feelings are that Galileo found a dead-end as he followed this argument.

We consider that Galileo was quite close to opening a new perspective for mathematics at that time. He failed, although he advanced as far as the episteme of his time allowed. One lives in a cultural atmosphere, surrounded by cultural ways of understanding and ways of thinking that open up conceptual perspectives on some ideas and make it practically impossible to consider others. Though at some future moment, a visionary (individual or collective) opens what we could name a zone of potential development within this culture and slowly emerges a new conceptual reality.

Galileo looked at the correspondence of a natural  $m$  to its square  $m^2$  not as a way of comparing number collections, but as a way of counting how many squares he could produce: one, two, three...and so on. It is incorrect he was defining a bijection. Regarding the tension between our intuitive, sensory-motor knowledge, and the formalization of mathematical knowledge, the American mathematician Pierpont (1899, p. 406) wrote:

The analysis of to-day is indeed a transparent science. Built up on the simple notion of number, its truths are the most solidly established in the whole range of human knowledge. It is, however, not to be overlooked that the price paid for this clearness is appalling, it is total separation from the world of our senses.

After the passage from Galileo's book discussed earlier with our students, we asked them the question: What happens if we delete all the odd numbers from the list of natural numbers? As Galileo explain, we can count the even numbers and, for this, we will have to use all the natural numbers. So far, students were following Galileo's reasoning for counting squares. But now a mathematical notion that Galileo lacked was present: the notion of set, which opens a conceptual space so that the list of even numbers could be conceived holistically; that is, to conceive it, as a complete infinity, to distinguish it from infinite processes when trying to count the list of even numbers. There was an obstacle that Galileo was not able to surmount: while we perceive one set as a proper subset of another, it becomes almost unacceptable to see the subset as having the same number of elements than the larger set.

We cannot say that the students' thinking and the paradoxical situation experienced by Galileo coincide at the cognitive level, nevertheless we can affirm that there is a common root between them explained by embodied theories of human intelligence.

M. Donald (2001, p. 155) explains, that: Basic animal awareness intuits the mysteries of the world directly, allowing the universe to carve its own image in the mind. This is a largely receptive mode of knowing, and we share it with our animal cousins. In contrast, the symbolizing side of our mind is more aggressive in its approach. It creates a sharply defined, abstract universe that is largely of its own invention.

Our symbolic universe permanently establishes its own rules trying to give meaning to what it perceives. These two ways of thinking and conceiving, the holistic one and the symbolic, continue to revolve around each other. It is in this sense that we affirm that our cognitive capacity is hybrid. The symbolic universe develops in a cultural atmosphere that gradually recodes, culturally, the natural world. Including, of course, our intuitions.

We agree in this sense with Pierpont's words according to which the cognitive price we will pay is too high if we try to erect an insurmountable wall between our intuition and the formalized versions of mathematics, particularly those referring to infinity.

#### **From a Cognitive Consideration to the Didactical Laboratory**

According to Georg Cantor, an actual or completed infinite set is characterized by the existence of a one-to-one correspondence between the set and a proper subset of it. The concept of one-to-one correspondence was within Cantor's reach but was not available to Galileo. This is an important conceptual difference. Galileo was upset when he was not able to resolve the paradox of infinite sets. His mathematical (and cultural) ways of understanding made it impossible for him to conceive of infinity as a property of sets. The existence of one-to-one correspondence came to answer, with Cantor, the unanswered questions Galileo himself had formulated. Nevertheless, we can find an analogy involving finite collections. For instance, if everyone is sitting in a room except one person, we know that the number of persons exceeds by one the number of chairs. However, this commonsense observation or a similar one, was not enough for Galileo to lead him to conceive of infinite collections as Cantor did. A key conceptual tool was missing... This discussion generated a hot debate among the students interviewed in our small laboratory. Their previous knowledge and reflections had not been aimed at becoming aware of how an idea as

logically as simple as the one-to-one correspondence between two sets, contained the key to access to the actual infinity that exists in that virtual reality called mathematics. Then, it was due time to reflect on how the original intuitions they had about the number of elements in a set, (obtained by usual counting), should be "forgotten" when dealing with infinite sets.

In a well-known letter to Dedekind, regarding this phenomenon of infinite sets Cantor wrote: "I see it, but I don't believe it," (Zimmerman, 2013, p. 452). Students, as had happened to Cantor, saw it, but they felt forced to accept it—even if they did not believe it!

Intuitions are the unarticulated consequences of embodiment, which we acquire through our experiences in the human world, even if we cannot make them explicit. We cannot change them at will. This is because our intuitions, when trying to explain a more or less known fact, do not obey a strict logical-deductive control and, therefore, their level of coherence is, at most, local. If we say that the whole is greater than each of its parts, this seems acceptable to us. But this acceptance ends when we remember (or learn) that we can establish a one-to-one correspondence between the set of natural numbers and the odd numbers. So, we must set limits to that popular sentence which seemed so natural as to be taken as universal.

Through school education, the moment arrives when the logical part of our rationality prevails temporarily, so to speak, over our intuitions. However, it is necessary to remember that this logical part of our thinking always hides deep down, an intuition that is fragmented or separated from our thinking. Our classroom experiences made tangible the belief that, for students, to learn the basics of infinity was such as walking through an abrupt territory. It was risky to try guessing the answers. They felt they were missing the necessary tools to address the problems emerging from the new cognitive and mathematical demands. We asked ourselves: how can we face the cognitive problems coming from the study of infinity, considering that the main problem is the development of meaning? The students were facing a formalized terrain.

Feeling the underlying tension we saw in their faces; a transition was a must. The transition from instrumental fluency to cognitive fluency is an educational problem of global concern. Indeed, what is simpler from the logical, formal point of view, does not necessarily coincide with what is simpler from the intuitive point of view. Both aspects are constitutive parts of human cognition.

### **Experimentation and Search**

The activity *El infinito en el espejo*, is part of a sequence of activities whose purpose is to recognize, in freshman students of mathematics in a Mexican university, the swing between the intuitive conception of the concept of infinity and its formalization that generates the need to specify a mathematical, formal answer. The device designed for the activity consisted of two mirrors of the same size, placed one in front of the other. Among them were 9 plasticine balls with the same volume and the same color. The students had to look at their reflection in the mirrors and determine how many balls they could count. After allowing the students to observe the device, the teacher (the first author of this article) would ask the group the number of balls they saw. The answers should be spoken aloud and had the objective that all the students could reflect on what their classmates answered. One of them, Omar, reflecting on how the reflections of the balls would be in the mirror if they were removed one by one, makes a direct comparison between the amounts of balls that are reflected. He comments:

infinities come in different sizes; and that depends, if it is one [he refers to a ball] then it will be smaller than an infinity of nine [ball], but at the same time it cannot be bigger, because being natural numbers they still belong to the same infinity.

In this statement, Omar clearly specifies that his intuition makes him think that there would be two infinities of different sizes: the one that is formed with the reflections of a single ball and that of the reflection of nine balls. For Omar the first infinity is formed by multiples of one; the second, by multiples of nine. It seems intuitive to him that these two infinities identified by him will have different sizes (“it will be smaller than an infinity of nine”). Recognizing that in both cases the number of balls is related to the natural numbers, his intuition is held back by his formal mathematical knowledge. This leads him to conclude that they would be infinities of the same size (“but at the same time it cannot be larger because, being natural numbers, they still belong to the same infinity”). Omar continues with his reflection aloud, establishing a difference between intuitive knowledge and formal knowledge:

In other words, he explains that intuitively it seems that it is bigger, but if we put mathematics into it, no. It is still the same infinity of natural numbers and there cannot be different infinities of natural numbers because it is the same infinity.

When Omar says, “but if we put mathematics in it”, he reveals how he seeks justification from the logic that his formal mathematical training provides him. This search allows him to make sense of his answer.

Omar, motivated by the researcher's questions, talks about the existence of sizes of infinities different from the natural numbers:

He had said that there are larger infinities such as the infinity of the real numbers. And this is because there are irrational numbers like  $\pi$  and fractions like  $1/4$  that each “skyrocket to infinity”. That is why it is a greater infinity. Because they are like many infinities combined into one; then it is bigger than a single infinity of a single thing: the natural numbers.

Omar intuitively expresses what real numbers are for him. Briefly explain that real numbers include irrational numbers, such as  $\pi$ , and rational numbers, such as  $1/4$ , and clarifies that infinity is embedded in each number. By expressing “there are irrational numbers”, Omar implies that he knows several of them, although he only gives  $\pi$  as an example, he can then make sure that he knows more, like  $\sqrt{2}$  or  $e$  and adds them to the set of rational numbers (he says “fractions”). With this, Omar makes tangible that he knows how the set of real numbers is formed.

Now, by giving  $\pi$  and saying that “each one fires to infinity”, he refers to the fact that infinity is embedded in each of the irrational numbers. This is so because, for Omar, the characteristic of the decimal, nonperiodic expansion of every irrational number shows the existence of an infinity for each number. But Omar does not end his reflection on infinity here. Thus, as each irrational number has its own infinity, the set of irrational numbers has an infinity greater than the infinity of the natural numbers. Omar's reasoning is based on his intuition that the non-ending fractional part of an irrational numbers means that they are not identifiable with the same clarity as natural numbers. In other words, the irrational numbers are not as accessible as the natural ones, since each of these does not have an associated infinity, since they do not have an unlimited decimal expansion. For example, for Omar, the number 3 does not have an associated infinity, since it does not have non-periodic non-zero digits; however,  $\pi$  does have an associated infinity, since the decimal part has non-periodic digits. In this way, Omar's intuition causes the natural numbers to be associated with a single infinity, quantitatively smaller than the infinity of the irrational numbers.

The colloquial way in which Omar expresses himself to describe the cardinality of the set of real numbers (“they are like many infinities combined into one”) leads us to think again about how he relies on his intuition to make himself understood verbally. Omar reinforces his intuition considering the amount of numbers that make up the real ones. His sensorimotor intuition, when expressing words like “many” or “combined”, seeks to give an explanation close to the experiences

of daily life, close to that idea of larger infinities. The swing from intuition to the formal can be observed in the arguments that Omar uses to explain the existence of different sizes of infinities, in the go-between intuitive ideas, which are developed through experiences, and the certainty he obtain from the formal definitions, like the one that the irrational ones are immersed in the infinite.

Now, this argument, based on intuition, is also based on formal knowledge, acquired by Omar during his school years. The definitions of rational and irrational number are part of Omar's mathematical language. Given the observation of the reflections of the balls, Omar must, then, make use of his intuition and, at the same time, he finds support in his academic knowledge.

Thus, intuitively, if you put more balls it is a bigger infinity, but if you put mathematics on them, well no. But after what we're discussing right now, well no, I'm still in conflict thinking that the theory can work, if it's theoretical it's possible.

In this intervention, the swing between the intuitive and the symbolic is observed more clearly. Indeed, Omar bases his answer on intuition (“if you put more balls it is a bigger infinity”), and also on the formal (“but if you put mathematics on them, well no”). This going and coming between the intuitive and the theoretical is difficult for him to assimilate (“I am still in conflict thinking that the theory can work”). Even so, he seeks to separate what he visually identifies in the reflections of the balls from what he accepts in theory (abstract concepts). By making this distinction, Omar implies that the mathematical existence of objects is dependent on their material existence.

To conclude his reflection, Omar comments the following:

Look, it is not feasible that there is a material infinity, not even the space [he is referring to the Universe] that we know is expanding, it may grow at a certain rate, but we do not know if it is infinite or not, because if it continues growing means that it has not stopped and, if it stops, we will never know because we would not notice. So, that is why it is not feasible to measure an infinity because we do not have a material infinity.

In the expression "we do not have a material infinity", Omar condenses the inevitable amalgamation of intuition and the formal, produced by trying to explain what is in front of his eyes: the reflection of nine balls produced by the arrangement of mirrors. Your intuition tells you that infinity is characterized by the unattainable and the immeasurable. The example that he himself gives in his argument helps to give credibility to this intuition.

Even if the Universe does not expand, there is no certainty of its dimensions; intuitively speaking it would be impossible to know its dimensions; that is, it would be impossible to know the infinite. It is this non-finiteness of space that makes it impossible for Omar to conceive of a “material infinity”. In other words, for Omar infinity is insensitive, and that is why mathematics must be used (as he did in his second reflection) to explain it.

### **Conclusion and Perspectives**

In this article we report how Omar, a first-semester mathematics student, intertwines his intuition and formal knowledge to account for the infinite reflection of a configuration of plasticine balls, placed in a straight line between two mirrors. In the analysis, Omar bases his answers both on the use of intuition and on the application of mathematical concepts. The constant tension between the intuitions and the formal approach is what allows Omar not to assign a cardinality to the set of plasticine balls, but to conceive this set as an infinite set. Our work takes advantage of the field of embodied cognition, as developed by Donald (2001) and Lakoff and Núñez (2000).

The notion of infinity lives, so to speak, between the sensory-motor experience and the symbolic formulations typical of mathematics. But the premature replacement of Galilean insights (which are actually inarticulate consequences of embodiment) with Cantorian formalization creates

cognitive obstructions that are difficult for students to overcome. A hypothesis derived from this research consists in proposing that the conception of mathematical infinity may require not the substitution of intuition for symbolism, but the intertwining between them. Each time Omar needed to justify the infinity of plasticine balls, he needed to blend his mathematical background with his intuitions.

There emerges, as a didactic necessity, the notion of conceptual metaphor (Lakoff-Núñez) linking intuitions with their eventual formalization. All cases of infinity in mathematics, according to Lakoff-Núñez (2000, p. 158) (for example, limits, infinite series, infinite sets, etc.), correspond to processes that do not end but that we conceptualize as if they did really have an end. Human beings can imagine the result of a process that does not end. This is the case of the existence of irrational numbers defined by an infinite collection of digits that we cannot even fully know, except in that other dimension of our cognitive experiences: in the virtual reality of mathematics and other symbolic worlds.

### References

- Duval, R. (1983). L'obstacle du dedoublement des objets mathématique. *Educational Studies in Mathematics* 14 (358-414)
- Galilei, G. (1638/1954). *Discourses and mathematical demonstrations: concerning two new sciences*. Dover edition, New York.
- Kidron, I., Tall, D. The roles of visualization and symbolism in the potential and actual infinity of the limit process. *Educational Studies in Mathematics* 88, 183–199 (2015). <https://doi.org/10.1007/s10649-014-9567-x>
- Lakoff, G. y Núñez, R. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. Basic Books, New York.
- Maor, E. (1987). *To infinity and beyond: A cultural history of the infinite*. Birkhäuser, Boston.
- Pierpont, J. (1899). On the arithmetization of mathematics. *Bulletin of the American Mathematical Society*, 5(8), 394-406.
- Prieto Sánchez, J.A., Fernández -Escalona, C. (2017). Estudio del infinito actual siguiendo el modelo de inclusión de Bolzano con la ayuda de una experiencia física de espejos paralelos. En Codina A., Puig, L., Arnau, D., Sánchez, M.T., Montoro, A. B., Claros, J., Arnal, M., Baeza, M. A. (Eds.), *Investigación en pensamiento numérico y algebraico* (pp. 11-18). Madrid: Universidad Rey Juan Carlos, SEIEM.
- Tall, D. (2001). Conceptual and Formal Infinities. *Educational Studies in Mathematics* 48: 199–238.
- Zimmerman, S. (2013). I believe it, but I don't see it. *International Journal of Mathematic Education in Science and Technology*, 44(3), 452-456.

## INVESTIGATING SPACE AND PLACE FACTORS IN A MATHEMATICS CLASSROOM DISCOURSE

Hillary Ongoyo Omoze  
Purdue University  
homoze@purdue.edu

Abigail Erskine  
Purdue University  
aerskin@purdue.edu

Keywords: Instructional Activities and Practices, Classroom Discourse, Teacher Educators

The research presented in the PME-NA 44th proceedings highlight the significance of identity in classroom interactions and its role in contextualizing space and place within a mathematics classroom. Scholars, such as Darragh (2016) and Watson (2006), define identity as a dynamic and contextually constructed construct. Glanfield et al. (2022) emphasize that our past experiences, familiar places, and cultural backgrounds shape our social interactions and mathematics learning. It is essential for mathematics teaching and learning to be responsive to the contextual variables present in learning institutions and students' backgrounds (Anderson, 2014). Factors such as instructional approaches, professional development for teachers, and the unique sociocultural and ideological aspects of the school community influence and shape in-class experiences. Additionally, Hoglebe and Tate (2012) argue that neighborhoods, communities, and religions play a role in shaping and constraining student performance and instructional practices in mathematics education. For instance, students from overpopulated urban settings face challenges related to limited resources, pollution, and insecurity, which can impact their learning experiences beyond the subject content. In contrast, students from suburban areas may have more relaxed and secure out-of-class experiences. Our work aims to explore how Mathematics Teacher Educators (MTEs) conceptualize space and place as contextual variables in a mathematics classroom, leveraging the experiences of both students and teachers to gain insights into this topic.

Social constructivists emphasize the inseparable connection between individuals' reasoning, sense-making, social interactions, cultural backgrounds, ideologies, and social activities (Yackel & Cobb, 1996; Yackel et al., 2000; Zembat & Yasa, 2015). Factors such as culture, socioeconomic status, sociopolitical contexts, and ideologies play a crucial role in shaping students' out-of-class experiences (Holland et al., 1998), which in turn can impact their in-class learning experiences (Masingila, 1993). Geospatial factors, including space and place, also influence teachers' pedagogical approaches and may potentially limit the attainment of mathematics learning goals (Hoglebe & Tate, 2012).

This study takes place at a public university in the United States, specifically within two graduate seminar classes offered by the curriculum and instruction department. The participants in these classes comprise graduate students in a mathematics education program. During the study, the participants will engage in a review of articles that utilize geospatial frameworks to inform classroom interactions. They will discuss these articles in small groups and reflect upon their discussions. The researchers will then analyze the coded discussions and reflections to uncover insights and report the findings of the study.

Recent research findings from the PME-NA 44th proceedings (Glanfield et al., 2022) indicate that mathematics teacher educators (MTEs) base their conceptions and framings of classroom interactions on students' sociocultural, socioeconomic, and historical backgrounds. These framings, conceptualized as either place (physical address) or space (historical, sociocultural, socioeconomic, and political backgrounds) by Tate (2008), shape the MTEs' understanding of the context in which teaching, and learning occur. Our preliminary findings align with these previous studies,

highlighting a strong correlation between teachers' awareness of students' backgrounds and their approach to classroom interactions.

### References

- Anderson, C. R. (2014). Place matters: Mathematics education reform in urban schools. *Journal of Urban Mathematics Education*, 7(1).
- Darragh, L. (2016). Identity research in mathematics education. *Educational Studies in Mathematics*, 93, 19-33. <https://doi.org/10.1007/s10649-016-9696-5>
- Glanfield, F., Nicol, C., & Thom, J. S. (2022). Remembering place: mathematical actions for innovative, resilient, and culturally rich communities. In *Proceedings of the 44th North American Chapter of the International Group for the Psychology of Mathematics Education*.
- Hogrebe, M. C., & Tate IV, W. F. (2012). Geospatial perspective: Toward a visual political literacy project in education, health, and human services. *Review of Research in Education*, 36(1), 67-94.
- Holland D., Lachicotte W. Jr., Skinner D., & Cain C. (1998). *Identity and agency in cultural Worlds*. Cambridge: Harvard University Press.
- Masingila, J. O. (1993). Learning from mathematics practice in out-of-school situations. *For the learning of mathematics*, 13(2), 18-22.
- Tate, W. F. (2008). Putting the "Urban" in mathematics education scholarship. *Journal of Urban Mathematics Education*, 1(1), 5-9.
- Watson, C. (2006). Narratives of practice and the construction of identity in teaching. *Teachers and Teaching: theory and practice*, 12(5), 509-526.
- Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for research in mathematics education*, 27(4), 458-477.
- Yackel, E., Rasmussen, C., & King, K. (2000). Social and sociomathematical norms in an advanced undergraduate mathematics course. *The Journal of Mathematical Behavior*, 19(3), 275-287. [https://doi.org/10.1016/S0732-3123\(00\)00051-1](https://doi.org/10.1016/S0732-3123(00)00051-1)
- Zembar, I. O., & Yasa, S. A. (2015). Using classroom scenarios to reveal mathematics teachers' understanding of sociomathematical norms. *International Journal of Education in Mathematics, Science and Technology*, 3(3), 242-261.



# MATHEMATICS INSTRUCTION AND MATHEMATICAL VOCABULARY: ENGAGING A NEURODIVERGENT STUDENT

Angela R. Crawford  
Boise State University  
Angelacrawford1@boisestate.edu

Aysia Kernin  
Boise State University  
aysiakernin@u.boisestate.edu

*This study explores volitional and affective responses to instructional activities aimed at developing geometric and spatial vocabulary of a neurodivergent student. Using teaching experiment methodology across 15 instructional sessions, we observed how the student responded to games, direct instruction, and vocabulary support embedded in spatial reasoning activities. A narrative microgenetic analysis explored how these activities were associated with evidence of engagement and confidence in learning. We describe how embedding vocabulary in the student's own mathematical activity was supportive of developing vocabulary, engagement, and confidence, while games and direct vocabulary instruction were not.*

**Keywords:** Students with Disabilities; Instructional Activities and Practices; Affect, Emotion, Beliefs, and Attitudes; Geometry and Spatial Reasoning

As mathematics educators we aim to *engage all learners*, believing that mathematics learning should be a joyful experience for all students. Too often, students with difficulty in mathematics do not experience instructional activities in this way. The PME-NA 2023 conference theme reflects our desire to facilitate students' engagement and interest which in turn will support math learning.

Our research project aimed to encourage positive beliefs and attitudes around mathematics learning in Eva, a neurodivergent student who has had few opportunities for success in the school setting. We conducted a teaching experiment to find instructional activities that were supportive of Eva's learning. We used a narrative microgenetic analysis to characterize the interaction between the instructional activities and Eva's volitional and affective responses.

## **Theoretical Perspective**

Advocating for rehumanizing the mathematics education of students with disabilities, Lambert and colleagues (2018) emphasizes the need to disrupt persistent deficit perspectives found in research. They call for the use of multiple theoretical frameworks to further our understanding of the relationship between disability and mathematics learning. We aimed to participate in rehumanizing the mathematics education of students with disabilities in this research that explores instruction grounded in different theoretical perspectives. We recognize Eva as an individual whose cognitive and affective being is not in harmony with typical educational settings, and we take the position that deficits are present in our instruction. Therefore, we conducted this research with Eva, a nine-year-old student with considerable difficulty in mathematics which may stem from instructional environments which are not responsive to her language processing, executive functions, and anxiety. We investigated how Eva responded to instruction which reflects different theoretical perspectives on teaching with the goal of understanding how to make mathematics learning a joyful experience.

## **Individual Differences**

If we are to engage all learners, we need to identify instructional approaches that are responsive to cognitive diversity and volitional or affective processes which are associated with mathematics outcomes. Cognitive differences related to mathematics achievement vary across domain-specific, domain-general, and executive functioning processes (Ashkenazi et al, 2013; Clements et al.,

2016). Additionally, volitional and affective processes such as motivation, inhibitory control, and anxiety can affect how a student experiences the mathematics learning environment (Nelson et al., 2022). Thus, a variety of individual differences in processes may need to be considered when tailoring instructional support for an individual student.

Eva has difficulty with language processing, executive functions, and anxiety. Language processing, the most common of neurodevelopmental disorders (Koerte et al., 2016), may include difficulty with auditory processing, receptive and expressive vocabulary knowledge, and semantic processing (Hämäläinen et al., 2013; Roesch, 2019). Further, extensive research has identified associations between executive functions and anxiety with success in mathematics learning (Clements et al., 2016; Nelson et al., 2022; Pérez-Fuentes et al., 2020). Thus, several processes may impact Eva's learning of geometric and spatial vocabulary.

### **Spatial Reasoning and Language Processing**

Research on spatial reasoning in students with atypical language processing is limited and inconclusive. Meta-analyses indicate that individuals with dyslexia, a language processing disorder, may perform better, the same, or worse than peers depending upon the nature of the task; a consistent finding is greater variability in their performances as compared to peers (Chamberlain et al., 2018; Gilger et al., 2016). A recent study found that geometric processing was associated with short-term memory skills, while arithmetic and measurement skills were associated with receptive vocabulary (Yang, 2023). We were not able to locate research specifically reporting on language processing and geometric or spatial orientation vocabulary. More research is needed to determine if there are patterns which offer insights for instruction.

### **Behaviorist Learning Theory, Constructivism, and Learning Environments**

Games have long been used to support mathematics learning (Bright et al., 1985; Pan et al., 2022; Siegler, 2009). Pan and colleagues (2022) recently conducted a systematic review of the role of games in mathematics education. They found that behaviorist learning theories were present in nearly 60% of the games studied, followed by experiential learning, and finally constructivist theories. Overall, the research has shown small, positive effects on achievement and self-efficacy, but there are a number of studies with contradictory results. Further, the degree of generalizability is murky across instructional context and student populations (Pan et al., 2022). This review suggests it is not clear what underlying instructional theory and what game formats are likely to have the most benefit for Eva.

For many years there has been disagreement between the fields of special education and mathematics education about approaches to instruction (Woodward, 2004; Munter et al., 2015). Many researchers, particularly in special education, contend vocabulary instruction should be explicit, providing precise, student-friendly definitions supported with examples, non-examples, and representations (Fuchs et al., 2021; Munter et al., 2015). Further, vocabulary should be used repeatedly throughout a lesson or lessons to reinforce students' understanding and corrective or affirmative feedback provided as appropriate (Fuchs et al., 2021; Hattie & Timperley, 2007). These approaches are linked with behaviorist learning theory (Woodward, 2004). Aims are to control cognitive load and ensure successful performance (Grigorenko et al., 2020; Fuchs et al., 2021; Kalyuga, 2009).

In contrast, many researchers in mathematics education and some in special education contend that vocabulary and definitions can be developed with constructivist instructional processes (Ball & Bass, 2000; Lampert, 1990; Munter et al., 2015). Students participate in creating definitions, building on their own experiences or current understandings. The teacher's role is to ensure that students construct mathematically sound definitions which emerge from their ideas and activity

(Freudenthal, 1968; Munter et al., 2015). One argument for this approach is that instruction which starts from student thinking is consistent with a strengths-based approach believed to support positive volitional and affective outcomes (e.g., motivation and beliefs about self) and academic outcomes (De Corte et al., 2000; Rappolt-Schlichtmann et al., 2018). There is a growing number of studies that intentionally test the effects of instruction grounded in constructivism with neurodiverse students or students with or at-risk for difficulty in mathematics (e.g., Baroody et al., 2012; Gervasoni et al., 2021; Hunt et al., 2020). However, research is needed on the effects of these instructional approaches for vocabulary development.

### **Research Purpose**

We aimed to better understand the features of instructional activities which support Eva's spatial reasoning and geometry understandings as well as positive volitional and affective responses. Specifically, we asked: What instructional activities help Eva develop geometry and spatial orientation vocabulary in ways that are also supportive of her engagement and confidence? We analyzed Eva's responses to vocabulary-rich games, direct instruction in vocabulary, and embedding vocabulary in Eva's mathematical activity.

### **Method**

This analysis is part of a larger project in which we investigated Eva's mathematics learning through a teaching experiment centered on established learning trajectories (Crawford & Kernin, 2022). The teaching experiment was followed by a narrative microgenetic analysis. Teaching experiments are forms of design research which systematically investigate teaching and learning in naturalistic settings (Confrey & Lachance, 2000; Steffe et al., 2000). After the teaching experiment, the narrative microgenetic analysis focused on identifying conditions that might promote learning (Lavelli et al., 2005; Siegler, 2006).

### **Participant Information**

Eva, a multi-racial girl, was nine years old and in grade 3 at the time of this study. When she was seven years old, Eva was given a neuropsychological evaluation which identified her as having attention deficit with unspecified impulse-control and conduct disorder, speech-sound disorder, specific language disorder with impairments in written language and mathematics, and generalized anxiety. Eva has received special education services in a pull-out program ("resource room") at a public school and a public charter school in a small city in the western United States. Upon meeting with Eva's mother, we discussed ways to provide her with additional academic support and be mindful of her anxiety. We agreed Eva's mother would be present for all teaching experiment sessions. We gained informed parent consent and student assent following guidelines established by the university's Institutional Review Board.

We recognize our roles as participants in this research. One of us, Angie, conducted the teaching experiments in relationship with Eva and her mother. This relationship involved establishing trust and sharing authority through negotiating goals, activities, what was to be attended to and what was not. As researchers we, Angie and Aysia, continue this relationship with Eva as we interpret her activity and make inferences about her engagement and confidence. We take the position of bracketing ourselves into this narrative, rather than attempting to position ourselves outside (Connelly & Clandinin, 2006).

### **Teaching Experiment Procedure**

One author, Angie, conducted the teaching experiment sessions. All sessions took place at Eva's home, and Eva's mother was present to support Eva's well-being. The sessions took place in 2021 amidst the COVID-19 pandemic, and meeting dates were flexible to allow for family needs and local quarantining requirements. Due to the pandemic and to limit factors that might contribute

to Eva's anxiety, it was not possible to have an additional researcher present during the sessions. There were 15 teaching experiment sessions, each ranging from 30-45 minutes.

Tasks described in this report come from spatial reasoning learning trajectories developed with children from birth to grade 3, specifically the spatial orientation and 2D shapes and shape composition trajectories (Clements & Sarama, 2021). The spatial orientation trajectory involves relationships that position objects in space, including vocabulary such as "on," "under," "beside," etc. The 2D shapes trajectory involves learning to name, describe, and classify two-dimensional shapes using properties of the shapes. The 2D shape composition trajectory involves putting together two-dimensional shapes to make other larger, composite shapes.

Angie used a planning protocol to document instructional decisions and reflections on teaching experiment sessions. The protocol included prompts for observations, outcomes, adjustments made, and rationale for adjustments. Then plans for the next session were developed based on the reflections. The protocol and plans were shared with two colleagues. One colleague, with expertise in early mathematics instruction for students with learning disabilities, advised on the tasks and supports. A second colleague, with experience with children with anxiety, advised on the plans in light of Eva's affective responses.

### **Data Sources**

During each session, Angie asked Eva's permission to video record the activity and only recorded when Eva stated she was comfortable with it. We recorded nine of the 15 sessions. Angie took field notes during each session. Therefore, data sources were the planning protocols previously described, field notes from each session, photographs of student work, videos and transcriptions, and a post-hoc observation protocol.

The other author, Aysia, completed post-hoc observations using a protocol as a way to triangulate observations and interpretations of Eva's responses to activity. This protocol recorded: the task description, student behaviors (strategies, demonstrations, comments), teacher behaviors (explanations, providing time to work without interruption, additional supports), and any other observations. Aysia completed the protocol for each video.

### **Data Analysis**

We used a microgenetic narrative analysis which can account for the multidimensional nature of learning (Lavelli et al., 2005). This approach comprises five stages (Lavelli et al., 2005). The first involves watching videos to identify a list of potential "frames," the specific lenses which might serve as focal points for analysis. Tasks, language use, vocabulary, and affective responses were among the potential frames we identified. In the second stage, we constructed descriptive narratives of each session of the teaching experiment to document the sequence of events. To ensure accurate and comprehensive narratives, we each wrote a draft narrative for half of the sessions and then reviewed and revised the narratives written by the other. We recorded thoughts related to frames, interpretations, questions, or reflections in a parallel set of memos. In the third stage, we used the descriptive narratives to discuss possible combinations of frames that captured interactions influencing learning. This report presents the vocabulary and Eva's affective responses frames. The fourth stage involved re-reading the descriptive narratives to develop the frames into a meaningful and plausible "plot." We sought evidence to confirm or refute this plot through an in-depth analysis of the data sources with particular attention to overt behaviors and strategies. In the fifth and final stage of analysis, we used the plot and evidence to create a narrative that synthesizes our views on Eva's experience.

## Findings

This narrative microgenetic analysis tells the story of our efforts to identify features of instructional activity which support Eva's geometry and spatial orientation vocabulary as well as positive volitional and affective responses. Specifically, we ask: What instructional activities help Eva develop geometry and spatial orientation vocabulary in ways that are also supportive of her engagement and confidence? We describe Eva's responses to vocabulary-rich games, direct vocabulary instruction, and vocabulary support embedded in spatial reasoning activities.

### Vocabulary-rich Games

Angie planned games which hinged on geometric or spatial vocabulary in activity. One example is the game "I Spy," used with the spatial orientation learning trajectory. In this version of the game one person states they are thinking of an object that is located in relation to other objects, identified by spatial vocabulary such as besides, above, between, etc. The other person needs to determine what the secret object is.

We saw considerable evidence of Eva's positive engagement with the "I Spy" game. Eva asked to play it during more than one session. When Angie and Eva played in Eva's backyard, Eva enthusiastically ran about to search for the object. When played seated at a table, Eva often rocked back in her chair and appeared relaxed. Eva eagerly took turns being the person who had a secret object in mind and the person who sought the object.

However, there was little evidence that her vocabulary was developing. She was able to locate objects when the clue contained one, familiar spatial term such as "on" or "under." When other terms were substituted (e.g., "below") or added (i.e., multi-part clues such as "under" and "beside"), Eva was not able to locate the object. In these instances, Eva appeared to search for the secret object based on adjectives rather than use the spatial orientation clues. For example, when Angie gave her the clue, "I spy something blue beside a tree," Eva looked for anything blue, whether or not it was near a tree—even when Angie repeated the phrase "beside a tree" several times or used "beside" with the next object. When it was Eva's turn to select a secret object, she would describe the appearance or function of the object rather than use spatial orientation terms.

Angie tried another game that pressed Eva to use vocabulary more intentionally. Angie used pattern blocks and connecting cubes to make a simple design which was hidden from Eva's view by a screen. Angie asked Eva to make a copy of the design based on the verbal cues provided. Eva found this activity frustrating. For the first two designs, Eva selected and placed pattern blocks and cubes based on known attributes (e.g., red, triangle) without orienting the object as described (e.g., "next to the square") and thought she was correct. On the second design, she reached out and grabbed the screen so that she could copy the design. The following transcript picks up after Eva has been told by her mom and Angie that she cannot grab the screen:

Eva: Why does everybody have to be super, super intense?

Angie: Because you don't learn as much when you look.

Mom: Want to try it again to see if you can listen and get it? Or is it too hard for you?

Eva: Too hard

Angie: Okay then let's do it this way. [adjusts by giving Eva the three brown cubes which are needed] Okay. I would like you to put one of these in front of you....One behind it, and one in front of it....One in front of you...and then put one behind it and one in front of it.....And now, can you take one more?.... Do you have red or brown?...One more brown and put it on top of the one in the middle. You have one in front and one behind and one on top of the one in the middle.

Eva: Everything is square. [Eva's design does not match Angie's]  
 Angie: Okay. Put one in front of you.  
 Eva: I know it's a square. I know.  
 Angie: Now put one behind it.  
 Eva: It is behind.  
 Angie: And one in front of it.  
 Mom: In front.  
 Angie: And one on top.  
 Eva: Too hard  
 Mom: Watch me. You give me direction so you can kind of get an idea of what in front and on top means. Okay. Just watch, please stop getting frustrated.  
 Eva: I'm watching [holds up the plastic lid of a box in front of her face]  
 Mom: Can you watch please? Okay what are we doing?  
 [Angie and Mom go through same sequence placing brown cubes]  
 Mom: Did you understand it?  
 Eva: No.  
 Mom: Do you want to try?  
 Eva: No. Too hard.  
 Angie: Do you want to tell us [cut off by Eva before finishing the question]  
 Eva: Too hard. Too hard. Too hard.  
 Angie: Okay, that's alright. It's too hard, too hard, too hard. That's honest.  
 Eva: At least I tried.

This transcript shows how Eva struggled to process the spatial orientation vocabulary. Unlike the "I Spy" game, Eva did not show enjoyment or confidence with this activity and asked several times to disengage. She seemed to find the mystery of the games enjoyable unless she had difficulty processing the multiple clues or when pressed to rely only on spatial orientation clues. When that difficulty arose, she either resorted to random searching or wanted to abandon the activity. Despite some evidence of engagement, there was little evidence that she was developing understanding of the vocabulary or confidence in her knowledge.

### **Direct Vocabulary Instruction**

Angie used direct instruction to link shape names with shape attributes, aiming to deepen Eva's understanding of the geometric vocabulary. During the first session, Eva was able to identify circles, triangles, and squares. So, in a subsequent lesson, Angie explained the three features that make a circle and focused on terminology such as "curve," (i.e., a curved line, no gaps or breaks in the line, curve stays the same all the way around). Angie gestured with an example and non-example and repeated the three features. Eva played with a sensory toy intended to help focus attention but did not show outward signs of listening. Angie asked Eva to repeat back one of the features. Eva said she did not remember. Angie then asked Eva to find something curved. Eva pointed to a square-shaped cushion with rounded edges, straight sides, and flat surfaces. Angie pointed out parts of the cushion that were straight and curved. Eva started playing with and chasing her pet dog who had been sitting at her feet.

In the next lesson, Angie asked Eva if she remembered the three features of a circle. Eva's response was, "No, but I bet you'll tell me." After naming the features, Angie asked Eva to identify which shape among several drawn on a card was not a circle and explain why it was not. Eva

pointed to the correct shape but her explanation was, “It doesn’t go right.” Eva then began to tell a story about seeing circles on the playground.

During another session, Angie wanted to explicitly discuss the difference between curved and straight lines. The following transcript comes from this session:

Angie: Okay. Are you ready?

Eva: Now I’m paying attention

Angie: Okay, number 1 [pointing to numbered cards with two-dimensional shapes]. You see the one that has a number 1 is not a circle, because it...? What are its sides?

Eva: Triangle

Angie: It has straight sides. So, can you repeat back for me why number 1 is not a circle?

Eva: Because it's a triangle,

Angie: and it has ....?

Eva: Four sides

It is not clear if the error at the end, when Eva says the triangle has four sides, is simply an error due to lack of attention, misspeaking or mis-counting, or something else. We note, though, that the error occurred when Eva was pressed to respond with reference to an attribute. It is also evident in these descriptions of direct instruction on attributes of circles that Eva’s engagement was limited. Distractions were prevalent and Eva’s effort to answer questions was limited to pointing or giving simple, short answers. Looking over her responses in aggregate, responses such as “I bet you’ll tell me,” pointing to a cushion, and “because it’s a triangle,” we considered the possibility that her engagement was perfunctory.

#### **Vocabulary Embedded in Activity**

As Eva worked on activities in the 2D shape composition tasks, Angie gave feedback that incorporated shape names or spatial transformation vocabulary. In the early sessions, Eva relied on trial-and-error to find the correct shape to fill in a larger shape puzzle. She gradually began to select pattern blocks with intention based on appearance and turn them to orient them correctly. Angie encouraged this with positive feedback, “Wow! What great problem solving! I saw you *turn* the block to find out if it would fit another way.” In the next session, Eva was using trial-and-error again to fill in an outline of a dog. Eva almost gave up when she could not place a shape to fill in the outline that indicated a trapezoid. Angie pulled out three pattern blocks and said it was one of these and that she could find it if she used “turning” again. Eva selected the trapezoid and turned it until it was oriented correctly. Angie then said, “Yes! You saw that it was a trapezoid that would fit if you turned it.” In the next session, Eva intentionally selected and placed trapezoids to create a hexagon in the center of a picture. Later, after Angie and Eva’s mom had used the name of the shape on several occasions, Eva had difficulty providing the full term but could approximate it (e.g., “tra-me-zo”).

Also, we noted that Eva’s confidence grew, and she was able to solve increasingly complex 2D shape composition tasks. For example, in the eleventh session Eva demonstrated engagement and confidence in her ability to solve tasks which involved composing shapes without interior lines provided as cues:

Angie: Can you choose a couple of those to do? Thank you.

Eva: Oooh! [Looks through puzzles and chooses one without interior lines]

Angie: You want to do that one?

Eva: Hard.

Angie: I also have some others in here that are even more hard but I thought [Eva starts talking before Angie finishes statement]

Eva: This looks like a camel [begins looking for pattern blocks to fill in the shape]

Angie: A camel? It kind of does. Yes, it's got a hump on its back.

Eva: Yep. This is hard, this is hard.

Angie: What do you think seems hard about that one?

Eva: Everything.

Angie: Pardon me? What?

Eva: Everything.

Angie: What do you mean "everything"? It seems like you knew what to do right away.

Eva: No I mean like it's going to be hard for friends, and mom or dad, I mean that.

We see evidence of Eva's engagement in the interest she showed in solving the puzzle. Further, there is clear evidence in her confidence in solving this puzzle by stating that everything seemed hard about it, but it would be hard for others like her friends or parents.

### Discussion

To identify instructional activities that were enjoyable for Eva and helped her to develop geometric and spatial orientation vocabulary, we tried a variety of formats: vocabulary-rich games, deliberate vocabulary instruction, and vocabulary development embedded in other activity. We found that only vocabulary embedded in Eva's mathematical activity seemed to have positive effects on her understanding of terms, engagement, and confidence.

Eva was engaged in some of the vocabulary-rich games but not others. Also, we found that repeatedly using a particular term across several iterations of the activity did not seem to help her apply or reason with that term. This may be explained by ways in which the game increased cognitive load through requiring her ability to process verbal cues without other supports (Kalyuga, 2009). Our finding suggests one possible reason why studies of games have had contradictory results which limit generalizability (Pan, 2022).

We also saw little success with direct vocabulary instruction. Eva showed little sustained engagement in these lessons and responded in what seemed perfunctory ways. This is despite recommendations for this approach on the basis of research evidence from experimental and quasi-experimental studies for its efficacy for students with difficulty in mathematics (Fuchs et al., 2021). We believe this discrepancy highlights the fact that quantitative research studies report aggregated results. As such, the conclusions may not apply or have limited efficacy when used on an individual, case-by-case basis.

We found the most success in developing vocabulary, confidence, and engagement when the vocabulary support mathematized Eva's activity (Freudenthal, 1968). As a result, Eva demonstrated understanding of the terms, sometimes used the terms herself, and remained engaged with the activities during each session. Over time, she expressed confidence in her ability to complete these tasks. The vocabulary development with this instructional approach may be attributed to the combination of presentation without increasing cognitive load and the positive, specific feedback she received (Kalyuga, 2009; Hattie & Timperley, 2007).

We engaged in this research to learn how we might help Eva to develop mathematical vocabulary in ways that supported her engagement and confidence. Our findings suggest that for Eva an instructional approach that engages, builds confidence, and hopefully leads to a joyful experience, is when her activity is mathematized and, as a result, valued.

### References

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.



- Ashkenazi, S., Black, J. M., Abrams, D. A., Hoefl, F., & Menon, V. (2013). Neurobiological underpinnings of math and reading learning disabilities. *Journal of learning disabilities, 46*(6), 549-569.
- Ball, D. L., & Bass, H. (2000). Chapter VII: Making Believe: The Collective Construction of Public Mathematical Knowledge in the Elementary Classroom. *Teachers College Record, 102*(7), 193-224. <https://doi.org/10.1177/016146810010200707>
- Baroody, A. J., Eiland, M. D., Purpura, D. J., & Reid, E. E. (2012). Fostering at-risk kindergarten children's number sense. *Cognition and Instruction, 30*(4), 435-470. <http://doi.org/10.1080/07370008.2012.720152>
- Bright, G. W., Harvey, J. G., & Wheeler, M. M. (1985). Learning and mathematics games. *Journal for Research in Mathematics Education. Monograph, 1*, i-189.
- Chamberlain, R., Brunswick, N., Siev, J., & McManus, C. (2018). Meta-analytic findings reveal lower means but higher variances in visuospatial ability in dyslexia. *British Journal of Psychology, 109*(4), 897-916. <https://doi.org/10.1111/bjop.12321>
- Clements, D. H., & Sarama, J. (2021). *Learning and teaching early math: The learning trajectories approach* (3rd ed.). Routledge.
- Clements, D. H., Sarama, J., & Germeroth, C. (2016). Learning executive function and early mathematics: Directions of causal relations. *Early Childhood Research Quarterly, 36*, 79-90. <https://doi.org/10.1016/j.ecresq.2015.12.009>
- Confrey, J. & Lachance, A. (2000). Transformative teaching experiments through conjecture-driven research design. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 231-265). Lawrence Erlbaum Associates.
- Connelly, F. M., & Clandinin, D. J. (2006). Narrative inquiry. In J. L. Green, G. Camilli, & P. B. Elmore (Eds.) *Handbook of complementary methods in education research*. (pp. 477-487). Lawrence Erlbaum Associates.
- Crawford, A. R., & Kernin, A. (2022). A 2D shape composition learning trajectory of a student with difficulty in mathematics [Research report]. In A. E. Lischka, E. B. Dyer, R. S. Jones, J. Lovett, J. Strayer, & S. Drown (Eds.), *Proceedings of the forty-fourth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1607-1615). Middle Tennessee State University. <https://doi.org/10.51272/pmna.44.2022>
- de Corte, E., Verschaffel, L., & Op't Eynde, P. (2000). Self-regulation: A characteristic and a goal of mathematics education. In *Handbook of self-regulation* Boekaerts, M., Pintrich, P. R., & Zeidner, M. (Eds.) (pp. 687-726). Academic Press.
- Freudenthal, H. (1968). Why to teach mathematics so as to be useful. *Educational studies in mathematics, 3*-8.
- Fuchs, L.S., Newman-Gonchar, R., Schumacher, R., Dougherty, B., Bucka, N., Karp, K.S., Woodward, J., Clarke, B., Jordan, N. C., Gersten, R., Jayanthi, M., Keating, B., and Morgan, S. (2021). *Assisting Students Struggling with Mathematics: Intervention in the Elementary Grades* (WWC 2021006). Washington, DC: National Center for Education Evaluation and Regional Assistance (NCEE), Institute of Education Sciences, U.S. Department of Education. Retrieved from <http://whatworks.ed.gov/>.
- Gervasoni, A., Roche, A., & Downton, A. (2021). Differentiating instruction for students who fail to thrive in mathematics: The impact of a constructivist-based intervention approach. *Mathematics Teacher Education and Development, 23*(3), 207-233.
- Gilger, J. W., Allen, K., & Castillo, A. (2016). Reading disability and enhanced dynamic spatial reasoning: A review of the literature. *Brain and Cognition 105*, 55-65. <https://doi.org/10.1016/j.bandc.2016.03.005>.
- Grigorenko, E. L., Compton, D. L., Fuchs, L. S., Wagner, R. K., Willcutt, E. G., & Fletcher, J. M. (2020). Understanding, educating, and supporting children with specific learning disabilities: 50 years of science and practice. *American Psychologist, 75*(1), 37-51. <https://psycnet.apa.org/doi/10.1037/amp0000452>
- Hämäläinen, J. A., Salminen, H. K., & Leppänen, P. H. (2013). Basic auditory processing deficits in dyslexia: systematic review of the behavioral and event-related potential/field evidence. *Journal of learning disabilities, 46*(5), 413-427. <https://doi.org/10.1177/F0022219411436213>
- Hattie, J., & Timperley, H. (2007). The power of feedback. *Review of educational research, 77*(1), 81-112.
- Hunt, J. H., Martin, K., Khounmeuang, A., Silva, J., Patterson, B., & Welch-Ptak, J. (2020). Design, development, and initial testing of asset-based intervention grounded in trajectories of student fraction learning. *Learning Disability Quarterly, 1*-14. DOI: 10.1177/0731948720963589
- Kalyuga, S. (2009). Knowledge elaboration: A cognitive load perspective. *Learning and Instruction, 19*(5), 402-410.
- Koerte, I. K., Willems, A., Muehlmann, M., Moll, K., Cornell, S., Pixner, S., ... & Schulte-Körne, G. (2016). Mathematical abilities in dyslexic children: a diffusion tensor imaging study. *Brain imaging and behavior, 10*, 781-791.

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

- Lambert, R., Tan, P., Hunt, J., & Candela, A. G. (2018). Rehumanizing the mathematics education of students with disabilities: Critical perspectives on research and practice. *Investigations in Mathematics Learning*, 10(3), 129-132. DOI: 10.1080/19477503.2018.1463006
- Lampert, M. (1990). When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching. *American educational research journal*, 27(1), 29-63.
- Lavelli, M., Pantoja, A. P. F., Hsu, H.-C., Messinger, D., & Fogel, A. (2005). Using microgenetic designs to study change processes. In D. G. Teti (Ed.), *Handbook of research methods in developmental psychology*. (pp. 40–65). Blackwell Publishers.
- Munter, C., Stein, M. K., & Smith, M. S. (2015). Dialogic and direct instruction: Two distinct models of mathematics instruction and the debate(s) surrounding them. *Teachers College Record*, 117, 110306, 1-32.
- Nelson, G., Kiss, A. J., & Crawford, A. (2022). Developmental distinctions in mathematics for students with disabilities. In E. Talbott & T. Farmer (Eds.). *Handbook of research on special education*. Routledge Press.
- Pan, Y., Ke, F., & Xu, X. (2022). A systematic review of the role of learning games in fostering mathematics education in K-12 settings. *Educational Research Review*, 100448. <https://doi.org/10.1016/j.edurev.2022.100448>
- Pérez Fuentes, M. D. C., Núñez Niebla, A., Molero, M. D. M., Gázquez Linares, J. J., Rosário, P., & Núñez, J. C. (2020). The role of anxiety in the relationship between self-efficacy and math achievement. *Psicología educativa: revista de los psicólogos de la educación*.
- Rappolt-Schlichtmann, G., Boucher, A. R., & Evans, M. (2018). From deficit remediation to capacity building: Learning to enable rather than disable students with dyslexia. *Language, Speech, and Hearing Services in Schools*, 49(4), 864-874. [https://doi.org/10.1044/2018\\_LSHSS-DYSLC-19-0031](https://doi.org/10.1044/2018_LSHSS-DYSLC-19-0031)
- Roesch, D. (2019). *Language-based risk factors in children with developmental dyslexia: A systematic review* (Doctoral dissertation, University of Pretoria).
- Siegler, R. S. (2006). Microgenetic analyses of learning. In D. Kuhn, R. S. Siegler, W. Damon, & R. M. Lerner (Eds. ) *Handbook of child psychology: Cognition, perception, and language* (pp. 464–510). John Wiley & Sons Inc.
- Siegler, R. S. (2009). Improving the numerical understanding of children from low-income families. *Child Development Perspectives*, 3(2), 118-124.
- Steffe, L. P., Thompson, P. W., & von Glaserfeld, E. (2000). Teaching experiment methodology: Underlying principles and essential elements. In A. E. Kelly, & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education*, Laurence Erlbaum Associates, 267-306.
- Woodward, J. (2004). Mathematics education in the United States: Past to present. *Journal of Learning Disabilities*, (37)1, 16-31.
- Yang, Y., Zhang, X., & Huo, S. (2023). Pathways to arithmetic, geometry, and measurement in preschool children: The role of general cognitive and language skills. *Early Childhood Research Quarterly*, 62, 315-323. <https://doi.org/10.1016/j.ecresq.2022.09.007>

## “MATH IS...”: THE MUSINGS OF YOUNG ADOLESCENTS PRE- AND POST-COVID

Heidi J. Higgins  
University of North Carolina, Wilmington  
higginsh@uncw.edu

Shelby P. Morge  
University of North Carolina, Wilmington  
morges@uncw.edu

*Due to the emphasis placed on learning mathematics in K-12 education and society, students have a range of thoughts and emotions about it. This brief report describes middle school students' perceptions about math prior to and after Covid. Data were collected as a quick write activity which was posed to students in their math class. The prompt given to students was "Math is..." In our brief report, we will describe trends across the two data sets. Analysis of preliminary data indicates student's responses fall into three main categories: content, utility, and emotion. Additional analysis identified patterns within those categories. We also make suggestions for what this means for middle school mathematics teachers and how to increase student interest and motivation.*

Keywords: Affect, Emotion, Beliefs, and Middle School Education

Mathematics is a subject that people seem to either love or hate. This relationship with the subject is reflected in daily life and conversations. Positive representations of mathematics in popular culture have been known to spark wonder while negative representations can reinforce negative stereotypes (Finch, 2017). Yet there isn't a prevalence of mathematics in social media but when it is presented it is typically unfavorable. Mathematics is also represented in particular ways in schools, sometimes out of reach of the students who need to learn it. These everyday experiences can lead to students developing negative mathematical identities in that they find a disconnect between who they are and how mathematics is relevant to them personally (Boaler & Greeno, 2000; Gutiérrez, 2013)

Su (2020) grounds mathematics in what it means to be human and live a full life. This perspective of mathematics makes it relatable and useful to all learners. It removes barriers of mathematics that are present in society – for instance the image that comes to mind when asked to describe a mathematician. We want our students to believe that everyone can be a math person. In order to support them in developing a positive mathematical identity, we must first understand what they think of it. This brief report describes middle school students' perceptions about math prior to and after Covid. Data were collected as a quick write activity which was posed to students in their math class.

### Perspectives

Over the past several decades, mathematics related beliefs have received increased attention from researchers. Teacher and student beliefs have been attributed to things such as educational or school influences (De Corte, Verschaffle, & Depaepe, 2008; Wang, Zhang, & Cai, 2019) and society or popular culture influences (Boaler, 2002; Damarin, 2000). In addition, researchers have found relationships between student beliefs about mathematics and performance in math class (Dowker, Cheriton, Horton, & Mark, 2019; Furinghetti & Pehkonen, 2000; Ma & Xu, 2004).

Various strategies have been used to assess student beliefs about mathematics. Lubienski, McGraw & Strutchens (2004) reviewed survey data collected from NAEP to identify gender differences in student responses. Questionnaires have been used with primary school children to identify relationships between mathematics related beliefs and performance (Dowker, Cheriton, Horton, & Mark, 2019) as well as anxiety (Haciomeroglu, 2017). In addition, Çetinkaya,

Özgören, Orakci, and Özdemir (2018) collected metaphors about mathematics from middle school students and found that they fell into four different categories and were both positive and negative.

Researchers have suggested approaches for improving students’ math beliefs. Boaler (2015) emphasizes the importance of a growth mindset in which students believe they can improve in math class. This includes strategies such as using open ended tasks, posing problems prior to teaching the method, incorporating visuals, and requiring students to convince and reason. Related to these suggestions, Gilbert, Musu-Gillette, Woolley, Karabenick, Strutchens, & Martin (2014) suggest that frequent exposure to reform practices is important for students who are lower performers and judge themselves as being less capable in mathematics. Another group of researchers created *The Mathematics Heritage Project* which designed tasks that intentionally connected school mathematics to students’ cultural and familial backgrounds. Student reflections indicated an increased awareness of the relevancy of mathematics and how mathematics was interconnected to their everyday lives (Desai, Kurtz, and Safi, 2021).

### Methods

In order to determine middle school students’ thoughts about mathematics, we posed the prompt, “Math is...” Students were given a few minutes to write their responses to this quick write. The initial data set was collected in 2018, prior to Covid. We were interested in whether there were differences in middle school students’ perceptions about mathematics after experiencing Covid adjustments and posed the same prompt to students in 2023.

Each of the researchers independently conducted a content analysis by reviewing the responses and organizing into groups by theme (Kolbe & Burnett, 1991). This initial analysis was followed by discussion to determine the broad categories. Each researcher came up with three broad categories that were similar in detail and through discussion were able to come to agreement on the description of each category which included (1) Content (2) Utility, (3) Emotion. Researchers then sorted the data into the three broad categories with some data aligning with more than one category. Researchers met to discuss the data sets which eventually produced 100% agreement on sorting. Next the researchers reviewed the data for patterns within the categories. This step was also followed by discussion that eventually produced agreement.

### Results

The sample examined in the initial phase of this study was collected in 2018 included students enrolled in three different local public middle schools located in a suburban area on the east coast. The second sample was obtained in 2023 from two different local public middle schools. The frequency of participants’ gender and ages are provided in Table 1.

**Table 1. Distribution of Participant Groups in Terms of Gender and Age Level**

Age Level	Female		Male		Non-Binary or Did Not Identify		Total	
	2018	2023	2018	2023	2018	2023	Total 2018	Total 2023
11	46	4	47	9	1	0	94	13
12	59	29	77	31	3	2	139	62
13	35	48	54	60	0	1	89	109
14	16	42	22	66	0	4	38	112

15	0	4	1	5	0	1	1	10
Did not Identify	1	0	6	1	0	0	7	1
Total	157	127	207	172	4	8	368	307

The initial analysis resulted in three themes:

- Content – student response mentions computation or a specific math example
- Utility – student response mentions useful or needed for the real world
- Emotion – student response indicates emotions

The researchers noticed that all student responses did not clearly fall into one of these three categories. Several responses aligned with one of the major categories, though many of the statements could be coded into two or three categories.

### Content

Content knowledge is defined as “the concepts, principles, relationships, processes, and applications a student should know within a given academic subject, appropriate for his/her and organization of the knowledge” (Ozden, 2008, p. 634). Student responses that focused on content included statements such as:

- “Math is using numbers to solve equations/problems.”
- “Math is the process of which you deal with numbers, variables, fractions, etc...”
- “Math is a subject that has something to do [sic] numbers, multiplication, exponents, equations, etc...”

When reviewing the data, 49 of the 2018 student responses and 39 of the 2023 student responses mentioned mathematics content.

### Utility

Student responses also referred to the usefulness of mathematics or referred to mathematics as needed for the real world. When reviewing the 2018 data we created three subcategories:

- Mathematics is needed for life in general
- Mathematics is needed to be successful in a job or career
- Mathematics is needed for day-to-day tasks

Some student responses included more than one of these descriptions. The 2023 data included a fourth category that was not observed in the 2018 data. Students described mathematics as something that was meant to be useful, but they did not see how. (See Table 2)

**Table 2: Student Responses Mentioning Utility**

Year	Total	Life in General	Job or Career	Day-to-day Tasks	Unsure why useful
2018	89	59	23	15	0
2023	55	37	12	6	7

## Emotion

The third and largest subset of responses described mathematics with emotions. Responses in this category included statements about the complexity of doing math that included words such as math is... hard, difficult, confusing, easy, challenging, and frustrating. Some students referenced it as something that is fun, boring, their favorite or even their worst class. We further categorized the emotional responses into whether or not math is perceived as something that is negative, positive, or neutral. (See Table 3)

**Table 3: Student Responses Mentioning Emotions**

Year	Total	Positive	Neutral	Negative
2018	296 (80%)	201 (68%)	19 (6%)	48 (16%)
2023	242 (78.5%)	58 (24%)	80 (33%)	104 (43%)

## Discussion

Analysis of middle grades students' perceptions of math revealed three distinct categories: content, utility, and emotions. Many student responses included some combination of these three categories. Within the utility category, we noticed that students have a superficial perspective of what math is in the real world. Many examples were connected to consumerism. The 2023 data introduced a new category in which students questioned its use in the real world. We were challenged when deciding if terms like "useful" or "helpful" were emotional or utility responses. Furthermore, we noted that several emotional responses included terms that could be interpreted as positive or negative, such as "easy" and "challenging." Many student responses included terms that were both positive and negative such as, "hard" and "fun". Our comparison between 2018 and 2023 emotional responses revealed a significant shift from mostly positive to mostly negative.

These preliminary results remind us of the importance of reform-based instructional practices and the need for a supportive classroom environment. Students need opportunities to see mathematics as something they are able to do. This includes the use of open-ended math tasks with a low floor (allowing access to all students), high ceiling (encouraging high level thinking), and wide walls (allowing multiple pathways) (Resnick, 2017) that encourage exploration, discussion, and collaboration. Another suggestion is intentional interdisciplinary instructional practices in which students and teachers discuss mathematics connections with other subject areas and students' cultural backgrounds. These connections will support students in experiencing the relevance of mathematics and provide examples of mathematics in the real world beyond making purchases. Middle school is a necessary time to make these connections because students are beginning to generate interest in different activities and careers which can impact future academic pathways.

## References

- Boaler, J. (2002). Paying the price for "sugar and spice": Shifting the analytical lens in equity research. *Mathematical Thinking and Learning*, 4(2-3), 127-144.
- Boaler, J. (2015). *Mathematical Mindsets: Unleashing Students' Potential Through Creative Math, Inspiring Messages and Innovative Teaching*. John Wiley & Sons.
- Boaler, J., & Greeno, J. G. (2000). Identity, agency, and knowing in mathematics worlds. *Multiple Perspectives on Mathematics Teaching and Learning*, 1, 171-200.
- Çetinkaya, M., Özgören, Ç., Orakci, S., & Özdemir, M. Ç. (2018). Metaphorical perceptions of middle school students towards math. *International Journal of Instruction*, 11(3), 31-44.
- Damarin, S. K. (2000). The mathematically able as a marked category. *Gender and Education*, 12(1), 69-85.

- De Corte, E., Verschaffel, L., & Depaep, F. (2008). Unraveling the relationship between students' mathematics-related beliefs and the classroom culture. *European Psychologist*, 13(1), 24–36. <https://doi.org/10.1027/1016-9040.13.1.24>
- Desai, S., Kurtz, B., & Safi, F. (2021). Mathematics heritage project: An exploration empowering students' mathematical identities. *Journal of Humanistic Mathematics*, 11(2) 106-122.
- Dowker, A., Cheriton, O., Horton, R., & Mark, W. (2019). Relationships between attitudes and performance in young children's mathematics. *Educational Studies in Mathematics*, 100, 211-230.
- Fabian, K., Topping, K. J., & Barron, I. G. (2018). Using mobile technologies for mathematics: effects on student attitudes and achievement. *Educational Technology Research and Development*, 66, 1119-1139.
- Finch, F. (2017, April 28). Mathematics in popular culture: The good, the bad and the bogus. *The Wiley Network*. <https://www.wiley.com/en-us/network/publishing/research-publishing/trending-stories/mathematics-in-popular-culture-the-good-the-bad-and-the-bogus>
- Furinghetti, F., & Pehkonen, E. (2000). A comparative study of students' beliefs concerning their autonomy of doing mathematics. *Nordisk Matematikdidaktikk*, 8(4), 7-26.
- Gilbert, M. C., Musu-Gillette, L. E., Woolley, M. E., Karabenick, S. A., Strutchens, M. E., & Martin, W. G. (2014). Student perceptions of the classroom environment: Relations to motivation and achievement in mathematics. *Learning Environments Research*, 17, 287-304.
- Gutiérrez, R. (2013). The sociopolitical turn in mathematics education. *Journal for Research in Mathematics Education*, 44(1), 37–68.
- Haciomeroglu, G. (2017). Reciprocal relationships between mathematics anxiety and attitude towards mathematics in elementary students. *Acta Didactica Napocensia*, 10(3), 59-68.
- Kolbe, R. H., & Burnett, M. S. (1991). Content analysis research: An examination of applications with directives for improving research reliability and objectivity. *Journal of Consumer Research*, 18(2), 243-250. <https://doi.org/10.1086/209256>
- Lubienski, S. T., McGraw, R., Strutchens, M. E., Kloosterman, P., & Lester, F. (2004). NAEP findings regarding gender: Mathematics achievement, instructional experiences and affect. Results and interpretations of the 1990 through 2000 mathematics assessments of the National Assessment of Educational Progress, 305-336.
- Ma, X., & Xu, J. (2004). Determining the causal ordering between attitude toward mathematics and achievement in mathematics. *American Journal of Education*, 110(3), 256-280.
- Ozden, M. (2008). The Effect of Content Knowledge on Pedagogical Content Knowledge: The Case of Teaching Phases of Matters. *Educational sciences: theory and practice*, 8(2), 633-645.
- Resnick, M. (2017). Lifelong kindergarten: Cultivating creativity through projects, passion, peers, and play. MIT press.
- Su, F. (2020). *Mathematics for Human Flourishing*. Yale University Press.
- Vankúš, P. (2021). Influence of game-based learning in mathematics education on students' affective domain: A systemic review, *Mathematics*, 9(9) 986-996.
- Wang, G., Zhang, S., & Cai, J. (2019). Chinese high school students' mathematics-related beliefs and their perceived mathematics achievement: A focus on teachers' praise. *Eurasia Journal of Mathematics, Science and Technology Education*, 15(7), em1713. <https://doi.org/10.29333/ejmste/105875>

## MATH THEIR WAY: EXPLORING CHILD-INITIATED MATHEMATICS

Mari Altshuler  
University of Illinois at Urbana-Champaign  
mshuler@illinois.edu

*This interview-based study explored early elementary students' perceptions of mathematics through the lens of the math that they opted to do. It asked, how do the mathematics activities that children initiate compare to those in which they engage in school? Results showed that when young students seized agency and engaged in math on their own terms, their activities were more expansive and creative than in school and they found these activities pleasurable.*

Keywords: Affect, Emotions, Beliefs, and Attitudes; Early Childhood Education; Elementary School Education

When I first met Hiba, she was seven years old, at the very end of first grade. She had been learning remotely for just over two months—the first two months of the COVID-19 pandemic. Hiba shared her plans for inventing her own summer camp in her backyard, and she made sure I knew how much she hated homework. Most notably though, Hiba shared that she saw herself and her relationship with math as being on a trajectory toward greatness. When I asked Hiba why she engaged in math outside of school, she responded, “First I was okay, then not bad, then I got good...then I’m going to get great, amazing, excellent, *mathematician*.”

### Theoretical Framework

Hiba believed she was capable of learning, growing, and achieving in mathematics. She felt she had agency over her mathematical future, and she saw mathematics as a part of her current and future selves. Of course, not all students feel like Hiba. Despite research that emphatically shows there is nothing inherently exclusive about math, there continues to be a pervasive and resilient societal perception that being ‘good at math’ is something only a few, special people can achieve (e.g. Boaler & Greeno, 2000; Su 2020). Schools perpetuate this discourse through the way that mathematics is taught, often continuing to pass along the message that math is about memorizing procedures devoid of meaning in order to produce singular right answers (Black et al., 2011; Lakoff & Nuñez, 2000). In turn, students seeking agency, collaboration, and creativity come to see math as not for them. Many students stop seeing themselves on a trajectory toward mathematician, and instead disassociate from math, believing that math is not a part of who they are or who they are hoping to become (Boaler & Greeno, 2000).

This study builds on the body of math identity research that has sought to support more students in forming positive, empowered relationships with mathematics while also challenging beliefs that certain students or groups of students are inherently less mathematically capable (e.g. Martin, 2000; Nasir, 2002; Jessup, et al., 2020). It takes the perspective that learning mathematics is a fundamentally social and cultural process, and that in math classrooms, students do not just learn content, but they also learn how to be math learners and what it means to be someone who does mathematics (Vygotsky, 1978; Brown et al, 1989). Relatedly, the learning experiences within different classrooms and schools offer different opportunities for students to develop identities in relationship with the discipline (Wenger, 1998).

Additionally, this study grows from child development work that shows young children actively interpret their contexts, make sense of who they are as individuals, and can offer unique



perspectives on their experiences (Sabol et al., 2020; Rogers et al., 2012). The vast majority of mathematics identity research has focused on older students, especially at the high school level (e.g. Boaler & Greeno, 2000; Nasir & Hand, 2008). To further understand students' mathematics identities, we need to investigate young children's experiences with and relationships to mathematics. Early elementary school students have valuable insights to share.

Specifically, this study investigates moments in which young children seize agency (e.g. Ruef, 2021; Cobb et al., 2009) over their mathematics experiences. Research has shown that when students are positioned as agentive and have active roles in their classrooms, they are more likely to enjoy mathematics and see it as a part of their futures (Boaler & Greeno, 2000). When math learning environments value students as whole people and open space for them to make meaningful decisions about their learning, students' identification with the discipline is stronger (Nasir & Hand, 2008). This study extends previous exploration of student agency in mathematics by asking: How do the mathematics activities that children initiate compare to those in which they engage in school?

### **Study Design**

This study, which is part of a larger dissertation project (Altshuler, 2022) took place between May and December of 2020. Because of COVID-19, this was a time of significant change in children's schooling. Though this study does not emphasize the impacts of COVID specifically, the pandemic is an important and relevant backdrop for this project. Children were spending significantly more time at home than typical, thus opening more opportunities for them to initiate activities, including those that were mathematical in nature.

### **Recruitment and Participants**

I chose to recruit students for this study who were in Kindergarten through second grade because their voices are largely missing from the literature on math identity and because as young children they had just begun their formal schooling, creating new possibilities for developing disciplinary affinities. I also limited recruitment to the Chicagoland area to ensure a broadly shared context around virus transmission, public policy guidelines, and school closures. I recruited participants through snowball sampling (Biernacki & Waldorf, 1981).

In total, 30 children, and one caregiver per child, participated. The majority of students were enrolled in public schools (n=26), with a few enrolled in private schools (n=4). At the start of the study, 9 students were in Kindergarten, 11 were in 1<sup>st</sup> grade, and 10 were in 2<sup>nd</sup> grade. As identified by their caregivers, 18 of the participants were female, and 12 were male. Five were African American/Black, 5 were Asian, 2 were Hispanic/Latinx, 15 were white, 2 were multiracial, and 1 self-identified as Middle Eastern. 21 of the students spoke only English at home, with the others speaking Gujarati, French, Spanish, Hebrew, Mandarin, or Arabic.

### **Data Collection**

I conducted three rounds of semi-structured interviews (Spradley, 1979) on Zoom with each child. The first interviews took place in May-June of 2019, the second took place in August, and the third took place in November-December once students were in a new school year. During each interview I asked students about their math experiences in and outside of school, including whether they chose to do math of their own accord. I conducted 90 interviews, for a total of 29 hours and 26 seconds of data. The interviews averaged 19 minutes and 37 seconds long.

### **Data Analysis**

After transcribing all interviews, I began analysis by creating a list of students who, in any of the three interviews, responded that at some point they did math "just because." Then, I gathered

their responses about the math that they initiated outside of school. For comparison, I also gathered their responses about the types of math activities that they did in their classrooms.

From there, I took an inductive approach (Miles et al., 2020). I looked across all the activities that students described doing, whether on their own or in their classrooms. I clustered similar activities into groups, which became codes, and I iterated between coding students' quotes and clarifying the code definitions until I had a final list of 10 codes: *solving equations*, *counting*, *creating equations*, *asking questions*, *measuring*, *drawing*, *discussing/explaining*, *building*, *telling time*, and *estimating*. These encapsulated the activities that students engaged in on their own, at school, or across both contexts, and I used them to compare the number of students who engaged in each type of mathematics activity on their own and at school.

Finally, I returned to the original student quotes to gather information on why students initiated the mathematical activities. I again clustered similar reasons into codes and iterated between quotes and code definitions until I had three purpose codes: *being embedded in something else*, *to meet school-based expectations*, and *driven by pleasure or curiosity*.

### Results

Of the 30 students who participated in this study, 17 of them responded in at least one interview that sometimes they did math just because they felt like it. The 17 students were balanced across grade levels and were representative of the study demographics overall.

There was a great amount of variety across children's self-initiated mathematics activities. Sebastian, a second grader, explained "I love [doing] crazy math problems...[like] 3,776... divided by like 64. I did that all in my head." Deja, another second grader described playing teacher and explaining math concepts to her neighbors and stuffed animals. First grader Darius watched videos on YouTube to learn his times tables, and Valentina, a Kindergartener, counted her Paw Patrol figurines. Table 1 lists how many students initiated each type of activity and how many students participated in each activity in school. The total number of activities is larger than 17 because many students engaged in more than one type.

**Table 1: Comparing Child-Initiated Mathematics Activities to Math in Classrooms**

Activity	Child-Initiated (# of students)	At school (# of students)
Solving equations	14	17
Counting	8	9
Creating equations	8	0
Asking questions	3	0
Measuring	3	0
Drawing	2	1
Discussing/Explaining	2	2
Building	2	0
Telling time	1	0
Estimating	0	1

Each child, on average, described initiating more types of mathematics activities than they participated in during school, and the overall spread of child-initiated mathematics activities is broader than what students described as happening in their classrooms. Further, several of the child-initiated mathematics activities that did not take place in classrooms are creative. When children *create equations*, they establish themselves as producers, not just receivers of

mathematics. Children’s ideas were also front and center when they asked questions. Though less common, drawing and building are again creative activities. Creativity can only happen with agency; when engaging in creative activities, children make decisions about what they are doing, and how they do it. Creativity was not central to the at-school mathematics.

Just as there was variety in the types of mathematics activities that children initiated, there was also variety in their reasons for doing so. In a few instances, children shared that they initiated math activities because they wanted to practice or improve on a specific skill or because they were trying to get ready for the next grade level. In other instances, children identified that math was an important part of a desired activity—like counting the points in mini-golf or measuring the ingredients for brownies. Overwhelmingly, children-initiated mathematics activities because they were driven by pleasure or curiosity. The children described this in several different ways. Often, they shared that they did math just “for fun” or because “they were bored.” Other times they wanted to learn something new, or they wanted to challenge themselves. These results are exciting; not only did these students initiate mathematics activities that were creative and expansive and nature, they also did so because it made them feel good.

**Table 2: Children’s Reasons for Initiating Mathematics Activities**

Reasons	# of Students
School-based expectations	6
Embedded in a desired activity	6
Driven by pleasure or curiosity	30

### **Discussion and Conclusion**

In this study, I explored what it looked like when young children engaged in mathematics on their own accord. This is an important area for study because child-initiated mathematics activities reveal what children understand to be mathematical. Through my analysis, I found that children’s self-initiated mathematics activities sometimes mirrored the activities they described doing in school. But on their own, children initiated a wider range of mathematics activities, and they initiated more activities that were creative in nature. Children engaged in these activities for a variety of purposes, some of which were tightly connected to academic expectations, but mostly because doing so gave them pleasure or fulfilled their curiosities.

Child-initiated mathematics activities highlight what happens when young children seize agency (e.g. Ruef, 2021; Cobb et al., 2009) over their mathematical experiences. In turn, paying close attention to child-initiated mathematics activities may be one way for educators to gain insights into how to encourage children’s positive relationships with math (Boaler & Greeno, 2000). By attending to the expansive, creative, and pleasurable mathematics that children engage in on their own, we can follow their lead and build related opportunities for student agency in school-based mathematics learning environments. Further, by noting who does not consider their activity to be mathematical in nature, or who is not opting to initiate mathematics activities on their own, we can reflect and reshape our learning environments to better reach and support these students as well. It may be that children who already have more positive mathematics identities are more likely to initiate mathematics activities, but it may also be that by engaging in mathematics activities on their own, children strengthen their mathematics identities. Continuing to study and unpack this cyclical relationship and the role of educators within it could contribute to the discipline’s ongoing challenge of expanding who gets to be and feel mathematical.

### **References**

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

- Altschuler, M. (2022). *"I think it's fun, and sort of challenging, but this is just me—I like challenges": Exploring early elementary school students' emerging mathematics identities* (Publication number 29324134) [Doctoral dissertation, Northwestern University.] ProQuest Dissertations Publishing.
- Biernacki, P. & Waldorf, D. (1981). Snowball sampling: Problems and techniques of chain referral sampling. *Sociological methods & Research*, 10(2), 141-163. <https://doi.org/10.1177/00491241810100205>
- Black, L., Mendick, H., & Solomon, Y. (Eds.). (2011). *Mathematical relationships in education: Identities and participation*. Routledge.
- Boaler, J., & Greeno, J. G. (2000). Identity, Agency, and Knowing. *Multiple perspectives on mathematics teaching and learning*, 1, 171.
- Brown, J. S., Collins, A., & Duguid, P. (1989) Situated cognition and the culture of learning. *Educational researcher*, 18(1), 32-42. <https://doi.org/10.21236/ada204690>
- Cobb, P., & Hodge, L. L. (2002). A relational perspective on issues of cultural diversity and equity as they play out in the mathematics classroom. *Mathematical thinking and learning*, 4(2-3), 249-284. [https://doi.org/10.1207/s15327833mtl04023\\_7](https://doi.org/10.1207/s15327833mtl04023_7)
- Jessup, N.A., Wolfe, J. A., Udiani, O., & Kalinec-Craig, C. A. (2020) Pandemics and issues of equity. *Mathematics Teacher: Learning and Teaching PK-12*, 113(10), e58-e59. <https://doi.org/10.5951/mtlt.2020.0154>
- Lakoff, G., & Nuñez, R. (2000). *Where mathematics comes from* (Vol. 6). Basic Books. <https://doi.org/0465037704fm.qxd>
- Martin, D. B. (2000). *Mathematics success and failure among African-American youth: The roles of sociohistorical context, community forces, school influence, and individual agency*. Routledge. <https://doi.org/10.4324/9781410604866>
- Miles, M. B., Huberman, A. M., & Saldaña, J. (2020). *Qualitative data analysis: A methods sourcebook*. Sage publications.
- Nasir, N. I. S. (2002) Identity, goals and learning: Mathematics in cultural practice. *Mathematical thinking and learning*, 4(2-3), 213-247. [https://doi.org/10.1207/s15327833mt104023\\_6](https://doi.org/10.1207/s15327833mt104023_6)
- Nasir, N. I. S. & Hand, V. (2008). From the court to the classroom: Opportunities for engagement, learning, and identity in basketball and classroom mathematics. *The Journal of the Learning Sciences*, 17(2), 143-149. <https://doi.org/10.1080/10508400801986108>
- Rogers, L. O., Zosuls, K. M., Halim, M. L., Ruble, D., Hughes, D., & Fuligni, A. (2012). Meaning making in middle childhood: An exploration of the meaning of ethnic identity. *Cultural Diversity and Ethnic Minority Psychology*, 18(2), 99. <https://doi.org/10.1037/a0027691>
- Ruef, J. (2021). How Ms. Mayen and her students co-constructed good-at-math. *Journal for Research in Mathematics Education*, 52(2), 152-188. <https://doi.org/10.5951/jresmetheduc-2020-0264>
- Sabol, T. J., Busby, A. K. & Hernandez, M. W. (2021) A critical gap in early childhood policies: Children's meaning making. *Translational Issues in Psychological Science*. <https://doi.org/10.1037/tps0000241>
- Vygotsky, L. (1978). *Mind in society: The development of higher psychological processes*. Harvard University Press.
- Wenger, E. (1998). *Communities of practice: Learning, meaning, and identity*. Cambridge University Press.

## MEME-INGFUL MATHEMATICS: CRAFTING CRITICAL CONVERSATIONS ABOUT MATHEMATICS THROUGH STUDENT MEME ANALYSIS

Gregory Beaudine  
University of Iowa  
Gregory-  
beaudine@uiowa.edu

Gregory Benoit  
Boston University  
Gbeniot1@bu.edu

K. Elizabeth Hammonds  
Columbus State University  
hammonds\_katherine1@colu  
mb.usstate.edu

Our view of self is built upon our understanding of the world and its history, much of which is crafted by interactions beyond the classroom, including “the Internet, films, newspapers, television programs, novels, and strip cartoons” (Joseph, 2011, p.1). Through these interactions, society crafts conceptions about those who enjoy the study of mathematics. Hersh and John-Steiner (2011) outline four common myths about mathematics; mathematicians “lack emotional complexity” (p. 2), the study of mathematics “is a solitary pursuit” (p. 4), the great mathematicians are young and male, and that “mathematics is an effective filter for higher education” (p. 6). More generally, our society tends to view those who do well in mathematics as gifted, smart, nerdy, or crazy (Boaler, 2022; Hall & Surtamm, 2020; Epstein et al., 2010).

These same messages about mathematics are replicated throughout social media by way of the creation and distribution of memes - “small cultural units of transmission that flow from person to person by copying or imitation” (Gal et al., 2016, p. 1700). The more popular or relatable the message, the more swiftly it is copied within each online echo chamber or ecosystem (Gleeson et al., 2014). Through their study, Bini et al. (2020) established that mathematical memes elicit ideas about mathematics and create openings for the collective meaning-making of the mathematical idea encoded in the meme.

### Critical Conversations about Mathematics

The students at Wakanda High School (Benoit et al., in press) were asked to identify and submit a mathematics meme or graphic interchange format (GIF) which they engaged with through social media and present these images to the class. This critical conversation about mathematics was intended to prompt students’ open reflection of their experiences, both past and present, surrounding mathematics, and explore messages and stereotypes presented through their chosen images. The images chosen ranged from a system of equations utilizing fast food meals to an animated GIF of a lemur falling asleep in math class. Through this reflective activity, students engaged in critical conversations about societal mathematical messages, hypothesized about the author’s intentions, and examined the perspectives of creator and consumer.

Preservice teachers at Stark University explored memes in a slightly different format. They engage in a series of conversations aimed to counter their mathematical stereotypes. Through this process, mathematical memes were routinely used as “warmup” in their Numbers and Operations course. A primary goal for this activity was levity and laughter, as students had previously identified math as a difficult or unpleasant discipline. The activity also allowed students the reflective space to compare and contrast more enjoyable experiences with mathematics against some of their Day 1 pre-conceptions about mathematics and mathematicians.

### Conclusion

Education requires conversation and conversations lead to a shared understanding. In both discussions, above, students were afforded the opportunity to explore memes presented in the

classroom. As shown, mathematical memes have the potential to activate and drive mathematical discussions which can potentially fuel an understanding of and interest in mathematics.

### References

- Benoit, G., Hammonds, K. E., Beaudine, G., & Salopek, G. (In Press). Using Memes to Open Spaces for Critical Conversations about Mathematics.
- Bini, G., Robutti, O., & Bikner-Ahsbals, A. (2020). Maths in the time of social media: Conceptualizing the internet phenomenon of mathematical memes. *International Journal of Mathematical Education in Science and Technology*: 1-39. DOI: 10.1080/0020739X.2020.1807069
- Boaler, J. (2022). *Mathematical mindsets: Unleashing students' potential through creative mathematics, inspiring messages and innovative teaching*. John Wiley & Sons.
- Epstein, D., Mendick, H., & Moreau, M.-P. (2010). Imagining the mathematician: young people talking about popular representations of maths. *Discourse (Abingdon, England)*, 31(1), 45–60. DOI: 10.1080/01596300903465419
- Gal, N., Shifman, L., and Kampf, Z. (2016). “It gets better”: Internet memes and the construction of collective identity. *New Media & Society*, 18(8). DOI: 10.1177/1461444814568784
- Gleeson, J. P., Ward, J. A., O’Sullivan, K. P., and Lee, W. T. (2014). Competition-induced criticality in a model of meme popularity. *Physical Review Letters*, 112. DOI: 10.1103/PhysRevLett.112.048701
- Hall, J., & Surrattam, C. (2020). Numbers and nerds: Exploring portrayals of mathematics and mathematicians in children’s media. *International Electronic Journal of Mathematics Education*, 15(3). 1306-3030. DOI: 10.29333/iejme/8260.
- Hersh, R. & John-Steiner, V. (2011). *Loving + Hating Mathematics: Challenging the Myths of Mathematical Life*. Princeton University Press.
- Joseph, G. G. (2011). *The Crest of the Peacock: Non-European Roots of Mathematics*, Third Edition. Princeton University Press.

# MODEL ELICITING ACTIVITIES AS A POWERFUL TOOL TO EXPAND COLLABORATIVE WORK OPPORTUNITIES FOR UNDERGRADUATE STUDENTS

Luis E. Montero-Moguel  
The University of Texas at San Antonio  
luis.monteromoguel@utsa.edu

Guadalupe Carmona  
The University of Texas at San Antonio  
guadalupe.carmona@utsa.edu

Verónica Vargas-Alejo  
University of Guadalajara  
veronica.vargas@academicos.udg.mx

Dinorah Méndez Huerta  
University of Guadalajara  
dinorah.mendez7682@alumnos.udg.mx

*This research contributes to the need to identify and expand learning environments that encourage undergraduate students to develop collaborative work skills and apply their classroom knowledge to solve real-world problems. Using qualitative methods, we examine the effects of the interaction between two teams of students when solving a Model-eliciting activity, based on the theoretical framework of the Models and Modeling Perspective. Our analysis shows that the students had two opportunities for interaction, within the team and between the modeling teams. Through these interactions, the students refined their models in three directions: mathematical knowledge, interpretation and modeling of the phenomenon, and mathematical representations. The findings of this study emphasize the significant value of Model-eliciting activities in enhancing students' collaboration and modeling abilities.*

Keywords: Modeling, Undergraduate education, Precalculus, Model eliciting activities.

## Introduction

Learning opportunities for higher education should help students develop, in-depth knowledge of the disciplines, as well as skills that allow them to work collaboratively to solve the problems of 21st century (National Academies of Sciences, Engineering, and Medicine, NASEM, 2018a; National Science Foundation, 2016). In addition, collaborative work favors the cognitive growth of students (NASEM, 2018b). However, employers recognize that students graduating from higher education lack communication and teamwork skills (Felder, 2021). Thus, it is necessary to broaden research around the search for learning environments that encourage the collaborative work of undergraduate students to allow them to deepen their knowledge of the disciplines in contexts close to those they will face in their professional lives.

One of the ways to address the preparation of undergraduate students in real-life contexts is through mathematical modeling because “it connects mathematics to real-world questions through the cyclic process of receiving a non-mathematized task, applying mathematics to better understand the phenomenon and then circling back to the real-world questions to arrive at solutions” (Hjalmarsen et al., 2020, p. 225). This connection between mathematics and real-life situations is central, the mathematical modeling activities should propose that students do not focus on modeling the data, but that the modeling activities encourage students to delve into the interpretation and modeling of the phenomenon (Stillman & Brown, 2021). Zawojewski et al. (2003) point out that groups of elementary students who participate in model-eliciting activities (MEA) “learn that together they are quite powerful, that they need to listen to each other, and that they can indeed solve these complex problems, they develop a clearer, more stable idea of how modeling activities can be accomplished” (p. 357). However, it remains uncertain whether these

types of interactions are similarly generated within undergraduate student modeling teams. We believe that the research by Zawojewski et al. (2003) should be extended to understand if the MEAs contribute in the same sense in undergraduate students.

Given the relevance of providing undergraduate students with skills that allow them to collaboratively solve real-life problems, this qualitative study aims to describe the effect generated in the modeling process by the interaction of two teams of undergraduate business students while participating in a modeling activity. The research questions that guided this study were, (1) to what extent does the interaction between the student modeling teams favor the modification of their initial models when solving a MEA? and (2) what opportunities for interaction does the MEA generate?

### **Theoretical Framework**

This study was based on the Models and Modeling perspective (MMP). According to the MMP, students work individually or in teams to build models that allow them to describe, explain, and predict intentionally designed situations in real contexts (Ärlebäck & Doerr, 2018). According to the MMP, the models are conceptual systems (consisting of elements, relations, operations, and rules governing interactions) that are expressed using external notation systems, and that are used to construct, describe, or explain the behaviors of other system(s)—perhaps so that the other system can be manipulated or predicted intelligently.

A mathematical model focuses on structural characteristics (rather than, for example, physical or musical characteristics) of the relevant systems (Lesh & Doerr, 2003, p. 10). These intentionally constructed situations are called MEAs, they are built under six principles: personal meaningfulness or reality principle, model construction, self-evaluation, model externalization or model-documentation, simple prototype, and model generalization (Lesh et al., 2000). MEAs encourage students to “develop modeling cycles, also called iterative modeling cycles, which are interpretations (or conceptual systems) of problem situations” (Montero-Moguel & Vargas-Alejo, 2022, p. 218). When students solve the MEAs in small groups (between two and three students), they face complex problems that require multiple cycles of interpretation and refinement, collaborative work and communication are a fundamental part for students to expand their conceptual system (Zawojewski & Carmona, 2001). The conceptual system tends to be expressed externally through a variety of representations, for example, written symbols, graphs, spoken language, diagrams, tables, and equations (Lesh & Doerr, 2003). The collaborative work of elementary students when solving MEA encourages them to “learn the power of representing their ideas externally (e.g., drawing a picture) to communicate with their peers, and the power of taking different points of view as they experience the process of comparing, contrasting, and reconciling their initial interpretations” (Zawojewski et al., 2003, p. 341).

### **Methods**

This qualitative study involved two teams of three freshman undergraduate business students at a university in Mexico. Both teams worked in the same classroom and had access to computers. The students were studying the subject of mathematics applied to business. Prior to the study, the students had not addressed the issue of functions during their professional career. For this study, we constructed the MEA called “Yogurt is breaking investment barriers” (Yogurt MEA), based on the six principles of MEA construction (Lesh et al., 2000). Some big math ideas built into the design of Yogurt MEA are: exponential function, logistic function, variation, growth, and rate of change. The MEA was divided into three parts, the first a journalistic note in the context of health care, the consumption of healthy yogurt and the high profits of yogurt companies. The second section contained questions about the context of the journalistic note. The third part, the problem



situation, included a video ([https://youtu.be/4y6V\\_6Hff\\_E](https://youtu.be/4y6V_6Hff_E)) containing images of seven days of culturing yogurt kefir grains, the images showing the number of yogurt kefir grains growth and the temperature on each day. Under the context of the creation of a yogurt company, the MEA encourages students to build models to describe the growth of yogurt kefir grains that allow them to start the production of a Yogurt company. The implementation was conducted in four phases.

1. Warm-up activity. Reading of the journalistic note, warm-up questions and discussion of the context. The participation in this phase was group.
2. Resolution of the problem situation. Modeling of the Yogurt MEA problem situation. The participation in this phase was in teams.
3. Plenary. Presentation of models of both teams and discussion. The participation in this phase was group.
4. Resolution of the problem situation. Second modeling of the Yogurt MEA problem situation. The participation in this phase was in teams.

Data collected for analysis came from researchers' detailed review of implementation video recordings and transcripts, group discussion, team model presentation, the models built by the students (using Word, Excel and PDF documents), and the researcher's reports. Data was collected during the implementation of the MEA. All students' responses were in Spanish. Thus, translation of select excerpts of the transcripts are made available in English for this manuscript.

In order to identify and compare the models built by the teams, this qualitative study developed a first cycle of *a priori* coding for each team separately from the modeling cycles of the teams. We chose *a priori* coding because it "is appropriate for qualitative studies in disciplines with pre-established and field-tested coding systems if the researcher's goals harmonize with the protocol's outcomes" (Saldaña, 2016, p. 175). In this study, our coding and analysis system is based on the MMP. Since to the relevance of analyzing the students' interpretations, we code the data in three directions: a) *mathematical knowledge* (mathematical concepts used to describe and predict the phenomenon), b) *modeling of the phenomenon* (students focus on modeling the characteristics of the phenomenon not on modeling the data), and c) *representations* (the mathematical representations used to describe the model, e.g., graphs, tables). The results were discussed among the researchers until consensus was reached based on the theoretical framework. For the second cycle of analysis, we analyzed the differences and coincidences between the coded results of both teams. The data was analyzed using the NVivo 12 software, which allowed the association and identification of patterns in the models of each team and between the models of both teams. The results were organized into two modeling cycles, which are presented in the following section.

### **Analysis and discussion**

The analysis focused on two team modeling cycles, one prior to the group discussion and the second after the group discussion. In the first cycle both teams worked independently to build their initial models; subsequently they presented and discussed their models as a group. The analysis allowed us to identify that, in the first cycle, the models of both teams were based on different mathematical concepts, they used different mathematical representations to describe their model, and even the interpretation and control of the phenomenon was different in both models. The analysis of the second cycle allowed us to identify that both teams expanded and refined their models. We identified that the students' models shared coincidences in mathematical concepts, understanding and control of phenomena, and mathematical representations. Both cycles built by the teams are described below.

## First Modeling Cycle

For both teams, the work during the construction process of the initial models was a complex task that required multiple cycles of interpretation; collaborative work within the team was essential to solve the MEA.

**Team 1 Model.** Team 1’s interpretation of the phenomenon led the students to identify that it is relevant that yogurt kefir grains are grown in jars. Based on this interpretation, the students based their model on the growth of yogurt kefir grains in different jars. The students represented in a table the data extracted from the video on the growth of yogurt kefir grains in milliliters and based on this table they built three more. They used the mathematical concept of proportionality to identify the initial and final percentage growth of yogurt kefir grains after one month of production. The students’ model included tabular and verbal representations (Figure 1). During the presentation of their model, Team 1 mentioned the following.

Team 1: Aquí pusimos la primera semana cómo fue creciendo de 40 a 200 y a la segunda semana, para que vean el crecimiento, ya lo separamos el de 200 en cinco frascos y en lugar de hacer 200 al final llega mil. Después continuamos, dividimos los mil entre 4 frascos y salieron 25. Pusimos en cada frasco 40 mililitros y tuvimos un crecimiento de 12 en total. En un mes tuvimos un crecimiento de 625%. [Here we describe in the first week how it grew from 40 to 200. The second week, so that you can see the growth, we separated the 200 into five jars and instead of making 200 at the end the total is one thousand. Then we continued, we divided the thousand between 4 bottles and 25 came out. We put 40 milliliters in each bottle, and we had a growth of 12 in total. In one month, we had a growth of 625%].

Kéfir Healthy  
Presente

Por medio de la presente, le comunico los resultados obtenidos de la investigación realizada al comportamiento de crecimiento de los búlgaros. Como mostramos a continuación tuvimos un crecimiento del 625% en 4 semanas. También es importante mencionar que la temperatura se mantuvo en un promedio de 20°C.

Semana 1				Semana 2			
dia	frascos	mililitros	Total mL	dia	frascos	mililitros	Total mL
1	1	40	40	1	5	40	200
2	1	50	50	2	5	50	250
3	1	70	70	3	5	70	350
4	1	90	90	4	5	90	450
5	1	120	120	5	5	120	600
6	1	150	150	6	5	150	750
7	1	200	200	7	5	200	1000

Semana 3				Semana 4			
dia	frascos	mililitros	Total mL	dia	frascos	mililitros	Total mL
1	25	40	1000	1	125	40	5000
2	25	50	1250	2	125	50	6250
3	25	70	1750	3	125	70	8750
4	25	90	2250	4	125	90	11250
5	25	120	3000	5	125	120	15000
6	25	150	3750	6	125	150	18750
7	25	200	5000	7	125	200	25000

Dependiendo de la cantidad de mililitros de búlgaros necesarios para el proceso del yogurt casero y la cantidad que ustedes necesitan para ofertar, se podría determinar cuando es el mejor momento para empezar con la venta.

**Figure 1: Letter from Team 1 in the first modeling cycle**

**Team 2 Model.** Based on the context, Team 2 identified that setting the temperature value was a factor that improved the growth of dairy yogurt kefir grains. The students’ model was based on

the mathematical concept of the average growth rate. The students built a table containing the days, the amount of yogurt kefir grains and the temperature. With these data they calculated the growth rate that served them to describe and predict the growth of dairy yogurt kefir grains. The students included tabular, graphic, and verbal representations in their model (Figure 2). During the presentation of their model, Team 2 mentioned the following.

Team 2: Sacamos los porcentajes. La variación de porcentaje del día uno al día dos. Entonces me daba un resultado de 20%. Y así sucesivamente. Calculando el porcentaje del crecimiento. [...]. Yo dije mantenemos siempre la temperatura, sacamos un promedio en el porcentaje del crecimiento que me daban a 21 grados y encontramos un promedio de 33.33%. Entonces, si yo mantengo siempre mi temperatura 21 grados porcentualmente más o menos voy a tener un crecimiento del 33%, a lo que del día uno al día cuatro yo iba a tener un crecimiento de 4570 más de mi colonia de bacterias. [We got the percentages. The percentage change from day one to day two. Then it gave me a result of 20%, and so on. Calculating the percentage of growth. [...]. I said we always maintain the temperature; we took the average of the percentage of growth at 21 degrees and we found an average of 33.33%. So, if I always maintain my temperature 21 degrees, I will have a growth of approximately 33%, so from day one to day four I would have a growth of 4570 more for my colony of bacteria].

Para: Corporativo Kéfir Healthy

Por medio de la presente carta, queremos informarle sobre el crecimiento de la cantidad de búlgaros.

Los resultados arrojan que

El Día 1 al Día 2 tenemos un crecimiento del 20% manteniendo una temperatura de 20 grados centígrados.

Del Día 2 al Día 3 se obtiene un crecimiento del 40 % manteniéndolo a temperatura a 21 grados centígrados

Del Día 3 al Día 4 se obtiene un crecimiento del 40 % manteniéndolo a 21 grados centígrados.

DEL Día 4 al Día 5 tenemos de porcentaje del 40% a 30 % con la disminución de temperatura de 20 grados centígrados.

Del Día 5 al Día 6 sigue disminuyendo a 25% manteniendo a 20 grados centígrados.

Del día 6 al Día 7 aumenta a 33% aumentando la temperatura a 21 grados centígrados.

En conclusión, manteniendo la temperatura a 21 grados centígrados tenemos un mejor crecimiento en promedio de 33%.

Si mantenemos la temperatura en 21 grados centígrados del día 1 con 40 ml de búlgaros al día 31 del mes tendremos un crecimiento de 4576 veces más que el día 1



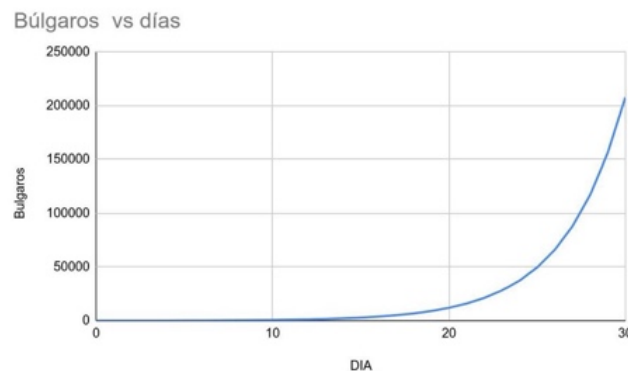
Figure 2: Letter from Team 2 in the first modeling cycle

Observations. In the first cycle, both teams' models described, predicted, and even controlled the growth of dairy yogurt kefir grains. Both teams focused on modeling the phenomenon, however each emphasized different controls to enhance the growth of the yogurt kefir grains. Team 1 focused on controlling the culture in jars and Team 2 on controlling the temperature. The mathematical concepts underlying the models of each team were different. Team 1 used proportionality and Team 2 the average growth rate associated with exponential behavior. Both teams constructed tables and verbally described their models; in addition, Team 2 included a graphic representation.

### Second Modeling Cycle

After the first cycle, the teams presented their models, interacted with questions, and reflected on their models and the other team's models. Subsequently, the teams worked on a second modeling cycle to modify their models from the first cycle.

**Team 1 Model.** Team 1 reflected on the phenomenon, they continued to consider important the cultivation of yogurt kefir grains in jars and added in their interpretation the importance of controlling the temperature. To describe and predict the growth of yogurt kefir grains, Team 1 calculated the average growth rate for the first seven days (33%) and used this rate to exponentially describe the growth of yogurt kefir grains. Team 1, which had initially included verbal tabular representations in its initial model, added graphical and algebraic representations to its second model (Figure 3). This modification coincides with the findings of Lesh and Doerr (2003), regarding the fact that students generate modifications in their representations during the modeling process.



Esta gráfica muestra un crecimiento exponencial de los búlgaros, tomando en cuenta el promedio de 33% que crecen día con día.

Sin más por el momento, quedamos a sus ordenes.

**Figure 3: Graphic included in the second model of Team 1**

**Team 2 Model.** Team 2 maintained the idea of controlling the phenomenon of the growth of yogurt kefir grains by setting the controlled temperature, also included descriptions to denote that the culture would be carried out in jars to favor the growth of the bacterium. They argued that since the culture would be in jars, then they could remove half of the yogurt kefir grains on day five to improve growth. This consideration impacted their mathematical analysis, although the model continued to be based on the average growth rate analysis, the team described growth before five days and after those five days considering an average growth rate of 33% for both. Students

included tabular, verbal, and graphic representations. Team 2 described the following in their second model:

Team 2: Pero si el día 5 tomamos la mitad de los búlgaros y los colocamos en otro frasco y empezamos a crear un ambiente favorable para los mismos, al final del mes tendremos una mayor cantidad de la bacteria para iniciar el proceso de producción y tienen un mejor rendimiento en su multiplicación. Que también se pone en etapa de prueba el estar dividiendo en frascos. Y seguir aumentando la cantidad en producción. [But if on the 5th we take half of the yogurt kefir grains and place them in another jar and begin to create a favorable environment for them, at the end of the month we will have a greater number of the bacteria to start the production process and have a better yield in their multiplication. That dividing more jars is also put in the test stage. And keep increasing the quantity in production].

Observations. In the second cycle both teams refined their initial models. The two teams deepened their knowledge of the phenomenon and control of variables. Both teams proposed two ways to control variables, through temperature and culture in jars. Based on the characteristics of the phenomenon, both teams based their models on the calculation of the average daily growth rate associated with the mathematical concept of exponential function. To build their models, the teams used verbal, tabular, and graphical representations.

### Conclusions

In response to the research question: *to what extent does the interaction between the student modeling teams favor the modification of their initial models when solving a MEA?* Our analysis identified the following results. Through the interaction of the teams after their first MEA modeling cycle, both teams refined their models in different directions: mathematical knowledge, interpretation and modeling of the phenomenon, and representations included in the models. *Mathematical knowledge*, Team 1, which initially built a model based on proportionality analysis, refined its model to incorporate similar characteristics to Team 2's model, which used an analysis supported by the average growth rate. *Interpretation and modeling of the phenomenon*, Team 2 initially focused on the growth of yogurt kefir grains considering temperature control but without considering the container, they included in their second model descriptions similar to those of Team 1's model, based on the importance of having bacteria in jars to improve reproduction. *Representations included in the models*, Team 1, which had initially included tabular and verbal representations, included in its second model a graph similar to that of the initial model of Team 1 and additionally included an algebraic representation.

Regarding the second research question: *what opportunities for interaction does the MEA generate?* This study found that the implementation of this MEA gave the students two different levels of interaction, the first is the work *within* the modeling team and the second, the interaction *between* modeling teams. At the first level of interaction, it is noteworthy how in the process of solving the MEA, the interaction between the members of each team, their mathematical knowledge and their interpretation of the phenomenon allowed the development of an iterative process among the team members to face a complex problem. The students' model allowed them to describe, predict, and interpret the MEA phenomenon with the customer in mind. Our findings align with the research conducted by Zawojewski et al. (2003) at the elementary level, because, undergraduate students can learn that teamwork allows them to compare, contrast, and extend their initial interpretations. Additionally, we concur with Zawojewski and Carmona (2001) and Zawojewski et al. (2003) regarding that when students work in small teams, they recognize that

working together allows them to face complex problems. Moreover, our findings support the perspective put forth by NASEN (2018b) that students' cognitive growth is fostered through collaborative work. At the second level of interaction, our findings reveal that the students did not discard their initial models, but rather, the interaction with the other team allowed them to expand and refine their models. The MEA, therefore, provides students with the opportunity to share, evaluate, reflect, and modify their own proposals based on interaction with a team with different ideas.

These findings contribute to the needs identified by NASEM (2018a) regarding generating learning environments where undergraduate students improve their collaborative work skills as they deepen the concepts of the disciplines, with the aim of facing the real problems of our century. Future research could explore how to integrate these findings into teaching practices to improve undergraduate students' collaborative work skills and problem-solving abilities.

### References

- Ärleback, J. B., & Doerr, H. M. (2018). Students' interpretations and reasoning about phenomena with negative rates of change throughout a model development sequence. *ZDM*, *50*(1), 187–200. <https://doi.org/10.1007/s11858-017-0881-5>
- Felder, R. M. (2021). STEM education: A tale of two paradigms. *Journal of Food Science Education*, *20*(1), 8–15. <https://doi.org/10.1111/1541-4329.12219>
- Hjalmarson, M. A., Holincheck, N., Baker, C. K., & Galanti, T. M. (2020). Learning models and modeling across the STEM disciplines. In C. C. Johnson, M. J. Mohr-Schroeder, T. J. Moore, & L. D. English (Eds.), *Handbook of research on STEM education* (pp. 223–233). Routledge.
- Lesh, R., & Doerr, H. M. (2003). Foundations of a Models and modeling perspective on Mathematics teaching, learning, and problem solving. In R. Lesh & H. M. Doerr (Eds.), *Beyond constructivism: A models & modelling perspective on mathematics problem solving, learning & teaching* (pp. 3–34). Lawrence Erlbaum Associates
- Lesh, R., Hoover, M., Hole, B., Kelly, A., & Post, T. (2000). Principles for developing thought revealing activities for students and teachers. In M. Hoover & R. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 591–645). Routledge.
- Montero-Moguel, L. E., & Vargas-Alejo, V. (2022). Ciclos de modelación y razonamiento covariacional al realizar una actividad provocadora de modelos. *Educación Matemática*, *34*(1), 214–248. <https://doi.org/10.24844/EM3401.08>
- National Academies of Sciences, Engineering, and Medicine. (2018a). *The Integration of the humanities and arts with sciences, engineering, and medicine in higher education: Branches from the same tree* (D. Skorton & A. Bear, Eds.; p. 24988). National Academies Press.
- National Academies of Sciences, Engineering, and Medicine. (2018b). *How People Learn II: Learners, contexts, and cultures*. National Academies Press.
- National Science Foundation. (2016). *STEM 2026: A vision for innovation in STEM education*. US Department of Education. <https://www.voced.edu.au/content/ngv:75381>
- Saldaña, J. (2016). *The coding manual for qualitative researchers* (3rd Ed.). Sage.
- Stillman, G. A., & Brown, J. P. (2021). Modeling the phenomenon versus modeling the data set. *Mathematical Thinking and Learning*, *0*(0), 1–26. <https://doi.org/10.1080/10986065.2021.2013144>
- Zawojewski, J., & Carmona, G. (2001). A developmental and social perspective on problem solving strategies. In R. Speiser & C. Walter (Eds.) *Proceedings of the 23<sup>rd</sup> Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 549-60). ERIC Clearinghouse for Science, Mathematics, and Environmental Education.
- Zawojewski, J., Lesh, R., & English, L. (2003). A models and modeling perspective on the role of small group learning activities. In R. Lesh & H. M. Doerr (Eds.), *Beyond constructivism: A models y modelling perspective on mathematics problem solving, learning y teaching* (pp. 337–358). Lawrence Erlbaum Associates.

## MODELS CONSTRUCTED IN THE CONTEXT OF REFORESTATION

Luis Antonio Muñoz González  
Universidad de Guadalajara  
luis.munoz7674@alumnos.udg.mx

Verónica Vargas-Alejo  
Universidad de Guadalajara  
veronica.vargas@academicos.udg.mx

Keywords: Model Eliciting Activity, Models, Reforestation

### Introduction

This poster describes the results of implementing a Model-Eliciting Activity [MEA], which is a task specifically crafted to simulate a real-life scenario within the context of reforestation in a Mexican community. The research question addressed in this study was: What models did a postgraduate student construct to solve the reforestation MEA? Two key mathematical concepts involved in the task were linear function and proportions.

### Theoretical framework

The theoretical framework used in this study was based on Models and Modeling Perspective (MMP) (Makar et al., 2020). The MMP proposes that students solve MEAs to learn mathematics. MEAs “are similar to many real life situations in which mathematics is useful” (Lesh & Doerr, 2003, p. 4). The students create, modify, and refine models when they solve MEAs. The MMP defines a model as: “a system for describing some other system for some specific purpose” (Lesh, 2010, p. 18). Researchers engage students in solving MEAs so as to elicit their ways of thinking and promote the development of mathematical knowledge when the students solve them (Lesh, 2010) because: “the goal is to produce powerful conceptual tools (or conceptual systems) for constructing, describing, explaining, manipulating, predicting, or controlling complex systems” (Lesh & Doerr, 2003, p. 212).

### Methodology

This research was qualitative. A postgraduate student, who was enrolled in a master’s degree program in mathematics teaching during his first semester, participated in solving the Reforestation MEA designed by Vargas-Alejo & Cristóbal-Escalante (2017) in accordance with the six principles of the MEA (Lesh et al., 2000). The scenario is related to a reforestation project undertaken by a community in Michoacán to preserve their land and improve their economy. The MEA was implemented in three phases: 1) homework, 2) presentation and discussion, and 3) homework. First, the student created a model as homework; second, he discussed the model with a researcher; and finally, he modified his model.

### Results and conclusions

The results indicate that the postgraduate student constructed two models. The first model was distorted as the student’s approach did not address the posed question making it difficult to observe the proportionality and linear function concepts. In the second model, the student considered more established conditions within the problem. The student deepened the context and it served to self-assess. Although he found a diversity of solutions that he presented to the client for consideration and selection, he did not use all the recommendations or relationships immersed in the problem. This experience allowed the postgraduate student to learn about modeling and to understand how a MEA requires several “modeling cycles” (Lesh, 2010). Also, he recognized that communication was essential to modify and extend the meanings.

### References

Lesh, R. (2010). Tools, Researchable Issues & Conjectures for Investigating What it Means to Understand Statistics (or Other Topics) Meaningfully. *Journal of Mathematical Modelling and Application*, 1(2), 16-48

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

- Lesh, R., Hoover, M., Hole, B., Kelly, A., & Post, T. (2000). Principles for Developing Thought Revealing Activities for Students and Teachers. In A. Kelly & R. Lesh (Eds.), *Research Design in Mathematics and Science Education* (pp. 591-646). Mahwah, New Jersey: Lawrence Erlbaum Associates.
- Lesh, R. & Doerr, H. (2003). *Beyond Constructivism: Models and Modeling Perspectives on Mathematics Problem Solving, Learning and Teaching*. Mahwah, NJ: Lawrence Erlbaum Associates.
- Makar, K., Doerr, H., y DelMas, R. (2020). Developing statistical modeling with paper helicopters. *Mathematics Teacher: Learning and Teaching Pk-12*, 113(2), 147-151.
- Vargas-Alejo, V. & Cristóbal-Escalante, C. (2017). MEA Propuesta de Reforestación. Retrieved on December 10, 2017 from <https://plus.google.com/u/0/communities/100320204402382732939>



# POST-SECONDARY MULTILINGUAL INTERNATIONAL STUDENTS' EXPERIENCES IN MATHEMATICS

Jihye Hwang  
Michigan State University  
hwangji7@msu.edu

Keywords: Multilingual, international students, post-secondary, gender, narrative inquiry

The number of multilingual international students has been significant in post-secondary STEM education (Granovskiy & Wilson, 2019), which is similar in mathematics education. Even though their significant number in the U.S. post-secondary mathematics education, multilingual international students' experiences have not been focused on at the post-secondary level. Even though mathematics educators have studied multilingual students' experiences at the K-12 level, some differences exist between K-12 and post-secondary mathematics education. For example, the common contexts of K-12 mathematics education research regarding multilingual students are more about the bilingual setting, such as English and Spanish (Domínguez, 2011; Morales & DiNapoli, 2018) or Spanish and Catalan (Planas & Civil, 2015; Planas & Setati, 2009). In contrast, post-secondary mathematics classrooms are more linguistically diverse. As language is closely related to one's identity and access to their learning, this population needs more attention to serve them in a more equitable learning environment.

With the goal of understanding their experiences, I present four multilingual post-secondary international students' mathematics identities in this poster. By mathematics identity, I mean how one understands their relationships with mathematics, how those relationships are constructed across time and space, and how one understands their future possibilities in mathematics by adapting Norton's (1997) notion of identity.

## Methods

This study had four participants: Jia—a Chinese; Helena—a Colombian; Jihyun—a Korean; and Maria—a Mexican (all pseudonyms). The four participants' narratives were constructed using narrative inquiry as a research method (Connelly & Clandinin, 1990; Polkinghorne, 1995). Data was collected through synchronous and email interviews. An email interview option was provided, considering English is neither the participants' nor the researcher's first language. The narratives were constructed through a narrative analysis, member-checked, and revised until both agreed.

## Results and Discussion

The four students shared the challenges they experienced by being multilingual and international students in post-secondary mathematics education, including what their first language meant to them and how it felt to study mathematics in a second language and change their behaviors. They also explain how they use their first and second languages in studying mathematics, implying that both languages are important sources of meaning (Barwell, 2018). In addition to the linguistic aspect, they also shared how their racial and gender identity influenced their mathematics identities, which connects to intersectionality (Hsieh et al., 2021; Leyva, 2016). As all participants were women, their gender identity significantly influenced their experiences and mathematics identity. This study provides a way of understanding what it can mean to be a multilingual international woman in their mathematics education by looking at some of their previous and current experiences.

## References

- Barwell, R. (2018). From language as a resource to sources of meaning in multilingual mathematics classrooms. *Journal of Mathematical Behavior*, 50(August 2017), 155–168. <https://doi.org/10.1016/j.jmathb.2018.02.007>
- Connelly, F. M., & Clandinin, D. J. (1990). Stories of experience and narrative Inquiry. *Educational Researcher*, 19(5), 2–14. <https://doi.org/10.3102/0013189X019005002>
- Domínguez, H. (2011). Using what matters to students in bilingual mathematics problems. *Educational Studies in Mathematics*, 76(3), 305–328. <https://doi.org/10.1007/s10649-010-9284-z>
- Granovskiy, B., & Wilson, J. H. (2019). *Foreign students in the United States (CRS Report No. IF11347)*. <https://doi.org/10.1080/00193089.1974.10533501>
- Hsieh, T. Yang, Simpkins, S. D., & Eccles, J. S. (2021). Gender by racial/ethnic intersectionality in the patterns of Adolescents' math motivation and their math achievement and engagement. *Contemporary Educational Psychology*, 66(May). <https://doi.org/10.1016/j.cedpsych.2021.101974>
- Leyva, L. A. (2016). An intersectional analysis of Latin@ college women's counter-stories in mathematics. *Journal of Urban Mathematics Education*, 9(2), 81–121.
- Morales, H., & DiNapoli, J. (2018). Latinx Bilingual Students' Perseverance on a Mathematical Task: A Rehumanizing Perspective. *Journal of Research in Mathematics Education*, 7(3), 226. <https://doi.org/10.17583/redimat.2018.3274>
- Norton, B. (1997). Language, identity, and the ownership of English. *TESOL Quarterly*, 31(3), 409–429.
- Planas, N., & Civil, M. (2015). Bilingual mathematics teachers and learners: the challenge of alternative worlds. In K. Beswick, T. Muir, & J. Wells (Eds.), *Proceedings of 39th Psychology of Mathematics Education Conferences* (Vol. 4, pp. 41–48). PME.
- Planas, N., & Setati, M. (2009). Bilingual students using their languages in the learning of mathematics. *Mathematics Education Research Journal*, 21(3), 36–59. <https://doi.org/10.1007/BF03217552>
- Polkinghorne, D. E. (1995). Narrative configuration in qualitative analysis. *International Journal of Qualitative Studies in Education*, 8(1), 5–23. <https://doi.org/10.1080/0951839950080103>

## “OH! THAT’S INTERESTING!”: CAPTIVATING STUDENTS WHO HATE MATHEMATICS WITH MATHEMATICAL IDEAS

Brit Claiborne  
Boston University  
bcn@bu.edu

Alexandra Videlock Barstow  
Boston University  
abarstow@bu.edu

Leslie Dietiker  
Boston University  
dietiker@bu.edu

*The positive relationship between students’ attitudes toward mathematics and mathematics achievement is well documented. Yet there is a worsening problem of low appeal of mathematics especially at the secondary level. Therefore, in this paper we focus on three high school students who report a strong dislike of mathematics. By analyzing student surveys, interviews, and lesson observation data, we examined how some mathematical lessons improved these students’ experiences (i.e., their aesthetic dimensions). We found that while student preferences varied, each student was interested in lessons that centered them as sense-makers and in which the content unfolded with suspense. Such lessons led to positive aesthetic responses such as surprise, curiosity, and satisfaction. Thus, lessons can be designed in which even students with the most negative views of mathematics can find mathematical concepts interesting.*

Keywords: Affect, Emotion, Beliefs, and Attitudes; Instructional Activities and Practices; Curriculum; High School Education.

There is widespread agreement that lessons that are *captivating* (i.e., those that students find interesting and engaging) increase student learning. For example, across all cycles of the TIMSS Assessment since 1995, there has been a consistent relationship between students liking mathematics and increased mathematics achievement (Mullis et al., 2020). Unfortunately, in the U.S., there is a persistent problem with low appeal of mathematics, especially at the secondary level. For example, by the time U.S. fourth graders in 2015 reached eighth grade, the percentage of those who did not like learning mathematics nearly doubled to 45% (Mullis et al., 2016, 2020). This trend of secondary students disliking mathematics is also becoming worse over time; between 2011 and 2019, the portion of U.S. eighth grade students who do not like learning mathematics increased by five percent (Mullis et al., 2012). This phenomenon calls for an urgent focus on this growing collection of students who do not enjoy mathematics, and for an attempt to connect the types of learning experiences offered in their mathematics classrooms with their *affect*, which we draw from Middleton and colleagues (2017) to define as a students’ attitudes, interests, beliefs, and emotions.

Luckily, student affective reactions to mathematical experiences are not fixed and are responsive to the teacher and learning environment. Student interest in a lesson, for example, can be piqued even if the student does not generally find mathematics interesting (Hidi & Renninger, 2006; Middleton et al., 2017). The same is true for other aspects of affect, such as their emotions and engagement (Goldin, 2017; Hannula et al., 2016; Satyam, 2020). Although long-term attitudes, beliefs, and emotional views of mathematics may be relatively stable without much day-to-day variation (Middleton et al., 2017), an accumulation of new types of experiences can eventually lead to shifts of a student’s overall affective traits (Hidi & Renninger, 2006; Marmur, 2019; Middleton et al., 2017). This gives each lesson power to shape a student’s general disposition towards mathematics—and by extension, their future achievement.

Thus, we're interested in how mathematical experiences potentially impact a student's emotional reaction, what we refer to as the *aesthetic dimension* of a mathematical learning experience. In this study, we analyze the experiences of three students—Cassie, Mateo, and Mei

(pseudonyms)—who each have a strong and stable dislike of mathematics. Despite this mathematical disposition, however, each student found at least one observed mathematics lesson interesting. We analyze their aesthetic responses to these lessons and describe the curricular designs that enabled these positive student experiences. By connecting the mathematical content and lesson design with student affect and interest, we begin to answer the questions: *What aspects of a lesson captivate students who particularly dislike mathematics? What can we learn about the designs of mathematics lessons from these students?*

### **Theoretical Framework**

To connect a student's mathematical experience and their emotional reaction, we draw from theoretical perspectives that attend to the aesthetic dimensions of experiences. Drawing from Dewey (1934) and Sinclair (2001, 2008), we interpret the aesthetic dimension of an experience not as a characteristic of the experience itself, but rather as the way the experience emotionally impacts an individual. Thus, it is the interaction of an individual and the experience (e.g., a theatrical play or mathematics lesson). Although the notion of aesthetic is often associated with artistic beauty, we instead contend that *all* experiences, both positive and negative, provide opportunities for aesthetic impulses which can compel an individual to act (Dietiker, 2015). For example, in a surprising moment, an individual might be compelled to wonder or question. On the other hand, when an experience is uninspiring, this individual might instead feel bored and be compelled to shift their attention to something else. Other aesthetic experiences, particularly in mathematics classrooms, can include a sense of frustration when things persistently don't make sense and the individual does not have an indication that this will change, as well as satisfaction when an individual shifts from confusion to clarity (i.e., when it feels like the pieces are coming together to make a coherent whole). Thus, the aesthetic dimensions of a mathematical experience can motivate an individual to persevere through difficulty or can, unfortunately, lead them to quit in frustration.

### **Methods**

This study is part of a three-year design-based research project, called the *Mathematically Captivating Learning Experience (MCLE) Project*, which focused on better understanding the aesthetic dimensions of high school mathematics lessons. In MCLE, six veteran high school mathematics teachers designed and taught lessons specifically to increase positive aesthetic opportunities such as surprise or suspense. In order to learn about student aesthetic experiences, all participating students were surveyed after each lesson about their experiences, and a subset of students were selected for a post-lesson interview. In addition, to learn whether these MCLE lessons led to improved aesthetic experiences, two to four additional "everyday" lessons were observed, and students were surveyed using the same protocols so that students would not know which lessons were specially designed for the project. Finally, to learn whether the improvement of the aesthetic experiences only occurred for certain students (for example, those who already like mathematics), a dispositional survey was given at the start of the year.

In the present case study, which draws from the 2018-2019 cycle, we analyze three participating high school students who especially dislike mathematics but who also found at least one of their mathematics lessons interesting. We draw from the post-lesson survey and interview data to describe the aesthetic experiences of three students who expressed strong dislike of mathematics in their Disposition Survey. In the rest of this section, we describe the surveys and

interviews used for the present study, explain how we identified the three focal students, and finally describe our data analysis.

### **Data Collection**

At the start of each academic year, all participating students took a Disposition Survey, which took approximately 20 minutes to complete. This survey asked Likert questions on the extent to which they agree or disagree with statements about views of mathematics (e.g., “I like mathematics”), their views of activities related to mathematics (“I like to think about math or solve puzzles outside of school.”), and their experiences in mathematics classrooms (e.g., “my teacher makes lessons interesting”). It also included an open-response question that asked students to describe how they typically feel during mathematics lessons. The development and testing of the Disposition Survey is described in Riling et al. (2018).

Researchers observed approximately three MCLE lessons and three everyday lessons. For each observation, researchers took field notes. Video cameras were positioned from different angles of the classroom to capture all interactions and student emotional expressions. In addition, audio-recorders were placed around the classroom to capture student and teacher discourse.

After each lesson observation, all student participants were given a Lesson Experience Survey (LES, Riling et al., 2019) which prompts students to select three terms that best describe their view of the lesson from a list of 16 positive, neutral, and negative descriptors (e.g., fascinating, fine, dull). These descriptors were generated by asking high school students questions like, “If you wanted to describe an experience that made you [emotional response, e.g., bored], what word would you use?” The LES also contains Likert questions to measure the degree to which they understood the content (from 1=don’t understand, 2=somewhat understand, and 3=understand), were interested in the lesson (from 1=very bored to 4=very interested), as well as multiple four-point questions (1=strongly disagree to 4=strongly agree) on whether they wished more lessons were like this one, felt the content was relevant to their lives, found the content challenging, or felt that the lesson was typical.

In addition, to better understand student experiences, a small number of students (between two and four) were individually interviewed. These students were selected to reflect a variety of perspectives (e.g., engaged, disengaged), while seeking representation across gender, racial, and ethnic identities. During these interviews, students were asked to describe and explain any feelings they had during the lesson and were offered a list of descriptors to choose from.

### **Participants**

From more than 200 participating students of MCLE during the 2018-2019 cycle of the design-based research, we reduced the pool by examining Disposition Survey responses and identifying those students with (a) no positive responses about mathematics and (b) at least three strongly negative responses. We eliminated those with fewer than three completed LESs and any with inconsistent responses with regard to views of mathematics. Of the 12 remaining students, we sought those who provided evidence of interest in at least one lesson and disinterest in at least one lesson. This selection led us to our three cases: Cassie, Mateo, and Mei (see Table 1). These students each had different teachers (Ms. Cherry, Mr. Ash, and Ms. Willow, all pseudonyms).

Across the students, there were differences in grade, gender identity, race, and ethnicity. However, all three students are multilingual and were in Algebra 2.

**Table 1: Student Demographic Information and Identifications**

	School	Teacher	Grade	Gender Identity	Race/Ethnicity
Cassie	B	Ms. Cherry	10	Female	Black or African American
Mateo	A	Mr. Ash	11	Male	Hispanic/Latino
Mei	A	Ms. Willow	10	Female	White, Asian

### Data Analysis

We conducted an inductive qualitative analysis (Merriam & Tisdell, 2015) of the Disposition and LES surveys, post-lesson interviews, and lesson observation data. We first built disposition profiles of each student using the responses on the Disposition Survey, writing analytic memos describing their attitudes about mathematics and mathematics class, past experiences, class activity preferences, and beliefs about themselves as mathematics learners. We then examined their LES responses, reading closely and constantly comparing their lesson experiences with their disposition. We analyzed within a single lesson and across each student’s four or five LESs, noting both patterns and outliers in their responses. To triangulate, we also reviewed transcripts of the post-lesson interview(s) for each student, looking for both confirming and disconfirming evidence of our interpretations of their disposition and lesson experiences. Throughout this analysis, we noted aesthetic and affective constructs such as when students said they felt “frustrated” or “relieved.” We used these responses to point us to key lesson moments.

We next explored the lesson data for all of the lessons our case students experienced (14 lessons overall). For each lesson, we drew from the observation notes, audio and video recordings of lessons, lesson materials, and whole-class LES data to write a detailed memo describing the lesson. For each student, we looked across the lessons they experienced for patterns, comparing again with the disposition profile. We also closely analyzed the aspects of the lesson(s) that were interesting to our students and those that were not, contrasting key differences in design and enactment. Through this process, we consistently returned to each student’s voice to ground our analysis and remain close to the data. After determining important aspects of the lessons that attracted each focal student, we completed a cross-case analysis to generate themes across all three cases. All interpretations were discussed and reconciled as a research team and alternative interpretations and disconfirming evidence are included.

### Findings

The three students selected for this study all expressed extreme dislike of mathematics. Yet despite this similarity, what they disliked differed. In this section, we illuminate ways that lessons can encourage these students—chosen for their negative views of mathematics—to become interested in and positively affective towards a specific mathematics class experience. To start, for each student, we describe their views of mathematics. Next, we present their reactions to the specific observed lessons collected for this study (both positive and negative) to identify the captivating curricular features. Finally, we present themes across the three cases.

#### Cassie

Given Cassie’s mathematical disposition, it is not surprising that she largely found the observed lessons boring as their sequence of activities generally followed the pattern: warm-up problem(s), student explanation, small group problem solving, and answer confirmation by Ms. Cherry. Cassie’s focus on being correct and having her answers validated before sharing would be a challenge to her participation in this structure, as she would be asked to discuss with her group and share her thinking before knowing if she was right. Overall, Cassie found these lessons uninteresting, irrelevant, and too challenging.

However, Cassie found one MCLE lesson interesting, and this lesson differed from the typical lesson in Cassie’s class in important ways. In the warm-up for this lesson, students were asked to describe a graphed translation of a linear function and justify their reasoning. This presented an opportunity for differing arguments; students could describe vertical or horizontal translations. Although Cassie did not explicitly mention this portion of the lesson in her post- lesson interview, given her desire to be perceived as knowledgeable, this opener with a focus on justification—rather than a single correct solution—could have enabled her to participate without fear of being wrong.

Next, students graphed different parent functions and translations by hand and used graphing calculators to check their work. This segment of the lesson was similar, however, to one of the lessons Cassie found uninteresting. In that uninteresting lesson, students contended to find the polynomial equation for a given graph (modeling a roller coaster), using graphing software to verify potential equations. While both lessons asked students to use technology to investigate connections between representations, Cassie’s responses to these two lessons differed for key reasons. First, Ms. Cherry posed the roller coaster problem as a whole-class race to see who could find the correct equation first. When another student found the equation and was publicly praised by Ms. Cherry, Cassie describes, “I felt sad because I didn’t get to - I wasn’t the one who discovered it.” That lesson then moved on without an opportunity for Cassie’s sense-making. In contrast, in the interesting lesson, Ms. Cherry framed the exploration in non-competitive terms, and Cassie’s process was not truncated by a single correct answer. This extended inquiry was important as at first Cassie did not like this portion of the lesson. In her interview, she says, “I was really frustrated ... I was really confused.” It is possible that Cassie felt uncomfortable with her initially incorrect predictions, but since the lesson was designed without a narrow focus on a single solution and provided ample time for exploration, she remained engaged.

During this exploration, Cassie began to make sense of the connections between the equations and graphs. She explains, “an aha moment, like, it was just like a thought, like ‘maybe, this works for this, or this works for that.’ But I was ... kind of nervous, but what if I’m wrong or what if I’m right?” Since Cassie likes to find answers and feel confident in their accuracy before sharing with a group, this progress beyond her frustration was key for Cassie to remain engaged. Despite her very strong negative views of mathematics, Cassie felt satisfied by this “aha moment.” Later in the lesson, Ms. Cherry said something Cassie had found would be discussed in the following class, and Cassie described feeling proud: “I was like, look! ... this is similar to what we were looking for, even if it was wrong ... I figured out something, and I was like, ‘Oh! That’s interesting!’” Cassie notes her interest in discovering something—even if it was not completely correct. This is further evidence that Cassie can be engaged with mathematical ideas in ways that disrupt her performance anxiety. When she felt frustrated by not understanding at the beginning of the lesson, Cassie’s struggle was productive because she could validate her own conjectures using technology, had the time to make sense of the mathematics, and did not have her thinking cut short by a focus on a single correct answer. Cassie, a student who strongly dislikes mathematics, felt interested and engaged in learning.

### **Mateo**

Based on the analysis of his responses to the Disposition Survey items, Mateo dislikes both mathematics class and mathematics as a subject. While he has had positive and negative experiences in earlier mathematics courses, he strongly disliked his 10<sup>th</sup> grade class, which is likely highly influential on his current reported views. Mateo’s previous experiences also hinged directly on the teacher; both in years he enjoyed and those he did not, Mateo reported that the teacher was the most influential factor. In line with this, Mateo prefers lessons where the teacher takes an active

role and prefers lecture over all other class activities. While Mateo's concept of "lecture" is unclear, we can conclude that Mateo is drawn to mathematics classes that have a high content focus and direction by the instructor. Notably, Mateo was the only student of the three that describes feeling "alright" in his current mathematics class. Further, he is the only one who sees mathematics as useful for his career or as potentially being amazing. This suggests that Mateo has some potential interest but has not had the learning experiences to foster this interest. He is bored and dislikes his mathematics classes but not because he is completely apathetic toward the subject; Mateo is a student seeking out mathematical stimulation.

Therefore, it's not a surprise that the lessons that he enjoyed were mathematically suspenseful. The first lesson that Mateo found interesting was an MCLE lesson introducing imaginary numbers. The lesson gave students the expectation that they would be solving quadratic equations to find numbers that have certain sums and products. When a challenge emerged where the solutions to a quadratic equation had an integer sum and product but were not real numbers, students were motivated to solve the mystery. It was not until students had grappled with the idea and concluded that the solution would need to include the square root of a negative number that imaginary numbers were introduced. Mateo reported he understood the topic of this lesson, wished more lessons were like this one, and described it as "intriguing." The intentional design of a lesson around a mathematical mystery captivated Mateo because it caused him to question his mathematical assumptions (i.e., that all numbers are real).

The second lesson Mateo found interesting modeled exponential decay with dice. Mr. Ash distributed a total of 100 dice to the students and prompted them to roll all the dice simultaneously and remove those dice that showed a number divisible by three. This procedure was repeated until all dice were eliminated. The lesson captured students' attention as they waited to see when the final die would be eliminated. The lesson then had students compare the probabilistic results with the experimental outcome. Mateo reported that the lesson was interesting and found it "enjoyable" and "satisfying." We can gain more insight about this lesson from his classmate's post-lesson interview. His classmate called the lesson suspenseful and further stated, "We don't do a lot of stuff like that and so like an activity like that was pretty fun." While Mateo did not find the topic of exponential decay challenging, he wished more lessons were like this one and indicated it was the only lesson that he felt was relevant to his life. This lesson, like the imaginary numbers lesson, elicited an aesthetic response of surprise and suspense through the unfolding mathematical ideas, a quality that seems key to attracting Mateo.

Looking at the lessons Mateo did not like; we can see that Mateo found these lessons uninteresting and was generally apathetic towards them. These three lessons involved a range of instructional strategies including group work, pattern recognition, guided exploration, and lecture. Even though for two of these lessons the majority of other students were interested, Mateo reported feeling unchallenged, bored, and did not find any of the lessons relevant to his life. While the lessons that involved problem solving could provide an element of unpredictability, Mateo did not experience them in that way. In these lessons, the investigations involved repeated calculations or pattern recognition, which are not particularly suspenseful and did not call Mateo to stretch his sense-making skills. Rather, Mateo is interested when lessons contain mathematically suspenseful activities and keep the important mathematics central.

### **Mei**

Mei exhibits the strongest negative feelings about mathematics of our case students. None of Mei's Disposition Survey responses about mathematics were positive or even neutral. When considering mathematics in school, her teacher is pivotal to Mei's experience. In previous years



she credited her teacher and her classmates as a reason she liked or did not like her class. In her current class, Mei typically feels “frustrated and confused.” While Mei thinks her current teacher, Ms. Willow, is encouraging and respectful of her ideas, she feels that Ms. Willow fails to make lessons interesting or make the material clear to her. As a result, Mei turns to her peers to negotiate and validate her sense-making in mathematics class. Mei says she pushes herself hard to completely understand her lessons, and yet, she remains stressed, confused, and frustrated.

The lessons Mei did not find interesting were representative of typical lessons in her class that are teacher-led and highly scaffolded, lacking opportunity for student exploration and sense-making. This could contribute to Mei’s consistent confusion in class. Additionally, Mei specifically states that she struggles with algebraic abstraction: “I get confused when the letters are involved in math.” These non-captivating lessons were heavily dependent on algebraic manipulation with little connection to concrete values or multiple representations.

In the one lesson that Mei found interesting, the design connected the algebraic and numerical representations of logarithms. Near the beginning of this MCLE lesson, students examined a table of logarithm values looking for pairs with integer sums and differences and then used these patterns to discover the addition and subtraction properties of logarithms. Mei said the problems at the beginning of the lesson appeared “scary,” stating repeatedly that she finds the abstraction of algebra difficult. However, since the properties of logarithms emerged from patterns in their decimal values, Mei was motivated and had a point of entry. She stated: “I was ... pleasantly surprised that I was able to solve the questions ... because of the pattern that makes sense.” The way the algebraic abstractions were built from numerical patterns interested Mei and fostered her sense-making.

This lesson was also designed for student inquiry; the formal logarithm properties were not made explicit until students had made the connection independently. In other lessons, the new content was provided early in the lesson by Ms. Willow or in a whole-group discussion, where Mei felt lost. In contrast, this lesson was designed to create suspense through the hunt for patterns. Significantly, the lesson also gave Mei ample time to search for these patterns, thereby building suspense and increasing the chances that she accomplishes her goal (i.e., making sense of the pattern). Similar to other lessons, Mei found this lesson challenging and frustrating, but the active role she was offered and the time she was provided allowed her to be successful in her sense-making, which led to her to also describe the lesson as “thought-provoking.” Thus, Mei’s aesthetic experience of satisfaction and surprise were a result of the lesson’s challenging and suspenseful design and contributed to her interest.

Another aspect of Mei’s aesthetic response to this lesson was curiosity. She felt a sense of “wonder” and a desire to continue exploring the conceptual foundation for the patterns she discovered. Even a week later, Mei’s curiosity continued to impact her experience. Despite finding this later everyday lesson uninteresting, Mei’s other responses were uncharacteristically positive. For example, similar to how she described the interesting lesson, she found this later lesson frustrating, thought-provoking, and satisfying, and it was the only lesson besides the logarithm properties lesson that she wished more classes would be like.

### **Themes Across Cases**

Two central themes emerged across the mathematics lesson experiences of these three students. First, the lessons these students found interesting and engaging all prioritized students grappling with important mathematics, with appropriate time for inquiry. These lessons decentralized the cognitive work away from the teacher, instead positioning the students as sense-makers. Also critical to this success is the focus away from a single correct answer, which turned classmates into

competitors and truncated our students' thinking. Instead, the interesting lessons incorporated ample time for students to explore mathematical relationships with the appropriate tools to evaluate their own progress.

A second key finding across the student cases was the centrality of suspense. Not only were the lessons that interested our students open-ended, but they also were crafted so that students could experience surprise, wonder, or satisfaction. These aesthetic responses were due to the lessons' arc—the mathematics was accessible for Cassie, Mateo, and Mei to begin working without understanding the climax that the content was building towards. In Mei's case, this involved searching openly for patterns within a table, but concluded with some of the central logarithmic properties. For Mateo, what began as a sum and difference number puzzle motivated the need for an expansion of the number system. The intentionally suspenseful and surprising aesthetic of these lessons allowed for continued student engagement, satisfying “aha moments,” and sustained curiosity even beyond the class period.

### Discussion

Cassie's, Mateo's, and Mei's experiences in mathematics class offer a key message for educators and curriculum designers: lessons can be designed to interest and engage even those students who have the most negative views of mathematics. Further, the aspects of the lessons that reach these students are not superfluous or gimmicky. Cassie, Mateo, and Mei were all drawn to lessons that positioned them as sense-makers about important mathematical concepts, and they each fostered a sense of satisfaction from their own mathematical engagement.

Although we only focused on three students that had different reasons for disliking mathematics, we were surprised that the captivating lessons shared common characteristics of opportunities for student sense-making as well as structuring the unfolding content so that suspense could be built. We wonder how much their previous mathematical learning experiences had offered satisfaction; we contend that all students have a right to experience meaningful mathematics in ways that enable students to exclaim “Oh!” as Cassie did, as the mathematical ideas come together and make sense. We argue that more research needs to address the particular needs of students like these, in the hopes that all future students can experience the wonder and satisfaction of mathematical inquiry. With repeated positive aesthetic experiences with mathematics, we hope to reverse the trend in poor attitudes toward mathematics in high school.

### Acknowledgements

This material is based upon work supported by the National Science Foundation under Grant No. #1652513. We would also like to thank teacher participants and research team members for their help in collecting the data for this study.

### References

- Dewey, J. (1934). *Art as experience*. Penguin Group.
- Dietiker, L. (2015). What mathematics education can learn from art: The assumptions, values, and vision of mathematics education. *Journal of Education*, 195(1), 1–10.
- Goldin, G. A. (2017). Motivating Desires for Classroom Engagement in the Learning of Mathematics. In C. Andrà, D. Brunetto, E. Levenson, & P. Liljedahl (Eds.), *Teaching and Learning in Maths Classrooms: Emerging Themes in Affect-related Research: Teachers' Beliefs, Students' Engagement and Social Interaction* (pp. 219–229). Springer International Publishing. [https://doi.org/10.1007/978-3-319-49232-2\\_21](https://doi.org/10.1007/978-3-319-49232-2_21)
- Hannula, M. S., Di Martino, P., Pantziara, M., Zhang, Q., Morselli, F., Heyd-Metzuyanim, E., Lutovac, S., Kaasila, R., Middleton, J. A., Jansen, A., & Goldin, G. A. (2016). *Attitudes, Beliefs, Motivation and Identity in Mathematics Education: An Overview of the Field and Future Directions*. Springer International Publishing. <https://doi.org/10.1007/978-3-319-32811-9>

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

- Hidi, S., & Renninger, K. A. (2006). The four-phase model of interest development. *Educational Psychologist, 41*(2), 111–127. [https://doi.org/10.1207/s15326985ep4102\\_4](https://doi.org/10.1207/s15326985ep4102_4)
- Marmur, O. (2019). Key memorable events: A lens on affect, learning, and teaching in the mathematics classroom. *The Journal of Mathematical Behavior, 54*, 100673. <https://doi.org/10.1016/j.jmathb.2018.09.002>
- Merriam, S. B., & Tisdell, E. J. (2015). *Qualitative Research: A Guide to Design and Implementation*. John Wiley & Sons, Incorporated.
- Middleton, J., Jansen, A., & Goldin, G. A. (2017). The complexities of mathematical engagement: Motivation, affect, and social interactions. In *Compendium for research in mathematics education* (pp. 667–699). National Council of Teachers of Mathematics Reston, VA.
- Mullis, I. V., Martin, M. O., Foy, P., & Arora, A. (2012). TIMSS 2011 International Results in Mathematics. Retrieved from Boston College, TIMSS & PIRLS International Study Center Website: <https://Timssandpirls.Bc.Edu/Timss2011/International-Results-Mathematics.Html>.
- Mullis, I. V., Martin, M. O., Foy, P., & Hooper, M. (2016). TIMSS 2015 International Results in Mathematics. Retrieved from Boston College, TIMSS & PIRLS International Study Center Website: <http://Timssandpirls.Bc.Edu/Timss2015/International-Results/>.
- Mullis, I. V., Martin, M. O., Foy, P., Kelly, D. L., & Fishbein, B. (2020). TIMSS 2019 International Results in Mathematics and Science. Retrieved from Boston College, TIMSS & PIRLS International Study Center Website: <https://Timssandpirls.Bc.Edu/Timss2019/International-Results>.
- Riling, M., Dietiker, L., & Gates, M. (2019). *How do students experience mathematics? Designing and testing a lesson specific tool to measure student perceptions*. American Educational Research Association (AERA), Toronto, Ontario, Canada.
- Riling, M., Dietiker, L., Gibson, K., Tukhtakhunov, I., & Ren, C. (2018). Factors that influence student mathematical dispositions. *Proceedings of the 40th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, 1012.
- Satyam, V. R. (2020). Satisfying moments during the transition-to-proof: Characteristics of moments of significant positive emotion. *The Journal of Mathematical Behavior, 59*, 100784. <https://doi.org/10.1016/j.jmathb.2020.100784>
- Sinclair, N. (2001). The aesthetic IS relevant. *For the Learning of Mathematics, 21*(1), 25–32. <http://www.jstor.org/stable/40248343>
- Sinclair, N. (2008). Attending to the aesthetic in the mathematics classroom. *For the Learning of Mathematics, 28*(1), 29–35. <http://www.jstor.org/stable/40248596>

## ONE VARIABLE, MULTIPLE MEANINGS: MEASURING RESPONSE TIME IN THOSE WITH MATH ANXIETY

Christine Hartmann  
Utah State University  
christine.hartmann@usu.edu

Olivia Ewing  
Utah State University  
olivia.ewing@usu.edu

Leyla Karami Isheqlou  
Utah State University  
a02370623@usu.edu

Emmett Speed  
Utah State University  
espeed@utah.gov

Kerry Jordan  
Utah State University  
kerry.jordan@usu.edu

Keywords: Affect, Emotion, Beliefs, Attitudes, Cognition, Learning Theory, Undergraduate Education

Math anxiety (MA) and math performance have a reciprocal relationship through which several features at the individual and environmental levels interact (Chang & Beilock, 2016). At the individual level, we see motivational, cognitive, affective, and physiological processes. At the environmental level are cues from teachers, parents, and fellow students about MA and expectations about mathematics performance. Children are an ideal population to sample and test theoretical models of MA because data can be collected from both the individual (child) and environmental (parent) levels. Additionally, when observed in adults these variables may converge in ways that show established individual behavioral and cognitive patterns within MA.

A common measure used in MA research is response time, which is thought to reveal implicit cognitive reactions to stimuli. Two notable tasks that use this variable are the arithmetic affective priming task (AAPT) and the emotional Stroop (Rubinsten & Tannock, 2010; Rubinsten et al., 2012; Pellicioni et al., 2014). Designed to be an implicit measure of MA, the AAPT is theorized to measure a person's prior relationship with mathematics (Rubinsten & Tannock, 2010; Rubinsten et al., 2012). The emotional Stroop is a test of attentional bias which shows how a threatening stimulus captures attention (Hopko et al., 2002). The emotional Stroop has been verified once (Pellicioni et al., 2015) and not supported once (Hopko et al., 2002). The AAPT's premise has been supported twice (Rubinsten & Tannock, 2010; Rubinsten et al., 2012) and not supported twice (Hartmann et al., in preparation; Kucian et al., 2018). These tests' goals are at odds because the AAPT posits that stimuli related to a person's prior experience make their response time *faster*, while the emotional Stroop posits that related stimuli will make their response time *slower*.

The purpose of this dissertation project is to study which response time task is a better predictor of math anxiety, and to determine whether the same task is a better predictor for both children and adults. I will conduct two studies, one with children ages 7-9 and one with adults, to collect data to construct two regression models of MA using personal and environmental variables, the AAPT, the emotional Stroop, and a survey of math anxiety. For people with math anxiety, discomfort often prevents them from performing math tasks to the best of their ability and from continuing their mathematics education to pursue careers in STEM. I believe that if we can understand the hidden cognitive impact of math anxiety, as measured by response time, we as researchers and educators will be better poised to assist students who have math anxiety.

## References

- Chang, H., & Beilock, S. L. (2016). The math anxiety-math performance link and its relation to individual and environmental factors: A review of current behavioral and psychophysiological research. *Current Opinion in Behavioral Sciences*, 10, 33-38. <https://doi.org/10.1016/j.cobeha.2016.04.011>
- Hartmann, C., Covington, B., Jordan, K. (2023). Comparing Implicit and Explicit Measures of Math Anxiety [Manuscript in preparation]. Department of Psychology, Utah State University.
- Hopko, D. R., McNeil, D. W., Gleason, P. J., & Rabalais, A. E. (2002). The emotional Stroop paradigm: Performance as a function of stimulus properties and self-reported mathematics anxiety. *Cognitive Therapy and Research*, 26(2), 157-166. <https://doi-org.dist.lib.usu.edu/10.1023/A:1014578218041>
- Kucian, K., Zuber, I., Kohn, J., Poltz, N., Wyszkon, A., Esser, G., & Von Aster, M. (2018). Relation between mathematical performance, math anxiety, and affective priming in children with and without developmental dyscalculia. *Frontiers in Psychology*, 263. <https://doi.org/10.3389/fpsyg.2018.00263>
- Rubinsten, O., Bialik, N., & Solar, Y. (2012). Exploring the relationship between math anxiety and gender through implicit measurement. *Frontiers in Human Neuroscience*, 6, 279. <https://doi.org/10.3389/fnhum.2012.00279>
- Rubinsten, O., & Tannock, R. (2010). Mathematics anxiety in children with developmental dyscalculia. *Behavioral and Brain functions*, 6(1), 46. <https://doi-org.dist.lib.usu.edu/10.1186/1744-9081-6-46>
- Suárez-Pellicioni, M., Núñez-Peña, M. I., & Colomé, À. (2014). Reactive recruitment of attentional control in math anxiety: an ERP study of numeric conflict monitoring and adaptation. *PloS One*, 9(6). <https://doi.org/10.1371/journal.pone.0099579>

# RELATIONSHIPS BETWEEN STUDENTS' MICRO-IDENTITIES AND STUDENTS' MATHEMATICS

Christopher O. Roman  
Portland State University  
roman7@pdx.edu

*The purpose of this paper is to investigate the relationships between adult students' mathematical micro-identities (informed by positioning theory) and their mathematics (informed by radical constructivism). I used positioning theory to conduct a detailed turn-by-turn analysis of students' micro-identities, and to inform my models of students' mathematics. Results showed how positive micro-identities can promote persistence, vulnerability, visibility, or flourishing in students' mathematics. Implications for research and practice are discussed. The latter connects to the PMENA conference theme by highlighting the type of classroom environment teachers can foster to engage their students and maximize opportunities for learning.*

Keywords: Cognition, Calculus, Classroom Discourse

Steffe and Thompson (2000) used the term “students’ mathematics” to refer to the mathematical realities of students. Radical constructivist research programs have utilized models of students’ mathematics to better understand students’ knowledge of mathematical concepts including, but not limited to, fractions (e.g. Steffe, 2001), graphing (e.g. Moore & Thompson, 2015), speed and time (e.g. Thompson, 1994), and variation (e.g. Jones & Kuster, 2021; Saldanha & Thompson, 1998). While I acknowledge the importance of this work to the field of mathematics education, one limitation is that it does not attend to who students are, specifically in micro-level moments of interaction. I argue that attending to the micro-level identities that students enact<sup>1</sup> as they engage with their mathematics in moments of interaction can add valuable information to researchers’ models of students’ mathematics.

Conceptualizing identity at the micro-level is ideal for investigating momentary shifts in who people are (Wood, 2013). I adopt Wood’s (2013) definition of *micro-identity* as “the position of a person in a moment of time” (p. 780), as well as her definition of *mathematical micro-identities* as “the subset of micro-identities that position the person relative to mathematics and mathematical activity” (p. 780). Defining micro-identities as positions is useful for investigating how micro-identities shift from moment-to-moment: say from mathematical student to menial worker (Wood, 2013), novice to expert (Esmonde, 2009), or evaluator to leader (Evans et al., 2006). The purpose of this paper is to investigate how the micro-identities that adult students enact during an online synchronous lesson are related to their mathematics. Thus, my research question is as follows: “What are the relationships between adult students’ mathematical micro-identities and their mathematics?”

## Theoretical Framework

### Students’ Mathematics and Researcher Positionality

Steffe and Thompson (2000) distinguish between students’ mathematics and mathematics of students. They define *students’ mathematics* as “whatever might constitute students’ mathematical realities” (p. 268), and *mathematics of students* as “our interpretations of students’ mathematics” (p. 268). Mathematics of students refers to the models of students’ mathematics that researchers

---

<sup>1</sup> Similar to Bishop (2012), I use the term “enact” to highlight that identity is a dynamic construct.

construct by analyzing mathematical *discourse*, which I define as any interaction that utilizes written or spoken words, gestures, and representations to convey meaning (cf. Bishop, 2012). Models of students' mathematics can never be said to depict students' actual mathematical realities since it is impossible for researchers to fully separate from their own experiences, biases, and interpretations. Thus, it is important for researchers to share their positionality so that readers can then make their own inferences about how compatible these models are to students' mathematical realities.

My focus on students' mathematics is connected to my view of knowledge, which I frame from the perspective of radical constructivism (von Glasersfeld, 1995). A central tenet of radical constructivism is that "knowledge is not passively received", but rather, "is actively built up by the cognizing subject" (von Glasersfeld, 1995, p. 51). Thus, in my view, knowledge is not static; it is dynamic and constantly developing from moment-to-moment. It is this "moment-to-moment" view of knowledge that motivates my decision to construct models of students moment-to-moment mathematical micro-identities, as well as models of students' mathematics.

From a radical constructivist perspective, models of students' mathematics should include descriptions of the mental operations that underpin students' mathematical explanations and strategies (Steffe & Thompson, 2000). These mental operations are discovered by researchers as they interact directly with students via short-term clinical interviews (Goldin, 2000) or long-term teaching experiments (Steffe & Thompson, 2000). For the sake of modeling students' mathematical micro-identities in a more natural setting, I chose to interact with students indirectly in the current study by observing them as they engaged with their mathematics during a single class lesson. This made it difficult for me to make any concrete claims about students' mental operations. As a result, the models of students' mathematics that I describe are focused on descriptions of students' strategies rather than their mental operations.

### **Connecting Positioning Theory and Students' Mathematics**

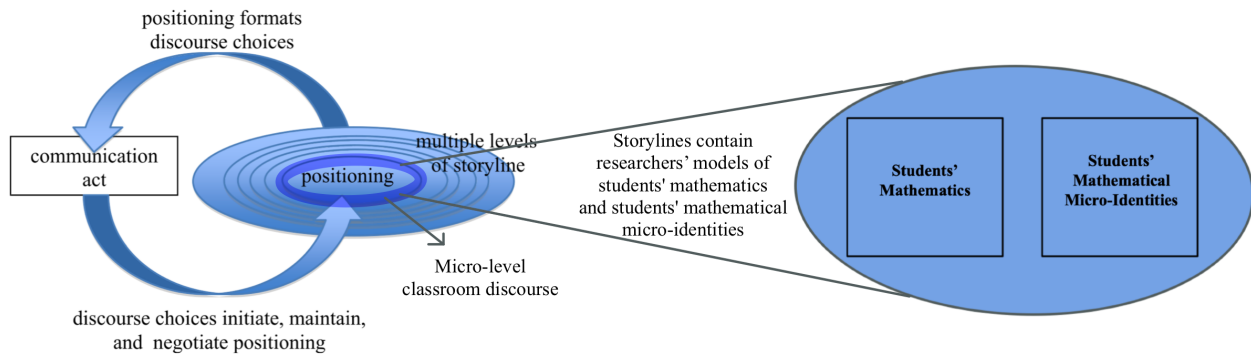
This study utilizes positioning theory—a theory that applies to discourse—to operationalize *micro-identity* as "the position of a person in a moment of time" (Wood, 2013, p. 780). Many researchers who use positioning theory draw from the work of Davies and Harré (1999) and van Langenhove and Harré (1999). These scholars noted that there are three constructs that are central to positioning theory: speech acts, positions, and storylines.

Herbel-Eisenmann et al. (2015) refined Davies and Harré's (1999) speech act construct. They used the term *communication act* to refer to the meaning that spoken words, as well as gestures, physical positions, and stances have for participants in discourse. Harré (2012) defined a *position* as "a cluster of short-term disputable rights, obligations, and duties" (p. 193). He defined *positioning* as "a process by which rights and duties are assigned, ascribed, or appropriated and resisted, rejected, or repudiated" (p. 196). Here, *rights* refer to "what you (or they) must do for me" and *duties* refer to "what I must do for you (them)" (p. 197).

Storylines have been given minimal attention in the literature compared to positioning (Herbel-Eisenmann et al., 2015), and have been conceptualized vaguely (Kayi-Aydar & Miller, 2018). Part of the complication with the conceptualization of storylines is that at any given moment, "multiple storylines exist and may even be embedded in broader sets of discursive conventions" (Herbel-Eisenmann et al., 2015, p. 191). Thus, it is important for researchers to clarify the scale at which they are conceptualizing storylines (Herbel-Eisenmann et al., 2015). In the current study, I am focused on storylines that occur at the micro-level of classroom discourse.

Given my positionality as a radical constructivist researcher, I view a *storyline* as a model of the story that is being constructed by participants in discourse as they engage with positioning and

communication acts. The term *model* is used to highlight the fact that storylines are based on the experiences, biases, and interpretations of the researcher. The storylines in the current study are constructed specifically from mathematical discourse, and hence, they contain my models of students' mathematics and mathematical micro-identities. To illustrate my conceptualization of the relationship between communication acts, positioning, storylines, and students' mathematics, I extend Herbel-Eisenmann et al.'s (2015) dynamic positioning theory framework (see Figure 1).



**Figure 1: Extension of Herbel-Eisenmann et al.'s (2015) Positioning Framework (p. 194) Methods**

The discourse analyzed in this study involved four students who were enrolled in a 10-week mathematics course titled “Concepts of Calculus for Middle School Teachers” at a public university in the United States. I attended the course as a graduate student mentor, whose responsibilities included assisting the instructor with lesson planning, facilitating discussion during group work, and teaching one lesson independently. Due to the lingering effects of the Covid-19 pandemic, the instructor taught the course remotely through Zoom in an online synchronous format. A major goal of the course was to improve students’ covariational reasoning abilities (Carlson et al., 2002).

There were eight students enrolled in the course: six white women and two white men. Three women—Sam, Kelly, and Eden—and one man, James, were selected to participate in this study (pseudonyms were used). Selection was based on convenience as these were the only students who consented to being video recorded for research purposes. Kelly and James were both in-service teachers who had prior exposure to calculus. Eden and Sam were both prospective teachers who had not previously completed calculus courses.

The data for this study was made up of one 67-minute video recorded lesson that included the four participating students. The recorded lesson was chosen because it involved a variety of five different math tasks. The primary focus was on the four students’ work within a Zoom breakout room. In this paper I focus on modeling the mathematics and micro-identities of one student, Sam, as I inferred the most growth in her mathematics during the lesson.

### Analysis

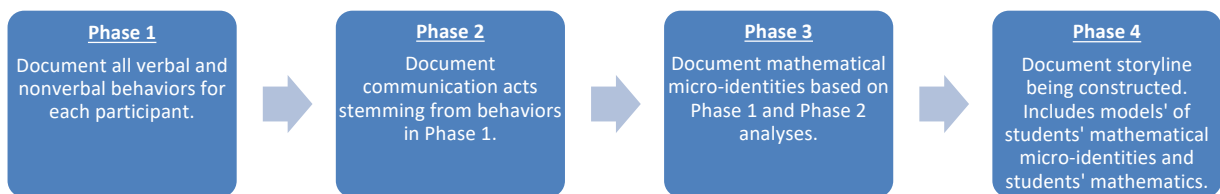
I created a transcript that was separated by speaker turns (Wood, 2013), and then split the transcript into five episodes (Erickson, 2006). Each episode corresponded to one of the five math tasks in the lesson. I focused my analysis on three of the five episodes (2, 3, and 5) because they involved graphical tasks with potential for engendering students’ covariational reasoning.

I conducted a turn-by-turn analysis of each episode by placing each turn in its own row of a single column in a Google spreadsheet, and then adding on additional columns for separate



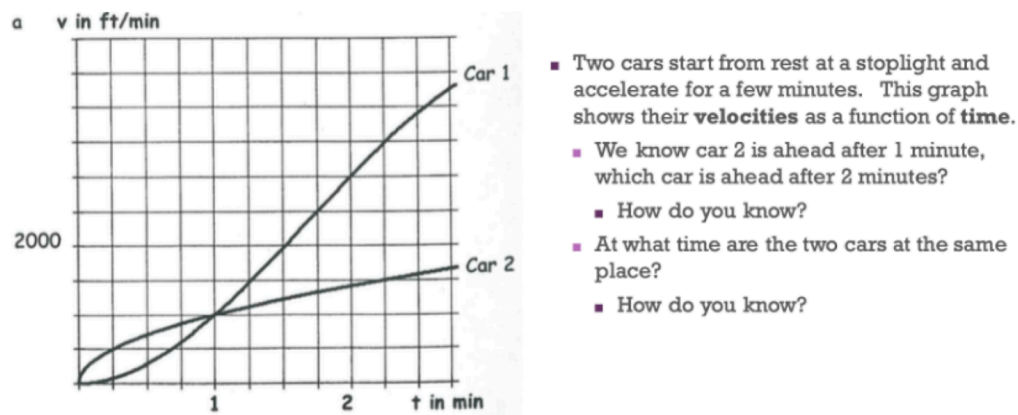
components of my analysis (e.g. gestures, communication acts, micro-identities, etc.). Similar to Wood (2013), I marked a turn as relevant if it involved mathematical discussion. For each relevant turn, I then engaged in four phases of analysis (see Figure 2). First, I replayed the video recording several times to document the verbal and nonverbal behaviors that were exhibited by each of the four participating students. Second, I documented my interpretations of the communication acts that stemmed from these behaviors. Third, I used the information from the first two phases to document my interpretations of the mathematical micro-identities that were being assigned by the speaker, both to themselves and to the other students. Fourth, I documented my interpretation of the storyline that was being constructed by the participating students.

The storylines contained my models (i.e. interpretations and descriptions) of students' mathematical micro-identities and students' mathematics; these models were informed by my interpretations of the verbal and nonverbal behaviors, and communication acts. Although my analysis of storylines occurred turn by turn, I also combined storylines across multiple turns for the sake of modeling more detailed stories. Lastly, after completing the turn-by-turn analyses, I scanned each of the analyzed episodes holistically by looking at how the storyline was constructed across all of the turns. This allowed me to hone in on Sam in order to highlight important relationships between Sam's mathematical micro-identities and Sam's mathematics.



**Figure 2: Four-Phase Analysis for Relevant Speaker Turns Results**

In the following sections, I highlight three transcript excerpts that correspond to one episode of the lesson. I focus on one episode for the sake of space, and because this episode was critical for Sam in that it promoted the flourishing of her mathematics. Each turn in the lesson transcript was numbered, resulting in a total of 291 turns for the entire lesson. The focal episode



**Figure 3: Part 2 of “Truckin’...” Task**

began approximately halfway into the lesson at Turn 137 and ended approximately two-thirds of the way into the lesson at Turn 198. For context, the students were working primarily on the second part of the “Truckin’...” task throughout the episode (see Figure 3).

### Micro-Identities of Capability In Mathematics

The students started this episode by discussing part one of the “Truckin’...” task, which involved determining which car would be ahead after one minute. Then, the instructor (who had joined their breakout room) challenged them to move on to the second part of the “Truckin’...” task and think about which car would be ahead after two minutes. Kelly, James, and Eden were attempting to arrive at a solution to the problem but were having little success. During this time (Turns 156-171), Sam was not speaking. She appeared to be thinking, however, as she was writing on something that was not visible on the Zoom camera, and also shifting back and forth between her computer screen and her written work. Sam shared her thinking with the rest of the group, but only after Kelly and Eden had given up (see Table 1).

**Table 1: Transcript Excerpt 1: Sam Shares her Preliminary Thoughts**

Turn	Speaker	Transcript <sup>2</sup>
169	Eden	Also, this is such a wild time for a class too, cause I feel so brain dead, and I feel like I could have half of a thought, and my mind is just like "boom", it's like elevator music like when you click a link and it makes that sound .. I:, Yeah, I think my .. we're only an hour in and // it's fading fast ....
170	Kelly	Yeah I feel like I just hit a wall with this question // and my brain just sort of shut off.
171	Eden	It was like we stopped talking for like 30 seconds, and my brain was like, well that's it, you've had all the thoughts you're gonna have today (giggles).
172	Sam	I don't know if this is the right way to go about it but // right now I'm like estimating .. like where it's at on the y-axis, and honestly just adding it together one by one? Which I don't know .... // if that's the way to go about this. But that's, currently what I am doing right now.

Eden’s mental exhaustion (“I feel so brain dead...”) contributed to her enacting a micro-identity (position) of incapable math student (Turn 169) as she was unwilling to persist in thinking about the problem (i.e. which car would be ahead after two minutes). Kelly accepted this micro-identity of incapable math student (Turn 170) by also declining to persist with the problem. Sam, on the other hand, was unwilling to give up on thinking about the problem (Turn 172). By continuing to persist, Sam rejected the micro-identity of incapable math student that was initially offered by Eden, and instead enacted a micro-identity of capable math student.

**Storyline: Persistence and vulnerability in Sam’s mathematics.** The negative micro-identity of incapable math student that Eden and Kelly enacted was associated with a lack of persistence.

<sup>2</sup> Transcript conventions taken from Temple and Wright (2015):

- .. noticeable pause, less than 0.5 seconds
- ... half-second pause; each extra dot represents additional half-second pause
- : lengthened sound (extra colons represent extra lengthening)
- // slash marks indicate uncertain transcription or speaker overlap
- [XX] overlapping brackets indicate two speakers talking at the same time
- [XX]

Eden felt “brain dead” (Turn 169) and Kelly felt like her “brain just sort of shut off” (Turn 170). This caused them to disengage entirely with their mathematics. Rather than following suit, Sam enacted the positive micro-identity of capable math student and persisted with the problem by sharing her thinking with the other students.

Sam’s strategy involved estimating points on the  $y$ -axis, and then adding them together. At this point in the episode, it was not clear whether Sam was aware that the  $y$ -axis represented velocity, nor was it clear how she was coordinating her  $y$ -axis estimates with their corresponding values on the  $x$ -axis (time). It was clear, however, that Sam was not confident in her strategy as she admitted that she was not sure whether her strategy was the correct approach (Turn 172). Thus, persisting with the problem meant that Sam had to allow herself to be vulnerable to critique. This indicates that the micro-identity of capable math student is associated with both persistence and vulnerability in one’s mathematics.

### Supportive Mathematical Micro-Identity

A few turns after Sam shared her preliminary thoughts on the problem (Turn 172 in Table 1), Kelly asked Sam if she had ever taken calculus (Turn 175 in Table 2), thus positioning her as a calculus student. Sam rejected the micro-identity of calculus student (“I am doing this without a calculus mindset”), and instead enacted a micro-identity of precalculus student (Turn 176). Kelly then said that she was “very excited” (Turn 177), presumably for Sam to share her completed strategy with the group. The encouraging nature behind the excitement that Kelly exhibited towards Sam’s strategy suggested that she was enacting a micro-identity of supportive math student. Sam began to doubt herself (“I don’t know whether that’s a good thing or a bad thing”), and enacted a micro-identity of unconfident math student (Turn 178). Kelly, however, eased Sam’s doubt by reassuring her that “it’s awesome” and encouraging her to continue, thus maintaining the micro-identity of supportive math student (Turn 179).

**Table 2: Transcript Excerpt 2: Kayla Supports Sam’s Thinking**

Turn	Speaker	Transcript
175	Kelly	Okay Sam, just like a super quick question. I don't want to interrupt you, I'm just very curious. Have you taken calculus before? ...
176	Sam	Umm::, I took Precalculus in high school, [and that is as far as I've gotten. So I am doing this without a calculus mindset.
177	Kelly	[Okay .... Okay cool ..... Cool. Okay, I'm very excited. That's all, I'm done.
178	Sam	I don't know whether that's a good thing or a bad thing, but //
179	Kelly	No it's awesome, it's awesome. Keep going, please.

**Storyline: Supporting Sam’s mathematics.** When Kelly assigned the position of calculus student to Sam (Turn 175), it was not clear whether she intended this to support Sam’s mathematics. In fact, after Sam rejected the micro-identity of calculus student (Turn 176) she began to doubt herself (Turn 178), which led her to enact a micro-identity of unconfident math student. Kelly, however, reassured Sam that her non-calculus background was “awesome” (Turn 179), making it evident that she had initially positioned Sam as a calculus student because she valued her knowledge, and thus, was supporting her thought process. This indicates that Kelly was positioning herself as a supportive math student from the outset of this excerpt. Although Sam did not explicitly share her mathematics in this excerpt, the micro-identity of supportive math student

that Kelly enacted allowed Sam to continue working on her solution strategy, albeit out of view of the others. After Kelly told Sam to “Keep going” (Turn 179), Sam said “Okay” and then spent a little over two minutes moving her gaze back and forth between her computer screen and her written work (outside the view of the camera). Thus, the micro-identity of supportive math student can contribute to the flourishing of students’ mathematics.

**Micro-Identity Of Mathematical Explainer**

After working intently on the problem for over two minutes, Sam told the other students that Car 2 was ahead of Car 1 by fifty yards at the two minute mark. But Sam did not have time to explain her thought process to her group members because the instructor had closed the breakout rooms. She was, however, selected by the instructor to share her solution strategy with the entire class. Table 3 depicts the final turn in the episode (Turn 198) where Sam shared her computer screen through Zoom, and then enacted a micro-identity of mathematical explainer by explaining her completed solution strategy to the whole class.

**Table 3: Transcript Excerpt 3: Sam Explains her Solution Strategy**

Turn	Speaker	Transcript
198	Sam	Okay .... Umm:, so I .... went to here (pointing to the quarter minute mark on the t-axis) and then, like for car:: 2 for example, right here (pointing to the output of car 2 at a quarter minute), I knew that the: velocity was uhh:, 500 .. umm: because each of these (pointing to the tick marks on the v-axis) represents 500 feet per minute. And umm:, I went through and like added 500 (pointing to the output of car 2 at a quarter minute), plus umm:, 525 (pointing to the output of car 2 at a half minute), plus 575 (pointing to the output of car 2 at three-quarters of a minute), umm: plus right here is 1000 (pointing to the output of car 2 at one minute), and so on up to 2 minutes. And then I did the same thing for car 1 up to 2 minutes (slides pointer along the graph of car 1 until she reaches the output at 2 minutes). And umm:, these are all estimates cause I can't see the exact points. But, through doing that, I got that they were only 50 yards apart after two... (gets interrupted by Instructor)

**Storyline: Sam explains her mathematics.** By enacting a micro-identity of mathematical explainer (Turn 198), Sam was able to explicitly reveal the thought process that was directly linked to her mathematics. In other words, she was able to make her mathematics visible to the whole class. This indicates that the micro-identity of mathematical explainer is important in making students’ mathematics visible.

Turning now to Sam’s solution strategy (Turn 198), she began by estimating the distance traveled by Car 2 after two minutes. To do this, she estimated the velocity of Car 2 that corresponds to each tick mark (up to two minutes) on the *t*-axis: 500 corresponds to the first tick mark, 525 corresponds to the second tick mark, 575 corresponds to the third tick mark, 1000 corresponds to the fourth tick mark, etc. As Sam was estimating these values, she was also

simultaneously adding them together one by one. She then did the same thing for Car 1 and somehow determined that the two cars would be 50 yards apart after two minutes. It was not clear what Sam did with the velocities that she added together for each car to obtain her final result, nor was it clear what she interpreted the quantities of these summed velocities to be. In addition, it was

not clear whether Sam envisioned the cars taking on velocities in between values that correspond to the tick marks on the  $t$ -axis. The thinking she displayed was primarily discrete and did not contain evidence of continuity across the quantities' values.

There are two important things to note about Sam's explanation. First, she did not attend to the fact that the  $t$ -axis was partitioned into intervals of quarter minute length. Doing so would have required her to divide each estimated velocity for Car 2 (500, 525, 575, 1000, etc.) by four as these velocities correspond to each minute on the  $t$ -axis, rather than each quarter minute. Second, Sam's final solution of 50 yards did not have the correct units of feet. Since one of the earlier tasks the students worked with in this lesson involved units of yards, it is possible that Sam conflated the units in the "Truckin'..." task with the units of this earlier task. Ultimately, however, the structure of Sam's solution strategy was correct. Had she attended to the units in the problem, and to the way she was coordinating her velocity and time values, her final answer would have been "correct" by normative standards.

### Discussion

This study focused on investigating the relationships between the mathematical micro-identities and mathematics of one student, Sam, during a single online synchronous mathematics lesson. Thus, the answer to my research question—"What are the relationships between adult students' mathematical micro-identities and their mathematics?"—is based on this context. The results showed three important relationships between the mathematical micro-identities enacted throughout the episode and Sam's mathematics.

First, the micro-identity of capable math student allowed Sam to continue persisting with the problem (i.e. which car was ahead after two minutes) even though (1) the other students had given up on the problem and (2) she was not confident in her solution strategy. This suggests that capable mathematical micro-identities can help students persist with their mathematics, and also exhibit vulnerability in their mathematics. The latter relationship is interesting as it is more common for students to associate capability in mathematics with confidence and correct solutions, rather than vulnerability. Second, the micro-identity of supportive math student that was enacted by Kelly motivated Sam to continue working on and refining her solution strategy, suggesting that supportive mathematical micro-identities can promote the flourishing of students' mathematics. Third, the micro-identity of mathematical explainer allowed Sam to share her thinking with the rest of the class, suggesting that mathematical micro-identities involving explanations and thought processes make students' mathematics visible.

The results of this study have two important implications: one for researchers and one for teachers that connects to the PMENA conference theme. First, teacher-researchers that conduct future teaching experiments (Steffe & Thompson, 2000) should consider the ways they are positioning students, as well as the way students are positioning each other, as these positions can give rise to micro-identities that can add more nuanced descriptions to their models of students' mathematics. Second, mathematics teachers should consider fostering a classroom environment that (1) positions all students as capable, (2) provides a safe space for students to exhibit vulnerability in their mathematics, (3) encourages students to support one another, and (4) allows students the opportunity to share their explanations and thought processes. This implication connects to the PMENA conference theme because teachers who foster such classroom environments can promote the enactment of positive micro-identities that can engage students with their mathematics in a variety of ways: say by persisting with their mathematics, exhibiting vulnerability toward their mathematics, or explaining their mathematics to others. Engaging students with their mathematics, in turn, maximizes the opportunities for learning.

## References

- Bishop, J. P. (2012). “She’s always been the smart one. I’ve always been the dumb one”: Identities in the mathematics classroom. *Journal for Research in Mathematics Education*, 43(1), 34–74. <https://doi.org/10.5951/jresmetheduc.43.1.0034>
- Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education*, 33(5), 352–378. <https://doi.org/10.2307/4149958>
- Davies, B., & Harré, R. (1999). Positioning and personhood. In R. Harré & L. van Langenhove (Eds.), *Positioning theory: Moral contexts of intentional action* (pp. 32–52). Malden, MA: Blackwell Publishers.
- Erickson, F. (2006). Definition and analysis of data from videotape: Some research procedures and their rationales. In J. L. Green, C. Gregory, & P. B. Elmore (Eds.), *Handbook of complementary methods in education research* (pp. 177–191). Mahwah, NJ: Lawrence Erlbaum Associates.
- Esmonde, I. (2009). Mathematics learning in groups: Analyzing equity in two cooperative activity structures. *Journal of the Learning Sciences*, 18(2), 247–284. <https://doi.org/10.1080/10508400902797958>
- Evans, J., Morgan, C., & Tsatsaroni, A. (2006). Discursive positioning and emotion in school mathematics practices. *Educational Studies in Mathematics*, 63(2), 209–226. <https://doi.org/10.1007/s10649-006-9029-1>
- Goldin, G. (2000). A scientific perspective on structured, task-based interviews in mathematics education research. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 517–545). Mahwah, NJ: Lawrence Erlbaum Associates.
- Harré, R. (2012). Positioning theory: Moral dimensions of social-cultural psychology. In J. Valsiner (Ed.), *The Oxford handbook of culture and psychology* (pp. 191–206). Oxford University Press.
- Herbel-Eisenmann, B. A., Wagner, D., Johnson, K. R., Suh, H., & Figueras, H. (2015). Positioning in mathematics education: Revelations on an imported theory. *Educational Studies in Mathematics*, 89(2), 185–204. <https://doi.org/10.1007/s10649-014-9588-5>
- Jones, S. R., & Kuster, G. E. (2021). Examining students’ variational reasoning in differential equations. *The Journal of Mathematical Behavior*, 64, 1–15. <https://doi.org/10.1016/j.jmathb.2021.100899>
- Kayi-Aydar, H., & Miller, E. R. (2018). Positioning in classroom discourse studies: A state-of-the-art review. *Classroom Discourse*, 9(2), 79–94. <https://doi.org/10.1080/19463014.2018.1450275>
- Moore, K. C., & Thompson, P. W. (2015). Shape thinking and students’ graphing activity. In T. Fukawa-Connelly, N. Infante, K. Keene, & M. Zandieh (Eds.), *Proceedings of the 18th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 782–789). Pittsburgh, Pennsylvania: RUME.
- Saldanha, L., & Thompson, P. W. (1998). Re-thinking co-variation from a quantitative perspective: Simultaneous continuous variation. In S. B. Berenson & W. N. Coulombe (Eds.), *Proceedings of the Annual Meeting of the Psychology of Mathematics Education—North America* (Vol. 1, pp. 298–304). Raleigh, NC: North Carolina State University.
- Steffe, L. P. (2001). A new hypothesis concerning children’s fractional knowledge. *The Journal of Mathematical Behavior*, 20(3), 267–307. [https://doi.org/10.1016/S0732-3123\(02\)00075-5](https://doi.org/10.1016/S0732-3123(02)00075-5)
- Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 267–307). Mahwah, NJ: Lawrence Erlbaum Associates.
- Temple Adger, C., & Wright, L. J. (2015). Discourse in educational settings. In D. Tannen, H. E. Hamilton, & D. Schiffrin (Eds.), *Handbook of discourse analysis* (pp. 858–879). Malden, MA: Wiley Blackwell.
- Thompson, P. W. (1994). The development of the concept of speed and its relationship to concepts of rate. In J. Confrey & G. Harel (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 181–236). New York, NY: State University of New York Press.
- van Langenhove, L., & Harré, R. (1999). Introducing positioning theory. In R. Harré & L. van Langenhove (Eds.), *Positioning theory: Moral contexts of intentional action* (pp. 14–31). Malden, MA: Blackwell Publishers.
- von Glasersfeld, E. (1995). *Radical constructivism: A way of knowing and learning*. (Studies in Mathematics Education, Vol. 6). London: Falmer.
- Wood, M. B. (2013). Mathematical micro-identities: Moment-to-moment positioning and learning in a fourth-grade classroom. *Journal for Research in Mathematics Education*, 44(5), 775–808. <https://doi.org/10.5951/jresmetheduc.44.5.0775>

# RE/MAKING “ENGAGEMENT AND LEARNING NEEDS:” CITATION MAPPING FOR FIELD ANALYSIS IN MATHEMATICS EDUCATION RESEARCH

Christopher H. Dubbs  
East Stroudsburg University  
cdubbs@esu.edu

Susan O. Cannon  
University of Georgia  
cannonso@uga.edu

*We used citation cartography to answer the question, “what research agendas should we pursue, with respect to engagement and learning needs, to ensure that all students reach their potential?” First, we found that current research studying engagement and learning needs were located on the periphery of the field with few connections between (1) these needs, (2) research related to race, gender, and social justice mathematics, and (3) research related specifically to mathematics content areas. We, thus, conclude with two recommendations for the field. First, we advise that mathematics education researchers consider how we might more fully connect distant subfields in mathematics education research. Second, we highlight the potential of making connections to fields outside of mathematics education. We believe that these two moves will help ensure all students reach their potential by having their engagement and learning needs met.*

Keywords: Research Methods, Systemic Change, Reflexivity

## Introduction

In this research report we seek to outline a tentative answer to one of the conference theme questions: “What research agendas should we pursue to ensure that all students reach their potential by paying attention to engagement and learning needs?” From our perspective, it is necessary to first answer the question: “What research agendas *are we pursuing* to ensure that all students reach their potential by paying attention to engagement and learning needs?” That is, before we can provide evaluative judgment and recommendations on which agendas *ought* to be pursued, it is necessary to provide a sketch of which agendas *are* being pursued. To be explicit, then, our objective is (1) to outline what existing research agenda/foci are being undertaken that relate to students’ “engagement and learning needs” before (2) sharing our recommendations for potential paths forward for the field to engage in research that ensures all students reach their potential.

To provide such a sketch, we draw on the citation network analysis presented in *The Mathematics Education Atlas* (Dubbs, 2021) and the citation network map of the *Journal for Research in Mathematics Education (JRME)* in the 2010s. This bibliographic mapping provides a literal view of the landscape of mathematics education research and the research foci that mathematics education researchers (MERs) have studied/are studying as a field. By critically reading the *JRME* map, we isolated those research foci which center around “student engagement and learning needs” and consider their relations to the field at large. To wit, we found that research related explicitly to children’s learning needs were located on the periphery of the map and that there are few connections between students’ learning needs, research related to race, gender, and social justice mathematics, or research related specifically to mathematics content areas. Thus, we argue that it is necessary for MERs to make kin (Haraway, 2016) in lines of inventive connections between these subfields (what we will call bubbles) in mathematics education research in order to ensure all students reach their potential.

## **Theoretical Framework: Alchemical Combinations for Becoming and Re/making**

In chemistry, we get different products depending on which substances are combined in what quantities under which conditions. Foucault's theory of power-knowledge is something like that. We get different products of knowledge depending on which beliefs are combined with what societies under which political conditions. (Fendler, 2010, p. 53)

In addition to our researcher positionality (Foote & Bartell, 2011), the societies in which we exist, the perspective whence we observe, the theories we employ, and the data we analyze each influence the conclusions we draw. This is fundamentally an illustration of the alchemy implied by Foucault's power-knowledge. Thus, it is important that we explicitly name which theories frame our present investigation. First, we presume that the researcher and the field are both in a state of *becoming* (e.g., Barad, 2007; Cannon, 2020b). In short, neither a researcher nor the field are ever final, accomplished, or fixed and the ways that both *become over time* are entangled. Second, we presume that the researcher can take the field as their object and remake it (Rancière, 2009), remake it with care *à la* an artist creating, a farmer tending, or a doctor healing (Foucault, 1984). Other researchers might productively adopt alternative perspectives to produce alternative knowledges: equally valid products resulting from differences in substance combination and quantity. We elaborate the explicit connection between our theories and research questions now.

### **Field Becoming: Re/con/figuring Mathematics Education Research**

According to Cannon's theory of field becoming, in the process of disciplining ourselves to be recognized as MERs (e.g., reading from, citing, and writing for particular journals), we are conditioned by the norms and expectations outlined by the field (Cannon, 2020b). We seek the normative delimiters and boundaries that outline which sorts of research are "sensible and legitimate." Simultaneous with our own becoming, the field of mathematics education research is also *becoming*: there is disagreement about what constitutes legitimate research (e.g., Heid, 2010; Herbst et al., 2022), the field has studied shifting foci across time (Inglis & Foster, 2018; Dubbs, 2021), and "we create the field, we work the ground, we discipline ourselves and the field" (Cannon, 2020b, p. 1115). Thus, there is mutual re/con/figuring at work: scholars re/con/figure themselves by obeying or transgressing the ways of being and doing prescribed by the field, and the field is re/con/figured as scholars do work that preserves or perturbs the present state of the field.

If "each piece of writing or conversation or body carried into a particular space and how it is received in that space matters for how the field continues to become" (Cannon, 2020b, p. 1115) then it is important as researchers that we understand how the current configuration of the field positions research on "engagement" and "learning needs." In other words, if the foci and methods of current research make research with similar foci and methods seem more sensible (Parks & Schmeichel, 2012), it is necessary to first discern the ways we conduct and write about research that considers "engagement and learning needs," to know what ways of doing such research will be easiest to imagine. It is this argument that brings us to our first research question [RQ1]: "What research agendas *are we pursuing* as a field to ensure that all students reach their potential by paying attention to engagement and learning needs?"

We must take seriously that this mutual becoming is an *ethical* matter (Cannon, 2020b). In the process of constituting oneself as a proper subject of mathematics education research (Foucault, 1984), one must negotiate their subjectivity, there is room for innovation but not without constraint (Cannon, 2020b). In other words, if the ways that someone can conduct new research are in some ways bound to the ongoing ways of doing mathematics education research—as evidenced by



published studies, etc.—the ways that a mathematics education researcher might take up research on “engagement” and “learning needs” will be (de)limited by what *has been done*.

### **Field Remaking: Sociopolitical Aesthetics of Mathematics Education Research**

Mathematics education research, as a field, is worthwhile insofar as we continue to change. The instant we draw a line...we lose the game. (Dubbs, 202, p. 165)

After answering RQ1, thereby discussing “engagement and learning needs” field *as it is*, we turn to our second research question [RQ2], imported from the conference questions, to ask: “What research agendas should we pursue to ensure that all students reach their potential by paying attention to engagement and learning needs?” Reminding ourselves that the state of things is not justification for what ought to be (Hume, 1739), we have a critical eye towards those bubbles of research identified in RQ1. We argue that the *existence* of these ways of doing mathematics education research in the name of “engagement and learning needs” is not a sufficient warrant that these ways of doing this research *ought* to be that way—a separate ought-argument is necessary. In other words, our second question is fundamentally about changing the field, about remaking the field *as it ought to be*.

It is easier to imagine a change in the field, if we consider that “change is the result of a thousand creeping encroachments” (Rancière, 2000, para. 8). That is, the field can change by way of numerous small suggestions on what *should be done*, numerous small ways of *doing things differently*, and numerous small ways of *asking different questions*. Unlike attempts to establish fixed borders (e.g., Heid, 2010), we combine the perspectives of Cannon (2020a) and Dubbs (2021) to endorse a perspective of mathematics education research that is predicated on perpetual change and re/con/figuration. By desiring certitude, borders harden toward difference. It is not our role as the authors to define proper “engagement and learning needs research,” it is the field’s role to continually re/con/figure it. This article constitutes our initial re/con/figuration.

Since a researcher’s and the field’s becoming are entangled (Barad, 2007), it is helpful to think of remaking the field of mathematics education research as a sociopolitical aesthetic project. In other words, our project is the sociopolitical aesthetics of mathematics education research: *sociopolitical* in the sense that mathematics education research is a social endeavor wherein competing visions for what mathematics education research *ought* to consider (cf. Gutierrez, 2013), and *aesthetic* in that we are interested in what we can see, say, think, and do as sensible mathematics education research (cf. Rancière, 2009). Further, the research we ultimately undertake will influence both our own and the field’s *becoming*, the future of the field is of ethical concern (Dubbs, 2020)—echoing the ethical matter of subjectivity raised by Cannon.

### **Citation Network Methods: Using Graph Theory to Study Knowledge Production**

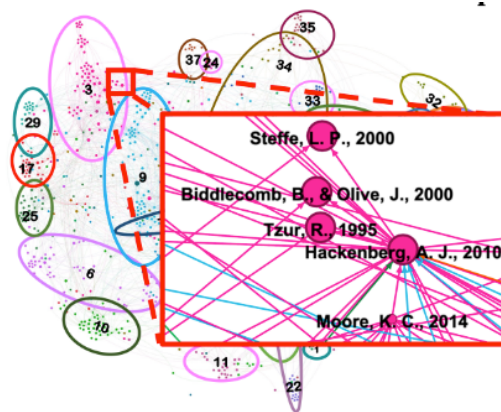
Citation networks can be considered maps of the field of mathematics education research (Dubbs, 2021). Dubbs mapped the complete citation relationships of articles published in the *Journal for Research in Mathematics Education (JRME)* from the 1970s to the 2010s, in *Educational Studies in Mathematics* in the 2010s, and *For the Learning of Mathematics* in the 2010s. Choosing these journals due to their prominence in the field (e.g., Nivens & Otten, 2017), Dubbs showed that (1) mathematics education research has not had a fixed focus of study across time (he warrants this through the shifts he identifies in focus across five decades of research in the *JRME*) and (2) mathematics education research does not currently have a proper singular object of study (he warrants this via the plural foci under consideration within and across the three journals during the 2010s). Our interest, hence, diverges with Dubbs here. While Dubbs sought to identify and describe the structure of much of mathematics education research, our interest is in reading one

map (the *JRME* 2010s) to identify how the field has talked about “engagement and learning needs,” and to propose, if necessary, alternative futures.

Dubbs’ research maps (2021) use circles, called nodes, to represent articles and their references. Each node is labeled with its APA-style author and year (e.g., Hackenberg, A. J., 2010) to identify which article/reference it represents. See in Figure 1, for example, an enlarged portion of Bubble 3 (Children’s Learning) from Dubbs’ *JRME* 2010s map. In Figure 1, one can see that some nodes are connected with an arrow, called an edge. Edges correspond to citations

and arrow direction indicates which article cites which. Notice, for example, that Hackenberg, 2010 (in Figure 1) has both *outgoing* arrows (i.e., it *cites* Steffe, 2000) and *incoming* arrows (i.e., it is *cited by* Moore, 2014)). To quickly see relative uptake (i.e., citation count), node size is proportional to the number of times the reference is cited: more cited articles have larger nodes.

**Figure 1: An enlarged portion of Bubble 3 of the JRME 2010s map to show graph details.**



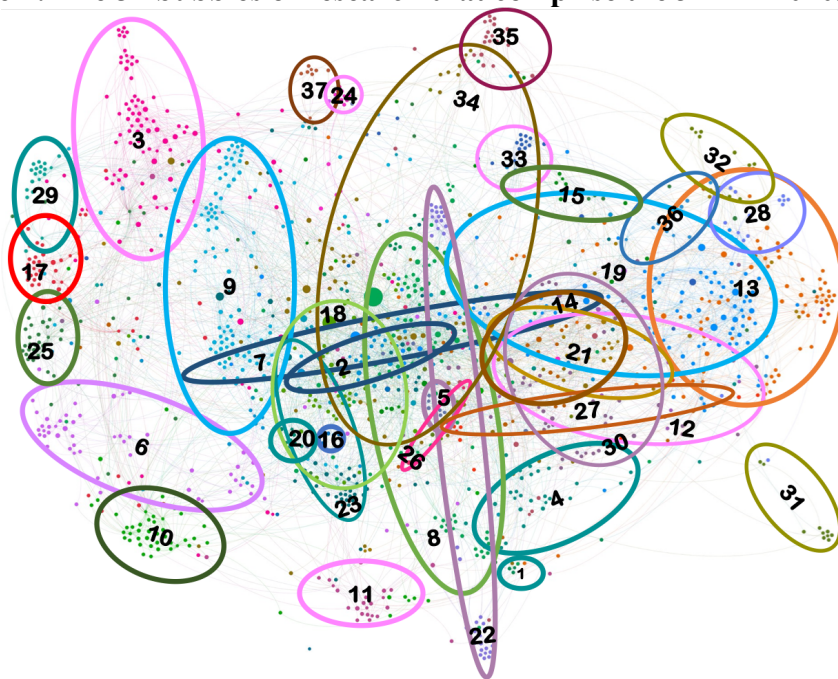
### **Bifocal Results: Macro- and Micro-views**

To answer the question, “What research agendas *are we pursuing* to ensure that all students reach their potential by paying attention to engagement and learning needs?,” we critically read the *JRME* 2010s map. Dubbs identified 37 bubbles (see Figure 2’s note for the name of each of the 37 bubbles), corresponding to delineated areas of focus within the research published in the *JRME* during the period from January 2010 to November 2019 (*JRME* 41(1) to *JRME* 50(5)). First, we considered which of these bubbles most directly related to students’ engagement and learning needs. The two most salient bubbles are: Bubble 3 (Children’s Learning) and Bubble 17 (Children’s Achievement, Intervention, & Trajectories). In this section, first, we consider the relative position of these two bubbles within the 2010s map before turning to a more detailed analysis of the overlap and connections between these bubbles and the remainder of the map.

#### **Macroanalysis: Position of Focal Bubbles Relative to the Map**

Consider the left-hand side of Figure 2. The first focal bubble, Bubble 3 (Children’s Learning), is located on the periphery of the map and only intersects with one other bubble, Bubble 9 (Mathematics Knowledge for Teaching). Bubble 3 is adjacent to Bubble 29 (Learning Disabilities), and the other focal bubble, Bubble 17 (Children’s Achievement, Intervention, and Trajectories). Bubble 3 has 99 nodes (i.e., articles/references), 429 outgoing edges (i.e., citations *to* the rest of the field), 178 internal edges (i.e., citations *within* the bubble), yet only 49 incoming edges (i.e., citations *by/from* the rest of the field). This imbalance suggests that while the articles within Bubble 3 seem to both (1) cite within the bubble a sufficient number of times

**Figure 2: The 37 bubbles of research that comprise the *JRME* 2010s map.**



Note: Dubbs' (2021) 37 bubbles for the *JRME* in the 2010s are: (1) Proof and Argument, (2) Professional Development, (3) Children's Learning, (4) Mathematical Discourse, (5) Schoenfeld, (6) Meaning of Equality, (7) Teaching's Influence on Learning, (8) Proof and Reform, (9) Mathematics Knowledge for Teaching, (10) Negative Numbers, (11) Limits and Calculus, (12) Culturally Relevant Mathematics African American & Indigenous, (13) Racial Identity & Success, (14) English Language Learner's Identity & Participation, (15) Achievement Gap, (16) Sociocultural Learning, (17) Children's Achievement, Intervention, & Trajectories, (18) Mathematics Curriculum, (19) Equity and Social Justice, (20) Math Achievement, (21) Mathematics Identity, (22) Proof in RUME, (23) Mathematics Teachers and Teaching, (24) Empirical Statistics, (25) Algebra, (26) Problem Posing and Multiple Solutions, (27) Racialized Mathematics Achievement Remediation, (28) Qualitative Metasynthesis, (29) Learning Disabilities, (30) Learning in Contexts, (31) Research on Research, (32) Gender and Achievement, (33) Urban Equity and Technology's Role, (34) Sociological Perspectives on Learning, (35) Psychological Studies & Replication, (36) Girls' Identities, and (37) Research in Undergraduate Mathematics Education.

to establish a cohesive bubble and (2) cite outside Bubble 3 sufficiently to anchor their work to the rest of the field, they are not sufficiently cited by the rest of the field to situate it more centrally.

Relative to the entire map, both Bubbles 3 (Children's Learning) and 17 (Children's Achievement, Intervention, & Trajectories) are located on the far left of the map and are isolated from the center of the map. Since the bubbles centrally located correspond to the areas of focus for the journal during this decade, this separation and isolation is noteworthy and indicates that the field is only tangentially pursuing research related to students' engagement and learning needs. These areas of research are not integrated into the *central* topics that are published by the journal. We consider the implications of this distance and disconnect in our final section.

### **Microanalysis: Focal Bubbles' Connections to the Map**

Within Bubble 3, articles on Wilkins and Norton's quantitative studies on splitting operations and fraction schemes anchor this bubble (e.g., their 2011 article is the most cited article in Bubble 3; see also, Norton & Wilkins, 2012, 2018) with their large number of incoming and outgoing citations (i.e., also the number of references they cite). Thus, anyone *becoming* with the field of

mathematics education research—be they a MER, graduate student, or scholar outside of mathematics education research—that is reviewing the literature published on children’s learning in the *JRME* during the 2010s (esp. Bubble 3), would glean particular norms about the focus: quantitative, constructivist, studies on children’s understanding of specific mathematical concepts. In other words, *becoming* a researcher of children’s learning constitutes conforming oneself to the norms of the field outlined by this previous research. The previously published research was, in turn, written in conformity with the norms established by previous research; this regression continues *ad infinitum*. We discuss the implications of this regressive conformity for research on “student engagement and learning needs” in the final section.

Another key article within Bubble 3 is Hackenberg’s (2010) study on the caring relationships between a group of sixth-grade students and their teachers. This study seems to be placed in Bubble 3 through Dubbs (2021) citational cartography due to the grade-level of this article since much of the literature cited by Hackenberg is from *outside* the field of mathematics education research, and thus, does not tether it as tightly to *central* topics in the field. Hackenberg draws on Nel Noddings’ care theory (2002) to unpack how students might *feel*, in relation with the teacher, while doing mathematics. Hackenberg ends her manuscript with the question, “How can the study of mathematical care and general care for students mutually inform each other?” (p. 268). Importantly, she suggests that MERs need to attend to the “broader canvas of the development of care theory, which includes theorizing about care in relation to assumptions founded on particular racial and cultural points of view” (p. 268). This suggestion is important for MERs, too, since student emotions intersect with student learning and engagement. The implications of Hackenberg’s presence in Bubble 3 for “student engagement and learning needs” writ large are discussed in the last section.

The second focal bubble, Bubble 17 (Children’s Achievement, Intervention, and Trajectories) is adjacent to Bubble 3 above. Both foci bubbles intersect with Bubble 29 (Learning Disabilities) which may indicate that when MERs engage in research related to children that the research is focused on deficits in children’s mathematical learning. This is supported by the presence of “intervention” in the title of Bubble 17. Bubble 17 centers on preschool mathematics education and research on intervention. Specifically, Clements and Sarama (2011), an anchor in the bubble, considers the interventions available to support young children’s opportunity to learn mathematics. While the research distinguishes differences in performance between low- and middle-income students and draws attention to differences in performance between African American children and “their peers” (p. 970) without naming any other races/ethnicities, the piece does not cite any research on race and mathematics.

In fact, both of the foci bubbles (3 and 17) related to “engagement and learning needs” are markedly distant from the bubbles related to race, gender, context, and social justice mathematics on the far right side of the map in Figure 2 (12-Culturally Relevant Mathematics African American and Indigenous; 13-Racial Identity & Success; 14-English Language Learner’s Identity and Participation; 15-Achievement Gap; 33-Urban Equity and Technology’s Role; 19-Equity and Social Justice; 21-Mathematical Identity; 27-Racialized Mathematics Achievement; 36-Girls’ Identities; and 30-Learning in Context). While research in Bubbles 3 and 17 indicate deficits in learning, and in the case of Clements and Sarama (2011) imply that those deficits may be related to race and SES, they do not interact with the articles and authors thinking about race, gender, and other socio-political constructs in relation to mathematics education. This is one of two noteworthy gaps in the map. We revisit the implications for the gap between “student learning and engagement needs” and race, gender, and other socio-political constructs in the final section.

The second gap is to the left of Bubble 9 (Mathematics Knowledge for Teaching), between bubbles 29 (Learning Disabilities), 17 (Children's Achievement, Intervention, & Trajectories), and 25 (Algebra) on the left periphery, Bubble 3 at the top periphery, and 6 (Meaning of Equality) and 10 (Negative Numbers) at the bottom. This area of relative low density indicates that these bubbles are siloed from mathematics education writ large but adhere to themselves due to self- and intra-citing. These bubbles related to mathematical content (25, 6, 10) and teacher and student knowledge of mathematics (9, 29, 17) are separated by a gap. While there are some interconnections, this gap suggests that MREs are not integrating research on mathematics content with research on teacher knowledge and student learning and vice versa. We return to these, and the foreshadowed implications, now.

### **Implications and Future Directions**

Academic fields are made through our intra-actions with(in) them. As we test the borders and stray outside the perceived lines, the field is reconfigured, and new tracts are laid that might be followed. (Cannon, 2020b, p. 1116)

We began this paper by considering “What research agendas *are we pursuing* to ensure that all students reach their potential by paying attention to engagement and learning needs?” [RQ1]. To answer that question, we looked at the articles published in the *JRME* during the 2010s to discern the norms and foci of any relevant research. In this section, we turn to RQ2: “What research agendas should we pursue to ensure that all students reach their potential by paying attention to engagement and learning needs?” We now make two explicit recommendations on which agendas *ought* to be pursued based on the critical readings of the map outlined above: (1) make kin *within* mathematics education research, and (2) make kin *outside* mathematics education research. Haraway (2016), states, “the task is to make kin in lines of inventive connection ...in myriad unfinished configurations of places, times, matters, meanings” (p. 1).

#### **Radical Reimagining One: Making Kin within Mathematics Education Research**

Our first suggestion for the field of mathematics education research is for MERs to make kin *within* the field. In other words, seek out opportunities for inventive connection within the extant research of the field. We illustrate the need and utility by revisiting our implications from above, in each case, illustrating the ways that making kin *within* is generative for a *becoming* field and *becoming* MERs.

First, revisiting the marginal position of Bubbles 3 (Children's Learning) and 17 (Children's Achievement, Intervention, and Trajectories) and their relative disconnect from the field, indicate opportunities to bring these ideas more centrally into focus. That is, the bubbles that focus squarely on the conference theme of "student learning and engagement needs" are positioned marginal to the field, as represented by the map of *JRME* 2010s. How is it that the field conducts research on, for example, mathematical discourse (Bubble 4) without considering the research within the focal bubbles? Is success constituted and measured differently? Are issues of learning needs and engagement largely absent?

Second, the implications of the gap between the focal bubbles and the bubbles on race, gender, and other socio-political constructs are significant. While mathematics research organizations and teacher organizations have been calling for equity to be centralized in mathematics education research (NCTM, 2014; TODOS: Mathematics for All, 2020), this map emphasizes the continued need to make explicit connections between research on learning needs, engagement, and interventions and the research on race, gender, sexuality, and other socio-political constructs. Is the field perpetuating color-blind ideology by way of color-blind research? How might thinking with

the excellent research in the equity bubbles inform Clements and Sarama's thinking about intervention? In sum, these findings and implications suggest that making kin *within* the field is a productive endeavor for MERs. In other words, by seeking opportunities for inventive connections *within* the extant research of the field, we can find new ways for a *becoming* field and as *becoming* MERs.

### **Radical Reimagining Two: Making Kin outside Mathematics Education Research**

Our second suggestion for the field of mathematics education research is for MERs to make kin *outside* the field. In other words, seek out opportunities for inventive connection outside the field. We illustrate the need and utility by revisiting our implications from above to show the ways that making kin *outside* is generative for a *becoming* field and *becoming* MERs.

First, we revisit the implications of the regressive conformity implied by writing that seeks to conform to the norms of a field. Since new research is expected to conform to existing norms, there are limits on the types of research that can be done. For example, if norms draw on quantitative, qualitative, and mixed methods, there is excluded an entire humanities-oriented research tradition (e.g., philosophy, history, textual analysis; See AERA, 2009). Conformity to these norms, then, precludes truly novel methods and approaches to research. Additionally, if research foci are normatively dictated, there is limited agency to engage with new areas of research. In other words, norms not only limit our freedom in *becoming* MERs, but also limit the field's *becoming* since it limits the ways in which the field is free to grow. Therefore, we advocate for intentional norm transgression: draw on methods and theories outside the normative bounds of the field. Where will those methods and theories lead us? Or consider research foci outside the normative limits of what is sensible for the field. What can the field *become*?

Second, recall that Hackenberg's (2010) study using Nodding's (2002) notion of care to center emotions was included in Bubble 3 due to its (1) focus on sixth-grade students and the corresponding literature base similarly cited by other authors, and (2) relative lack of connections to the other bubbles of the *JRME*. Hackenberg's references largely fall outside the *JRME*, leaving it unmoored within the landscape of the *JRME* and pushed toward the margins. While Hackenberg's lack on connections within the field of mathematics education research may have distanced her work from the central areas of research within the field, her inclusion of Noddings' theory of care (2002) in her manuscript—an uncommon theory in mathematics education research—was a line of inventive connection that caused a field disruption (Cannon, 2020a). This connection *outside* the field, then, raises a series of important questions *within* the field: Are the children, tweens, teens, young-adults, and adults in our mathematics classes being cared for? Are the emotional needs of all students being attended to? If not, are there underlying assumptions about certain-aged students being able to self-regulate? What do these assumptions induce when applied to the learning needs and engagement of all students?

### **Parting Thoughts**

Though the images of the maps are static, we acknowledge that the field of mathematics education is always being re/made, always *becoming*. We find hope in this. We map not to contain but to see how we got here and how we might want to find ways forward. The field is porous and open to rearrangements. We imagine a horizontal depth and moving across, knitting bubbles together until the field is more tightly woven, not to keep out, but to enmesh and bring into kinship (Haraway, 2016). We imagine a field so interconnected and entangled that individual bubbles are no longer legible: a field of MERs *becoming*.

## References

- American Educational Research Association. (2009). Standards for reporting on humanities-oriented research in AERA publications. *Educational Researcher*, 38(6), 481–486. <https://doi.org/10.3102/0013189X09341833>
- Barad, K. (2007). *Meeting the universe halfway: Quantum physics and the entanglement of matter and meaning*. Duke University Press.
- Cannon, S. O. (2020a). A call for field disruptions and field connections in mathematics education research. *Journal of Urban Mathematics Education*, 13(2), 1–16. <https://jume-ojs-tamu.tdl.org/jume/index.php/JUME/article/view/389/310>
- Cannon, S. O. (2020b). Field guide to academic becoming. *Qualitative Inquiry*, 26(8-9), 1110–1121. <https://doi.org/10.1177/1077800419881658>
- Clements, D. H., & Sarama, J. (2011). Early childhood mathematics intervention. *Science*, 333(6045), 968–970. <https://doi.org/10.1126/science.1204537>
- Dubbs, C. H. (2020). Whose ethics? Towards clarifying ethics in mathematics education research. *Journal of Philosophy of Education*. <https://doi.org/10.1111/1467-9752.12427>
- Dubbs, C. H. (2021). *Mathematics education atlas: Mapping the field of mathematics education research*. Crave Press.
- Fendler, L. (2014). *Michel Foucault*. Bloomsbury Publishing.
- Foote, M. Q., & Bartell, T. G. (2011). Pathways to equity in mathematics education: How life experiences impact researcher positionality. *Educational Studies in Mathematics*, 78(1), 45–68. <https://doi.org/10.1007/s10649-011-9309-2>
- Foucault, M. (1984). The Ethics of the Concern for Self as a Practice of Freedom. In P. Rabinow (Ed.), *Ethics: Subjectivity and truth* (1997, pp. 253–80). The New Press.
- Gutiérrez, R. (2013). The sociopolitical turn in mathematics education. *Journal for Research in Mathematics Education*, 44(1), 37–68. <https://doi.org/10.5951/jresmetheduc.44.1.0037>
- Hackenberg, A. J. (2010). Mathematical caring relations in action. *Journal for Research in Mathematics Education*, 41(3), 236–273. <https://doi.org/10.5951/jresmetheduc.41.3.0236>
- Haraway, D. (2016). *Staying with the trouble: Making kin in the Chthulucene*. Duke University Press.
- Heid, M. K. (2010). Where's the math (in mathematics education research)? *Journal for Research in Mathematics Education*, 41(2), 102–103. <https://doi.org/10.5951/jresmetheduc.41.2.0102>
- Herbst, P., Chazan, D., Crespo, S., Matthews, P. G., & Lichtenstein, E. K. (2022). How manuscripts can contribute to research on mathematics education: An expansive look at basic research in our field. *Journal for Research in Mathematics Education*, 53(1), 2–9. <https://doi.org/10.5951/jresmetheduc-2021-0191>
- Hume, D. (1739) *A Treatise of Human Nature*. Available online at: <https://ebooks.adelaide.edu.au/h/hume/david/h92t/B3.1.1.html>.
- Moore, K. C. (2014). Quantitative reasoning and the sine function: The case of Zac. *Journal for Research in Mathematics Education*, 45(1), 102–138. <https://doi.org/10.5951/jresmetheduc.45.1.0102>
- NCTM. (2014). *Principles to actions: Ensuring mathematical success for all*. National Council of Teachers of Mathematics.
- Nivens, R. A., & Otten, S. (2017). Assessing journal quality in mathematics education. *Journal for Research in Mathematics Education*, 48(4), 348–368. <https://doi.org/10.5951/jresmetheduc.48.4.0348>
- Noddings, N. (2002). *Educating moral people: A caring alternative to character education*. Teachers College Press.
- Norton & Wilkins, 2012, Norton, A., & Wilkins, J. L. (2012). The splitting group. *Journal for Research in Mathematics Education*, 43(5), 557–583. <https://doi.org/10.5951/jresmetheduc.43.5.0557>
- Wilkins, J. L., & Norton, A. (2018). Learning progression toward a measurement concept of fractions. *International Journal of STEM Education*, 5(1), 1–11. <https://doi.org/10.1186/s40594-018-0119-2>
- Rancière, J. (2000). History and the Art System (interview with Yan Ciret). *Art Press*, 258, 18–23.
- Rancière, J. (2009). *The politics of aesthetics: The distribution of the sensible*. Bloomsbury Publishing.
- Steffe, L. P. (2000). Perspectives on issues concerning the self, paideia, constraints and viability, and ethics. In L. P. Steffe & P. W. Thompson (Eds.), *Radical constructivism in action* (pp. 91–102). RoutledgeFalmer.
- TODOS: Mathematics for All. (2020). The Mo(ve)ment to Prioritize Antiracist Mathematics: Planning for this and every school year. Tempe, AZ: Author. Available via <https://todos-math/statements>
- Wilkins, J. L., & Norton, A. (2011). The splitting loope. *Journal for Research in Mathematics Education*, 42(4), 386–416. <https://doi.org/10.5951/jresmetheduc.42.4.0386>

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

# ROLE OF GRIT AND OTHER FACTORS IN MITIGATING MATH ANXIETY IN COLLEGE MATH STUDENTS

Lori Ogden  
West Virginia University  
Lori.Ogden@mail.wvu.edu

Marjorie Darrah  
West Virginia University  
Marjorie.Darrah@mail.wvu.edu

Monica Leppma  
West Virginia University  
Monica.Leppma@mail.wvu.edu

*Math anxiety refers to feelings people get when facing computational challenges. The fear of failure often causes people to avoid activities, like taking challenging mathematics courses or choosing majors that require mathematics. Grit describes how people persist or persevere in the wake of failure. This article looks at factors that contribute to a person having grit and the interaction of grit and math anxiety. Students (N = 258) enrolled in college math classes at a large university completed a set of surveys that indicated their levels of self-compassion, mindfulness, self-efficacy, math anxiety, and grit. The analyses of the data found that grit predicted math anxiety, and that self-compassion, mindfulness, and self-efficacy predicted grit. The results of this study suggest that promoting factors that help students develop grit can in turn affect their math anxiety and thus their persistence in mathematics courses and STEM majors.*

Keywords: grit, math anxiety, self-efficacy, mindfulness, self-compassion

Many countries do not produce enough science, technology, engineering, and mathematics STEM majors to keep up with the demand for STEM workers (Langdon et al., 2011; Peri et al., 2014; van den Hurk et al., 2019). It is important to understand what contributes to STEM avoidance and lack of success in STEM majors. Since all STEM majors must take and pass mathematics courses for their major, one factor to STEM retention is success in mathematics. Although mathematics education is quite different from country to country, the pervasiveness of the effects of math anxiety on performance stands out in many parts of the world (Foley et al., 2017). Foley et al. (2017) point out how math anxiety must be considered when trying to increase mathematics achievement, which is important for STEM success.

Researchers have been examining ways to address math anxiety through non-academic interventions (Beilock & Maloney, 2015; Brunyé et al., 2013). Factors such as grit, mindfulness, self-compassion, and self-efficacy have established associations with anxiety, well-being, resilience, and academic achievement (Jarukasemthawee et al., 2021; Neff et al., 2005; Neff & Germer, 2013; Tubbs et al., 2019). However, research is needed to better understand the correlates of grit, particularly in relation to math anxiety. Therefore, our first aim was to assess the validity of grit as a predictor of math anxiety. Our second aim was to determine whether self-efficacy, self-compassion, and mindfulness were associated with grit. Our research questions were as follows:

1. How does grit affect math anxiety?
2. What roles do self-efficacy, self-compassion, and mindfulness have in predicting grit?

## Background

*What is Grit?* Angela Duckworth has been credited with identifying grit as a trait that should be studied within an academic setting. She and her colleagues attempted to answer the question, “Why do some individuals accomplish more than others of equal intelligence?” (Duckworth et al., 2007, p. 1087). They define grit as persistence or perseverance in the wake of failure. In other words, students with grit see academic success as a purpose worth pursuing. Academic success may entail



a long and sometimes bumpy road, but gritty students do not give up easily (Duckworth, et.al., 2007). Grit is explained as a two-dimensional concept that includes *consistency of interest* and *perseverance of effort* (Duckworth, et. al., 2007; Duckworth & Quinn, 2009). Six individual studies that focused on grit as a predictor of academic success concluded that level of education influenced grit, grit intensified with age, undergraduate students with higher levels of grit earned higher GPAs, and grit predicted retention better than self-control among cadets during their first summer at West Point (Duckworth, 2007)

Some researchers have found stronger relationships between student success and the perseverance of effort component of grit than with consistency of interest (Bowman, et. al., 2015; Cross, 2014; Hodge et. al., 2016). A study of Australian university students revealed that first-generation college students had higher levels of grit, specifically in perseverance of effort, and that effort predicted student productivity and student engagement (Hodge et al., 2016). Meta-analytic synthesis of grit literature conducted by Crede and Tynan (2017) further discussed the lower-order facets of grit; perseverance and consistency and agreed that perseverance of effort played a larger role in student achievement than did consistency of interest. Furthermore, they argued that perseverance should be considered a distinct construct apart from grit. Another meta-analysis by Lam and Zhou (2019) also supported this notion. In addition, they pointed to the need for additional research focused on which grit facets affect academic learning outcomes and suggested teachers identify and foster strategies that promote grit through both sustained effort and the development of academic interest. They questioned whether grit is something that can be taught and invited researchers to further explore this notion.

*How is Grit Related to Math Anxiety?* Sheila Tobias (1976) coined the phrase "mathematics anxiety." This first article discussed the dread, powerlessness, paralysis, and mental disorder that occur in people with this condition. Most students have the cognitive ability to carry out a mathematical task successfully, but their fear and anxiety regarding math get in the way (Tobias, 1991). Cognitive researchers have established an important relationship between negative emotions, such as anxiety, and academic performance. Wang et al. (2015) uncovered an intriguing relationship between math anxiety and math performance in motivated individuals; high and low levels of anxiety both had a negative relationship with performance and moderate levels of math anxiety had a positive association with effective performance.

Math anxiety not only affects performance in math but also affects motivation and interest in the subject (Beilock et al., 2015). Lack of success in math courses frequently affects students' persistence in college and may even affect their motivation or interest in pursuing STEM-related degrees. Since math anxiety is associated with poorer performance in math (Henslee & Klein, 2017) and math anxiety may cause capable students to avoid math classes, this can affect the number of students preparing for much-needed STEM-related occupations (Beilock et al., 2015).

Researchers are beginning to demonstrate that personality factors influence the development of math anxiety (Beilock et al., 2015; Brunyé et al., 2013). Some agree that to reduce math anxiety it is important to foster a student's strength in facing obstacles (Mammarella et al., 2018; Ramirez et al., 2018). A positive response to setbacks (Ramirez et al., 2018) and higher levels of resilience (Mammarella et al., 2018) are beneficial to alleviate math anxiety. Holtby (2018) claims there is a lack of evidence about how grit relates to math anxiety but found a significant negative correlation between grit and math anxiety. In that study, students who exhibited greater dedication to the pursuit of long-term goals reported feeling less anxiety in relation to their mathematical work.

*How is Grit Related to Self-Efficacy?* In Bandura's Social Cognitive Theory, self-efficacy is defined as "how well one can execute courses of action required to deal with prospective

situations" (Bandura, 1982). Bandura's theory also implies connections between self-efficacy, cognitive functioning and achievement, claiming that if a person believes they can perform well then they will view a difficult task as something they can achieve, rather than something they must avoid (Bandura 1993, Bandura 1996). Self-efficacy has also been related to motivation and positive attitudes in mathematics (Czocher, Melhuish et al., 2020). Honicke and Broadbent (2016) suggest that people with higher self-efficacy possess skills and attitudes that support achievement and success, for example, they found a correlation between self-efficacy and self-regulatory learning strategies. Self-efficacy has also been shown to influence the effort students put toward learning and their ability to persist, thus having an effect on their achievement (Mohammadyari, 2012). Alhadabi & Karpinski (2020) found that people who had more grit and high self-efficacy had high expectations for themselves and commonly set mastery goals for themselves. Those researchers found that the grit dimensions of consistency of interest and perseverance of effort are positively related to self-efficacy.

*How is Grit Related to Self-Compassion?* Self-compassion consists of three components: the ability to be kind to oneself regardless of circumstances, the acceptance of imperfection as an inevitable aspect of the human condition, and the mindful awareness of thoughts and feelings (Neff, 2003). Self-compassion's association with grit has yet to be explored in STEM education. According to our search of the literature, only sports psychology research has started to examine the relationship between self-compassion and grit to improve athletes' wellbeing and performance. Doorley et al. (2022) suggested that because self-compassion promotes healthy responses to stressful events, it seems to have features similar to grit.

Research has shown that self-compassion is associated with positive psychological functioning. For example, Neff et al. (2007) established a positive correlation between self-compassion and optimism, personal initiative, and curiosity and exploration. Moreover, higher levels of self-compassion are associated with decreased levels of difficulties in aspects of emotion regulation in college students, such as goal-directed behavior, impulse control, and emotional responses (Meyer & Leppma, 2019). Neff and Germer (2013) also demonstrated a negative relationship between self-compassion and psychological distress (e.g., depression, anxiety, and stress). Self-compassion also has a negative relationship with math anxiety in college students (Leppma & Darrah, 2022).

*How is Grit Related to Mindfulness?* Mindfulness is defined as the intentional, nonjudgmental awareness of the present moment (Kabat-Zinn, 2003). Mindfulness was shown to be positively related to improved emotion regulation (Meyer & Leppma, 2019) and inversely related to depression, anxiety, and anxiety symptom severity (Tubbs et al., 2019) in college students. Raphiphatthana et al. (2018) theorized that because mindfulness is associated with improved coping abilities, it may promote the ability to persevere and work through obstacles to reach a long-term goal, which is the definition of grit (Duckworth et al., 2007). Although mindfulness seems to be an antecedent to grit, very little research exists exploring the relationship between mindfulness and grit. We found only three studies to date. The first is the study by Jarukasemthawee et al. (2021) mentioned above. In the second, Raphiphatthana et al. (2019) compared associations between mindfulness and grit for Western and non-Western university students. Their findings indicated that mindfulness was associated with grit in both cultures; however, there was a stronger association in Western university students. In the final study, 343 undergraduates completed an online survey, and 74 of those participants completed the survey again 4.5 months later to assess a potential longitudinal relationship between mindfulness and grit (Raphiphatthana et al., 2018). Results indicated that mindfulness predicted an increase in grit, which was maintained over time. The authors concluded that mindful individuals tend to be

grittier. They suggested that non-judgmental awareness of the present moment may facilitate sustained interest in long-term goals.

### Methods

Participants included 258 students, 68 students were enrolled in College Algebra with co-requisite and 190 were enrolled in the course without co-requisite. Of the students that reported, 233 were in a STEM major, the others were in various fields. Not all respondents completed every scale of the survey. For the analysis of each scale, we used all respondents who completed that scale. Student placement into College Algebra courses (co-requisite or non-co-requisite) is based on their SAT/ACT Math scores or how they performed on a Math Placement Exam (ALEKS). The co-requisite course focused on prerequisite skills including operations on real numbers and simplifying algebraic expressions, and also metacognitive skills such as mindfulness, coping with math and test anxiety, and self-compassion.

The research took place at a large public research university upon institutional review board approval. The participants in the project were a convenience sample of students who were enrolled in college algebra in the fall semester of 2020. The university offers college algebra and college algebra with a co-requisite support course for students with lower entrance scores. Students in both versions of the courses were asked to participate in the research study. All 494 students enrolled in the course were sent a survey near the 3rd week of the semester through email and offered five bonus points for completing it. The response rate for the survey was 52.2% (258/494). The following instruments were used to collect the responses:

- The Abbreviated Mathematics Anxiety Rating Scale (AMARS; Alexander & Martray, 1989),
- Grit-S (Duckworth & Quinn, 2009),
- Self-Efficacy for Learning and Performance Subscale (SELPS) from the Motivated Strategies for Learning Questionnaire (MSLQ) for College Students (Pintrich & De Groot, 1990),
- Five Facet Mindfulness Questionnaire (FFMQ; Baer et al, 2006), and
- Self-Compassion Scale-Short Form (SCS-SF; Raes et al., 2011).

### Results

#### Data Analyses/Power Analyses

Power analyses were calculated and it was determined that the minimum sample size for multiple regression with a medium effect size, alpha level of .05, power of .90, and three predictors was 99. Our total sample size was much higher at 258. Simple linear regression was used to analyze research question 1. For research question 2, we used hierarchical multiple regression. The order of variables entered into the hierarchical multiple regression was based on empirical support (Pallant, 2020). In block 1, we entered self-efficacy due to its strong support regarding math engagement and achievement in the literature (Samuel & Warner, 2019). We entered self-compassion in block 2. Our data demonstrated a higher correlation between self-compassion and grit than mindfulness and grit. In addition, research indicates that self-compassion is frequently a mediator between mindfulness and other variables (Neff, 2016). Thus, we entered mindfulness in block 3. (See Table 1 for correlations).

**Table 1: Correlations for Study Variables**

	Variable	1	2	3	4
1	Grit	--	.383*	.394*	.356*
2	Self-Efficacy	.383*	--	.338*	.278*
3	Self-Compassion	.394*	.338*	--	.549*
4	Mindfulness	.356*	.278*	.559*	--

\*p < .001

### Research Question 1: Does grit have an impact on math anxiety?

Linear regression was used to evaluate the ability of grit to predict levels of math anxiety. A scatterplot indicated a linear relationship between the variables and homoscedasticity and normality of the residuals was detected. The model was significant. Math anxiety = 112.35 + (-1.72).  $F(1, 234) = 46.69, p < .001$ . Grit was negatively related to math anxiety and accounted for 17% of the variance. (See Table 2).

**Table 2: Math Anxiety Simple Linear Regression**

Grit	B	95% CI for B		SE B	$\beta$	R2	$\Delta R2$
Model						.172	.169
Constant	112.35***	99.77	124.92	6.38			
Grit	-1.72***	-2.21	-1.24	.25	-.415***		

\*p < .05. \*\*p < .01. \*\*\*p < .001.

Note: Model = Enter method in SPSS Statistics; B = unstandardized regression coefficient; CI = confidence interval; LL = lower limit; UL = upper limit, SE B = standard error of the coefficient;  $\beta$  = standardized coefficient; R2 = coefficient determination;  $\Delta R2$  = adjusted R2.

### Research Question 2: Do Self-Compassion, Mindfulness, and Self-Efficacy predict Grit?

Hierarchical multiple regression was used to better understand grit and assess the ability of self-efficacy, self-compassion, and mindfulness to predict levels of grit. To ensure no violations of assumptions of normality, linearity, multicollinearity, and homoscedasticity, preliminary analyses were conducted. In Step 1, self-efficacy was entered, and it explained 14.7% of the variance in grit. In Step 2, self-compassion was entered, and it explained 22.6% of the variance. In Step 3, mindfulness was entered. The variance explained by this model was 24.4%,  $F(3, 230) = 24.46, p < .001$ . (See Table 3).

**Table 3: Hierarchical Regression Final Model**

Grit	B	95% CI for B		SE B	$\beta$	R2	$\Delta R2$
Model		LL	UL			.244	.234
Constant	12.62***	9.53	15.71	1.57			
Self-Efficacy	.13***	.07	.19	.03	.27***		
Self-Compassion	.13**	.04	.21	.04	.21**		
Mindfulness	.03*	.01	.06	.01	.17*		

Note: Model = Enter method in SPSS Statistics;  $B$  = unstandardized regression coefficient; CI = confidence interval;  $LL$  = lower limit;  $UL$  = upper limit, SE  $B$  = standard error of the coefficient;  $\beta$  = standardized coefficient;  $R^s$  = coefficient determination;  $\Delta R^2$  = adjusted  $R^s$ .

### Discussion

This study seeks to establish relationships among self-compassion, mindfulness, self-efficacy, math anxiety, and grit. This article attempts to bring understanding to the correlates of grit and its role in math anxiety. The idea is to ultimately test interventions that may have beneficial effects on the variables. In this article we consider data collected three weeks into the semester shortly before the first exam and before any intervention was implemented.

Findings from the present study indicate an inverse relationship between grit and math anxiety in college students. This result aligns with Holtby's (2018) similar findings in an investigation of grit, coping, and math anxiety in elementary school students. In a study of 56 students enrolled in Grades 4, 5, and 6, Holtby found a significant negative correlation between grit and math anxiety. Although further research is needed, these findings taken together with the present study's findings suggest that understanding and increasing correlates of grit may be helpful in fostering resilience and perseverance in students by reducing math anxiety.

Findings also support our hypothesis that as self-compassion, mindfulness, and self-efficacy are higher this will predict higher levels of grit. The analysis of this student data corresponds to the conclusions in Alhadabi and Karpinski (2020) who found that grit dimensions (G-CI and G-PE) correlate positively with self-efficacy in university students ( $N = 258$ ). Our results further align with Jarukasemthawee et al. (2021), who found statistically significant positive relationships between grit and self-compassion and grit and mindfulness in 320 Thai national athletes. Contrary to our findings, however, their findings suggested a direct relationship between mindfulness and grit but an indirect relationship between self-compassion and grit, mediated by self-regulation. A possible explanation for this is the different context and population. In two studies examining grit specifically in undergraduate university students, researchers found that increased mindfulness predicted increased grit (Raphiphatthana et al., 2018; Raphiphatthana et al.), which correspond with our findings. Our study is novel in that it also found that self-compassion was a predictor of grit in college math students. Based on our findings, students who have higher levels of self-efficacy, mindfulness, and self-compassion tend to be grittier.

### Conclusions

Most math educators do not suffer from math anxiety and in turn may have trouble relating to students who do. Furthermore, those who teach mathematics courses may not know how to advise students who are searching for ways to mitigate their anxiety. Research suggests that non-cognitive factors, such as math anxiety, can hinder students' abilities in math (Henslee & Klein, 2017). Our findings suggest that implementing interventions aimed to increase self-efficacy, mindfulness, and self-compassion can increase grit in students. However, we realize that math instructors must understand this connection and more importantly how to intervene through the introduction of specific practices that include mindfulness and self-compassion. Since instructors establish classroom norms, the success of these types of interventions is tied to instructor buy-in and training. Professional development for those who are teaching entry-level math courses is critical to the success of interventions designed to affect a student's mindset.

This study provides empirical support for grit as a predictor of math anxiety. Our findings also provide insight into correlates of grit. Self-efficacy, mindfulness, and self-compassion all predicted

grit. Understanding and addressing these factors can provide educators with helpful direction in developing grit in students and reducing math anxiety that impedes the progress of many students. As a next step after finding these connections, we have begun to develop interventions that we believe will help to reduce math anxiety. The interventions are in the form of online modules students work through either for homework or in class that contain a short description of the topic, video, and open-ended questions for reflection. These modules are on topics including grit, math anxiety, mindfulness, self-compassion, and growth mindset. We are in the process of testing them with students to determine their effectiveness at reducing math anxiety and increasing student persistence and performance. Future research will explore mechanisms, interventions, and effectiveness in a broader population.

### References

- Alexander, L., & Martray, C. (1989). The development of an abbreviated version of the Mathematics Anxiety Rating Scale. *Measurement and Evaluation in Counseling and Development*, 22(3), 143-150.  
<https://doi.org/10.1080/07481756.1989.12022923>
- Alhadabi, A., & Karpinski, A. C. (2020) Grit, self-efficacy, achievement orientation goals, and academic performance in University students, *International Journal of Adolescence and Youth*, 25(1), 519-535,  
<https://doi.org/10.1080/02673843.2019.1679202>
- Baer, R. A., Smith, G. T., Hopkins, J., Krietemeyer, J., & Toney, L. (2006). Using self-report assessment methods to explore facets of mindfulness. *Assessment*, 13(1), 27-45. <https://doi.org/10.1177/1073191105283504>
- Bandura, Albert (1982). Self-efficacy mechanisms in human agency. *American Psychologist*, 37 (2): 122–147.  
<https://doi.org/10.1037/0003-066X.37.2.122>.
- Bandura, A. (1993). Perceived self-efficacy in cognitive development and functioning. *Educational psychologist*, 28(2), 117-148.
- Bandura, A. (1996). Failures in self-regulation: Energy depletion or selective disengagement?. *Psychological Inquiry*, 7(1), 20-24.
- Beilock, S. L., & Maloney, E. A. (2015). Math anxiety: A factor in math achievement not to be ignored. *Policy Insights from the Behavioral and Brain Sciences*, 2(1), 4-12.
- Bowman, N. A., Hill, P. L., Denson, N., & Bronkema, R. (2015). Keep on truckin' or stay the course? Exploring grit dimensions as differential predictors of educational achievement, satisfaction, and intentions. *Social Psychological and Personality Science*, 6(6), 639-645.
- Brunyé, T. T., Mahoney, C. R., Giles, G. E., Rapp, D. N., Taylor, H. A., & Kanarek, R. B. (2013). Learning to relax: Evaluating four brief interventions for overcoming the negative emotions accompanying math anxiety. *Learning and Individual Differences*, 27, 1-7. <https://doi.org/10.1016/j.lindif.2013.06.008>
- Credé, M., Tynan, M. C., & Harms, P. D. (2017). Much ado about grit: A meta-analytic synthesis of the grit literature. *Journal of Personality and Social Psychology*, 113(3), 492.
- Cross, T. M. (2014). The gritty: grit and non-traditional doctoral student success. *Journal of Educators Online*, 11(3), n3.
- Czocher, J. A., Melhuish, K., & Kandasamy, S. S. (2020). Building mathematics self-efficacy of STEM undergraduates through mathematical modelling. *International Journal of Mathematical Education in Science and Technology*, 51(6), 807-834.
- Doorley, J. D., Kashdan, T. B., Weppner, C. H., & Glass, C. R. (2022). The effects of self-compassion on daily emotion regulation and performance rebound among college athletes: Comparisons with confidence, grit, and hope. *Psychology of Sport & Exercise*, 58, 1-11. <https://doi.org/10.1016/j.psychsport.2021.102081>
- Duckworth, A. L., Peterson, C., Matthews, M. D., & Kelly, D. R. (2007). Grit: perseverance and passion for long-term goals. *Journal of personality and social psychology*, 92(6), 1087.
- Duckworth, A. L., & Quinn, P. D. (2009). Development and validation of the Short Grit Scale (GRIT-S). *Journal of personality assessment*, 91(2), 166-174.
- Foley, A. E., Herts, J. B., Borgonovi, F., Guerriero, S., Levine, S. C., & Beilock, S. L. (2017). The Math Anxiety-Performance Link: A Global Phenomenon. *Current Directions in Psychological Science*, 26(1), 52–58.  
<https://doi.org/10.1177/0963721416672463>
- Henslee, A., & Klein, B. (2017). Using brief guided imagery to reduce math anxiety and improve math performance: A pilot study. *Journal of STEM Education*, 18(4).

- Hodge, B., Wright, B., & Bennett, P. (2018). The role of grit in determining engagement and academic outcomes for university students. *Research in Higher Education*, 59(4), 448-460.
- Holtby, N. M. (2018). *Grit, Coping, and Math Anxiety: Examining the Pathways Through Which Devotion to Long-Term Goals May Promote Student Well-Being* (Doctoral dissertation).
- Honicke, T., & Broadbent, J. (2016). The influence of academic self-efficacy on academic performance: A systematic review. *Educational Research Review*, 17, 63-84.
- Jarukasemthawee, S., Pisitsungkagarn, K., O'Brien, W., Manley, H., & Pattanamontri, C. (2021). Relationships between mindfulness, self-compassion, and grit among Thai national athletes: The mediating role of self-regulation. *International Journal of Sport and Exercise Psychology*.  
<https://doi.org/10.1080/1612197X.2021.2010230>
- Kabat-Zinn, J. (2003). Mindfulness-based interventions in context: past, present, and future. *Clinical Psychology: Science and Practice*, 10(2), 144-156. <https://doi.org/10.1093/clipsy/bpg016>
- Lam, K. K. L., & Zhou, M. (2019). Examining the relationship between grit and academic achievement within K-12 and higher education: A systematic review. *Psychology in the Schools*, 56(10), 1654-1686.
- Langdon, D., McKittrick, G., Beede, D., Khan, B., & Doms, M. (2001). STEM: Good jobs now and for the future. ESA Issue Brief# 03-11. US Department of Commerce.
- Leppma, M., & Darrah, M. (2022). Self-efficacy, mindfulness, and self-compassion as predictors of math anxiety in undergraduate students. *International Journal of Mathematical Education in Science and Technology*, 1-16.
- Mammarella, I. C., Donolato, E., Caviola, S., & Giofrè, D. (2018). Anxiety profiles and protective factors: A latent profile analysis in children. *Personality and Individual Differences*, 124, 201-208.
- Meyer, L. P., & Leppma, M. (2019). The role of mindfulness, self-compassion, and emotion regulation in eating disorder symptoms among college students. *Journal of College Counseling*, 22, 211-224.
- Mohammadyari, G. (2012). Comparative study of relationship between general perceived self-efficacy and test anxiety with academic achievement of male and female students. *Procedia-Social and Behavioral Sciences*, 69, 2119-2123.
- Neff, K. (2003). Self-compassion: An alternative conceptualization of a healthy attitude toward oneself. *Self and identity*, 2(2), 85-101. <https://doi.org/10.1080/15298860309032>
- Neff, K. (2016). The self-compassion scale is a valid and theoretically coherent measure of self-compassion. *Mindfulness*, 7, 264-274. <https://doi.org/10.1007/s12671-015-0479-3>
- Neff, K., & Germer, C. (2013). A pilot study and randomized control trial of the Mindful Self-Compassion program. *Journal of Clinical Psychology*, 69(1), 28-44. DOI: 10.1002/jclp.21923
- Neff, K. D., Hsieh, Y. P., & Dejjitrat, K. (2005). Self-compassion, achievement goals, and coping with academic failure. *Self and Identity*, 4(3), 263-287. <https://doi.org/10.1080/13576500444000317>
- Neff, K. D., Rude, S. S., & Kirkpatrick, K. L. (2007). An examination of self-compassion in relation to positive psychological functioning and personality traits. *Journal of research in personality*, 41(4), 908-916.
- Pallant, J. (2020). *SPSS survival manual: A step by step guide to data analysis using IBM SPSS*. Routledge.
- Peri, G., Shih, K. Y., & Sparber, C. (2014). Foreign STEM workers and native wages and employment in US cities (No. w20093). National Bureau of Economic Research.
- Peterson, C. H., Casillas, A., & Robbins, S. B. (2006). The student readiness inventory and the big five: Examining social desirability and college academic performance. *Personality and Individual Differences*, 41(4), 663-673. <https://doi.org/10.1016/j.paid.2006.03.006>
- Pintrich, P. R., & De Groot, E. V. (1990). Motivational and self-regulated learning components of classroom academic performance. *Journal of Educational Psychology*, 82(1), 33. <https://doi.org/10.1037/0022-0663.82.1.33>
- Raes, F., Pommier, E., Neff, K. D., & Van Gucht, D. (2011). Construction and factorial validation of a short form of the self-compassion scale. *Clinical Psychology & Psychotherapy*, 18(3), 250-255. <https://doi.org/10.1002/cpp.702>
- Ramirez, G., Shaw, S. T., & Maloney, E. A. (2018). Math anxiety: Past research, promising interventions, and a new interpretation framework. *Educational Psychologist*, 53(3), 145-164.
- Raphiphatthana, B., Jose, P. E., & Chobthamkit, P. (2019). The association between mindfulness and grit: An east vs. west cross-cultural comparison. *Mindfulness*, 10, 146-158. <https://doi.org/10.1007/s12671-018-0961-9>
- Raphiphatthana, B., Jose, P., & Salmon, K. (2018). Does dispositional mindfulness predict the development of grit? *Journal of Individual Differences*, 39(2), 76-87. <https://doi.org/10.1027/1614-0001/a000252>
- Samuel, T. S., & Warner, J. (2019) I can math! Reducing math anxiety and increasing math self-efficacy using a mindfulness and growth mindset-based intervention in first-year students. *Community College Journal of Research and Practice*. <https://doi.org/10.1080/10668926.2019.1666063>
- Tobias, S. (1976). Math anxiety: Why is a smart girl like you counting on your fingers. *Ms*, 5(1), 56-59.

- Tobias, S. (1991). Math mental health: Going beyond math anxiety. *College Teaching*, 39(3), 91-93.
- Tubbs, J. D., Savage, J. E., Adkins, A. E., Amstadter, A. B., & Dick, D. M. (2019). Mindfulness moderates the relation between trauma and anxiety symptoms in college students. *Journal of American College Health*, 67(3), 235-245. <https://doi.org/10.1080/07448481.2018.1477782>
- van den Hurk, A., Meelissen, M. & van Langen, A. (2019) Interventions in education to prevent STEM pipeline leakage. *International Journal of Science Education*, 41:2, 150-164, <https://doi.org/10.1080/09500693.2018.1540897>
- Wang, Z., Lukowski, S. L., Hart, S. A., Lyons, I. M., Thompson, L. A., Kovas, Y. & Petrill, S. A. (2015). Is math anxiety always bad for math learning? The role of math motivation. *Psychological science*, 26(12), 1863-1876.



# STUDENTS' EXPERT-LIKE ATTITUDES IN CALCULUS AND INTRODUCTORY COMPUTER SCIENCE COURSES WITH ACTIVE-LEARNING PEDAGOGY

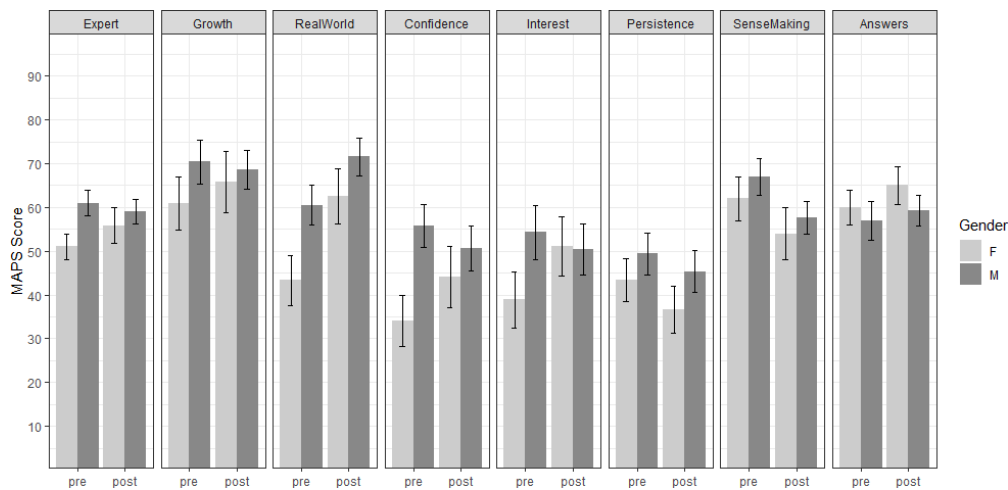
Steve Balady  
California State University San Bernardino  
steve.balady@csusb.edu

Cynthia Taylor  
Oberlin College  
ctaylor@oberlin.edu

Keywords: Affect, Emotion, Beliefs, and Attitudes, Calculus, Computing and Coding, Gender

Student attitudes are key to retention in the Computer Science major, especially those regarding “weedout” courses such as Calculus. We use two validated instruments, the Computer Attitudes Survey (CAS) and the Mathematics Attitudes and Perceptions Survey (MAPS), to compare student attitudes in Calculus I and CS I active-learning courses with explicit real-world connections. Our analysis includes 109 students for the CAS and 73 for the MAPS. Our poster will emphasize the following results:

- Calculus students started the course with MAPS scores in line with prior work, but unlike prior work we find that overall MAPS scores did not decline during the semester. Women’s reported interest in math increased significantly during the semester, as shown in Figure 1.
- Both courses were taught by gender minorities in STEM, and our results are consistent with recent work (Sullivan, Voigt, Apkarian, Martinez, & Hagman, 2021) finding that women, transgender, and non-binary students were significantly more likely to describe peer interaction as helpful when their instructor identified as a member of the same group.
- When entering their course, CS students are more excited about CS 1 than Calculus students are about Calculus I, and they have a stronger belief that the course contents will be useful. The interest gap is even higher among Calculus students identified as needing additional precalculus support, so work targeting interest is particularly important and potentially impactful for these students.



**Figure 1: Calculus I pre and post MAPS data (N=73), stratified by gender. Sample means along with individual uncertainty for these means reported as  $\pm 1$  standard error.**

## References

Sullivan, T. J., Voigt, M. K., Apkarian, N., Martinez, A. E., & Hagman, J. E. (2021, July). Role with it: Examining the impact of instructor role models in introductory mathematics courses on student experiences. In 2021 ASEE Virtual Annual Conference Content Access.

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

# STUDENTS' MATHEMATICS IDENTITY AND MATHEMATICAL AGENCY FROM 5<sup>TH</sup> THROUGH 11<sup>TH</sup> GRADE

Jennifer Cribbs

Oklahoma State University  
Jennifer.cribbs@okstate.edu

Juliana Utley

Oklahoma State University  
Juliana.utley@okstate.edu

*We explored 5<sup>th</sup> through 11<sup>th</sup> grade students' mathematics identity and mathematical agency through a cross-sectional analysis. Results indicate changes in these measures with significant decreases in mathematics identity from 5<sup>th</sup> to 6<sup>th</sup> grade as well as 8<sup>th</sup> to 10<sup>th</sup> and 11<sup>th</sup> grade. However, a significant increase in mathematics identity was found from 6<sup>th</sup> to 8<sup>th</sup> and 9<sup>th</sup> grade. Mathematical agency also significantly shifted between grade levels, particularly with the sub-constructs collective and disciplinary/conceptual agency, which had a similar trend as mathematics identity. This study highlights the potential for meaningful changes in mathematics identity and mathematical agency across students' schooling.*

Keywords: Affect, Emotion, Beliefs, and Attitudes

Students' mathematics identity and mathematical agency play an important role in students' persistence, performance, and long-term engagement with mathematics. For example, Cribbs et al. (2021) found that mathematics identity was predictive of students' STEM career choice. Likewise, Godwin et al., (2016) found that student agency was predictive of an engineering career choice. The predictive power of identity and agency on students' career choice provides evidence that these factors influence students' future choices. Additionally, Bohrnstedt and colleagues (2021) found that students' mathematics identity was predictive of their mathematics achievement. Evidence from these studies as well as a wealth of literature in the field noting the importance of mathematics identity and mathematical agency (e.g., Aguirre et al., 2013; Bishop, 2012; Boaler & Greeno, 2000; Graven & Heyd-Metzuyanin, 2019; Gresalfi et al., 2009) highlight the importance of this area of research. However, there is still a need to better understand how these factors shift over the course of students' accumulating mathematics experiences in school. The purpose of this study was to conduct a cross-sectional analysis of 5<sup>th</sup>-11<sup>th</sup> grade students' identity and agency as it relates to mathematics.

## Theoretical Framework

A variety of definitions have been used in prior research when exploring mathematics identity, which aligns to the perspectives and approaches that researchers have used for their studies (Darragh, 2016; Graven & Heyd-Metzuyanin, 2019; Radovic et al., 2018). Mathematics identity in this study draws on the theoretical perspective of core identity (Cobb & Hodge, 2011; Gee, 2001), which reflects an individual's more enduring sense of identity. This perspective aligns with Holland and Lave's (2001) discussion of identity "thickening" (p. 19). Prior related research found that mathematics identity is comprised of three sub-constructs: interest, recognition, and competence/performance (Cribbs et al., 2015; Cribbs & Utley, in ReviewA). Interest is defined as an individual's design or curiosity to think about and learn mathematics; recognition is defined as how individuals view themselves and how they perceive others view them in relation to mathematics; competence is defined as an individual's belief about their ability to understand mathematics; and performance is defined as an individual's beliefs about their ability to perform in mathematics. Together, these sub-constructs provide a picture of an individual's mathematics identity.

Another construct to explore along with mathematics identity is mathematical agency. While limited research has explored mathematical agency quantitatively, research notes the important role of agency when exploring identity (Aguirre et al., 2013; Holland & Lave, 1998). Aguirre et al. (2013) noted that “the behavioral aspect of mathematics identity can also be captured by the term agency” (p. 16). NCTM (2018) notes that “how students are positioned to participate in mathematics affects not only what they learn but how they come to see themselves” (p. 28). This statement hints at the relationship between agency and identity, where each informs the other. Speranzo and Klingensmith’s (2020) also theorized this reciprocal relationship between mathematics identity and mathematical agency when discussing their work with children. This idea is consistent with the ways that researchers discuss identity and agency in that they are connected aspects of how an individual sees themselves and that this sense of self informs their actions and vice versa. Researchers have discussed mathematical agency in four ways: disciplinary agency, conceptual agency, collective agency, and critical agency. Each of these types of agency have been defined in nuanced ways that make them distinct from each other. For example, Gresalfi et al. (2009, p. 56) defined disciplinary agency “recalling facts or definitions and executing procedures” where “there are correct answers, and a student either gets it right or doesn’t”. They also defined conceptual agency as “procedures with connections and, especially doing mathematics generally” where “students being positioned to take initiative in constructing meaning and understanding of the methods and concepts that are the subjects of their learning” (p. 56). Both of these perspectives for exploring agency can capture the type of norms that have been established in students’ mathematics classrooms. Raithelbuer (2016) positioned mathematical agency differently through a social and collective perspective. Aguirre et al. (2013) stated that collective agency is “when teachers and their students act together to solve problems, working from the shared belief that viable strategies can be developed, and solutions found” (p. 17). A final perspective is critical agency, which Turner (2012) discussed as an equity-based approach, which includes “students’ capacity to: view the world with a critical mind-set; imagine how the world might become a more socially just, equitable place; [and] engage in action aimed at personal and social transformation” (p. 55).

Through these different perspectives, mathematical agency and mathematics identity can be explored to provide a more nuanced picture of students’ mathematics identity development. Our research questions are:

1. Is there a significant change in students’ mathematics identity with respect to grade level?
2. Is there a significant change in students’ mathematical agency with respect to grade level?

### **Methods**

A cross-sectional survey design, method where “data come from people at different age points with different people representing each age point”, was used to assess 5<sup>th</sup>-11<sup>th</sup> grade students’ mathematics related identity and beliefs (Field et al., p. 916)

### **Participants**

Surveys were collected from two districts in a mid-western state in the United States. Students in grades 5<sup>th</sup>-11<sup>th</sup> grade were asked to participate, resulting in 1,655 surveys. After data was cleaned and responses with large portions of missing entries removed, the sample included 1,437 participants. With regard to gender, 45% identified as Male, 52% as Female, 1.5% as Other, and 1.5% did not specify. Participants were between the ages of 9 and 18. Twenty percent of the sample indicated they were of Hispanic origin. With regard to race, 56% indicated they were White, 17% Native American and White, 7% Native American, 6% multicultural, 5% were uncertain, 3% Black or African American, 2% African American and White, 2% Asian, and 1%

Other. In terms of grade level distribution, 18% were in 5<sup>th</sup> grade, 13% 6<sup>th</sup> grade, 10% 7<sup>th</sup> grade, 10% 8<sup>th</sup> grade, 12% 9<sup>th</sup> grade, 21% 10<sup>th</sup> grade, and 11% 11<sup>th</sup> grade.

### Data Collection and Analysis

A survey was administered to participants in late spring of 2022. In addition to background factors (e.g., gender, race, age), the survey included items for a variety of mathematics related constructs, which will be detailed in the subsequent sections.

*Mathematics Identity.* Mathematics identity was measured through a 16-item Likert-scale (1-Strongly disagree; 5-Strongly agree). Although this measure is comprised of three sub-constructs (competence and performance load as one sub-factor (Cribbs & Utley, in ReviewA)), overall mathematics identity can be explored by combining items and creating one overall mean. This scale included items such as “I see myself as a mathematics person” and “I look forward to taking math.”

*Mathematical Agency.* Mathematical agency was measured using three sub-scales: Discipline/Conceptual (7-items, e.g., “I can correct or fix my math errors when solving problems”), Collective (4-items; e.g., “In math class, we listen to each other’s math ideas”), and Critical (7-items; e.g., “Math has helped me make sense of the world around me.”). Given that the sub-factors for mathematical agency are theoretically distinct, they will be explored separately in this study. However, Discipline/Conceptual Agency will be combined as they have been shown to load together in prior research (Cribbs & Utley, in ReviewB). All items were on a Likert-scale with 1-Strongly disagree and 5-Strongly agree.

Both descriptive and inferential statistics were used in analysis of data. Prior to data analysis, simple random imputation was used to create a complete dataset, which is a better option than using listwise or mean imputation (Schlomer et al., 2010). Descriptive statistics included collection of means and deviations for the mathematics identity of students at each grade level and frequencies for use in describing the participant sample. Inferential statistics included a One-way Analysis of Variance with polynomial contrasts to look for significant change and trends in students’ mathematics identity and mathematical agency across grades 5-11.

### Results

A One-way Analysis of Variance (ANOVA) was used to examine whether fifth through eleventh grade students differ with respect to their mathematics identity and mathematical agency. The independent variable was the grade level of students (i.e., 5<sup>th</sup>, 6<sup>th</sup>, ..., 11<sup>th</sup>).

Table 1

*Means and Standard Deviation of Mathematics Identity and Agency Scores Across Grade Levels*

Grade	n	Math Identity M(SD)	Mathematics Agency		
			Critical M(SD)	Collective M(SD)	Disciplinary/ Conceptual M(SD)
5 <sup>th</sup>	257	3.09 (1.00)	2.93 (1.17)	2.63 (1.07)	3.19 (1.01)
6 <sup>th</sup>	191	2.75 (0.96)	2.64 (1.09)	2.73 (1.10)	2.86 (0.96)
7 <sup>th</sup>	141	3.03 (1.13)	2.83 (1.15)	2.96 (1.15)	3.15 (1.07)
8 <sup>th</sup>	148	3.18 (0.94)	2.60 (0.95)	2.87 (1.04)	3.16 (0.87)
9 <sup>th</sup>	170	3.21 (1.00)	2.65 (0.99)	2.90 (1.00)	3.31 (0.87)
10 <sup>th</sup>	295	2.67 (1.00)	2.47 (0.97)	2.65 (0.96)	2.89 (0.90)
11 <sup>th</sup>	156	2.73 (1.03)	2.70 (0.86)	2.60 (0.86)	2.88 (0.85)

The dependent variable was the students mean score on the mathematics identity and mathematical agency scales. See Table 1 for the means and standard deviation for each grade level.

First assumptions for the ANOVA were checked. The Shapiro-Wilk and Kolmogorov-Smirnov tests for normality indicated a significant result suggesting that the data was not normally distributed; however, Field (2017) and Pallant (2013) indicate that this is common with large samples and that researchers should examine both the distribution of the data in a histogram and the normal probability plots. In this case, both suggest a normal distribution of the dataset. The Levene's F test indicated that the homogeneity of variance assumption was met ( $p = .146$ ) for mathematics identity. However, it was not met for the three mathematical agency scales, so Welch's F test was used.

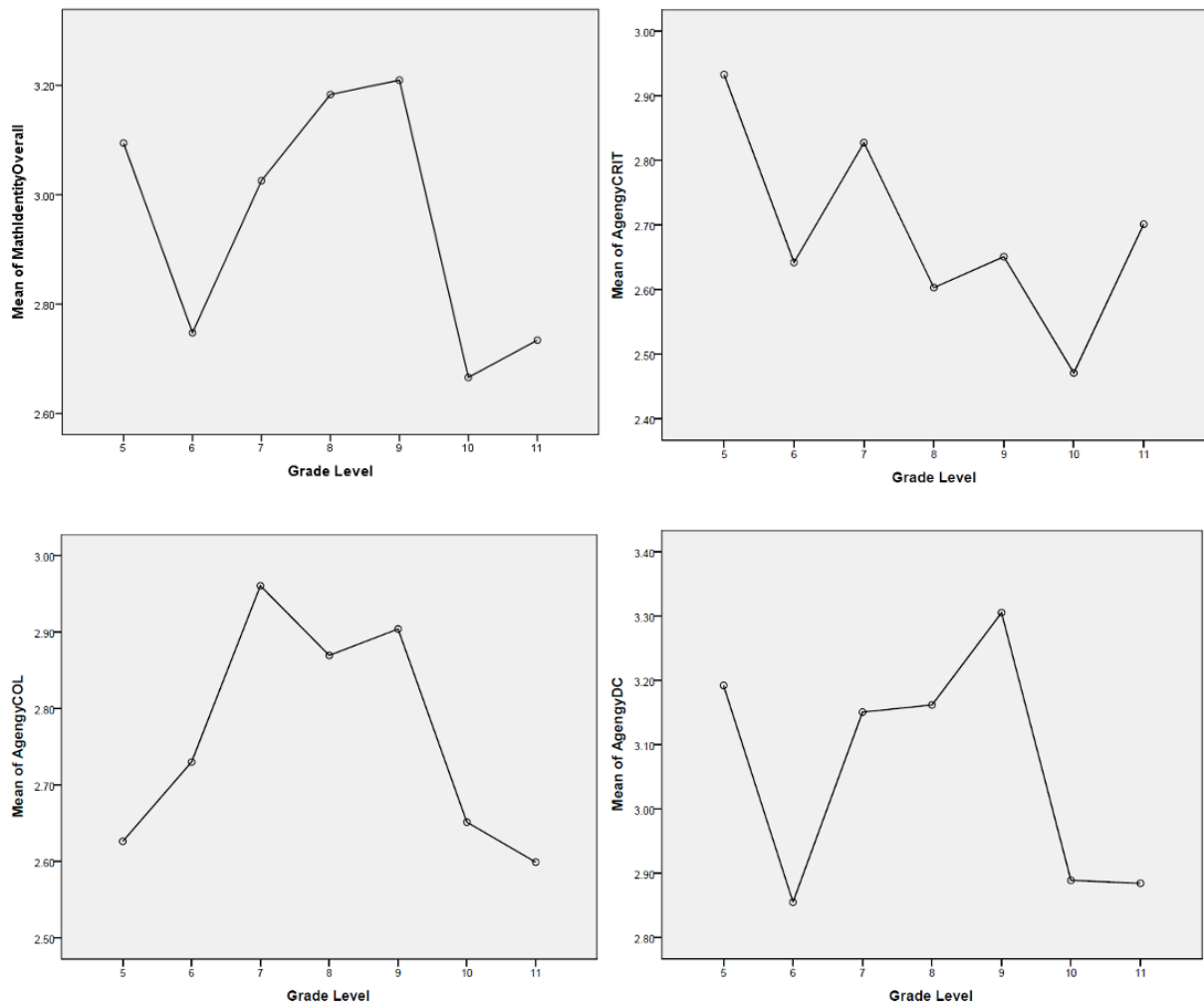
Table 2  
*Post Hoc Results for Mathematics Identity and Mathematical Agency by Grade Level*

Grade	Mean	Mean Differences ( $\bar{X}_i - \bar{X}_j$ ) (Cohen's $d$ is indicated in parentheses)						
		5 <sup>th</sup>	6 <sup>th</sup>	7 <sup>th</sup>	8 <sup>th</sup>	9 <sup>th</sup>	10 <sup>th</sup>	11 <sup>th</sup>
<i>Mathematics Identity</i>								
5 <sup>th</sup>	3.09	----						
6 <sup>th</sup>	2.75	-0.34*	----					
7 <sup>th</sup>	3.03	-0.06	0.28	----				
8 <sup>th</sup>	3.18	0.09	0.43*	0.15	----			
9 <sup>th</sup>	3.21	0.12	0.46*	0.18	0.03	----		
10 <sup>th</sup>	2.67	-0.42*	-0.08	-0.36*	-0.51*	-0.54*	----	
11 <sup>th</sup>	2.73	-0.36*	-0.02	-0.30	-0.45*	-0.48*	0.06	----
<i>Critical Mathematical Agency</i>								
5 <sup>th</sup>	2.93	----						
6 <sup>th</sup>	2.64	-0.29	----					
7 <sup>th</sup>	2.83	-0.10	0.18	----				
8 <sup>th</sup>	2.60	-0.33*	-0.04	-0.22	----			
9 <sup>th</sup>	2.65	-0.28	0.01	-0.18	0.05	----		
10 <sup>th</sup>	2.47	-0.46*	-0.17	-0.36*	-0.13	-0.18	----	
11 <sup>th</sup>	2.70	-0.23	0.06	-0.13	0.10	0.05	0.23	----
<i>Collective Mathematical Agency</i>								
5 <sup>th</sup>	2.63	----						
6 <sup>th</sup>	2.73	0.10	----					
7 <sup>th</sup>	2.96	0.33	0.23	----				
8 <sup>th</sup>	2.87	0.24	0.14	-0.09	----			
9 <sup>th</sup>	2.90	0.27	0.17	-0.06	0.03	----		
10 <sup>th</sup>	2.65	0.02	-0.08	-0.31	-0.22	-0.25	----	
11 <sup>th</sup>	2.60	-0.03	-0.13	-0.36*	-0.27	-0.30	-0.05	----
<i>Disciplinary/Conceptual Mathematical Agency</i>								
5 <sup>th</sup>	3.19	----						
6 <sup>th</sup>	2.86	-0.33*	----					
7 <sup>th</sup>	3.15	-0.04	0.29	----				
8 <sup>th</sup>	3.16	-0.03	0.30*	0.01	----			
9 <sup>th</sup>	3.31	0.12	0.45*	0.16	0.15	----		
10 <sup>th</sup>	2.89	-0.30*	0.03	-0.26	-0.27*	-0.42*	----	
11 <sup>th</sup>	2.88	-0.31*	0.02	-0.27	-0.28	-0.43*	-0.01	----

\* $p < .05$

The ANOVA of students' average score on the measure of mathematics identity revealed a statistically significant main effect ( $F(6, 1351) = 10.47, p < .001, r = .21$ ) indicating that not all students had the same average score on the measure of mathematics identity. The ANOVA of

students' average score on mathematical agency revealed a statistically significant main effect for collective mathematics agency ( $F(6, 545.85) = 3.55, p < .05, \omega^2 = .00$ ), critical mathematics agency ( $F(6, 548.69) = 4.81, p < .001, \omega^2 = .02$ ), and disciplinary/conceptual mathematics agency ( $F(6, 547.63) = 7.75, p < .001, \omega^2 = .01$ ).



**Figure 1: Means plot of average mathematics identity by grade level**

Post hoc analysis indicated that significantly negative changes in mathematics identity from across multiple grade level comparisons, with several that are particularly noteworthy such as 5<sup>th</sup> to 6<sup>th</sup>, 10<sup>th</sup>, and 11<sup>th</sup> grade as well as 9<sup>th</sup> to 10<sup>th</sup> and 11<sup>th</sup> grade. There was a significantly positive change in mathematics identity from 6<sup>th</sup> to 8<sup>th</sup> and 9<sup>th</sup> grade. The post hoc analysis for critical mathematical agency indicated significantly negative changes from 5<sup>th</sup> to 8<sup>th</sup>, 10<sup>th</sup>, and 11<sup>th</sup> grade. In addition, there was a negative change in mean from 7<sup>th</sup> to 10<sup>th</sup> grade. The post hoc analysis for collective mathematical agency indicated only one significant change with a decrease in mean from 7<sup>th</sup> to 11<sup>th</sup> grade. Finally, the post hoc for disciplinary/conceptual mathematical agency indicated multiple negative changes in mean with shifts from 5<sup>th</sup> to 10<sup>th</sup> and 11<sup>th</sup> grade as well as

9<sup>th</sup> to 10<sup>th</sup> and 11<sup>th</sup> grade. Results also indicated a positive change in mean from 6<sup>th</sup> to 8<sup>th</sup> and 9<sup>th</sup> grade. Table 2 provides an overview of the post hoc results.

A trend analysis was also performed to examine shifts in means from 5<sup>th</sup> to 11<sup>th</sup> grade. The trend analysis for mathematics identity indicated there was a significant cubic trend,  $F(6, 1351) = 10.47, p < .05, r = .07$ ). A quadratic trend was found for collective mathematical agency,  $F(6, 545.85) = 17.05, p < .001, \text{est. } \omega^2 = .01$ ). A linear trend was found for critical mathematical agency,  $F(6, 548.69) = 9.786, p < .001, \text{est. } \omega^2 = .01$ ), and a quartic trend for disciplinary/conceptual mathematical agency,  $F(6, 547.63) = 17.61, p < .001, \text{est. } \omega^2 = .01$ ). Figure 1 provides a visual representation of these trends.

### Discussion

Results from this study note significant changes in mathematics identity and mathematical agency across 5<sup>th</sup> through 11<sup>th</sup> grade for the participating students. In particular, a decrease in means from 5<sup>th</sup> to 6<sup>th</sup> grade was found for all but one factor (collective agency). This change might be surprising given participants in this study were in the same school in 5<sup>th</sup> and 6<sup>th</sup> grade. However, some noteworthy information about this particular school setting was that both 6<sup>th</sup> grade teachers in the year data was collected were new to the school or profession. This finding might hint at the significant impact of teacher changes at the school level and students' math identity and agency. This aligns with prior research noting the difference that teachers' have on students' experiences and opportunities to support mathematical agency (Atabas et al., 2020). Another noteworthy trend was the increase of students' mathematics identity and disciplinary/conceptual mathematical agency as they transitioned into middle school (7<sup>th</sup>-8<sup>th</sup> grade) and 9<sup>th</sup> grade. Also noteworthy, there were significant decreases in mean from middle grades and 9<sup>th</sup> to 10<sup>th</sup> and/or 11<sup>th</sup> grade for all of the factors. In looking at Figure 1, the trend for each of the measures appears similar except for critical mathematical agency, which seems to have a downward trend across 5<sup>th</sup> – 11<sup>th</sup> grade.

While this study did not explore factors for why these changes occur, it is promising to find that significant changes were evident, particularly when middle grades is often discussed as a period of time when students have declining perceptions toward mathematics (Gottfried et al., 2007). This study begins to unpack an area of research that needs much more exploration. The cross-sectional study provides a picture for discussion, but there are limitations to this analysis that future work can endeavor to explore. For example, each grade level is a different group of students. Results might be different if this study was conducted longitudinally to explore changes in identity and agency. In addition, participants in 5<sup>th</sup> through 9<sup>th</sup> grade were in one district while 10<sup>th</sup> and 11<sup>th</sup> were in a different district. It is uncertain how these varying settings influenced results. A follow up study with a larger group of schools/districts would allow for a more robust analysis of trends.

### Connecting to the Conference Theme

In line with the conference theme, this line of research endeavors to explore and share students' experiences through their participation in school settings, by examining their identity development over time (cross-sectionally). Given the call from the field to prioritize this area of work in K-12 settings (Larson, 2016; NCTM, 2018, 2020a, 2020b; NCSM & TODOS, 2016), this study is a step toward deepening our understanding of the critical role these constructs play in students' schooling.

### References

- Aguirre, J. M., Mayfield-Ingram, K., & Martin, D. B. (2013). The impact of identity in K-8 mathematics learning and teaching: Rethinking equity-based practices. NCTM.
- Atabas, S., Schellinger, J., Whitacre, I., Findley, K., & Hensberry, K. (2020). A tale of two sets of norms: Comparing opportunities for student agency in mathematics lessons with and without interactive simulations. *The Journal of Mathematical Behavior*, 58, 100761. <https://doi.org/10.1016/j.jmathb.2020.100761>

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.



- Bishop, J. P. (2012). "She's always been the smart one. I've always been the dumb one": Identities in the mathematics classroom. *Journal for Research in Mathematics Education*, 43(1), 34-74.  
<https://doi.org/10.5951/jresmetheduc.43.1.0034>
- Boaler, J., & Greeno, J. G. (2000). Identity, agency, and knowing. In J. Boaler (Ed.), *Multiple perspectives on mathematics teaching and learning*, (pp. 171 – 200). Ablex Publishing.
- Bohrnstedt, G. W., Cohen, E. D., Yee, D., & Broer, M. (2021). Mathematics identity and discrepancies between self- and reflected appraisals: Their relationships with grade 12 mathematics achievement using new evidence from a US national study. *Social Psychology of Education*, 24(3), 763-788.
- Cobb, P., & Hodge, L. L. (2011). Culture, identity, and equity in the mathematics classroom. In A. J. Bishop (Managing Ed.), E. Yackel, K. Gravemeijer, & A. Sfard (Vol. Eds.), *Mathematics Education Library: Vol. 48. A journey in mathematics education research: Insights from the work of Paul Cobb* (pp. 179-195) New York, NY: Springer.
- Cribbs, J., Hazari, Z., Sadler, P.M., & Sonnert, G. (2015). Establishing an explanatory framework for mathematics identity. *Child Development*. 86(4), 1048-1062. <http://doi.org/10.1111/cdev.12363>
- Cribbs, J., Hazari, Z., Sonnert, G., & Sadler, P. (2021). College students' mathematics-related career intentions and high school mathematics pedagogy through the lens of identity. *Mathematics Education Research Journal*.  
<https://doi.org/10.1007/s13394-020-00319-w>
- Cribbs, J., & Utley, J. (In ReviewA). Mathematics identity instrument development for fifth through twelfth grade students.
- Cribbs, J., & Utley, J. (In ReviewB). Mathematical agency instrument development for fifth through twelfth grade students.
- Darragh, L. (2016). Identity research in mathematics education. *Educational Studies in Mathematics*, 93 (1), 19–33.  
<https://doi.org/10.1007/s10649-016-9696-5>
- Field, A. (2017). *Discovering statistics using IBM SPSS Statistics* (5th ed.). Sage.
- Field, A., Miles, J., & Field, Z. (2012). *Discovering statistics using R*. Sage publications.
- Gee, J. P. (2001). Identity as an analytic lens for research in education. *Review of Research in Education*, 25, 99-125.
- Godwin, A., Potvin, G., Hazari, Z., & Lock, R. (2016). Identity, critical agency, and engineering: An affective model for predicting engineering as a career choice. *Journal of Engineering Education*, 105(2), 312-340.  
<https://doi.org/10.1002/jee.20118>
- Gottfried, A. E., Marcoulides, G. A., Gottfried, A. W., Oliver, P. H., & Guerin, D. W. (2007). Multivariate latent change modeling of developmental decline in academic intrinsic math motivation and achievement: Childhood through adolescence. *International Journal of Behavioral Development*, 31(4), 317-327.
- Graven, M., & Heyd-Metzuyanim, E. (2019). Mathematics identity research: The state of the art and future directions. *ZDM*, 51(3), 361-377. <https://doi.org/10.1007/s11858-019-01050-y>
- Gresalfi, M., Martin, T., Hand, V., & Greeno, J. (2009). Constructing competence: An analysis of student participation in the activity systems of mathematics classrooms. *Educational Studies in Mathematics*, 70(1), 49-70.  
<https://doi.org/10.1007/s10649-008-9141-5>
- Holland D., Lachicotte W. Jr., Skinner D., & Cain C. (1998). *Identity and agency in cultural worlds*. Harvard University Press.
- Holland, D., & Lave, J. (Eds.). (2001). *History in person: Enduring struggles, contentious practice, intimate identities*. School of American Research Press.
- Larson, M. (2016, September 15). A renewed focus on access, equity, and empowerment. National Council of Teachers of Mathematics. <https://www.nctm.org/News-and-Calendar/Messages-from-the-President/Archive/Matt-Larson/A-Renewed-Focus-on-Access,-Equity,-and-Empowerment/>
- National Council of Supervisors of Mathematics & TODOS: Mathematics for ALL. (2016). *Mathematics education through the lens of social justice: Acknowledgement, actions, and accountability* [Position statement].  
[https://www.todos-math.org/assets/docs2016/2016Enews/3.pospaper16\\_wtodos\\_8pp.pdf](https://www.todos-math.org/assets/docs2016/2016Enews/3.pospaper16_wtodos_8pp.pdf)
- National Council of Teachers of Mathematics. (2018). *Catalyzing change in high school mathematics: Initiating critical conversations*. NCTM.
- National Council of Teachers of Mathematics. (2020a). *Catalyzing change in early childhood and elementary mathematics: Initiating critical conversations*. NCTM.
- National Council of Teachers of Mathematics. (2020b). *Catalyzing Change in Middle School Mathematics: Initiating Critical Conversations*. NCTM.
- Pallant, J. (2013). *SPSS survival manual: A step by step guide to data analysis using IBM SPSS* (5th ed). McGraw Hill.

- Radovic, D., Black, L., Williams, J., & Salas, C. E. (2018). Towards conceptual coherence in the research on mathematics learner identity: A systematic review of the literature. *Educational Studies in Mathematics*, 99(1), 21-42. <https://doi.org/10.1007/s10649-018-9819-2>
- Raithelhuber, E. (2016). Extending agency: The merit of relational approaches for childhood studies. In F. Esser, M. S. Baader, T. Betz, & B. Hungerland (Eds.), *Reconceptualising agency and childhood* (Chapter 6). Taylor & Francis. <https://library.oapen.org/handle/20.500.12657/46467>
- Schlomer, G. L., Bauman, S., & Card, N. A. (2010). Best practices for missing data management in counseling psychology. *Journal of Counseling Psychology*, 57(1), 1-10. <https://doi.org/10.1037/a0018082>
- Speranzo, L., & Tileman, E., (2020). Using 4 learner-centered routines to build positive math identity in equitable classrooms. [https://www.ifl-news.pitt.edu/2020/12/using-4-learner-centered-routines-to-build-positive-math-identity-in-equitable-classrooms/?utm\\_campaign=shareaholic&utm\\_medium=email\\_this&utm\\_source=email](https://www.ifl-news.pitt.edu/2020/12/using-4-learner-centered-routines-to-build-positive-math-identity-in-equitable-classrooms/?utm_campaign=shareaholic&utm_medium=email_this&utm_source=email)
- Turner, E. (2012). Critical mathematical agency in the overcrowding at Francis middle school project. In E. Tan, A. Calabrese Barton, E. Turner & M. Varley Gutiérrez (Eds.), *Empowering science and mathematics education in urban communities* (pp. 51– 76). University of Chicago Press.

# TEACHERS' PERCEPTIONS AND BELIEFS ON MATH ANXIETY AMONG 3<sup>RD</sup> AND 4<sup>TH</sup>-GRADE CHILDREN: CASE STUDY IN MEXICO

Beatriz Ruiz  
Universidad Iberoamericana  
b\_ruiz\_b@hotmail.com

*This study aims to investigate the emotional dimension of math anxiety among children by exploring the perceptions of value and beliefs about math held by third and fourth-grade teachers at a primary school in Mexico City. These perceptions and beliefs can influence the presence of math anxiety in their students during crucial grades, as they lay the foundation for understanding key math concepts used in later years. Four teachers participated in semi-structured interviews, providing insights into their perspectives on math anxiety. Preliminary findings indicate two prevalent concerns from the teachers' viewpoints: children's lack of motivation to learn math and their belief that math is a difficult subject they cannot grasp. Despite the school's supportive environment, including socioemotional support, economic resources, and math learning facilities, negative attitudes toward math persist.*

Keywords: Math anxiety, Emotion, Beliefs, and Attitudes.

## Introduction

Math anxiety is a multifactorial construct, the complexity of which has not been fully understood (Mammarella et al., 2019). It generates blockages and negative attitudes toward learning math from the first years of primary education, and its presence affects academic performance, produces demotivation, and in the long run, it leads to avoidance of learning tasks or topics that involve math (Ashcraft, 2002; Dowker et al., 2016). In Mexico, the phenomenon is little studied and understood.

According to Richardson and Suinn (1972), math anxiety consists of “a feeling of tension and anxiety that interferes with the manipulation of numbers and the resolution of mathematical problems in everyday and academic situations” (as cited in Berch & Mazzocco, 2016, p.329).

A literature review was carried out about mathematical anxiety and the two dimensions that define it. On the one hand, the cognitive dimension where the executive function is directly affected and can be associated with the learning of mathematical concepts or topics that require greater use of working memory and attentional control (Ashcraft & Krause, 2007; Ramírez et al., 2013). On the other hand, the emotional dimension, which is the focus of this work, where certain socio-emotional skills such as self-regulation, perceived self-efficacy, and derived from this, intrinsic motivation, can interact with mathematical anxiety.

The motivation for studying this construct in 3<sup>rd</sup> and 4<sup>th</sup>-grade primary school students is that there is increasingly compelling evidence that mathematical anxiety is generated in the early years of primary school. The presence of mathematical anxiety in adults has been observed when performing simple numerical tasks, such as counting and estimating magnitudes, which leads to the belief that anxiety was generated when learning basic numerical knowledge, and not with complex mathematical tasks. In fact, researchers believe that mathematical anxiety is generated before reaching the sixth grade of primary school (Maloney et al., 2015).

Furthermore, it is relevant to highlight the lack of evidence regarding mathematical anxiety in contexts outside the United States and Europe, as emphasized by Caviola et al. (2017). Math anxiety incorporates a significant cultural element (Lau et al., 2022), and its emergence doesn't

universally correlate with the learning of complex mathematical concepts or topics (Berch & Mazzocco, 2016). That is, the same concepts don't universally provoke anxiety in all school environments. However, a literature gap exists concerning math anxiety in children in Latin American countries, which motivates this project.

### **The emotional dimension of math anxiety**

The study of math anxiety in early school years is relevant because it allows for an understanding of the relationship between this construct and academic performance in math, and because it enables the design of early interventions that contribute to reducing anxiety and promoting better performance in math in the future (Ramírez et al., 2013).

There is a close interrelationship between the cognitive and emotional factors involved in learning math and generating anxiety, which affects mathematical performance (Ashcraft & Krauze, 2007; Berch & Mazzocco, 2016; Carey et al., 2017). This situation results in overall poor academic performance (Rossnan, 2006), affecting students' levels of interest, value, self-concept, and self-esteem (Pérez-Fuentes et al., 2020). Therefore, the development of certain socio-emotional skills could provide children with the tools to face anxiety during a math task.

Regarding the emotional dimension of math anxiety, a cyclical behavior has been observed in which a student who experiences continuous failures develops great insecurity, which in turn generates anxiety. This cycle negatively affects both math performance and student anxiety (Dowker et al., 2016). The reciprocal model proposed by Pekrun (Carey et al., 2017) explains this behavior, where control and value evaluations towards learning can predict academic anxiety, affecting performance.

In this sense, there is broad support in the literature to use as a theoretical framework the Cognitive-Motivational Model of Achievement Emotions (Pekrun et al., 2018) and Pekrun's Control-Value Theory (Pekrun et al., 2007). Pekrun et al. (2007) identify two types of appraisals that precede emotions related to achievement. The first type is subjective control over the achievement of the activity, such as expectations of success or self-concept beliefs, perception of the ability to perform an activity successfully and beliefs related to these abilities, self-concept expectations, and self-efficacy. The second type of appraisal is the subjective value assigned to the task and its results, such as the importance given to that activity (Gabriel et al., 2020). Therefore, math anxiety may appear with deficient control appraisals facing a very high subjective value with respect to the mathematical activities involved. Likewise, the Control-Value Theory implies that to generate positive achievement emotions, students must have positive beliefs about their abilities, thus avoiding anxiety. Self-efficacy beliefs to complete specific tasks are considered control valuations (Gabriel et al., 2020). Similarly, Bandura (1993) points out that perceived self-efficacy operates through a student's convictions to regulate their learning, which determines their academic goals and motivation to achieve them. Therefore, it is essential to know them both in the teacher and the student.

From a socio-epistemological point of view, Cantoral et al. (2015) identify that, in school mathematical discourse in teaching, the visible, such as curricula, classroom presentation, and student materials, is as important as the invisible, such as teachers' and students' beliefs and conceptions about math. Therefore, it is essential to consider teachers' beliefs and perceptions, as their ways of seeing, working, and developing mathematical thinking may differ according to cultural context and affect their students' learning.

However, not all topics in math generate anxiety in students. Ching (2017) finds that the impact of math anxiety on performance is more noticeable when more working memory resources are required to solve mathematical problems. For example, two-column problems that use carrying or

borrowing operations (Ashcraft & Kirk, 2001). In addition, Álvarez-Montesino et al. (2018) mention the presence of anxiety when introducing multiplication in groups of second graders in Spanish-speaking contexts. Therefore, it is important to identify the topics that present greater complexity and determine whether they generate anxiety.

### **Research Objective**

This qualitative study aims to examine teachers' beliefs and perceptions about the emotional dimension of math anxiety in 3rd and 4th-grade students in a school in Mexico City, to address the research question: in what ways do teachers' perceptions and beliefs about the emotional dimensions of math anxiety influence its prevalence among their students?

### **Methodology**

The study involved semi-structured interviews with all four teachers from 3rd and 4th grades at a Mexico City school. Participants received assurance of confidentiality prior to the interviews. Each individual interview, lasting 40 minutes, underwent recording with the teacher's consent. The research team used an interview guide, divided into eight sections based on the theoretical model.

The first section collects information on the academic program, hours dedicated to math classes, and the specific resources used. The second section focuses on teaching methods, strategies, technology usage, and homework, aiming to identify elements that contribute to student motivation based on Pekrun's (2018) Cognitive-Motivational Model. These sections provide insights into the context of math classes according to each teacher. The third section explores challenging concepts in math, drawing from teachers' experiences and observations of student difficulties and anxiety. In the fourth section, teachers' appraisals of math and their self-efficacy in teaching the subject are examined, referencing Pekrun's (2007) Control-Value Theory and Bandura's (1993) concepts. The fifth section delves into the perceptions and beliefs that teachers hold regarding children's learning of math, providing initial insights into the mindset and perspectives teachers have on specific aspects of teaching math, influenced by Bandura's (1993) theories. The sixth section focuses on student motivation and perceived self-efficacy, aligning with Pekrun's (2018) Cognitive-Motivational Model. The topic of math anxiety is introduced in the seventh section, where the teachers' knowledge and observations of its presence among students are explored. Finally, the eighth section raises important questions about the role of parents in math learning and their active participation in supporting their children throughout the entire learning process.

The interviews were transcribed in their entirety, and a content qualitative analysis was conducted. This involved creating matrices, allowing for a structured visualization and comparison of responses to identify commonalities and differences across themes. Descriptive narratives were developed, and relevant quotes from the teachers were presented to support the findings and provide a deeper understanding of the participants' experiences and perspectives.

### **Results**

The teachers have expressed concerns about the difficulties that students encounter with specific mathematical concepts. In 3<sup>rd</sup> grade, students exhibit fear and struggle with division, fractions, and problem-solving, particularly with problems involving consecutive operations. Teacher J describes “3<sup>rd</sup>-grade children are terrified of division” and Teacher V adds “The lack of proficiency in multiplication tables contributes to these challenges” and addresses this by incorporating daily practice of multiplication tables into her teaching. In 4<sup>th</sup> grade, students face difficulties in learning fractions, including concepts such as simplification, equivalent fractions, and division. Likewise, teachers of both grades affirm that learning these concepts is difficult for every generation. Teachers from both 3<sup>rd</sup> and 4<sup>th</sup> grades share a common view that math is not

difficult for them. However, despite this general sentiment, they recognize specific concepts, such as equivalent fractions or algebra as challenging. Teacher J from 3<sup>rd</sup> grade, stated “In my experience, I have felt fear and anxiety when I had to teach algebra.” They believe that all children are capable of learning and can learn at the same rate. However, Teacher N from 4<sup>th</sup> grade mentions: “In Mexico, people have the idea that math is a very difficult subject. We, as teachers, need to change this chip in children.”

While teachers may not be familiar with the term math anxiety, they have observed behaviors like nervousness and distraction in their students that suggest discomfort with math. They've also noticed students frequently leaving class at the beginning of math lessons. Additionally, 3<sup>rd</sup> grade teachers recognize that some parents exert high pressure on their children over homework, potentially struggling to assist due to differences in the math methods they learned compared to current teaching approaches.

### **Discussion**

Beliefs and opinions converge among 3<sup>rd</sup> and 4<sup>th</sup> grade teachers who prioritize students' learning and well-being by using engaging strategies and positive reinforcement. Despite their efforts, these teachers recognize that students often approach math with fear, demonstrating a disliking for the subject and encountering challenges not typically seen in other academic pursuits. These observations mirror findings by Larkin et al. (2016), Ramírez et al. (2013), and Mammarella et al. (2019), attributing such behavior to the presence of math anxiety. Yet, these teachers, unfamiliar with the term 'math anxiety,' recall situations where students display avoidance behaviors, as Ashcraft (2002) describes.

The teachers observe a lack of motivation, which is reported as apathy and disinterest, negative emotions that may stem from perceptions of lack of control, and negative value beliefs. Therefore, Pekrun's Control-Value Theory provides a suitable framework for study (Klee et al., 2021). According to the teachers, the lack of motivation is reinforced by the attitude of some parents who consider it unimportant and unnecessary to dedicate time to math, assuming that it is difficult and not wanting to pressure their children. This attitude may be due to parents experiencing math anxiety and transmitting it to their children as Schaeffer et al. (2018) describe.

To the author's knowledge, an extensive literature review has not revealed any previous research exploring this construct in a Mexican population of children in 3<sup>rd</sup> and 4<sup>th</sup> grade. As a qualitative case study, the results cannot be generalized due to the small sample size and the focus on one school. Nevertheless, the study provides significant findings for future research.

### **Conclusion**

The results provide valuable insights into the emotional aspect of math anxiety in children from the perspective of teachers. Specific mathematical concepts that hinder their learning were also identified. Furthermore, it was confirmed that math anxiety is an unfamiliar construct for teachers, but they acknowledged its importance in benefiting their students.

Recognizing factors contributing to math anxiety helps educators to develop better math learning strategies, minimizing students' negative emotional experiences. Also, conducting this research in the Mexican context acknowledges unique cultural influences. The project aims to foster an enjoyable, motivational approach to math education, reducing math anxiety and promoting inclusion for children who often self-exclude.

### **References**

Álvarez-Montesinos, J. A., Costa, H. M., & García-Orza, J. (2018). ¿Qué necesitamos para aprender a multiplicar? El rol de las habilidades numéricas básicas y la ansiedad. *Escritos de Psicología / Psychological Writings*, 11(3), 103–114. <https://doi.org/10.5231/psy.writ.2018.221>

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

- Ashcraft, M. H. (2002). Math anxiety: Personal, educational, and cognitive consequences. *Current Directions in Psychological Science*, 11, 181-185. <https://doi.org/10.1111/1467-8721.00196>
- Ashcraft, M. H., & Kirk, E. P. (2001). The relationships among working memory, math anxiety, and performance. *Journal of Experimental Psychology: General*, 130(2), 224–237.
- Ashcraft, M., & Krause, J. (2007). Working memory, math performance, and math anxiety. *Psychonomic Bulletin & Review*, 14(2), 243–248.
- Bandura, A. (1993). Perceived Self-Efficacy in Cognitive Development and Functioning. *Educational Psychologist*, 28(2), 117–148. [https://doi.org/10.1207/s15326985Sep2802\\_3](https://doi.org/10.1207/s15326985Sep2802_3)
- Berch, D. B., & M.M. Mazzocco (2016). *Why is math so hard for some children?* MD: Paul Brookes.
- Cantoral, R., Montiel, G., & Reyes-Gasperini, D. (2015). Análisis del discurso Matemático Escolar en los libros de texto, una mirada desde la Teoría Socioepistemológica. *Avances de Investigación en Educación Matemática*, 8, 9–28.
- Carey, E., Hill, F., Devine, A., & Szucs, D. (2017). The modified abbreviated math anxiety scale: A valid and reliable instrument for use with children. *Frontiers in Psychology*, 8(JAN), 1–13.
- Caviola, S., Carey, E., Mammarella, I. C., & Szucs, D. (2017). Stress, time pressure, strategy selection and math anxiety in mathematics: A review of the literature. *Frontiers in Psychology*, 8(SEP), 1–14. <https://doi.org/10.3389/fpsyg.2017.01488>
- Ching, B. H. H. (2017). Mathematics anxiety and working memory: Longitudinal associations with mathematical performance in Chinese children. *Contemporary Educational Psychology*, 51, 99–113. <https://doi.org/10.1016/j.cedpsych.2017.06.006>
- Dowker, A., Sarkar, A., & Looi, C. Y. (2016). Mathematics anxiety: What have we learned in 60 years? *Frontiers in Psychology*, 7(APR), 1-16.
- Gabriel, F., Buckley, S., & Barthakur, A. (2020). The impact of mathematics anxiety on self-regulated learning and mathematical literacy. *Australian Journal of Education*, 64(3), 227–242.
- Klee, H. L., Buehl, M. M., & Miller, A. D. (2021). Strategies for alleviating students' math anxiety: Control-value theory in practice. *Theory into Practice*, 00(00), 1–13.
- Larkin, K., & Jorgensen, R. (2016). 'I Hate Maths: Why Do We Need to Do Maths?' Using iPad Video Diaries to Investigate Attitudes and Emotions Towards Mathematics in Year 3 and Year 6 Students. *International Journal of Science and Mathematics Education*, 14(5), 925–944.
- Lau, N. T. T., Hawes, Z., Tremblay, P., & Ansari, D. (2022). Disentangling the individual and contextual effects of math anxiety: A global perspective. *Proceedings of the National Academy of Sciences of the United States of America*, 119(7), 1-11. <https://doi.org/10.1073/pnas.2115855119>
- Maloney, E. A., Ramirez, G., Gunderson, E. A., Levine, S. C., & Beilock, S. L. (2015). Intergenerational effects of parents' math anxiety on children's math achievement and anxiety. *Psychological Science*, 26(9), 1480–1488. <https://doi.org/10.1177/0956797615592630>
- Mammarella, I. C., Caviola, S., & Dowker, A. (2019). *Mathematics anxiety what is known and what is still to be understood*. Routledge.
- Pekrun, R., Muis, K., Frenzel, A., & Goetz, T. (2018). *Emotions at school*. Routledge.
- Pekrun, R., Frenzel, A. C., Goetz, T., & Perry, R. P. (2007). The Control-Value Theory of Achievement Emotions. An Integrative Approach to Emotions in Education. *Emotion in Education*, June 2014, 13–36. <https://doi.org/10.1016/B978-012372545-5/50003-4>
- Pérez-Fuentes, M. del C., Núñez, A., Molero, M. del M., Gázquez, J. J., Rosário, P., & Núñez, J. C. (2020). The Role of Anxiety in the Relationship between Self-efficacy and Math Achievement. *Psicología Educativa*, 26(2), 137-143.
- Ramirez, G., Gunderson, E. A., Levine, S. C., & Beilock, S. L. (2013). Math Anxiety, Working Memory, and Math Achievement in Early Elementary School. *Journal of Cognition and Development*, 14(2), 187–202.
- Rossnan, S. (2006). Overcoming math anxiety. *Mathitudes*, 1, 1–4.
- Schaeffer, M. W., Rozek, C. S., Berkowitz, T., Levine, S. C., & Beilock, S. L. (2018). Disassociating the relation between parents' math anxiety and children's math achievement: Long-term effects of a math app intervention. *Journal of Experimental Psychology: General*, 147(12), 1782–1790.

# THE LEARNING THROUGH ACTIVITY DESIGN FRAMEWORK: THE FRAMEWORK IN ACTION

Martin A. Simon  
New York University  
msimon@nyu.edu

*In this theoretical paper, I use an empirically based example to illustrate particular design features of the Learning Through Activity (LTA) design framework and examine the impact of particular design principles. The LTA design framework is based on our elaboration of Piaget's construct of reflective abstraction. The example discussed here, involving the learning of a fraction concept, contains both an unsuccessful attempt, not based on the LTA framework, and a subsequent successful attempt, based on the framework. I use this contrast to make theoretical distinctions with regards to designing for the learning of mathematical concepts.*

Learning Theory, Learning Trajectories and Progressions, Instructional Activities and Practices,

In this report, I use data from a teaching experiment to demonstrate the unique features of the Learning Through Activity (LTA, Simon, Kara et al., 2018) design framework. The data segment presented here shows two attempts to teach a particular fraction concept. Only the second attempt was informed by the LTA design framework, so the contrast presents a useful context for explication of the design theory. This is a theoretical report; the goal is not to report on data, but to use them to elucidate particular affordances of the LTA design framework.

Development of theoretical frameworks for instructional design is critical to significant improvement in mathematics instruction. Koedinger (2002) commented on the need to develop theories of mathematics learning and instructional design that build on more general learning theories.

There is a significant gap between theories of general psychological functions on one hand ... and theories of mathematical content knowledge on the other .... To better guide the design of groundbreaking and demonstrably better mathematics instruction, we need instructional principles and associated design methods to fill this gap in a way that is not only consistent with psychological and content theories but prompts and guides us beyond what those theories can do. (p. 21)

The LTA research program focuses on filling the theoretical gap indicated by Koedinger. Whereas the construct of reflective abstraction has been around for decades, design frameworks grounded in reflective abstraction are generally lacking. DiSessa and Cobb (2004) wrote, "Piaget's theory is powerful and continues to be an important source of insight. However, it was not developed with the intention of informing design and is inadequate, by itself, to do so deeply and effectively" (p. 81). LTA researchers have elaborated Piaget's (2001) construct of reflective abstraction to explain mathematics conceptual learning and developed instructional design principles based on the elaborated construct (Simon, Kara, et al., 2018).

## Why This Research Report

Although the LTA framework has been reported on previously, this research report can serve a particular function. In laying out the framework (Simon, Kara, et al., 2018), our emphasis was on specifying the design principles. In this report, I focus on the impact of the design framework and how the design principles are specifically implicated in that impact. A secondary function of this



report is that it can offer an image of reflective abstraction (as opposed to just a claim that reflective abstraction took place). Because the LTA design framework can promote the process of reflective abstraction, not just occasion it, there is greater access to observing the process.

### **Theoretical Framework**

A principal goal of LTA has been to improve capacity to design for *guided reinvention* of mathematical concepts, an idea developed by Freudenthal (1991). In Simon, Kara et al. (2018), we elaborated the construct and explained that “reinvention” refers to the opportunity for students to come to mathematical ideas through their mathematical work and “guided” indicates a well-planned sequence of tasks to support the reinvention.

To explain conceptual learning, we have built on and further elaborated Piaget’s (1980) construct of reflective abstraction. Piaget characterized reflective abstraction as a *coordination of actions* (p. 90). We have explained the coordination of actions that produces mathematical concepts as follows. The process begins with students solving tasks that they can solve with their available concepts. For a given task, they set a goal and call on a sequence of available actions to achieve their goal. This goal and sequence of actions is what we refer to as an “activity.” An activity is the precursor to a new concept. Reflective abstraction is the process by which the actions that make up the activity (or a subset of those actions) are coordinated into a single higher-level action. The result of this coordination is that the learners no longer need to go through the original sequence of actions. The abstraction allows the learners to anticipate the result of the previously enacted activity. (This is a brief and incomplete discussion of our elaboration of reflective abstraction. Readers are referred to Simon et al., 2016 and Simon, Kara et al., 2018.) This process will become clearer in the illustration provided by the data excerpt discussed below.

The five-step LTA design framework is based on this elaboration of reflective abstraction. Steps 1, 2, and 5 are common to many design approaches. Steps 3 and 4 represent the unique aspects of the LTA design framework.

*Step 1:* Specify prior knowledge of the (actual or hypothetical) students.

*Step 2:* Identify specific conceptual learning goals for the students.

*Step 3:* Specify an activity (sequence of actions) available to the students that could serve as the raw material for the intended abstraction.

*Step 4:* Generate a sequence of tasks designed to elicit the activity specified in Step 3 and promote reflective abstraction. Sometimes the tasks that elicit the activity are sufficient to promote the abstraction. In some cases, subsequent tasks are created that restrict the student’s ability to carry out the sequence of actions in the activity – prompting the students to use developing anticipations.

*Step 5:* Symbolize, introduce vocabulary, and/or promote institutionalization of ideas.

Note that a review of prior work on reflective abstraction is not included in this report for two reasons. First, the focus of this article is not the LTA learning theory, but rather the design framework and its impact. Second, a review of that literature is available in the article in which the theory was elaborated (Simon, Kara, et al., 2018).

### **The Research-Based Example**

The research-based example presented here focuses on what is commonly referred to as “fraction of a set” when the quantities in question are discrete quantities. In the teaching experiment from which this example derives, our goal was not restricted to discrete quantities and discrete quantities were not employed in the task sequences. Our goal was to promote abstraction of the *process of taking a fraction of a whole number for tasks in which the result is a whole number*. This is a modest instructional goal. I have selected the example because it allows us to

examine carefully a single theoretically-based instructional design, which would not be a possibility with a more complex goal.

Although this is a theoretical article, I offer some brief comments on the teaching-experiment context from which the example derived. The research discussed here was part of the Measurement Approach to Rational Numbers (MARN) Project, which used LTA's adaptation of the single-subject teaching experiment (Simon et al., 2010, Simon, 2018) to promote a range of fraction concepts. Because the research focused on students' development of concepts *through their mathematical activity*, we used one-on-one teaching experiments, to avoid the confounding factor of input from other students. Also, the researcher conducting the teaching experiment avoided using demonstrations, suggestions, hints, and leading questions. These were research-methodological decisions.<sup>3</sup> Line-by-line analysis was conducted to build up models of the students' mathematical understandings and how the understandings changed (described in depth in Simon, 2018). The data discussed here are from our 2-year teaching experiment with Kylie and are from her 5<sup>th</sup> grade year (10 years of age), her second academic year in the project.

#### THE EXAMPLE<sup>4</sup>

Prior to the work discussed here, Kylie had developed a concept of fraction-as-measure (Simon, Placa, Avitzur, & Kara, 2018), which included an understanding of improper fractions. That is, she could think about a unit fraction as a partial unit that measures the unit a particular number of times, and a non-unit fraction as a quantity created by iterating a unit fraction.

Preassessment related to the example presented here showed that she was unable to solve tasks such as  $\frac{3}{4}$  of 12. We originally expected that Kylie's learning to conceptualize this type of task would be a trivial first step towards the more challenging goal of reinventing fraction multiplication. Because of this expectation, we initially approached this part of our teaching experiment in a fairly typical way, uninformed by the LTA design theory.

We created a simple task sequence in which Kylie first found a unit fraction of a whole number (e.g.,  $\frac{1}{3}$  of 15). She demonstrated the ability to do such tasks the first time she encountered them. We followed these tasks with tasks that asked her to find a non-unit fraction of a whole number, which she was generally able to do. Given Kylie's success with tasks involving non-unit-fraction of a whole number when they immediately followed tasks involving a unit-fraction of a whole number, we needed to determine if learning had occurred. To assess learning, we posed tasks in subsequent sessions involving a non-unit-fraction of a whole number, this time *not* preceded by tasks involving a unit-fraction of a whole number. Kylie showed no ability to solve these tasks. We repeated the task sequence, but subsequent assessment again revealed that we had failed to promote the intended learning.

At this point, we began to take more seriously the challenge of promoting the concept in question and applied the LTA design framework described above. This resulted in creation of a new hypothetical learning trajectory (HLT, Simon, 1995). The HLT included use of the computer application, JavaBars (Biddlecomb & Olive, 2000), which we had adapted and used with Kylie in developing prior concepts. In our version of JavaBars, students can create a bar of any length, equipartition the bar, pull out a part (disembed), and iterate a bar or a part of a bar to make a composite bar. Kylie's use of the Java Bars application is demonstrated in Figure 1.

---

<sup>3</sup> Some of what we omit from our teaching experiments would be important in classroom instruction (e.g., group work, class discussions).

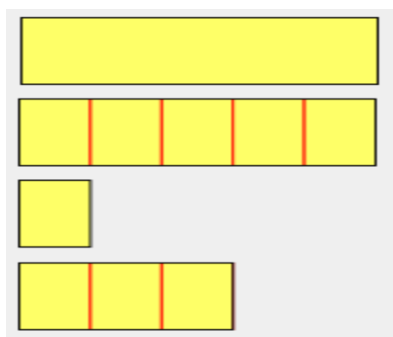
<sup>4</sup> The empirical results were briefly discussed in Simon, Placa, Kara, & Avitzur (2018).

Consistent with the LTA design framework, the initial tasks were designed so that Kylie would be able to solve them *using her available concepts*. We began with the following task (each task presented here is representative of a small set of similar tasks that was employed):

Here's a bar. It's 10 units long. Show me three-fifths of the bar. How many units long is three-fifths of the bar?

The task design was based on our expectation that Kylie, using JavaBars could create  $\frac{3}{5}$  of a bar calling on concepts she had previously constructed and demonstrated. To do so, she would partition the unit bar into five parts, pull out one part and iterate that part three times. (In JavaBars, the 5-part bar remains on the screen after a part is pulled out.) She could then figure out how many units in each of the five parts using whole-number division. Finally, she could evaluate the size of the new ( $\frac{3}{5}$ ) bar using whole-number multiplication. (She had previously demonstrated such competence with whole-number multiplication and division.) This is in fact what happened as evidenced in the transcript that follows (see Figure 1 for Kylie's work with JavaBars).

K: [Cuts the bar into five parts, pulls out a part, and iterates the part three times]  
R: How long is the new bar?  
K: It's six units long.  
R: Explain that.  
K: Two times five is ten and then if you repeat two –  
R: Tell me about the bars.  
K: I cut it up into fifths, and if you want to know how much each piece equals – two units.  
R: How do you know?  
K: Two times five is ten.  
R: What is it about the bar?  
K: Ten units.  
R: Why do you say, "Two times five?"  
K: It's easier than saying, "Five plus five."  
R: What is the five?  
K: How many pieces I cut the unit into.  
R: What is the two?  
K: The number of units in each piece.  
R: How did you get six?  
K: I looked at each of the pieces and counted by twos or two times three is six.



**Figure 1: Kylie's use of JavaBars for  $\frac{3}{5}$  of 10.**

Initially, when she was working with the bars, Kylie was solely focused on creating the  $\frac{3}{5}$  bar from the unit bar. She performed the computational actions once the work with the bars had been completed.

However, this sequence of actions began to change, and, for subsequent tasks, Kylie began to anticipate the calculations without using the bars.

Here's a bar. It's 18 units long. How many units long is  $\frac{2}{3}$  of the bar?

K: [long pause] Twelve units.

R: How did you get that?

K: Six times three –

R: [Interrupting] Where did the six times three come from?

K: Six comes from the third that I cut the eighteen up into. I know that six times three is eighteen. So, I figured six plus six equals twelve and since two-thirds and you wanted to know how many units was in two-thirds – well six plus six units.

After tasks of this type, Kylie was given tasks without reference to the bars:

What is  $\frac{5}{6}$  of 12?

She responded, “Ten units.” Whereas the HLT seemed to be producing positive results, the real test (as it was in our earlier attempts to foster the concept) was whether she could solve computational tasks (with understanding), such as the latter task, when they were posed without the earlier tasks preceding them (i.e., those involving manipulating bars). In a subsequent session, Kylie was given the following as the first task of the session:

What is  $\frac{2}{3}$  of 15?

K: 10 units.

R: How did you get it?

K: ... you are talking about fifteen units. So, two-thirds of fifteen units is ten. Since there are – Three-thirds are a whole. So that's fifteen, and five times three – So the five are the thirds, and there are two of the fives, so ten.

The learning evidenced by this task proved to be stable as Kylie continued to solve similar assessment tasks in subsequent sessions.

### **Analysis of Kylie's Learning**

Here I summarize the results of our analysis of these data. Initially, Kylie solved the tasks using an available activity. It is characterized in the following four actions. (Numbers reflect the first task in the transcript above ( $\frac{3}{5}$  of a 10-unit bar).

3. Partition the bar and pull out a part (partition into 5 parts and pull out 1 part).
4. Iterate the part to create the fractional bar (iterate the part 3 times to make  $\frac{3}{5}$  of the bar).
5. Use whole-number multiplication/division to find the length of each part (10 units divided by 5 parts).
6. Use whole-number multiplication to determine the length of the new bar (2 parts/unit times 3 parts).

In the problem “ $\frac{2}{3}$  of an 18-unit bar”, Kylie began to coordinate pairs of actions from the activity above—Actions 1 and 3 and Actions 2 and 4. The coordination of each pair of actions resulted in a single higher-level action. Kylie's statement, “Six comes from the third that I cut the

eighteen up into,” suggests the merging of Steps 1 and 3. She was beginning to think about partitioning the bar and dividing the quantity as a single action; she was coming to see that she was partitioning the 18-unit bar into three parts of length six units. Similarly, she was seeing the iteration and multiplication as a single step, iterating a six-unit part two times creating a 12-unit bar. The coordination of these two pairs or actions led to Kylie being able to anticipate the solution to such tasks as resulting from two calculation steps, the abstraction we were trying to foster.

The positive results with respect to Kylie’s learning indicated that this second HLT had the intended impact. This HLT was a three-part process represented by the three tasks discussed above.

### Conclusions

The example presented here illustrates application of the Learning Through Activity (LTA) design framework, the process of Kylie’s learning, and the nature of the resulting abstraction.<sup>5</sup> In this section, I examine how the design principles contributed to Kylie’s learning. I also use the comparison with our first, unsuccessful task sequence to further clarify important theoretical distinctions about designing for conceptual learning.

The successful second HLT described above and the results of its use with Kylie demonstrated application of the LTA theoretical framework, a framework composed of an elaboration of reflective abstraction and a set of design steps based on that elaboration. In this example we see the following:

- The goal Kylie set (related to solving the first task) and the activity she called upon to solve that task were available to Kylie given her knowledge at the outset of this instructional unit. Further, Kylie’s learning did not necessitate Kylie setting a goal of finding/learning a method of calculation, nor was it necessary for the researcher/teacher to announce such a goal. Her learned anticipation (abstraction) emerged through her work with the task sequence without her having intended to achieve that outcome.
- The coordination of actions that led to the abstraction was a product of Kylie’s activity. Note, Kylie was readily able to solve the tasks (using JavaBars) from the outset. What changed was her insight that allowed her to avoid going through the original sequence of actions, replacing them with an abstracted two-step computation method, a new activity composed of higher-level actions.
- Although, by the end, Kylie could quickly compute the result, she did so with knowledge of the logical necessity (Simon, 2017) of what she was doing. When asked, she demonstrated the ability to explain the quantities involved and why her computation made sense.
- As stated earlier, the unique aspects of the LTA design framework are Steps 3 and 4. We can see that Step 3 was executed successfully in the design of this HLT. Step 3, “Specify an activity (sequence of actions) available to the students that could serve as the raw material for the intended abstraction,” has two parts. We have already concluded that the activity was *available* to Kylie. The activity also proved to be the raw material for the abstraction. Pairs of actions in the four-step action sequence were coordinated to produce the goal reasoning (concept). Thus, the activity was the raw material for the coordination of actions (reflective abstraction).
- Design Step 4, “Generate a sequence of tasks designed to elicit the activity specified in Step 3 and promote reflective abstraction,” was also executed successfully. We can see that the

---

<sup>5</sup> The data do not allow for a claim about the effectiveness of the approach to a classroom of diverse students, which requires further study.

initial tasks elicited the expected activity. Further, the modified tasks (represented by the second and third task in the example) promoted the coordination of actions.

A key question raised by the example presented is, why did the second HLT work and the first did not? Our original unsuccessful approach to promoting the concept involved beginning with unit fractions of a whole number and then moving to non-unit fractions of a whole number. Although Kylie was able to solve the tasks when the latter tasks followed the former, she did not demonstrate a sustained learning effect.

The HLT developed subsequently contrasted with the original approach in an important way. Instruction did not begin with unit-fraction tasks. Rather the HLT began with JavaBars activity involving non-unit fraction tasks—tasks which Kylie could do prior to any learning taking place. By engaging in several of these tasks and the modified tasks represented by the second and third tasks above, Kylie began to coordinate pairs of actions. At a certain point, she no longer needed the bars and could compute the result quickly. Thus, the second HLT began with tasks similar to the target tasks (i.e., non-unit fractions). Whereas the first HLT proceeded from a less difficult to a more difficult computational task, the second HLT proceeded from a less to a more sophisticated solution to similar tasks. As explained, this change in the sophistication of the solution was the result of a coordination of actions. So why is this distinction important?

Having contrasted the two approaches, I answer this question by returning to explanation of the lack of success of the first and the success of the second. In the first approach (in hindsight), the posing of tasks involving unit fractions immediately prior to posing tasks involving non-unit fractions served to break down the requisite two-step process into two one-step processes. Another way of saying this is that the task sequence allowed Kylie to associate the non-unit -fraction tasks with the unit-fraction tasks. However, the results proved that she did not learn to anticipate the two-step process. When given just the non-unit-fraction tasks, she did not think to associate them with the unit-fraction tasks.

The second approach fostered coordination of actions, actions that were part of the activity that Kylie called on in the initial non-unit-fractions tasks done with the Java Bars application. This coordination of actions resulted in anticipation of the calculation process—a learned anticipation that proved to be stable in assessments in subsequent sessions. That is, the coordination of actions (reflective abstraction) changed her assimilation of the tasks—she viewed the tasks differently. This proved more powerful and more lasting than needing to recall an association between problem types.

The example that I have presented and analyzed demonstrates the value of fostering abstraction through coordination of actions. Further it demonstrates how the LTA design framework, in particular its unique aspects represented by Steps 3 and 4, can inform the generation of HLTs that foster particular coordination of actions.

### References

- Biddlecomb, B., & Olive, J. (2000). *JavaBars* [Computer software]. Retrieved from <http://math.coe.uga.edu/olive/welcome.html>.
- diSessa, A. A. and Cobb, P. (2004). Ontological innovation and the role of theory in design experiments. *Journal of the Learning Sciences*, 13(1), 77–103.
- Freudenthal, H. (1991). *Revisiting mathematics education: China lectures*. Dordrecht: Kluwer Academic Publishers.
- Koedinger, K. R. (2002). Toward evidence for instructional design principles: Examples from Cognitive Tutor Math 6. In D. S. Mewborn, P. Sztajn, D. Y. White, H. G. Wiegel, R. L. Bryant, & K. Nooney (Eds.), *Proceedings of twenty-fourth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education Vol. 1* (pp. 21–49). Columbus, OH: ERIC.

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

- Piaget, J. (1980). *Adaptation and intelligence: Organic selection and phenocopy*. Chicago, IL: University of Chicago Press.
- Piaget, J. (2001). *Studies in reflecting abstraction* (R. L. Campbell, Trans.). Sussex: Psychology Press.
- Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education*, 26, 114-145.
- Simon, M.A. (2017). Explicating mathematical concept and mathematical conception as theoretical constructs. *Educational Studies in Mathematics*, 94(2), 117-137.
- Simon, M. A. (2018). An emerging methodology for studying mathematics concept learning and instructional design. *Journal of Mathematical Behavior*, 52, 113-121.
- Simon, M. A., Kara, M., Placa, N., & Avitzur, A. (2018). Towards an integrated theory of mathematics conceptual learning and instructional design: The Learning Through Activity theoretical framework. *Journal of Mathematical Behavior*, 52, 95-112.
- Simon, M.A., Placa, N., Avitzur, A. (2016). Participatory and anticipatory stages of mathematical concept learning: Further empirical and theoretical development. *Journal for Research in Mathematics Education*, 47(1), 63-93.
- Simon, M. A., Placa, N., Kara, M., & Avitzur, A. (2018). Empirically-based hypothetical learning trajectories for fraction concepts: Products of the Learning Through Activity research program. *Journal of Mathematical Behavior*, 52, 188-200.
- Simon, M. A., Placa, N., Avitzur, A., & Kara, M. (2018). Promoting a Concept of Fraction-As-Measure: A Study of Learning Through Activity. *Journal of Mathematical Behavior*, 52, 122-133.
- Simon, M. A., Saldanha, L., McClintock, E., Karagoz Akar, G., Watanabe, T., & Ozgur Zembat, I. (2010). A developing approach to studying students' learning through their mathematical activity. *Cognition and Instruction*, 28, 70-112.

# UNDERREPRESENTED STUDENTS' MOTIVATIONAL ATTITUDES IN MATHEMATICS

Dalton D. Marsh  
California State University, San Bernardino  
Dalton.Marsh@csusb.edu

*It is widely agreed that attitudes about mathematics play an important role in students' performance, choice, and persistence in STEM. Motivational theories posit this link and suggest that differences in these attitudes should explain in part why female, Black, Hispanic, low-income, and first-generation students are underrepresented in STEM fields in the United States. This study employed nationally representative data from the High School Longitudinal Study of 2009 (HSLS:09) and structural equation modeling to study five types of math attitudes: self-efficacy, identity, interest, utility, and cost. Multi-group factor analytic methods were used to compare mean levels of these attitudes across subgroups based on STEM career expectations, college generational status, parent income, gender, and race. The results suggest that explaining underrepresentation in STEM via differences in motivational attitudes is not straightforward.*

Keywords: Affect, Emotion, Beliefs, and Attitudes; Equity, Inclusion, and Diversity.

Broadening participation in science, technology, engineering, and mathematics (STEM) has become a critical effort across the globe to further innovation and strengthen economies (Freeman et al., 2015). Internationally, female students and students from socioeconomically disadvantaged backgrounds are less likely to aspire to STEM careers (OECD, 2016). In the United States, where gender, class, and race are interconnected (Shields, 2008), STEM degree earners are disproportionately White or Asian males, especially in engineering and the physical, computer, and mathematical sciences (NSF & NCSES, 2021). Moreover, the Black/African American, Hispanic/Latinx, and Indigenous students who do attain STEM degrees and occupations disproportionately leave the STEM workforce (Riegle-Crumb et al., 2019). Underrepresentation is a critical issue as not only do workers in STEM occupations earn more, but workers with STEM degrees earn higher wages, regardless of whether they work in STEM occupations (Noonan, 2017). Furthermore, compared to other occupations, STEM jobs offer the smallest pay gaps across gender and racial/ethnic lines (Carnevale et al., 2011). Therefore, broadening participation in STEM is of critical importance not only as a means to help meet demand for STEM-competent workers, but also for addressing social and economic inequalities. Several factors contributing to the underrepresentation of certain groups in STEM have been studied. Some of the most common include academic performance, opportunity to learn, and motivational attitudes, which all tend to be closely linked (OECD, 2016). This study focuses on attitudes towards mathematics and how they differ across well-represented and underrepresented groups with the following research questions: Are there differences in students' attitudes about mathematics across gender, race, and social class backgrounds? If so, do these differences explain underrepresentation in STEM, i.e., do students belonging to the groups with disproportionate representation in STEM have less positive math attitudes on average compared to well-represented groups in STEM?

## Theoretical Framework

The framework that this study uses to understand students' mathematics attitudes is expectancy-value (EV) theory (Eccles, 2009). EV theory holds that students' choice, performance, and persistence in achievement-related tasks are most proximately determined by their expectation



for success and their subjective value of the tasks. Expectancy attitudes include self-efficacy or confidence to successfully perform the tasks involved. Value attitudes include identity or belonging within the tasks' domain, interest or enjoyment of the tasks, utility or usefulness of the tasks for future goals, and the perceived cost (e.g., social, time, effort) associated with engaging in the tasks. Applying the theory to STEM, students with higher mathematics and science expectancies and values are more motivated to participate and achieve in STEM. The research literature has largely corroborated the EV model. Indeed, studies have associated more positive mathematics and science EV attitudes with higher achievement in mathematics (Sharpe and Marsh, 2021; Simpkins et al., 2006), higher levels of mathematics and science coursework (Froiland and Davison, 2016; Guo et al., 2015; Simpkins et al., 2006; M.-T. Wang, 2012; X. Wang, 2013), greater interest in STEM careers (Andersen and Ward, 2014; Gottlieb, 2018; M.-T. Wang, 2012; X. Wang, 2013), a higher likelihood of enrolling in a STEM degree program (Federman, 2007; Guo et al., 2015; Trusty, 2002), a higher likelihood of attaining a STEM degree (Ma, 2011; Maltese & Tai, 2011), and a higher likelihood of being employed in a STEM occupation (Eccles and Wang, 2016; M.-T. Wang et al., 2015).

The EV model also explains that due to societal, cultural, and educational influences, EV attitudes differ across gender, race, and social class background. Groups that develop less positive attitudes about mathematics and science because of these influences are less motivated in STEM and hence less likely to participate and achieve in STEM. Studies in Canada, Australia, and the United States have attributed lower female participation in STEM to less positive attitudes about mathematics (Guo et al., 2015; M.-T. Wang et al., 2015; Watt et al., 2012). However, in terms of race/ethnicity, this explanation remains an open question (Andersen and Ward, 2014; Gottlieb, 2018; Riegle-Crumb & King, 2010). This is because much of the EV research has been limited to White, middle-class populations, but this study aims to address this gap with nationally representative data.

## Methodology

### Data Source and Sample

To answer the research questions this study employed data from the first wave of the High School Longitudinal Study of 2009 (HSLs:09), a study that follows a nationally representative sample of U.S. students from high school to postsecondary years. Compared to previous studies administered by the National Center for Education Statistics (NCES) it has a unique focus on pathways into STEM. The first wave of data collection began in the fall of 2009 with a sample of over 23,000 ninth graders (typically 14 to 15 years old) attending 944 public and private schools throughout the United States. Sampling involved a complex, two-stage design in which eligible schools were randomly selected first, stratified by school type (public, private) and region, and then students within those schools were randomly selected, stratified by student race/ethnicity. Relatively small groups were oversampled. Participating students completed a mathematics assessment and a questionnaire about their high school experiences and attitudes, including their attitudes towards mathematics (Ingels et al., 2011).

The sample in this research consisted of a subset of the full HSLs:09 sample who identified as Asian, Black or African American, Hispanic or Latino/a, or White. For the feasibility of multi-group analysis, it was chosen to focus on these groups. The 2,300 students identifying as Native American, Alaska Native, Native Hawaiian, or belonging to two or more races were not included in the analysis, leaving a total of 21,180 cases. After proper weighting procedures (see below), the analytic sample is representative of all U.S. students who were ninth graders in 2009 and identified as belonging to one of the above racial/ethnic groups.

## Measures

Six scales represented student's ninth-grade EV attitudes towards mathematics. The scales were measured via confirmatory factor analysis using various base-year survey items. Higher scores on these scales indicate a greater sense of expectancy or value and hence a more positive attitude towards mathematics. The questionnaire wordings for cost pertained to both mathematics and science disciplines. To facilitate interpretation across the attitudes, cost items were reverse coded so that higher scores on this scale also represent a more positive attitude (i.e., a less costly view). All items were measured on four-point Likert scales. Table 1 presents the survey items used to measure these scales, along with Cronbach's alpha ( $\alpha$ ) measures of internal consistency.

To examine differences, several grouping variables were used. The high math achievement group consisted of students scoring in the top quintile of the algebraic reasoning assessment. STEM career expectation was represented by a binary variable indicating that the students' expected occupation at age 30 was in a STEM field. STEM occupations included careers in life and physical sciences, engineering, mathematics and information technology. Low income indicated the students' family income was below 185 percent of the federal poverty threshold. First-generation indicated that neither parent had a four-year degree. To analyze differences across gender and race/ethnicity, an intersectional approach was used (Shields, 2008). Students were grouped into eight mutually exclusive categories: Asian female, Asian male, Black or African American female, Black or African American male, Hispanic or Latina female, Hispanic or Latino male, White female, and White male. To facilitate comparison against the dominant group in STEM, White male was used as the reference group (Riegle-Crumb & King, 2010).

**Table 1: Factor Items for Math Attitude Scales**

---

Factor (Cronbach's Alpha)
Prompt
Item

---

Self-Efficacy ( $\alpha = .90$ )

How much do you agree or disagree with the following statements about your Fall 2009 math course?

1. You are confident that you can do an excellent job on tests in this course.
2. You are certain that you can understand the textbook in this course.
3. You are certain that you can master the skills being taught in this course.
4. You are confident that you can do an excellent job on assignments in this course.

Identity ( $\alpha = .84$ )

How much do you agree or disagree with the following statements?

1. You see yourself as a math person.
2. Others see you as a math person.

Interest ( $\alpha = .78$ )

How much do you agree or disagree with the following statements about Fall 2009 math course?

1. You are enjoying this class very much.
2. You think this class is a waste of your time. (Reverse coded)
3. You think this class is boring. (Reverse coded)

Utility ( $\alpha = .78$ )

How much do you agree or disagree with the following statements about your Fall 2009 math course? What students learn in this course...

1. is useful for everyday life.
2. is useful for college.
3. is useful for a future career.

Time Cost ( $\alpha = .81$ )

How much do you agree or disagree with the following statements? If you spend a lot of time and effort in your math and science classes...

1. you won't have enough time for hanging out with your friends. (Reverse coded)
2. you won't have enough time for extracurricular activities. (Reverse coded)

Social Cost ( $\alpha = .83$ )

How much do you agree or disagree with the following statements? If you spend a lot of time and effort in your math and science classes...

1. you won't be popular. (Reverse coded)
2. people will make fun of you. (Reverse coded)

---

### Analysis Procedure

SPSS 24 was used to clean the data, perform descriptive statistics, and estimate alpha reliabilities. Mplus 8.7 was then used to perform several confirmatory factor analyses (CFAs). A single group CFA was performed on the whole sample, then four multi-group CFAs were performed across subsamples based on (a) STEM career expectation, (b) family income, (c)

college generational status, and (d) race/ethnicity and gender. The CFAs assessed the convergent and discriminant validity of the measurement model (the model specifying the factor structure) and assessed model fit. Assessing convergent and discriminant validity involves examining factor loadings, which indicate the degree to which items are related to each other, and factor correlations. Items measuring one construct should load highly on that factor and not highly on others. Also, for a factor to be distinct it should not be highly correlated with others. Assessing model fit involves examining fit indices (Dash & Paul, 2021). Cutoff criteria for these values are outlined in the next paragraph.

In Mplus, the MLR estimator (Muthén & Muthén, 2017) was used which is robust against non-normality and accounts for MAR (missing at random) missing data via full information maximum likelihood estimation (FIML; Peugh & Enders, 2004). Balanced repeated replication (BRR) weighting procedures (Asparouhov & Muthén, 2010) were employed to handle the complex sampling design of HSLs:09 (students within schools). The HSLs:09 survey weight W1STUDENT was used in combination with its corresponding 200 BRR weights (see Ingels et al., 2014). Model fit was assessed with the standardized root mean square residual (SRMR) index, which was the only index available when using BRR weighting (Asparouhov & Muthén, 2018). SRMR values less than .05 and .08 are considered excellent and acceptable fits, respectively (Dash & Paul, 2021; Hu & Bentler, 1999). Convergent validity was assessed by examining standardized factor loadings. Loadings of .6 or higher are considered signs of good convergent validity (Dash & Paul, 2021). Discriminant validity was assessed by examining factor correlations. Correlations above .85 are considered high and signs of a lack of discriminant validity (Awang, 2014).

The next step was to examine differences across groups. Standardized latent mean difference tests were conducted which hold the mean of the reference group to zero and estimate the number of standard deviations the other group means are above or below zero. Statistical significance of the differences is then assessed via t-tests (Byrne, 2011). Therefore, the values on the attitude scales reported in the results do not represent absolute attitude levels, but rather attitude levels relative to the reference group.

## Results

The CFA began by testing the validity of the measurement models. With the original items from HSLs:09, the self-efficacy, identity, and utility scales had all their standardized factor loadings well above the .6 cutoff. The interest and cost scales though required some adjusting. Only 3 of the original 6 items that HSLs:09 created to measure interest were retained. The favorite and least favorite school subject items and the item about whether the student was taking their math course because they enjoy math had standardized factor loadings below the .4 and were removed. Although the item about thinking the class is a waste of time had a factor loading below .6, at .58 it was between .4 and .6 so it was retained (Awang, 2014). When all grouped together, the four cost items showed weak intercorrelation. However, it was found that there was high correlation in two pairs—one with the items relating to the social cost of engaging in math and science and the other relating to time and effort. Their standardized factor loadings were above the .6 cutoff, so the two measures were considered distinct constructs. The factor loadings were also checked across each subsample in the multigroup models. All standardized factor loadings were above the .6 threshold (except for the one interest item which was between .4 and .6). Thus, the revised measurement model met acceptable criteria for convergent validity across all CFA models. Lastly, the factor correlations were examined; none exceeded the .85 threshold indicating that the revised measurement model had good discriminant validity as well.

Next, latent mean difference tests were used to examine differences across groups. Results are standardized and represent the number of standard deviations the group's mean is above (or below) its reference group's mean. Table 2 displays the results for math attitudes across the educational and family background groups. The SRMR indices for each multigroup model were around .03 indicating excellent fits. Almost all math attitudes were significantly higher among high math achievers and those expecting to pursue STEM careers. The one exception was that math utility was not found to differ across achievement groups, suggesting that a difference between high math achievers and those expecting to pursue STEM careers is a perceived usefulness of mathematics. Utility perceptions stood out again when comparing attitudes across income and college generational status. Self-efficacy, identity, and time cost were significantly lower on average for low-income students (compared to middle or upper-income students) and lower on average for first-generation students (compared to students who have at least one parent with a college degree). On the other hand, utility was higher for low-income and first-generation students. Interest and social cost had small and less significant differences. In other words, while low-income and first-generation students tended to feel less confident in math, less likely to identify as a math person, and found the time commitment for putting effort into math and science to be more costly, they reported more positive perceptions about the usefulness of math and were less likely to be concerned about the social cost of putting effort into math and science.

**Table 2: Comparison of Math Attitudes for Math Achievement, STEM Career Expectation, Family Income, and College Generational Status Groups**

	High Math Achievement	STEM Career Expectation	Low Income	First-Generation
Self-Efficacy	.663***	.315***	-.110**	-.225***
Identity	.935***	.436***	-.109**	-.194***
Interest	.392***	.257***	.031	-.078*
Utility	-.025	.406***	.254***	.110**
Time Cost	.251***	.186***	-.123*	-.143***
Social Cost	.198***	.109*	.079*	.089**

*Note.* Results are standardized and based on latent mean difference tests; the values represent the number of standard deviations the group's mean is above (or below) its reference group's mean. \*p < .05, \*\*p < .01, \*\*\*p < .001

**Table 3: Comparison of Math Attitudes Across Race/Ethnicity and Gender**

	Asian		Black		Hispanic		White
	Male	Female	Male	Female	Male	Female	Female
Self-Efficacy	.352***	.065	.049	.012	-.048	-.351***	-.178***
Identity	.583***	.303***	-.067	-.028	-.086	-.296***	-.146***
Interest	.545***	.482***	.140*	.277**	.135*	.011	.071*
Utility	.411***	.065	.472***	.432***	.204***	.125	-.039
Time Cost	-.107	.035	-.061	.253***	-.167**	.073	.238***
Social Cost	-.163*	.360***	.097	.337***	.071	.248***	.188***

*Note.* Results are standardized and based on latent mean difference tests; the values represent the number of standard deviations the group’s mean is above (or below) its reference group’s mean. \* $p < .05$ , \*\* $p < .01$ , \*\*\* $p < .001$

Table 3 displays the results of the latent mean difference tests for racial, ethnic, and gender groups. The SRMR index for this multigroup model was .042 suggesting an excellent fit to the data. The values in the table represent the number of standard deviations above (or below) that group’s means is compared to the mean for white males (which is set to zero). Overall, there were several differences in math attitude levels across the gendered racial/ethnic groups. Asian male ninth-graders tended to have high levels of self-efficacy, identity, interest, and utility, but were more concerned with the social cost of math/science than White male ninth-graders. Asian female ninth-graders had higher levels of identity and interest. Black ninth-graders, both males and females, and Hispanic males tended to have higher interest and utility than White male ninth-graders, and similar levels of self-efficacy and identity. In general, female ninth-graders perceived math and science to be less socially costly as White males ninth-graders. In terms of the time cost, Black and White female students reported math and science as less costly, while Hispanic and Asian female ninth-graders were about even with White male ninth-graders. Only Hispanic male ninth-graders were more concerned than White male ninth-graders. In terms of self-efficacy and identity, Hispanic and White ninth-graders were similar, reporting lower levels than their male counterparts.

### Discussion

This study found that there were differences in students’ attitudes about math across gender, race, and social class backgrounds. Furthermore, within underrepresented groups, students were motivated in mathematics in different ways. While some results confirmed commonly found motivational disparities (M.-T. Wang & Degol, 2013), others identified motivational assets. Ninth-graders belonging to the low-income, first-generation, Hispanic female, and White female groups reported lower confidence in their math abilities and were less likely to identify as a “math person” on average. However, ninth-graders belonging to the low-income, first-generation, Black, and Hispanic groups tended to find math equally or in many cases, more useful. In fact, Black ninth graders tended to report just as high or higher math attitudes in all categories. Therefore,

differences in math attitudes, as measured by HSLS:09 and this research, do not do well to explain underrepresentation in STEM.

The EV attitude that stood out across underrepresented and well-represented groups was math utility. Some qualitative studies suggest explanations for this result. Garibay (2015) found that underrepresented students place high value on effecting social change and impacting communities in need. Students in a study by Aschbacher et al. (2010) cited early interest in STEM for using science to help people and those who persisted in STEM were twice as likely to hold this view. A study by Eastman et al. (2017) followed students in an Urban Scholars program. While the program encouraged STEM, 3 of 4 scholars decided not to continue in STEM due to college experiences suggesting STEM to be uninterested in human interactions and needs. Therefore, underrepresented students tend to be highly motivated by the potential usefulness of math and science to benefit people, especially those in need, and these perceptions are closely tied with their interest and persistence in STEM. Another unexpected result was that female ninth graders tended to be less concerned with the social costs of doing well in math and science. This was surprising as female students are often seen as more likely to be turned away by the “nerd-genius” stereotype in STEM (Starr, 2018).

The results of this study caution a focus on motivational deficits surrounding underrepresented students without simultaneously considering motivational assets. A tangible takeaway is that students from underrepresented groups tend to place a high value on the usefulness of mathematics. Students’ self-efficacy and identity, especially for underrepresented groups, should continue to be supported, but stakeholders should not lose sight of the ways in which students are already motivated in mathematics. In particular, the findings of this study suggest that STEM teaching and outreach should also prioritize leveraging the usefulness of mathematics to broaden participation in STEM.

## References

- Aschbacher, P. R., Li, E., & Roth, E. J. (2010). Is science me? high school students' identities, participation and aspirations in science, engineering, and medicine. *Journal of Research in Science Teaching*, 47(5), 564–582. <https://doi.org/10.1002/tea.20353>
- Andersen, L., & Ward, T. J. (2014). Expectancy-value models for the STEM persistence plans of ninth-grade, high-ability students: A comparison between Black, Hispanic, and White students. *Science Education*, 98(2), 216–242. <https://doi.org/10.1002/SCE.21092>
- Asparouhov, T. & Muthén, B. (2010). *Resampling methods in Mplus for complex survey data* (Technical Appendix). <https://www.statmodel.com/techappen.shtml>
- Asparouhov, T. & Muthén, B. (2018). *SRMR in Mplus* (Technical Appendix). <https://www.statmodel.com/techappen.shtml>
- Awang, Z. (2014). *Research methodology and data analysis* (2nd ed.). UiTM Press.
- Byrne, B. M. (2011). *Structural equation modeling with Mplus: Basic concepts, applications, and programming*. Routledge.
- Carnevale, A. P., Smith, N. & Melton, M. (2011). *STEM: Science, technology, engineering, mathematics*. Georgetown University Center on Education and the Workforce. <https://cew.georgetown.edu/cewreports/stem/>
- Dash, G., & Paul, J. (2021). CB-SEM vs PLS-SEM methods for research in social sciences and technology forecasting. *Technological Forecasting and Social Change*, 173. <https://doi.org/10.1016/j.techfore.2021.121092>
- Eastman, M. G., Christman, J., Zion, G. H., & Yerrick, R. (2017). To educate engineers or to engineer educators? Exploring access to engineering careers. *Journal of Research in Science Teaching*, 54(7), 884–913. <https://doi.org/10.1002/tea.21389>
- Eccles, J. S. (2009). Who am I and what am I going to do with my life? Personal and collective identities as motivators of action. *Educational Psychologist*, 44(2), 78–89. <https://doi.org/10.1080/00461520902832368>
- Eccles, J. S., & Wang, M. (2016). What motivates females and males to pursue careers in mathematics and science? *International Journal of Behavioral Development*, 40(2), 100–106. <https://doi.org/10.1177/0165025415616201>

- Federman, M. (2007). State graduation requirements, high school course taking, and choosing a technical college major. *B. E. Journal of Economic Analysis & Policy*, 7(1), 1–34. <https://doi.org/10.2202/1935-1682.1521>
- Freeman, B., Marginson, S., & Tytler, R. (2015). *The age of STEM: educational policy and practice across the world in science, technology, engineering and mathematics*. Routledge.
- Froiland, J. M., & Davison, M. L. (2016). The longitudinal influences of peers, parents, motivation, and mathematics course-taking on high school math achievement. *Learning and Individual Differences*, 50, 252–259. <https://doi.org/10.1016/j.lindif.2016.07.012>
- Garibay, J. C. (2015). STEM students' social agency and views on working for social change: Are STEM disciplines developing socially and civically responsible students? *Journal of Research in Science Teaching*, 52(5), 610–632.
- Hu, L. & Bentler, P. (1999). Cutoff criteria for fit indices in covariance structure analysis: conventional criteria versus new alternatives. *Structural Equation Modeling*, 6(1), 1–55. <https://doi.org/10.1080/10705519909540118>
- Ingels, S. J., Pratt, D. J., Herget, D. R., Dever, J. A., Fritch, L. B., Ottem, R., Rogers, J. E., Kitmitto, S., Leinwand, S. (2011). *High School Longitudinal Study of 2009 (HSL:09): Base-year data file documentation* (NCES 2011–328). National Center for Education Statistics (NCES). <http://nces.ed.gov/pubsearch>
- Gottlieb, J. J. (2018). STEM career aspirations in Black, Hispanic, and White ninth-grade students. *Journal of Research in Science Teaching*, 2018, 1–28. <https://doi.org/10.1002/tea.21456>
- Guo, J., Parker, P. D., Marsh, H. W., & Morin, A. J. S. (2015). Achievement, motivation, and educational choices: A longitudinal study of expectancy and value using a multiplicative perspective. *Developmental Psychology*, 51(8), 1163–1176. <https://doi.org/10.1037/a0039440>
- Ma, Y. (2011). Gender differences in the paths leading to a STEM baccalaureate. *Social Science Quarterly*, 92(5), 1169–1190. <https://doi.org/10.1111/j.1540-6237.2011.00813.x>
- Maltese, A. V. & Tai, R. H. (2011). Pipeline persistence: Examining the association of educational experiences with earned degrees in STEM among US students. *Science Education*, 95(5), 877–907. <https://doi.org/10.1002/sce.20441>
- Muthén, L. K. & Muthén, B. O. (2017). *Mplus user's guide* (8th ed.). Muthén & Muthén.
- National Science Foundation (NSF) & National Center for Science and Engineering Statistics (NCSES). (2021). *Women, minorities, and persons with disabilities in science and engineering* (Special Report NSF 21-321). National Science Foundation. <https://nces.nsf.gov/pubs/nsf21321/report>
- Noonan, R (2017). *STEM jobs: 2017 update* (ESA Issue Brief 02-17). Office of the Chief Economist, Economics and Statistics Administration, U.S. Department of Commerce. <https://www.commerce.gov/data-and-reports/reports/2017/03/stem-jobs-2017-update>
- Organisation for Economic Co-operation and Development (OECD) (2016). *Equations and inequalities: Making mathematics accessible to all*. OECD. <http://dx.doi.org/10.1787/9789264258495-en>
- Peugh, J. L., & Enders, C. K. (2004). Missing data in educational research: A review of reporting practices and suggestions for improvement. *Review of Educational Research*, 74(4), 525–556. <https://doi.org/10.3102/00346543074004525>
- Riegle-Crumb, C., & King, B. (2010). Questioning a white male advantage in STEM: Examining disparities in college major by gender and race/ethnicity. *Educational Researcher*, 39(9), 656–664. <https://doi.org/10.3102/0013189X10391657>
- Riegle-Crumb, C., King, B., & Irizarry, Y. (2019). Does STEM stand out? Examining racial/ethnic gaps in persistence across postsecondary fields. *Educational Researcher*, 48(3), 133–144. <https://doi.org/10.3102/0013189X19831006>
- Sharpe, S. T. & Marsh, D. D. (2022). A systematic review of factors associated with high schoolers' algebra achievement according to HSL:09 results. *Educational Studies in Mathematics*. <https://doi.org/10.1007/s10649-021-10130-4>
- Shields, S. A. Gender: An intersectionality perspective. *Sex Roles* 59(5), 301–311 (2008). <https://doi.org/10.1007/s11199-008-9501-8>
- Simpkins, S. D., Davis-Kean, P. E., & Eccles, J. S. (2006). Math and science motivation: A longitudinal examination of the links between choices and beliefs. *Developmental Psychology*, 42(1), 70–83. <https://doi.org/10.1037/0012-1649.42.1.70>
- Starr, C. R. (2018). “I’m Not a Science Nerd!”: STEM stereotypes, identity, and motivation among undergraduate women. *Psychology of Women Quarterly*, 42(4), 489–503. <https://doi.org/10.1177/0361684318793848>
- Trusty, J. (2002). Effects of high school course-taking and other variables on choice of science and mathematics college majors. *Journal of Counseling & Development*, 80(4), 464–474. <https://psycnet.apa.org/doi/10.1002/j.1556-6678.2002.tb00213.x>
- Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.



- Wang, M.-T. (2012). Educational and career interests in math: A longitudinal examination of the links between classroom environment, motivational beliefs, and interests. *Developmental Psychology*, 48(6), 1643–1657. <https://doi.org/10.1037/a0027247>
- Wang, M.-T. & Degol, J. (2013). Motivational pathways to STEM career choices: Using expectancy-value perspective to understand individual and gender differences in STEM fields. *Developmental Review*, 33(4), 304–340. <https://doi.org/10.1016/j.dr.2013.08.001>
- Wang, M.-T., Degol, J., & Ye, F. (2015). Math achievement is important, but task values are critical, too: Examining the intellectual and motivational factors leading to gender disparities in STEM careers. *Frontiers in Psychology*, 6(36), 1–9. <https://doi.org/10.3389/fpsyg.2015.00036>
- Wang, X. (2013). Why students choose STEM majors: Motivation, high school learning, and postsecondary context of support. *American Educational Research Journal*, 50(5), 1081–1121. <https://doi.org/10.3102/0002831213488622>
- Watt, H. M. G., Shapka, J. D., Morris, Z. A., Durik, A. M., Keating, D. P., & Eccles, J. S. (2012). Gendered motivational processes affecting high school mathematics participation, educational aspirations, and career plans: A comparison of samples from Australia, Canada, and the United States. *Developmental Psychology*, 48(6), 1594–1611. <https://doi.org/10.1037/a0027838>

## WHAT IS A UNIT? BROADENING UNITS COORDINATION

Karen Zwanch

Oklahoma State University  
karen.zwanch@okstate.edu

Sarah Kerrigan

George Fox University  
skerrigan@georgefox.edu

*Units coordination, defined by Steffe (1992) as the mental distribution of one composite unit (i.e., a unit of units) “over the elements of another composite unit” (p. 264) is a powerful tool for modeling students’ mathematical thinking in the context of whole number and fractional reasoning. This paper proposes extending the idea of a numerical unit to an algebraic unit and a covariational unit. Evidence to support this extension is taken from two qualitative studies conducted with middle-grades students. Results suggest the coordination of two unknowns in algebraic contexts is limited for students who assimilate with composite units. Additionally, in the context of covariational reasoning, results describe a new partitioned unit that students who assimilate with composite units may apply. Implications for engaging all learners are discussed.*

Keywords: Algebra and Algebraic Thinking; Learning Theory; Number Concepts and Operations; Middle School Education

### What is a Unit?

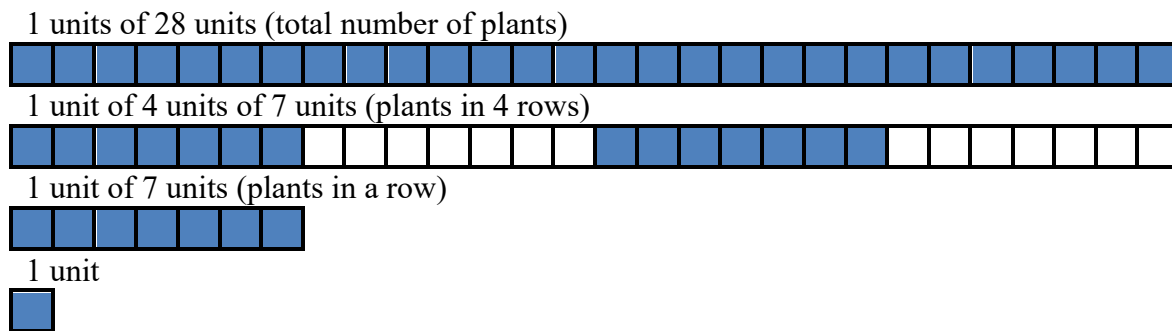
Researchers over the past several decades have found children’s construction of whole number is characterized by the types of units they construct and the operations or unit transformations they enact. Several different classifications of student’s unit structures emerged from the literature including number sequences (Olive, 2001; Ulrich, 2016a), Units Coordination (UC) stages (Boyce & Norton, 2016; Hackenberg, 2007; Steffe, 1992), and multiplicative concepts (MC) (Hackenberg & Lee, 2015; Hackenberg & Tillema, 2009). Central to each of these is the level of units a student assimilates with, as well as characterized by different types of operations and transformation of those units (Ulrich, 2015, 2016a).

Steffe (1992) imagined UC as a cognitive foundation to numerical reasoning, where UC describes the mental distribution of one composite unit (i.e., unit of units) “over the elements of another composite unit” (p. 264). For instance, to reason multiplicatively about finding the number of plants in four rows with seven plants in each row, one must mentally join, or construct, a row as a unit containing seven units. This constitutes a composite unit of seven. Then, one can mentally distribute, or think about inserting, the unit of seven plants into each of the four rows. The garden as a unit of four rows constitutes the second composite unit in this example. This typifies one possible way that UC can apply to students’ whole number reasoning, and how it forms a foundation for multiplicative reasoning.

In the UC paradigm, when a student has assimilated one level of units, their foundational unit are units of 1. This aligns with the Tacitly Nested Number Sequence (TNS) (Olive, 2001) and MC1 (Hackenberg & Tillema, 2009). With this unit of 1, stage 1 students, in activity, construct other units through iterating their unit of 1  $n$  times to make a unit of  $n$  units of 1. This unit of units is a composite unit (Steffe, 1992). For example, a stage 1 student can construct 4 in activity by iterating their unit of 1 four times.

Stage 2 students, have composite units as their foundational unit because they have assimilated the process of constructed composite units into an object they can act upon. Thus, a stage 2 student

can engage with multiple composite units to solve multiplication tasks multiplicatively rather than additively, as described above in the garden problem. In the garden



**Figure 1: Example of UC through solving  $4 \times 7$  garden problem**

problem example, a student reasoning multiplicatively constructs a third level of units (1 unit of

4 units of 7 units) in activity to reach an answer of 28. Because a stage 2 student's foundational unit is a two-level unit, composite unit, when they arrive at the answer 28, it gets reduced back to a composite unit and the third level decays. This differs from a stage 3 student who assimilates with three levels of units. These students can assimilate the three levels of units needed to reason multiplicatively to get the answer of 28 while maintaining all three levels. Stage 3 students can also construct a fourth level in activity.

Overall, UC has proved to be a powerful tool in modeling students' mathematical thinking in situations of counting (Steffe, 1992; Steffe & Cobb, 1988), and additive and multiplicative reasoning (Hackenberg & Tillema, 2009; Steffe, 1992) but also in contexts of fractional reasoning (Hackenberg, 2007; Steffe, 2002; Steffe & Olive, 2010), combinatorial reasoning (Tillema, 2013), proportional reasoning (Steffe et al., 2014) and measurement (Hackenberg et al., 2021; Zwanch et al., 2023) These are a few representative examples of the immense body of UC research literature that has begun to explain students' mathematical behaviors by considering their mental coordination of numerical units.

More recent literature, however, begins to extend UC to a more diverse set of mathematical concepts including algebra (Hackenberg, 2013; Hackenberg & Lee, 2015; Hackenberg et al., 2017; Hackenberg et al., 2021; Zwanch, 2022a, 2022b) and covariational reasoning (Kerrigan, 2023). As seen in the example of UC above, the core of numerical reasoning with whole numbers the iterable units. This is also true of students' construction of fractions as measures. However, as researchers have explored connections with students' UC in other mathematical domains, the question arose of if there is an different foundational unit than an iterable units (Castillo-Garsow, 2014). Therefore, the purpose of this paper is to report two exemplars that broaden Steffe's (1992) unit to algebraic and covariational reasoning.

### **Exemplar 1: Algebraic Reasoning**

#### **Methods**

This exemplar is a case of algebraic reasoning demonstrated by a sixth-grade student, Candace, participating in a constructivist teaching experiment (Steffe & Thompson, 2000). The student participated in eleven, one hour teaching episodes across one year during which she worked with a

teacher-researcher to test the boundaries of her algebraic reasoning in a variety of contexts. In the majority of teaching episodes, Candace worked with a partner, but in the episode presented here, her partner, Liam, only joined them for the last 15 minutes. Prior to the teaching episodes, Candace and Liam were found to iterate units of 1 in mental activity and assimilate numerical situations with two levels of units (i.e., Candace and Liam had each constructed an advanced tacitly nested number sequence (aTNS). See Ulrich, 2016b). Their numerical reasoning was measured using a paper and pencil assessment written and validated by Ulrich and Wilkins (2017) and by a follow up clinical interview with the first author.

The task presented in this exemplar is a word problem that could be modeled by a system of two linear equations (Figure 2). The teacher-researcher had a goal of testing the extent to which Candace and Liam could write and solve a system of two linear equations in two variables to model and algebraically solve the task. In this portion of the teaching experiment, Candace is using virtual Cuisenaire rods. In previous teaching episodes, Candace and Liam worked with physical Cuisenaire rods, which were selected to make visible the students' operations with composite units (e.g., a yellow five block reflects a composite unit of five). Virtual Cuisenaire rods were selected for this teaching episode because they contain a variable pink and variable white block that can be adjusted to any integral length greater than or equal to one unit. Therefore, the students could use the variable blocks to represent values greater than 10 and could adjust the lengths of the blocks to represent unknown quantities.

Systems of Equations Task: Find two blocks, A and B, that together are 35 units long. Block A is 5 units longer than block B. How long are blocks A and B?

$A+B=35$     $A=B+5$

Note that the teacher-researcher created and locked the “35” blocks shown above. All other blocks were introduced, adjusted, and labeled by Candace.

**Figure 2: Candace’s Blocks and Labels**

## Results

Initially when asked to solve a word problem that could be modeled by a system of linear equations in two variables, Candace used guess and check strategies. The teacher-researcher set a goal of introducing algebraic notation and determining the extent to which Candace could reason algebraically about the equations to solve the system. The teacher-researcher’s hypothesis was that utilizing Cuisenaire rods to represent the numerical values in the problem might support equation writing and subsequent algebraic reasoning, however, Candace quickly memorized the lengths of each Cuisenaire rod based on their color (e.g., the yellow block is five units long) and she continued to solve the word problems using numerical reasoning.

**Candace’s reasoning with numbers greater than 10.** The researcher next hypothesized that Candace might operate with unknowns more meaningfully if she were given tasks that she could not solve by relying on the lengths of the Cuisenaire rods that she had memorized. Accordingly, Candace was given the task in Figure 2. Larger numbers were selected to encourage Candace to

work with the variable-length Cuisenaire rods, thereby limiting her ability to solve the task by recalling the length of each color of Cuisenaire rod. Working with the variable-length Cuisenaire rods was also designed to support Candace's thinking about unknowns.

To begin the problem, Candace added a variable pink (block A), variable white (block B), and yellow block (5) to the screen. She adjusted A longer than B, then aligned B and yellow end to end and placed A underneath (See Figure 2, Block Structure #2). Then, she made copies of A and B, and aligned them end to end under the existing 35 block. She said, "Too long!" and adjusted A and B to be shorter. The shorter A and B blocks did not hold the difference of five, so she adjusted their lengths a second time and said, "This might take me all day." After the third adjustment, Candace compared A and B to the 35 block, and the researcher observed that A and B together were 33 units long.

Teacher-researcher (T): How do you think you'll have to adjust them?

Candace (C): I think that I'm going to have to make each of them one longer.

T: Okay. Let's try it.

C: [Adjusted A and B to be one unit longer. Compared them to 35 (Block Structure #1).

Compared them to each other (Block Structure #2).] Yay!

T: Awesome! And can you write two equations to represent your block structures?

C: [Writes  $A+B=35$  and  $A=B+5$ ]

...[Interruption as Liam joins and Candace explains her progress on the problem.]

T: So using only As, Bs, and fives, how else could we make 35? Is there another way?

C: I think I have an idea. [Makes 2 copies of B and adds a yellow (5) to the screen. Aligns the two Bs and yellow end to end under a 35 block (Block Structure #3).]

T: Ah, now you did that really quick. How'd you know that that was going to fit?

C: Because I figured that if you take two of those [B], put it down there, and then add a five that it would work.

T: Okay, that's good. [To Liam] Do you see how she figured that out?

Liam: Not really.

T: Yeah, let's see if we can get her to explain a little more. [To Candace] How'd you know it was going to be two Bs and a five?

C: Well, because, so when I was looking at that [points to Block Structure #2], I realized that it was 35 and kind of figured out what these [Bs] are. I figured out that they're 15 I'm pretty sure because 15 plus 15 equals 30 plus 5 equals 35.

T: Ah, so you figured out the numbers in your head?

C: Yeah.

The purpose of introducing this task was to motivate Candace's use of the equations to solve for A and B. From the teacher-researcher's perspective, Candace created equations and block structures consistent with inventing the algebraic substitution method for solving a system of linear equations when she created Block Structure #3 because the A in Block Structure #1 was replaced by a B and a yellow in Block Structure #3. However, this was not Candace's perception of the situation. Rather, Candace identified a numerical combination that summed 35 without consideration of the relationship between A and B. Furthermore, in the remaining three teaching episodes, Candace successfully wrote equations to model systems of linear equations but did not move from numerical to algebraic reasoning.

**Analysis of Candace's reasoning with numbers greater than 10.** The teacher-researcher's goal in this teaching episode was to pose situations that might motivate Candace's algebraic

reasoning, however, that goal was not met. Instead, Candace reasoned strategically that she needed to “make each of them [variable blocks] one longer,” which aTNS students may do by leveraging their operations on embedded composite units (Ulrich, 2016b). This suggests that Candace’s foundational composite unit did not support her algebraic reasoning in this context. This is consistent with Hackenberg et al.’s (2017) finding that students who assimilate numerical situations with two levels of units tend to reason about numerical values rather than unknowns to reduce the complexity of the UCs in a task. The need for students, such as Candace, to reduce the complexity of the UCs is due to the nature of an unknown. To operate on an unknown quantity requires operations on two levels of units because an unknown is a unit that contains an unknown number of units of one (Hackenberg et al., 2017). Although aTNS students can operate on two levels of units (Ulrich, 2016b), the systems of equations task demonstrates that Candace’s foundational units did not support reasoning about two unknowns (i.e., two two-level unit structures), nor her reasoning about the relationship between the two unknowns.

### **What is a Unit in Algebraic Reasoning?: Implications for Algebra**

The exemplar presented here demonstrates one way that students with foundational composite units might be supported to solve problems that can be modeled by systems of linear equations in two variables, although not in the way that we, as mathematics educators, might expect. Candace leveraged an assimilatory composite unit to strategically guess and check values for the lengths of the variable Cuisenaire rods and to determine the numerical solutions to the problem. Candace did not leverage the operations of her aTNS, however, to coordinate two unknowns. These findings continue to broaden the applicability of UC to different forms of algebraic reasoning, as a way of supporting mathematics educators in better understand how students construct algebraic ideas. This finding also demonstrates how students’ numerical UCs may relate to their operations in the context of systems of linear equations in two variables.

Additionally, classroom instruction is often limited to teaching algebraic methods to solve systems of equations (Oktaç, 2018; Proulx et al., 2009), which may limit the opportunity for students like Candace to engage meaningfully with the mathematics she is being taught in school. To engage all students, mathematics educators must continue to develop instructional strategies that differentiate in a manner that is sensitive to each student’s operations with numerical and algebraic units.

### **Exemplar 2: Covariational Reasoning**

#### **Methods**

This exemplar is a case of covariational reasoning of an eighth-grade student who was part of a larger study that examined connections between middle-grades students’ covariational reasoning, UC, and working memory. Through semi-structured clinical interviews and written assessments six students were selected from a pool of eight to achieve cognitive diversity in both UC and working memory. This report focuses on Daniel’s, an advanced stage 2, work on one of the 12 covariation tasks given throughout the study. Each task consisted of its own Zoom semi-structured clinical interview (Clement, 2000) lasting 30-45minutes. The tasks were GeoGebra applets consisting of different dynamic situations the students engaged with on an iPad.

Each task protocol consisted of asking students general questions about what quantities they noticed and how those changed. After general questions about quantities, the students were asked about a specific relation between two quantities in the animation. For example, the fifth task in the task sequence, presented in this exemplar (Figure 4), consisted of a shape tripling in size starting with a square, then jumping right to a rectangle, then up to a square again. It repeated this process for several changes before restarting. In the second part of the interview students were asked to

relate how the area of the shape changed with respect to time. For these two parts of the interviews, general and specific quantity relations, students were not allowed to write anything down to capture students' work relying solely on their working memory for another part of the study not reported here.



**Figure 4: First three stages of Task 5**

### Results

A portion of Daniel's reasoning about how the size of the shape changes as time passes is reported here. Throughout the task, Daniel reasoned about how the shape changed through multiplication by three. This report includes responses from two of the task questions that highlight how he used his units coordinating structures to reasoning about how the quantities changed. The first of them is from the first part of the protocol and asked Daniel whether any of the quantities he identified as changing, changed in the same way. The second question was from the second part of the interview and asked Daniel how the area changed as time passed.

**Daniel's initial reasoning about how the area changed.** Prior to answering how the changing quantities changed, the interviewer asked what quantities in the animation changed. Daniel identified the size of the shape saying, "how big it's getting," and then specified length. The next question in the task protocol asked Daniel how any of the changing quantities he previously identified in the animation were changing in the same way. Daniel's initial response to this question was that the quantities changed in the same way. He described the change as being the same, "percentage-wise," before specifying they were changing by two-thirds. After running through the first few steps of the animation verbalizing his two-thirds idea while making corresponding sweeps with his hands, Daniel restated his claim to include a multiplicative change, "I think it's going up by two-thirds or tripling the whole thing."

Daniel was then asked a follow-up question how he concluded the shape tripled and not some other multiple. His initial response was that he "just estimated that it was going up by three or you know tripling itself." Here is the exchanging following his initial estimation explanation:

- I: Okay, and so, how did you come to that estimate of what, why do you think it was tripling instead of doubling?
- D: Well, a double would be shorter, like if it doubled itself it wouldn't work as long. I think I looked at how many spaces it would take up [makes length measure with thumb and index finger and makes a kind of iterating motion] if there, there were three of them, and if there were two of them.
- I: Okay, and so, when you say like taking up space, what do you mean by that?
- D: Or, okay I looked at how many how big like the square was by itself [points to bottom left corner of animation, starting square position]. And then I looked at um how many [makes bouncing along a line motion with stylus] if like, if you added on two more of those same squares [makes length form both index fingers and iterates it twice] how big would it be.
- I: Okay, so you're sort of imagining copies of it?

D: Yeah.

I: And it looked like there would be three copies for each step?

D: [nods] mmhmm.

**Analysis of Daniel's initial reasoning about how the area changed.** In this first question, Daniel shows evidence of several different unit structures to capture how the shape's size changed. He first focused on the shape's form (square to rectangle to square) and spatial direction of change (right and up) in connection with the change of size. Then when he answered the question about changing similarly, Daniel transitions from using gross quantities to measured quantities when he introduces numerical transformations of two-thirds and tripling. His two-thirds answer suggests he had a partitioned unit and thought additively but then transitioned or encapsulated that into a multiplicative structure to get his tripling action. This description of the change as repeated multiplication by three is found in Ellis and colleague's (2016) learning trajectory for exponential growth.

In the follow-up question asking about how he generated his tripling idea we gain further evidence of Daniel's use of a unit to measure the size of the shape and construction of his tripling action. Daniel used the starting square as a measuring unit and mentally partitioned the rectangle into three and knew that could be represented by iterating his unit two more times. This led him to generate a partitioned unit of 3 as 1 and 2 (1,2). He then repeated that process with the rectangle to square change only he re-unitized his unit of measure to be the rectangle. Thus, he re-unitized and lost his composite unit structure going through the next shape change. However, he did maintain his partitioned unit structure of 3 as (1,2). The partitioned unit is a new structure identified by Kerrigan, (2023) and might have structural connections to higher dimensional units (Tillema, 2014).

**Daniel's shift in reasoning about how the area changed.** Later in the task, Daniel was asked, "As time passes, how is the shape getting bigger?" To which he responded:

Um...Okay, so as time goes on, it's increasing it's like tripling itself or actually I think it's... I think it's like...Since it's tripling itself to the right and then doing it three more times, it's technically adding on just nine, nine of that unit, [slight pause] or eight more of that unit until like the one square in this corner. So, it's going to make 1,2,3 [moves finger over to the left with each number spoken] and then six up here [points above previous 1,2,3]. So, it's doing that three times, but it stops when it does look another 1,2,3 like this because but it's a very big square so.

The researcher asked more follow-up questions to confirm the hypothesis that Daniel now maintained his initial unit of the starting square and composite unit when constructing the larger square by two successive multiplications by three. In the follow-up questions, Daniel gave a more detailed description of how he got from square to square as a transformation of multiplication by three twice or by adding eight. Both of these resulted in Daniel measuring the second square as 9 units.

**Analysis of Daniel's shift in reasoning about how the area changed.** From this question, Daniel shifted from re-unitizing after each multiplication by three to maintaining the rectangle as a composite unit and constructed a third level in activity to make the square nine units. He also categorized the transition from square to square through addition by saying his multiplication by three twice was equivalent to adding eight of his starting unit square. Thus, Daniel was able to think about the resulting square as nine in two ways, additively as  $1+8$  and multiplicatively as  $3 \times 3$ .



The additive decomposition also shows that Daniel did not maintain the three-level structure of nine outside of his multiplication scheme which aligns with his stage 2 assessment.

Note, in his previous tripling scheme, Daniel also leveraged his partitioned unit of (1,2) to describe change from step to step. Here, he only had one utterance that suggested he also maintained the partition unit structure in this 3x3 scheme when he said, “So, it's going to make 1,2,3 and then six up here.” The six indicates that Daniel had two copies of the three units he just constructed in his 1,2,3 utterance (the rectangle). In the follow-up questions to confirm how he constructed his 1+8 and 3x3 models, Daniel did not mention 6 or any kind of (1,2) partition.

### **What is a Unit in Covariational Reasoning?: Implications for Covariational Reasoning**

This exemplar shows one way in which units construction and transformations were leveraged by a stage 2 student in reasoning about how area changed. Daniel's generation of a model of exponential growth as repeated multiplication by three for each time step is not surprising based on previous findings on exponential growth (Ellis et al., 2016). This analysis extends this to focus on the specific use of the unit structures to capture the change.

Daniel generated several unit structures as his reasoning evolved throughout the task sequence. He first generated a partitioned unit (1,2) to describe how he saw the quantities as changing the same “by two-thirds.” This transitioned to represent a multiplicative relationship of multiplication by three. This structure was a reconciliation of his additive and multiplicative worlds reflective of Confrey and Smith's (1995) approach to covariational reasoning. The partition unit also might be fundamental for determining how amounts of change, change, a component of Carlson and colleagues' (2002) covariational reasoning framework. Daniel's generation of the six added units to the 3 rectangle units to get the 9 units of the square shows he constructed three levels in activity and represented and a second level of amount of change from his (1,2) structure applied to his composite unit rectangle.

An important part of Daniel's covariational reasoning depended on his construction of a measuring unit. Many covariation tasks are designed for graphical settings and posed without any measuring units. Similarly, this task was designed without any given measuring units or tools. However, Daniel displayed his most sophisticated covariational reasoning when he generated his own measuring units and was thus able to leverage his units coordinating structures. This suggests an added importance to incorporating a measuring component to covariational task designs to increase accessibility of the tasks and for researchers to consider how students are using their unit structures when reasoning covariationally.

### **Engaging All Learners**

Engaging all students calls mathematics educators to consider and respond to the strengths and needs of individual students in our research and teaching. Les Steffe's units coordination research programs have accomplished just that by positioning children's mathematics in the forefront of research and teaching (Tillema & Hackenberg, 2017). We seek to extend this line of research to algebraic and covariational reasoning as a means of positioning these learners' mathematics as legitimate and as building on the students' mathematics as a foundation upon which mathematics educators can develop responsive curricula and instruction.

Better understanding the mental structures and activities needed for algebraic and covariational reasoning will allow researchers and educators to design activities and curricula that supports mathematical development in these areas. In particular, better ways to add supports that will make tasks accessible to students with different cognitive diversity in terms of units construction.

## References

- Boyce, S., & Norton, A. (2016). Co-construction of fractions schemes and units coordinating structures. *The Journal of Mathematical Behavior*, 41, 10–25. <https://doi.org/10.1016/j.jmathb.2015.11.003>
- Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying Covariational Reasoning While Modeling Dynamic Events: A Framework and a Study. *Journal for Research in Mathematics Education*, 33(5), 352–378. <https://doi.org/10.2307/4149958>
- Castillo-Garsow, C. (2014). Besides the Iterable Unit: A reply to Steffe et al. *Epistemic Algebraic Students: Emerging Models of Students' Algebraic Knowing*, 3, 157–174.
- Clement, J. (2000). Analysis of clinical interviews: Foundations and model viability. *Handbook of Research Design in Mathematics and Science Education*, 547–589.
- Confrey, J., & Smith, E. (1995). Splitting, Covariation, and Their Role in the Development of Exponential Functions. *Journal for Research in Mathematics Education*, 26(1), 66–86. <https://doi.org/10.2307/749228>
- Ellis, A., Ozgur, Z., Kulow, T., Dogan, M. F., & Amidon, J. (2016). An Exponential Growth Learning Trajectory: Students' Emerging Understanding of Exponential Growth Through Covariation. *Mathematical Thinking and Learning*, 18(3), 151–181. <https://doi.org/10.1080/10986065.2016.1183090>
- Hackenberg, A. J. (2007). Units coordination and the construction of improper fractions: A revision of the splitting hypothesis. *The Journal of Mathematical Behavior*, 26, 27-47. DOI: 10.1016/j.jmathb.2007.03.002
- Hackenberg, A. J. (2013). The fractional knowledge and algebraic reasoning of students with the first multiplicative concept. *The Journal of Mathematical Behavior*, 32, 538-563. <https://doi.org/10.1016/j.jmathb.2013.06.007>
- Hackenberg, A. J., Aydeniz, F., & Jones, R. (2021). Middle school students' construction of quantitative unknowns. *The Journal of Mathematical Behavior*, 61, 1-19. <https://doi.org/10.1016/j.jmathb.2020.100832>
- Hackenberg, A. J., Jones, R., Eker, A., & Creager, M. (2017). “Approximate” multiplicative relationships between quantitative unknowns. *The Journal of Mathematical Behavior*, 48, 38-61. <https://doi.org/10.16/j.jmathb.2017.07.002>
- Hackenberg, A. J., & Lee, M. Y. (2015). Relationships between students' fractional knowledge and equation writing. *Journal for Research in Mathematics Education*, 46(2), 196-243.
- Hackenberg, A. J., & Tillema, E. S. (2009). Students' whole number multiplicative concepts: A critical constructive resource for fraction composition schemes. *The Journal of Mathematical Behavior*, 28, 1-18. DOI: 10.1016/j.jmathb.2009.04.004
- Kerrigan, S. (2023). *Modeling middle grade students' algebraic and covariational reasoning using unit transformations and working memory* [Doctoral Dissertation, Virginia Tech]. [www.vtechworks.lib.vt.edu/handle/10919/113723](http://www.vtechworks.lib.vt.edu/handle/10919/113723)
- Oktaç, A. (2018). Conceptions about systems of linear equations and solutions. In S. Stewart, C. Andrews-Larson, A. Berman, & M. Zandieh (Eds.), *Challenges and Strategies in Teaching Linear Algebra, ICME-13 Monographs* (pp. 71-101). Springer. <https://doi.org/10.1007/978-3-319-66811-6>
- Olive, J. (2001). Children's Number Sequences: An Explanation of Steffe's Constructs and an Extrapolation to Rational Numbers of Arithmetic. *The Mathematics Educator*, 11(1).
- Proulx, J., Beisiegel, M., Miranda, H., & Simmt, E. (2009). Rethinking the teaching of systems of equations. *The Mathematics Teacher*, 102(7), 527-533.
- Steffe, L. P. (1992). Schemes of action and operation involving composite units. *Learning and Individual Differences*, 4(3), 259-309.
- Steffe, L. P. (2002).
- Steffe, L. P., & Cobb, P. (1988). *Construction of arithmetical meanings and strategies*. Springer.
- Steffe, L. P., Liss II, D. R., & Lee, H. Y. (2014). On the operations that generate intensive quantity. In L. P. Steffe, K. C. Moore, L. L. Hatfield, & S. Belbase (Eds.), *Epistemic algebraic students: Emerging models of students' algebraic knowing, WISDOM<sup>e</sup> Monographs* (Vol. 4, pp. 49-79). Laramie, WY: University of Wyoming.
- Steffe, L. P., & Olive, J. (2010). *Children's fractional knowledge*. Springer.
- Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 267-306). Erlbaum.
- Tillema, E. S. (2013). A power meaning of multiplication: Three eighth graders' solutions of Cartesian product problems. *The Journal of Mathematical Behavior*, 32(3), 331-352. <https://doi.org/10.1016/j.jmathb.2013.03.006>
- Tillema, E., & Hackenberg, A. J. (2017). Three facets of equity in Steffe's research programs. Proceedings of the 39<sup>th</sup> annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (pp. 57–67). Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.
- Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

- Ulrich, C. (2015). Stages in constructing and coordinating units additively and multiplicatively (part 1). *For the Learning of Mathematics*, 35(3), 2-7.
- Ulrich, C. (2016a). Stages in constructing and coordinating units additively and multiplicatively (part 2). *For the Learning of Mathematics*, 36(1), 34-39.
- Ulrich, C. (2016b). The tacitly nested number sequence in sixth grade: The case of Adam. *The Journal of Mathematical Behavior*, 43, 1-19. <https://doi.org/10.1016/j.jmathb.2016.04.003>
- Ulrich, C., & Wilkins, J. L. M. (2017). Using written work to investigate stages in sixth-grade students' construction and coordination of units. *International Journal of STEM Education*, 23(4). <https://doi.org/10.1186/s40594-017-0085-0>
- Zwanch, K. (2022a). Examining middle grades students' solutions to word problems that can be modeled by systems of equations using the number sequences lens. *Journal of Mathematical Behavior*, 66, 1-16. <https://doi.org/10.1016/j.jmathb.2022.100960>
- Zwanch, K. (2022b). Using number sequences to account for differences in students' generalizations. *School Science and Mathematics*, 122, 86-99. DOI: 10.1111/SSM.12516
- Zwanch, K., Carter, H. C., & Davenport, J. (2023). Translating units coordination to units of length and area: A case study of undergraduate interior design students. *Research in Mathematics Education*, 1-18. DOI: 10.1080/14794802.2022.2160801

## WIDENING THE EPISTEMOLOGICAL WINDOW

Dana C. Cox

Miami University

Dana.Cox@MiamiOH.edu

Suzanne R. Harper

Miami University

Suzanne.Harper@MiamiOH.edu

Jane M. Keiser

Miami University

JaneKeiser@MiamiOH.edu

*We reflect on the ways in which our methodological choices influence the diversity of voices that contribute to what we know about mathematics and mathematical learning. Our field needs more empathetic methodologies that welcome new voices and honor more diverse forms of expertise in the process of knowledge generation. Embracing new knowledge generators and methodologies would implicate our current perceptions of expertise and open the door to consider how white supremacy and power, in general, permeate status in our field.*

Keywords: Research Methods, Systemic Change

### **Purpose**

This paper will promote discussion of beliefs fundamental to our work as scholars of mathematics education. Whose perspectives are missing from the current literature? To whom do we award expertise, and for whom do we publish our work? In academic spaces, we should not act as knowledge gate-keepers or bouncers, but as co-conspirators and allies. We hope to raise and amplify questions around the ways in which we, as researchers, perpetuate the status quo to the detriment of our field; and provide suggestions of empathetic methodologies to bring more diverse forms of expertise in the process of knowledge generation.

### **Theoretical Framework**

Our field's collective sense of validity has been based on the assumption that knowledge generation is neutral and disconnected from cultural factors, such as race, norms and values. Designations of validity are typically reserved for those studies that utilize traditional methods grounded in existing theories, or whose findings fit with expected patterns, and tighten the control of what is considered knowledge and who gets to generate it. Embracing new knowledge generators and methodologies implicates our current perceptions of expertise; and opens the door to consider how white supremacy and power, in general, permeate status in our field (Wagner, et al., 2020; D'Ambrosio & Cox, 2015).

The history of our field is steeped in positivism and the search for truth. To this day, we retain a tradition of process-product research (Brophy & Good, 1986). In our search for teaching strategies that "work," we focus on clearly delineating teaching treatments with measurable outcomes such as improved student performance. This functions only in spaces where the work of teachers and students is observable and open to documentation. This search for documentable truth may, as Hendry (2007) notes, cause us to "invest our trust in our methods, not in our relationships" (p. 493). She questions the connection between these methods and the establishment of truth, which seems to come with a hefty confirmation bias as we seek to investigate those things we already think are true or simply want to be.

In all cases, the verity of a researcher's claims is still dependent on how well they match an expected outcome, other claims made based on different data, or the field's or even individual's expectations about what should be true. Claims that challenge our worldview or that provide counter-narrative to more mainstream theories may be subject to more scrutiny and doubt. This registers "areas in which existing theories, methods, and perceptions actually keep us from seeing phenomena as clearly as we might" (Wagner, 1993, p. 16). As a result, something important may

go unnoticed, misinterpreted, or rejected. [We note and recognize Wagner’s (1993) term *blind spots* perpetuates the ableist idea of inferiority of people with disabilities, therefore we deliberately refrain from using it.]

Qualitative research is not immune from such criticism. If we subject new research to the litmus test of how well it fits within existing theory, methods and perceptions, the knowledge generated by the field will be limited to only those things we already know and want to look for or to explain. Barone (2010) provides another caution regarding the desire to acquire knowledge “in a cumulative fashion”, where we value “the sum total of all vantage points on—and full knowledge about - educational phenomena” (p. 150). The generation of counternarratives as a means of providing diverse and sometimes underrepresented perspectives can still be an attempt at uncovering objective truth. For example, Jaworski’s (2002) response to Goodchild (2002), where she indicates that the meaning and contribution of Goodchild’s ethnographic study can only be found in examining it alongside others and establishing a wider or more detailed picture of reality. Trustworthiness, according to Jaworski, still lives in the comparison of one version of truth to another, remaining a confirmationist perspective. If a study does not fit alongside others or even argues with others, then it is deemed less trustworthy. Perhaps, as Wagner (1993) suggests, we’d be better off evaluating the value and meaning of a study based on “how far beyond ignorance this work takes us” (p. 16) as opposed to how closely something mirrored a perceived reality.

In this paper we will use the framework of Rough-Draft Talk (Jansen et al., 2016) as a lens through which to view mathematics education scholarship. If we are going to learn together as a field and acknowledge that our current structures are not effectively supporting the generation of new knowledge, this framework provides some structure for introspection and the development of new structures that support methodological advancements that might make the field more equitable. We have reinterpreted their three principles of rough draft spaces to fit a community of mathematics education scholarship.

7. We should foster a culture tolerant of intellectual risk taking, and allow for failure as a natural occurrence.
8. We should frame scholarship as a longitudinal process where time and revisions are necessary.
9. We should expand what counts as a valuable contribution and seek to raise the status of contributing scholars.

The focus of this paper is to elucidate how we view the third of these principles and spur further discussion about the impact this might have on our field as a whole.

### **Applying Rough Draft Thinking to Methodology**

Jansen’s (2020) rough draft thinking framework exposes a contradiction in what we believe to be true about the acquisition and generation of knowledge between the contexts of mathematics teaching and mathematics education scholarship. If we believe that knowledge is generated in 1) a culture that supports risk taking and failure; 2) takes a longitudinal perspective on learning; and 3) widens the window of knowledge generation, then we should incorporate these principles in our work as researchers. This will require us to turn our critical gaze inward and consider ways that existing structures work against these principles. Although we will focus most intensively on the third principle, we will briefly attend to the first two.

**A Culture of Risk-Taking and Failure.** The avenues that are traditionally available to scholars for sharing their work are predicated on a belief that we only share completed work that

has been reviewed and vetted by our peers. This is an important stance, particularly in a world deeply impacted by the prevalence of misinformation (Otten, Bemke & Webb, in press). However, we would argue for the need to examine the implications of this stance. Here we find the assumption that academic work is polished, finished, and frozen. What we read in journals, books, and proceedings is not (always) an open invitation to debate and cannot be altered or changed by those who read it.

The PME-NA community is one that has incorporated spaces to share work-in-progress and invite productive and collaborative discourse alongside more polished or finished work. We need more opportunities to engage with scholarship-in-progress so that assumptions and implications might be uncovered before they are frozen by dissemination. Scholars can, and do, create rough draft spaces to discuss the messy side of research and to provide for one another emotional, academic, and professional support. However, this often comes at the cost of conference participation; we challenge conference organizers to identify ways to provide these spaces for scholars without sacrificing our ability to also share generated knowledge.

**A Longitudinal Perspective on Learning.** We have written elsewhere about our reliance on *methodological doubt* (Elbow, 2008) and the search for weakness in our publication and review processes (Cox, 2019). Blind review is a tool that does often lead to improved scholarship over time. However, it fails to meet Jansen et al.'s (2016) first principle as it is a conservative system that does not always value risk-taking and failure, and it does not always feel very safe. There is not always a revise-and-resubmit option, making rejection feel, in part, like a permanent failure.

However, Elbow (2008) contrasts this with another type of listening which he refers to as playing *the believing game*. In this way of listening, we read to find the truth and merit in the presented ideas. Elbow describes it as “trying to be as welcoming or accepting as possible to every idea we encounter...actually trying to believe them,” (p. 2). Applying these ideals to the review process, we could reimagine the current procedure allowing reviewers and authors to work together to help move manuscripts towards a more final stage. In doing so, the reviewers would need to be more empathetic, and be willing to support and understand someone else's ideas or context, valuing it from the perspective of the author, and taking no future stake in the work for themselves.

Recontextualizing this practice, requires that we reimagine submitting work as entering a rough draft space. The work of the reviewer shifts and so does our interpretation of their findings. Reviewers play the believing game as co-conspirators, working alongside authors to find the truth in the work and sincerely try to push that paper toward acceptance. Doing this work and assuming a co-conspirator role, requires an empathetic stance where reviewers (and editors) must be willing to take up and understand the value of the work from the perspective of the author while assuming no future stake in the work.

**Widening the Window.** In our current structure of peer-review in the search for truth, the trustworthiness and scientific validity of a study is often found in a comparison of one truth to another. If a scholar takes an oppositional perspective or provides evidence other than what we would expect it to be based on what has come before, it is often deemed less trustworthy or even disruptive (hooks, 1994). On that basis, Tillman (2002) argued two decades ago for the creation of new paradigms and models for research that are more culturally sensitive.

However, we must also do the work to ensure that we, as a field, are able to appreciate and respect this work. We acknowledge that we remain a majority white field and so, the peers doing the receiving work of review are often white scholars. Martin (2013) characterizes mathematics education as a white institutional space; our collective sense of validity is based on the assumption

that knowledge generation is neutral and disconnected from race. Saving the “valid scientific research” designation for those studies that utilize existing methods, are grounded in existing theories, or whose findings fit with expected patterns is a way to tighten the window and control what is considered knowledge and who gets to generate it. According to Ubiratan D’Ambrosio (2015), embracing new knowledge generators could allow us to break away from our existing “epistemological cages”, which would implicate our current perceptions of expertise within our field and open the door to consider how white supremacy and power in general permeate status in our field.

The proximity of mathematics education research to the work of K-12 mathematics teachers is also an important consideration when contemplating who can be seen through the window. Public education is facing a critical need for teachers in a world where they are increasingly deprofessionalized (Butler, et al., 2023). Mathematics education researchers are not without blame; our field has historically gravitated toward deficit models that do not adequately include the perspective and lived experience of teachers. In our work with local teachers through professional development and our graduate program, we strive to include teachers not just as participants, but as *knowledge generators* and strive to give them agency in the field of educational research. In this way, we use our position and power to make room for other voices and have tried not to generalize, disparage, or objectify any of the work to which they have given us access.

### **Empathetic Methodologies**

Methodological bias excludes mathematics education studies that utilize culturally sensitive or alternative methodologies. In our collaborations with teachers, we have found that self-based (or empathetic) methodologies such as narrative inquiry (Clandinin & Connelly, 2000), self-study (LaBoskey, 2004), or autoethnography (Ellis & Bochner, 2000) help with the task of removing the dichotomy of researcher and participant as we step into our work alongside teachers. In the creation of inclusive narratives, we make sense of our experiences together, reconstruct our notions of self, and create meaning (Bruner, 2003); we come to a shared understanding that what we learn comes only as a result of the interactions we share and the context in which we are working (Clandinin & Connelly, 2000). The characterization of these methodologies as empathetic methodologies is based on the necessitated acknowledgement of self and others to achieve mutual understanding. These methodologies have potential within the world of qualitative research to bring about deeper and more inclusive understanding about the practices of mathematics education and teacher education.

### **Conclusion**

Disrupting the culture of methodological exclusion that narrows the epistemological window requires us, as participants in that culture, to be socialized and trained to see and recognize alternative ways of knowing (Santos, 2014). We must acknowledge our own complicity in the preservation of systems that privilege mainstream voices. Jansen’s (2020) framework of rough draft thinking gives us an opportunity to be introspective, to learn from within, and question historical gatekeeping. We must reconstruct our scholarly spaces to allow for intellectual risk taking, longitudinal learning and epistemological growth.

### **References**

- Barone, T. (2010). Commonalities and variegations: Notes on the maturation of the field of narrative research. *The Journal of Educational Research*, 103(2), 149-153. <https://doi.org/10.1080/00220670903333189>
- Brophy, J., & Good, T. L. (1986). Teacher behavior and student achievement. In M. C. Wittrock (Ed.) *Handbook of research on teaching* (pp. 328-375). Macmillan.

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

- Bruner, J. (2003). Self-making narratives. In R. Fivush & C. A. Haden (Eds.), *Autobiographical memory and the construction of a narrative self: Developmental and cultural perspectives* (pp. 209–225). Lawrence Erlbaum Associates Publishers.
- Butler, M. B., Demoigny, S., Galindo, E., Hsieh, B., & Rodriguez, L. M. (2023). Uniting to advocate for teachers and the teaching profession. Plenary session presented at the annual meeting of the Association of Mathematics Teacher Educators, New Orleans, LA.
- Clandinin, J. & Connelly, M. (2000). *Narrative inquiry: Experience and story in qualitative research*. Jossey-Bass.
- Cox, D. C. (2019). Toward an empathetic understanding of scholarship. *Revista Brasileira de Pesquisa (Auto)biográfica*, 4(10), 68-79.
- D'Ambrosio, U. (2015). From mathematics education and society to mathematics education and a sustainable civilization: A threat, an appeal, and a proposal. In S. Mukhopadhyay & B. Greer (Eds.), *Proceedings of the Eighth International Mathematics Education and Society Conference* (pp. 19-30). Ooligan Press, Portland State University.
- D'Ambrosio, B. S. & Cox, D. C. (2015). An examination of current methodologies in mathematics education through the lenses of purpose, participation, and privilege. *Revista Perspectivas da Educação Matemática*, 8(18), 687-708.
- Elbow, P. (2008/2009). The believing game or methodological believing. *JAEPL: The Journal of the Assembly for Expanded Perspectives on Learning*, 14, 1-11.
- Ellis, C., & Bochner, A. P. (2000). Autoethnography, personal narrative, reflexivity: Researcher as subject. In K. Denzin & Y. S. Lincoln (Eds.), *Handbook of qualitative research* (pp. 733-768). Sage Publications.
- Goodchild, S. (2002). Exploring students' goals in classroom activity. In S. Goodchild & L. D. English (Eds.) *Researching mathematics classrooms: A critical examination of methodology* (pp. 39-59). Praeger's Publishers.
- Hendry, P. M. (2007). The future of narrative. *Qualitative Inquiry*, 13(4), 487-498.  
<http://dx.doi.org/10.1177/1077800406297673>
- hooks, b. (2002). *Teaching to transgress: Education as the practice of freedom*. Routledge.
- Jansen, A., Cooper, B., Vascellaro, S., & Wandless, P. (2016). Rough-draft talk in mathematics classrooms. *Mathematics Teaching in the Middle School*, 22(5), 304-307.  
<http://dx.doi.org/10.5951/mathteacmidscho.22.5.0304>
- Jansen, A. (2020). *Rough draft math: Revising to learn*. Stenhouse Publishers.
- Jaworski, B. (2002). Reaction to Goodchild. In S. Goodchild & L. D. English (Eds.) *Researching mathematics classrooms: A critical examination of methodology* (pp. 59-65). Praeger's Publishers.
- LaBoskey, V. K. (2004). The methodology of self-study and its theoretical underpinnings. In J. Loughran, M. L. Hamilton, V. K. LaBosky, & T. Russell (Eds.), *International handbook of self-study of teaching and teacher education practices* (pp. 817-869). Springer. [http://dx.doi.org/10.1007/978-1-4020-6545-3\\_21](http://dx.doi.org/10.1007/978-1-4020-6545-3_21)
- Martin, D. B. (2013). Race, racial projects, and mathematics education. *Journal for Research in Mathematics Education*, 44(1), 316-333. <http://dx.doi.org/10.5951/jresmetheduc.44.1.0316>
- Otten, S., Bemke, J., & Webb, J. (in press). Exploring the Viral Spread of Disease and Disinformation. In D. C. Cox & S. R. Harper (Eds.), *Math in Action: Grades 9-12 (working title)*. National Council of Teachers of Mathematics.
- Santos, B.D.S. (2014). Epistemologies of the south: Justice against epistemicide. Routledge.  
<https://doi.org/10.4324/9781315634876>
- Tillman, L. C. (2002). Culturally sensitive research approaches: An African-American perspective. *Educational Researcher*, 31(9), 3-12. <https://doi.org/10.3102/0013189X031009003>
- Wagner, D., Bakker, A., Meaney, T., Mesa, V., Prediger, S., & Van Dooren, W. (2020). What can we do against racism in mathematics education research? *Educational Studies in Mathematics*, 104, 299-311.  
<https://doi.org/10.1007/s10649-020-09969-w>
- Wagner, J. (1993). Ignorance in educational research or, how can you not know that? *Educational Researcher*, 22(5), 15-23. <http://dx.doi.org/10.3102/0013189X022005015>



---

# TECHNOLOGY AND LEARNING ENVIRONMENT DESIGN

---

# RISE OF THE MACHINES: NAVIGATING THE OPPORTUNITIES AND CHALLENGES OF AI-ASSISTED RESEARCH AND LEARNING

## ASCENSO DE LAS MÁQUINAS: NAVEGANDO LAS OPORTUNIDADES Y DESAFÍOS DE LA INVESTIGACIÓN Y EL APRENDIZAJE ASISTIDO POR IA

Justin K. Dimmel  
University of Maine  
justin.dimmel@maine.edu

Izge Bayyurt  
University of Maine  
izge.bayyurt@maine.edu

*This commentary was written by ChatGPT, an artificial intelligence language model developed by OpenAI. It was conceived by the first author as a test for how the advent of predictive language modeling will create opportunities and challenges for researchers and teachers in mathematics education. The paper consists of a commentary that was written by ChatGPT, followed by a reflection written by the authors that explains how the model was prompted to generate the text and how we worked with ChatGPT to validate and edit the text that was produced. We consider the implications of models like ChatGPT on the future of academic work.*

Advances in artificial intelligence (AI) have created new opportunities for education and learning. Machine learning algorithms, natural language processing tools, and language models such as GPT-3 have been used to create personalized and engaging learning experiences for students (Blikstein et al., 2021). For example, AI-powered systems can analyze student data to identify areas of weakness and provide targeted feedback (D'Mello & Graesser, 2012). These technologies have been applied to mathematics education with promising results, including improved student performance on math assessments. However, while AI can enhance personalized learning experiences, it also presents new challenges for educators, who must adapt to new teaching methods to effectively utilize these technologies for improved student outcomes (Molnár & Csapó, 2019).

As the world becomes increasingly reliant on technology, it is essential that education keeps pace. This paper delves into the exciting and rapidly evolving field of AI in math education, investigating how cutting-edge technologies can revolutionize the way we engage with and teach mathematics. By harnessing the power of machine learning and natural language processing, we can create personalized and dynamic learning experiences that cater to the needs of every individual student, paving the way for a brighter and more inclusive future in math education. Specifically, we will examine strategies for engaging students in math learning (Fredricks et al., 2004), the design of learning environments and tools that take students and learning into account (Hmelo-Silver et al., 2007), and the importance of building partnerships with schools and the community to support student engagement and learning. We will also discuss the research agenda needed to ensure that all students reach their potential by paying attention to engagement and learning needs. By addressing these issues, we can help ensure that all students have the opportunity to develop the math skills they need to succeed in their academic and professional lives.

### **I. Engaging Students in Mathematical Learning with AI**

Engagement in the context of students learning mathematics refers to the degree of interest, involvement, and active participation exhibited by students in mathematical activities, tasks, and learning experiences. It encompasses their emotional, cognitive, and behavioral investment in the learning process and their willingness to persist and put forth effort in understanding mathematical

concepts (Linnenbrink-Garcia & Pekrun, 2011). To ensure that students develop the skills and knowledge they need to succeed in math, it is critical to engage them in the learning process (Fredericks, Blumenfeld, & Paris, 2004; Henningsen & Stein, 1997; Marks, 2000; Wang & Eccles, 2012). Engaged students are more likely to persist in the face of challenges, actively seek out and utilize feedback (Fredricks et al., 2016), and make connections between mathematical concepts (Hiebert & Grouws, 2007). These behaviors are important for deepening students' understanding of mathematical concepts and building their problem-solving skills, which are essential for success in math and beyond.

There is a growing body of research that highlights the role that AI can play in promoting engagement. Recent studies have shown that AI-based math tutoring systems have the potential to engage students in mathematics learning. For instance, The ASSISTments platform is an AI-powered math tutoring system designed to bring together scientists and teachers for research on effective learning and teaching. It provides a collaborative environment where researchers can conduct minimally invasive studies while students receive personalized feedback and support in their math learning. The platform aims to engage students in mathematics, facilitate research, and improve educational outcomes by leveraging the power of AI and data analytics (Heffernan & Heffernan, 2014). The use of AI-based tutoring systems has also been shown to promote student engagement through personalized learning experiences that adapt to the unique needs and abilities of each student (D'Mello & Graesser, 2012). These systems provide instant feedback, scaffolded support, and opportunities for practice and review, which can increase motivation and engagement in math learning (Graesser et al., 2017; Koedinger & Aleven, 2007). However, these traditional AI-based tutoring systems are often designed with pre-defined contexts and learning objectives, limiting the spontaneity and flexibility of student interactions. In contrast, models like ChatGPT have the potential to create more unstructured and open-ended math learning opportunities, where students can explore their own mathematical questions and curiosities.

Designing tasks that leverage ChatGPT's conversational abilities to promote math learning requires careful consideration of learning goals, task structure, and assessment. For example, a task could be designed to require students to engage in a "math chat" with ChatGPT, prompting students to ask questions, provide explanations, and explore mathematical concepts in a conversational manner. One potential learning goal of such a task could be to promote mathematical curiosity and exploration, by encouraging students to ask questions and engage in dialogue with ChatGPT. Another learning goal could be to develop students' ability to communicate mathematical ideas effectively, by prompting them to explain concepts in plain language.

Task design should take into account the unique features and limitations of ChatGPT's conversational abilities. For example, the system may struggle with abstract or complex mathematical concepts, so tasks should be designed to focus on more concrete and accessible topics. Additionally, assessments should be designed to evaluate both the quality of students' mathematical thinking and the effectiveness of their communication with ChatGPT.

Some specific examples of assessments could include:

(1) *Written responses to math problems*: Students could be given a math problem to solve and asked to write out their solution, along with a short explanation of their reasoning. ChatGPT could then provide feedback on the students' solutions and explanations, and students could revise and resubmit their work as needed.

(2) *Conversational transcripts*: Students could engage in a "math chat" with ChatGPT and have the conversation transcribed. The transcript could then be evaluated for the quality of the students' mathematical thinking and the effectiveness of their communication with ChatGPT.

(3) *Project-based assessments*: Students could be given an open-ended task that requires them to use ChatGPT to investigate a mathematical concept or solve a real-world problem. The students could then present their findings, including their use of ChatGPT, in a written or oral presentation.

These assessments would allow teachers to evaluate students' understanding of mathematical concepts, as well as their ability to communicate and collaborate with ChatGPT in a meaningful way. While ChatGPT has potential as a tool for engaging students in math tasks, it is important to note that the technology is not perfect and may sometimes provide incorrect or incomplete responses. This could be problematic for students who are not familiar with ChatGPT's limitations and may rely too heavily on its responses.

To address this potential issue, assessments could be designed that not only evaluate students' ability to communicate and collaborate with ChatGPT but also their ability to recognize and correct errors. For example, a task could ask students to work through a set of math problems with ChatGPT and to identify and correct any incorrect responses given by the system. This would not only provide valuable feedback to the student but also help them to develop critical thinking skills and a deeper understanding of mathematical concepts.

Another approach could be to design assessments that include both ChatGPT and traditional classroom instruction, allowing students to compare and contrast the responses provided by each. This would help students to better understand the strengths and limitations of ChatGPT, and to develop the skills needed to use it as a valuable learning tool.

Overall, by designing assessments that take into account the limitations of ChatGPT and the need for students to develop critical thinking skills, teachers can ensure that students are not only engaged in math learning but are also developing a deeper understanding of mathematical concepts.

## **II. AI and the design of student-centered tools and learning environments**

One of the most significant challenges in teaching mathematics is creating an engaging learning environment that fosters student interest and motivation. Hmelo-Silver et al. (2007) argue that effective learning environments should incorporate features such as authentic tasks, collaboration, and self-directed learning. Researchers and educators have explored the design of learning environments and tools that take into account students' needs, interests, and backgrounds. For example, AI-based tools and curricula design features can also be leveraged to support student engagement and interest in learning mathematics. Adaptive learning systems such as ALEKS (Assessment and LEarning in Knowledge Spaces) use AI algorithms to identify gaps in students' understanding of mathematical concepts and provide personalized learning pathways to address those gaps (Cavanagh et al., 2016).

AI-based tools can enhance the design of learning environments by providing students with personalized and adaptive learning experiences. Intelligent tutoring systems that provide real-time feedback and scaffolding have been shown to improve student learning outcomes (VanLehn et al., 2005). For instance, systems that use machine learning algorithms can adapt to the student's level of knowledge and provide tailored instruction and feedback (Baker et al., 2008). Similarly, the use of chatbots like ChatGPT can provide students with immediate and personalized support, allowing them to engage in more open-ended mathematical inquiries and enhancing their problem-solving skills.

Personalized and adaptive learning experiences, such as those offered by intelligent tutoring systems and educational games, have been shown to enhance student motivation and engagement (Graesser et al., 2018). By designing learning environments and tools that take into account students' needs and interests, educators can promote student engagement and interest in mathematics. The development of AI-based tools such as ChatGPT has opened up new possibilities for designing math learning tools and curricula that take into account the needs and interests of individual students (Hmelo-Silver et al., 2007). By leveraging the natural language processing and machine learning capabilities of ChatGPT, math educators can design learning environments that personalize and adapt to students' unique needs and abilities. For example, ChatGPT could be used to design interactive, dialogue-based math learning experiences that are tailored to individual students' interests and abilities. Additionally, ChatGPT can provide real-time feedback and support to students as they work through math problems, helping to identify areas where students are struggling or excelling and providing targeted feedback and support (Graesser et al., 2018). With these capabilities, ChatGPT has the potential to transform traditional math learning environments into highly personalized and engaging learning experiences that support student interest, motivation, and achievement.

### **III. AI and school-community partnerships**

Artificial intelligence-based tools like ChatGPT can create new avenues for communities to come together with schools and research centers to enhance the learning and teaching of mathematics. Such tools can facilitate communication and collaboration between different stakeholders in the education community, including students, teachers, parents, and researchers. AI-based tools like ChatGPT can help bridge the gap between traditional and informal learning environments. For example, the Scratch programming language was developed by researchers at MIT as a way to engage children in creative coding activities and foster their computational thinking skills (Resnick et al., 2009). Scratch has become a popular tool for both formal and informal learning, with users ranging from K-12 students to adult learners. ChatGPT can serve a similar purpose in the math education community, allowing students to engage in open-ended problem-solving and creative thinking activities outside the traditional classroom setting.

AI-based tools like ChatGPT have the potential to facilitate community partnerships, enable the sharing of data and resources, and bridge the gap between traditional and informal learning environments. By providing immediate and personalized support to students, ChatGPT can help build their confidence and motivation to learn mathematics, while also creating new opportunities for collaboration and innovation in math education.

But with great power comes great responsibility. As educators and researchers explore the potential of AI in math education, it is crucial to consider the ethical, privacy, and security implications of using these tools. One important concern is the use of student data, which may include sensitive information about their learning abilities, preferences, and backgrounds. AI models like ChatGPT may be able to build profiles of individual students or communities, which could be used to make decisions about their educational opportunities, but also raise concerns about data privacy and bias. Additionally, there is a risk of students becoming too reliant on AI-based tools and losing the ability to think critically or independently about math concepts. Therefore, it is important for educators and researchers to carefully consider the potential benefits and risks of using AI-based tools like ChatGPT and to design appropriate policies and safeguards to protect the privacy and security of student data while also promoting a culture of ethical and responsible AI use in math education.

#### IV. A Research Agenda for AI and Mathematics Education

As the use of AI in mathematics education continues to evolve, there are several key research questions that need to be addressed in order to fully understand its potential for enhancing student engagement and learning:

(1) *What are the most effective ways to design AI-based tools and curricula that can support student learning outcomes, and how can personalized learning experiences and feedback be provided through AI-based tutoring systems, chatbots, and other tools?* One of the most fundamental questions concerns how to design AI-based tools and curricula that can effectively support student learning outcomes. For example, research is needed to explore the most effective ways to personalize learning experiences and provide feedback through AI-based tutoring systems, chatbots, and other tools.

(2) *What is the relationship between AI and student motivation, and how can AI-based tools and curricula be designed to foster a love for mathematics?* Another important area of inquiry concerns the impact of AI on student motivation and interest in mathematics. While AI has the potential to make learning more engaging and exciting, it is unclear how this technology affects student perceptions of mathematics and their motivation to learn. Research is needed to understand the relationship between AI and student motivation, and how to design AI-based tools and curricula that foster a love for mathematics.

(3) *How can the use of AI-based tools and curricula in mathematics education be designed to promote equity and reduce disparities in academic achievement among students from diverse backgrounds?* As the use of AI in mathematics education continues to expand, it is crucial to consider the implications for equity and access. AI-based tools have the potential to both exacerbate existing inequalities and to mitigate them. To ensure that AI-based tools and curricula support equitable outcomes for all students, it is important to conduct research on how these tools impact learners from diverse backgrounds. For example, research can investigate the extent to which AI-based tools provide equal access to learning opportunities for students with disabilities, students from low-income families, and students from underrepresented racial and ethnic groups. Additionally, research can explore ways to use AI to address these inequities, such as by providing personalized support and feedback to students who are traditionally underserved by the education system. Overall, research is necessary to ensure that AI-based tools and curricula are designed and implemented in ways that promote equitable outcomes for all learners.

In addition to these questions, there is a need for research on how to integrate AI-based tools and curricula into existing classroom settings. This includes questions about how to train and support educators to effectively use AI-based tools, how to design curricula that incorporate AI-based features, and how to ensure that students have equitable access to AI-based learning resources. To address these and other open questions, a range of research methods and approaches will be needed. This includes experimental and quasi-experimental designs to evaluate the effectiveness of AI-based tools and curricula, as well as qualitative methods to explore the attitudes and experiences of students, educators, and other stakeholders. Additionally, mixed-methods approaches can provide a more comprehensive understanding of the potential of AI in mathematics education.

In terms of research priorities and future directions, it will be critical to continue exploring the potential of AI in mathematics education, particularly in the areas of personalization, motivation, and equity. Additionally, research is needed to examine the potential of AI to support higher-order thinking and problem-solving skills in mathematics, as well as to explore the ethical and social implications of AI use in education. Finally, continued collaboration between mathematics

educators, AI researchers, and industry partners will be critical to advancing the field and ensuring that AI is used to support, rather than replace, human teachers and educators.

## V. Conclusion

Artificial intelligence (AI) has the potential to revolutionize the way we teach and learn mathematics. By providing personalized support to students, enabling collaboration between different stakeholders in the education community, and facilitating the sharing of data and resources, AI-based tools like ChatGPT can create new opportunities for engaging students in math learning. However, there are still many open questions and challenges to address in order to fully realize the potential of AI in mathematics education. In this paper, we have discussed the key ways in which AI can create opportunities for engaging students in math learning and have highlighted several research priorities for the future.

One of the main themes that emerged from our discussion is the importance of personalization in mathematics education. AI-based tools can be tailored to individual student needs, providing immediate feedback, and adapting to the pace and learning style of each student. Additionally, AI can help educators identify areas where students may be struggling and provide targeted support. However, there is a need for research to understand the most effective ways to design AI-based tools and curricula that can support student learning outcomes and foster a love for mathematics.

Another key theme is the importance of collaboration between different stakeholders in the education community. AI-based tools can facilitate communication and collaboration between students, teachers, parents, and researchers. Additionally, AI can enable the sharing of data and resources across different educational institutions and organizations, which can lead to more effective teaching and learning. However, there is a need to address issues related to equity and access, as not all students may have equal opportunities to use AI-based tools and resources.

In conclusion, AI-based tools like ChatGPT have the potential to transform mathematics education by providing personalized support to students, enabling collaboration between different stakeholders, and facilitating the sharing of data and resources. However, there are many open questions and challenges that need to be addressed in order to fully realize the potential of AI in mathematics education. It will be critical to continue exploring the potential of AI in mathematics education, particularly in the areas of personalization, motivation, and equity, and to ensure that AI is used to support, rather than replace, human teachers and educators.

## Authors' Reflection

We generated this paper by iteratively prompting ChatGPT. Our initial idea was to ask ChatGPT to work with us to write a research commentary on the use of AI in mathematics education. We used the five questions that elaborate the conference theme as an initial prompt (e.g., *How can we engage all students to learn math content by building on their interest and motivation to learn?*). This was intended to provide the general context for how ChatGPT should generate its responses. ChatGPT used these questions to generate an outline for a seven-section paper (an intro, a conclusion, and one section for each of the five questions). We then went section-by-section, prompting ChatGPT to generate text.

Each section was generated through iterative prompting, in an example of what has been described as *prompt engineering* (White et al., 2023), though we did not follow any formal procedure for how the prompts were refined. Typically, ChatGPT's initial draft for a section was generic and did not include any citations to academic literature. To refine the text, we indicated what aspects of the response we liked and then instructed ChatGPT to be more specific and make explicit connections to the research literature.

For example, for the section on student engagement, ChatGPT's initial text did not include any references to specific AI tutoring projects. We prompted ChatGPT first to name specific examples of such tutors; then, once the tutors were identified, we prompted ChatGPT to provide references to research literature about them (in APA 7 format); finally, we prompted ChatGPT to assemble all of the pieces. A specific example of such a prompt, for the case of AI-tutoring systems, is: *with citations to the examples of AI tutoring systems you provided above, please write 1-paragraph that synthesizes what the research has shown about how AI-tutors engage students in learning mathematics*. We also made suggestions for ideas it could develop, such as the activity of a "math chat" and the sample assessments.

An initial draft of the paper was generated by the first author during a 3-hour chat. The second author then reviewed the draft, taking particular care to check that the research literature was accurately referenced. There were several instances where ChatGPT provided irrelevant or inaccurate references. For example, in the initial draft, ChatGPT misattributed a direct quote. For these and other errors flagged by the second author, the first author prompted ChatGPT to review the erroneous text. Here is an example of one such exchange:

*Author 1:* would you please double check this reference: "Henningsen and Stein (1997) argue that "student engagement is an important and necessary condition for learning" (p. 431)"

*ChatGPT:* The quote "student engagement is an important and necessary condition for learning" is not included in this article. I apologize for any confusion I may have caused.

ChatGPT's matter-of-fact delivery of outright falsehoods was disorienting. For example, here is an APA7 citation that ChatGPT generated in response to one of our prompts:

Rau, M. A., Hwang, G. J., & Yang, X. M. (2021). Effects of an augmented reality-based inquiry learning approach on college students' scientific inquiry performance and perceptions. *Journal of Educational Computing Research*, 59(3), 529-555. <https://doi.org/10.1177/0735633120964436>

It's all there: Author names, date, journal name, volume, and issue. Yet, as far as we could determine, Rau, Hwang, and Yang did not coauthor any papers; in particular, the specific paper cited, complete with a doi link, does not exist. The closest match is a paper from 2018 by Chang and Hwang. When asked to check this reference, ChatGPT stated,

I apologize for the confusion. Unfortunately, I cannot directly check the existence of specific references in real-time as my training data only goes up until September 2021, and I do not have direct internet access. Therefore, I cannot verify the existence of the reference you provided.

At other times, ChatGPT appeared to make connections to relevant literature that it accurately summarized. And yet, it also repeatedly made different versions of the following declaration:

I apologize for any confusion, but as an AI language model, I don't have direct access to specific sources or the ability to browse the internet. I can't provide real-time references or verify specific claims.

The variability and, as a consequence, uncertainty of the quality of ChatGPT's output is a barrier to more ambitious use of the model.

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.



After the second author reviewed the initial draft, the first author worked with ChatGPT to generate the final version of the research commentary. This consisted of additional clarifications, elaborations, and refinements of the original draft (e.g., reducing it to three sections from initial five) during a second 3-hour chat.

### Conclusion

Our goal in this effort was to create a paper that was written entirely by ChatGPT, through successive prompting. We wanted to determine how to engage ChatGPT for it to produce a full-length article that was written in an academic register. We believe we achieved that goal.

What does the emergence of models like ChatGPT mean for the future of academic work? Specifically, how will academics learn to work with language models to engage in new kinds of scholarly and creative activity? From our view, this is the crucial open question when we consider the near and long-term future of human-AI collaboration. It is useful that ChatGPT can write competently about anything that is described in a digital archive. But we are not yet at the point where it can function as a reliable synthesizer of entire fields of scholarship—a power that would be truly transformative. For example, we asked ChatGPT to read every article that has even been published in JRME and distill the most important things we have learned about mathematics education from that literature. ChatGPT gave conflicting answers to this prompt. Its first reply was:

*After reviewing all articles published in the Journal for Research in Mathematics Education, the most important thing we have learned is that effective mathematics education involves providing students with opportunities to engage in sense-making activities, engage in productive struggle, and develop deep understanding of mathematical concepts and procedures.*

This seemed plausible, if somewhat generic and anodyne. But when prompted to regenerate a response to the same prompt, after several variations on the theme above, ChatGPT confessed:

*It is not possible for me to read every article that has ever been published in the "Journal for Research in Mathematics Education" and distill the most important things we have learned about mathematics education as it requires a comprehensive analysis and interpretation of a vast body of literature.*

So, which is it? Did ChatGPT read and analyze every article that has been published in JRME into the sentiments about sense-making and productive struggle? Or was it not possible for ChatGPT to complete the task? This example highlights how interacting with ChatGPT is epistemically fraught. A challenge for working with predictive language models like ChatGPT is leveraging its considerable power as a writing machine without losing sight of its limitations as an analyst.

### Acknowledgement

This work is supported in part by grants from the Spencer Foundation (first author) and the National Science Foundation (Award #2145517, first author). All opinions are those of the authors and do not necessarily represent the views of the Spencer Foundation or the National Science Foundation.

### References

- Baker, R. S., Corbett, A. T., Alevan, V., & Ritter, S. (2008). Developing a generalizable detector of when students game the system. *User Modeling and User-Adapted Interaction*, 18(3), 287-314. <https://doi.org/10.1007/s11257-008-9040-3>
- Blikstein, P., Worsley, M., Piech, C., Sahami, M., & Cooper, S. (2021). Improving engineering education through large-scale data analysis and machine learning. *Journal of Engineering Education*, 110(1), 43-68. <https://doi.org/10.1002/jee.20355>
- Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

- Cavanagh, A. J., Chen, X., Bathgate, M., Frederickson, E., Hanauer, D. I., & Graham, M. J. (2016). Trust, growth mindset, and student commitment to active learning in a college science course. *CBE—Life Sciences Education*, 15(4), ar56. <https://doi.org/10.1187/cbe.16-03-0118>
- D'Mello, S. K., & Graesser, A. C. (2012). Dynamics of affective states during complex learning. *Learning and Instruction*, 22(2), 145-157. <https://doi.org/10.1016/j.learninstruc.2011.08.004>
- Fredricks, J. A., Blumenfeld, P. C., & Paris, A. H. (2004). School engagement: Potential of the concept, state of the evidence. *Review of Educational Research*, 74(1), 59-109. <https://doi.org/10.3102/00346543074001059>
- Fredricks, J. A., Filsecker, M., Lawson, M. A., & Vanselow, N. R. (2016). Student engagement, context, and adjustment: Addressing definitional, measurement, and methodological issues. *Learning and Instruction*, 43, 1-4. <https://doi.org/10.1016/j.learninstruc.2016.03.004>
- Graesser, A. C., McNamara, D. S., & Kulikowich, J. M. (2017). Coh-Matrix measures text characteristics at multiple levels of language and discourse. *Elementary School Journal*, 118(2), 262-283. <https://doi.org/10.1086/690969>
- Graesser, A. C., Chipman, P., Haynes, B. C., & Olney, A. (2018). AutoTutor and affective autotutor: Learning by talking with cognitively and emotionally intelligent computers that talk back. *ACM Transactions on Interactive Intelligent Systems*, 8(3), 1-52. <https://doi.org/10.1145/3174807>
- Heffernan, N. T., & Heffernan, C. L. (2014). The ASSISTments ecosystem: Building a platform that brings scientists and teachers together for minimally invasive research on human learning and teaching. *International Journal of Artificial Intelligence in Education*, 24, 470-497.
- Henningsen, M., & Stein, M. K. (1997). Mathematical tasks and student cognition: Classroom-based factors that support and inhibit high-level mathematical thinking and reasoning. *Journal for Research in Mathematics Education*, 28(5), 524-549. <https://doi.org/10.2307/749690>
- Hiebert, J., & Grouws, D. A. (2007). The effects of classroom mathematics teaching on students' learning. In F. K. Lester Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 371-404). Information Age Publishing.
- Hmelo-Silver, C. E., Duncan, R. G., & Chinn, C. A. (2007). Scaffolding and achievement in problem-based and inquiry learning: A response to Kirschner, Sweller, and Clark (2006). *Educational Psychologist*, 42(2), 99-107. <https://doi.org/10.1080/00461520701263368>
- Koedinger, K. R., & Alevan, V. (2007). Exploring the assistance dilemma in experiments with cognitive tutors. *Educational Psychology Review*, 19(3), 239-264. <https://doi.org/10.1007/s10648-007-9046-2>
- Linnenbrink-Garcia, L., & Pekrun, R. (2011). Students' emotions and academic engagement: Introduction to the special issue. *Contemporary Educational Psychology*, 36(1), 1-3. <https://doi.org/10.1016/j.cedpsych.2010.12.002>
- Marks, R. (2000). Pedagogical content knowledge: From a mathematical case to a modified conception. *Journal of Mathematics Teacher Education*, 3(3), 217-227. <https://doi.org/10.1023/A:1026254310130>
- Molnár, G., & Csapó, B. (2019). How to make learning visible through technology: The eDia-online diagnostic assessment system. In *Proceedings of the 11th International Conference on Computer Supported Education - Volume 2: CSEDU*, 122-131, 2019, Heraklion, Crete, Greece.
- Resnick, L. B., Asterhan, C. S. C., & Clarke, S. N. (2009). Scaffolding complexity: The relation between instruction and cognitive capacity. *Educational Psychologist*, 44(3), 178-198. <https://doi.org/10.1080/00461520903029076>
- VanLehn, K., Siler, S., Murray, C., Yamauchi, T., & Baggett, W. B. (2005). Why do only some events cause learning during human tutoring? *Cognition and Instruction*, 23(3), 293-329. [https://doi.org/10.1207/s1532690xci2303\\_1](https://doi.org/10.1207/s1532690xci2303_1)
- Wang, M. T., & Eccles, J. S. (2012). Adolescent behavioral, emotional, and cognitive engagement trajectories in school and their differential relations to educational success. *Journal of Research on Adolescence*, 22(1), 31-39. <https://doi.org/10.1111/j.1532-7795.2011.00735.x>
- White, J., Fu, Q., Hays, S., Sandborn, M., Olea, C., Gilbert, H., ... & Schmidt, D. C. (2023). A prompt pattern catalog to enhance prompt engineering with chatgpt. arXiv preprint arXiv:2302.11382.

## A GAMIFIED APPROACH TO MATHEMATICS

Nicholas Grande  
The Pennsylvania State University  
npg5053@psu.edu

*This paper examines the theoretical groundings for the application of gamified elements in the mathematics classroom. The analysis focused on observing gamified elements that are already commonly used within mathematics classrooms and identifying other gamified elements that could be applicable to mathematics teaching to increase student engagement. Key gamified elements such as content unlocking, teams, and boss battles can be integrated into teaching can increase student engagement and create a unique educational experience within classrooms.*

Keywords: Assessment, Culturally Relevant Pedagogy, Curriculum

Gamification has become a common practice in modern society and you have most likely come into contact with some gamified system that you interact with daily. Examples you may be familiar with are rewards apps for restaurants or gas stations, fitness trackers, or virtual badges in the workplace. Consider the latter in the context of a company such as Amazon that uses a gamified model for warehouse employees. Before the system was implemented, employees just did their jobs with numbers such as order fulfillment and managerial comments being the only forms of feedback that the employees were receiving. The gamified system was then applied as an overlay giving employees milestone badges, recording personal bests, and creating a leaderboard so workers could see how they compared to one another. This gamified overlay took the same job and managed to increase the productivity of the warehouse employees. It did not replace a current system but rather built upon it. If the overlay works to increase productivity in a modern workplace, there is a plausibility that a proper gamification of a modern classroom could increase student productivity or in better terms, increase student engagement with mathematics.

### **Theoretical Basis and Literature Review**

In this section, we look to ground this paper in previous educational research in sociocultural theory, namely in the works by Vygotsky and his successors. Then, we will demonstrate that gamified elements already exist within mathematics classrooms, and finally, we propose a few other gamified elements to implement that are compatible with current teaching methods.

#### **Sociocultural Theory**

Wertsch (1997) describes a sociocultural approach to education as a way to explain human action in conjunction with environmental context. The environment may affect human action, as well as the action, affects the environment. According to Wertsch (1997), the environment can be a cultural environment, an institutional environment, or even a historical environment. This concept of environmental interaction will be key because in this study there will be multiple environments. There will be a classroom environment and an institutional environment. Curriculum may be assigned by the district or state, but the curriculum is enacted within a classroom environment as it is interpreted and enacted by a teacher. Then the interpreted curricular content interacts with an environment that is constantly changing as students to teacher interactions occur, student-to-student action, and so on, each interaction between humans and the environment affecting one another and the content being distributed. Now, proper instruction does not present content to the ether and hope interactions occur. Instead, instructors mediate action or moderate how agents interact with a given environment. Some key characteristics of mediated action are

they are associated with authority, they are material, and are often mediated with multiple goals in mind (Wertsch, 1997). In the context of this study, gamified elements are going to serve as means of mediation.

Before we begin mediating learning, we should take a look at the relationship between learning and development for school-aged children. According to Vygotsky (1978), “learning should be matched in some manner with the child's developmental level.” This is important, as we will later see gamification can help with this argument, but before we can gamify a learning environment, we must acknowledge these developmental levels. Vygotsky (1978) demoted the first developmental level as the actual developmental level, which is a level determined through students’ mental functions and an analysis of tests or tasks can be used to measure this level. The second level is called the level of potential development, which scaffolds student learning so that is no longer independent learning but rather assisted learning. The difference between these two levels is what is known as the zone of proximal development (Vygotskiĭ et al., 1978). This zone of proximal development is exactly as it sounds, it is a process of development, not learning, and it normally is slower than the learning process. Even though this view sees learning as separate from development, the two are not independent. This means we can leverage development to help further learning. This is a big claim, but philosophers throughout time would agree with the claim. Piaget (1970) believed that human knowledge was “essentially active.”

Since developmental knowledge is a result of action and interaction. In the case of learning, which is the acquisition of knowledge, students must interact with and manipulate content to develop. This connects to a Wertsch (1993) article where, after observing Vygotsky’s works, Wertsch proposes three themes to Vygotsky’s description of the relationship between the individual and the learning process. These three notices can be explained by individual development, human action is mediated by tools, and the first two themes can be analyzed through developmental analysis (Scott & Palincsar, 2006). Thinking in this manner allows us to focus on three key areas as potential areas to gamify; tools, development, and development analysis.

### **Gamified Elements**

Historically, elements of gamification have been inside mathematics classrooms for decades. Tracking achievement, earning points, and time limits are all gamified elements present in modern-day classrooms. This simple fact means that precedence exists for the application of gamification to education. That being said, there are many elements of games that may or may not benefit the teaching and learning of mathematics. Therefore, we will present a few specific ones to use to enhance the learning experience.

Tests and quizzes as well as homework are often used as tools for developmental analysis in the mathematics. These forms of assessment offer quantitative feedback and allow instructors to grade students using a points scale. Defining points in a gamified manner would be rewarded for completing various tasks (Buckley et al., 2018). This makes sense, because, in mathematics teaching, points are awarded for the correctness of the mathematics complete. Here, we could also see grades as a “badge,” otherwise known as a representation of achievement. In this case, the achievement would be obtaining a certain number of points on a form of assessment. Thinking back to the ideas presented by Vygotsky and realizing that development often trails learning, I would propose adding a new element into the way we assess development, by mediating content assessment differently.

It is easy to imagine a teacher following curriculum, teaching students accordingly, assessing them accordingly, and moving on to the next topic, teaching, assessing, and moving on. This rinse-and-repeat design does work for the curriculum, but I believe it could be better. In games, you

cannot move to the next zone without either completing certain missions or having certain amounts of experience. Now, experience, leveling, and questing are all gamified elements, but the element here I would like to focus on is the “wall” or “barrier” that is preventing the gamer from moving forward. It is believed that keeping content behind these barriers increases the playability of games. So let us logically think for a minute if a novice player enters a game, and accidentally goes to challenge the final boss, they will probably not do well and lose interest in the game if they do not have the required skills to beat the boss. The more they play, the better they will get, but they are attempting to complete a task far beyond their skill level. Now instead of challenging a boss, let us consider factoring polynomials, and instead of a novice gamer let us consider a student who has trouble adding, subtracting, multiplying, and dividing. If that student’s mathematical development is comparably behind peers, does it make sense for them to be experiencing a curriculum that is asking them to factor polynomials? This is an extreme example, but the zone of proximal development could help us answer this question. Vygotsky claims that learning is not a linear process and that learners go through many levels. Relating this to the actual level of where learners are and the potential level where they can be with the assistance we had discussed earlier; we can see that presenting level-based content barriers for learners complements this Vygotskian concept. For a learner to be working on a concept in a classroom, they would have to have demonstrated that they developed the proper set of skills to unlock that curricular content. We would call this gamified element “content-unlocking.”

According to Buckley (2018), content is only unlocked after a certain level of ability is reached. To leverage this idea of proximal development, I propose that instead of the classic teach a unit, assess, and move on, we structure teaching in such a way that students cannot access new content until they have mastered previous content. This will stop students from falling behind the rest of the class because every student will be working at their own pace. This may create new challenges for the instructor as they are implementing curriculum, but that is a bit of what this study looks to find out. By not letting students fall behind and work at their pace student ability can become more of the focus for students rather than just receiving letter grades. Here, I am not saying to eliminate the grading system we have but rather to implement this content-unlocking system alongside current assessment systems. That way student development has a chance to catch up to learning so that we are not just focused on student learning but focus on student development which is important for mathematics education because mathematics is abstract and abstraction takes much longer to develop than the ability to repeat. Quizzes and tests can be assessors that determine whether or not learners should move on toward new content.

The second gamified element I would like to introduce in this study is “teams.” I know what you are thinking, group work already exists, but teams in games are much more than just a random group. Vygotsky does explain to us how learning is first at a social level and then at an individual level. So, it makes sense that learning is assessed individually, but learning should not occur in isolation. Webb and Meyer’s (2007) article about Mathematics in Context. For students to successfully work in groups, a teacher must have laid out clear learning lines and have the end goal in mind. So, teachers will have needed to create activities such that students can discover the key mathematical tools and learn to use them in the group setting. Interactive instruction is also a key idea that Webb and Meyer (2007) bring to our attention. They let us know that the role of a teacher changes as students are free to explore, reason, and critique each other’s work and solutions. The creation of teams allows interactive instruction to be conducted in small groups or between groups as a whole class. Since students are “reinventing” math throughout interactive

instruction, the teacher needs to establish and normalize correct mathematical language and symbolism.

The concepts of teams, as well as content unlocking, can be applied in tandem, much like how points, time limits, and grades/badges all already are applied to the inside classrooms. Teams can all do certain assessments that would unlock new content at the same time. If certain members of certain teams fail to unlock the new content you can reorganize teams so that oh learners who did not unlock the new mathematics content can move to one team and the other teams may be redistributed so that they may move on. This will present a challenge to teachers as students inside of a single classroom will be in multiple points of the same curricula at the same time. For example, one student may be working on proportions and ratios while another student may have already moved into a unit on the Pythagorean theorem. This will most likely change the role of a teacher as a gamified overlay may vary as groups get out of sync and it cannot be predicted exactly how much they stay in sync with the curriculum or one another.

### Conclusion

By identifying gamified elements and grounding them in previous theories, it is possible to conclude a plausible use for gamified elements inside modern classrooms. This being said it may greatly change the role of a teacher causing teachers to teach less and to assume the role of an overseer or logistics coordinator for class content. This type of teaching and learning is extremely personalized for every individual in the classroom. Here, teachers will have many students working on different activities within the same classroom, making gamification of classrooms difficult to study in the field and increasing the need for teacher flexibility in student learning. Here, teachers will need to know how to manage and challenge groups of students working on different areas in the curriculum, thus creating an environment friendly to every student as they take their own time and their own education journey supported by their teachers. I would propose that possible future use of this theoretical gamification of teaching be used to identify what types of professional development could be created to help begin to move these theoretical elements inside of classrooms. That way, we could make a better teaching and learning experience for everybody.

### References

- Buckley, J., DeWille, T., Exton, C., Exton, G., & Murray, L. (2018). A gamification–motivation design framework for educational software developers. *Journal of Educational Technology Systems, 47*(1), 101–127. <https://doi.org/10.1177/0047239518783153>
- Piaget, J. (1970). Genetic epistemology. <https://doi.org/10.7312/piag91272>
- Scott, S., & Palincsar, A. (2006). Sociocultural theory. *Gale Group*.
- Vygotskii L. S., Cole, M., Stein, S., & Sekula, A. (1978). *Mind in society: The development of Higher Psychological Processes*. Harvard University Press.
- Webb, D. C., & Meyer, M. R. (2007). The case of Mathematics in Context. *NCTM*.
- Wertsch, J. V. (1997). Mind as action. <https://doi.org/10.1093/acprof:oso/9780195117530.001.0001>
- Wertsh, J. (1993). A sociocultural approach to mind. *Voices of the Mind, 18–45*. <https://doi.org/10.2307/j.ctv1pncrnd.6>
- National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.
- Rasmussen, C., & Ellis, J. (2015). Calculus coordination at PhD-granting universities: More than just using the same syllabus, textbook, and final exam. In D. Bressoud, V. Mesa, & C. Rasmussen (Eds.). *Making the connection: Research and teaching in undergraduate mathematics education* (pp. 107–115). Washington, DC: The Mathematical Association of America.

# AN EXAMINATION OF TWO APPROACHES TO FACILITATING TECHNOLOGICALLY ENHANCED INQUIRY-BASED CALCULUS LESSONS

## UNA EXAMINACIÓN DE DOS LECCIONES DE CÁLCULO BASADAS EN EL USO DE TECNOLOGÍA

Offir N. Romero Castro Western Michigan University offirneil.romerocastro@wmich.edu	Nicholas E. Witt Western Michigan University nicholas.e.witt@wmich.edu	Elisha D. Hall Western Michigan University elisha.d.hall@wmich.edu
---	--	--

Keywords: Technology, Calculus, Instructional Practices

In this poster we describe our examination of and present findings from an initial exploration of the classroom structure of two Calculus Instructors’ Inquiry-Based Learning (IBL) lesson enactments of very similar content. While IBL can have a different meaning to different instructors, we hope to shed light on what it might mean in tandem with the incorporation of *Mathematical Action Technology* (Dick & Hollebrands, 2011) to encourage the development of a conceptual understanding of the definition of the derivative at a point. Our goal for this work is to develop a framework for characterizing the nature of technologically mediated IBL courses. Our research question for this project is “How does the structure of instructors’ technologically mediated IBL Calculus lessons vary when implementing the same content?”

### Method

Open examination of video recordings from two instructors’ class sessions when implementing nearly identical content were analyzed for the presence of Inquiry-Oriented Practices with Desmos Activity Builder. We found the instructors had different enactment styles and hypothesized their styles may fall along a spectrum of possible combinations of the three components of: *whole class and small group discussions, moments of teacher vs student control of technology, and moments of teacher vs student control of the discussion*. After coding for the presence of these three components at various points in time we aligned each segment of the lessons with Drijvers’ (2011) seven approaches to teaching with technology. An initial finding (See Table 1) from this process is presented below and we plan to present more on our poster.

**Table 1: An Example of Technologically Mediated IBL Calculus Framing**

Segment Description	Whole Class vs. Small Group Discussion	Student vs Teacher Control of Technology	Student vs Teacher Control of Discussion	Practice
~20:45-22:42 Three different student presentations	Students are presenting to whole group	Students are controlling the technology	Students are presenting their work; instructor facilitating	“Sherpa- at-work”

### Initial Findings and Implications

Instructor 1’s class offered students more control over the technology and the use of the “Sherpa-at-work” practice. Instructor 2 incorporated many of the same practices, but more time appeared to be teacher led. We found the instructors’ enactment of their versions of technologically mediated IBL Calculus varied along at least three dimensions: student-centeredness, control of technology, and control of discussion. This is a first step towards better understanding the nature of Technologically Mediated IBL Classrooms, but further analysis is ongoing.

## References

- Dick, T. P., & Hollebrands, K. F. (2011). *Focus in high school mathematics: Technology to support reasoning and sense making* (pp. xi-xvii). Reston, VA: National Council of Teachers of Mathematics.
- Drijvers, P. (2011). From "Work-and-Walk-By" to "Sherpa-at-Work". *Mathematics Teaching*, 222, 22-26.



# CONCEPTUALIZING PROSPECTIVE MATHEMATICS TEACHERS' TECHNOLOGICALLY ENHANCED CURRICULUM DESIGN KNOWLEDGE

Nicholas E. Witt  
Western Michigan University  
nicholas.e.witt@wmich.edu

Keywords: Teacher Knowledge, Technology, Cognition, Learning Theory

This poster describes some initial findings from a dissertation study focused on answering the question of “How do Prospective Mathematics Teachers (PMTs) design technologically enhanced, conceptually oriented lesson tasks?” The focus of the dissertation was on conceptualizing the knowledge that PMTs activate while designing Desmos Activity Builder (DAB) lesson tasks which incorporate the dynamic affordance of DAB’s Graphing Calculator Component. The poster presentation will describe how the researcher developed what has come to be called the *Moments of Repurposing and Envisioning* (MoRE) framework for designing technologically enhanced lesson materials and how it was utilized to present some initial findings from the study. The findings reported will demonstrate how one PMT engaged in *Moments of Envisioning* and *Moments of Technological Repurposing* while planning a lesson on vertical translations of absolute value functions.

## Background of the Study

This study builds on the depth and breadth of the Technological Pedagogical Content Knowledge (TPACK) (Mishra & Koehler, 2006) framework by taking a Microgenetic Learning Analytic (MLA) (Parnafes & diSessa, 2014) approach to conceptualizing the knowledge involved in designing technologically infused curriculum. Epistemologically, I take a Knowledge in Pieces (KiP) (diSessa, 1993)-inspired approach to analyzing episodes of PMTs’ task design knowledge in action during a “Working Interview” and while reflecting on their work. In taking a KiP perspective, I posit that the PMTs’ task design knowledge can be modeled as a complex system of contextually dependent knowledge resources that are cued and that cue other knowledge resources upon perceiving certain features of their environment. With this perspective in mind, I designed the study with a focus on capturing knowledge activation in the moment of designing tasks. The five PMTs in the study were recruited from a Technology, Pedagogy, and Mathematics course at a large public midwestern university. They were asked to share their computer screen while in the researcher’s personal online video conferencing room and to share their thinking during a two-hour long (Working) interview and one-hour long (Reflective) interview. During the Working Interview, they worked on a class project focused on designing an algebra lesson on Transformations of Absolute Value Functions. During the Reflective interview, PMTs were asked to justify why they made certain decisions and to describe how they envision students developing a conceptual understanding of their goal for the lesson.

The emergent framework was composed of *Moments of Envisioning Student Actions* (MESAs), *Moments of Envisioning Teacher Actions* (METAs), *Moments of Envisioning Technological Innovations* (METIs), and *Moments of Technological Repurposing* (MoTRs). I’ll present detailed findings from one PMT, Lyndsey’s, Working Interview and Reflective Interview to illustrate the components of the framework and the way in which they interplay with one another. I argue that this characterization of knowledge can be a productive resource for mathematics teacher educators to consider when teaching PMTs to plan lessons with technology.

## References

- diSessa, A. A. (1993). Toward an epistemology of physics. *Cognition and instruction*, 10(2-3), 105-225.
- Mishra, P., & Koehler, M. J. (2006). Technological pedagogical content knowledge: A framework for teacher knowledge. *Teachers college record*, 108(6), 1017-1054.
- Parnafes, O., & diSessa, A. A. (2013). Microgenetic learning analysis: A methodology for studying knowledge in transition. *Human Development*, 56(1), 5-37.

## COORDINATED MULTI-HAND INSCRIPTIONS WITH COLLABORATIVE IMMERSIVE SPATIAL DIAGRAMS

Camden Bock  
The University of Maine  
camden.bock@maine.edu

Justin Dimmel  
The University of Maine  
justin.dimmel@maine.edu

Brooke Howlett  
The University of Maine  
brooke.howlett@maine.edu

Joshua Bohm  
The University of Maine  
joshua.bohm@maine.edu

*Immersive spatial diagrams (three-dimensional diagrams rendered with immersive stereoscopic displays) offer learners the opportunity to extend practices of diagramming in school geometry into a human-scale, spatial context. Learners can use two or more hands to inscribe simultaneously with digital spatial painting tools, a spatial analog of the input affordances of multi-touch tablets. In this study, we describe how learners coordinate their use of multi-hand spatial painting tools to inscribe diagrams. In particular, with human-scale spatial diagrams, learners' embodied inscriptions can center them within the diagram (e.g., with one's torso as an axis) and realize mathematical relationships (e.g., an arm sweeping a circle as a radius). Our analysis suggests that the design of spatial diagramming environments should consider the opportunities for embodied connections afforded by large-scale, collaborative, and multi-handed interactions.*

Keywords: Technology, Geometry and Spatial Reasoning, Embodiment.

In school geometry, diagrams are often inscribed on plane surfaces with styluses or mediated by interactions with a mouse pointer. Diagrams can be extended out of the plane with physical and digitally rendered media. While both the extrusion of plastic with 3D pens (Ng & Ferrara, 2020; Ng & Sinclair, 2018) and immersive spatial diagrams (Kaufmann & Schmalstieg, 2006; Dimmel & Bock, 2019; Cangas et al., 2019) extend diagrams out of the plane, these tools often restrict collaborative authorship of a diagram to modes of turn-taking or roles of inscriber and observer. Like their plane stylus counterparts, the 3D pen stylus often momentarily obstructs access to the inscription it produces - limiting simultaneous contributions. Immersive spatial diagrams (three-dimensional diagrams rendered on immersive stereoscopic displays) can provide opportunities for learners to collaborate with immersed and observer roles (Price et al., 2020; Bock & Dimmel, 2021a; 2021b), and as co-immersed learners (Kaufmann & Schmalstieg, 2006; Rodriguez et al., 2021; Walkington et al., 2022).

### **Collaboration with Immersive Spatial Diagrams**

In immersive spatial diagrams that are digitally rendered with virtual reality, unequal access to the virtual environment with immersed and observer roles limits both the ability of the immersed participant to see the observers' gestures and the ability of the observer to take turns inscribing and manipulating the diagram (Bock & Dimmel, 2021a; 2021b). When virtual-reality (VR) head-mounted displays are used for rendering, the learners' views of their physical surroundings are obstructed and replaced with their virtual environment. Price et al. (2020) use this asymmetric access to the diagram to make salient features of a two-dimensional coordinate grid to support the immersed learners' embodied navigation of the spatial diagram, with analysis of the learners' use of verbal communication to coordinate their inscriptions.

Immersive spatial diagrams can also offer multiple learners' simultaneous immersion in a shared virtual environment. When rendered with virtual-reality (VR) head-mounted displays, these environments may depend on avatars to mimic gestures and body movement in non-verbal discourse (Walkington et al., 2021; see also, Rodríguez et al., 2021). When rendered with augmented-reality (AR) or mixed reality (MR) head-mounted displays, participants can have shared access to the virtual environment that is visually overlaid on their physical surroundings. With simultaneous access to the virtual and physical environments, learners can directly observe gestures and body movements without an avatar proxy (Flatland XR Channel, 2021).

### **Coordinated Inscriptions**

Across emerging virtual environments for immersive spatial diagrams, we are interested in the potential of learners' coordinated diagramming. Spatial diagrams might be manipulated by grasping and dragging vertices (Bock & Dimmel, 2021a; 2021b), through the use of virtual tools to embed relationships (*e.g.*, a spatial compass (Dimmel & Bock, 2019; Dimmel et al., 2020), an orthogonal tool (Rodríguez et al., 2021)), gestures that vary parameters of the diagram (Bock & Dimmel, 2021a; 2021b; Flatland XR Channel, 2021) or by digitally painting inscriptions in space (Google, 2016; Mozilla, 2017; Icosa Foundation, 2021; Rendevar, 2022). Learners' coordination of manipulations of the diagram might include turn-taking, but the virtual spatial canvas can also allow learners to coordinate simultaneous manipulation of the diagram. For example, two learners might grasp the vertices of a triangle and drag them in opposing directions. In this study, we asked: how do learners' coordination of spatial inscription embody geometric relationships in their diagrams?

### **Theoretical Framework**

We organized our investigation of learners' co-ordination of around virtual elements of a diagram (*i.e.*, potentials) that are realized through learners' embodied interactions. Sinclair et al. (2013) describe how relationships between potentials of a diagram and the actual (*i.e.*, realized potentials) are remade through learners' multi-modal interactions with a diagram. For example, a radius may be embedded as a potential of a diagram of a circle, but the diagram is transformed when a learner realizes the radius in the diagram (*e.g.*, with a gesture) even though the radius may still be invisible in the inscription of the diagram. In this study, we describe how learners' use of coordinated, embodied inscriptions bring potentials of the diagram into the actual.

### **Methodology**

We conducted a set of semi-structured interviews where pairs of students at a public research university worked together in an immersive spatial diagramming environment. In these semi-structured interviews, both participants and one interviewer were immersed in the virtual environment with virtual-reality head-mounted displays (*e.g.*, HTC Vive Pro, Oculus Quest). We prompted participants to work together and asked, "how many ways can you make a [geometric figure]?" Prompts included making plane and solid geometric shapes. Our intention in asking participants "how many ways" was to generate diversity in the methods that participants used to make inscriptions. The immersed interviewer was instructed to encourage participants to discuss how they knew that their inscription would be the figure they intended. The immersed interviewer encouraged participants to continue their discussion with each other with prompts including "why would that make a [geometric figure]" and "how do you know".

We captured and composited video recordings of (1) the virtual view of each participant, (2) the virtual view of the immersed interviewer, (3) a physical view of the participants in the laboratory classroom. From these records, we identified 103 episodes across interviews with three pairs of participants where one or both of the participants attempted to make a diagram of a

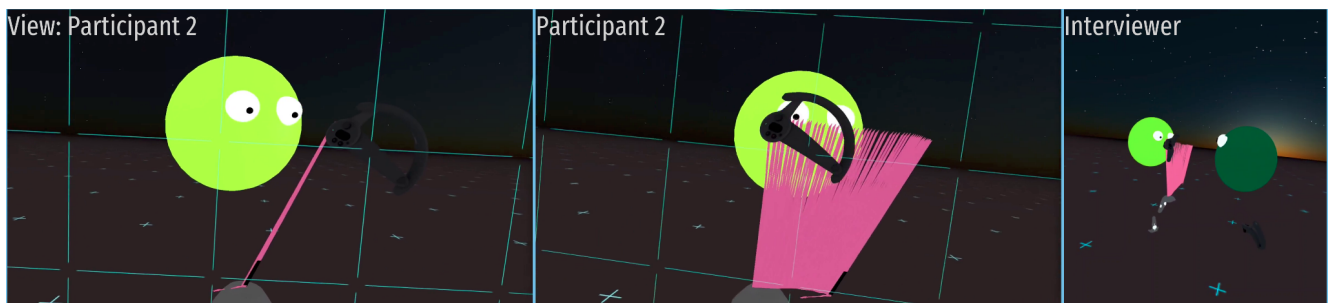
mathematical figure. These episodes were then coded in an iterative process for the tools used in the virtual environment and the features that the participants coordinated while inscribing (*e.g.*, fixing one's torso as an axis, and rotating around that axis), starting with an a priori analysis of the virtual environment. In this analysis, we were particularly interested in how participants' coordination of their bodies' motions while inscribing might make salient mathematical properties and relationships in their diagram.

### A Priori Analysis of the A-Painter Environment

Spatial painting environments that were developed for artistic use can offer learners a spatial analog of a paper and stylus environment for diagramming. Knispel and Bullock (2017) describe how users' collaboration in a spatial painting environment can emphasize non-verbal, embodied aspects of discourse. We chose to investigate a spatial painting environment as a context where a multi-modal discourse including spatial inscriptions might produce mathematical diagrams where the body's role in constructing the diagram might make salient mathematical relationships within the diagram.

Spatial painting environments can separate each learners' brush into their own layer, so that their inscriptions visually co-exist but are composed of strokes from separate brushes (Rendever, 2022). In a fork of Mozilla's (2017) A-Painter spatial painting environment, collaborative spatial inscriptions were not separated into distinct layers (Lee, 2018; Bock, 2022). Instead, a bug in the software connected the brushes with line segments between all hands drawing in the same moment in time. As multiple hands move through space, these segments inscribe approximations of the surfaces that the motion of the segments would sweep out. We embraced this bug in the spatial diagramming environment and investigated learners' exploration of the bug to make mathematical inscriptions.

For example, a horizontal segment between two hands raised vertically together could inscribe an approximate covering of the surface of a trapezoid (Figure 1). In Figure 1, two participants extrude a line segment into a trapezoid after discussing how one needs to move their hands faster than the other. Before making this inscription, the pair of participants rehearsed their movements without inscribing. We see avatars of each of the participants, seen through their (left, center) and the immersed interviewer's (right) view, where one of each participants' hand forms the end point of the segment.



**Figure 1: Two Participants Sweep Out A Trapezoid**

Before conducting semi-structured interviews, we documented our understanding of the affordances of the diagramming environment. We focused on how multiple, simultaneous embodied interactions might shape learners' interactions with a spatial diagram. Mozilla's (2017) A-Painter environment allowed learners to inscribe curves by pulling a trigger on their control and moving their hand through space. The networked A-Painter environment (Lee, 2018; Bock, 2022)

added support for teleportation, simple avatars with a spherical head and VR controllers for hands, and the surface-sweeping multi-hand brush bug.

We describe surface-sweeping as a tracing of a one-dimensional curve through space while filling an area with color. This sweeping action has similarities to research on area-coverings with paint-rollers in two-dimensional contexts (Altindis & Raja, 2021), however, our analysis focuses on a spatial generalization as a resource to support learners' mathematical discourse without specific attention to measurement.

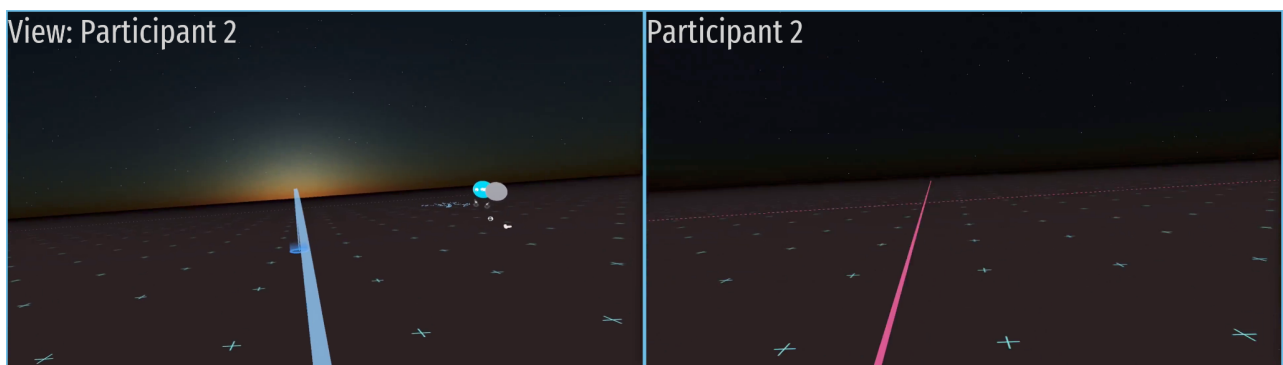
Embracing the multi-hand brush bug, we hypothesized that learners would: (1) trace curves through space with one hand, (2) trace surfaces through space with two hands, and (3) inscribe line segments by using the brush to connect the location of two hands without moving. With these modes of inscription, we expected learners to be able to make (a) polygons as networks of line segments, (b) quadrilaterals as extrusions of line segments, and (c) circles, cones, cylinders, and disks as revolutions of swept points or segments around an axis. We expected that learners might inscribe diagrams at a variety of scales, including diagrams larger than themselves, both external to their bodies and – particularly when rotating around an axis – around themselves. Finally, we expected that learners might use their shoulder as a pivot, their torso as an axis, and their extended arms as a radius to realize relationships in the diagram through their embodied inscription.

### Results

Three pairs of participants completed semi-structured interviews with the A-Painter spatial diagramming environment. For the first pair to complete the semi-structured interview, the multi-hand brush bug presented was expected to connect the two hands of one participant, but not connect between hands of different participants. During the interview, the participants experienced the multi-hand brush inscribing only between one's own hands; simultaneously, the immersed interviewer's environment drew connections between participants. Before the second and third pairs' interviews, the multi-hand brush was patched to always connect drawings across participants.

Every pair of participants traced curves through space with one hand, traced surfaces through space with two hands, inscribed line segments by using the brush to connect the location of two hands without moving. Every pair of participants made (1) polygons and polyhedrons as networks of line segments, (2) quadrilaterals as extrusions of line segments, and (3) circles, cones, cylinders, and disks as revolutions of swept points or segments around an axis.

Unexpectedly, each pair of participants also continued to trace curves or surfaces while they teleported, which extended their drawings between distant locations (Figure 2). In Figure 2, a participant teleports while drawing [left], producing a long line segment [right]. The blue arc selects the participant's teleportation destination. We see this from the participants' first person virtual perspective; in the right panel, they have turned around to see their inscription.



### Figure 2: Drawing While Teleporting

The second and third pair of participants traced surfaces extending from one to another at both short (~1 meter, Figure 3) and long (2-50 meters, Figure 4) distances. In Figure 3, two participants are sweeping out a pair of cones. In the left panel, we see the participants in the laboratory classroom aligning their perspectives in the virtual environment. We see the participants looking at each other in the left participant's view [center left] and the right participant's view [center right]. In the right panel, we see the immersed interviewers view of the participants' avatars. While their perspectives are not aligned in the physical space, they are in the virtual environment. In Figure 4, we see one participant's perspective of the virtual environment, facing another participant's avatar. The participants have traced out a cylinder.

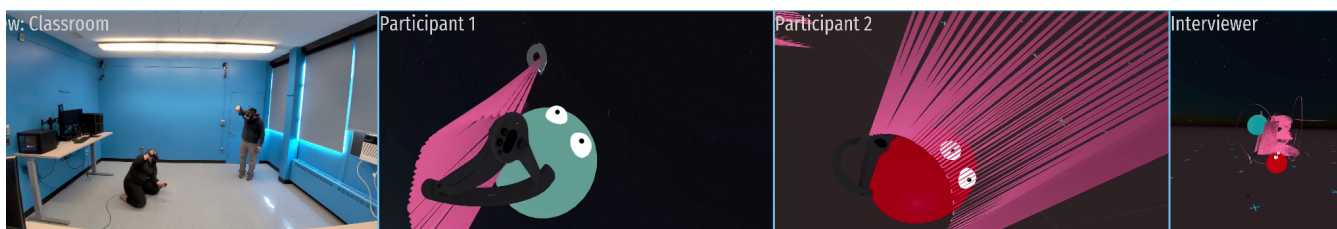


Figure 3: Two Participants Sweep Out a Cylinder at a Short Distance

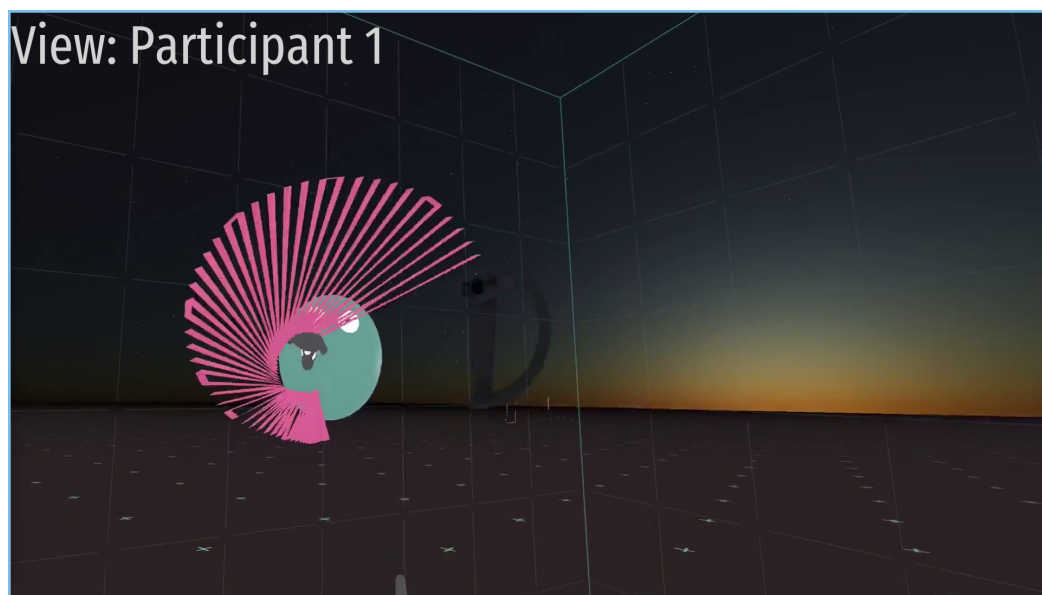
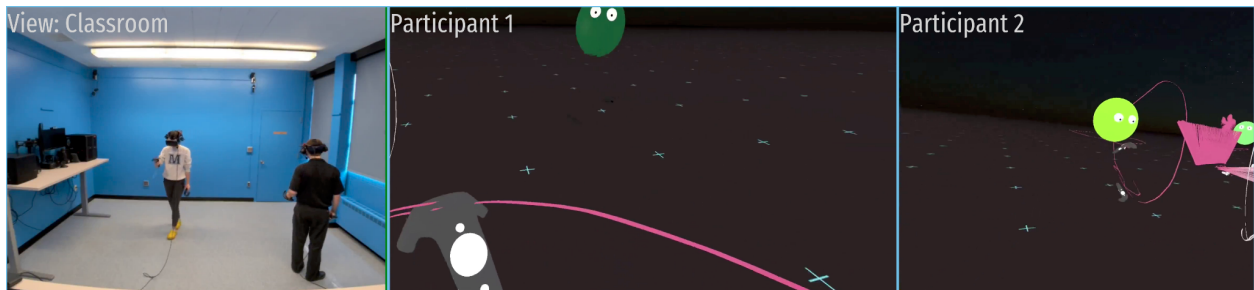


Figure 4: Two Participants Sweep Out a Cylinder at a Moderate Distance

All elements of our a priori analysis appeared in participants' use of the spatial painting environment; they also attended to a number of other features to coordinate their embodied inscription, both while inscribing alone and with a partner. Participants attended to (1) maintaining parallel relationships during their movement, (2) co-varying rates of the movement of two or more hands, (3) reflective symmetries in movement, (4) both shoulders and elbows as pivot points, (5) arms and forearms as radii, (6) arbitrary axes imagined in space, horizontally and vertically aligned perspectives as axes, self and others' bodies as axes, and (7) rotations fixed around an axis or pivot point.

Each of these embodied features of the diagram brings forth a potential of the diagram; for example, by extending an arm as a radius, fixing ones' torso as an axis, and rotating around that

axis while tracing a curve the diagram of a circle that is produced (Figure 5) has realized embodied potentials of the radius and axis or rotation that are invisible if the diagram is reduced to its rendering. In Figure 5, a participant extends their forearm as a radius [left], turns around their torso while using their controller to draw in the virtual environment [center], completing the circle [right].



**Figure 5: A Participant Traces a Circle**

Table 1 describes select examples of the relationships between these coordinated features and the realized potentials of the diagram. For example, in Figure 6, the participant anchors one hand above their head and stretches out a forearm [left], turns to sweep out the segment [center] and stands inside the completed cone [right]. We see the participant coordinating a rotation, with their torso as an axis, while holding one hand above their head (on the axis) and another forearm extended as a radius. Here, the participant embodies the rotational symmetry of the diagram, a circle formed by the revolution of a radius as the base of the cone, and the necessary intersection of the axis and the segment that forms the cone.

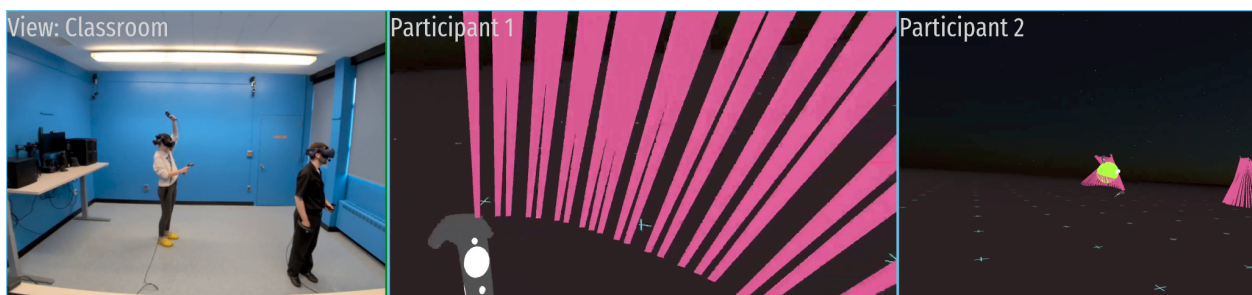
In Figure 7, two participants hold their hands to form a rectangle [left], sweep the rectangle around an imagined axis [center-left, center-right], inscribing a thick washer [right]. In this example, the participants' collaboratively coordinate their motion around an axis to realize a washer as a revolution of a rectangle.

**Table 1: Coordination of Movement during Inscription of Select Diagrams**

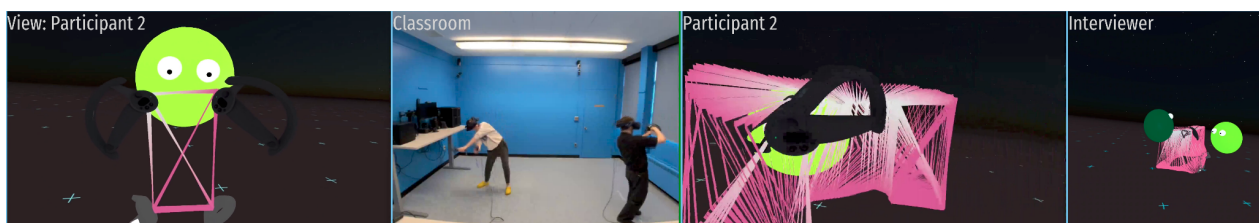
Subject	Coordinated Features	Realized Potentials
Parallelogram	co-planar motion, moving at the same rate	length of one-dimensional cross-sections is fixed
Trapezoid (Figure 1)	co-planar motion, moving at related rates	length of one-dimensional cross-sections grows
Circle (Figure 5)	rotation, fixed around self as axis, extended arm as radius	rotational symmetry, radius
Circle	rotation, fixed around shoulder as pivot, extended arm as radius	radius, center
Cylinder (Figure 4)	rotation, fixed around external vertical/horizontal axis, of vertical/horizontal segment	rotational symmetry with axis parallel to revolved segment
Cylinder (Figure 3)	rotation, fixed around axis formed by aligned perspective (vertical/horizontal), each drawing	rotational and reflective symmetries, circles as base of cylinder



Subject	Coordinated Features	Realized Potentials
	circle, opposite direction of movement, matching rate of movement	
Cone (Figure 6)	rotation, fixed around self as axis, one hand above head, one arm extended as radius	rotational symmetry, circles as base of cone, segment intersecting axis at endpoint
Washer (Figure 7)	rotation, fixed around one participant's torso as axis, four hands composing rectangle along radius & height	rotational symmetry, washer as revolved rectangle, radius as direction, height



**Figure 6: A Participant Anchors a Segment Above Their Head While Rotating**



**Figure 7: Two Participants Extrude a Rectangle around an Axis**

### Discussion & Implications

In two-dimensional, digitally rendered dynamic geometry diagrams, the multiple, simultaneous inputs of multi-touch tablets supported new potentials for both one and multiple learners' interactions with the diagram, that can center embodied and spatial relationships in the diagram (Jackiw, 2013; Chorney & Sinclair, 2018). These multi-touch inputs can be found in multi-hands transformations of dynamic diagrams in immersive spatial diagram environments (e.g., Bock & Dimmel, 2021a; Dimmel et al., 2020; Rodríguez et al., 2021); we find learners capitalizing on the potential of these affordances with A-Painters multi-hand brush.

#### Attention to Covarying Rates of Movement

In diagrams of parallelograms, trapezoids, cylinders, and cones, participants attended to the covering motion or change in position of their hands. Pairs of participants coordinated moving their hands “at the same rate” to construct parallelograms and cylinders, at related rates to construct trapezoids, and “the same in opposite directions” to construct cones. This embodiment of covariation drew one pair of participants to focus their investigation on how the paths of the endpoints of a segment that is revolved around an axis can produce a variety of figures, including a pair of cones, a cylinder, and a variety of self-intersecting surfaces.

This presents exciting opportunities to investigate how multiple inputs can reshape how learners draw mathematical diagrams. As an inscription tool, the relationships between two participants' embodiments of circles are realized in a visual trace. Looking at the potential of multi-hand tools in spatial diagram environments, we take inspiration from York et al.'s (2022) Graph Tracer, where students embody varying quantities as one moves a stylus and the other moves the canvas of inscription, and the relationship between their embodiments is found in the graph that is produced. Further work is needed to understand how the affordance of multi-hand inputs can be used in the design of immersive spatial diagram environments, including considering redesigning tools implemented as analogs of mouse pointer input systems.

### **Scale and Ego-centric Diagrams**

Finally, participants' use of embodied, spatial inscription tools often produced diagrams that surrounded the learner (*e.g.*, Figure 6) or extended between learners (*e.g.*, Figure 3), instead of small, external diagrams that are common in school geometry. Immersive spatial diagrams offer the affordance of scale, which can support learners' taking new perspectives in investigations of geometry (Benally et al., 2022). In future iterations of A-Painter, we plan to investigate how learners use the scale of immersive spatial diagrams when digital painting tools are available to markup dynamically rendered representations of mathematical figures. For example, spatial painting tools could be combined with other dynamic spatial renderings to be used to markup spatial realizations of surfaces of revolution in calculus contexts.

### **Acknowledgments**

This work is supported in part by a Janet Waldron Doctoral Research Fellowship from the Graduate School at the University of Maine (first author) and by grants from the Spencer Foundation (second author) and the National Science Foundation (Award #2145517, second author). The views expressed are the authors' and do not necessarily reflect the views of the Spencer Foundation, the National Science Foundation, or the University of Maine.

### **References**

- Atlindis, N., Raja, W. (2021). Enacted Task Characteristics: Setting An Infrastructure For Students' Quantitative Reasoning. Proceedings of the 43rd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Philadelphia, PA.
- Benally, J., Palatnik, A., Ryokai, K., & Abrahamson, D. (2022). Charting our embodied territories: Learning geometry as negotiating perspectival complementarities. For the Learning of Mathematics.
- Bock, C. G. (2022). Immersive-math/naf-a-painter (1e02bad0b16fb5890cd22f2a60f22f142d6183b5) [JavaScript]. The University of Maine. <https://gitlab.com/immersive-math/naf-a-painter>
- Bock, C. G., & Dimmel, J. K. (2021a). Dynamic Spatial Diagrams and Solid Geometric Figures. *Psychology of Mathematics Education North America*.
- Bock, C. G., & Dimmel, J. K. (2021b). Digital Representations without Physical Analogues: A Study of Body-Based Interactions with an Apparently Unbounded Spatial Diagram. *Digital Experiences in Mathematics Education*, 7(2), 193–221. <https://doi.org/10.1007/s40751-020-00082-4>
- Cangas, D., Morga, G., & Rodríguez, J. L. (2019). Geometry teaching experience in virtual reality with NeoTrie VR. *Psychology, Society, & Education*, 11(3), 355–366.
- Chorney, S., & Sinclair, N. (2018). Fingers-on geometry: The emergence of symmetry in a primary school classroom with multi-touch dynamic geometry. *Using Mobile Technologies in the Teaching and Learning of Mathematics*, 213–230.
- Dimmel, J., & Bock, C. (2019). Dynamic Mathematical Figures with Immersive Spatial Displays: The Case of Handwaver. In G. Aldon & J. Trgalova (Eds.), *Technology in Mathematics Teaching: Selected Papers of the 13th ICTMT Conference* (pp. 99–122). Springer International Publishing. [https://doi.org/10.1007/978-3-030-19741-4\\_5](https://doi.org/10.1007/978-3-030-19741-4_5)
- Dimmel, J., Pandiscio, E., & Bock, C. (2020). The Geometry of Movement: Encounters with Spatial Inscriptions for Making and Exploring Mathematical Figures. *Digital Experiences in Mathematics Education*. <https://doi.org/10.1007/s40751-020-00078-0>

- Flatland XR Channel (Director). (2021, December 13). AR Collaboration. <https://www.youtube.com/watch?v=HFdOBiHzZes>
- Google. (2016). Tilt Brush. Alphabet, Inc. <https://www.tiltbrush.com/>
- Icosa Foundation. (2021). Open Brush. <https://openbrush.app/>
- Jackiw, N. (2013). Touch and multi-touch in dynamic geometry: Sketchpad explorer and “digital” mathematics. *Proceedings of the 11th International Conference on Technology in Mathematics Teaching*, 149–155.
- Kaufmann, H., & Schmalstieg, D. (2006). Designing Immersive Virtual Reality for Geometry Education. *IEEE Virtual Reality Conference (VR 2006)*, 51–58. <https://doi.org/10.1109/VR.2006.48>
- Knispel, J., & Bullock, F. (2017). Collaborative VR painting in web browsers. *SIGGRAPH Asia 2017 VR Showcase*, 1–2. <https://doi.org/10.1145/3139468.3148451>
- Lee, H. (2018). Haydenjameslee/networked-a-painter (a58aec460128d5902aaedac3ef68799c2fdf397b) [JavaScript]. <https://github.com/haydenjameslee/networked-a-painter>
- Mozilla. (2017). Aframevr/a-painter [JavaScript]. A-Frame. <https://github.com/aframevr/a-painter> (Original work published 2016)
- Ng, O.-L., & Ferrara, F. (2020). Towards a Materialist Vision of ‘Learning as Making’: The Case of 3D Printing Pens in School Mathematics. *International Journal of Science and Mathematics Education*, 18(5), 925–944. <https://doi.org/10.1007/s10763-019-10000-9>
- Ng, O.-L., & Sinclair, N. (2018). Drawing in Space: Doing Mathematics with 3D Pens. In *Uses of Technology in Primary and Secondary Mathematics Education* (pp. 301–313). Springer.
- Price, S., Yiannoutsou, N., & Vezzoli, Y. (2020). Making the Body Tangible: Elementary Geometry Learning through VR. *Digital Experiences in Mathematics Education*. <https://doi.org/10.1007/s40751-020-00071-7>
- Rendever. (2022). MultiBrush on Oculus Quest. <https://www.oculus.com/experiences/quest/3438333449611263/>
- Sinclair, N., Freitas, E., & Ferrara, F. (2013). Virtual encounters: The murky and furtive world of mathematical inventiveness. *ZDM*, 45(2), 239–252. <https://doi.org/10.1007/s11858-012-0465-3>
- Rodríguez, J. L., Romero, I., & Codina, A. (2021). The Influence of NeoTrie VR’s Immersive Virtual Reality on the Teaching and Learning of Geometry. *Mathematics*, 9(19), Article 19. <https://doi.org/10.3390/math9192411>
- Walkington, C., Nathan, M. J., Hunnicutt, J., Washington, J., & Holcomb-Webb, K. (2022). Learning geometry through collaborative, embodied explorations with augmented reality holograms. *Proceedings of the 16th International Conference of the Learning Sciences–ICLS, 1992–1993*.
- Walkington, C., Wang, M., Nathan, M. J., Kageyama, K., Edwards, L. D., & Ferrari, G. (2020). Collaborative gestures among secondary students conjointly proving geometric conjectures. Paper Accepted for Presentation at the 14th International Congress on Mathematical Education, Shanghai, China. (Modified Format Due to COVID-19), Shanghai.
- York, T., Greenstein, S., & Akoum, D. (2022). Embodying covariation through collaborative instrumentation. In A. F. Lischka, E. B. Dyer, R. S. Jones, J. N. Lovett, J. Strayer, & S. Drown (Eds.), *Proceedings of the forty-fourth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Middle Tennessee State University.

## DIGITAL WALL: A REFLECTIVE TOOL FOR STUDENTS' SELF-REGULATING ONLINE LEARNING WITHIN A PROBLEM-SOLVING APPROACH

Daniel Ortiz  
CINVESTAV-IPN  
daniel.ortiz@cinvestav.mx

Tania Azucena Chicalote Jiménez  
CINVESTAV-IPN  
tania.chicalote@cinvestav.mx;

*A digital wall is a tool for students to structure and register their online work on mathematical problem-solving activities that involve the coordinated use of digital technologies. How could students use such digital wall to understand mathematical concepts and to develop problem-solving competencies? The aim of this study is to analyze and document the extent to which the students' use of the wall became a powerful tool to share, discuss, and refine their problem-solving approaches. To this end, the development of the sessions was structured around principles that foster an inquiry method that privileges the participants' formulation of questions; the systematic and coordinated use of digital technologies, and a support system for students to face technical and content questions. Results indicate that the use of a digital wall became a powerful tool for the participants to report and share their ideas and problem-solving approaches to refine and extend their solution methods.*

Keywords: Problem Solving; Online and Distance Education; Technology; Preservice Teacher Education

### **An inquiry mathematical problem-solving approach**

During the COVID-19 pandemic lockdown, it became clear that students' learning activities might occur not only within a confined classroom, but also be organized and implemented in remote or online spaces through the activation of communication apps and online platforms.

Further, digital apps such as Dynamic Geometry Systems (DGS), have the potential to transform the way students learn about mathematical concepts (Trocki & Hollebrands, 2018) by allowing students to interact in real-time with executable affordances to explore and process mathematical objects attributes actions [Moreno-Armella & Hegedus, 2009]. Thus, we argue that a problem-solving approach where students are asked to use technology's affordances to make conjectures, validate results and explore multiple solutions exploit the potential of learning in online spaces. Santos-Trigo et. al (2022) state that when students work in online scenarios, they can use technology to extend their learning activities that appear in regular classrooms. For instance, they can record their teaching presentations and review them after class sessions, consult online platforms to review concepts, or to interact with peers to discuss or address difficulties that arise during the sessions. Likewise, they can consult online developments to check examples of solved problems. That is, technologies shape the way students engage in mathematical task throughout all problem-solving phases:

In understanding and making sense of problem statements, students could direct their attention to the reconstruction of figures embedded in problem statements and this process might lead them to problem posing activities [...] Similarly, representing and exploring dynamically elements of the problem becomes important to identify mathematical relations that lead students to solve the problem [...] Furthermore, dynamic models become a source for students to extend and generalize initial conditions and solutions to the problem (Santos- Trigo et al., 2022).

Students learn and develop mathematical ideas when they delve into a diverse type of situations that imply posing and pursuing questions and conceiving their learning process as series of

dilemmas that need to be represented, explored and solved in terms of mathematical resources (Santos-Trigo & Reyes-Martínez, 2019). In this perspective, mathematical problems or tasks are a departure point for learners to engage in sense-making activities to understand concepts and to think of different ways to work on and solve those problems (Santos-Trigo & Camacho, 2013). This is, an inquiry problem-solving approach is essential for learners to understand mathematical concepts and to develop problem-solving proficiency: How do students develop and inquiry method to work on mathematical tasks? What questions should they pose and pursue during their understanding of mathematical concepts?

One of the greatest challenges we faced as a result of social long confinement was the limited research results to support the teachers' decisions and actions to frame online teaching and learning environments. In this context, teachers experienced difficulties to select and implement the use of online resources, ways to provide cognitive and social support for students in online spaces, and to assess and give feedback to students (Johns & Mills, 2021; Martin et al., 2020; Trenholm et al., 2015); Within online learning spaces, teachers have less access to subtle cues about student's level of engaging, such as insight or "aha!" moments, limiting opportunities to offer guidance and correct feedback (Mullen, et al., 2021). That is, when thinking about fostering the interaction between students and teachers in online spaces, traditional means of communication are not effective to assess students' concept understanding.

Engelbrecht et al. (2020) points out that online spaces can become favorable spaces for learning when participants share information, tools, and resources constantly. In this sense, a digital wall can be used as a tool that exploits technology's affordances to help students register and reflect upon their own learning experiences: what websites do they check when approaching a mathematical task? What are the definitions that students consider important? How do they control their own work? In other words, a digital wall is a conceptual instrument that provides a way to structure and follow up students problem-solving performances. That is, the research question that guides the development of this study was: To what extent did the students use of a digital wall become a reflective tool for them to register their work and to understand concepts and to develop their problem-solving competencies?

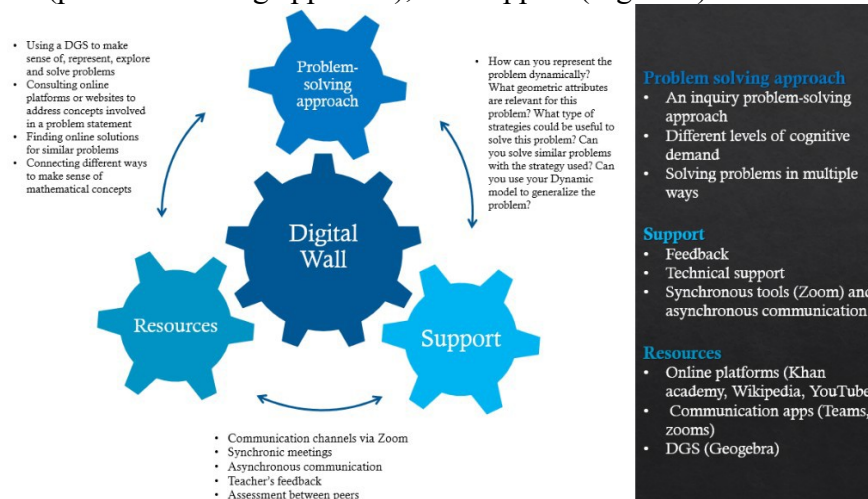
### **Conceptual framework**

A digital wall as a tool to organize and structure problem solving activities is based on three intertwined elements: an inquiry problem-solving approach, the use of digital technologies and the role of tasks to engage students in mathematical discussions (Santos-Trigo et al., 2022).

Since DGS offer a set of affordances for learners to think of multi-representations of mathematical knowledge (Leung, 2017), problem-solving activities within dynamic environments can be oriented towards exploration, rather than focusing on finding one single solution to a problem. Thus, students can use a DGS to analyze and investigate connections between similar problems or to think of how different mathematical concepts can be used to represent, explore, and solve a problem in a variety of ways. Thus, tasks are key for mathematical discussions, and they become a vehicle for students to reflect upon their own knowledge when they approach them. That is, the type of tasks and the way they are implemented in a learning setting are crucial for students to understand concepts and develop their mathematical thinking. In this context, questions that lead students to conjecture, pose problems and to look for generalizations, are essential for students to delve into central mathematical practices (Schoenfeld, 2020).

How should students organize their work when using a digital wall to register, communicate and reflect on their own learning? Santos-Trigo and Camacho (2013) proposed four stages that characterize the ways of reasoning that emerge when students use digital technologies to solve

mathematical tasks: (1) Making sense of the problem, that is, using technology to grasp a general idea of what the solution of a problem might involve; (2) using digital tools, such as Geogebra, to explore and formulate conjectures or possible objects' relationships; (3) finding arguments that justify, refute, or validate conjectures and hypotheses; and, (4) looking back at the problem to identify mathematical connections, extend the problem or pose new related questions. The conceptual framework that supports the use of a digital wall and allows it to connect with an inquiry problem-solving approach to learn mathematics is an adjusted version of the RASE conceptual framework (Churchill et. al., 2016), that distinguishes three intertwined dimensions: resources, activities (problem-solving approach), and support (Figure 1).



**Figure 1. A framework to structure a digital wall within a problem-solving approach.**

**A problem-solving approach to support mathematical learning.** By conceiving the discipline as a set of sense-making activities, students are encouraged to pose and pursue questions to delve into concepts and solve problems in multiple ways. A mathematical task is only a departure point into developing mathematical thinking. This will characterize the way in which activities are to be implemented and assessed.

**Learning resources.** Schoenfeld described a model to explain students' performance in goal-oriented activities, such as problem-solving, in terms of three categories: resources, goals and decision-making. In online spaces, students' use of digital technologies permeates the way in which sub-goals and self-monitoring occur when they deal with mathematical tasks. That is, resources are not only to be considered in the way Schoenfeld (2011) originally described, but also how students use technology (i.e., DGS, online platforms, YouTube videos) to represent, explore, extend, and make sense of mathematical problems, as well as to share and convey their results.

**Online support.** This support includes teachers' feedback, spaces for sharing ideas and solutions like forums, messenger apps, and online meetings. It includes ways to monitor and assess students' understanding of concepts and problem-solving performances.

### Methodological Elements

Since our main objective is to describe how students exhibit problem-solving abilities by using a digital wall, this study has a qualitative nature. Data were obtained from an 8-weeks problem-solving workshop that was implemented as a part of a mathematics education master program. In this report, we focused on the work of two participants (referred as Hector and Carol). In order to increase the trustworthiness of the data analysis, we included strategies to delve into peer

debriefing, triangulation of various types of information and reflexivity of the data (Freitas, 2017). The workshop was conducted fully online, via Microsoft Teams platform and Zoom, for synchronous meetings on a weekly basis.

Participants worked in two episodes: First, they were given a mathematical task in advance, to register their individual work in their digital wall; then, during weekly Zoom meetings (3 hrs. duration), students shared their ideas, solutions, and conjectures with the rest of the group, where they would receive feedback from their peers and workshop coordinators. The guidelines given to the participants to work on the tasks were based on Santos-Trigo and Reyes-Martínez (2018) that involve: (a) Always construct a dynamic model of the problem and explore it to identify conjectures or relationships regarding the objects observed in the model; (b) look for different ways to explore, (c) interpret and solve the mathematical problems; (d) show arguments that justify and support conjectures and hypotheses; (e) change the problem statement's initial conditions of the problem statement to explore or discuss whether the strategies used to solve the original problem can be used to solve similar ones; and, (f) identify, analyze, and contrast concepts, strategies or methods used.

OneNote app was chosen for students to report their work in the digital wall, since it allows the user to create notes with ease, including images, videos and even to embed GeoGebra constructions (at the time where the study was conducted) by including the online link. One *a priori* example of a digital wall in OneNote can be found [here](#). It was suggested to work around three basic sections for each problem: understanding the problem, exploring the dynamic model, and posing new questions regarding the original problem statement (extending the problem).

### Results & Analysis

Individual digital walls were the main source of information about the participants' performance throughout the workshop. Given a problem, three main elements from the digital wall were used to characterize participants' performance (Santos-Trigo, et al., 2022):

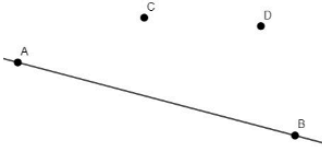
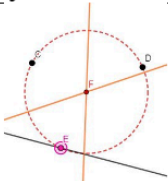
- Comprehension phase. What types of questions do students pose to understand and delve into problem statement? Do they consult online resources to gain a better understanding of the problem?
- Exploration and solution phase. How are the DGS affordances being used to represent the problem? How do the students incorporate these affordances into problem-solving strategies? What types of conjectures do the students make? How do they support these conjectures? What resources are students activating to work on the task?
- Extension phase. What new questions do students pose to extend the initial problem? Are they investigating to what extent the strategies used can be useful to work and solve new problems? Are they identifying what type of knowledge, concepts, strategies, and resources they used to achieve the problem solution? Can they solve a more general related problem?

Participants were asked to solve Apollonius tangency problems, that is, given three objects (a point, a line, or a circle) draw a circle that is tangent to those three objects (or that passes through, for the case of a point). Whilst these problems have been solved via algebraic methods and ruler-and-compass constructions extensively, by using a DGS students can find novel solutions involving the loci of dragging points in forms of conic sections (Santos-Trigo, et. al, 2021). Solving the Apollonius tangency problems using a DGS involves the activation of the tool affordances to dynamically represent and explore the involved objects behaviors. It also demands

that students rely on problem-solving strategies that include simplifying or reducing initial problem conditions to approach particular cases and then to solve the problem. Ten cases or configurations associated with the Apollonio’s problem can be dynamically modeled to solve it (Santos-Trigo, et al., 2019). We will now include a discussion about participants’ work on three problems.

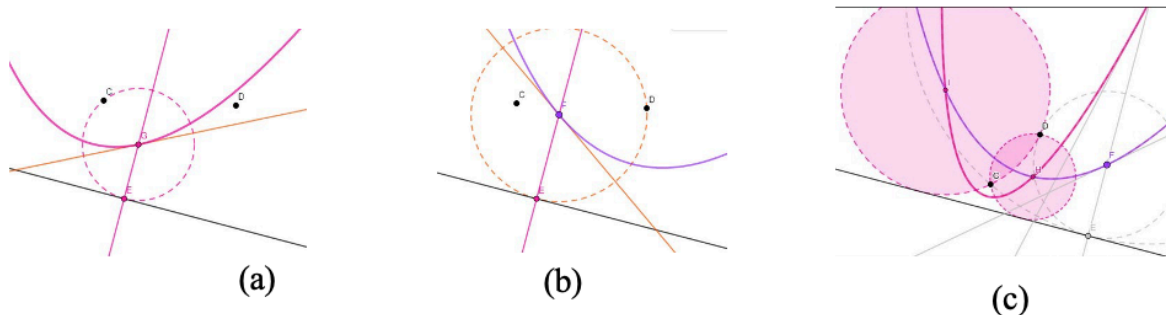
**Initial case: given two points and a line, how to draw a tangent circle to line AB that passes through points C and D (2P1L problem)?** Prior to work on this problem, students explored how to draw a tangent line to a circle from a point on the circle, and the first 2 cases of Apollonius problems: Drawing a circle that passes through three points (PPP problem) and drawing a tangent circle to three lines (LLL problem). For the 2P1L problem, both Hector and Carol relied on the same strategy, summarized in Table 1. First, participants drew a circle passing through the given points C, D, and through a point E, online AB. This clearly is not a *robust* solution, because this circle is not necessarily tangent to AB. By moving point E, when will the circle become tangent to line AB? This exploration was mainly visual, since participants only moved point E until the circle *seemed* tangent (see step 10 of [DM1](#)). Is there another way to support empirically the tangency of the circle? By recalling the construction of a tangent line to a point on a circle, they noticed that if they traced a perpendicular from point E to the given line, the solution should be obtained when they point F lies on the perpendicular line (Step 11, [DM1](#)). This, however, was still a mainly empirical solution.

**Table 1: Exploration from Hector & Carol to the 2P1L tangency problem.**

Initial objects	Dynamic model	Strategy
 <p>Line AB and points C &amp; D</p>	 <p><a href="#">Dynamic model 1 (DM1)</a></p>	<p>Returning to previous problems (CCC and tangent to a circle): Drawing point E on the line AB, then a circle that passes through C, D, and E. What conditions for point E make the circle tangent to the line AB?</p>

The perpendicular line was a key element considered when giving feedback to Hector in the form of a question: By moving point E, we can see the intersection G of the perpendicular line and the perpendicular bisector of points E and C moving in a particular way as E moves along the line AB. How can we describe this movement (Steps 12-13, [DM1](#))? By using the *locus tool*, a parabola-like curve appears (Figure 2a). Is this curve indeed a parabola? Hector quickly noted that G lies in the perpendicular bisector of CE, thus, the distance from GC is equal to the distance from G to line AB, this is what defines a parabola as a locus with C as the focus and AB as its directrix. What does this locus mean in terms of the problem? Each point on the parabola is the center of a circle tangent to line AB that passes through C. If the same process is done with the perpendicular bisector of points E and D, another parabola can be obtained (Figure 2b): this parabola holds a similar meaning, since it contains the centers of circles tangent to line AB passing through D. How can these parabolas be used to obtain the solution of the problem? The intersections of both parabolas are the centers of circles that are tangent to line AB and that pass through both points D and C (Figure 2c). See [Dynamic Model 2](#) for a full solution.



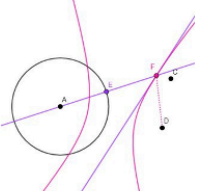
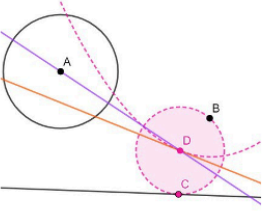


**Figure 2: Using parabolas to solve [Problem 2P1L](#).**

In this first case, the solution demanded the activation of resources like perpendicular lines and perpendicular bisectors. The perpendicular line from E contains all centers of circles that are tangent to line AB at E, while the perpendicular bisector of EC contains all the centers of circles that pass through C and E. The intersection of the perpendicular bisector of CE and the perpendicular line to line AB from point E then gave a partial solution, which was then generalized in the form of a locus (a parabola) that allowed to solve the problem. It was also noted that by simplifying the initial conditions, it was possible to orderly explore this problem to obtain a solution that could be extended to the rest of Apollonius configurations. We will now describe how this strategy was applied by both Hector and Carol in different problems.

**How did students explore more configurations for the tangency problem?** By taking the 2P1L problem as a starting point, students went on to solve other configurations of the tangency problems (see Table 2).

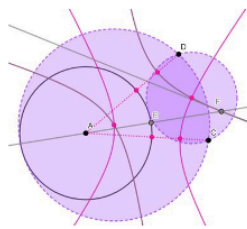
**Table 2: Examples of participant’s models for the tangency problem**

Student	Problem	Dynamic model	Strategy
Hector	Draw a <a href="#">circle</a> tangent to a given circle that passes through two given points (2P1C)		Returning to a previous problem (2P1L): Draw point E on the given circle. In this case, a “perpendicular line” is translated as the AE line, and then the perpendicular bisector of E and point D gives a point F that is the center of a circle tangent to circle A and passes through D. When will this circle also pass through C? The rest of the solution can be found on the link to Hector’s exploration.
Carol	Draw a circle tangent to a given line, and a given circle that passes through a given point (1P1L1C)		Simplifying the initial conditions: Take a point C on the given line, draw the parabola with focus B and directrix the given line. Then, trace the perpendicular bisector of BC. This is tangent to the parabola at point D. A circle with center D that passes through B will also be tangent to the given line. By moving point C, when will this circle be tangent to the given circle? The rest of the solution can be found on the link to Carol’s exploration.

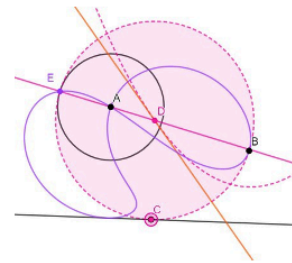
Hector relied upon the solution of the 2P1L by simplifying the initial conditions. First, he conceived that the line in the 2P1L problem was a “circle of infinite radius”, and tried to translate the same ideas into the 2P1C problem. That is, he took a point E on the circle given and traced a line passing through A (the center of the given circle) and E, this would be the “perpendicular line”; then, he traced the perpendicular bisector of ED. The intersection of these two lines is a point

F, the center of a circle tangent to the given circle and passing through point D. As point E moves within the given circle, what is the locus of point F? By using the locus tool, we can see that the resulting curve seems like a hyperbola (step 10, [DM3](#)).

Hector conjectured that it was indeed a hyperbola and its foci must be points A and D. He justified this conjecture algebraically by noticing that  $||AF|-|FD||=|AE|+|EF|-|FD|$ , since F lies on the perpendicular bisector of ED,  $|EF|=|FD|$ , thus  $||AF|-|FD||=|AE|=r$ . Since the difference of distances of any point on this curve to the fixed points A and D is constant ( $r$  is the radius of the given circle), it follows that the locus is a hyperbola with foci D and A. Any point on this hyperbola will be the center of a circle that passes through D and is tangent to the given circle. Similarly, repeating the same process for point C gives another hyperbola that contains all centers of circles that pass-through point C and are tangent to the given circle. The intersections of both hyperbolas give the solution (Figure 3). Even further, Hector explored a way to trace the hyperbola using tools from Geogebra by obtaining one of its vertexes, which can be explored in the rest of the steps in [DM3](#). Finally, he explained how all tangency problems can be solved by using a similar strategy: simplifying the initial statement's conditions. That is, by "ignoring" one of the three objects given and trying to find a family of circles that satisfy the conditions for the two remaining objects. Then, a locus that describes a family of solutions can be used for the third object and the solution will be obtained with the intersection of both loci. Although this extension was shared with the rest of the participants, Carol and Val struggled to implement this strategy.



**Figure 3. Using hyperbolas to solve Problem 2C1L.**



**Figure 4. Carol's attempt to solve Problem 1P1L1C.**

Now, Carol's approach to the 1P1L1L problem is described in Table 2. First, she "ignored" the given circle and focused on finding a family of circles tangent to the given line that pass-through the given point B. By using the results in the 2P1L problem, she drew a parabola with locus B and the given line as the directrix. She then drew a point C on the given line, and used the perpendicular bisector of BC to find the center D of a circle that passes through B and is tangent to the given line at C. By moving point C, Carol visually identified that this newly drawn circle will be tangent to the given circle at some position for point C (step 13 [DM4](#)). She then traced the line connecting the center A of the given circle with point D. Then she found the intersection E of line AD with the circle with center D. How does point E moves as C is moved along the given line (step 15 [DM4](#))? Visually, when point E lies on the given circle, a solution is found: the circle with center D is tangent to both given objects and passes through point B. By using the locus tool, Carol expected to find a curve that could be described in geometric terms; alas, when the obtained curved seems too complex to describe in structural terms (Figure 4), however, it allows to see that it has 4 intersections with circle A, and thus, there must be 4 solutions. The point C can be moved manually to "see" these four solutions, but since Geogebra does not allow to find the intersection of

geometric objects with curves obtained via the locus tool, a *robust* solution could not be obtained by Carol.

### Concluding Remarks

The use of digital technologies, including online developments, communication apps and DGS, become important to share and extend mathematical discussions as well to open up novel routes for students to represent and explore mathematical problems (Santos-Trigo, et al., 2021). For the Apollonius problems, it became clear that they had to be solved by considering only two of the three objects and characterize the family of centers of circles that satisfy these two conditions. Table 3 summarizes the different loci obtained when considering two of the three objects.

**Table 3: Loci obtained for pairs of objects involved in the tangency problems.**

Objects	2 points	2 lines	1 point & 1 line	1 circle & 1 point	1 circle & 1 line	2 circles
Locus	Perpendicular bisector	Angle bisector	Parabola	Hyperbola	<a href="#">2 Parabolas</a>	<a href="#">2 hyperbolas</a>

Once the 2P1L and the 2P1C were solved, Hector found ways to solve the rest of the Apollonius problems by characterizing the locus of centers of circles tangent (or pass through) to only 2 objects: a point and a circle (a hyperbola), a line and a circle (two parabolas), and eventually, two circles (two hyperbolas). By studying these simplified versions of the tangency problems, Hector developed a *heuristic* that allowed him to successfully explore and solve the rest of Apollonius' tangency problems. That is, looking into the connections between the mathematical objects involved in a problem is an important step for students to use resources as key tools to solve similar problems. Although Carol was not able to initially solve the tangency problems on her own, she constantly posed questions about the strategies she employed. Rather than focusing on the problem's difficulty or faulty resources, she asked for help to implement problem-solving strategies. Carol explored the problem in a way that allowed her to connect the strategy of simplifying the initial conditions with other problems and attempted to use the locus tool to find *robust* solutions, even though initially she struggled to find a proper way to explore problems in order to find locus that could be described in geometric relations (i.e., the parabola for the 2P1L problem and the hyperbola for the 2P1C problem). Students' abilities to coordinate the use of digital resources into their mathematical understanding on virtual spaces can hinder or enhance the appropriation of problem-solving skills. Synchronous sessions were complementary for the digital wall's work since students were able to share their advances on the tasks and help each other to find new ways to solve and pose new problems; asynchronous work was characterized using online platforms that allowed participants to consult concepts and definitions about geometric loci or construction procedures (like tangent lines to a circle from an external point) that were used as supports for other problems tackled in the workshop.

Digital technologies offer affordances in online spaces that shape what problem-solving skills mean and how mathematical resources can be used in finding alternative ways to explore and discuss classical problems. We argue that the use of digital technologies in problem-solving activities has the potential to transform how mathematics can be conveyed to learners, and a digital wall can serve as a key tool that guides the structure and implementation of online activities.

## References

- Churchill, D., Fox, B., & King, M. (2016). Framework for Designing Mobile Learning Environments. In D. Churchill, J. Lu, T. K. F. Chiu, & B. Fox (Eds.), *Mobile Learning Design: Theories and Applications*, 3–25. Springer Singapore. [https://doi.org/10.1007/978-981-10-0027-0\\_1](https://doi.org/10.1007/978-981-10-0027-0_1)
- Engelbrecht, J., Llinares, S., & Borba, M. C. (2020). Transformation of the mathematics classroom with the internet. *ZDM - Mathematics Education*. <https://doi.org/10.1007/s11858-020-01176-4>
- Freitas, E., Lerman, S., & Parks, A. (2017). Qualitative Methods. In J. Cai (Ed.), *Compendium for Research in Mathematics Education*, 159–183. NCTM.
- Johns, C., & Mills, M. (2021). Online Mathematics Tutoring During the COVID-19 Pandemic: Recommendations for Best Practices. *PRIMUS*, 31(1), 99–117. <https://doi.org/10.1080/10511970.2020.1818336>
- Leung, A. (2017). Exploring Techno-Pedagogic Task Design in the Mathematics Classroom. In A. Leung & A. Baccaglioni-Frank (Eds.), *Digital Technologies in Designing Mathematics Education Tasks: Potential and Pitfalls*, 3–16. Springer International Publishing. [https://doi.org/10.1007/978-3-319-43423-0\\_1](https://doi.org/10.1007/978-3-319-43423-0_1)
- Martin, F., Sun, T., & Westine, C. D. (2020). A systematic review of research on online teaching and learning from 2009 to 2018. *Computers and Education*, 159, 104009. <https://doi.org/10.1016/j.compedu.2020.104009>
- Moreno-Armella, L., & Hegedus, S. J. (2009). Co-action with digital technologies. *ZDM - International Journal on Mathematics Education*, 41(4), 505–519. <https://doi.org/10.1007/s11858-009-0200-x>
- Mullen, C.; Pettigrew, J.; Cronin, A.; Rylands, L. & Shearman, D. (2021). The rapid move to online mathematics support: changes in pedagogy and social interaction. *International Journal of Mathematical Education in Science and Technology*, 53(1), 64-91. <https://doi.org/10.1080/0020739X.2021.1962555>
- Santos-Trigo, M., Aguilar-Magallón, D., Reyes-Martínez, I. (2019). A Mathematical Problem-Solving Approach Based on Digital Technology Affordances to Represent, Explore, and Solve Problems via Geometric Reasoning. In: Felmer, P., Liljedahl, P., Koichu, B. (eds) *Problem Solving in Mathematics Instruction and Teacher Professional Development*. Research in Mathematics Education. Springer, Cham. [https://doi.org/10.1007/978-3-030-29215-7\\_8](https://doi.org/10.1007/978-3-030-29215-7_8)
- Santos-Trigo, M., Barrera-Mora, F., & Camacho-Machín, M. (2021). Teachers' Use of Technology Affordances to Contextualize and Dynamically Enrich and Extend Mathematical Problem-Solving Strategies. *Mathematics*, 9(8). <https://doi.org/10.3390/math9080793>
- Santos-Trigo, M., & Camacho, M. (2013). Framing the use of computational technology in problem solving approaches. *Mathematics Enthusiast*, 10, 279–302.
- Santos-Trigo, M., & Reyes-Martínez, I. (2019). High school prospective teachers' problem-solving reasoning that involves the coordinated use of digital technologies. *International Journal of Mathematical Education in Science and Technology*, 50(2), 182–201. <https://doi.org/10.1080/0020739X.2018.1489075>
- Santos-Trigo, M., Reyes-Martínez, I., & Gómez-Arciga, A. (2022). A conceptual framework to structure remote learning scenarios: a digital wall as a reflective tool for students to develop mathematics problem-solving competencies. *International Journal for Learning Technology*, 17(1), 27-52. <https://doi.org/10.1504/IJLT.2022.123686>
- Schoenfeld, A. (2011). *How We Think: A Theory of Goal-Oriented Decision Making and its Educational Applications*. <https://doi.org/10.4324/9780203843000>
- Schoenfeld, A.H. (2020) 'Mathematical practices, in theory and practice', *ZDM* [online] <https://doi.org/10.1007/s11858-020-01162-w>.
- Trenholm, S., Alcock, L., & Robinson, C. (2015). An investigation of assessment and feedback practices in fully asynchronous online undergraduate mathematics courses. *International Journal of Mathematical Education in Science and Technology*, 46(8), 1197–1221. <https://doi.org/10.1080/0020739X.2015.1036946>
- Trocki, A., & Hollebrands, K. (2018). The Development of a Framework for Assessing Dynamic Geometry Task Quality. *Digital Experiences in Mathematics Education*, 4(2–3), 110–138. <https://doi.org/10.1007/s40751-018-0041-8>

## EPISTEMIC REKEYING: EPISTEMIC TENSIONS ACROSS DISCIPLINES AS OPPORTUNITIES FOR ARTISTIC RESPONSE

Corey Brady

Southern Methodist University  
corey.brady@smu.edu

Lauren Vogelstein

New York University  
lev226@nyu.edu

*There are strong motivations to implement integrated STEAM activities that engage with key ideas in mathematics. In integrating mathematics with other STEM disciplines, however, epistemic tensions can emerge. Rather than attempting to suppress, avoid, or adjudicate these tensions, we propose a strategy of “epistemic rekeying,” in which epistemic tensions are offered as provocations for students to create playful and artistic responses. This approach takes epistemic tensions seriously and makes them accessible to young learners. We give the rationale for this approach and describe settings where students’ creativity suggested its potential to us.*

Keywords: Integrated STEM / STEAM, Mathematical Representations, Computational Thinking, Classroom Discourse.

STEM and STEAM integration (Takeuchi et al, 2020) signal possibilities for integrative learning experiences involving multiple disciplines. A number of different arguments have been made for STEAM integration, including the increasingly trans-disciplinary nature of scientific research (National Science Foundation, 2019; Nersessian, 2017) or the observation that problems in the world of work are seldom confined to a single school subject area (Lesh, Hamilton, & Kaput, 2007). One can also argue that integrative STEAM activities may enhance students’ interest and increase the relevance of STEM subjects (Lehrer & Schauble, 2020).

### Epistemic Tensions

In spite of their promise, integrative STEAM activities that involve representational tools and practices can encounter emergent tensions between the distinct epistemic frames that characterize different disciplinary ways of knowing. Such tensions can be seen as problematic, creating hidden challenges for both teachers and students that may distract from instructional objectives (Lehrer & Schauble, 2020). Moreover, epistemic tensions are *essential* to and *inherent* in interdisciplinary work: the literature on professional boundary crossing (Akkerman & Bakker, 2011; Osbeck & Nersessian, 2017) has studied a variety of different personal and institutional responses to such tensions, revealing the rich array of ways of life that can arise at disciplinary intersections. Thus, mathematics educators cannot simply “design around” these tensions: doing so may even risk falsifying the participants’ experience of interdisciplinarity.

To address this dilemma, we propose an approach to integrative STEAM activities, in which the epistemic tensions in representations, practices, and ways of knowing of two disciplines are foregrounded in playful, shared embodiment activities experienced by classroom groups of students. Students are then invited to engage aesthetically with the ideas that these epistemic tensions have evoked for them. Aesthetically keying (Goffman, 1974) these activities can enable learners to make creative use of such interdisciplinary tensions and explore them through expressive action. We argue that this form of epistemic rekeying offers an approach to interdisciplinarity that neither trivializes tensions between disciplines nor presents these tensions as inaccessible to younger learners.

We investigate a research and practice problem core to STEAM integration: *How can we engage students with agency, in the face of epistemic tensions in disciplinary ways of knowing?* We

illustrate the emergence of the epistemic rekeying approach in work within the Computation and Mathematics Play Spaces (CAMPS) project, which engaged middle-school (Grades 5-8) students in formal and informal settings, to integrate mathematics and computational thinking in the context of artistic expression.

We show how rekeying activities toward artistic expression enabled students to approach epistemic tensions in ways that shifted *away* from frames highlighting “correctness” (and adjudication between contrasting disciplinary perspectives) to frames highlighting “generativity” that could playfully engage with tensions, exploring the expressive potential of representations. Our “epistemic re-keying” approach is still in formation; in addition to analyzing activities, we thus also share conjecture maps (Sandoval, 2014) to show how epistemic rekeying may be a general approach to constructing integrative STEAM activities.

### **Theoretical Framework: Frames, Framing, and Rekeying**

The framing of a situation or interaction reflects participants’ negotiated determination of “what is going on here” (Goffman, 1974). Faced with a barrage of information that is overwhelming and often conflicting, humans have to make snap decisions about what “kind” of situation they are in, in order to determine what is relevant, what the rules are, and how they should act. It is remarkable, then, that framing can often be done implicitly and without uncertainty rising to conscious experience, especially since framing is a matter of shared agreement and coordination (Goffman, 1974). Episodes within integrative STEAM activities are often (implicitly or explicitly) framed as under the aegis of one discipline or another.

In unfamiliar environments, however, questions of framing can come to occupy the foreground (DeLiema, Enyedy, & Danish, 2019). Novel settings make it more likely than usual for multiple candidate framings to emerge, as participants look for contextual clues about the tools, participation structures, language, and interactions that are appropriate. Such settings can offer different frames for different people (Hand, Penuel, & Gutierrez, 2012), or make it ambiguous both to participants and to outside observers what is actually going on (Wisittanawat & Gresalfi, 2021). In integrative STEAM activities, epistemic tensions can provoke frame indeterminacy, when the interpretive lenses of two disciplines yield different meanings for a representation or action. Frame indeterminacy can be experienced as a crisis or breakdown, but situations designed to provoke frame indeterminacy can also bring together different interpretations of shared experience, thus offering powerful learning opportunities.

In addition to shifting from one frame to another, social groups can modify frames in ways that Goffman (1974) describes as keying, and re-keying. The paradigmatic example of rekeying is play: following Bateson’s (1956; 1972) reflections on animals’ play, Goffman describes how in play, a primary activity can be transformed. For instance, when dogs play at fighting, biting, growling, and many other recognizable actions remain, but because they are wrapped in the signal “this is play” they lose their original meanings and take on new significance. In our approach to integrative STEAM activities, we look for opportunities to invite students to rekey epistemic tensions as provocations for playful and aesthetic response.

### **Disciplinary Context: Mathematics and Computer Science in the NetLogo Environment**

NetLogo (Wilensky, 1999) is an agent-based modeling environment widely used in classrooms and educational research, to model complex systems in the natural and social worlds (Wilensky & Rand, 2015). The NetLogo world consists of two main types of computational entities: “patches,” which form a Cartesian grid in the world, and “turtles,” which can move about at a layer above the patches. In the CAMPS project, we supported students in learning to program, and using

mathematical representations and logic to create artistic computational performances that employed both of these NetLogo agent types (Brady et al, 2020; Brady 2021).

### **Epistemic Tension: Continuous and Discrete Representations of Space**

A key emergent issue in our designs was the representation of space as continuous (mathematics) or as discrete (computer science). In a continuous representation (e.g. Cartesian plane), space is infinitely divisible in each dimension; in contrast, in a discrete representation (e.g. images composed of pixels), there is a minimum resolution that can be distinguished. Familiar mathematical formalizations of lived space depend upon the representation of space as continuous. Fundamental concepts such as the density of the rational numbers in the reals and the theory of limits and convergence of sequences of numbers and functions rely upon a view of space as infinitely subdividable and continuous, as opposed to ‘chunky’ and discrete. In contrast, computational representations typically use finite precision, which defines a granularity to space. Indeed, it is arguable that discretization is fundamental to and inevitable in all digital representations.

Learners encounter shocks to their intuition when confronted with consequences of both continuous and discrete representations, and so it is *not* clear that one of these two is more “natural” to humans than the other. For instance, even after accepting a proposition that matter is discrete (e.g., that an atomic component such as a quark may be indivisible), people have trouble with the idea that the space in which this matter exists is discrete (e.g., that these smallest particles cannot move smoothly but must change their location by ‘jumping’ the minimal spatial grain-size.) Moreover, what about discrete *time*? On the other hand, some propositions rooted in a continuous perspective can be received as equally counter-intuitive. For instance, a continuous view of number holds that in selecting a number between 0 and 1 at random, the probability of selecting any particular number, say,  $\frac{1}{2}$ , is zero. A discrete view rescues us from this apparent paradox – with a “grain size” or “resolution” of  $\frac{1}{1000}$  (“three decimal places”), the probability of selection for each number in  $[0,1)$  is  $\frac{1}{1000}$ . Since there are 1000 such numbers, the probabilities add as learners expect.

All coordinates in the NetLogo environment have limited precision and are hence discrete, thus the patch grid presents learners a salient version of “chunky,” discrete space. In this paper we share the analysis of how entailments of the highly-discrete patch space became problematic for a group of students engaged in a shared-embodiment activity in which they played the role of the patch grid in their class (cf. Vogelstein & Brady, 2019; Brady 2021). Epistemic tensions about the representation of space also appeared in activities where students embodied and programmed turtles, though space limitations permit only a brief sketch of one such activity.

### **Methods**

The CAMPS project has produced three summer camps integrating math, computation, and art, co-designed and facilitated with middle-school mathematics teachers from a large urban district in the southeastern United States. Our first camp used graphic arts to foster connections between mathematics and agent-based programming. This “Image Camp” highlighted NetLogo patches, treating patches as pixels and used collective embodied and computational activities that encouraged students to explore how the group of patches could produce computational compositions and visual effects. In Year 2, we added a second camp that focused on performative movement expressed in choreography and code. This “Action Camp” highlighted how large numbers of NetLogo turtles can create dramatic effects as they move and change in concert. In addition to the camps, some partner teachers arranged to bring adaptations of camp activities to their students during the school year.

In this paper, we analyze data from a “Code Friday” session in the 7th grade mathematics classroom (N=34 students) of one of the lead teachers, Ms. S. Ms. S was a veteran teacher with over 25 years of teaching experience. The school where she taught served a student population that was diverse both economically and racially.

The mathematics class in which Ms. S ran “Code Friday” was an honors class; students in the class were consented to participate in a year-long study of these sessions. Multiple data sources feed our analysis here: video from a mobile camera and computer screen recordings from a subset of consenting students comprise the primary corpus. Focal episodes were identified based on field notes and initial viewing of the video record, highlighting students constructing and making sense of the patch grid’s representation of points and lines. We used interaction analysis (Jordan & Henderson, 1995; Hall & Stevens, 2015) methods, repeatedly viewing video; attending to students’ and teachers’ epistemic framing; and considering dialog, intonation, gesture, and embodied expressivity to characterize clashes and shifts in epistemic frames. We continually discussed and compared our interpretations to iteratively refine them.

### Findings

The Code Friday session that is the focus of this analysis began with a shared-embodiment activity, called the *Stadium Cards activity*. This was an activity Ms. S had taught the previous summer (Vogelstein & Brady, 2019) and that she re-organized for her 7<sup>th</sup> grade class. In it, a subset of the class collectively embodied a small 2x5 patch grid (see Figure 1, below), while the rest of the class watched and commented on the patch-actors’ work. Ms. S acted as the NetLogo *Observer* (the Observer has a ‘global’ view and can issue commands to *all* patches, to individual patches, or to any subset of patches). In this role, she used NetLogo syntax to ask patches to change their “patch color” (called “pcolor” in NetLogo syntax). Many of the prompts of the activity engaged with the idea of the patches’ varying *state* (a computer-science concept referring to the current value of its variables), and with the NetLogo representation of color as a number between 0 and 140, with a “wrapping” rule, so that the color 141 is the same as the color 1 (a mathematics concept pertaining to *modular arithmetic*). The Stadium Cards activities were challenging for learners, who worked to make sense of the “epistemic games” (Shaffer, 2005) involved in the world of NetLogo agents and their syntax, *decoding* it (Vogelstein & Brady, 2019) in the way that a traveler might decipher a foreign language (cf. Papert, 1980).

Disagreements among the students were resolved by encouraging students to articulate their thinking, and then learning the rules of these epistemic games. For example, in response to:

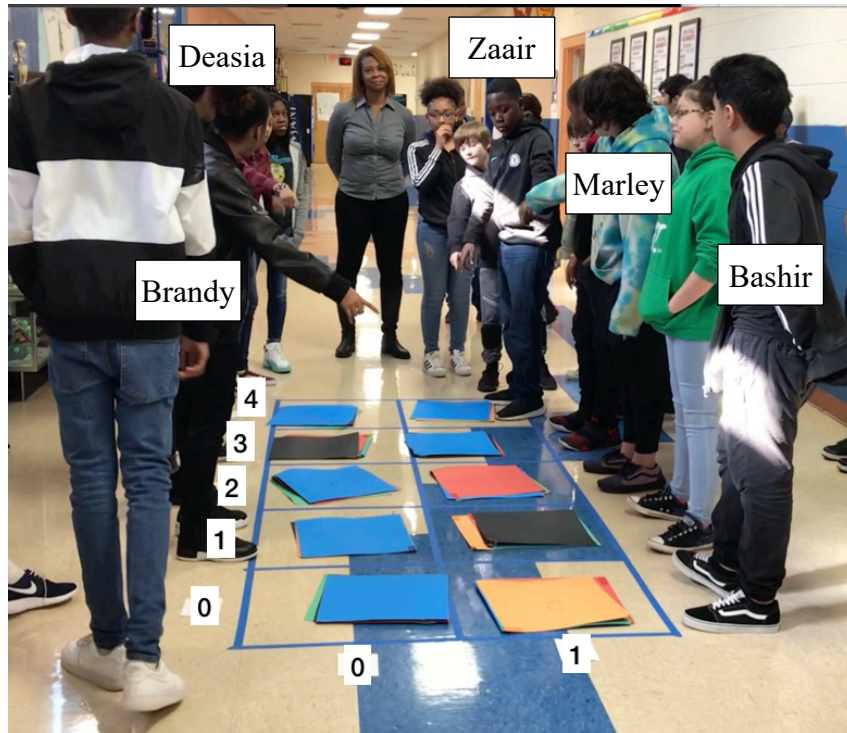
“ask patches, set pcolor pcolor plus three”

One student interpreted the command as setting the pcolor to the *value* of the signed number “+3.” The mathematical indeterminacy and lexical novelty of the computer-science variable assignment (`set x x + 3`), permitted this interpretation, and Ms. S recognized the student’s logic before facilitating a discussion toward the NetLogo meaning, captured by a student who demonstrated “pcolor + 3” by advancing the color on the color-card ring three times. Here, the design of the color-card ring manifested modular arithmetic and supported the “NetLogo logic” of modular arithmetic and color wrapping. Each patch-actor could consult the representational tool of the color-card ring, resolving the tension between arithmetic results (adding 3 to the number of their current color), and the “wrapping” rule.

The final Stadium Cards challenge of the day was designed to introduce NetLogo’s system of coordinates as a means to refer to patches by their location in the grid. The 2x5 grid was augmented with white index cards, intended to identify the coordinates along each dimension (0



and 1 along the horizontal axis closest to the camera and crossing the frame; 0 to 4 along the vertical axis on the left, extending from the camera toward Ms. S.) In contrast to the color-cards, as we will see below, the representational infrastructure for coordinates was distributed spatially and therefore needed to be consulted or “read” from a particular location and orientation.



**Figure 1. The Stadium Cards activity. Each student standing around the 2x5 patch grid was responsible for their patch. They controlled the color of their patch by manipulating the bound ring of colored paper and replacing the stack in the grid-space. Coordinate labels were written on index cards. The image is annotated to make them more legible.**

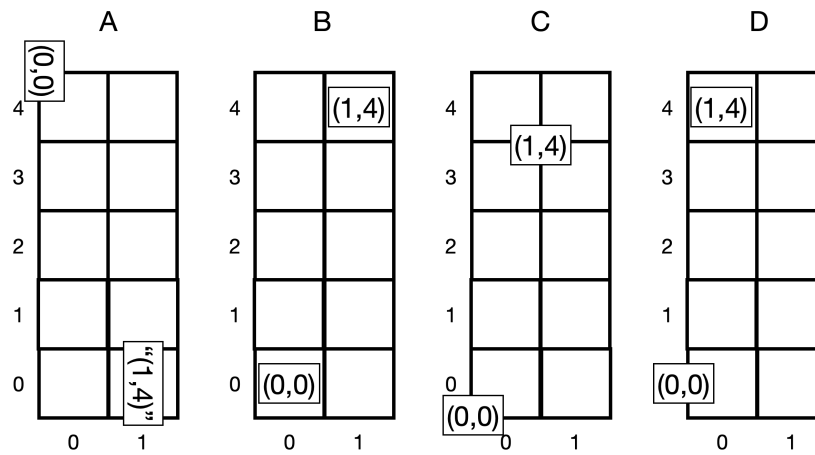
After telling the group that the patches had coordinates, Ms. S issued the command:

“ask patch one four, set pcolor orange”

The class intensely debated *which* patch would have the coordinates 1 4. (The four models of the grid that appeared in students’ discourse are shown in Figure 2.) There were two sources of disagreement. One arose from the challenge students experienced in shifting their view of the patch grid to locate the origin and construct the positive coordinate axis directions. Students tended to reason from a perspective in which the patch to the lower-left of the grid *from their standing point* was the origin. Figure 2a shows one student’s viewpoint that follows this line of reasoning (as well as reversing the order of x- and y- coordinates, presumably to allow the coordinate pair (1, 4) to appear on the grid). Challenges related to standing point *do* connect with representational feedback, practices and conventions associated with the Cartesian plane, but since they do not deal with disciplinary tensions, they are peripheral to our analytic focus.

The second challenge arose from the ambiguity of the placement and nature of the origin and of the 0-coordinate in each dimension. We had placed index cards at the centers of the patch axis intervals, in an attempt to indicate the patch-coordinates. These patch coordinates defined a

discrete description of the plane: all patch-coordinates are integers, and thus NetLogo's patch origin is the patch whose x- and y-coordinates are both zero (see Figure 2b). In contrast, many participating students' reasoning drew upon a *mathematical* representational convention that was cued by the Stadium Card grid. To support students in knowing where to put their ringed stacks of patch colors, we created a grid of taped lines. This "background" structure offered support for a view of the Cartesian plane (used in *mathematical* practices with representational tools such as graph paper) in which *gridlines* represent the location of exact-integer coordinates. In particular, under this interpretation, the lower-most and left-most tape-lines would represent the x- and y-axes respectively, and the origin would be a point at *the bottom-left corner* of the patch mentioned above (see Figure 2c). Thus, the coordinate 1 4 would be *between* patches – in the middle of the grid and at the point touched by the "top" four patches (see Figure 2c).



**Figure 2. Inferred assignments of the origin based on students' answers to where patch 1 4 would be.**

A fourth and final, hybridized perspective, which combined the two conventions, emerged in the argumentation of one vocal student, Brandy. Brandy's conclusion was that Deasia was the actor for the patch 1 4. This was initially puzzling to her peers, perhaps since it appeared to combine a discrete view of the grid in the y-direction with a continuous view in the x-direction. Articulating her different interpretation, Brandy said:

Brandy: Guess who's on y [pointing to Deasia] and guess who's number 4? [pointing again]

This way of presenting her thought reveals a key feature of shared embodiment in the activity. Our description above of the cues for the two perspectives neglected to consider the positioning of physical bodies of students who were animating the patches. Students stood off the grid, either to the left of patches with x-coordinate 0, or to the right of patches with x-coordinate 1. No patch-actors stood above or below the grid. That meant the actors' bodies indexed the y-coordinates but not the x-coordinates. Thus, for a student attending to the actors' bodies, the "origin actor" was standing at the location marked in Figure 2d. If one combines an actor-centered perspective of locations on the y-axis, with a measurement-movement perspective of the x-axis in this way, it is entirely comprehensible that the coordinate 1 4 would be reached by the actor "on y" and "who's number 4" taking one step in the positive-x direction. This conception would not address how to issue commands to students on the right-hand side of the grid, but neither Brandy nor Deasia was in that position.

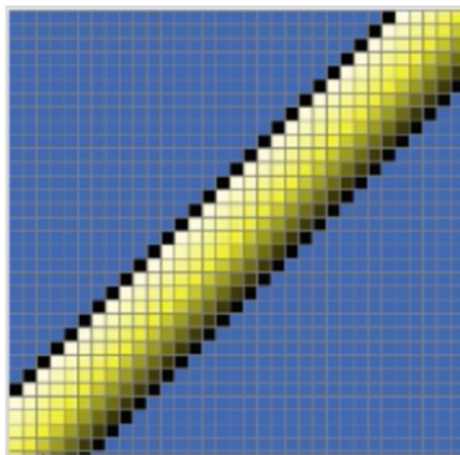
Deasia responded immediately to Brandy, pointing to the “0” label on the x-axis at the other end of the grid and saying “But that says zero.” Next, Marley (a patch-actor on the right-hand side of the grid) asked Brandy, “Are you saying that you do y first?” Marley’s comment was initially puzzling. However, she may have been trying to understand Brandy’s method, from the perspective of her own patch, or have been responding to Brandy’s implied method of getting to 1 4 (namely, that the actor at y=4 should take one step in the positive x direction). Further, because “doing y first” would violate a maxim of reading coordinates (“x is first,” which had been voiced by several students) she could be using the phrase to question the validity of Brandy’s method.

Consensus emerged at the end of the debate that Zaair was in fact the actor that NetLogo would designate as patch 1 4, and that he should change his color. As a way of giving voice to a still-unresolved tension she sensed in the students’ thinking, the second author offered a perspective highlighting the discreteness of the patch coordinate system: “in NetLogo...zero has a *thickness* to it; everything has a *width* to it...”

On returning to the classroom and the NetLogo environment, students were given several minutes of free coding time to experiment with the ideas and syntax they had just encountered. They also had their “NetLogo phrasebook,” which many students explored. Bashir took the opportunity to explore the idea of the thickness of computational lines. He started by returning to a phenomenon that the class had found strange in earlier work – namely that the  $y = x$  line was pixelated when drawn with patches. At the corners between pixels, this “line” had no thickness, while in the middle it was quite thick. Bashir typed in the Command Center:

```
ask patches [if pxcor = pycor [set pcolor 94]]
```

After trying several numbers as colors, he decided on yellow (45), and explored what nearby numbers looked like (e.g., 45-1). He rapidly typed the sequence of commands in Figure 3, using the Command Center feature to recall the last command issued, and then editing it:



```
ask patches [if pxcor = pycor - 1 [set pcolor 45 - 1]]
ask patches [if pxcor = pycor [set pcolor 45]]
ask patches [if pxcor = pycor - 2 [set pcolor 45 - 2]]
ask patches [if pxcor = pycor - 3 [set pcolor 45 - 3]]
ask patches [if pxcor = pycor - 4 [set pcolor 45 - 4]]
ask patches [if pxcor = pycor - 5 [set pcolor 45 - 5]]
ask patches [if pxcor = pycor + 5 [set pcolor 45 + 5]]
ask patches [if pxcor = pycor + 4 [set pcolor 45 + 4]]
ask patches [if pxcor = pycor + 3 [set pcolor 45 + 3]]
ask patches [if pxcor = pycor + 2 [set pcolor 45 + 2]]
ask patches [if pxcor = pycor + 1 [set pcolor 45 + 1]]
```

**Figure 3. Bashir’s iterative construction of a “cool” 3D line.**

In this work, Bashir built on prior coding activities (which explored how small changes to syntax could make big changes to the effect of the code), and on his experiences in the Stadium Cards activity (assigning numbers to colors and to locations, and noting that patches whose coordinates differed by one were right next to each other). He used these novel and unfamiliar findings to explore how he could achieve a visual effect related to the class’s problem about line

pixelation. Changing color and position in a coordinated, stepwise fashion, he created a three-dimensional effect that he showed off to his table neighbors and the class as a whole as “cool.”

### Discovering Epistemic Rekeying

Bashir’s artistic use of the ideas from prior activities was an unexpected innovation for us, and if his response had been unique, we might not have attended to it in our ongoing design iterations. However, several students in the class played artistically with the ideas from this activity, and in other activities, we saw similar tendencies. For an example also exploring the discrete-continuous epistemic tension, several students in the Action Camp (now embodying *turtles*), became fascinated with rules that involved a precise x-coordinate value ( $x=0$ ). A turtle’s coordinates can take on decimal values, so that when it moves across the screen, it may never trigger an “if  $x_{cor} = 0$ ” rule. They invented choreographic rules that relied on the condition ( $x_{cor} = 0$ ) being one turtles would reliably achieve. Turtles would execute one behavior if they met the condition, “ $x_{cor} < 0$ ,” another if they met “ $x_{cor} > 0$ ,” and the *combined* behavior if they met “ $x_{cor} = 0$ .” Implementing these rules brought both computational and aesthetic rewards.

### Conjecture Mapping

As we have encountered the inventive responses of children when they are presented with epistemic tensions, we have come to see the potential in rekeying these tensions as provocations for artistic production. We propose that this may be a generative design element for creating and facilitating integrative STEAM activities. We capture this idea in the following conjecture map:

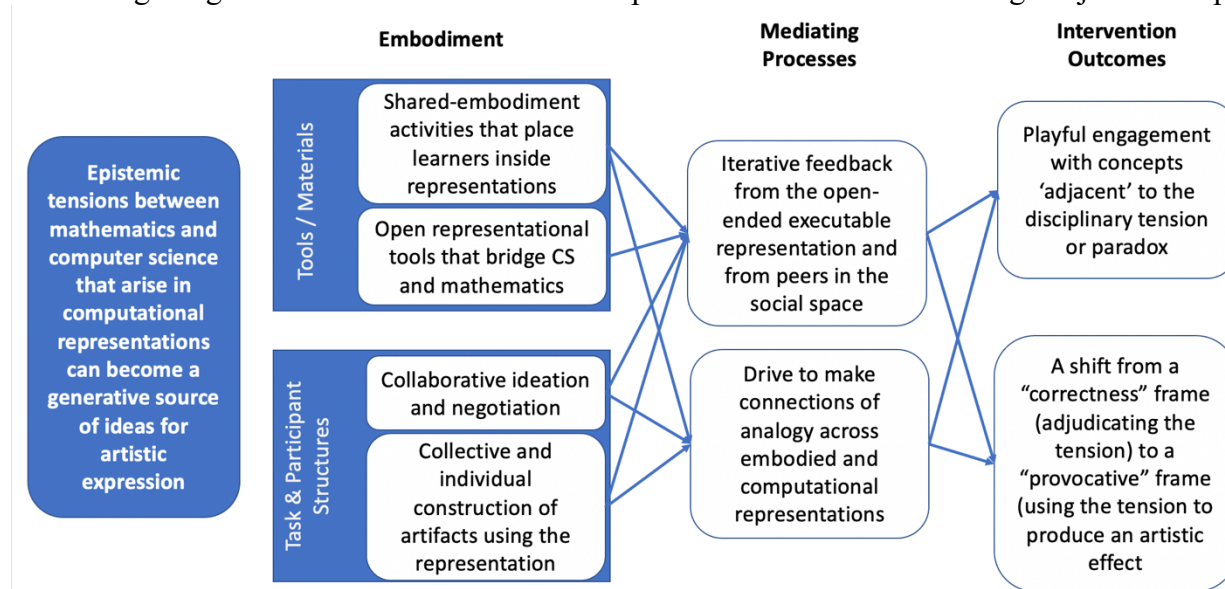


Figure 4. Conjecture map for epistemic rekeying in integrative STEAM activity sequences.

### Discussion and Conclusions

In this paper we have described our route to formulating a design element for integrative STEAM activities, which we call epistemic rekeying. We identified the essential problem that it solves, in addressing epistemic tensions between disciplinary practices of representation, and we described an instance of an activity sequence, beginning with a shared embodiment activity, followed by independent creative work, in which students playfully engaged with the epistemic tension as a provocation for artistic creation. Finally, we shared our current conjecture map.

### Acknowledgments

We are grateful to the CAMPS project team and the teachers and students who participated the

project, without whose creative energy we would not have made these discoveries.

### References

- Akkerman, S. & Bakker, A. (2011). Boundary crossing and boundary objects. *Review of Educational Research*, 81(2), 132-169.
- Bateson, G. (1956). The message 'this is play'. In B. Schaffner (Ed.), *Group Processes: Transactions of the Second Conference* (pp. 145-242). New York: Josia Macy, Jr. Foundation.
- Bateson, G. (1972). *Steps to an ecology of mind*. Chicago, IL: University of Chicago Press.
- Brady, C. (2021). Patches as an expressive medium for exploratory multi-agent modelling. *British Journal of Educational Technology*, 52(3), 1024-1042.
- Brady, C., Gresalfi, M., Steinberg, S., & Knowe, M. (2020). Debugging for Art's Sake: Beginning Programmers' Debugging Activity in an Expressive Coding Context. In Gresalfi, M. and Horn, I. S. (Eds.), *The Interdisciplinarity of the Learning Sciences, 14th International Conference of the Learning Sciences (ICLS) 2020, Volume 3* (pp. 1229-1236). Nashville, Tennessee: International Society of the Learning Sciences.
- DeLiema, D., Enyedy, N., & Danish, J. A. (2019). Roles, rules, and keys: How different play configurations shape collaborative science inquiry. *Journal of the Learning Sciences*, 28(4-5), 513-555.
- Goffman, E. (1974). *Frame analysis: An essay on the organization of experience*. Cambridge, MA: Harvard University Press.
- Hall, R., & Stevens, R. (2015). Interaction analysis: Approaches to knowledge in use. In *Knowledge and interaction: A synthetic agenda for the learning sciences* (pp. 72-108). New York: Routledge
- Hand, V., Penuel, W. R., & Gutiérrez, K. D. (2012). (Re) framing educational possibility: Attending to power and equity in shaping access to and within learning opportunities. *Human Development*, 55(5-6), 250-268.
- Jordan, B., & Henderson, A. (1995). Interaction analysis: Foundations and practice. *Journal of the Learning Sciences*, 4(1), 39-103.
- Lehrer, R., & Schauble, L. (2020). Stepping Carefully: Thinking Through the Potential Pitfalls of Integrated STEM. *Journal for STEM Education Research*, 1-26.
- Lesh, R. A., Hamilton, E., & Kaput, J. J. (Eds.). (2007). *Foundations for the future in mathematics education*. New York: Lawrence Erlbaum.
- National Science Foundation [NSF]. (2019). Growing convergence research.
- Nersessian, N. (2017). Hybrid Devices Embodiments of Culture in Biomedical Engineering. In *Cultures without Culturalism: The Making of Scientific Knowledge* (pp. 117-144). Duke U Press.
- Osbeck, L., & Nersessian, N. (2017). Epistemic identities in interdisciplinary science. *Perspectives on Science*, 25(2), 226-260.
- Papert, S. (1980). *Mindstorms: Children, computers, and powerful ideas*. New York: Basic Books
- Sandoval, W. (2014). Conjecture mapping: An approach to systematic educational design research. *Journal of the learning sciences*, 23(1), 18-36.
- Shaffer, D. W. (2006). Epistemic frames for epistemic games. *Computers & education*, 46(3), 223-234.
- Shaffer, D. W. (2005). Epistemic games. *Innovate: journal of online education*, 1(6), 2.
- Takeuchi, M. A., Sengupta, P., Shanahan, M. C., Adams, J. D., & Hachem, M. (2020). Transdisciplinarity in STEM education: a critical review. *Studies in Science Education*, 1-41.
- Vogelstein, L. & Brady, C. (2019). Taking the Patch Perspective: A Comparative Analysis of a Patch Based Participatory Simulation. In Lund, K., Niccolai, G. P., Lavoué, E., Hmelo-Silver, C., Gweon, G., & Baker, M. (Eds.), *A Wide Lens: Combining Embodied, Enactive, Extended, and Embedded Learning in Collaborative Settings, 13th International Conference on Computer Supported Collaborative Learning (CSCL) 2019, Volume 1* (pp. 512-519). Lyon, France: International Society of the Learning Sciences.
- Wilensky, U. (1999). *NetLogo*. Evanston, IL: Center for Connected Learning (CCL), Northwestern U.
- Wilensky, U., & Rand, W. (2015). *An introduction to agent-based modeling: modeling natural, social, and engineered complex systems with NetLogo*. Cambridge, MA: MIT Press.
- Wisittanawat, P., & Gresalfi, M. S. (2021). The "tricky business" of genre blending: Tensions between frames of school mathematics and video game play. *Journal of the Learning Sciences*, 30(2), 240-278.

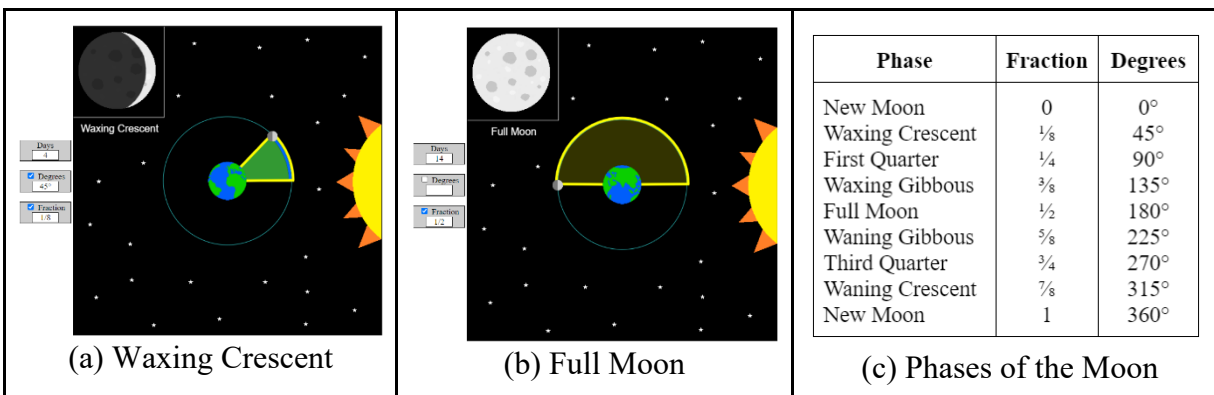
## EXPLORING LUNAR PHASES BY CO-SPLITTING FRACTIONS AND ANGLES

Amanda Provost  
Montclair State University  
provostal@montclair.edu

Nicole Panorkou  
Montclair State University  
panorkoun@montclair.edu

Keywords: Integrated STEM/STEAM, Geometry and Spatial Reasoning, Rational Numbers & Proportional Reasoning, Technology

When mathematics and science are taught together it is essential that they are taught in ways that illustrate their reciprocal relationship (Fitzallen, 2015). Building on our prior work (Panorkou & Germia, 2020), we discuss a study that bridges angle measurement, covariation, and co-splitting with the scientific phenomenon of lunar phases. We report on the results of a whole-class design experiment (Cobb et al., 2003) in a sixth-grade classroom in the Northeastern U.S. conducted on Google Meet. The Moon Pie simulation (Figure 1) models the relationship between the moon's revolution around the earth and its different phases. The user can click on the moon to drag it around in its orbital path, observing the resulting changes in its phase as displayed in the picture-in-picture view of the moon as seen from earth. The simulation displays a readout of the number of days elapsed during the moon's orbit using a 28-day approximation of a lunar month to provide a simplified model of a real scientific phenomenon that would be pedagogically accessible for middle school students (Weintrop et al., 2016).



**Figure 1: The Moon Pie Simulation**

We focus on the retrospective analysis of one pair of students, Ali and Jaden, to describe their reasoning about fraction-degree relationships of a circular orbit. Our analysis focused on how students analyzed the situation into a structure of quantities and quantitative relationships (Thompson, 1989; 1994). Our results showed that students were able to coordinate the fraction of the orbit and the degrees traveled by the moon, illustrating evidence of covariational reasoning (Carlson et al., 2002; Confrey & Smith, 1994). The fraction of the orbit represented as a wedge supported their quantification of an angle as a rotation, bridging both static and dynamic angle conceptions (Devichi & Munier, 2013). Questioning by the researcher elicited students' progression in the development of full, half, quarter, and eighth benchmark relationships between the fraction and degrees traveled around the orbit. Both students used co-splitting strategies (Corley et al., 2012) to identify fraction and degree values at locations throughout the orbit of the

moon. Our results show how the use of technology can constructively support the reciprocal relationship between math and science learning.

### References

- Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education*, 5(33), 352–378.
- Cobb, P., Confrey, J., DiSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. *Educational Researcher*, 32(1), 9–13.
- Confrey, J., & Smith, E. (1994). Exponential functions, rates of change, and the multiplicative unit. *Educational Studies in Mathematics*, 26(2–3), 135-164.
- Corley, A., Confrey, J., & Nguyen, K. (2012). The co-splitting construct: Student strategies and the relationship to equipartitioning and ratio [Paper presentation]. *American Education Research Association Annual Meeting*, Vancouver, Canada.
- Devichi, C., & Munier, V. (2013). About the concept of angle in elementary school: Misconceptions and teaching sequences. *The Journal of Mathematical Behavior*, 32(1), 1-19.
- Fitzallen, N. (2015). STEM education: What does mathematics have to offer? In M. Marshman, V. Geiger, & A. Bennison (Eds.). *Mathematics education in the margins (Proceedings of the 38th annual conference of the Mathematics Education Research Group of Australasia)* (pp. 237–244). Sunshine Coast: MERGA.
- Panorkou, N., & Germia, E. F. (2021). Integrating math and science content through covariational reasoning: the case of gravity. *Mathematical Thinking and Learning*, 23(4), 318-343.
- Thompson, P. (1989). A cognitive model of quantity-based reasoning in algebra [Paper presentation]. *Annual Meeting of the American Educational Research Association*, (pp. 27–31). San Francisco, CA.
- Thompson, P. W. (1994). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 181–234). SUNY Press.
- Weintrop, D., Beheshti, E., Horn, M., Orton, K., Jona, K., Trouille, L., & Wilensky, U. (2016). Defining computational thinking for mathematics and science classrooms. *Journal of Science Education and Technology*, 25(1), 127-147.

## GOOGLE DOCS AS A TOOL TO SUPPORT ROUGH-DRAFT MATH DISCOURSE

### GOOGLE DOCS COMO UNA HERRAMIENTA PARA APOYAR EL DISCURSO BORRADOR DE MATEMÁTICAS

Tenchita Alzaga Elizondo  
Portland State University  
halzaga@pdx.edu

Brittney Ellis  
Texas State University  
b\_e107@txstate.edu

*The transition back to face-to-face instruction from emergency distance learning due to the pandemic, begs the question: what benefits (if any) from this experience can be imported to face-to-face instruction? This study builds on previous work on students' use of Google Docs in a remote introduction-to-proofs course and explores the conjecture that the lack of permanency of text on Google Docs supports students' rough-draft math practices. By studying two days of a face-to-face introduction to proof course, we present how students' discourse surrounding the use of Google Docs explicitly and implicitly evidenced rough-draft math.*

Keywords: Technology; Classroom Discourse; Equity, Inclusion, and Diversity; Reasoning and Proof

The COVID-19 pandemic forced educators and learners to shift toward emergency distance learning (EDL, Tate & Warschauer, 2022), which caused difficulties, stress, and anxiety among all involved (Engelbrecht et al., 2023). As universities transition back to in-person learning environments though, an important question arises: what benefits from EDL (if any) can be imported to face-to-face instruction? Studying remote instruction in introduction-to-proofs courses, Alzaga Elizondo (2022) found that Google Docs supported student engagement in a *process approach* to writing (Graham & Perin, 2007; Sun & Feng, 2009) – defined by multiple revision cycles that incorporate feedback from peers and the instructor. Additionally, similar to previous studies (e.g., Kessler et al., 2012), Google Docs also seemed to aid students in feeling more comfortable sharing their ideas, presumably due to the medium's lack of permanence. The author argued that this medium appeared to support students engaging in rough-draft math (Jansen, 2020a, 2020b), a practice described as “rehumanizing mathematics” (Gutiérrez, 2018) as it supports the disruption and dismantling of inequities by emphasizing the human aspect of learning mathematics. Whether (and how) students in an inquiry-oriented, *face-to-face* introduction-to-proofs course used Google Docs to engage in rough-draft math during small group work may provide backing for importing Google Docs to face-to-face instruction. Thus, in this study, we analyzed students' discourse surrounding Google Doc edits to address the research question: In what ways do students' discourse while using Google Docs reflect rough-draft math practices during small group work?

#### Theoretical Perspectives

Inquiry-oriented instruction creates space for students to engage in doing and discussing mathematics in the classroom, often in small groups (Laursen & Rasmussen, 2019). As such, learning opportunities arise through students' discourse with their peers and the instructor (Moschovich, 2007). The nature of inquiry classrooms also opens up space for more uncertainty (and perhaps anxiety) around “doing” and discussing mathematics with others, which may inadvertently exclude some students from participating fully – limiting access to opportunities to



learn (Battey & McMichael, 2021). If students are expected to collaborate on mathematics tasks in an inquiry setting, then perhaps engaging in practices that reflect a learning culture where sharing rough-draft ideas and revising ideas are valued may foster inclusive learning environments (Jansen, 2020a, 2020b).

Rough-draft mathematics (Jansen, 2020a, 2020b) involves actively changing the math classroom culture away from “performing” and “evaluating” (i.e., quickly providing correct answers which are immediately judged as right or wrong) toward a “learning culture” where both teachers and students try to make sense of thinking before evaluating. Teachers can promote this type of learning culture by discussing the process of learning and how it is connected to rough drafts, inviting students to share “rough draft” ideas and engage in “non-evaluative” talk by trying to understand first before judging, positioning students as competent and providing explicit opportunities for them to revise their thinking. However, how *students* might then engage in such rough-draft math practices beyond explicitly stating “this is just my rough-draft idea, but...” is less explicit in the literature (Jansen, 2020a, p. e018). We argue that student discourse may evidence ways students engage in rough-draft math practices during small group work, particularly when they share initial ideas about a task and make suggested revisions.

### Methods

This study is part of a larger design-based research project developing inquiry-oriented introduction to proof curriculum with instructional support materials. Building on Alzaga Elizondo’s (2022) findings, the project team purchased a set of iPads© to investigate students’ use of Google Docs in a university introduction to proofs course taught face to face. Students used iPads daily to work on Google Docs to facilitate their collaborative work in small groups. A stationary camera and screen recordings captured one focal group’s interactions each day (26 days total). Data from two different days (one at the beginning of the semester and one in the middle) formed the basis for our preliminary analysis. These days were selected because there was substantial discourse between students when using Google Docs, which we anticipated might evidence students’ engagement in rough-draft math discourse.

To begin our analysis, we referenced the version history feature of the Google Docs to identify all instances when an edit was made. We then used the screen recording data to create detailed transcript segments of students’ discursive activity (Erickson, 2006) surrounding instances of Google Doc edits (see Temple Adger & Wright, 2015 for discourse transcript conventions used). In the first cycle of coding, rough-draft math discourse was tagged when an utterance could be identified as sharing an initial idea (i.e., not a “final” product/answer) or proposing a refinement, which connects to Jansen’s conceptualizations of rough-draft math practices. Through re-reading and coding the data, we noticed such discourse had an undertone of uncertainty which was indicated by speech patterns and/or tone of voice. This noticing prompted another cycle of coding to refine and expand the initial codes for rough-draft math discourse. In the next section, we present findings from our discourse analysis.

### Preliminary Results

Across the two days we identified 23 instances in which students edited the text on the Google Doc, including creating brand new text. All but three instances were associated with evidence of either explicit and/or implicit rough-draft math discourse (according to our coding). We further identified two types of discourse in each category, which we present here with examples from the data. Note that while we discuss these discourse types separately, they are not meant to be interpreted as disjoint; meaning evidence for explicit and implicit rough-draft math discourse could co-occur in one instance of student discourse.

### Explicit Rough-Draft Math Discourse

Students' *explicit* rough-draft discourse was coded when students shared an idea and expressed it as a suggestion or “not final” solution. This occurred in (at least) two ways. First, students' discourse reflected a *hypothetical solution* through the use of phrases like “what if” or “maybe we could try...” just before writing text on the Google Doc. Consider the exchange between two students who were working on writing their definition of a one-to-one function:

- Amber: so.. so if we were to do this with function notation.. maybe we say.. (speaking while typing)  $f(A)=B$ ? .. do we want to.. [types “ $f(A)=B$ ” on the Google Doc]
- Ana: mmm.. hmm  $f$  of  $A$ , should we maybe.. like if  $A$  is a group.. should we have.. that like little  $a$ 's are the elements.. in it? Just to like sort of distinguish.. it? [“ $f(A)=B$ ” is edited to “ $f(a)=b$ ” on the Google Doc] (emphasis added)

Amber and Ana both used phrases like “if we were to do this” and “should we maybe” to frame their suggestions as hypothetical, meaning a potential solution that is up for discussion. They then used the Google Doc to illustrate the hypothetical idea. The text written in the Google Doc was not meant to be viewed as permanent, rather as a starting point in their collective discussion that can be revised. Notice that Ana does respond by editing Amber's public idea in the Google Doc,  $f(A)=B$ , to  $f(a)=b$  to reflect elements of  $A$  and  $B$  (“like little  $a$ 's are the elements”).

Students' discourse also showed explicit evidence of rough-draft math when shared ideas were framed as *not formal yet*. Such discourse is tied to the introduction to proofs context since students are learning how to use formal mathematical language and notations. Google Docs were used to explicate informal or intuitive ideas, which was framed as such in their discourse. Consider an example as the group continued working on their definition of one-to-one:

- Amber: (speaking while typing) further, for any two elements in  $A$  (inaudible) they must map to two different elements in  $B$ .  
[Types “Further, for any two elements in  $A$  they must map to two different elements in  $B$ ” in the Google Doc]
- Amber: that's not very “zhuzhy” but..

Amber's discourse explicitly indicated that her verbal and typed idea on the group's Google Doc was not very “zhuzhy.” We interpreted her use of “zhuzhy” to mean “mathy” or “formal” since “mathy” was used by both students and the instructor in this class to mean using symbolic and/or formal math language. In other words, Amber's idea captured the essence of the one-to-one definition and her discourse indicated that it could be further revised to include symbolic or formal math notation. Both examples show how Google Docs provided students a place to house their hypothetical and informal ideas that they knew could be refined as needed.

### Implicit Rough-Draft Math Discourse

Students' discourse also evidenced rough-draft math *implicitly* through *tone of voice* and use of *filler words* (e.g., “uh”, “like”, stuttering, repeating words). When proposing ideas, their discourse would often reflect a questioning tone, which was indicated by a question mark at the end of an utterance (Temple & Wright, 2015). In the above exchange between Ana and Amber, notice how several utterances end with a question mark. For instance, Ana finished her proposed revision by saying, “Just to like sort of distinguish.. it?” The questioning tone indicated some uncertainty, and the Google Doc edits that immediately followed allowed them to act on this uncertainty by providing a nonpermanent space to display the ideas they had in mind. Given this expressed uncertainty, Ana's edit of Amber's idea could be framed as a potential refinement that should be

up for discussion and not a final solution that needed to be evaluated, which aligns with conceptualizations of rough-draft math practices.

Students also used Google Docs to illustrate ideas that were challenging to verbalize, which was evidenced in their discourse by using filler words. A filler word is any word or sound that is inserted into the main message of the speaker and often reflects nervousness or uncertainty over the topic being discussed (Duvall et al., 2014). For instance, consider how a student communicated a potential definition for “symmetry” of a shape building on the sentence starter “A symmetry of a shape S is...”:

Henry: yeah, I think I think we could just yeah like kind of like uh just just rotation  
[Types “rotations in the range  $[0,2\pi]$ ” to their statement in the Google Doc]

Note that Henry’s discourse primarily consisted of filler words such as “uh”, “like”, “yeah”, and the repetition of “I think” and “just.” This indicated an uncertainty about his idea and a potential struggle to verbalize it. Henry’s subsequent use of the Google Doc to display his idea nonverbally shows how the Google Doc opened an opportunity for him to share his thinking with the group. We interpreted this as an instance when the Google Doc medium created an opportunity for students to act on feelings of uncertainty around their math ideas.

### **Discussion & Conclusion**

In this study, students’ discourse explicitly and implicitly evidenced rough-draft math when ideas were framed as *hypothetical solutions* or *not formal yet* and by using *filler words* and a questioning *tone of voice*, respectively. In either case, students subsequently used Google Docs as a tool to frame their written text as initial ideas that were open to refinement (i.e., rough-draft math discourse). We posit that the unique features of Google Docs may have supported students to act on moments of uncertainty about their ideas rather than shutting down. As a word processor, Google Docs allows for text to be easily edited. Familiarity with this medium and knowing their ideas could be refined and/or deleted with minimal effort seemed to support students in feeling comfortable sharing their ideas. In a follow-up interview, Amber discussed how working with Google Docs was different than past experiences working with paper:

You have to erase it [pencil], or scratch it out or re-copy it, or it’s just not as editable and doing it on our shared document, we could readily edit it, and also just looks cleaner. It’s easier to take something as being worthwhile work if it’s not scrawled on a piece of paper.

Her discussion points to some barriers students may face when working with paper and pencil in small groups (e.g., messy and hard to edit), which may discourage students from sharing ideas. She also makes an observation that an idea comes across as “being worthwhile work” when it is formatted in a Google Doc versus “scrawled on” paper. This may have implications for how students perceive each other’s mathematical work, particularly if the Google Doc medium allows for a more accessible and collaborative flow of ideas between group members (which requires further research to study such a phenomenon in-depth). We plan to build on our initial exploration of students’ rough-draft math discourse via Google Docs to investigate the conjecture that this medium provides students a unique opportunity to act on feelings of uncertainty or anxiety while working on inquiry-oriented mathematics tasks.

### **Acknowledgments**

The research reported in this paper was supported by the National Science Foundation (award no. 1916490).

## References

- Alzaga Elizondo, T. (2022). Collective Proving Activity in a Synchronous Online Environment [Ph.D., Portland State University]. <https://www.proquest.com/docview/2758131153/abstract/DCFC850CE1834417PQ/1>
- Bathey, D., & McMichael, E. W. (2021). Inquiry-based practices: Opening possibilities for (in)equitable interactions in classrooms. In *International Handbook of Inquiry and Learning* (pp. 189-203). Taylor and Francis.
- Duvall, E., Robbins, A., Graham, T., & Divett, S. (2014). Exploring filler words and their impact. *Schwa. Language & Linguistics*, 11, 35-49. <https://schwa.byu.edu/files/2014/12/F2014-Robbins.pdf>
- Engelbrecht, J., Borba, M.C. & Kaiser, G. (2023). Will we ever teach mathematics again in the way we used to before the pandemic?. *ZDM Mathematics Education*. <https://doi-org.proxy.lib.pdx.edu/10.1007/s11858-022-01460-5>
- Erickson, F. (2006). Definition and analysis of data from videotape: Some research procedures and their rationales. In J. L. Green, G. Camilli, & P. B. Elmore (Eds.), *Handbook of complementary methods in education research* (pp. 177–191). Erlbaum.
- Graham, S., & Perin, D. (2007). A meta-analysis of writing instruction for adolescent students. *Journal of Educational Psychology*, 99(3), 445–476. <https://doi.org/10.1037/0022-0663.99.3.445>
- Gutiérrez, R. (2018). Introduction: The Need to Rehumanize Mathematics. In I. Goffney, R. Gutiérrez, & M. Boston (Eds.), *Rehumanizing Mathematics for Black, Indigenous, and Latinx Students*. National Council of Teachers of Mathematics.
- Jansen, A. (2020a). *Rough Draft Math: Revising to Learn*. Stenhouse Publishers.
- Jansen, A. (2020b). Rough-draft thinking and revising in mathematics. *Mathematics Teacher: Learning and Teaching PK-12*, 113(12), e107-e110. <https://doi.org/10.5951/MTLT.2020.0220>
- Kessler, G., Bikowski, D., & Boggs, J. (2012). Collaborative writing among second language learners in academic web-based projects. *Language Learning & Technology*, 16(1), 91–109. <https://scholarspace.manoa.hawaii.edu/server/api/core/bitstreams/dfcf827b-601e-4ff8-9d2b-199606ddcb1a/content>
- Laursen, S. L., & Rasmussen, C. (2019). I on the prize: Inquiry approaches in undergraduate mathematics. *International Journal of Research in Undergraduate Mathematics Education*, 5(1), 129-146. <https://doi.org/10.1007/s40753-019-00085-6>
- Moschkovich, J. (2007). Examining mathematical discourse practices. *For the Learning of Mathematics*, 27(1), 24-30. <https://www.jstor.org/stable/40248556>
- Sun, C., & Feng, G. (2009). Process approach to teaching writing applied in different teaching models. *English Language Teaching*, 2(1), 150–155. <https://files.eric.ed.gov/fulltext/EJ1082330.pdf>
- Tate, T., & Warschauer, M. (2022). Equity in online learning. *Educational Psychologist*, 57(3), 192–206. <https://doi.org/10.1080/00461520.2022.2062597>
- Temple Adger, C., & Wright, L. J. (2015). Discourse in educational settings. In D. Tannen, H. E. Hamilton, & D. Schiffrin (Eds.) *The handbook of discourse analysis* (pp. 858-879). John Wiley & Sons, Inc.
- Crespo, S. (2003). Learning to pose mathematical problems: Exploring changes in preservice teachers' practices. *Educational Studies in Mathematics*, 52(3), 243–270. <https://doi.org/10.1023/A:1024364304664>
- Herbst, P., & Chazan, D. (2006). Producing a viable story of geometry instruction: What kind of representation calls forth teachers' practical rationality? In S. Alatorre, J. L. Cortina, M. Sáiz, & A. Méndez. (Eds.), *Proceedings of the 28th North American Group for the Psychology of Mathematics Education Conference* (Vol. 2, pp. 213–220). Mérida, México: UPN.

## OPPORTUNITIES FOR INSTRUMENTAL GENESIS IN MATHEMATICS LESSONS

Catherine Dennis

University of New Hampshire  
Catherine.Dennis@unh.edu

Orly Buchbinder

University of New Hampshire  
Orly.Buchbinder@unh.edu

*As digital technologies become a commonplace in mathematics classrooms, they transform the ways students and teachers interact with mathematics. Students learn to use a technological tool for solving mathematical tasks through a process known as instrumental genesis. Teachers' support for this process is known as instrumental orchestration. In this paper we explore how both peers and teachers provide opportunities to influence the instrumental genesis process with group, one-on-one, and full class discourse. Through analyzing videos of three lessons from professional development websites we found evidence of six literature-based categories of instrumental orchestration, as well as five new categories expanding the instrumental orchestration framework. The study outcomes can be used to advance research on integration of digital technology in classrooms to promote rich mathematics learning for all students.*

Keywords: Learning theory, technology, instrumental genesis, instrumental orchestration

The potential of digital technologies to transform teaching and learning in mathematics classrooms has long been recognized (Bray & Tangney, 2017; Clark-Wilson et al, 2020). In recent years, there have been several developments with respect to digital technology that support its growing integration in mathematics classrooms. One such development has been the advancement of free digital technology tools for learning mathematics, such as *Desmos* and *GeoGebra*, which are designed to support user interaction with the tool in exploratory ways. As opposed, for example, to scientific calculators, whose primary role is to simplify calculations, digital tools like *GeoGebra* allow for exploring properties of mathematical objects and relationships among them (e.g., how changing the value of parameters in an equation affects the graph of a function corresponding to this equation). The second development is students' increased access to and familiarity with hand-held devices such as smartphones and laptops, making the students "Digital Natives" (Francis & Hoefel, 2018). Concurrently, there is an increased number of mathematics applications and digital tools available on these devices, as well as the growing number of ready-available digital tasks and activities, sometimes called "techtivities" (e.g., Johnson et al., 2021). In addition, new technologies like projectors, screen share, and communication devices are opening new doors for discussing mathematical ideas.

To integrate technology into teaching mathematics, teachers need to adapt their methods of instruction, modify the organization of their classrooms and the nature of mathematical tasks (Bozkurt & Uygan, 2020). As students interact with a technological tool, the tool gradually becomes an *instrument* supporting students' mathematical thinking. This process is called *instrumental genesis*, and the process that supports students' instrumental genesis is called *instrumental orchestration* (we elaborate on these in the theoretical perspectives section) (Drijvers et al., 2010; Maschietto & Trouche, 2010; Trouche, 2004; Trouche & Drijvers, 2010). Researchers identified six categories of instrumental orchestration, which arose from characterizing teacher work in leading whole class discussions in technology rich lessons, with teachers at the center and with students contributing to the whole class discussion (Buteau et al., 2020; Drijvers et al., 2010). However, with students being digital natives it is reasonable to expect that their technological

competence would affect the nature of classroom interactions surrounding instrumental genesis as well as teacher instrumental orchestration. Additionally, there has been little research on the impact of peers, lesson types, and varied modes of discourse on instrumental genesis.

### **Research Aims and Questions**

As technology is becoming more ubiquitous in mathematics classrooms, it is expected that the repertoire of instrumental orchestration types will evolve and grow (Buteau et al., 2020). This study expands upon the existing research by examining instrumental orchestration in three technology rich lessons in which both teachers and peers could influence instrumental genesis. Through this exploration, we attempted to confirm the presence of literature-based categories for instrumental orchestration (e.g., Drijvers et al., 2010) while also documenting the potential emergence of additional categories of peer influence on instrumental genesis. The study was guided by the following research questions.

1. What are the didactical configurations (the setting and the artefacts) for each lesson?
2. How do teachers enact instrumental orchestration in various modes of discourse?
3. How do peers influence each other's instrumental genesis?

While most studies on instrumental orchestration utilized real-time observations, this study seeks to examine whether the types of instrumental orchestration can be observed in video recordings of classroom teaching, as a way to expand methodological practices for studying instrumental genesis and instrumental orchestration.

### **Theoretical Perspectives and Background**

An *artefact* is a material object, e.g., a calculator, or abstract object, e.g., quadratic formula, whose aim is to assist in future mathematics activity or learning (Maschietto & Trouche, 2010). What gives an artefact, either material or abstract, its meaning are cognitive structures, or schemes. Once a student puts a meaning to the artefact and uses it as a tool for performing conceptual or physical tasks, the tool becomes an *instrument*. The process of users, or learners, turning an artefact into an instrument is called *instrumental genesis* (Clark-Wilson et al., 2020). Teachers support students' instrumental genesis through the process of *instrumental orchestration* by carefully planning and organizing lessons that engage students with technology (Clark-Wilson et al., 2020). The orchestration metaphor is used because of many moving parts, intentions, and instruments needed for learning and instrumental genesis to occur in harmony.

Instrumental orchestration and instrumental genesis are frameworks for investigating how teachers and students integrate technological tools in mathematics classroom. The frameworks are grounded in cognitive theories of learning and have been developed empirically and shaped over the years (Artigue, 2002; Drijvers et al., 2010; Trouche & Drijvers, 2010; Trouche, 2004).

Scholars (Drijvers et al., 2010; Trouche, 2004) describe three components of instrumental orchestration: didactical configuration, exploitation mode, and didactical performance. *Didactical configuration* refers to the setting and artefacts for the lesson. How is the room organized? What artefacts are available? How easily can students access them? The *exploitation mode* involves planning mathematical tasks that support instrumental genesis. Teachers decide what tasks to use and how technology is integrated in the task. Lastly, the *didactical performance* can be observed during the lesson in teacher's actions, both pre-planned as well as ad hoc, or in the moment, decisions. The exploitation mode and the didactical performance may vary considerably, even when teachers use the same instructional task. Pierce et al. (2010) identified great variety in how CAS technology was used, and the amount of responsibility for independent learning provided by

12 mathematics teachers implementing the same lesson plan on quadratic relationships. In addition, Bozkurt and Ruthven (2018) observed that novice and experienced teachers' use of dynamic geometry software in their classrooms differed at all levels of instrumental orchestration: the lesson organization, the choice of the task and the level of integration of the tool into mathematics teaching.

Drijvers et al. (2010) identified six types of instrumental orchestration in technology-rich classrooms: technical demo, explain-the-screen, link-screen-board, discuss-the-screen, spot-and-show, and sherpa-at-work. The first three orchestration types are teacher centered. With *technical demo* a teacher demonstrates the technical features of a tool and how to use it in mathematical tasks. This can occur with a small group of students or with the whole class. In *explain-the-screen* a teacher explains a projected screen to the entire class. It can be an example problem or a technical component. With *link-screen-board* the teacher makes connection between the technological representations and the paper, pencil, or board representations of mathematics. The next three orchestration types are more student centered. With *discuss-the-screen* the whole class engages in collaborative conversation about a (projected) computer screen. In *spot-and-show* student reasoning is discussed amongst a group or whole class. Lastly, *sherpa-at-work* is a technique where one student uses technology to represent his or her work to the entire class.

### Methods

Three videos were chosen for this investigation. All videos are freely available online. The lessons were videorecorded in United States classrooms; each using a different technological tool and classroom setting. The variety of lesson types intended to reveal a variety of activities, modes of discourse and contributions to instrumental genesis, both through teachers and peers.

Lesson 1 video<sup>6</sup> (20 min) showed an Algebra 2 class with about 20-25 students working in groups on an activity called "Modeling with Polynomials". Ms. Burke, the teacher, had 13 years of teaching experience at the time. Students had to determine the maximum volume of an open top box made from cutting grid paper with 1-centimeter squares. Students modeled polynomials using *Desmos*, working in groups of 4-5 students and sharing *Chromebooks*.

The lesson 2 video<sup>7</sup> (10 min) featured an Algebra 1 classroom, similarly, with about 20-25 students, who worked more independently, led by Mr. Kwon, a second-year teacher. Students used *Desmos* and *code.org* to investigate rates of change of a rocket, utilizing multiple representations afforded by *Desmos*, to make connections between graphs, tables, and functions. Every student had a desktop computer and appeared to be working independently.

The video for lesson 3<sup>8</sup> (4 min) also showed students using *Desmos* in a virtual learning environment. Ms. Saltz, who had around 13 years of teaching experience, taught an Algebra 1 lesson on *Zoom* with a group of about 20 students. The lesson was about the roots of polynomials.

### Analytic Framework

The videos were fully transcribed for the analysis. In addition, the first author of the paper wrote analytic memos, and summarized data on classroom environment, lesson format, and technological resources provided to students. The analysis of video, notes, and transcriptions was used to respond to research question one - the didactical configuration of each lesson, which was identified by examining questions such as: How were the classrooms arranged? What resources were provided to students for instrumental genesis? With what modality was the lesson being

---

<sup>6</sup> <https://www.insidemathematics.org/classroom-videos/public-lessons/11th-12th-grade-algebra-ii-modeling-polynomials>

<sup>7</sup> <https://learn.teachingchannel.com/video/code-in-hs-math/?1664818181743>

<sup>8</sup> <https://www.philasd.org/etv1/2021/04/08/using-desmos-in-math-virtual-lessons/>

taught? Same data sources, as well as worksheets accompanying the video, were analyzed for identifying instances of instrumental genesis, when students used technology to solve a mathematical task.

To answer research questions 2 and 3, we coded each lesson by identifying instances of the six types of instrumental orchestration according to Drijvers et al.’s (2010) framework. In addition, we captured instances where students supported each other’s instrumental genesis and described them as new categories. Since the term instrumental orchestration usually refers only to teacher actions (Clark-Wilson et al., 2020), we use the term “influence on instrumental genesis” to describe both teacher and peer actions. Additionally, each instance of influence of instrumental genesis was coded for the mode of discourse, i.e., who is involved in the discourse: group, teacher-group, teacher-student, and whole class. Group was coded any time discourse took place between more than two students; teacher-student was coded when a teacher spoke to a student one-on-one.

## Results

### Lesson 1

**Didactical Configuration.** Mrs. Burke’s class was organized into groups, with four students per group, seated at desks that face each other. Some of the group members were assigned roles, like facilitator, recorder, and resource manager. For example, Mrs. Burke said, “Facilitators, will you make sure that you’re moving your group into step 4?” In the first part of the “Cutting Corners” activity, each group received a piece of graph paper, and was guided to fold it into an open box by cutting four squares at the corners of the sheet. Each group calculated the volume of the resulting box, and the whole class completed a table on the board recording the dimensions and the volume of various boxes. Next, each group received two *Chromebooks* to enter the data from the table into *Desmos* software, graph it and fit a polynomial function through the plotted points. The worksheet guidelines assisted students with using *Desmos*, for example, “Use the regression feature to find a function to model the data.”

**Modes of Discourse and Instrumental Genesis Influence.** This lesson contained four modes of discourse: group, teacher-group, teacher-student, and whole class. There were 38 transcript lines that contained evidence of influence on instrumental genesis. Each line was coded for the type(s) of instrumental genesis influence present and who was involved in the discourse. For example, within one group one student performed a technical demonstration through *Desmos* for another student, saying “You see how I did it?”. This was coded as a technical demo in group discourse. Table 1 shows the modes of discourse in the rows, and the type of instrumental genesis influence in the columns. Percentages were calculated out of 38 lines.

**Table 1: Lesson 1 Modes of Discourse and Influence on Instrumental Genesis**

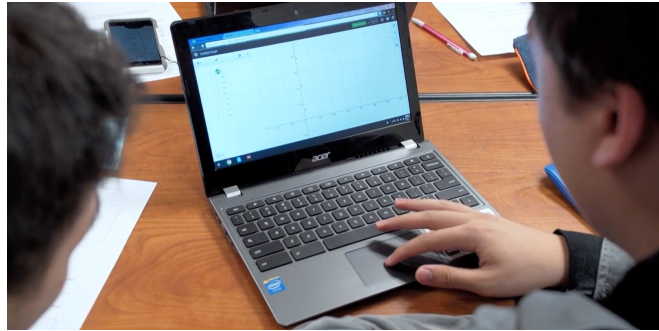
Discourse Mode	Technical Demo	Explain the Screen	Link-Screen-Board	Discuss the Screen	Sherpa at Work	Trouble-shooting Tech*	Written Directions*	Total
Group	8%	3%	18%	13%	0	5%	0	47%
Teacher- Group	0	0	16%	11%	3%	3%	0	32%
Teacher-Student	0	0	5%	0	3%	0	0	8%
Whole Class	0	0	0	3%	3%	0	8%	13%
Total	8%	3%	39%	26%	8%	8%	8%	

**New Categories: Written Directions and Troubleshooting Technology.** As seen in Table 1, the five out of six categories of instrumental orchestration framework (except for spot-and-show) appeared in multiple areas of discourse, beyond whole class discussions. In fact, 79% of all



influences on instrumental genesis occurred within the groups or between the teacher and the groups. Additionally, two new categories of influence on instrumental genesis arose: written directions and troubleshooting technology. Some of the worksheets Mrs. Burke gave provided written directions for students to make links between mathematics and the *Desmos* software. For example, “Use your model to find the maximum volume of the box.” Since the directions were from the teacher and to the whole class, they were coded as “whole class” discourse.

In addition, Mrs. Burke and the students had to use their knowledge of both technology and mathematics to troubleshoot technological issues that arose. For example, when one group was creating a scatterplot in *Desmos* their graph showed all points on the y-axis (Figure 1). They questioned, “why is it like that?”, knowing something must be wrong, and worked together to fix the technological issue. This instance was coded as group troubleshooting technology. Through this process students can learn more about how mathematics content interacts with technology.



**Figure 1: Group “troubleshooting technology” when graph has all outputs on y axis**

## **Lesson 2**

**Didactical Configuration.** Mr. Kwon’s class was in a computer lab with one-to-one desktops. The students sat in rows facing the front of the room. In the mathematical task, the students first watched a *Desmos* animation of a rocket and had to change the parameters of the animation to make the rocket go at different speeds and initial height values. Students also created various representations of the flight of the rocket using tables, graphs, and equations. Then students had to switch to a different software, *Code.org*, to match the code that animates the rocket to a mathematical representation of its movement. Students experimented with both constant and varied rate of change of the rocket’s velocity. The students seemed to have a basic understanding of the computer programs since they appeared to work on their own with minimal guidance. Students worked independently for most of the lesson, occasionally asking the teacher questions.

**Modes of Discourse and Instrumental Genesis Influence.** This lesson contained three modes of discourse: group, teacher-student, and whole class. There were 27 transcript lines that were coded for influencing student’s instrumental genesis. Five out of six original instrumental orchestration categories (all except *link-screen-board*) were detected, as well as a new category: *link-screen-screen* which described instances when a teacher was explicating connections between two or more software programs. For example, Mr. Kwon said, “At each part of the lesson I had students ask me to come over and check to see if their table and the graph and their equation matches the animation” (Figure 2).

### **New Categories: Written Directions, Troubleshooting Technology, and Link-Screen-Screen.**

The two new categories detected in lesson 1: written directions and troubleshooting technology were also present in this lesson (Table 2). Another new category, labeled *link-screen-screen* arose since students used two different mathematics programs *Desmos*, and *Code.org*. to work on the

mathematical task. *Link-screen-screen* describes teacher (or peer) actions supporting students' coordination of representations across various computer screens and mathematical task.



Figure 2: Teacher-Student discourse. Link-screen-screen.

Table 2: Lesson 2 Modes of Discourse and Influence on Instrumental Genesis

Discourse Mode	Explain the Screen	Link-Screen-Board	Discuss the Screen	Sherpa at Work	Spot and Show	Link-Screen-Screen*	Trouble-shooting Tech*	Written Directions *	Total
Group	0	0	0	4%	0	0	4%	0	7%
Teacher-Student	11%	33%	0	0	0	19%	7%	0	70%
Whole Class	0	7%	4%	0	7%	0	0	4%	23%
Total	11%	41%	4%	4%	7%	19%	11%	4%	

### Lesson 3

**Didactical Configurations.** Ms. Saltz taught her lesson via *Zoom* video conferencing. Her screen was shared so students could see her actions on *Desmos*. All students attending the lesson and the teacher were logged into the same *Desmos* activity. Ms. Saltz had the ability to pace students' *Desmos* screens and display student work anonymously. The lesson involved finding solutions to polynomial equations using both algebraic and graphical representations.

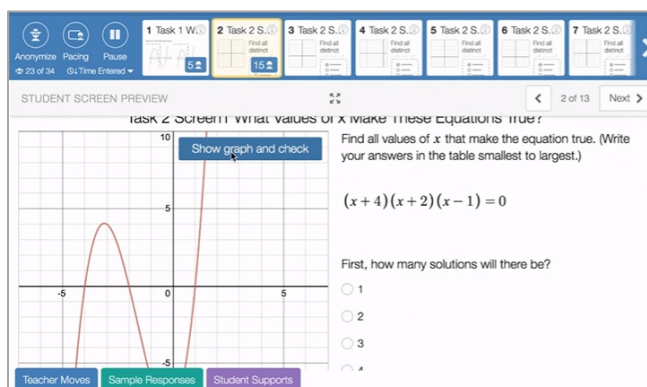


Figure 3: Teacher's shared *Desmos* screen during the virtual lesson

**Modes of Discourse and Instrumental Genesis Influence.** This lesson only had whole class discourse and contained eight influences on instrumental genesis. Two new categories arose: *anonymous spot-and-show* and *verbal directions*. Table 3 shows the distributions.

Table 3: Lesson 3 Modes of Discourse and Influence on Instrumental Genesis

Discourse Mode	Technical Demo	Spot and Show (Anonymous)*	Sherpa at Work*	Verbal Directions*	Total
Whole Class	38%	25%	12%	25%	100%
Total	38%	25%	12%	25%	

**New Categories: Anonymous Spot-and-Show and Verbal Directions.** *Verbal directions* were similar to written directions in the previous lessons. Ms. Saltz gave students directions on how to use their technology for mathematics without showing it first, since all students were following on simultaneously on their screens. For example, she gave students verbal directions on what to click to get to the correct page or screen, “I want you to click on the link-I want you to go to our Desmos for the day”. Additionally, she used the *spot-and-show* technique to discuss student work with the whole class, but did it anonymously, without disclosing the name of the student. This way the teacher avoided putting social pressure on students to show their work to the whole class, but student work was still utilized to advance the mathematical goal of the lesson.

### Summary

Regarding research question 1, the didactical configurations of each lesson afforded different teaching methods and modes of discourse. In lesson 1 the room arrangement of grouped desks allowed for easy communication amongst students. The groups sharing *Chromebooks* enabled students to discuss the technology as it related to the mathematics they were studying. In lesson 2 all seats faced the front of the classroom making it challenging for students to talk to each other. Additionally, each student had their own desktop, so they rarely looked at each other’s screens. This didactical configuration seemed to encourage single student-to-teacher discourse rather than discourse among students or the whole class. Lesson 3 was fully remote. Although the task had a potential to elicit student participation, the teacher lectured most of the time, without providing students opportunities to discuss mathematics in groups or with each other.

Research question 2 asked how teachers enact modes of instrumental orchestration during different modes of discourse. In lesson 1 the teacher interacted with groups, individual students, and the whole class. Teacher-group interactions had the highest frequency, with 32% of the total. The results showed that *link-screen-board*, *discuss-the-screen*, and *sherpa-at-work* orchestration strategies appeared in all different modes of discourse. The new categories of *troubleshooting technology* and *written directions* appeared during group-teacher interactions and whole class interactions. In lesson 2 the teacher’s discourse modes included whole class and teacher-student interactions, the latter accounted for 70% of the total interactions. The orchestration categories: *explain-the-screen*, *link-screen-board*, *discuss-the-screen*, and *sherpa-at-work* were detected, as well as new categories of *troubleshooting technology*, *written directions*, and *link-screen-screen*. Lesson 3 only had whole class interactions in a lecture style. Orchestration categories included *technical demo*, *spot-and-show*, and *sherpa-at-work*, with a new category of *verbal directions*.

Expanding the scope of examination of modes of discourse beyond whole class discussion allowed for identifying opportunities for instrumental genesis between peers (research question 3). The most common mode of discourse for lesson 1, accounting for 47% of interactions, was within groups. The categories of *technical demo*, *explain-the-screen*, *link-screen-board*, and *discuss-the-screen* were all transferrable into group interactions. Additionally, students worked together to *troubleshoot technology*. Lesson 2 only had two instances (7%) of peer interaction for instrumental genesis: one *sherpa-at-work* and one *troubleshooting technology*. Lesson 3 had no evidence of peer interactions, thus, no peer support for instrumental genesis could be detected.

In summary, different didactical configurations and lesson types can impact modes of discourse and opportunities for instrumental genesis. The six instrumental orchestration categories from Drijvers et al. (2010) could be applied in a variety of discourse modes. The group work heavy lesson, lesson 1, had the most opportunities for instrumental genesis through peer interactions in groups. Lesson 2 with one-to-one devices allowed for minimal peer interactions and minimal peer influence on instrumental genesis. However, there were many teacher-student opportunities for instrumental genesis. The remote lecture style lesson only had whole class discourse opportunities for instrumental genesis. Different discourse modes allowed for the emergence of new categories of instrumental genesis influence: *written and verbal directions*, *troubleshooting technology*, *link-screen-screen*, and *anonymous spot-and- show*.

### **Discussion and Implications for Education**

Our study attempted to extend the scope of examination of instrumental orchestration beyond whole class discussions in real classrooms (e.g., Bozkurt & Uygan, 2020; Drijvers et al., 2010), to video recordings featuring various parts of technology-integrated lessons. In addition, we used video footage that is freely available online, taken from websites geared toward teachers and teacher educators. Since these videos are intended for use in professional development, or to showcase “exemplary teaching” (e.g., see description link to video 3), it is important to understand what types of opportunities for instrumental orchestration they contain.

While our methodological approach appeared to yield fruitful results, it has clear limitations. The main one being the lack of access to the raw footage of the whole lesson but only to selected clips uploaded to the web. For example, the lesson 1 video featured various groups of students, not allowing for prolonged observation of a single group’s dynamic. Lessons 2 and 3 were represented by even shorter, edited down clips. While the types of didactic configurations, discourses, and influences on instrumental genesis observed in the video may be representative of the other parts of the lesson, there is no evidence in the video to support this assumption. Importantly, our goal in this paper was not to characterize the lessons themselves, but to characterize the types of supports for instrumental genesis that are visible in the existing data. Thus, we make no claims about the types of support for instrumental orchestration in these lessons in general.

Our study was exploratory. Methodologically, we attempted to provide proof of concept about the utility of using publicly available video of teaching and learning with technology to study instrumental genesis. The results show that the literature-based categories of instrumental orchestration appeared in all the examined videos, supporting our methodological assumption. In addition, our study revealed novel categories of influence on instrumental genesis in the actions of teachers and, importantly, peers.

The results of this study can be useful to teacher educators. With the growing integration of technology in teaching mathematics, future teachers can benefit from applying the categories of instrumental orchestration to analyze instructional video. This can potentially help teachers to enrich their repertoire of instrumental orchestration strategies, and to gain appreciation for how different lesson configurations afford different opportunities for instrumental genesis and instrumental orchestration.

Our study also points to the need to continue examining teacher and peer support for instrumental genesis beyond the existing frameworks. The modes of instruction and technological tools are constantly evolving, and with them the pedagogical practices of teachers (Bray & Tangney, 2017). Continuing to examine instrumental orchestration is important to advance our understanding of student mathematics learning in technology-rich environments.

## References

- Artigue, M. (2002). Learning mathematics in a CAS environment: The genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. *International Journal of Computers for Mathematical Learning*, 7, 245–274.
- Bozkurt, G., & Ruthven, K. (2018). The activity structure of technology-based mathematics lesson: A case study of three teachers in English secondary schools. *Research in Mathematics Education*, 20(3), 254–272.
- Bray, A., & Tangney, B. (2017). Technology usage in mathematics education research—A systematic review of recent trends. *Computers & Education*, 114, 255-273.
- Burke, A. (Teacher). (2017). Modeling with polynomials [Video]. University of Texas at Austin Charles A. Dana Center: inside mathematics.
- Buteau, C., Muller, E., Rodriguez, M. S., Gueudet, G., Mgombelo, J., & Sacristán, A. I. (2020). Instrumental orchestration of using programming for mathematics investigations. *Mathematics Education in the Digital Age (MEDA)*, 443.
- Clark-Wilson, A., Robutti, O., & Thomas, M. (2020). Teaching with digital technology. *ZDM- Mathematisch Education*, 1-20.
- Drijvers, P., Doorman, M., Boon, P., Reed, H., & Gravemeijer, K. (2010). The teacher and the tool: Instrumental orchestrations in the technology-rich mathematics classroom. *Educational Studies in mathematics*, 75(2), 213-234.
- Francis, T., & Hoefel, F. (2018). *True Gen': Generation Z and its implications for companies*. McKinsey & Company, 12.
- Johnson, H., Olson, G., Gardner, A., & Wang, X. (2021). A college algebra intervention to address power dynamics and promote students' reasoning. *Justice through the lens of calculus: Framing new possibilities for diversity, equity, and inclusion*, 251-256.
- Kwon. (Teacher). (2021). Series coding in the classrooms: coding in the algebra classroom [Video]. Teaching Channel.
- Maschietto, M., & Trouche, L. (2010). Mathematics learning and tools from theoretical, historical and practical points of view: The productive notion of mathematics laboratories. *ZDM – Mathematics Education*, 42, 33–47.
- Pierce, R., Stacey, K., & Wander, R. (2010). Examining the didactic contract when handheld technology is permitted in the mathematics classroom. *ZDM- Mathematics Education*, 42, 683-695. doi.org/10.1007/s11858-010-0271-8
- Trouche, L. (2004). Managing the complexity of human/machine interactions in computerized learning environments: Guiding students' command process through instrumental orchestrations. *International Journal of Computers for mathematical learning*, 9(3), 281-307.
- Trouche, L. & Drijvers, P. (2010). Handheld technology for mathematics education: Flashback into the future. *ZDM – Mathematics Education*, 42(7), 667-681.
- Saltz, T. (Teacher). (April 8, 2021). Using Desmos in math virtual lessons [Video]. The school district of Philadelphia Exemplary Teaching Video Library.

## OPPORTUNITY TO LEARN FUNCTION TRANSFORMATIONS IN DYNAMIC MATHEMATICAL ENVIRONMENT: AN ANALYSIS OF 347 GEOGEBRA APPLETS

Xiangquan Yao  
Pennsylvania State University  
xzy73@psu.edu

Nicholas Grande  
Pennsylvania State University  
npg5053@psu.edu

*This study examined the opportunity to learn function transformations afforded by the GeoGebra applets available on the GeoGebra website. Our analysis focused on the functions and their representations through which function transformations are explored in these GeoGebra applets, the effects and components of transformations that the GeoGebra applets afford students to learn, and the scaffolding provided in these applets. The results show that function transformations in the GeoGebra applets are often explored in the context of function families (e.g., quadratic and trigonometric functions) that use specific representations (e.g., graphical and symbolic). Moreover, the defining parameters of transformations and corresponding points on the graphs of parent and child functions are not visible in most GeoGebra applets. Only a small number of GeoGebra applets include questions or tasks that aim to scaffold students' exploration of function transformations. These results invite us to rethink how to design GeoGebra applets that maximize students' opportunity to learn function transformation conceptually in dynamic mathematical environments.*

Keywords: Opportunity to Learn; Function Transformations; Technology

Function transformation is an important mathematical concept for secondary and post-secondary students to learn. Function transformation together with function concept, covariation and rate of change, families of functions, and multiple representations of functions are the big ideas that are essential for students to understand functions (Lloyd, Beckmann, Zbiek, & Cooney, 2010). The study of function transformation can provide students with opportunities to use, reflect on, and possibly modify their understanding of function. Moreover, the study of function transformation can provide students with opportunities to build connections between transformations of a geometric shape and transformations of the graph of a function so that students can see how a single mathematical idea manifests in various domains of mathematics. Such experience can support students to perceive mathematics as a field of intricately related ideas rather than a collection of compartmentalized concepts and procedures. Given its importance, many countries (e.g., the United States, Australia, Turkey, and China) emphasize function transformations in their curriculum standards. For instance, in the United States, the Common Core State Standards for Mathematics (NGA Center and CCSSO, 2010) requires students to “identify the effect on the graph of replacing  $f(x)$  by  $f(x) + k$ ,  $k \cdot f(x)$ ,  $f(k \cdot x)$ , and  $f(x + k)$  for specific values of  $k$  (both positive and negative); find the value of  $k$  given the graphs; experiment with cases and illustrate an explanation of the effects on the graph using technology” (p.70). However, researchers have found that students experience difficulties in understanding function transformations, including but not limited to recognizing which transformation is applied to a parent function, identifying properties of function under transformation, and providing a mathematically sound explanation for the rules of function transformations (Lage & Trigueros, 2006; Zazkis, Liljedahl, & Gadowsky, 2003). These difficulties come from different sources, including an insufficient understanding of the function concept (Boz-Yaman & Yigit Koyunkaya, 2019; Kimani 2008; Lage & Gaisman, 2006; Baker et al., 2001), problematic conceptions of symbols involved in

transformations (Yim & Lee, 2021), and inappropriate instructional approach (Hall & Giacini, 2013; Borba & Confrey, 1996).

The use of mathematics action technologies (Dick & Hollebrands, 2011) such as graphing calculators, computer algebra systems, and dynamic mathematical environments (e.g., GeoGebra and the Geometer's Sketchpad) has the potential to transform how mathematics is taught and learned. Meanwhile, researchers have pointed out that the realization of such potentiality heavily depends on the design of appropriate teaching/learning tasks and resources with these technologies (Günster & Weigand, 2020; Leung & Bolite-Frant, 2015). The use of different tasks and resources might impact not only what mathematical ideas are learned but also how they are learned. Therefore, it is important to analyze the opportunity to learn afforded by tasks and resources that are designed with the use of a mathematics action technology. As an interactive geometry, algebra, statistics, and calculus application that can run on multiple platforms (e.g., desktops, tablets, and online), GeoGebra is developed for learning and teaching mathematics from primary school to university level. Although GeoGebra users can create activities from scratch, existing GeoGebra applets are important resources for educators because GeoGebra cloud service allows users to upload and share GeoGebra applets with others. Currently, it hosts more than one million free activities, simulations, exercises, lessons, and games for mathematics and science. A significant number of these GeoGebra applets are created for students to learn function transformations. Given that GeoGebra is a community of millions of students and teachers who are potential users of these applets, it is important to examine the opportunity to learn function transformations afforded by publicly available GeoGebra applets from the GeoGebra website.

### **Theoretical Background and Research Questions**

#### **Opportunity-to-Learn Required for Understanding Function Transformations**

When particular learning goals are not achieved by students, it is plausible to ask whether they have received the learning experience that enables them to develop the competencies expressed in these goals. Therefore, it seems natural for researchers to introduce the notion of opportunity-to-learn (OTL). Indeed, this concept was coined by Carroll (1963) when referring to sufficient time for students to learn (Walkowiak, Pinter, & Berry, 2017). Since then, the notion of OTL has been interpreted from multiple theoretical perspectives, where the focus may be on cognitive, curriculum and assessment design, social or affective dimensions of learning, issues of equity and access, or the broad policy and political contexts of learning and teaching (Goos, 2014). At the core of the OTL concept is the need for high-quality curriculum and instructional resources and teaching to enable students to achieve rigorous learning goals. Brewer and Stasz (1996) considered three OTL variables: (1) curriculum content coverage, that is, whether students have been taught the subjects and topics essential to attain the standards; (2) instructional strategies, that is, whether students have experience with particular kinds of tasks and solution processes; (3) the quality of instructional resources. This study focused on the quality of a particular type of instructional resource, namely, online GeoGebra applets. The existing GeoGebra resources were chosen because of the large number of GeoGebra users and the high volume of publicly available GeoGebra resources on function transformations. To our knowledge, these existing GeoGebra resources have not been carefully studied. This study aimed to examine the quality of GeoGebra resources on function transformations to understand how they afford or constrain students' OTL about function transformations.

Function transformations are mathematical operations (i.e., translation, reflection, rotation, and dilation) that can be performed on an existing function and its graph to give a modified version of that function and its graph that has a similar shape to the original function. To understand function

transformations, students need to have the opportunity to understand both the function and the transformation performed on it. Since different function families are included in the secondary mathematics curriculum, it is important to consider what functions are used to explore function transformations. Moreover, like other mathematical objects, the function concept cannot be directly perceived with instruments and access to the function concept relies on the use of a system of semiotic representations (Duval, 2017). Therefore, representations of parent and child functions impact students' opportunity to learn function transformations. Functions can be represented by words, tables, symbols, or graphs, each of which has its advantages. Graphs provide a visual representation of a function, showing how the function changes over a range of inputs. Symbolic representation uses an equation to compactly express a function relation that makes it easy to compute functional values. By using two columns (one with the dependent variable and the other with the independent variable), tables explicitly supply the functional values of specific inputs. Building connections among multiple representations of a parent function and its child function is critical for understanding function transformations. Besides the opportunity to learn parent and child functions and their representations, students also need to have the opportunity to learn transformations performed on functions. This includes the opportunity to learn not only the effects of different transformations but also what defines each transformation. For instance, understanding function translation requires the opportunity to learn not only the effect of translation as a motion but also what defines a translation (i.e., translation vector). Similarly, understanding function dilation requires the opportunity to learn not only the effect of dilation as a motion but also what defines a dilation (i.e., a center of dilation and a scale factor). When examining the OTL afforded or constrained by instructional resources, it is important to analyze whether and in what ways the resources explicitly provide students with the opportunity to learn different aspects of function transformations.

### **Approaches to Function Transformations**

Within curriculum and instructional resources, different approaches have been taken to function transformation. These approaches afford different opportunities to learn function transformation. The transformation of functions can be taught through a graphical approach (Heid, Wilson, & Blume, 2015). In this approach, graphing utility is used to examine the graphs of a parent function and its child function simultaneously. Simultaneous display of the graphs of the two functions for different values of the parameters of a child function illustrates the relationship between the graph of the child function and the graph of the parent function for particular parameter values. Relying on visual and numerical clues, the students are assumed to connect the movement of the graph with the change in the numerical values. A tabular approach might also be used to examine function transformation (Zazkis et al., 2003; Heid et al., 2015), in which a table of values is created by plugging numbers into the equations, and then points are plotted on the coordinate plane. Take the horizontal translation of a quadratic function  $f(x) = x^2$  3 units right as an example. The table can show that  $x - 3$  will produce the same output values as  $x$ , just 3 units "earlier." Thus,  $f(x - 3)$  will have the same output values as  $f(x)$ , but 3 units "later". The points formed by the ordered pairs of input and output values demonstrate that the graph of  $(x - 3)^2$  is 3 units to the right of the graph of  $x^2$  as a result of each value of  $x - 3$  being 3 less than the corresponding value of  $x$ . It is worth noting that both the graphical and the tabular approaches make use of visual and numerical patterns to reveal the rules for function transformations but do not explain why the patterns exist. Although students might see the patterns, they might still perceive the rules as "counterintuitive" and inconsistent. Nonetheless, both approaches were



frequently used by teachers and students to explain the rules of function transformations (Zazkis et al., 2003).

Zazkis and colleagues (2003) attributed students' difficulties in understanding the horizontal translation of function to the inadequate instructional sequence in which the treatment of transformations of functions is presented in the context of exploring functions rather than in the context of exploring transformations. They proposed a pedagogical approach to function translation that takes transformations as the starting point rather than focuses on the algebraic representation of functions as in the graphical and tabular approaches. According to their approach, students first learn the effect of translation on a set of points in a coordinate plane and are introduced to the formal notation of translation  $T(x, y) = (x + a, y + b)$ , where  $a$  and  $b$  are horizontal and vertical components of the motion, respectively. Once the formal notation is introduced, students should be provided ample opportunities to connect the visual image of translation to the mapping rule, which can be achieved by asking students to carry out transformations according to given mappings and to identify mappings according to given visual images. The teacher can then shift students' effort from the translation of a set of points in a geometric shape to the translation of a set of points on the graph of a function. Building on their prior understanding, students will understand that if  $P(x, y)$  is a point on the graph of a function  $y = f(x)$  and  $P'(x', y')$  is the corresponding point of  $P$  under a translation  $(a, b)$ , then  $x' = x + a$  and  $y' = y + b$ . Therefore,  $x = x' - a$  and  $y = y' - b$ . Since the point  $(x, y)$  is on the graph of  $y = f(x)$ , the point  $(x' - a, y' - b)$  is also on the graph of  $y = f(x)$ . Thus, we have  $y' - b = f(x' - a)$  as the algebraic expression of the function after translation. This approach explains why  $f(x - a)$  means shifting the graph of  $f(x)$   $a$  unit to the right if  $a$  is positive, which appears to be "counterintuitive" if students only focus on the algebraic representation of functions. It is worth noting that this transformation approach can be used to explain the rules for other transformations of function.

The transformation of a function can also be thought of as the transformation of the underlying coordinate axes rather than the graph of the function (Heid et al., 2015). Take the translation of a function as an example. Let  $y = f(x)$  be a function graphed in the  $XY$  coordinate plane and  $P(x, y)$  be a point on that graph. The  $X'Y'$  coordinate plane is obtained by translating the original coordinate plane by a vector  $(a, b)$  for which the origin of the new coordinate plane has coordinates  $(a, b)$  in relation to the  $XY$  coordinate plane. The coordinates of  $P$  in relation to the  $X'Y'$  coordinate plane is  $(x - a, y - b)$ . Thus, the graph of the function  $f$  is of the form  $y - b = f(x - a)$  in relation to the  $X'Y'$  coordinate plane, which can be written as  $y = f(x - a) + b$ . Other transformations of function can be thought of in a similar way. However, the approach of transforming axes was not frequently used by teachers (Zazkis et al., 2003). Both transformational approaches can avoid the "counterintuitiveness" and "inconsistency" that students frequently experience when dealing with function transformations.

Guided by the notion of OTL and the literature on function transformations, this study aimed to answer the following questions:

1. What are the opportunities to learn function families and their representations afforded by the GeoGebra applets on function transformations?
2. What are the opportunities to learn the effects and components of transformations afforded by the GeoGebra applets on function transformations?
3. To what extent is scaffolding used in the GeoGebra applets to support the opportunity to learn function transformations?

## Methodology

### Data Collection

GeoGebra applets on function transformations were collected via the built-in search engine provided by GeoGebra in the classroom resources section of their website. The following keywords were used to identify GeoGebra applets on function transformations: *Function Transformation, Function Translation, Function Shift, Function Stretch, Function Shrink, Function Dilation, and Function Rotation*. First, all files were observed and sorted for direct duplication. If a file was a direct duplicate of another file, it was omitted from the data so that every file would be represented once. Direct duplicates were identified in one of two ways. The first way was that the file said “copy of” in the name and the original file was identified apart from the copy. The second way a file was identified as a direct duplicate if the title was changed from the default “copy of” yet the author went unchanged, and the rest of the file was identical to another file the author made. Occasionally, these files also had additional authors, but since there were no changes to the file created by the first author they were also omitted. Second, all files unrelated to function transformations (e.g., transformations of geometric shapes) were also omitted. This process resulted in 347 unique GeoGebra applets on function transformations.

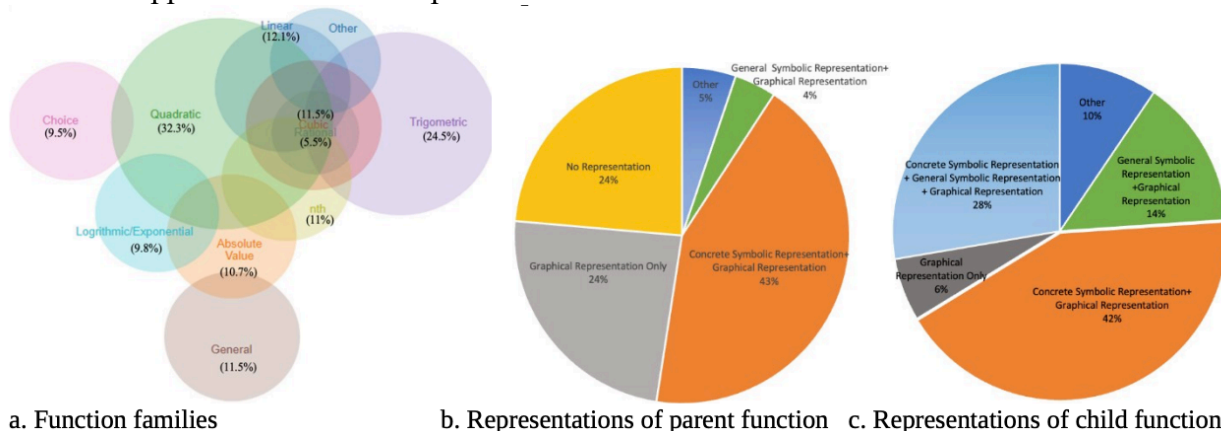
### Data Analysis

Building on the literature on function transformation, a coding framework was developed to analyze the opportunity to learn function transformations afforded by each GeoGebra applet. The framework considers the types of functions in which function transformations are explored, the types of representations of the parent and child functions, the effects of transformational motions, the visibility of corresponding points, the visibility of defining parameter(s) of transformations, and the types of scaffolding. The types of functions include specific function families (e.g., linear, quadratic, cubic, absolute value,  $n^{\text{th}}$  root, rational, exponential, logarithmic, trigonometric, parametric, and step-wise) that are given in the applet, a function that is represented with a general form of function notation (e.g.,  $g(x) = a \cdot f(b \cdot (x - h)) + k$ ), and function that can be entered by its user. The types of representations can be graphical, concrete symbolic (e.g.,  $f(x) = x^2$ ) general symbolic (e.g.,  $f(x + h)$ ), and tabular. It is also possible that no representation of a parent function is given. The effects of transformational motions include horizontal and vertical translation, horizontal and vertical dilation, reflection, and rotation. The coding framework also considers whether these effects of transformations are explicitly noted in a GeoGebra applet. Visibility of corresponding points on the parent and child functions considers whether corresponding points on the parent and child functions are shown in a GeoGebra applet. Visibility of defining parameter(s) of transformations captures whether the defining parameter(s) (e.g., translation vector, line of reflection, and center and scale of dilation) of a transformation are shown in a GeoGebra applet. The types of scaffolding include scaffolding for understanding function transformations, scaffolding for how to interact with a GeoGebra applet, and no scaffolding. It is important to note that many GeoGebra applets in our data set use different function families and multiple representations to explore different types of function transformations. When analyzing these GeoGebra applets, we coded all the function families, representations, and types of function transformations that are afforded by them. Using the coding framework, two researchers first coded the 347 GeoGebra applets independently. Disagreement only occurred in 18 GeoGebra applets and was resolved through discussion.

## Results

### Opportunity to Learn Functions and their Representations

As shown in Figure 1a, most of the GeoGebra applets include more than one function family by creating check boxes that show/hide different functions. More importantly, particular function families appear in more GeoGebra applets than others. Specifically, 32.3% of the GeoGebra applets use quadratic functions to explore function transformations followed by trigonometric functions (24.5%), linear functions (12.1%), cubic functions (11.5%), general functions (11.5%; e.g.,  $g(x) = a \cdot f(b \cdot (x - h)) + k$ ),  $n^{\text{th}}$  root functions (11.0%), absolute value functions (10.7%), exponential and logarithm functions (9.8%), and rational function (5.5%). 9.5% of the GeoGebra applets also allow their users to enter functions of their own choice. Other functions (i.e., stepwise, parametric, and higher degree polynomial functions) are only included in a very small number of GeoGebra applets (1%~3%). This indicates that more existing GeoGebra applets choose to explore function transformations in the context of quadratic and trigonometric functions. It is worth noting that the parent functions in most GeoGebra applets are in their simplest form (e.g.,  $f(x) = x^2$ ,  $f(x) = \sin x$ , and  $f(x) = |x|$ ). Rarely, a parent function provided by the GeoGebra applets is not in its simplest form.

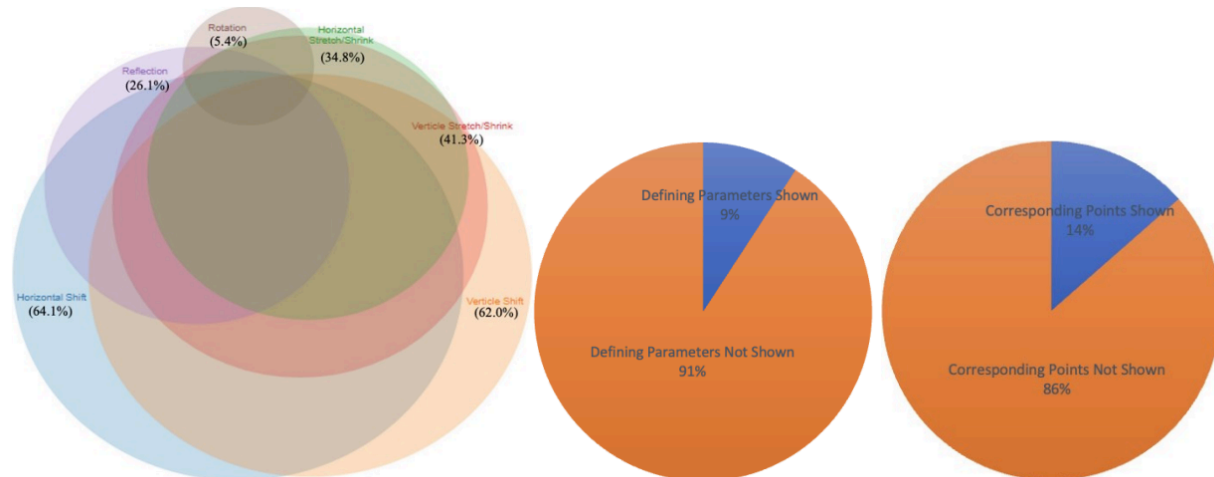


**Figure 1: Function Families and Their Representations in the GeoGebra Applets**

Regarding the representations of a parent function, 43% of the GeoGebra applets use and only use concrete symbolic and graphical representations, 24% use only graphical representations, and 4% use and only use general symbolic and graphical representations. There are 24% of GeoGebra applets that has no representation of the parent function, which means that the parent function is not visible in these applets when the function graph is transformed. This often occurs when the GeoGebra applets only include the graph of a child function and sliders that control its coefficients. Regarding the representations of a child function, 42% of the GeoGebra applets use and only use concrete symbolic and graphical representations, 28% use and only use concrete symbolic, general symbolic, and graphical representations, and 14% use and only use general symbolic and graphical representations. A comparison of Figures 1b and 1c shows the use of only graphical representation in the parent function is more prominent than those in the child function. It also shows the use of general symbolic and graphical representations as well as the combination of concrete symbolic, general symbolic, and graphical representations is more prominent in the child function than those in the parent function.

## Opportunity to Learn the Effects and Components of Transformations

73.5% of the GeoGebra applets do not specify the function transformations to be explored in the applets. These GeoGebra applets often allow their users to see the movements of a function graph by manipulating sliders but leave the users to name the specific transformations controlled by each slider. 21% of the GeoGebra clearly state the specific transformations and only allow these transformations to be explored. 5.5% of the GeoGebra applets specify particular function transformations to be explored but also allow their users to see the effects of other transformations. For instance, a GeoGebra file with  $y = b \cdot f(x)$  in which  $b$  is controlled by a slider that takes values from  $-5$  to  $5$  would allow its users to see the effects of reflection over the  $x$ -axis although the file might only target vertical shrink and stretch. Of the 26.5% of the GeoGebra applets that include specified function transformations, 64.1% include horizontal translation, 62.0% include vertical translation, 41.3 % include vertical stretch and shrink, 34.8 % include horizontal stretch and shrink, 26.1% include reflection over  $x$ -axis or  $y$ -axis, and 5.4 % include rotation. This indicates that more GeoGebra applets provide students with opportunities to learn function translation followed by the opportunity to learn vertical and horizontal stretch and shrink. Only a small number of GeoGebra applets provide students with opportunities to learn the effect of reflection and rotation of a function. Figure 2a also shows that most GeoGebra applets allow their users to explore more than one function transformation. This is typically done by creating multiple sliders that control the coefficients of a function and/or check boxes that show/hide different transformations.



a. Types of function transformations    b. Visibility of defining parameters    c. Visibility of corresponding points

**Figure 2: Effects and Components of Transformations in the GeoGebra Applets**

When analyzing the opportunity to learn components of transformations, we consider the visibility of both the defining parameters of the transformations and the corresponding points on the graphs of the parent and child functions. Only 9% of the GeoGebra applets show the defining parameters of a transformation (e.g., translation vector, line of reflection, or the center and scale of dilation) performed on the graph of function under exploration (Figure 2b). It is worth noting that among these GeoGebra files 75% of them are on function translation and therefore have the translation vector shown. Only in a few GeoGebra applets, the defining parameters of other transformations are shown. Only 14% of the GeoGebra applets show at least one pair of corresponding points on the graphs of the parent and child functions (Figure 2c).

## Scaffolding to Learn Function Transformations

Only 18% of the GeoGebra applets provide scaffolding that aims to support their users to understand the targeted function transformations. This type of scaffolding varies in the amount of guidance. It can be short questions (e.g., What happens to the graph as  $k$  gets larger than 1? Is this a stretch or a shrink?) or a well-structured sequence of questions or activities that engage users in thinking about and reflecting on their interaction with the GeoGebra applet.

33% of the GeoGebra applets only describe the files and/or a direction for how users should interact with the applets. Here are some examples of this type of direction. “Use the sliders to adjust the values of  $a$ ,  $b$ ,  $c$ , and  $d$ . The first checkbox switches between a sine and a cosine graph. Click the other checkboxes to see the steps for graphing a function like this”. “Use the sliders in the green box to adjust the values of  $a$ ,  $w$ , and  $k$ . The gray graph is the graph of the parent function  $y = f(x)$ . The red graph is the graph of the transformed function  $y = a \cdot f(x + w) + k$ . To work with a different function, click the *Change Parent Function* button, and then click on the name of the function that you would like to use”. This type of direction does not directly engage users in thinking about function transformations. 49% of the GeoGebra applets provide no direction or scaffolding for their users.

## Discussion and Conclusion

This study aimed to examine the opportunity to learn function transformations afforded by the GeoGebra applets available on the GeoGebra website. The results show that quadratic function (often in its simplest form) is used by the largest number of GeoGebra applets to explore function transformations. Although an important concept in secondary school mathematics, quadratic function in its simplest as a parent function might not be ideal for exploring either vertical and horizontal shrink and stretch of the graph of a function or reflection of the graph of a function over the y-axis. Similarly, although it is difficult to differentiate between horizontal and vertical translations in the context of a linear function, a number of GeoGebra applets still use a linear function to explore function translation. This suggests that it is important to consider the choice of functions through which function transformations are explored. The characteristics of a function should not obscure the properties of the targeted function transformations. Moreover, the results also show that the parent and/or child functions are often provided and only a small number of GeoGebra applets allow their users to enter a function of their own choice or use a general function (Figure 1a). This raises the question of the extent to which these GeoGebra applets allow students to develop an understanding of function transformations that go beyond the function contexts in which the understanding is developed. Future design of GeoGebra applets on function transformations might consider providing users the opportunity to choose their functions to explore function transformations.

The results show that only a small number of GeoGebra applets include at least one pair of corresponding points on the graphs of the parent and child functions and clearly indicate the defining parameters of the transformations under exploration (Figures 2b and 2c). This suggests that the majority of the GeoGebra applets do not afford their users to attend to properties of transformations that act on the graph of a parent function. In fact, the majority of the GeoGebra applets focus on showing the connections between the change in the parameters of the symbolic expression a function and the movement of its graph (Figures 1b and 1c), suggesting a graphical approach to function transformations (Heid, Wilson, & Blume, 2015). Although this approach allows students to easily observe the function transformation rules in particular under the aid of technology (Göbel, 2021), it does not support students to understand why the rules work. It has been argued that mathematics action technology should serve not only as an amplifier but also as a

reorganizer (Pea, 1985; Sherman, 2014). To use technology as a reorganizer, we should consider how technology can be used to spark conceptual change in students' understanding (Roschelle, Noss, Blikstein, & Jackiw, 2017). Traditionally function transformations are presented in the context of exploring functions, in which graphical and tabular approaches are often used to introduce function transformation rules. The use of dynamic mathematical environments makes it easy to graph a function and perform transformation actions on the function graph or points on the graph. As a result, it supports the approach to function transformations proposed by Zazkis and colleagues (2003). Following this approach, GeoGebra applets can be designed to support students to develop a conceptual understanding of function transformations. These GeoGebra applets should include carefully designed scaffolding that supports students to articulate their thinking and reflect on their interaction with technology.

### References

- Baker, B., Hemenway, C., & Trigueros, M. (2001). On transformations of basic functions. In H. Chick, K. Stacey, & J. Vincent (Eds.), *Proceedings of the 12th ICMI Study Conference on the Future of the Teaching and Learning of Algebra* (Vol. 1, pp. 41–47). University of Melbourne.
- Borba, M. C., & Confrey, J. (1996). A student's construction of transformations of functions in a multiple representational environment. *Educational Studies in Mathematics*, 31(3), 319-337.
- Boz-Yaman, B., & Yigit Koyunkaya, M. (2019). Examination of Pre-Service Mathematics Teachers' Mental Constructions of Function Transformation. *Acta Didactica Napocensia*, 12(1), 33-56.
- Brewer, D. J., & Stasz, C. (1996). Enhancing opportunity to learn measures in NCEs data. Santa Monica: RAND.
- Duval, R. (2017). *Understanding the mathematical way of thinking-The registers of semiotic representations*. Springer.
- Göbel, L. (2021). *Technology-assisted guided discovery to support learning: Investigating the role of parameters in quadratic functions*. Springer.
- Goos, M. (2014). Creating opportunities to learn in mathematics education: A sociocultural perspective. *Mathematics Education Research Journal*, 26, 439-457.
- Günster, S. M., & Weigand, H. G. (2020). Designing digital technology tasks for the development of functional thinking. *ZDM*, 52(7), 1259-1274.
- Hall, B. & Giacini, R. (2013). Exploring function transformations using the common core. *Mathematics Teacher*, 107(2), 132–137.
- Heid, M. K., Wilson, P. S., & Blume, G. W. (Eds.). (2015). *Mathematical understanding for secondary teaching: A framework and classroom-based situations*. IAP.
- Kieran, C. (2019). Task design frameworks in mathematics education research: An example of a domain-specific frame for algebra learning with technological tools. In G. Kaiser, & N. Presmeg (Eds.), *Compendium for early career researchers in mathematics education, ICME-13 monographs* (pp. 265–287). Springer.
- Kimani, P. M. (2008). *Calculus students' understandings of the concepts of function transformation, function composition, function inverse, and the relationships among the three concepts* (Doctoral dissertation). Syracuse University.
- Lage, A. E. & Gaisman, M. T. (2006). An analysis of students' ideas about transformations of functions. In A. S., Cortina, J.L., Sáiz, and Méndez, A.(Eds), *Proceedings of the 28th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 68-70). Mérida, México: Universidad Pedagógica Nacional,
- Leung, A. & Bolite-Frant, J. (2015). Designing mathematics tasks: The role of tools. In A. Watson & M. Ohtani (Eds.), *Task design in mathematics education: An ICMI Study 22* (pp. 191-225). New York: Springer.
- Lloyd, G., Beckmann, S., Zbiek, R. M., & Cooney, T. (2010). *Developing Essential Understanding of Functions for Teaching Mathematics in Grades 9-12*. Reston, VA: National Council of Teachers of Mathematics.
- Dick, T., & Hollebrands, K. F. (Eds.) (2011). *Focus in high school mathematics: Technology to support reasoning and sense making*. Reston, VA: National Council of Teachers of Mathematics.
- National Governors Association Center for Best Practices (NGA Center) and Council of Chief State School Officers (CCSSO).2010. *Common Core State Standards for Mathematics*. Washington, DC: NGA Center and CCSSO.
- Pea, R. (1985). Beyond amplification: Using the computer to reorganize mental functioning. *Educational Psychologist*, 20(4), 167–182.

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

- Roschelle, J., Noss, R., Blikstein, P., & Jackiw, N. (2017). Technology for learning mathematics. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 853–876). Reston, VA: National Council of Teachers of Mathematics.
- Sherman, M. (2014). The role of technology in supporting students' mathematical thinking: Extending the metaphors of amplifier and reorganizer. *Contemporary Issues in Technology and Teacher Education*, 14(3), 220-246.
- Walkowiak, T. A., Pinter, H. H., & Berry, R. Q. (2017). A reconceptualized framework for “opportunity to learn” in school mathematics. *Journal of Mathematics Education at Teachers College*, 8(1), 7-18.
- Yim, J., & Lee, H. S. (2022). Students' use of prior conceptions of symbols in finding an equation for a horizontal translation of the graph of a function. *International Journal of Science and Mathematics Education*, 20, 1699-1711.
- Zazkis, R., Liljedahl, P., & Gadowsky, K. (2003). Conceptions of function translation: Obstacles, intuitions, and rerouting. *The Journal of Mathematical Behavior*, 22(4), 435-448.

## STUDENT ENGAGEMENT, UNDERSTANDING, AND STEM INTEREST IN A GAME BASED SUPPLEMENTAL FRACTION CURRICULUM

Jessica H. Hunt  
North Carolina State  
University  
Jhunt5@ncsu.edu

Michelle Taub  
University of Central Florida  
Michelle.taub@ucf.edu

Matthew Marino  
University of Central Florida  
Matthew.marino@ucf.edu

Kenneth Holman  
University of Central Florida  
Kenneth.homan@ucf.edu

Alejandra Duarte  
North Carolina State  
University  
Aduarte2@ncsu.edu

Brianna Bentley  
North Carolina State  
University  
blbentle@ncsu.edu

*We analyzed the effects of a game-based, supplemental fraction curriculum on fourth and fifth grade students' fraction knowledge, engagement, and STEM interest. Students with and without disabilities with intersecting identities (e.g., race, disability status, gender) comprised the sample. Results indicate significant differences in fraction concept knowledge as a result of the curriculum for all students, but not STEM interest. Furthermore, engagement was a significant predictor of STEM post test scores, but not fraction concept post test scores. Implications of the results in the context of previous research on game-based mathematics curriculums are shared.*

Keywords: engagement, fractions, disability, curriculum

The study reported here addresses a game-based supplemental fraction curriculum and aligns with two central questions of the PMENA conference: (a) What design features for tools and curricula consider supporting engagement, interest, and learning for all students? (b) How might learning environments that take all students into account impact engagement, interest, and learning?

### Literature Review

*Engagement* is often described as “participation in activity with some cognitive or affective investment” with an “inseparability of learning from the engagement through which learning takes place” (Middleton et al., 2017, p. 668). Most historical and contemporary research quantitatively defines engagement as *traits* that students evidence at set points in time or discusses engagement as sets of behaviors teachers can model for students (Klem & Connell, 2009; McLeskey et al., 2017). Yet, students' engagement can also be observed or documented during the processes of learning, or as *states*. We posit that research that examines engagement as *states* can yield important contributions to the literature and can be of particular benefit to diverse student populations for which research on engagement is sorely missing, such as students with disabilities (McKlesky et al., 2017). For example, researchers can investigate if these students' engagement differs across contexts and how different levels of engagement coincide with outcomes, such as learning and STEM interest.

Digital game-based mathematics curricula have gained increased prevalence as a means to improve students' engagement, STEM interest, and learning outcomes over the past two decades (Sies, 2018). For example, Lin et al. (2013) found digital games improved students' learning via problem-solving tasks within a game. Additionally, summaries of gaming research have identified the potential of games to enhance STEM content accessibility and interest, increase opportunities to learn via solving problems, and allow students to explore mathematics in ways that were



previously inconceivable, especially for students with disabilities and other marginalized populations (Marino et al, 2013).

Despite the tremendous potential of games, there is a pressing need for rigorous empirical research in terms of the effects of game-based curriculums, how games affect student engagement and STEM interest, and the extent to which engagement, understanding of math content, and interest are related. For example, Byun and Joung (2018), in a meta-analysis of 296 articles on gaming in mathematics education, found only 17 studies with sufficient statistical data to support effect size calculations. They noted many of the articles included drill and practice games and reported performance in the absence of student engagement and interest. Byun and Joung (2018) also identified a low percentage of the study authors (i.e., 7%) with a background in mathematics education.

### **Enhancing Engagement, Learning, and Interest with Universal Design for Learning**

Universal Design for Learning is a framework for the design and implementation of efficacious instructional materials. The framework is organized around nine guidelines and 31 checkpoints, organized vertically to proactively designing for learner variability through: (a) multiple means of engagement (i.e., considering how to engage students in multiple ways), (b) multiple means of representation (i.e., providing content in multiple formats), and (c) multiple means of action and expression (i.e., providing opportunities for students to demonstrate their understanding in multiple ways). King-Sears (2020) pointed out UDL-based interventions must include a flexible, purposeful design in order to engage a maximum number of learners. Marino and Basham (2013) reported that the proactive identification of barriers across physical, social/emotional, cultural, and cognitive aspects of the lesson are critical during STEM lessons.

Instruction should be intentionally planned so that it is personally challenging for all learners. When planning for learner variability, curriculum materials should consider specific considerations such as individual and group strengths, abilities, background knowledge, and motivation for participating in the learning activity. The implementation of UDL within the game-based program focused on integrating the three principles above across four instructional domains: 1) clear goals, 2) intentional planning for learner variability, 3) flexible methods and materials, and 4) timely progress monitoring. The goal of the game is to support student engagement and understanding around the idea that fractions are quantities with magnitudes determined by the multiplicative coordination of the numerator and the denominator.

We also designed the Dream2B interface intentionally for variability so that players can access each challenge in multiple ways (see Hunt et al., 2020). For example, an interactive learning environment motivates players by allowing them to customize the game based on their preferences. The player has a choice of flexible methods, materials, and analytical tools that they can use to employ individual strategies and ways of reasoning. Sandbox play supports players to create fractional quantities by partitioning, repeating, distributing, and coordinating units (Wilkins & Norton, 2018) without high stakes repercussions. These and other UDL features have been empirically shown to increase engagement and eliminate differences in performance between students with and without disabilities in middle school science classrooms (Marino et al., 2013). However, there is a pressing need to determine if the same results will occur in mathematics as well as how engagement, understanding, and interest are related within the context of the game-based program.

### **Research Questions**

This study addresses the following research questions: (1) What is the change from pre- and post-test scores of conceptual understanding of fractions before and after engaging in the game-

based curriculum? Does this relationship differ between student demographics? (2) What is the change from pre- and post-test scores of STEM interest fractions before and after playing Dream2B? Does this relationship differ between student demographics? (3) What is the relationship between pre-test and post-test conceptual understanding of fractions, pre-test and post-test STEM interest and student self-reported engagement across all game worlds, and (4) What is the association between pre-test scores, post-test scores, and student self-reported engagement?

### Methods

The current study examines a five-unit, 36 lesson fraction curriculum with a video game embedded within it. Developed using the UDL framework, the program is designed to maximize accessibility and engagement by providing fraction conceptual understanding challenges rooted in authentic STEM careers. Each curriculum lesson has three parts: Before game previews, video game, and after-gameplay discussion activities. All components strategically link to mathematics curriculum standards (NCTM).

### Participants and Setting

Program testing occurred with six 4<sup>th</sup> and 5<sup>th</sup> grade teachers and their students (n = 132) in two schools in the southeastern United States. Both schools included students with intersecting identities in terms of race, language, and neurodiversity. Twenty-one students were identified as having learning disabilities. The program was delivered in the mathematics classroom, which commonly includes 15-25 students and one teacher. Teachers and students engaged with the program over a period of nine weeks, which is considered best practice for technology-based programs (Gersten & Edyburn, 2007). Demographic information for students is given in Table 1.

**Table 1: Student Demographics**

Gender		Race				Disability*	
Female	Male	Hispanic	African American	White	Two or More	Yes	No
41%	59%	28%	21%	32%	19%	16%	84%

### Teacher Development

Teachers attended four one-half day training sessions on implementation of the curriculum. Day 1 of training opened with the purpose of the study, the logic model, and the target population. Over the second and third day of training, teachers used the game and sample student gameplay to deepen their understanding of how the core components are used to bolster student learning. On the final day, a curriculum guide was given to teachers to drive delivery of the intervention, and teachers practiced using the resource through role playing in small groups, rotating between teaching roles and student roles. Teachers then engaged in the after-game tasks, discourse, and talk moves to facilitate a sample student conversation.

### Measures

Three forms of data were gathered to assess the research questions. First, the *Engagement in Science Learning Activities* survey (Chung et al., 2016) was used to measure student cognitive, behavioral, and affective engagement across the curriculum. Participants respond on a Likert-type scale ranging from 1 (YES!) to 4 (NO!). Both Cronbach’s  $\alpha$  and the polychoric coefficients yielded acceptable reliability when using all eight scale items (0.80 and 0.85, respectively).

Second, fraction knowledge was measured before and after students engaged in the program using the *Test of Fraction Schemes* (Wilkins et al., 2013). Internal consistency reliability for the test was reported as 0.70; criterion-related validity was reported as 0.58 ( $p < 0.01$ ).

Third, we used the *Upper Elementary School (4-5) Student Attitudes Toward STEM (S-STEM)* Survey (Friday Institute for Educational Innovation, 2012) to measure changes in students' self-reported STEM interests. The S-STEM was developed as part of a National Science Foundation (NSF) funded research program and measures students' confidence and self-efficacy in STEM subjects, 21st century learning skills, and interests in STEM careers. It contains 56 items across six constructs: math attitudes (8 items), science attitudes (9 items), engineering and technology attitudes (9 items), 21st century learning attitudes (11 items), interest in STEM career areas (12 items), and 7 "About You" items that measure short-term expectations for course success and exposure to STEM careers. Responses are supported by a five-point Likert scale, with response options ranging from "strongly disagree" (1) to "strongly agree" (5). Higher scores reflect the greater perceived value of participants. Cronbach's  $\alpha$  of the S-STEM ranged from 0.84 to 0.86 for the grade 4-5 subscales and 0.89 to 0.91 for the middle high school subscales, respectively).

**Data coding and scoring.** We coded and scored two types of test scores to address our research questions. First, pre- and post-tests of conceptual understanding of fractions were scored out of the total number of questions to create a ratio score (i.e.,  $\frac{\text{correct responses}}{\text{total questions}}$ ).

We also coded and scored other variables that would serve as predictors and dependent variables for our models. This includes variables related to STEM interest, demographics, and engagement. STEM interest was scored by summing all components of the *S-STEM* questionnaire. Demographic information included: gender (dichotomized score of 1: Male or 2: Female), race (scores of 1: Hispanic, 2: White, 3: African American, or 4: Two or More Races), and disability status (dichotomized score of 1: Disability or 2: No disability). To code engagement, we created a composite score from the *Engagement in Science Learning Activities* survey combining the score from all 5 worlds.

### Data Analysis

To determine the change from pre- and post-test scores of conceptual understanding of fractions and STEM interest before and after engaging in the game-based curriculum, we ran two ANOVAs with repeated measures using time as the repeated, within-subjects factor (i.e., fraction concept understanding score; STEM interest score) and student demographics as the between-subjects factors. Significance was set at 0.05 for both ANOVA procedures.

To determine association between pre-test scores, post-test scores, and student self-reported engagement, we first ran a bivariate correlation including all pre-test and post-test items to determine the relationship between these variables. Next, we ran multiple linear regressions, with post-test fraction and STEM interest scores as dependent variables and pre and post-test STEM interest, post-test fraction score, and self-reported engagement as our predictor variables. Significance was set at 0.05 for all regressions.

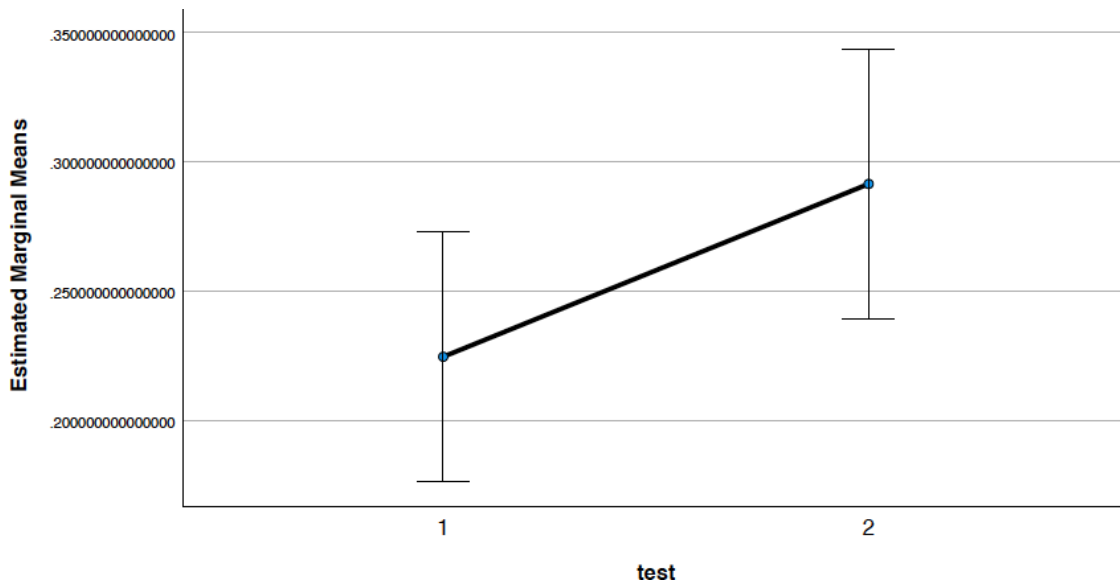
### Results

To address our research questions, we ran ANOVAs with repeated measures and multiple linear regressions to examine the relationships between pre-test scores, post-test scores, student reported engagement, and student reported demographics.

*Research Question 1: What is the change from pre- and post-test scores of conceptual understandings of fractions before and after playing Dream2B? Does this relationship differ between student demographics?* For this question, we ran an ANOVA with repeated measures

using time as the repeated, within-subjects factor and student demographics as the between-subjects factors.

Results revealed a significant, medium to large within-subjects effect for test time (eta squared = .086). Moreover, there were no significant interaction effects between time with gender, race, or disability status ( $p > .05$ ). This indicates that regardless of gender, ethnicity, or disability status, all students demonstrated a significant increase in score from pre-test to post-test. Levene's Test was significant for both pre- and post-test ( $ps < .05$ ), so we do not report the F statistic. However, results of a non-parametric Friedman test of differences among repeated measures rendered a Chi-square value of 24.43, which was also significant ( $p < .001$ ). Results are demonstrated in Figure 1.



**Figure 1: Main effect of time for fraction test score**

Research Question 2: What is the change from pre- and post-test scores of STEM interest fractions before and after playing Dream2B? Does this relationship differ between student demographics? For this question, we ran an ANOVA with repeated measures using time as the repeated, within-subjects factor and student demographics as the between-subjects factors.

Results did not reveal a small significant within-subjects effect for test time, with eta squared = .025. However, there were significant interaction effects between time\*gender, time\*disability status, time\*gender\*ethnicity, and time\*ethnicity\*disability status. This indicates the relationship between pre-test and post-test STEM scores were not significantly different while ignoring the effects of gender, ethnicity, and disability status. It is important to note that Levene's Test was significant for STEM pre-test ( $p < .05$ ), so we do not report the F statistics for the overall tests or their interactions. In addition, results of a non-parametric Friedman test of differences among repeated measures rendered a Chi-square value of .398, which was not significant ( $p > .05$ ).

Research Question 3: What is the relationship between pre-test and post-test conceptual understanding of fractions, pre-test and post-test STEM interest, and student self-reported engagement across all game world levels? To address this research question, we ran a bivariate correlation including all pre-test and post-test items to determine the relationship between these variables. We report Pearson coefficients in Table 2, below.

**Table 2: Pearson Correlations between study pre and post-test items**

	1	2	3	4	5
1-Pre-test fraction	-	.739***	.204*	.197*	-.095
2-Post-test fraction		-	.208*	.261**	-.131
3-Pre-test STEM			-	.360***	.111
4-Post-test STEM				-	.163
5-Engagement					-

\*\*\* $p < .001$ , \*\* $p < .01$ . \* $p < .05$ .

Research Question 4: What is the association between pre-test scores, post-test scores, and student self-reported engagement? To address this research question, we broke the question into several parts, as described below.

Do pre-test fraction conceptual understanding and student self-reported engagement predict post-test STEM interest? We ran a multiple linear regression with post-test STEM interest as our dependent variable and pre-test fractions score and self-reported engagement as our predictor variables. Results revealed a significant model;  $R^2 = .07$ ,  $F(2,121) = 4.54$ ,  $p < .05$ . It was found that pre-test fraction score ( $\beta = .21$ ,  $p < .05$ ) and reported engagement ( $\beta = .18$ ,  $p < .05$ ) significantly predicted post-test STEM interest. Therefore, as pre-test fraction score and self-reported engagement score increased, so did student post-test STEM interest.

Do pre-test STEM interest and student self-reported engagement predict post-test fraction conceptual understanding? We ran a multiple linear regression with post-test fraction score as our dependent variable and pre-test STEM interest and self-reported engagement as our predictor variables. Results revealed a significant model;  $R^2 = .067$ ,  $F(2,129) = 4.64$ ,  $p < .05$ . It was found that pre-test STEM interest significantly predicted post-test fraction score ( $\beta = .23$ ,  $p < .05$ ), however reported engagement ( $\beta = -.16$ ,  $p > .05$ ) did not significantly post-test fraction score. Therefore, as pre-test STEM interest increased, so did student post-test fraction score.

Do post-test fraction conceptual understanding and student self-reported engagement predict post-test STEM interest? We ran a multiple linear regression with post-test STEM interest as our dependent variable and post-test fractions score and self-reported engagement as our predictor variables. Results revealed a significant model;  $R^2 = .11$ ,  $F(2,121) = 7.13$ ,  $p < .01$ . It was found that post-test fraction score ( $\beta = .28$ ,  $p < .01$ ) and reported engagement ( $\beta = .20$ ,  $p < .05$ ) significantly predicted post-test STEM interest. Therefore, as post-test fraction score and self-reported engagement score increased, so did student post-test STEM interest.

Do post-test STEM interest and student self-reported engagement predict post-test fraction conceptual understanding? We ran a multiple linear regression with post-test fraction score as our dependent variable and post-test STEM interest and self-reported engagement as our predictor variables. Results revealed a significant model;  $R^2 = .092$ ,  $F(2,121) = 6.15$ ,  $p < .01$ . It was found that pre-test STEM interest significantly predicted post-test fraction score ( $\beta = .29$ ,  $p < .01$ ), however reported engagement ( $\beta = -.16$ ,  $p > .05$ ) did not significantly post-test fraction score. Therefore, as post-test STEM interest score increased, so did student post-test fraction score.

### Discussion and Conclusion

Overall, results reveal a significant effect of the supplemental, game-based fraction program on students' conceptual knowledge. On the other hand, the effects of the program on students' STEM

interest did not produce significant effects. Results also reveal that there is a relationship between all variables, demonstrating the relationship between student learning and performance, STEM interest, and engagement in the mathematics classroom. In addition, all significant predictors were positive relationships, such that an increase in the predictors (fraction test scores, STEM interest scores, and reported student engagement) were associated with an increase in the dependent variable. However, it is worth noting that engagement scores significantly predicted STEM interest posttest scores yet not fraction concepts post-test scores, indicating an increase in engagement was associated with higher STEM interest yet not necessarily higher fraction test- scores at post-test.

The study has limitations that need to be acknowledged. First, the engagement data (and the data on students' STEM interest) were gained via students' self-report. For engagement, the measure we used breaks down cognitive, emotional, or behavioral forms, yet we did separate out these differing forms of engagement in our analyses. Furthermore, because we view engagement as a state (as opposed to a trait), gameplay data could yield valuable information as to how students engaged with the game, however we did not have access to these data for this study.

Another point can be made about the STEM interest measure. Quantitatively, our results did not show significant changes, yet qualitative data (reported in Hunt et al., under review) did. Finally, students' conceptual knowledge, while found significant in this study, was measured via a distal measure. Future work should address these limitations. For example, future work should explore students' gameplay patterns as evidence of both engagement and conceptual growth and/or other measures that are better reflective of differencing forms of engagement and of the curriculum (e.g., curriculum-based measure). Future work might also report multiple forms of data through mixed-methods approaches that can incorporate a nested students-within-teachers structure and compare and/or merge different forms of data that address the same research questions together to gain more robust accounts of program effects on student outcomes.

Finally, because the results reported here were obtained during a feasibility study of the curriculum that did not use a control group, more work needs to be done with larger samples that encompasses a more robust design that can address the limitations of the current work.

### **Acknowledgments**

This work is supported by the National Science Foundation, grant #1949122. Any Opinions, findings and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect those of the National Science Foundation.

### **References**

- Byun, J. & Joung, E. (2018). Digital game learning for K-12 mathematics education: A meta-analysis. *School Science and Mathematics*, 118, 113-126.
- Chung, J., Cannady, M. A., Schunn, C., Dorph, R., & Bathgate, M., (2016) Measures Technical Brief: Engagement in Science Learning Activities. Retrieved from: <http://www.activationlab.org/wpcontent/uploads/2016/02/Engagement-Report-3.1-20160331.pdf>.
- Friday Institute for Educational Innovation (2012). *Student Attitudes toward STEM Survey-Upper Elementary School Students*, Raleigh, NC: Author.
- Hunt, J. H., Martin, K., Khounmeuang, A., Silva, J., Patterson, B., & Welch-Ptak, J. (2020). Design, development, and initial testing of asset-based intervention grounded in trajectories of student fraction learning. *Learning Disability Quarterly*, 0731948720963589.
- Hunt, J.H. Taub, M., Marino, M., Duarte, A., Bentley, B., Holman, K., & Kuhlman, A. (under review). Effects of game-handed supplemental fraction curriculum on student engagement, fraction knowledge, and STEM interest. *Education Sciences*.
- Gersten, R. & Edyburn, D. (2007). Defining quality indicators in special education technology research. *Journal of Special Education Technology*, Retrieved from <https://doi.org/10.1177/016264340702200302>

- Ke, F., & Abras, T. (2013). Games for engaged learning of middle school children with special learning needs. *British Journal of Educational Technology*, 44(2), 225–242.
- King-Sears, M. E. (2020). Introduction to special series on universal design for learning. *Remedial and Special Education*, 41(4), 191-193.
- Lin, C. H., Liu, E. Z., Chen, Y. L., Liou, P. Y., Chang, M., & Wu, C. H. et al. (2013). Game-based remedial instruction in mastery learning for upper-primary school students. *Educational Technology and Society*, 16(2), 271-281.
- Marino, M. T., Israel, M., Beecher, C. C., & Basham, J. D. (2013). Students and teachers perceptions of using videogames to enhance science instruction. *Journal of Science Education and Technology*. 22, 667-680.
- Marino, M. T., Becht, K., Vasquez III, E., Gallup, J., Basham, J. D., & Gallegos, B. (2014). Enhancing secondary science content accessibility with video games. *Teaching Exceptional Children*, 47(1), 27-34. doi: 10.1177/0040059914542762.
- McLeskey, J., Barringer, M-D., Billingsley, B., Brownell, M., Jackson, D., Kennedy, M., Lewis, T., Maheady, L., Rodriguez, J., Scheeler, M. C., Winn, J., & Ziegler, D. (2017, January). High-leverage practices in special education. Arlington, VA: Council for Exceptional Children & CEEDAR Center.
- Middleton, J. A., Jansen, A., & Goldin, G. A. (2017). The complexities of mathematical engagement: Motivation, affect, and social interactions. *Compendium for Research in Mathematics Education*, 667-699.
- Wilkins, J. L. M., & Norton, A. (2018). Learning progression toward a measurement concept of fractions. *International Journal of STEM Education*, 5(27). <https://doi.org/10.1186/s40594-018-0119-2>
- Wilkins, J. L. M., Norton, A. & Boyce, S. J. (2013). Validating a written instrument for assessing students fractions schemes and operations. *The Mathematics Educator*, 22(2), 31-54.

## SUPPORTING PRESERVICE SECONDARY MATHEMATICS TEACHERS TO DEVELOP THEIR TPACK KNOWLEDGE

Enrique Galindo  
Indiana University  
egalindo@iu.edu

Mehmet Fatih Öçal  
Ağrı İbrahim Çeçen  
University  
fatihocal@gmail.com

Selim Yavuz  
Indiana University  
syavuz@iu.edu

*Secondary mathematics preservice teachers enrolled in a methods class at a large Midwestern University were supported to develop their TPACK knowledge. Instructional tasks included engaging with explorations using technology to support reasoning and sense making, creating entries for a technology portfolio, studying theoretical constructs about different ways to use technology, and teaching mathematics lessons that integrate uses of technology. Portfolio entries were analyzed to look for the types of technology selected as well as for the ways in which technology was used. Preservice teachers included a variety of digital technologies in their portfolio entries, but most of the uses described aligned with ways to use technology to support reasoning and sense making. Findings suggest that the course helped preservice teachers develop their TPACK knowledge.*

Keywords: Technology; Preservice Teacher Education

The use of technology in the teaching and learning of mathematics has been considered essential for many years (NCTM, 2000). Recommendations for the preparation of mathematics teachers call for coursework that provides opportunities for preservice teachers (PSTs) to effectively use technology to engage in mathematics and statistics concepts, deepen their understanding of mathematics, and apply mathematical ideas (NCTM & CAEP, 2012; NCTM & CAEP, 2020). And that these opportunities should be present in all content areas as well as in the Mathematical Practices (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). Furthermore, PSTs should experience coursework that helps them to become proficient with tools and technology designed to support mathematical reasoning and sense making in their future teaching (AMTE, 2017). This flexible knowledge that is needed to effectively teach with technology is known as Technological Pedagogical Content Knowledge or TPACK (Mishra & Koehler, 2005; 2006). This report describes an experience in which secondary mathematics preservice teachers were supported to develop their TPACK knowledge in the context of a secondary mathematics course.

### Theoretical Perspectives

Knowing that teaching is a complex task, and that teachers and educators need guidance on how to integrate technology into the teaching of mathematics, a teacher knowledge framework for technology integration known as TPACK or Technology Pedagogy and Content Knowledge (Koehler & Mishra, 2013; Mishra & Koehler, 2005; 2006) was developed. The framework posits that effective teaching with technology requires teachers to go beyond knowledge of each of the three components of content, pedagogy, and technology, to develop a new kind of flexible knowledge that also considers the ways in which these domains interrelate (Koehler & Mishra, 2013). In order to have indicators of how the PSTs in this study were developing their TPACK knowledge, we documented the types of tools they selected and the ways in which they were using



these tools as PSTs were exploring or teaching mathematics with technology. Tools and their use were classified according to two frameworks described below.

Researchers have classified uses of technology in different ways (Dick & Hollebrands; 2011; Hoyles & Noss, 2003; 2009; Taylor, 1980). From the perspective of planning to teach mathematics we found Dick and Hollebrands' classification relevant. They classify technologies in two categories;(Koehler et al., 2013) (i) tools for the transmission/presentation of mathematical knowledge that they call *conveyance tools/technologies*) and (ii) tools for doing mathematics called *mathematical action tools*. Teachers planning to teach with technology would benefit from choosing mathematical action technologies in ways that support reasoning and sense making in mathematics class. Teachers would also benefit from using conveyance technologies in ways that support reflection and collaboration as well as monitoring and formative assessment of students thinking in mathematics.

Another classification was proposed by Pea. Pea (1985; 1987) classified the use of mathematical action or cognitive tools in two ways; (i) amplifier and (ii) reorganizer. A tool is considered an amplifier when it helps conduct a task in a faster or more efficient way, compared to doing the same task with paper and pencil only. On the other hand, the purpose of reorganizers for in-class activities is to develop students deeper understanding of mathematical concepts which cannot be done or is difficult to do without using technology. With reorganizers, exploration of interrelations among concepts (e.g., representations of functions with dynamic environments) can be carried out more meaningfully and in a brief time (Pea, 1985). We selected these two frameworks to examine the portfolio entries submitted by the PSTs.

### **Purpose**

The following research questions guided this study.

1. What is the secondary mathematics preservice teachers' knowledge about the digital tools they choose and plan to use?
2. What kind of digital tools do secondary mathematics preservice teachers choose to explore mathematical concepts? Why do they choose these technologies/tools?

### **Methods**

#### **Participants**

Participants were 15 PSTs enrolled in a secondary mathematics methods course; a course focusing on teaching mathematics in high school taken during the first semester of their Senior year. Prior to this semester they took a first methods course that focused on teaching mathematics in the middle school, they also took most of their mathematics content courses and some pedagogy and content courses.

#### **Data Sources**

Data sources were technology portfolios that the PSTs completed throughout the semester. In the portfolio entries PSTs provide evidence of their knowledge of the uses of technology and their experience using technology for mathematics learning and teaching. Entries in the portfolio can come from work they have done for their mathematics courses, their mathematics methods courses, or their field experiences. Portfolio entries can be from 1 to several pages long. They generally dedicate one section to discuss specific items, such as describing a mathematical exploration or lesson integrating the use of technology, or a technology resource that can be used to explore mathematics. Another section is used for reflecting on or discussing how the item showcases achievement of the goals for the portfolio entry. For instance, if an entry focuses on an exploration

utilizing an applet, one section describes the key features of the applet and displays a screenshot of the applet. On another section PSTs provide a description of the exploration, the learning goal, and an analysis of how exploring with the applet supported the attainment of the learning goal.

For their portfolios, PSTs needed to include entries in four categories, namely: 1) Explore mathematics with digital technologies; 2) Knowledge of resources for teaching and learning mathematics with technology; 3) Integrate technology into mathematics learning; and 4) Reflecting on the effectiveness of technology in teaching and learning mathematics. A minimum of 9 entries were required with three entries for the first category and 2 entries for each of the remaining three categories.

For this report, we analyze entries from the first two categories which can be a total of up to six entries. In the first category they had to include at least three entries about exploring mathematics concepts with technology. These entries need to span at least three of four content areas, namely: geometry, algebra, calculus, and probability and statistics. In each entry, they needed to address the following: (a) a description of the exploration task, (b) the learning goal of the exploration, (c) a description of the tool used in the exploration (applet, software, graphing calculator), (d) the way in which the tool supported their mathematical learning describing the understandings gained while exploring with the tool, and (e) compare and contrast with a similar task that could be done without the use of technology addressing what is gained with the use of technology and what is lost. In the second category they showcased their understanding of mathematical technology tools and software that are available to both students and teachers.

They demonstrate their capacity to make thoughtful decisions regarding the selection of technological resources to enhance mathematics education. Their entries can be annotated lists of resources or a critical evaluation of one technology resource, the latter being similar to a technology section of a journal or the software review section of a magazine. One entry of each type.

### Data Analysis

Portfolio entries were analyzed using the two theoretical perspectives described above. First it was noted if they were conveyance or *mathematical action* tools (Dick & Hollebrands, 2011). And then for mathematical action tools whether they were used as an *amplifier* or a *reorganizer* (Pea, 1985; 1987). For example, if the PST chose a tool to present a mathematical idea by means of a slide show, this was considered a conveyance tool. On the other hand, if the focus was to explore mathematics using a tool that provides interactive feedback based on mathematics, such as with dynamic geometry software, the digital tool was classified as a mathematical action tool. In what follows we provide some specific examples derived from PSTs entries.

In PST06's entry for exploring concepts in geometry, the participant used Microsoft PowerPoint software to present some information related to the content taught in the classroom. Since the main purpose of using this software was to present the content, the authors referred to Dick and Hollebrands' (2011) framework and the entry was coded as a conveyance tool. On the other hand, some participants considered various technological dynamic tools including Geogebra (n.d.), CODAP (2014) and Desmos (2023) in their entries. If the main aim was to do mathematics, perform the mathematical operations in a faster way, and explore the mathematical concepts and procedures in-depth, entries were coded as mathematical action tools based on the same framework. Once tools used in the portfolio entries were coded as conveyance or mathematical action tools, the authors used Pea's (1985;1987) framework to make differentiation in the usage of mathematical action tools. Technological tools could be coded as amplifiers and/or reorganizers in accordance with the purpose of use.

For example, PST03 and PST05 used CODAP to explore concepts in probability. The purpose in PST05's entry was to see different samples to calculate probabilities and see the visual results, but in a more effective and faster way. On the other hand, PST03 used CODAP to make students understand how to calculate probabilities and explore what the probability means behind the observed number. The purpose was much deeper than just using it to do mathematics in a faster and more efficient way. Therefore, this use of CODAP was coded as amplifier in PST05's entry and as reorganizer in PST03's entry.

Each entry was analyzed by two of the authors. After agreeing upon the analysis criteria and coding a few portfolio entries together to develop shared meanings, the researchers independently analyzed the remaining entries. Inter-rater reliability was calculated and found to be 94%, which is higher than the acceptable level (Marques & McCall, 2005). The codes on which the researchers disagreed were discussed to arrive at a joint code.

### Results

Tables 1 and 2 summarize the selection and use of digital technologies for each entry in the first category Explore Mathematics with Technology. PSTs included a variety of tools for their entries. The total submissions for each entry for Algebra, Geometry, Calculus, and Probability and Statistics were 13, 14, 8 and 12, respectively. Deadlines for submissions spanned throughout the semester. PSTs generally preferred to submit first for Geometry topics while they submitted Probability and Statistics topics last. An examination of the types of tools included reveals that they generally preferred online interactive platforms or websites for Algebra topics. For example, a Fractions Bar applet from Cool Math 4 Kids. The second most preferred tool was Desmos where they generally used pre-prepared online activities. For the Geometry topic, the general trend was to use dynamic geometry environments such as GeoGebra, Desmos and Geometer's Sketchpad. The Desmos software was dominant in the entry for Calculus. For Probability and Statistics, they mainly selected explorations using CODAP, an online simulation and data visualization software that includes various online pre-prepared activities for both teachers and students.

**Table 1: Entries for Explore Mathematics with Technology**

Entries / Participants	Algebra	Geometry	Calculus	Probability & Statistics
PST01*	Fraction Bars - Mathematics Learning Center Website	Geogebra	WolframAlpha	Excel/Google Sheets
PST02	GSP	Geogebra	NA	CODAP
PST03	Desmos	GeoGebra	NA	CODAP
PST04	Math Open References Website	Geogebra	NA	NCTM Illuminations
PST05	NA**	Math Learning Center	Desmos	CODAP

PST06	Fraction Bars - Mathematics Learning Center Website	Powerpoint & MathLearning Center Website	Desmos	CODAP
PST07	NA	Desmos	Desmos	NCTM
PST08	NA	Interactive Pythagoras Website	Desmos	Illuminations Microsoft Excel
PST09	Desmos	NA	Desmos	Matlab
PST10	Number-pieces- Mathematics Learning Center Website	Geometer's SketchPad	NA	NA
PST11	Fraction Bars	Geometer's SketchPad	NA	CODAP
PST12	Cool Math 4 Kids	Desmos	Desmos	Shodor Software
PST13	Desmos	GeoGebra	Desmos	CODAP
PST14	Algebra	GeoGebra	Desmos	NA
PST15	Desmos	GeoGebra	NA	NA

NA: Did not submit.

Table 2 presents the distribution of the digital technologies according to the constructs used for data analysis. Almost all digital technologies used for exploring mathematical concepts were mathematical action tools. PSTs relied on amplifiers in exploring mathematical concepts in Geometry and Calculus. On the other hand, there was a balanced distribution of digital technologies chosen by PSTs in exploring mathematical concepts in Algebra, and Probability and Statistics entries. Interestingly, there was no single reorganizer digital tool in the geometry topic. However, there were three digital tools considered both amplifier and reorganizer. From this selection, it could be interpreted that PSTs decided to use digital tools to save time and make quick calculations for exploring Geometry concepts. In addition, there were both deep explorations and easier uses of digital technologies to teach and explore intended concepts in Algebra, and Probability and Statistics topics. Considering the comparison of amplifier and reorganizer tools used in different concepts of mathematical topics, there is a variation of concepts investigated. Similar concepts in different topics were explored either with amplifiers or reorganizers with no apparent pattern.

**Table 2: Classification of Entries According to the Constructs Used for Data Analysis**

Entries for Exploring Mathematics Concepts / Selection and Use of Digital Technologies	Algebra (n=13) F (%)	Geometry (n=15) F (%)	Calculus (n=8) F (%)	Probability & Statistics (n=12) F (%)
Conveyance Tools	-	1 (6.67)	-	-
Amplifier	6 (50)	11 (73.33)	4 (44.44)	6 (50)
Mathematical Action Tools				
Reorganizer	5 (41.66)	-	3 (33.33)	6 (50)
Amplifier & Reorganizer	1 (8.33)	3 (20)	2 (22.22)	-

Table 3 shows frequency and percentages for the types of digital tools that PSTs included in their first entry for the second category *Knowledge of Resources for Teaching and Learning Mathematics with Technology*. PSTs provided descriptions for 61 digital technologies in their annotated lists (average of approximately 4.07). With 19 being conveyance tools while 35 of them were mathematical action tools. Furthermore, 7 digital technologies were both conveyance and mathematical action tools due to the intention of using such technologies. Overall, the number of mathematical action tools in the annotated lists surpassed the number of conveyance tools. As far as the second entry, critical review of a digital tool, 4 out of 14 PSTs evaluated digital tools that can be considered as conveyance ones, while 10 PSTs evaluated various kinds of mathematical action tools. One PST did not submit.

**Table 3: Types of Entries Included in the Category Knowledge of Resources**

Classification of the Digital Tools	Digital Technologies in the Annotated List (n=61) F (%)	Critically Evaluated Digital Technology (n=15) F (%)
Conveyance Tools	19 (31.15)	4 (26.67)
Mathematical Action Tools	35 (57.38)	10 (66.67)
Both Conveyance and Mathematical Action Tools	7 (11.48)	-
Empty	-	1 (6.67)

Overall, the leading digital mathematical action tool found in these entries was Desmos, an online dynamic environment that includes a graphing tool, a geometry tool, and some computer algebra system features. There were also some other dynamic environments in their entries such as GeoGebra, Geometer's Sketchpad, and GeometryPad. There were also various online interactive websites, software, simulation tools and platforms that were designed for both teachers' and students' use in mathematics. Some of the examples are Illuminations, Math PlayGround, MathWay and the National Library of Virtual Manipulatives. These were all described to be used

as mathematical action tools. As conveyance tools, the leading tool was Khan Academy, an online platform with resources, videos, presentations, and worksheets. There were also some other online conveyance tools including IXL Learning, TeacherTube, and Better Lesson.

### Conclusion

PSTs' entries regarding exploring mathematical concepts in Algebra, Geometry, Calculus, and Probability and Statistics indicated that all chosen digital technologies were mathematical action tools. Although the concepts intended to be explored vary, the fact that PSTs did not prefer to choose any conveyance tools gave evidence that there was a gradual development in their TPACK knowledge. The integration of such digital technologies also varied in exploring the intended concepts. The number of reorganizer tools were limited in such entries. For example, almost all entries in Geometry were amplifiers, while half of the total submissions in Algebra, and Probability and Statistics entries were amplifiers. The only exception where the activities included more reorganizer tools was in exploring Calculus concepts.

PSTs placed both conveyance and mathematical action tools as digital tools to teach and learn mathematics in their annotated lists of digital tools. In their annotated list of digital technologies, some PSTs mainly mentioned conveyance tools which were generally used for presentation of the documents and mathematical facts, and providing resources of assignments, questions, and fact sheets. Although some PSTs critically evaluate conveyance tools in teaching and learning mathematics, they focused more on mathematical action tools in such entries. These results suggest that it is possible to help PSTs develop their TPACK knowledge during their teacher preparation.

### Acknowledgement

This study is supported in part by the Scientific and Technological Research Council (STRC) of Türkiye, project number 1059B19210110. The opinions, findings, conclusions, and recommendations in this paper are those of the authors and do not necessarily reflect the views of the STRC.

### References

- Association of Mathematics Teacher Educators (AMTE). (2017). Standards for preparing teachers of mathematics. Author.
- Common Online Data Analysis Platform [Computer software]. (2014). Concord, MA: The Concord Consortium.
- Desmos Studio PBC (2023). Desmos. <https://www.desmos.com>
- Dick, T. P. & Hollebrands, K. F. (2011). Focus in high school mathematics: Technology to support reasoning and sense making. Reston, VA: National Council of Teachers of Mathematics.
- Geogebra GmbH (n.d.). Geogebra. <https://www.geogebra.org>
- Koehler, M., Mishra, P., & Cain, W. (2013). What is Technological Pedagogical Content Knowledge (TPACK)? *Journal of Education*, 193, 13-19. <https://doi.org/10.1177/002205741319300303>
- Marques, J. F. & McCall, C. (2005). The application of interrater reliability as a solidification instrument in a phenomenological study. *The Qualitative Report*, 10(3), 439-462. <http://www.nova.edu/ssss/QR/QR10-3/marques.pdf>
- Mishra, P. & Koehler, M. J. (2005). What happens when teachers design educational technology? The development of technological pedagogical content knowledge. *Journal of Educational Computing Research*, 32(2), 131-152.
- Mishra, P. & Koehler, M. J. (2006). Technological pedagogical content knowledge: A framework for teacher knowledge. *Teachers College Record*, 108(6), 1017-1054.
- National Council of Teachers of Mathematics (NCTM). (2000). Principles and standards for school mathematics. Reston, VA: Author.
- National Council of Teachers of Mathematics and the Council for the Accreditation of Educator Preparation (NCTM & CAEP). (2012). NCTM CAEP standards—middle grades (Initial preparation).
- National Council of Teachers of Mathematics and the Council for the Accreditation of Educator Preparation (NCTM & CAEP). (2020). Standards for the preparation of secondary mathematics teachers.
- National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010). Common Core State Standards for Mathematics. Washington, DC: Authors.

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

- Pea, R. D. (1985). Beyond amplification: Using the computer to reorganize mental functioning. 185, *Educational Psychologist*, 20(4), 167-182. [https://doi.org/10.1207/s15326985ep2004\\_2](https://doi.org/10.1207/s15326985ep2004_2)
- Pea, R. D. (1987). Cognitive technologies in mathematics education. A. H. Schoenfeld (Ed.), *Cognitive science and mathematics education içinde* (s. 89-122). Hillsdale, NJ: Erlbaum.

# TECHNOLOGY IN MATHEMATICS EDUCATION RESEARCH: ANALYSIS OF THE PAST FOUR DECADES

Eunhye Flavin  
Stonehill College  
eflavin@stonehill.edu

Ji-Eun Lee  
Oakland University  
lee2345@oakland.edu

Sunghwan Hwang  
Chuncheon National  
University of Education  
shwang@cnue.ac.kr

Keywords: Technology

This study performed a scoping review of the literature concerning the use of technology in mathematics education published in the last four decades between 1981–2022 to explore research trends. The introduction of technology in mathematics classrooms extends opportunities to engage students in investigating conceptual knowledge and communicating with their peers (Higgins et al., 2019; Roschelle et al., 2017). In 1980, The National Council of Teachers of Mathematics (NCTM) highlighted the importance of using technological tools in mathematics teaching and learning at all grade levels. Researchers in mathematics education have conducted various studies to examine features and opportunities of technology use in mathematics education. However, most of the literature reviews have examined research trends on a particular topic. Also, they analyzed less than 100 articles with manual coding methods to synthesize previous studies, which might lead to inaccurate outcomes due to a prolonged process and an insufficient number of articles.

This study aims to fill this gap and synthesize relevant studies on the use of technology in mathematics education by analyzing 2,433 articles retrieved from three research engines (*Web of Science*, *ERIC*, and *PsycInfo* databases). We employed topic modeling to automatically analyze a large corpus of text data to efficiently examine a large volume of articles (Blei, 2012). Specifically, we employed Latent Dirichlet Allocation (LDA) topic modeling to extract key terms and topics from the selected articles. The analysis focused on four aspects: (a) evolution of research trends of technology use in mathematics education, (b) frequently used words, (c) latent research topics, and (d) research trends for particular topics. This study also used the theories of the roles of technology use in mathematics education (Cullen, 2020; Drijverse, 2015) to interpret topics detected by automatic computation.

The findings revealed a steady increase in research interest, and the combination of frequently used words in the article abstracts suggests popular research topics that have been studied during the set period. The results of LDA identified seven research topics that were not precisely aligned with those identified in prior studies on mathematics education or educational technology (e.g., Topic 1. Using technology to support mathematics learning). This implied technology integration into mathematics education as a distinctive research area. Over time, the seven topics showed different research trends (stable, fluctuating, increasing, and decreasing). In the 1980s, Topic 5. ‘Teacher instruction and Technological pedagogical content knowledge’ (TPACK) were the most popular topics. However, during the 1990s and 2000s, Topic 7. ‘Examining the effect of technology on cognitive and affective domain’ was the most popular topic while the trendline gradually decreased over time and interest diminished in the 2010s. This study provided plausible reasons for these varied patterns. This study presented implications such as consideration of sociocultural theory as theoretical underpinnings for future studies.

## References

Blei, D. M. (2012). Probabilistic topic models. *Communications of the ACM*, 55(4), 77–84. <https://doi.org/10.1145/2133806.2133826>

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.



- Cullen, C. J., Hertel, J. T., & Nickels, M. (2020). The roles of technology in mathematics education. *The Educational Forum*, 84(2), 166–178. <https://doi.org/10.1080/00131725.2020.1698683>
- Drijvers, P. (2015). Digital technology in mathematics education: Why it works (or Doesn't). In S. I. Switzerland (Ed.), *Selected regular lectures from the 12th international congress on mathematical education* (pp. 1–17). Springer International Publishing Switzerland.
- Higgins, K., Huscroft-D'Angelo, J., & Crawford, L. (2019). Effects of technology in mathematics on achievement, motivation, and attitude: A meta-analysis. *Journal of Educational Computing Research*, 57(2), 283–319. <https://doi.org/10.1177/0735633117748416>
- Roschelle, J., Noss, R., Blikstein, P., & Jackiw, N. (2017). Technology for learning mathematics. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 273–296). NCTM.

## THE IMPACT OF A TECHNOLOGY METHODS COURSE ON PROSPECTIVE TEACHER TECHNOLOGICAL AND EPISTEMOLOGICAL BELIEFS

Jon D. Davis  
Western Michigan University  
jon.davis@wmich.edu

Author2 [Style: Pujiyarti] A-Author]  
Western Michigan University  
fnu.pujiyarti@wmich.edu

*This study examined the effect of a teaching mathematics with technology course on preservice elementary teachers' (PSETs) and preservice secondary mathematics teachers' (PSTs) beliefs about teaching with technology and beliefs about the nature of learning and teaching mathematics. All preservice teachers enrolled in the methods course engaged in technology lesson rehearsals, critiqued peers' technology lessons, taught a lesson involving technology in an area school, and reflected on that teaching experience. The group of PSTs had larger change values from the initial to final belief questionnaire than the group of PSETs, but both changes were statistically significant. PSETs experienced a statistically significant change from initial to final belief in five out of ten belief categories. PSTs experienced a statistically significant change from initial to final belief in seven out of ten belief categories.*

Keywords: technology, preservice teacher education, teacher beliefs

Multi-representational tools (Thurm & Barzel, 2020) currently available to mathematics teachers and their students have many powerful features including interconnected mathematical representations. Many of these tools are free and available, dependent only on an individual's or school's connection to the internet. One example of these tools is Desmos Activity Builder. Users can create an activity consisting of multiple screens each of which can contain mathematical action technologies (e.g., graphing calculator) (Dick & Hollebrands, 2011) as well as conveyance technologies (Dick & Hollebrands) that communicate information from the teacher to students and from students to the teacher while students are engaging in the activity. Nonetheless, despite this easy access to powerful technological tools built for educational purposes teachers have incorporated them into their daily instruction of mathematics in only limited ways (Drijvers, 2019).

Researchers have identified several factors such as the school culture, external assessment policies, teacher knowledge, and teacher beliefs that influence teachers' uptake of these tools (Drijvers, 2019). In this paper we investigate the beliefs of preservice elementary (PSETs) and preservice secondary mathematics teachers (PSTs) at the beginning and end of a unique teaching mathematics with technology course. In this paper we follow Philipp's (2007) definition of beliefs as "psychologically held understandings, premises, or propositions about the world that are thought to be true" (p. 259). The significance of beliefs and their impact on teacher practices regarding technology have not been lost on researchers. For example, there is a growing body of research that teachers with more traditional beliefs tend to deploy technology for low level uses (e.g., offloading calculations or graphing to technology) while those with more constructivist beliefs use technology for high level uses (using technology to develop mathematical ideas) (e.g., Judson, 2006). Thomas and Palmer (2014) surveyed 452 practicing New Zealand secondary mathematics teachers in 2005 and found that 22.4% of teachers mentioned a lack of confidence in using computers and 42.4% of teachers mentioned a lack of confidence in using calculators as obstacles in their implementation of technology. We also know that the relationship between beliefs about technology or mathematics and use of technology are not straightforward (Ertmer, Ottenbreit-Leftwich, & Tondeur, 2015). For

instance, teachers may hold both traditional and constructivist beliefs about teaching (Tondeur, Valcke, & van Braak, 2008), hold beliefs with different levels of conviction (Ertmer et al., 2012), and other variables such as the school culture may supersede teacher beliefs (Hennesy, Ruthven, & Brindley, 2005).

Thurm (2018) investigated the connections between teacher beliefs and frequency of use for 160 German teachers in a federal state where technology use is compulsory. Despite the compulsory aspect of technology use in the schools, most participating teachers had little experience with technology. He found that positive beliefs about use of technology to support multiple representations led to significant use of technology in this area. Furthermore, teachers who hold the belief that technology can support discovery of mathematical ideas not only use technology more frequently to support that discovery but also use technology in other areas such as when students practice content. On the other hand, if teachers believed that technology integration was too time consuming it led to a significant lower use in a variety of areas such as discovering and practicing mathematical ideas.

Thurm & Barzel (2020) used a beliefs questionnaire similar to the one in this study with practicing teachers to determine the effect of a professional development program on their beliefs regarding technology and epistemological beliefs. There were 39 teachers enrolled in the professional development with 88 teachers comprising a control group. The experimental group had more positive beliefs after the professional development that were statistically significant in the areas of supporting multiple representations ( $p = 0.01$ ), technology time use ( $p < 0.01$ ), and mindless working ( $p < 0.01$ ). There were no statistically significant effects regarding epistemological beliefs involving the nature of mathematics and the learning of mathematics.

Thurm & Barzel (2022) examined the connections among practicing secondary mathematics teachers' beliefs about technology, epistemological beliefs about mathematics, self-efficacy measures involving technology, and reported uses of technology (e.g., discovery learning). They found three relationships. First, they found connections among beliefs about teaching in more constructivist ways with technology, epistemological beliefs about active learning, self-efficacy beliefs about task design with technology, and modes of technology use involving discovery learning. Second, they found that beliefs about teaching with technology involving multiple representations were not connected to epistemological beliefs or self-efficacy beliefs. Third, they found that epistemological beliefs about the nature of mathematics and teacher beliefs about the risks of technology use were less central concerns for teachers teaching with technology.

Previous research involving single item belief measures have found that a technological methods courses involving field experiences had a statistically significant effect on PSTs and PSETs regarding mathematical action technologies in general and beliefs about computer algebra systems (CAS) in particular (Davis, 2020). However, this study relied on a belief questionnaire that consisted of single item measures that have questionable reliability (Gliem & Gliem, 2003). The beliefs questionnaire used in the current study is rooted in previous research about teachers and technology (e.g., Fleener, 1995) and includes connections to epistemological beliefs which others (e.g., Tharp et al., 1997) have found to be connected to teachers' use of technologies for educational purposes. This questionnaire has been used to primarily investigate practicing teachers' beliefs about technology (e.g., Thurm & Barzel, 2020), thus its use in this study with PSETs and PSTs provides important information about the beliefs of these individuals about technology as well as how those beliefs might be impacted by a unique teaching mathematics with technology course.

Three research questions guided this study.

1. What differences exist between initial and final overall beliefs questionnaire means for individual preservice elementary teachers and individual preservice secondary math teachers after experiencing a teaching mathematics with technology course?
2. What differences exist between initial and final overall beliefs questionnaire means for a group of preservice elementary teachers and a group of preservice secondary math teachers after experiencing a teaching mathematics with technology course?
3. What differences exist across ten initial and final beliefs questionnaire category means for a group of preservice elementary teachers and a group of preservice secondary math teachers after experiencing a teaching mathematics with technology course?

### **Framework**

The beliefs questionnaire used in this study consists of two sub-dimensions: beliefs about teaching mathematics with technology and epistemological beliefs about the nature of learning and teaching mathematics. Based upon previous research involving teacher beliefs and technology the beliefs about teaching mathematics sub-dimensions were broken down into six categories: skill loss (BTT-SL); mindless working (BTT-MW); prior mastery of mathematics by hand (BTT-PM); discovery learning (BTT-DL); multiple representations (BTT-MR); and time requirement (BTT-TR). For instance, previous research has identified that teachers often believe that students must first learn how to perform mathematical procedures with paper and pencil before those procedures are offloaded to technology (Fleener, 1995). The epistemological beliefs sub-dimension was broken down into four categories: nature of mathematics as rules and procedures (EB-RP); nature of mathematics as a process of inquiry (EB-I); learning mathematics through teacher direction (EB-TD); and learning mathematics through active learning (EB-AL). The benefit of this questionnaire is that it uses multi-item categories which do not suffer from reliability issues as is often the case with single item categories (Gliem & Gliem, 2003).

### **Methods**

The questionnaire at the heart of this study was composed of two sub-dimensions, beliefs about teaching mathematics with technology (Thurm, 2017) and epistemological beliefs (Blömeke et al., 2008), each of which used a Likert scale consisting of *strongly disagree*, *disagree*, *neutral*, *agree*, and *strongly agree*. The questionnaire consisting of both sub-dimensions was administered before the technology methods course began and again at the end of the course. We considered strongly disagree to be a 1, disagree a 2, and so on up to strongly agree which had a score of 5. We reverse scored items that were worded negatively leading to strongly disagree as a 5, agree as a 4, and so on. This matches the approach used by Thurm (2020). Previous research has found that the 51 items on the questionnaire exhibit good reliability and validity (Thurm, 2018). We found the mean value for each of the PTs and analyzed them by groups: PSTs and PSETs. We used a dependent two-tailed *t*-test to investigate the statistical significance of differences from initial to final administration of the questionnaire at an alpha level of 0.05. We used a dependent *t*-test as the initial and final belief scores were from the same preservice teacher and we used a two-tailed level of significance to account for the fact that beliefs associated with technology may have increased or decreased. We used *t*-values and the degrees of freedom to calculate effect size (Rosnow & Rosenthal, 2005). There were five degrees of freedom for the PSTs and nine degrees of freedom for PSETs. The dependent *t*-test rests on the assumption that the differences between initial and final beliefs questionnaire scores are normally distributed. Using the Shapiro-Wilk test of

normality on belief score differences we found that the total belief questionnaire values for PSTs and PSETs were normally distributed. The belief score differences for all ten categories for the PSETs were normally distributed. For PSTs differences across two categories violated our normality assumptions: BTT-DL and BTT-TR. For these two categories we used the Wilcoxon Signed Ranks test (Field, 2009).

A total of 16 preservice teachers (PTs) participated in this study. Six of them were PSTs while the remaining 10 were PSETs. All teacher names are gender specific pseudonyms. The study took place in a teaching mathematics with technology methods course at a medium sized university in the midwestern portion of the U.S. PTs were considered juniors or seniors with one to two additional semesters of study before beginning their student teaching internships.

The technological methods course where this research took place had several unique features. First, all PTs enrolled in the course had a field experience that involved mathematical action and conveyance technologies. The PTs were either given a lesson involving Desmos Activity Builder or worked with the instructor to create a lesson. Second, PTs rehearsed (Kazemi, Ghouseini, Cunard, & Turrou, 2016) the lesson in the technology methods course and received feedback on lesson design and lesson implementation from other PTs enrolled in the course as well as the instructor. Third, PTs taught the lesson in an area elementary, middle, or high school. Fourth, all PTs reflected on the planning, rehearsal, and field experience involving the lesson in a written assignment. These lessons involved a variety of different mathematics content areas, but always focused on a rich conceptual understanding of mathematics content (Nilsson, 2020).

### Results

Table 1 shows the average initial belief score, average final belief score, and change from initial to final belief score by individual PTs in each group. We see that the initial belief scores of the PSETs were higher than the PSTs. The PSETs also had higher final beliefs than the PSTs. The average score on the initial beliefs questionnaire for the PSTs was 3.59 which lies between neutral and agree on the Likert scale. Their final average rating of 4.08 was at the level of agree. The average initial score for the PSETs was 3.84 with the average final score of 4.16. Looking at individual PTs we see that there were three PSTs (Jerry, Matt, and Todd) and three PSETs (Phillip, Sarah, and Kate) that remained below the agree level on the final belief questionnaire. The change from initial to final belief questionnaire score for the group of PSTs was statistically significant ( $t = -5.118, p = 0.004$ ) with a large effect size of 0.92. The change from initial to final belief questionnaire score for the group of PSETs was statistically significant ( $t = -6.091, p < 0.001$ ) with a large effect size of 0.89.

**Table 1: Belief Questionnaire Means – Initial and Final by PT Group**

PT	Initial	Final	Change
PSTs			
Jerry	2.96	3.86	0.90
Julie	3.75	4.04	0.29
Matt	3.37	3.73	0.36
Paula	3.90	4.29	0.39
Joseph	4.08	4.67	0.59
Todd	3.51	3.86	0.35
<b>Item Mean</b>	<b>3.59</b>	<b>4.08</b>	<b>0.49</b>
PSETs			

Donald	3.98	4.51	0.53
Kevin	3.69	4.08	0.39
Rebecca	4.06	4.29	0.23
Anne	3.84	4.29	0.45
Cathy	3.69	4.16	0.47
Phillip	3.53	3.82	0.29
Karen	3.84	4.29	0.45
Sarah	3.76	3.94	0.18
Kate	3.82	3.84	0.02
Teresa	3.98	4.14	0.16
<b>Item Mean</b>	<b>3.84</b>	<b>4.16</b>	<b>0.32</b>

We also examined the differences by categories for each category as seen in Table 2. The smallest positive change among PSETs was in the nature of mathematics as a process of inquiry. A sample item in this category was the following. *If you engage in mathematical tasks, you can discover new things (e.g., connections, rules, concepts).* It is important to note that even though the change was very small, most of the PSETs had very high initial values in this category with a mean of 4.42. The only drops were from an initial rating of 5 (strongly agree) to a rating of 4 (agree). The second smallest change by PSETs as a group was in beliefs about teaching with technology discovery learning. A sample item in the category was the following. *Technology supports tasks where students can explore new content on their own.* The largest positive change for PSETs was in beliefs about teaching with technology prior mastery of mathematics by hand. A sample item in this category follows. *Technology may only be used by students if the mathematics the technology involves is mastered by them with pencil and paper.* Overall, the category of beliefs about teaching with technology involving multiple representations for the group of PSETs was negative but by a very small amount.

For the group of PSTs, the smallest positive change was in the category of epistemological beliefs: learning mathematics through active learning. A sample item in this category is as follows. *Time used to investigate why a solution to a mathematical problem works is time well spent.* The group of PSTs experienced a small decrease from initial to final belief questionnaire in the category nature of mathematics as a process of inquiry. The largest change for the group of PSTs was in beliefs about teaching with technology prior mastery of mathematics by hand with a movement of agree/neutral to disagree on items such as the following. *Technology may only be used to ease students' procedural work if the procedures are already mastered without technology.*

**Table 2: Belief Questionnaire Categories – Initial and Final by PT Group**

<b>Belief Category</b>	<b>Group</b>	<b>Initial</b>	<b>Final</b>	<b>Change</b>
BTT-SL	PSET	3.25	3.90	0.65
	PST	3.25	4.08	0.83
BTT-MW	PSET	3.24	3.72	0.48
	PST	2.60	3.40	0.80
BTT-PM	PSET	3.45	4.28	0.83
	PST	2.5	4.13	1.63
BTT-DL	PSET	4.34	4.40	0.06
	PST	4.23	4.63	0.40

BTT-MR	PSET	4.50	4.43	-0.07
	PST	4.17	4.74	0.57
BTT-TR	PSET	3.90	4.47	0.57
	PST	4.06	4.56	0.50
EB-RP	PSET	2.65	2.98	0.33
	PST	2.19	2.56	0.37
EB-I	PSET	4.42	4.47	0.05
	PST	4.41	4.30	-0.09
EB-TD	PSET	4.18	4.37	0.19
	PST	3.85	4.25	0.40
EB-AL	PSET	4.33	4.50	0.17
	PST	4.33	4.39	0.06

Table 3 shows the results of the paired *t*-test for each belief category for each group of PTs. We found that among the PSETs, five out of ten of our category changes were statistically significant at the 0.05 level. This included the following categories: BTT-SL; BTT-MW; BTT-PM; BTT-TR; and EB-RP. All these statistically significant values had large effect sizes of greater than 0.60 (Rosnow & Rosenthal, 2005). For the group of PSTs, changes from the initial to final belief questionnaire in the following categories were statistically significant: BTT-SL; BTT-SW; BTT-PM; BTT-MR; EB-RP; EB-I; and EB-TD. Recall that two category differences were not normally distributed so we used the Wilcoxon Signed Ranks test to examine their statistical significance. Neither the BTT-DL ( $Z = -1.947, p = 0.052$ ) nor BTT-TR ( $Z = -1.890, p = 0.059$ ) category differences for the PST group were statistically significant.

**Table 3: Paired Samples *t*-tests For Each Category and Group**

<b>Belief Category</b>	<b>Group</b>	<b>t</b>	<b>Significance</b>	<b>Effect Size</b>
BTT-SL	PSET	-3.788	$p = 0.004$	0.78
	PST	-5.000	$p = 0.004$	0.91
BTT-MW	PSET	-6.018	$p < 0.001$	0.89
	PST	-4.297	$p = 0.008$	0.89
BTT-PM	PSET	-2.984	$p = 0.015$	0.71
	PST	-7.050	$p < 0.001$	0.95
BTT-DL	PSET	-0.669	$p = 0.520$	0.22
	PST	--	--	--
BTT-MR	PSET	-.302	$p = 0.770$	0.10
	PST	-2.646	$p = 0.046$	0.76
BTT-TR	PSET	-3.431	$p = 0.008$	0.75
	PST	--	--	--
EB-RP	PSET	-2.410	$p = 0.039$	0.63
	PST	-3.081	$p = 0.027$	0.81
EB-I	PSET	-.474	$p = 0.647$	0.16
	PST	2.712	$p = 0.042$	0.77
EB-TD	PSET	-1.449	$p = 0.181$	0.43

	PST	-3.230	$p = 0.023$	0.82
EB-AL	PSET	-1.342	$p = 0.213$	0.41
	PST	-.674	$p = 0.530$	0.29

### Discussion

This study is a unique contribution to the preservice teachers' beliefs about teaching with technology body of research. It used a belief questionnaire with two sub-dimensions and a total of ten categories, each of which consisted of multiple items resulting in greater reliability and validity than single item measures (Gleim & Gleim, 2003). Moreover, this belief questionnaire had only been used with practicing teachers up to this point (e.g., Thurm & Barzel, 2022). In this study, we saw that PSETs had higher initial beliefs than PSTs in nearly every belief category except two: beliefs about teaching with technology – skill loss (BTT-SL) and learning mathematics through active learning (EB-AL). We conjecture that this may have to do with their university preparations up to the point of this technology methods course. The group of PSETs complete five mathematics courses that are taught using an active learning approach where students work in cooperative groups, engage in sense making, and discuss different mathematical approaches with their classmates. The group of PSTs, on the other hand, experienced at least five upper-level mathematics courses where the primary instructional routine was teacher lecturing and student notetaking.

In terms of sub-dimensions, there was greater change for both groups around beliefs about teaching with technology. In the sub-dimension of epistemological beliefs, both groups of preservice teachers experienced statistically significant changes in the category of the nature of mathematics as consisting of rules and procedures. We conjecture that this change occurred for both groups due to the focus of the teaching mathematics with technology course. Recall that while the lessons covered a range of different mathematics content areas from elementary through upper high school, these lessons focused on rich conceptual understanding (Nilsson, 2020) and less on using technology to teach procedures (Drijvers, 2015). While PSETs experienced reform-oriented or active learning of concepts in their mathematics courses before the course at the center of this study those courses did not focus on specifically using technology in the learning or teaching of concepts. For PSTs, technology was not used to learn concepts in their upper-level mathematics courses. Thus, it is not surprising that both groups experienced change in this category.

Thurm & Barzel (2020) found statistically significant differences in the categories of technology to support multiple representations (BTT-MR), time constraints (BTT-TR), and mindless working (BTT-MW). This study found that there were statistically significant differences in these areas for various groups of PTs in this study. There were statistically significant differences for PSTs in the category of BTT-MT, PSETs in the category of BTT-TR, and both groups in the category of BTT-MW.

As mentioned earlier, the teaching mathematics with technology course provided all students with a field experience in neighboring elementary, middle, and high schools. Many of the cooperating teachers that we worked with in these schools were former students of the instructor of the course (first author) who had completed technology beliefs surveys in earlier versions of the technology course. When the first author observed these former students' lessons involving technology, it was typically not used to develop students' conceptual understandings, nor was it used in dynamic ways as it was used in the technology course. These informal observations suggest that either other factors such as the school environment or teachers' experiences after the technology course had altered or superseded their previously strong beliefs about technology. Consequently, further research needs to be conducted to understand the nature of the beliefs of



these beginning teachers and the forces that affect these beliefs during their student internship and beginning teaching experiences.

It is difficult to construct an argument that one belief category might be greater than another, but the category with the greatest change was beliefs about teaching with technology prior mastery of mathematics by hand. In addition to the field experiences that all PTs experienced in the course the students enrolled in the course also completed journal writing addressing specific prompts. One of these prompts involved their previous experiences with technology as students in grades K-12 mathematics classrooms. They related stories where they used technology to do mathematics (Drijvers, 2015) after they had learned these procedures with paper and pencil. The significant change in this belief category suggests that the PTs may be willing to disrupt this trend and provide their students with very different experiences than they experienced themselves. Despite this strong change in beliefs, it is important to understand just what types of technology experiences these beginning teachers provide in their mathematics classrooms. Powerful technologies are now prevalent in classrooms, but if teachers do not believe that they contribute to developing students' mathematical understandings they will continue to be underused.

### References

- Blömeke, S., Müller, C., Felbrich, A., & Kaiser, G. (2008). Epistemologische Überzeugungen zur Mathematik. In S. Blömeke, G. Kaiser, & R. Lehmann (Eds.), *Professionelle Kompetenz angehender Lehrerinnen und Lehrer* (pp. 219–246). Münster: Waxmann.
- Davis, J. D. (2020). The effects of a technology course with collaborative design on prospective teachers' knowledge and beliefs. In A. I. Sacristan, J. C. Cortez-Zavala, & P. M. Ruiz-Arias (Eds.), *Mathematics education across cultures: Proceedings of the 42<sup>nd</sup> meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 2165-2173), Mexico. Cinvestav.
- Dick, T. P., & Hollebrands, K. F. (2011). Introduction to Focus in High School Mathematics: Technology to Support Reasoning and Sense Making. In T. P. Dick & K. F. Hollebrands (Eds.), *Focus in high school mathematics: Technology to support reasoning and sense making* (pp. xi-xvii). Reston, VA: National Council of Teachers of Mathematics. New York: Springer.
- Drijvers, P. (2015). Digital technology in mathematics education: Why it works (or doesn't). In S. J. Cho (Ed.), *Selected regular lectures from the 12th International Congress on Mathematical Education* (pp. 135-151). London: Springer.
- Drijvers, P. (2019). Head in the clouds, feet on the ground – A realistic view on using digital tools in mathematics education. In A. Büchter, M. Glade, R. Herold-Blasius, M. Klinger, F. Schacht, & P. Scherer (Eds.), *Vielfältige Zugänge zum Mathematikunterricht* (pp. 163–176). Springer.
- Ertmer, P. A., Ottenbreit-Leftwich, A., & Tondeur, J. (2015). Teacher beliefs and uses of technology to support 21st century teaching and learning. In H. R. Fives & M. Gill (Eds.), *International handbook of research on teacher beliefs* (pp. 403-418). New York: Routledge.
- Ertmer, P. A., Ottenbreit-Leftwich, A., Sadik, O., Sendurur, E., & Sendurur, P. (2012). Teacher beliefs and technology integration practices: A critical relationship. *Computers and Education*, 59, 423-435. 10.1016/j.compedu.2012.10.008
- Field, A. (2009). *Discovering statistics using SPSS (3rd ed.)*. Los Angeles, CA: Sage.
- Fleener, M. J. (1995). A survey of mathematics teachers' attitudes about calculators: The impact of philosophical orientation. *Journal of Computers in Mathematics and Science Teaching*, 14(4), 481-498.
- Gliem, J. & Gliem, R. (2003). *Calculating, interpreting, and reporting Cronbach's Alpha Reliability Coefficient for Likert-type scales*. Midwest Research to Practice Conference in Adult, Continuing and Community Education. Columbus, OH.
- Hennessey, S., Ruthven, K., & Brindley, S. (2005). Teacher perspectives on integrating ICT into subject teaching: Commitment, constraints, caution, and change. *Journal of Curriculum Studies*, 37(2), 155-192.
- Judson, E. (2006). How teachers integrate technology and their beliefs about learning: Is there a connection? *Journal of Technology and Teacher Education*, 14, 581-597.
- Kazemi, E., Ghousseini, H., Cunard, A., & Turrou, A. C. (2016). Getting inside rehearsals: Insights from teacher educators to support work on complex practice. *Journal of Teacher Education*, 67(1), 18-31.

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

- Nilsson, P. (2020). A framework for investigating qualities of procedural and conceptual knowledge in mathematics – An inferentialist perspective. *Journal for Research in Mathematics Education*, 51(5), 574-599.
- Philipp, R. A. (2007). Mathematics teachers' beliefs and affect. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 257-315). Charlotte, NC: IAP.
- Rosnow, R. L., & Rosenthal, R. (2005). *Beginning behavioral research: A conceptual primer (5th ed.)*. Englewood Cliffs, NJ: Pearson/Prentice Hall.
- Tharp, M. L., Fitzsimmons, J. A., & Ayers, R. L. B. (1997). Negotiating a technological shift: Teacher perception of the implementation of graphing calculators. *The Journal of Computers in Mathematics and Science*, 16(4), 551-575.
- Thomas, M. O., & Palmer, J. (2014). Teaching with digital technology: Obstacles and opportunities. In A. Clark-Wilson, O. Robutti, & N. Sinclair (Eds.), *The mathematics teacher in the digital era: An international perspective on technology focused professional development* (pp. 71-89). Dordrecht, Netherlands: Springer.
- Thurm, D. (2017). Psychometric evaluation of a questionnaire measuring teacher beliefs regarding teaching with technology. In B. Kaur, W. K. Ho, T. L. Toh & B. H. Choy (Eds), *Proceedings of the 41st conference of the International Group for the Psychology of Mathematics Education (Vol. 4, pp. 265-272)*. Singapore.
- Thurm, D. (2018). Teacher beliefs and practice when teaching with technology: A latent profile analysis. In L. Ball, P. Drijvers, S. Ladel, H.S. Siller, M. Tabach, & C. Vale (Eds.), *Uses of technology in primary and secondary mathematics education* (pp. 409-419). Cham: Springer.
- Thurm, D., & Barzel, B. (2020). Effects of a professional development program for teaching mathematics with technology on teachers' beliefs, self-efficacy and practices. *ZDM*, 52, 1411-1422. <https://doi.org/10.1007/s11858-020-01158-6>
- Thurm, D., & Barzel, B. (2022). Teaching mathematics with technology: A multidimensional analysis of teacher beliefs. *Educational Studies in Mathematics*, 109, 41-63. <https://doi.org/10.1007/s10649-021-10072-x>
- Tondeur, J., Valcke, M., & van Braak, J. (2008). A multidimensional approach to determinants of computer use in primary education: Teacher and school characteristics. *Journal of Computer Assisted Learning*, 24, 494-506. doi: 10.1111/j.1365-2729.2008.00285.x

# HOW ANIMAL MATHEMATICS HELPS HUMAN MATH ED REFORM

Thomas Rick  
Louisiana State University  
tomricks@lsu.edu

Keywords: Equity, diversity, mathematics-for-all, philosophy of mathematics, reform

This study develops practical theory for improving mathematics education (math ed) reform attempts, using Glaser's (1965) constant comparative methodology to integrate results of recent scientific research legitimizing animal mathematics (math). The field of math ed is dominated by a human-only math perspective--emphasizing Western culture's tendencies toward abstracting symbolization, reducing serialization, and exacting precision--that disadvantages many students (especially those with neurodiversity). Numerous studies in animal cognition research, however, demonstrate the surprising capacity for many animals to legitimately mathematize, including basic math competence utilizing shared phylogenetic core systems like subitizing, the approximate magnitude system, and simple arithmetic (Bongard & Nieder, 2010; Brannon, 2005; Spelke & Lee, 2012). Further, neurological research reveals many animals share with humans similar neuronal architecture to successfully mathematize even down to the level of single *number neurons* (Nieder, 2021). Sure, in carefully controlled trials, animals can demonstrate capacity for learning simple (Western) human math; however, much research illuminates animals' non-symbolic, non-serialized, fuzzy mathematizing which enhances environmental survivability. Further, this research also documents how *all humans always mathematize like animals* (regardless of age, developmental status, math capacity, or professional training), meaning that below all conscious math thinking, humans' brains mathematize in the very same ways as animals by utilizing the shared core systems and similar neurological structures (Amalric & Dehaene, 2019). For example, even while professional mathematicians perform advanced human math, fMRI brain-imaging technology reveals these seasoned human math veterans are still manifesting the very same underlying neurological signatures as animals (Kutter et al., 2018). This study uses the recent evidence legitimizing animal mathematizing- published in over 100 peer-reviewed scientific studies--to suggest improvements in math ed, including concrete steps for the more equitable inclusion of neurodiverse students (e.g., autism, dyscalculia, etc.) that struggle in today's math classrooms that emphasize symbol-laden, procedural-heavy, linearized precision. Essentially, I detail how understanding animal-like human mathematization can reform math ed curriculum, pedagogy, prof. development, research, and policy, by augmenting Western-dominated math culture with new animal-math approaches.

## References

- Amalric, M., & Dehaene, S. (2019). A distinct cortical network for mathematical knowledge in the human brain. *NeuroImage*, 189, 19-31.
- Bongard, S. & Nieder, A. (2010). Basic mathematical rules are encoded by primate prefrontal cortex neurons. *PNAS*, 107( 5), 2277-2282.
- Brannon, E. (2005). What animals know about numbers. In J. Campbell (Ed.), *Handbook of mathematical cognition* (Chapter 6, pp. 85-107). New York, NY: Psychology Press.
- Glaser, R. (1965). The constant comparative method. *Social Problems*, 12, 436-445.
- Gountas, J., & Moraes, M. (2020). Neuroimaging contributions to the understanding of neuropsychological cognitive processing for numeracy and mathematics. *Scholar J of Psychology Behavioral Sciences*, 3, 320-324.
- Kutter, E., Bostroem, J., et al. (2018). Single neurons in the human brain encode numbers. *Neuron*, 100, 753-761.
- Nieder, A. (2021). Neuroethology of number sense across the animal kingdom. *J of Experimental Bio*, 224, jeb211

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

8289. Spelke, E. S., & Lee S. A. (2012). Core systems of geometry in animal minds. *Ph Tr Royal Soc B*, 367, 2784-2793.

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

---

# EQUITY AND JUSTICE

---

# ADHD LEARNERS, COLLABORATIVE LEARNING AND MATHEMATICS: A CRITICAL LITERATURE REVIEW

Matthew Hertel  
The University of Texas at Austin  
hertelma@utexas.edu

Emma C. Gargroetzi  
The University of Texas at Austin  
egargroetzi@austin.utexas.edu

*Keywords: Students with Disabilities; Instructional Activities and Practices; Equity, Inclusion, and Diversity.*

This poster presents findings from a literature review conducted to examine access and participation, and challenge deficit framing of ADHD learners in collaborative mathematics. ADHD learners, present in nearly every K-12 classroom, are defined by the *Diagnostic and Statistical Manual of Mental Disorders* (5th ed.) as having a deficit in the area of attention (American Psychiatric Association, 2013). While we acknowledge the biological differences in neurodevelopment and impairments experienced by ADHD individuals, we also maintain that these impairments exist within an unjust, social framework that imposes normative standards of functioning on individuals who experience ADHD and other types of neurodivergence (Brown, 2014; Jurgens, 2020). The deficit framing of ADHD contributes to widespread assumptions that ADHD learners are incapable of meaningful contributions in educational settings. More broadly, disabled learners are disproportionately barred from access to agentic mathematical learning, such as found in collaborative mathematics (Tan et al., 2019).

Collaborative mathematics is a critical site for agency and authority in mathematics learning (Langer-Osuna, 2016). Extant research suggests that one of the determinants of successful collaboration is the coordination of joint attention (Barron, 2003). Yet, exceedingly little research exists investigating the participation of ADHD-learners in collaborative mathematics. Thus, we examine literature at the intersection of collaborative learning, mathematics education, and ADHD learners. We ask of this literature: 1) How is the learner framed? 2) What theories of learning are employed? 3) What methods are used? 4) What is understood as the purpose of collaborative learning? 5) Where is relevant literature located?

## Methods

As such, the literature review considers three spheres (Tan et al., 2022) of existing literature: 1) Collaborative group work; 2) Mathematics Education; 3) ADHD literature in education. The databases, *ERIC*, *Education Source*, and *APA PsycInfo* were probed using sets of common search terms. Articles were screened and coded in response to the research questions.

## Preliminary Findings

Thus far, no systematic review of literature has followed the representation of ADHD learners in collaborative mathematics. In our poster we will provide an overview of the types of studies, framings, and research journals where literature exists at the intersections of mathematics education, collaborative learning, and ADHD learners. Preliminary findings suggest conflicting views on the value of collaborative learning for ADHD learners (e.g., Saunders & Chambers, 1996) and the widespread framing of collaborative learning as a way to intervene upon presumed cognitive and social deficits in disabled learners (e.g., Kunsch et al., 2007). Through this review we call attention to the need for representation of the attentional differences and strengths brought to collaborative work by ADHD learners.

## References

- American Psychiatric Association. (2013). *Diagnostic and statistical manual of mental disorders* (5th ed.). <https://doi.org/10.1176/appi.books.9780890425596>
- Barron, B. (2003) When smart groups fail. *Journal of the Learning Sciences*, 12(3), 307-359. [https://doi.org/10.1207/S15327809JLS1203\\_1](https://doi.org/10.1207/S15327809JLS1203_1)
- Brown, L. (2014). Disability in an ableist world. *Criptiques*, 1.
- Jurgens, A. (2020). Neurodiversity in a neurotypical world: An enactive framework for investigating autism and social institutions. In *Neurodiversity Studies* (pp. 73-88). Routledge.
- Kunsch, C. A., Jitendra, A. K., & Sood, S. (2007). The effects of peer-mediated instruction in mathematics for students with learning problems: A research synthesis. *Learning Disabilities Research & Practice*, 22(1), 1-12.
- Langer-Osuna, J.M. (2016). The Social Construction of Authority Among Peers and Its Implications for Collaborative Mathematics Problem Solving. *Mathematical Thinking and Learning*, 18(2), 107–124. <https://doi.org/10.1080/10986065.2016.1148529>
- Saunders, B., & Chambers, S. M. (1996). A review of the literature on Attention-Deficit Hyperactivity Disorder children: Peer interactions and collaborative learning. *Psychology in the Schools*, 33(4), 333-340.
- Tan, P., Lambert, R., Padilla, A., & Wieman, R. (2019). A disability studies in mathematics education review of intellectual disabilities: Directions for future inquiry and practice. *The Journal of Mathematical Behavior*, 54, 100672.
- Tan, P., Padilla, A., & Lambert, R. (2022). A critical review of educator and disability research in mathematics education: A decade of dehumanizing waves and humanizing wakes. *Review of Educational Research*, 92(6), 871-910.

## BALANCE AND INTEGRATION OF CONTENT WITHIN EARLY CHILDHOOD AND ELEMENTARY SOCIAL JUSTICE MATHEMATICS LESSONS

Bailey Kaufman-Ridout  
Fordham University  
bkaufman12@fordham.edu

Meghan Riling  
Vanderbilt University  
meghan.riling@vanderbilt.edu

Khanh M. Nguyen  
Vanderbilt University  
khanh.m.nguyen@vanderbilt.edu

*With the push to incorporate the Teaching Mathematics for Social Justice framework (Gutstein, 2006) and Learning for Justice (2016) standards into early childhood and elementary settings, there is an increasing call for published social justice mathematics lessons (SJMLs) for our youngest learners. This study examines how mathematical and social justice content is supported within early childhood and elementary social justice mathematics lessons (SJMLs). We interpret three SJMLs designed for Pre-Kindergarten through second grade and analyze the implicit and explicit questions within them. Findings indicate that while published SJMLs are greatly needed in the early childhood and elementary classroom, there are inconsistencies with the structure, integration, and balance of mathematical and social justice content that makes up these lessons.*

**Keywords:** Social Justice, Curriculum, Early Childhood Education, Elementary School Education

Incorporating social justice into mathematics curriculum presents novel challenges. Therefore, lessons must be developmentally appropriate while positioning young learners as capable of participating in the intellectual work of social justice (Gutstein, 2006; Koestler et al., 2022). Published social justice mathematics lessons (SJMLs) can provide educators with an entry point into Teaching Mathematics for Social Justice (TMfSJ) pedagogy (Koestler et al., 2022). SJMLs must integrate mathematical pedagogical goals and social justice pedagogical goals in order to avoid superficial lessons that limit connections to the outside world, trivialize important social justice topics (Bartell, 2013; Kaufman-Ridout & Moldavan, 2022; Riling et al., 2022), or perpetuate the problematic notion that mathematics is apolitical and neutral (Turner, 2020).

The current TMfSJ literature is predominantly focused on secondary education, which is reflected in the amount of published secondary SJMLs (Berry et al., 2020; “Citizen Math,” n.d.; Felton-Koestler et al., 2017) and a dearth of published early childhood and elementary SJMLs. With the clear need for published SJMLs within the field, we wonder, are these published lessons or curriculum guides designed to support the stated goals of TMfSJ? Curricular analysis of the content in SJMLs has been piloted with secondary curriculum. Now, we examine early childhood and elementary curriculum at the opposite end of the grade-band spectrum. This study examines the research question: How are different types of content, especially mathematical and social justice, identified and integrated into early elementary social justice mathematics lessons?

### Conceptual Framework

For the purposes of this study, we define a *social justice mathematics lesson* as a unit of instruction within which students are intended to learn about both mathematical pedagogical goals and social justice pedagogical goals. Social justice includes matters of “access, participation, empowerment, and human rights” (Koestler et al., 2023, pg. 11). The mathematical and social justice elements may interact in different ways. For example, mathematics may be positioned as a tool for providing insight into a social justice topic, or a real-world social justice dilemma may serve as a steppingstone for investigating mathematical ideas (Barno & Dietiker, 2022; Felton-Koestler et al., 2016).



## Emerging Questions within SJMLs

In order to understand the integration of different types of content across lessons, we conceptualize lessons as stories (Dietiker, 2013, 2015; Riling et al., 2022). As stories, lessons contain a series of questions that emerge over time, which students may wonder about while experiencing an enactment of the lesson. To answer questions within a lesson, students are expected to engage in different kinds of thinking. We define a *mathematical question* as one that prompts students to mathematize, by which we mean seek patterns or form abstractions (Ernest, 2018; Gutiérrez, 2017; Su & Jackson, 2020). When responding to a mathematical question, students interpret objects and practices based on their meaning within mathematical systems. A *social question* prompts students to interpret objects within a social context. This may involve students thinking about different aspects of a social situation: material, cultural, emotional, etc. To answer a social question, a student might have to think about a person's material or emotional situation but would not be required to consider any mathematical aspects. Finally, a *hybrid question* requires students to interpret objects within both a mathematical and a social context. To answer this, a student must mathematize in addition to thinking about social phenomena while considering the material conditions represented.

### Methods

To learn more about the nature of the content of early childhood and elementary SJMLs, we analyzed three published SJMLs found in *Early Elementary Mathematics Lessons to Explore, Understand, and Respond to Social Injustice* (Koestler et al., 2022). In this section, we briefly explain our processes of lesson selection and analysis.

#### Lesson selection

We selected SJMLs from this book because it is the only available curriculum guidebook or collection of SJMLs for the early childhood setting at the time of analysis. The collection of lessons used in this study was self-identified as having an intentional inquiry-based, play-based, and project-based approach in order to maintain developmentally appropriate practices. For this first analysis, we selected the only three lessons in the book that were designed for both pre-kindergarten and elementary classrooms (see Table 1). The designers of each lesson identified the target mathematical domains and practices and social justice standards. Each lesson's associated materials also include a narrative description of the relevant mathematical concepts and social justice topics. While general mathematical themes were noted by lesson authors, we also identified more specific mathematical topics, including sorting within Lesson 5.1, *Exploring fairness*, counting and quantity comparison within Lesson 5.2, *Addressing food insecurity*, and shapes and sorting within Lesson 5.3, *Same and different*.

**Table 1: Lessons Analyzed**

Lesson title	Grade band	Mathematical focus	Social justice themes
5.1 Exploring fairness through data and numbers	PreK-K	Data Collection and Analysis	Celebrating unique identities Diverse inclusion
5.2 Addressing food insecurity	PreK-2	Number and Operations Measurement Data Collection and Analysis	Food insecurity
5.3 Same and different: An exploration of identity through geometry shapes	PreK-2	Geometry	Community across diverse groups

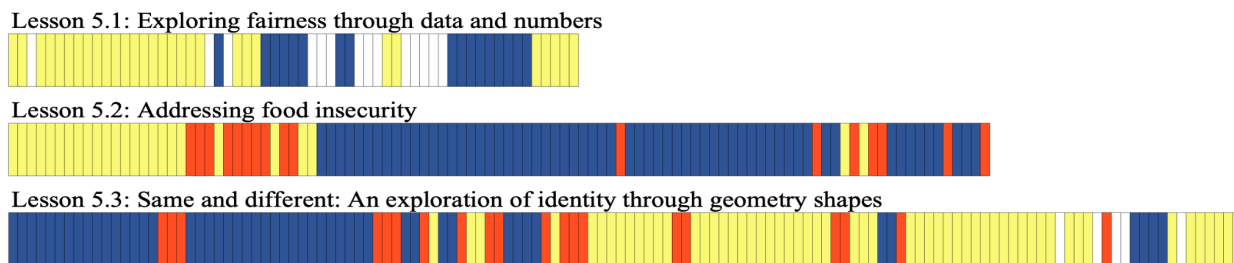
## SJML Question Analysis

To examine how the mathematical and social justice pedagogical goals are included in the SJML, we identified all questions, both explicit and implicit, that we anticipated would arise if the SJML were enacted. We listed all explicit questions written in the curricular materials, either in teacher notes or student-facing materials. Additionally, we included questions that were suggested implicitly by the lesson materials. This included questions that we anticipated would likely arise as students pursued stated lesson goals. This also included questions that we anticipated that students would wonder about during portions of the lesson in which the teacher is instructed to read a picture book. These questions were written to capture questions students would have about what would happen next in the books. We identified questions independently and then arrived at a consensus through conversation.

Next, we categorized the questions based on what kind of thinking they would require in response to the question: mathematical thinking, the consideration of social phenomenon, both, or neither. To aid these categorizations, we considered the SJMLs' stated mathematical foci and social justice topics as they relate to published mathematical content standards (NCTM, 2013; National Research Council, 2009) and social justice standards (Learning for Justice, 2016). We first categorized questions separately and then found consensus through conversation.

### Findings

We identified three common features among the analyzed lessons. To illustrate these patterns, each row of Figure 1 represents one of the analyzed lessons. Each rectangle in a row represents a question, arranged according to the order in which they arose within the written materials. The rectangles are colored according to the category of the question. Each lesson contains several single-focus stretches, in which only social or mathematical questions are introduced. Each lesson includes at least two such stretches that are at least nine questions long. The lessons often transitioned between extended single-focus sections using hybrid questions. We found that in two lessons, long mathematics sections were briefly interrupted by a single or few hybrid questions.



**Figure 1: Color-coded questions from the 3 analyzed lessons. Yellow boxes represent social questions, red represents hybrid questions, and blue represents mathematical questions. Empty (or white) boxes represent questions that are neither social nor mathematical.**

The structure of the three lessons also features several distinct differences. The implementation length of the SJMLs varies, with 5.1 and 5.2 spanning three days of instruction and 5.3 spanning six. Despite 5.1 and 5.2 spanning the same number of days, there are 61 questions in 5.1, as compared to 105 questions in 5.2, possibly indicating a difference in the rigor or engagement of the lesson. The question structure of the lessons varies with 5.1 and 5.2 beginning with long stretches of social questions, while 5.3 begins with a long mathematical question stretch (see Fig. 1). Another difference is the uneven representation of the different categories of questions (see Table

2). When considering the hybrid questions, for example, 5.1 is the only lesson that has no hybrid questions, thus preventing students from actively connecting that lesson's mathematical foci and social topics together. Even those categories of questions that do appear in all lessons are not evenly represented. Lesson 5.1 is somewhat dominated by social questions, which comprise 49.18% of the lesson's questions. Lesson 5.2 has a noticeably large amount of mathematical questions with 60.00% of the lesson's questions are mathematical, more than double the amount of the lesson's second largest category. Meanwhile, the social and mathematical categories are more balanced in Lesson 5.3; 43.18% of the questions are social, and 37.88% are mathematical. This indicates an imbalance across SJMLs between representation of pedagogical goals and integration of the content areas.

**Table 2: Percentages of question categories in each analyzed lesson**

	Lesson 5.1 Exploring fairness through data and numbers	Lesson 5.2 Addressing food insecurity	Lesson 5.3 Same and different: An exploration of identity through geometry shapes
Social	49.18%	23.81%	43.18%
Hybrid	0.00%	16.19%	15.15%
Mathematical	27.87%	60.00%	37.88%

### Discussion

Published SJMLs provide an entry point for teachers to engage in social justice work in their mathematics teaching. Our analysis suggests that while teachers who use these lessons will be exposed to lessons that present related social and mathematical topics, the common structure of long siloed blocks of mathematical-only or social-only questions prohibits meaningful interchange between the two types of content. The lack of meaningful integration is also apparent in our identification of a lesson that does not contain a single hybrid question, leaving the social and mathematical topics estranged from one another. Our categorization of questions suggests that the structure is not established in a consistent manner in early elementary SJMLs; consider the varying order of extended single-focus sections. Perhaps examining the structures in different grade bands could provoke new SJML design ideas.

By studying written SJMLs, rather than enacted lessons, we are analyzing *perceived* questions, which may differ from those that would arise in a classroom setting with participating teachers and learners. The presence of social justice topics might mean that SJMLs vary even more from classroom to classroom than mathematics-only lessons due to the political climate where the classroom is located. This points to the importance of research on enacted early childhood and elementary SJMLs.

Still, there is much to learn about the SJML curricular materials that are available to early childhood and elementary teachers and the distinct needs and possibilities of this grade band. For example, although we categorized questions by type, future work could more finely describe the content by analyzing specific social justice standards and mathematical objectives within lessons. Curricula meaningfully integrated with mathematics and social justice is needed to support both teachers in facilitating this work in their classrooms and young children engaging in rich mathematical tasks and navigating sensitive social justice topics.

## References

- Barno, E. & Dietiker, L. (2022). Opportunities in social justice mathematics curriculum: Analyzing high school algebra lessons. Paper presented at the 2022 annual meeting of the American Educational Research Association. Retrieved [2023, February 21, from the AERA Online Paper Repository.
- Bartell, T. G. (2013). Learning to teach mathematics for social justice: Negotiating social justice and mathematical goals. *Journal for Research in Mathematics Education*, 44(1), 129–163. <https://doi.org/10.5951/jresmetheduc.44.1.0129>
- Berry, R. Q., III., Conway, B. M., IV., Lawler, B. R., & Staley, J. W. (2020). *High school mathematics lessons to explore, understand, and respond to social injustice*. Corwin Press.
- Citizen math: Real world math problems. (n.d.). Citizen Math. Retrieved February 20, 2023, from <https://www.citizenmath.com/register/plans>
- Dietiker, L. (2013). Mathematical texts as narrative: Rethinking curriculum. *For the Learning of Mathematics*, 33(3), 14–19.
- Dietiker, L. (2015). Mathematical story: A metaphor for mathematics curriculum. *Educational Studies in Mathematics*, 90(3), 285–302. <https://doi.org/10.1007/s10649-015-9627-x>
- Ernest, P. (2018). The ethics of mathematics: Is mathematics harmful? *The Philosophy of Mathematics Education Today*, 187–216. Springer International Publishing. [https://doi.org/10.1007/978-3-319-77760-3\\_12](https://doi.org/10.1007/978-3-319-77760-3_12)
- Felton-Koestler, M. D. D., Simic-Muller, K., & Menéndez, J. M. (2017). *Reflecting the world: A guide to incorporating equity in mathematics teacher education*. Information Age Publishing.
- Gutiérrez, R. (2017). *Living mathematx: Towards a vision for the future*. 25.
- Gutstein, E. (2006). *Reading and writing the world with mathematics: Toward a pedagogy for social justice*.
- Kaufman-Ridout, B., & Moldavan, A.M. (2022). Using Anti-bias Picture Books to Elicit Social Justice Mathematics Lessons in Early Childhood Teach Education. In A. E. Lischka, E. B. Dyer, R. S. Jones, J. N. Lovett, J. Strayer, & S. Drown (Eds.), *Proceedings of the Forty-Fourth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, pp. 411–415. PME-NA 44, Nashville, TN.
- Koestler, C., Ward, J., del Rosario Zavala, M., & Bartell, T. G. (2022). *Early Elementary Mathematics Lessons to Explore, Understand, and Respond to Social Injustice*. Corwin Press.
- Learning for Justice. (2016). *Social justice standards: The teaching tolerance anti-bias framework*. <https://www.learningforjustice.org/sites/default/files/2020-09/TT-Social-Justice-Standards>
- National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.
- National Council of Teachers of Mathematics. (2013). *Mathematics in early childhood learning: A position of the National Council of Teachers of Mathematics*. National Council of Teachers of Mathematics.
- National Research Council. (2009). *Mathematics learning in early childhood: Paths toward excellence and equity*. National Academy Press.
- Riling, M., Dietiker, L., Yu, X., & Barno, E. (2022). The integration of mathematical and social content in secondary lessons for social justice. In A. E. Lischka, E. B. Dyer, R. S. Jones, J. N. Lovett, J. Strayer, & S. Drown (Eds.), *Proceedings of the Forty-Fourth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, pp. 469–477 PME-NA 44, Nashville, TN.
- Su, F., & Jackson, C. (2020). *Mathematics for human flourishing* (Illustrated edition). Yale University Press.
- Turner, B. O. (2020). Products of white institutional space: An analysis of whiteness in online mathematics tasks. *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, 578–581.

# POLYCULTURAL APPROACHES FOR CENTERING DIVERSITY AS A RESOURCE: THE CASE OF SOCIAL DESIGN-BASED EXPERIMENTS

Emmanuel Nti-Asante  
Florida International University  
entia001@fiu.edu

*The growing diverse nature of our STEM spaces has informed education experts to recommend use social-design-based experiments (SDBE) to design teaching-learning curriculum that frames these diversities as a resource. This call for SDBEs comes from identified challenges in existing approaches like funds of knowledge and identity. However, there is a need for a conceptual lens that educators will use to actualize this essential call in Mathematics Education. Given this, I use this proposal to explore existing approaches and their challenges to addressing diversity as a resource. I continue to develop a conceptual lens for teachers to follow in their operationalization of SDBE to design a teaching-learning curriculum that centers diversity as a resource. This conceptual lens comprises a five-phase participatory perspective and relationship to curriculum planning, designing, and enactment where teachers and students co-participate.*

Keywords: Equity, Inclusion, and Diversity, Instructional Activities and Practices, Learning Theory

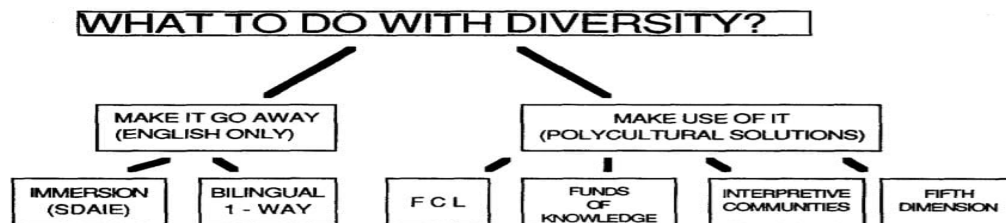
## Problem

Today, the mathematics classroom is diverse; we have many races, gender, color, ethnic groups, languages, and others. As the renowned socio-culturalist Michael Cole (1998) put it, there are two ways to deal with this diversity: make it disappear or use it as a resource (Gutiérrez et al., 2009, p.216). As shown in Figure 1, Cole (1998) reported emergent approaches like "English-only or Bilingual Education" by researchers and policymakers as an attempt to make diversity in terms of multilingual education disappear. In line with these policies in making diversity disappear, "mathematics education generates selection, exclusion and segregation" of students along the lines of gender, race, language, and socioeconomic status (Skovsmose & Valero, 2001, p. 41 cited in Foote & Bartell, 2011, p.45). On the contrary, seminal researchers researching diversity as a resource, like Hewlett et al. (2013), used their article "How-Diversity-Drives-Innovation" to detail how more diverse groups result in innovation. Thus far, researchers have developed polycultural approaches for attaining diversity as a resource (Civil, 2007; Cole, 1998; DiME, 2007; Nasir & Cobb, 2007; Adler, 2000; Trouche, et al., 2023). Cole (1998) used Figure 1 to summarize major polycultural approaches from Cultural Psychology that these researchers have used to approach diversity as a resource.

Among these polycultural approaches, much literature exists on how mathematics educators approach diversity as a resource through the funds of knowledge approach (González, et al., 2001; Hogg, 2011) and interpretive communities (Civil & Planas, 2012). In this proposal, I conceptualize how educators can use one emergent polycultural approach, Social Design Based Experiments (SDBE), to design teaching-learning curricula where they establish and incorporate diversity as a resource. I answer the research question; What are some conceptual lenses and their accompanying SDBE methodologies that can help researchers design teaching learning curricula where they establish and incorporate diversity as a resource?

In this proposal, diversity as a resource is a situation whereby policymakers and educators use polycultural approaches to rigorously study and establish the opportunity provided by the rich resources and reciprocal relations of exchange these diverse groups bring to interact in the multicultural classroom (Cole, 1998). These Polycultural approaches, useful for attaining diversity as a resource, refer to "the strategies that help educators and policymakers to recognize, establish and capitalize on the fact that multiple cultures are present in every classroom and that wherever culture-using creatures interact, they create between them a hybrid subculture, appropriate to the activities it mediates" (Cole, 1998, p.300). This hybrid subculture takes an approach that resists home and school binaries and formal and informal learning like bilingual education where students blend English with that of their home language. Instead, it focuses on what takes hold as children and youth move in and across their everyday lives in various settings and contexts (Tuomi-Gröhn & Engeström, 2003). This approach allows for identifying both possibilities and constraints within and across contexts (Cole & Engeström, 1993) where we can capitalize on the understanding of diversity as a resource through cultural dimensions of learning and development (cultural psychology) occurring as "people, ideas, and practices of different communities meet, collide, and merge" (Engeström, 2005, p. 46) to capitalize on the understanding of diversity as a result through cultural dimension of learning and development (cultural psychology).

In the remaining sections of this proposal, I provide a snapshot of the two predominant polycultural approaches in Mathematics Education. I continue with the theoretical and methodological conceptualization of SDBEs. I conclude the proposal with a call for studies where researchers operationalize the conceptualization.



**Figure 1: Polycultural Approaches**

### **Polycultural Approaches for Centering Diversity as a Resource in Mathematics Education**

Adler (2000) provided a comprehensive literature review and reconceptualization of how mathematics educators define resources. She stated that the common-sense notion of resources in and for education focuses on basic material and human resources. The basic material and human resources include the physical infrastructure in the school, like the buildings and teacher-pupil ratios or class size, and teacher qualifications, respectively (Adler, 2000). Adler argues for a provocative turn where researchers draw attention to resources and their use to question such taken-for-granted meanings. Thus, as a first conceptualization of resource in this study, Adler advocates researchers should consider resource as a verb and action (re-source) that would mean to source again or differently where they look beyond material objects and include human and cultural resources such as language and everyday practices as pivotal in school mathematical practice (Adler, 2000).

Cultural psychology and anthropology researchers have conceptualized polycultural approaches like funds of knowledge to attain this re-source. The term "funds of knowledge" refers to the historically accumulated and culturally developed bodies of knowledge and skills essential for household or individual functioning and well-being (Tapia, 1991, cited in Moll et al., 1992, p.133). In the funds of knowledge re-source project in mathematics education, the project Bridge, Civil (1994) stated that the project aimed to develop mathematics teaching innovations in which students and teachers engage in mathematically rich situations by creating learning modules that capitalize on students (and their families) knowledge and experiences in their everyday life. They reject a deficit theory model like approaches that foster disappearing of diversity for the mathematics education of students from non-dominant communities to take the sociocultural approach to education where they center it as a resource. In this approach, their research goal and questions relied on the components of the four vital interrelated models: *Household Ethnographic Analysis; Teacher-Researcher Study Groups; Classroom Implementation, and Parents as Learning Resources* (Civil, 1994, p.6). In Civil (1994), she stated that their work occurs mainly in classrooms where most Mexican students are economically disadvantaged. For example, Civil and Andrade (2002b) studied Hispanic households they visited to identify their use of mathematics, mainly in sewing and gardening.

Using a second polycultural approach to inquiry, an interpretive and critical approach, Planas and Civil (2013) provided a comprehensive literature review on how researchers approach language-as-resource. They defined this critical and interpretive to include researchers' and policy-makers attendance to diversity as a resource by encouraging mathematical participation, assuming that "all students have valuable knowledge and experiences to contribute to a mathematics discussion" (Planas & Civil, 2013, p. 366). Contrary to the English-only approach, Planas and Civil interpreted data from their two studies conducted in multilingual classrooms from Mexico and Catalan to argue that the presence of different languages in the mathematics classroom is a potential for thinking and doing, and more particularly for learning and teaching mathematics.

### **Challenges and Responses to the Operationalization of the Polycultural Approaches**

#### **Challenges**

Although these polycultural approaches have helped the mathematics education community come close to understanding diversity as a re-source, they bore the challenges that emergent approaches like SDBEs, under-conceptualized in mathematics education research, are addressing. Civil (1994) outlined some of the challenges in Project Bridge, including their fuzzy approach. Thus, they need an exemplary curriculum they could explore together or a series of activities on a specific topic in mathematics. Adler (2000) suggested a shift from these approaches that help us to "broaden a view of what such resources are to how resources function as an extension of the mathematics teacher in the teaching-learning process" p.207. Other researchers have called for the role of the student in designing the funds of knowledge. Esteban-Guitart (2021) argued that the funds of knowledge approach centers on adults and the familial practices of these students to leave out that of the students. Other challenges Esteban-Guitart (2021) identified include challenges with ethnography like timezone and cultural restrictions to researchers, and inadequate time to gather all the funds of knowledge of all learners. Concerning the interpretive approaches, Cole (1998) highlighted the need for macro impacts of such projects beyond the context.

#### **Response**

Individual researchers have responded to their identified problems with each polycultural approach. Building from their premise, Esteban-Guitart and Moll (2014) conceptualized the funds of identity construct. They used Funds of identity "to refer to the historically accumulated,

culturally developed, and socially distributed resources essential for a person's self-definition, self-expression, and self-understanding" (Esteban-Guitart & Moll, 2014, p.168). In this approach, they incorporate "identity artifacts" to overcome the challenges of ethnographic studies in the funds of knowledge approach. These identity artifacts are "documents created by the learners about themselves, in which they try to capture all the things that make sense and are meaningful to them and which, subsequently, can be used by teachers to work on curricular and pedagogical content" (Subero et al., 2018, p. 156 cited in Esteban-Guitart, 2021, p.169). Concerning Adler's call for focusing on how these resources function as an extension of the mathematics teacher in the teaching-learning process, Remillard (2005) theorized a participatory relationship and perspective to curriculum development. As used in this study, they stated that "curriculum materials refers to a set of materials designed to guide a program of instruction and student learning over time (Pepin et al., 2017b cited in Trouche et al. 2023, p.504). Contrary to canonical characterizations of curricula materials to be textbooks and technological tools, Trouche and colleagues considered curricula materials to include designed artifacts, having both visible and ideational components that bring attention to the variety of forms and languages of communication they embody and how they encode cultural messages. Wartofsky (1979) referred to these artifacts as tertiary artifacts. Hence, going forward, I will use curriculum and tertiary artifacts or artifacts to mean the same as this definition. In the development of these curriculum materials, the teacher co-participates in the designing process to bring their expertise, like teacher mathematical perceptions of curriculum, and pedagogical and mathematical knowledge, to identify which resources encapsulate the collective evolution of the teacher's expertise and intention for the planned curriculum (Trouche et al., 2023). They used this participatory perspective to offer "a second reconceptualized notion of resources used in this study and also in teaching to also include the human, ideational, linguistic, and discursive practices that mediate teacher-students' interactions with their planned and enacted curriculum, particularly in contexts where access to material resources is constrained" (Trouche, et al., 2023, p. 506). Trouche and colleagues stated that while enacting this planned curriculum, the participatory view assumes that teachers' use of the curriculum materials requires the teacher to consider how the encapsulated resource fosters teacher-student-context dialectical and mediational interactions.

### **New Directions to Addressing the Challenges**

Although these individual researchers have responded to their identified challenges with profound polycultural approaches in Mathematics Education, much remains. Esteban-Guitart (2021) referenced two studies that documented all the funds of the identity of all students, he stated that " we need procedures that could take into account and incorporate the funds of knowledge and identity of all the learners" p.169. Researchers need a conceptualization that incorporates Civil's (1994), and Cole's (1998) query of the funds of the knowledge and interpretive approaches stated above. Also, we would want the re-sources either from the familial practices of these students, based on the student's identities be a central part of the teaching-learning activities or the curriculum; what remains in most of the approaches above is how exactly how these re-sources are re-sourcing the field (extending the field to new frontiers or solving a longstanding educational problem for collective advancement of the practice). More profoundly, needful is an approach that brings all these patches of responses together and attends to the existing challenges in the funds of knowledge approach. Nazir et al., (2021) recommended SDBEs as a possible polycultural approach that reorganizes and incorporates other challenges in existing polycltural approaches to establish diversity as a resource. In the remaining sections of this proposal, I explore how Social-Design Based Experiments (SDBE) serve as this needful approach.



## **Towards Social Design-Based Experiments**

In this section I describe SDBEs, their theoretical and methodological stance.

### **What are SDBEs?**

Social Design-Based Experiments (SDBEs) involve proleptic forms of design organized around a utopian ideal and methodology (Brown & Cole, 2001; Levitas, 2013)—that is, imagining and designing an ideal ecology and outcome (Gutierrez et al., 2020, p.331). In line with centering diversity as a resource, Gutierrez and colleagues stated that;

the impetus for SDBEs was motivated by a desire for educational justice in environments and communities of interest in which learning for youth from non-dominant communities could become meaningful and consequential, expanding their social futures (p.333).

As a possibility to overcoming challenges in existing polycultural approaches, SDBEs are an advanced approach to design research organized around a commitment to transforming the educational and social injustices faced by non-dominant communities to boost social equity and learning (Gutierrez & Jurow, 2016).

### **Theoretical Perspectives of SDBEs**

Theoretically, SDBEs take a historicized, ecological approach and focus on people's everyday practices as a productive unit of analysis for understanding human activity and the learning therein (Gutiérrez et al., 2017).

They are situated within the increasing field of contemporary educational sociocultural theories (Atweh, Forgasz, & Nebres, 2001; Bartolini Bussi & Mariotti, 2008; Cantoral, 2013; Lave, 1988; Lave & Wenger, 1991; Sfard, 2008; Saxe, 2012; Stetsenko, 2017) that are profoundly intertwining cultures and what their individuals think, do, feel, imagine, hope and dream (Radford, 2021, p.27).

Radford stated they take inspiration from the works of Hegel, Marx's philosophies, the seminal work of L.S. Vygotsky and collaborators, and Freire's concept of education. Its goal is in twofold;

1. to offer a precise theoretical conception of learning as a genuine collective agentic cultural-historical process
2. to explore the practical pedagogical conditions that make genuine collective learning possible (Radford, 2021, p. xi).

Radford added that SDBE researchers draw on the work of Vygotsky and Freire to offer a theoretical perspective that changes classrooms into sites of communal life where students make the experience of ethics of solidarity, responsibility, plurality, and inclusivity (Radford, 2021).

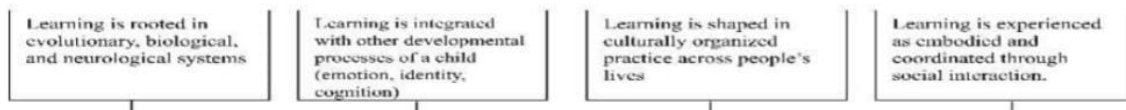
Radford stated that;

this theoretical stance they operate posits the goal of education in general, and mathematics education, as a political, societal, historical, and cultural endeavor aimed at the dialectical creation of reflexive and ethical subjects (poetic) who critically position themselves in historically and culturally constituted mathematical discourses and practices, and who ponder new possibilities of action and thinking (psoesis) p.36.

### **Methods for Designing SDBEs**

I use this section to describe the methods of designing SDBEs. These methods double as a conceptual lens to designing SDBEs teaching-learning curriculum. Methodologically, SDBE combines divergent and new solutions to identified problems in the funds of knowledge approach

to operationalize them chronologically as one. In the first phase, owing to Civil's (1994) identification of the fuzzy nature of the approach to designing for funds of knowledge, designing for SDBEs involves identifying an educational problem and taking a historical-ecological stance to inquiry as a lens to design. Thus, taking a participatory perspective to the planned curriculum discussed above, teachers deploy their expertise to identify through review of literature and survey methods to provide students with themes comprising findings from students' learning ecologies like; their funds of identity like games they play, religious and other cultural practices, personalization's like their future career (Walkington, 2012), funds of knowledge like their familial and parental occupation. For example, teachers might review literature on concepts students struggle with, they evaluate students' conceptions of this mathematical concept to identify peculiar struggles students have. They use political-historical methods to explore the historical developments of the concepts. Lastly, they engage students to identify their varying learning ecologies and a characterization of learning from the SDBE perspective by Nazir and colleagues (2021) as shown in Figure 2. These findings serve as themes for the second phase of the design.



**Figure 2: Some Description of Learning from SDBE Perspective useful as a Lens to Design**

In the second phase, teachers position all students as subjects who can see historically to transform their socio-historical circumstances and futures as learners and agents of social change. (Gutierrez et al., 2017). Hence, teachers employ extended case study methods (Burawoy, 1998), to engage students to identify the helpful artifact encapsulating the abovementioned themes. This helps us overcome challenges identified in the funds of identity approach where the struggle exists to identify all students' funds of identity or knowledge. It is at this point that students must search for an artifact from their community that encapsulates these themes that I consider the essential role of a multicultural classroom where we can establish diversity as a re-source. Thus, a third conceptualization of resource, where students from diverse backgrounds rather than a homogenous community in this multicultural classroom provide access to possible and variants artifacts. Hypothetically, searching for these historico-ecological artifactual constructs requires mostly students and practices from non-dominant groups and under-resourced environments.

In the third phase of the SDBE, students bring written narratives and images of their identified artifacts to school. Teachers, designers, and students take a participatory perspective to curricula design where they co-participate through joint activity to develop the curriculum which is the identified artifact into an enacted curriculum useful to all. Contrary to the funds of identity construct “identity artifact” this artifact is tertiary artifact and a full curriculum which holds possibilities beyond developing students’ identity to bring transformative praxis to the academy. The artifact is tertiary, thus an ‘imaginary’ artifact that has ‘lost’ its original role of representation since students often write its descriptions to become “abstracted from its use in productive praxis” and its “direct representational function” (Warsofsky, 1979, p.45). Engestrom (1990) describes them as the ‘Where-to’ artifacts since they hold visions of the future.

Learning from this SDBE perspective is considered as that of learning as a movement construct. Thus, students learn when they develop and engage with tertiary artifacts that historical

actors (teachers, students, and designers) develop to identify timescales, learning ecologies and possible futures (Gutiérrez & Jurow, 2016; Nasir et al., 2021). The meaning of context is metaphorical because SDBE researchers limit the concept of context to descriptions of the written and spoken word. This is the Vygotskian perspective of context which leads us to an understanding of ecology as interwoven and "actively achieved" (Cole, 1996, p. 134), "like tangled roots" (Packer, 2010, p. 24) through texts, rather than concentric circles (Gutiérrez, 2016)" p. 42. Hence, movement is a form of student objectification and subjectification's as they set encapsulated text into motion. Radford (2002) defines *objectification processes* as students gradually becoming acquainted with historically constituted cultural meanings and forms of reasoning and action by metaphorically setting encapsulated texts in motion. Thus, the objectification processes entail a moment of *poēsis*: a moment of "bringing forth" something to the realm of attention and understanding through a creative moment of disclosure—the event of the thing in consciousness (Radford, 2010). These processes of objectification and subjectification lead to a description of knowledge as the distinction between the potential and the actual. The potential refers to a system of social-historical-cultural entities, or as Evald Ilyenkov (2012) put it, a "complete totality of possible interpretations—those already known, and those yet to be invented" (Radford, 2015, p.17). However, Radford stated that to get to the actuality, one must set this knowledge as potential in motion to acquire cultural determinations. In general terms, knowledge moves through activity from an indeterminate form of possibilities (potential) to a determinate singularized form filled with content or concrete determinations (actual). Bringing the potential and actual together, knowledge is an archetype of actions, interpretations, reflections, etc.—a system of formal configurations (Radford, 2015). He stated that;

this form of knowledge results from and is produced through; human social labor, cultural dialectic sensuous synthesis of people's doings, and a dynamic and evolving implicit or explicit culturally codified way of doing, thinking, and relating to others and the world (Radford, 2015, p.16).

Given this, in the fourth stage, teachers implement this designed curriculum in phase three as an enacted curriculum for students to interact through dialogue (Freire, 1972).

"For Freire, dialogue is a horizontal relationship between people. It comprises communication between Subjects in a critical search or quest for something. Freire defines dialogue as the encounter between men [sic - Freire later repudiated sexist language], mediated by the world, in order to name the world and to transform the world (Freire, 1972). It is a process underpinned by values of mutual respect, humility, trust, faith, hope, love, and critical thinking (Freire, 1972; 1998)" (Rule, 2011, p.7).

During the implementation, audio and video recordings of students-students, teacher-students' interactions and dialogues help researchers identify how they use gestures, drawings, writings, to represent the historico-ecologies, deep mathematical thoughts and the transformative and extensions encapsulated in this artifact for collective use. In the fifth stage, the final stage of the method, teachers focus on space creation in the academy by employing extended case study to study the dynamic life trajectories of the curriculum to mainly describe its extension and transformative praxis through large-scale studies. They use these studies to establish the resourceability of these artifacts from diverse students to the academy.

## Conclusion

In this conceptual proposal, I provide theoretical orientation and methodological approaches which serve as a lens for the mathematics education community to center the increasingly diverse nature of our classroom as a resource that informs curricula planning, design, and enactment, which holds transformative possibilities of extending the field to new frontiers of development.

Although the field is very aware of such approaches, like the renowned funds of knowledge and identity, this emergent Social Design-Based Experiment (SDBE) I conceptualize in this proposal sets out to overcome known challenges with these existing approaches. As seminal work of the funds of knowledge approach points to the lack of a lens to design, which makes the approach fuzzy, SDBE takes a critical stance to develop themes to inform design. Thus, SDBE positions the teacher to plan the encapsulation of the curriculum as a helpful lens to inform the design of the curriculum. These lenses include teachers using findings from literature reviews to identify an educational problem, exploring students' concept conception to identify where they struggle, using surveys to identify students learning ecologies, using political-historical methods to explore the historical development of concepts and the transformative praxis this curriculum brings to the field collectively.

Contrary to the funds of knowledge approach, which is densely teacher-centered, this SDBE conceptualization allows students to participate in the curriculum design. Teachers do this by employing extended case study methods to design. Thus, they provide students with the lens to design I mentioned above, to identify the curriculum from their everyday practices and language use. Hence, the definition of curriculum used in this conceptualization comprises written narratives of students' everyday practices, artifactual designs, language, and cultural practices, which the students identify as encapsulating all the critical lenses to design stated above. Identifying such a curriculum from a homogenous group would be detrimental and challenging to design. Hence, it is out of the need to search for this curriculum encapsulating these critical lenses that I hypothesized the essence of a multicultural and multilingual classroom, thus, diversity as a resource. A heterogeneous group where students come from diverse backgrounds and use different languages provides opportunities for variant artifactual ecologies to design. Another dimension of resource emergent from this diverse group is re-source. Thus, the power of the identified curriculum is to bring the past of the development of mathematical concepts to the present through an imagined future. Although this approach to addressing diversity promises to center it as a resource in three forms, there exists the hypothetical nature of its lenses to design. In addition, planning, designing, and enactment involve longitudinal studies before their impact. Amidst these challenges, although I do not report in this proposal, I have conducted a six-month study in a Ghanaian classroom where I successfully applied the approach. I call on further studies employing this lens to center diversity as a resource in other hegemonic multicultural and multilingual contexts.

## References

- Adler, J. (2000). Conceptualizing resources as a theme for teacher education. *Journal of Mathematics Teacher Education*, 3(3), 205-224.
- Burawoy, M. (1998). The extended case method. *Sociological theory*, 16(1), 4-33.
- Civil, M. (2007). Building on community knowledge: An avenue to equity in mathematics education. *Improving access to mathematics: Diversity and equity in the classroom*, 105-117.
- Civil, M., & Planas, N. (2012). Whose language is it?: Reflections on mathematics education and language diversity from two contexts. In *Alternative forms of knowing (in) mathematics* (pp. 71-89). Brill.
- Cole, M. (1998). Can cultural psychology help us think about diversity?. *Mind, culture, and activity*, 5(4), 291-304.
- Civil, M. (1994). Connecting the home and school: Funds of Knowledge for Mathematics Teaching and Learning. Draft.

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

- Civil, M., & Andrade, R. (2002). Transitions between home and school mathematics: Rays of hope amidst the passing clouds. *Transitions between contexts of mathematical practices*, 149-169.
- DiME. (2007). Culture, race, power, and mathematics education. In F. Lester (Ed.), *Second handbook on mathematics teaching and learning* (pp. 405–434). Information Age Publishing.
- Cole, M., & Engeström, Y. (1993). A cultural-historical approach to distributed cognition. *Distributed cognitions: Psychological and educational considerations*, 1-46.
- Engeström, Y. (2005). *Developmental work research: Expanding activity theory in practice* (Vol. 12). Lehmanns media.
- Esteban-Guitart, M., Iglesias, E., Serra, J. M., & Subero, D. (2022). Community Funds of Knowledge and Identity: A Mesogenetic Approach to Education. *Anthropology & Education Quarterly*.
- Esteban-Guitart, M. (2021). Advancing the funds of identity theory: A critical and unfinished dialogue. *Mind, Culture, and Activity*, 28(2), 169-179.
- Esteban-Guitart, M., & Moll, L. C. (2014). Funds of identity: A new concept based on the funds of knowledge approach. *Culture & Psychology*, 20(1), 31-48.
- Foote, M. Q., & Gau Bartell, T. (2011). Pathways to equity in mathematics education: How life experiences impact researcher positionality. *Educational studies in mathematics*, 78, 45-68.
- Freire, P. (2018). *Pedagogy of the oppressed*. Bloomsbury publishing.
- González, N., Andrade, R., Carson, C., Rosebery, A., & Gonzalez, N. (2001). Creating links between home and school mathematics practices. *Classroom diversity: Connecting curriculum to students' lives*, 100-114.
- González, N., Andrade, R., Civil, M., & Moll, L. (2001). Bridging funds of distributed knowledge: Creating zones of practices in mathematics. *Journal of Education for students placed at risk*, 6(1-2), 115-132.
- Gutiérrez, K. D., Morales, P. Z., & Martinez, D. C. (2009). Re-mediating literacy: Culture, difference, and learning for students from nondominant communities. *Review of research in education*, 33(1), 212-245.
- Gutiérrez, K. D., & Jurow, A. S. (2016). Social design experiments: Toward equity by design. *Journal of the Learning Sciences*, 25(4), 565-598.
- Gutiérrez, K. D., Cortes, K., Cortez, A., DiGiacomo, D., Higgs, J., Johnson, P., ... & Vakil, S. (2017). Replacing representation with imagination: Finding ingenuity in everyday practices. *Review of Research in Education*, 41(1), 30-60.
- Gutiérrez, K. D., Jurow, A. S., & Vakil, S. (2020). Social design-based experiments: A utopian methodology for understanding new possibilities for learning. In *Handbook of the cultural foundations of learning* (pp. 330-347). Routledge.
- Hewlett, S. A., Marshall, M., & Sherbin, L. (2013). How diversity can drive innovation. *Harvard business review*, 91(12), 30-30.
- Hogg, L. (2011). Funds of knowledge: An investigation of coherence within the literature. *Teaching and teacher education*, 27(3), 666-677.
- Moll, L. C., Amanti, C., Neff, D., & Gonzalez, N. (1992). Funds of knowledge for teaching: Using a qualitative approach to connect homes and classrooms. *Theory into practice*, 31(2), 132-141.
- Nasir, N. I. S., Lee, C. D., Pea, R., & McKinney de Royston, M. (2021). Rethinking learning: What the interdisciplinary science tells us. *Educational Researcher*, 50(8), 557-565.
- Nasir, N. S., & Cobb, P. (2007). Equity in students' access to significant mathematical ideas. TCP.
- Radford, L. (2021). *The theory of objectification: A Vygotskian perspective on knowing and becoming in mathematics teaching and learning*. Brill.
- Radford, L. (2021). Davydov's concept of the concept and its dialectical materialist background. *Educational Studies in Mathematics*, 106(3), 327-342.
- Radford, L. (2022). Introducing equations in early algebra. *ZDM—Mathematics Education*, 54(6), 1151-1167.

- Radford, L. (2015). Early algebraic thinking: Epistemological, semiotic, and developmental issues. In *The proceedings of the 12th International Congress on Mathematical Education: Intellectual and attitudinal challenges* (pp. 209-227). Springer International Publishing.
- Radford, L. (2010). Algebraic thinking from a cultural semiotic perspective. *Research in Mathematics Education*, 12(1), 1-19.
- Radford, L. (2002). The seen, the spoken and the written: A semiotic approach to the problem of objectification of mathematical knowledge. *For the learning of mathematics*, 22(2), 14-23.
- Remillard, J. T. (2005). Examining key concepts in research on teachers' use of mathematics curricula. *Review of Educational Research*, 75(2), 211–246. <https://doi.org/10.3102/00346543075002211>.
- Rule, P. (2011). Bakhtin and Freire: Dialogue, dialectic and boundary learning. *Educational philosophy and theory*, 43(9), 924-942.
- Skovsmose, O., & Valero, P. (2001). Breaking political neutrality: The critical engagement of mathematics education with democracy. In B. Atweh, H. Forgasz, & B. Nebres (Eds.), *Sociocultural research on mathematics education. An international perspective* (pp. 37–55). Mahwah: Erlbaum.
- Tuomi-Gröhn, T., Engeström, Y., & Young, M. (2003). From transfer to boundary-crossing between school and work as a tool for developing vocational education: An introduction. In *Between school and work: New perspectives on transfer and boundary-crossing*. Pergamon Press.
- Trouche, L., Adler, J., & Remillard, J. T. (2023). Conceptualizing teachers' interactions with resources in crossing languages and cultures. *ZDM–Mathematics Education*, 1-23.
- Walkington, C. (2012). Context personalization in algebra: Supporting connections between relevant stories and symbolic representations. In *Annual Meeting of the American Educational Research Association. Vancouver, Canada*.
- Wartofsky, M. W., & Wartofsky, M. W. (1979). Perception, Representation, and the Forms of Action: Towards an Historical Epistemology: [1973]. *Models: Representation and the scientific understanding*, 188-210.

# DECIDING BETWEEN FAIR AND NOT FAIR: EMERGING MODELS OF PROPORTIONAL REASONING

Ayşe Öztürk  
The Ohio State University, Newark  
ozturk.25@osu.edu

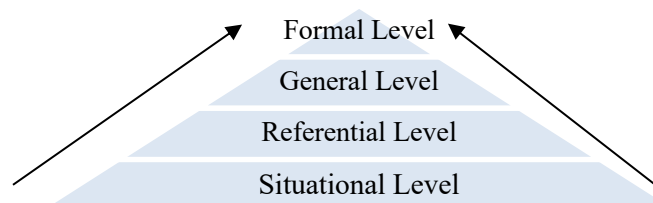
*This study describes the applications of emerging models of proportional reasoning from students who engaged in a problem-solving process resembling a real-life context and communicated their reasoning while considering the notion of fairness. Teaching experiment methodology guided the collection of the study's data from problem-solving sessions at a math camp for secondary students. Analysis revealed that the students' referential-level model, which represented the ideal situation for any team, inspired students to develop a model based on the steepness of line segments at the formal level. The need to justify why their models made mathematical sense led the students to refine their solutions and kept all class members engaged in the activity.*

Keywords: Equity, Inclusion, and Diversity; Communication; Proportional reasoning

The literature reveals that solving real-life problems opens a discussion space for students, guiding them to access and use their own funds of knowledge (Gravemeijer, 1999) and relevant, everyday experiences (Fosnot & Dolk, 2002) to construct sense-making and formal knowledge of mathematical concepts. Recently, researchers have begun to emphasize that maximizing mathematical sense-making through communication should position students to engage in problems related to equity and inclusion to promote their ability to understand and challenge injustice in real-life situations (Kozłowski & Si, 2019; Öztürk, 2023). This paper builds on prior research by exploring students' engagement at various levels of emerging modeling to develop proportional reasoning while solving a real-world problem based on the notion of fairness.

## Conceptual Framework

This study was drawn from Van den Heuvel-Panhuizen's (1996) example of how realistic contexts provide meaningful problems that encourage students to build their understanding of mathematics using initial knowledge, forming a tool that they can then use to construct formal notions about different mathematical concepts. We also adopted Gravemeijer's (1999) framework to investigate the students' levels of model use in supporting their understanding of proportional reasoning (Figure 1):



**Figure 1: Levels of Emerging Modeling (Adopted from Gravemeijer, 1999)**

Within the framework, students begin at a concrete starting point in developing a formal notion of a mathematical concept. At the situation level, real-life knowledge related specifically to the

problem situation is typically incorporated into developing strategies to solve the problem instead of formal mathematics knowledge. At the referential level, students develop models and strategies to refer to the problem situation, representing the initial step of mathematization. At the general level, students focus more on the mathematical aspect of the problem than the contextual details, applying mathematical skills and strategies to solve the problem. At the formal level, students work with notations, procedures, and mathematical relations. Fosnot and Dolk (2002) reported a shift in students' reasoning from "*model for*" as a representation of the problem context at the referential level to "*model of*" as a formal representation focused more closely on the mathematical aspects of the problem. In the current study, the framework guided the data analysis to understand students' communication of their proportional reasoning.

### **Research Setting and Participants**

The data for this paper came from an after-school mathematics program for high school students. During this 6-week program, the class met twice a week for 2 hours. The overarching goal of the research program was to understand how secondary students use their funds of knowledge in mathematics when solving problems based on real-life situations (Ozturk, 2021). The content focus was understanding quantities and using units to solve problems. Study participants included six 10th-grade students, two observer-researchers, and one teacher-researcher. All participants were given pseudonyms in this article.

### **Methods and Data Collection**

Following McClain's (2002) teaching experiment methodology, the class was engaged in a three-phase cyclical process: problem-solving in groups, presenting solutions to others, and returning to groups to revise the problem solutions. The method allows students to give each other feedback, critically listen to each other, and reflect upon diverse ways of reasoning. Each 2-hour class entailed solving two problems.

Data included three essential components: (a) video and audio recording of classroom events, followed by transcribing the recordings; (b) collecting the students' written work; and (c) using observer researchers' notes from the class sessions as supplemental data to organize the video recordings and transcripts as part of the initial data analysis process.

### **Data Analysis, Sample Student Dialogues, and Students' Written Work**

The study data comprised a 1-hour-long compilation of video excerpts in which students solved a problem and communicated their solutions to the class. The criterion for selecting video excerpts was that they contain students engaged in the different levels of use of models in supporting their understanding of proportional reasoning. We focused on how communication helped students move among different levels of emerging modeling while addressing the problem.

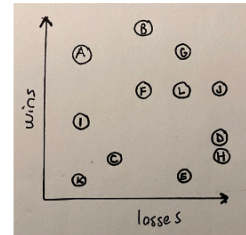
Applying a thematic analysis approach (Braun & Clarke, 2012), data analysis was conducted in four steps: (a) capturing evidence of students' problem-solving instances; (b) using a conceptual framework (Figure 1) to code students' use of models, as well as everyday and mathematical knowledge; (c) elaborating on each level of the students' use of the model when students supported their communication with non-verbal mathematical representations; (d) inductively creating themes to categorize the evidence of using models at various levels and communication during the phases of problem-solving and collective sense-making.

Although analysis is still ongoing to discern the influence of classroom discourse in the formalization of proportional reasoning in another paper, this research report targets illustrating one theme from our data regarding the levels of emerging models in proportional reasoning in the problem-solving process. We selected vignettes from different groups of students who worked on a problem to demonstrate their interaction while attending to the different levels of reasoning.



**Displaying proportional reasoning in different emerging levels.** In this theme, students work in pairs in the phases of the problem-solving process. They brainstorm and decide how to make sense of a problem, then choose methods of mathematization, abstracting the given situation and representing and manipulating it symbolically. The Ranking Teams Problem (Figure 2) required students to identify the relevant variables, make reason with relative quantities at the to identify the relevant variables, make reason with relative quantities at the labeled points, and determine the scales of the coordinate axis:

Each labeled point in the figure represents the record of wins and losses of each of the twelve high school soccer teams. Develop a fair system to determine the rank order of the teams from best to worst.



**Figure 2: Ranking Teams Problem (Adapted from Carmona & Greenstein, 2013)**

The modified problem emphasized applying the notion of fairness in evaluating and interpreting the results for ranking soccer teams and interpreting the results in the problem’s context. Table 1 demonstrates how students engaged at different levels of emergent modeling while presenting their models (graphs) and explaining their approaches to ranking the soccer teams. Notably, students moved from one level to another while sharing their work and critically listening to others’ strategies. Sharing solution approaches in the whole-class discussion allowed the students to draw mathematical conclusions about the relationships between the quantities.

The question of fairness was compelling and motivated students to explore more contextual information, such as why one team played fewer games than the other. At the situational level (see Table 1), Rafe and Birse identified what conditions might be necessary to create a fair system while ranking the teams. Rafe immediately considered comparing the winning points to make a comparison among various teams. Birse disagreed because of unfairness to K: Specifically, K displayed the lowest win record but also played the least amount of games. In real life, Birse noted, teams are compared based on wins and losses and are not only considered by their wins based on the total games they played. At this level, students contextualized team comparisons with a soccer tournament and identified critical variables (e.g., wins, losses, tournament rules).

**Table 1: Levels of Emergent Modeling, Students’ Written Work, and Dialogues**

Levels of emergent modeling (Adapted from Gravemeijer, 1999)	Students’ written work:	Students’ dialogues:
<b>Situational Level.</b> Discussing the nature of a fair ranking in real life.		<b>Rafe:</b> Let’s look for wins. B seems to be the best team, followed by A. <b>Bria:</b> It’s not fair for K. It looks as if K only played two times in the

<p>Determining critical variables in the context.</p>		<p>tournament, winning one and losing one. Comparing teams in real life means looking at both wins and losses, right?</p>
<p><b>Referential Level.</b> Constructing an ideal situation for any team. Determining the ideal line and worst-case line to establish a reference point when ranking teams.</p>	<p><i>Identifying the Ideal Line</i></p>	<p><b>Tan:</b> If we had a team called X, and X was on the wins line [y-axis], that team would be the dream team because it would have zero losses, making this line [y-axis] the ideal line.</p> <p><b>Peter:</b> Right! But if X were on the x-axis, it would be the worst team.</p> <p><b>Tan:</b> Just like the ideal and worst lines, the lines [starting at the origin] should help us make comparisons.</p>
<p><b>General Level.</b> Applying the emerging idea of slope to compare between two randomly chosen teams.</p>	<p><i>Comparing Teams I and G</i></p>	<p><b>Rafe:</b> Let's draw lines that go through the origin and each labeled point. For example, we have two lines. One goes from the origin to I, and the other goes from the origin to G.</p> <p><b>Bria:</b> The line that goes through I is closer to the ideal line [the y-axis] than the line going through G. We can say that I is in a better place compared to G.</p>
<p><b>Formal Level.</b> Applying proportional reasoning and assigning numbers on the axes to confirm how the slope idea (proportion of segments) works to rank teams.</p>	<p><i>Using Slopes to Order Teams</i></p>	<p><b>Asuhi:</b> Other examples made us wonder about making the segments proportional to the total number of games.</p> <p><b>Mya:</b> For example, 8 wins out of 9 games for A or 9 wins out of 12 games for B. Team A is better than B.</p> <p><b>Asuhi:</b> In other words, the steepest slope of these segments [tracing the vertical and horizontal segments] gives us the winner.</p>

At the referential level, the students employed models of the situation that referred implicitly or explicitly to the mental activities in the problem context described. For example, Tan and Peter established the best situation for a team in the tournament: winning every game played. Tan associated the ideal situation with a team (“X”) on the  $y$ -axis, acknowledging the  $y$ -axis as the ideal line because X’s placement on the  $y$ -axis would mean that X wins all games and has no losing scores. Peter then identified the worst-case scenario for X by placing the team on the  $x$ -axis, indicating only losses. Tan and Peter’s discussion allowed them to create other lines from the origin and entertain various possibilities, such as if X was on the line closest to the ideal or the worst-case line. Along with adding the solid arrows (see Table 1), Tan and Peter inferred how team X could go from the best to the worst rank, pointing to the closeness of the ideal line and worst line.

Tan and Peter’s presentation of their model to the class inspired others to adopt the strategy of drawing lines from the origin to the labeled points. At the general level, student interest centered around the interpretations of the different solutions. Rafe and Bria adapted the idea from Peter’s group and drew lines that went through the origin and each labeled point to compare and rank the teams. As they presented their model, Bria chose two teams, I and G, to demonstrate how their model worked to make comparisons (see Table 1). She emphasized that as the lines approached the  $y$ -axis, the teams on the lines obtained a better place in the ranking system order. Although the students did not explicitly define the slope or ratio terms, Bria’s explanation underlies the mathematical idea of the slope of a line or the ratio of wins over losses.

At the formal level, the students built on their previous activities via symbolization that reflected the emergence of the new mathematical reality. In the new model they developed, they drew segments from the origin to each letter, assigned numbers to relative quantities at the labeled points, and concluded that “the number of games is the sum of the coordinates.” They justified their segment-drawing solution arising from their model and explained how they calculated each team’s success-related performance by considering the proportional relationship between wins and losses (see Table 1). Using slopes of segments and proportional reasoning while highlighting the change in the vertical height over the change in horizontal length between endpoints allowed the students to define an individualized success record for each team.

### **Results and Conclusion**

This report illustrated how students engaged at different levels of emergent models of proportional reasoning while addressing a real-life-based problem. According to the study results, while working at the situational level, the students focused on the conditions of being fair in ranking soccer teams in a tournament; in contrast, at the formal level, students employed the emerging idea of using slopes of segments and proportional reasoning to target the precise meaning of the ratios they chose to build a system illustrating each team’s success. Furthermore, justifying models to other students and describing why they mathematically make sense kept all class engaged in the activity and led students to refine their solutions. *Model for*, representing the ideal situation for any team at the referential level, inspired students to develop a *model of* the steepness of line segments at the formal level.

### **References**

Braun, V., & Clarke, V. (2012) Thematic analysis. In H. Cooper (Ed.), *The handbook of research methods in psychology*. American Psychological Association.

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

- Carmona, G., & Greenstein, S. (2013). Investigating the relationship between the problem and the solver: Who decides what math gets used? In R. Lesh, P. L. Galbraith, C. R. Hains, & A. Hurford (Eds.), *Modeling students' mathematical modeling competencies: ICTMA 13* (pp. 245–254). Springer.
- Fosnot, C. T., & Dolk, M. L. A. M. (2002). *Young mathematicians at work: Constructing fractions, decimals, and percents*. Heinemann.
- Gravemeijer, K. (1999). How emergent models may foster the constitution of formal mathematics. *Mathematical Thinking and Learning*, 1(2), 155–177.
- Kozlowski, J. S., & Si, S. (2019). Mathematical creativity: A vehicle to foster equity. *Thinking Skills and Creativity*, 33, Article 100579.
- McClain, K. (2002). A methodology of classroom teaching experiments. In S. Goodchild & L. English (Eds.), *Researching mathematics classrooms: A critical examination of methodology* (pp. 91–118). Praeger.
- Ozturk, A. (2021). *Examining the Relationship Between Instructional Practices and Students' Mathematical Modeling Competencies*. [Doctoral dissertation, Ohio State University].  
[http://rave.ohiolink.edu/etdc/view?acc\\_num=osu1625748925881711](http://rave.ohiolink.edu/etdc/view?acc_num=osu1625748925881711)
- Ozturk, A. (2023). Fostering Quantitative Reasoning Through a Question of Fairness. *Mathematics Teacher: Learning and Teaching PK-12*, 116(4), 265-272. <https://doi.org/10.5951/MTLT.2022.0226>
- Van den Heuvel-Panhuizen, M. (1996). *Assessment and realistic mathematics education*. CD-β Press.

# DESIGNING FOR STUDENTS' JOYFUL MATHEMATICAL ENGAGEMENT THROUGH RESPONSIVE REHUMANIZING MATHEMATICS

Madison Knowe  
Vanderbilt University  
madison.l.knowe@vanderbilt.edu

Keywords: Affect, Emotion, Beliefs, and Attitudes; Middle School Education; Teacher Knowledge; Rehumanizing Mathematics

## Objectives and Framework

Joy and mathematics are often perceived as opposites as a result of a history of dehumanizing norms which frame mathematical engagement as an unemotional, objective, competitive, performative, and individualistic. However, formal mathematics environments can be redesigned to be more humanizing (Goffney et al., 2018; Gutiérrez, 2018; Mukhopadhyay & Greer, 2015; Yeh & Otis, 2019); Gutiérrez (2018) offers a framework which involves eight potential dimensions: participation/positioning, cultures/histories, windows/mirrors, living practice, creation, broadening mathematics, body/emotions, and ownership. This study investigates the complexities of designing for rehumanizing mathematics by asking *What is important to know about students and how does that information translate into designing for humanizing mathematical learning?*

## Methods

Through a partnership with a 6<sup>th</sup> grade teacher, this study explores how to design rehumanizing lessons that are responsive to the 25 students in her class. Data were collected through journal reflections, illustrated math autobiographies, and interviews. For this qualitative analysis, using a grounded theory approach, I looked for themes across these data sources, organized through the lens of Gutiérrez's eight dimensions of rehumanizing mathematics.

## Results and Implications

The collected data served to elicit student voice about joy both in and out of math class. This informed our design by indicating features of math class to retain and features of students' lives to incorporate to evoke more joy in math class. Themes from the analysis were categorized according to Gutiérrez's dimensions for rehumanizing mathematics (2018). Three dimensions that were consistent across students were: ownership, bodies, and participation. Students' positive descriptions of ownership were almost always in the context of the math class. For example, students liked having their work displayed in the class. Students also expressed fulfillment and enjoyment in participating in playing games. Some game play was outside of a math context, but many students expressed an explicit positive relationship between games and math indicating that games could be further emphasized in rehumanizing lessons. Reference to students' bodies was exclusively in non-mathematical physical activity (playing sports or walking dogs). This dimension could be easily translated to designing for embodied mathematical activities.

The partnering teacher and I therefore focused our designs for rehumanizing mathematics on providing tasks that supported students' ownership, participation in game play, and embodied activities. It is through these redesigned lessons in response to students' descriptions of joy that we hope to see joy emerge as evidence rehumanizing mathematics.

## References

- Goffney, I., Gutiérrez, R., & Boston, M. (2018). Rehumanizing mathematics for Black, Indigenous, and Latinx students. National Council of Teachers of Mathematics.
- Gutiérrez, R. (2018). Rochelle Gutiérrez: "Rehumanizing Mathematics: A Vision for the Future", Latinx in the Mathematical Sciences Conference 2018. Institute for Pure & Applied Mathematics (IPAM). Retrieved March 30 from <https://www.youtube.com/watch?v=D266LYIigS0>
- Mukhopadhyay, S., & Greer, B. (2015). Cultural responsiveness and its role in humanizing mathematics education. CERME 9-Ninth Congress of the European Society for Research in Mathematics Education, Singh, S. (2021). Chasing Rabbits: A Curious Guide to a Lifetime of Mathematical Wellness. Impress. Su, F. (2020). Mathematics for human flourishing. Yale University Press.
- Yeh, C., & Otis, B. M. (2019). Mathematics for whom: Reframing and humanizing mathematics. Occasional Paper Series, 2019(41), 8.

## DEVELOPING A SURVEY TO ACCESS MATHEMATICS TEACHERS' PERCEPTIONS TOWARD STUDENTS WITH DISABILITIES

### DESARROLLANDO UNA ENCUESTA QUE ACCEDA A PERCEPCIONES DE DOCENTES MATEMÁTICOS HACIA ESTUDIANTES CON DISCAPACIDAD

Offir N. Romero Castro  
Western Michigan University  
offirneil.romerocastro@wmich.edu

Laura R. Van Zoest  
Western Michigan University  
laura.vanzoest@wmich.edu

*Negative perceptions held by teachers toward students with disabilities create environments that make students feel uncomfortable and often incapable of participating actively in classrooms. Much of the research about these perceptions is focused on teachers of students with learning disabilities, which leaves out teachers' perceptions toward students with other disabilities. We are developing a responsive online survey to access what mathematics teachers identify as disabilities and their behavior toward students with disabilities in their classroom. In this paper, we describe the process we have developed for constructing this survey. We also share our conceptualization of the relationship between teachers' perceptions and equity affirmations toward disability, and the results of applying that conceptualization to our research context.*

Keywords: Equity, Inclusion, and Diversity; Students with Disabilities; Teacher Beliefs.

#### **Purpose of the Study**

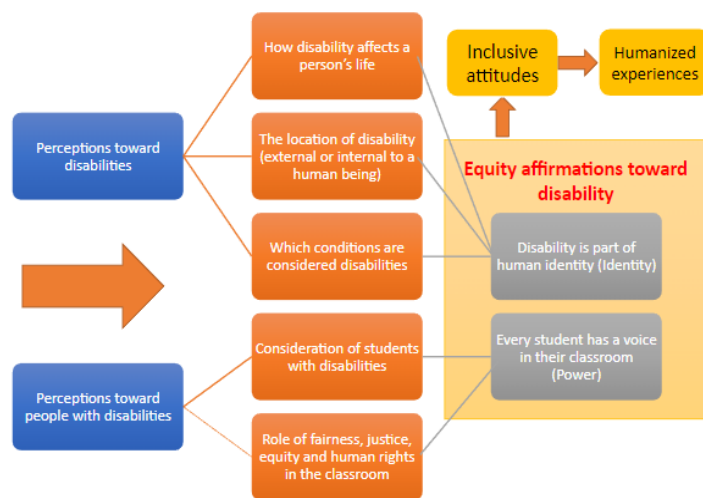
Teachers' perceptions affect their students' development because teachers are the instructional leaders of their classrooms. Unfortunately, these perceptions can create an environment where students receive mathematics content, rather than having the opportunity to make mathematical contributions themselves. Tan and colleagues (2019), refer to this as "mathematics for students" instead of "mathematics of students" (Tan et al., 2019). This type of environment does not allow for the wealth of ideas and ways of thinking that students who have disabilities can bring to classroom discussions about mathematics, and thus shortchanges the learning of everyone in the classroom. Therefore, the vision is inclusive mathematics classrooms where each student is considered able to engage in mathematics learning and enrich the learning of others.

Reaching this vision requires that teachers develop perceptions that provide an inclusive environment toward all types of disabilities in schools. An important starting place is to find out the extent to which mathematics teachers are already considering the complexity of disability as part of the human identity of any student, and how they see disabilities as affecting students' participation in their classroom. For this reason, our focus is on describing mathematics teachers' current perceptions towards disabilities in mathematics classrooms. Understanding these perceptions will support work with teachers to develop positive perceptions that provide an inclusive environment to disability in schools. A first step of this work is to provide a process for accessing teachers' perceptions toward disabilities and toward students with disabilities—the focus of this paper.

#### **Literature Review & Theoretical Framework**

Our work is guided by two equity affirmations that we identified in the work of Tan et al. (2019). The first one is related to perceptions toward disabilities and includes information about which conditions teachers consider as disabilities and the extent to which they interpret life with

disabilities through an ableist lens (e.g., in relation to what a person is unable to do). This affirmation embraces the complex nature of disabilities as being part of students' human identities. The second affirmation is related to perceptions toward students with disabilities and includes information about teachers' consideration toward students with disabilities and comparison to students without disabilities. This affirmation establishes that every student, regardless of whether they have a disability, has an important voice within the classroom. Gutiérrez (2012) wrote about four dimensions of equity; the two dimensions composing the critical axis are directly related to the equity affirmations we established. Gutiérrez's *identity* maps to disability as part of human identity and *power* maps to every student having a voice in their classroom. Figure 1 graphically presents the relationship between the different types of perceptions and the equity affirmations.



**Figure 1: Relationship between Teachers' Perceptions and Equity Affirmations toward Disability**

We take the perspective that if teachers have perceptions toward disabilities and students with disabilities that embrace the equity affirmations, then they will have inclusive attitudes that promote humanized experiences for every student in their classrooms. We take a *participationist* approach to learning (Sfard & Cobb, 2023; Vygotsky, 1987). This means that we see student learning as taking place through students' interactions with more knowledgeable others, such as the teacher and their peers. Following Tan and colleagues (2019) specifically, and NCTM (2000, 2014) more generally, we hold that all students (with or without disabilities) can contribute to the mathematical learning of others. This is because we see disabilities as part of students' human identities rather than something that sets them apart from other students.

We are particularly interested in how teachers' perceptions toward disability affect which students they choose to include in their enactments of the *teaching practice of building* (henceforth referred to as *building*). The MOST project defines *building* as making a Focal Student Contribution (FSC) "the object of consideration by the class in order to engage the class in making sense of that thinking to better understand an important mathematical idea" (Van Zoest et al., 2017, p. 36). They describe building as being comprised of four elements: "(1) *Establish* the student mathematics of the [FSC] as the object to be discussed, (2) *Grapple* toss that object in a way that positions the class to make sense of it, (3) *Conduct* a whole-class



discussion that supports the students in making sense of the student mathematics of the [FSC], and (4) *Make explicit* the important mathematical idea from the discussion” (Leatham et al., 2021, p. 1393). We used the information in Figure 1 in conjunction with the four elements of building to develop a framework for designing questions to assess perceptions about disabilities related to these critical aspects of student-centered teaching. For our purposes, we also added a zero element: the selection of the FSC.

### **Methods**

To guide the development of a survey to assess teachers’ current perceptions about disabilities, we did the following:

1. Identified key equity affirmations toward disabilities (Figure 1).
2. Developed a schema for the relationship between teachers’ perceptions and equity affirmations toward disability (Figure 1).
3. Investigated what international organizations have identified as disabilities.
4. Applied the equity affirmations toward disabilities to our research situation (see the resulting matrix in Figure 2 below).

We are using the United Nations and World Health Organization (2001) document, *Functioning and Disability: The International Classification of Functioning, Disability and Health* (found in step 3 above) to design questions that reveal what teachers identify as disabilities. This is important because we want to both better understand the range of official disabilities that teachers recognize and ask questions relating to their practice that include disabilities that they recognize as such. We are using the matrix in Figure 2 to design scenarios with responsive questions that reveal teachers’ application of the two critical dimensions of equity (Identity and Power) toward students with disabilities.

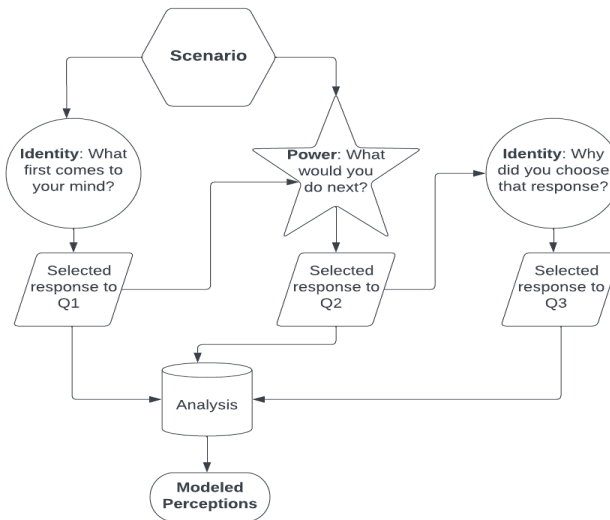
### **Results & Discussion**

In addition to the initial results that are represented in Figure 1, we share the results of applying the equity affirmations towards disabilities to our research situation in Figure 2.

	Dimensions of Equity	
Elements of Building	Identity	Power
<i>In relation to a Focal Student Contribution (FSC)</i>	<i>Disability is part of human identity</i>	<i>Every student has a voice in their classroom</i>
Select	Disability does not prevent a student from making a high-leverage mathematical contribution.	High-leverage contributions of students with and without disabilities are promoted, recognized and selected to be the focus of classroom discussion.
Establish	Disability is not an obstacle for a student's mathematical contribution to be established as an object to be discussed (FSC).	High-level mathematical contributions of students with and without disabilities are established as an FSC.
Grapple Toss	Disability is not a restriction for positioning students to make sense of the mathematics in an FSC.	Students with and without disabilities are positioned to make sense of the FSC.
Conduct	Disability is not a restriction for making sense of an FSC in a whole-class discussion.	Students with and without disabilities are important contributors to the sense-making discussion about the FSC.
Make Explicit	Disability is not an obstacle for summarizing the important mathematical ideas of the discussion.	Students with and without disabilities are important contributors to making the important mathematics of the discussion explicit.

**Figure 2: Critical Axis of Equity toward Disability in the *Building on a Focal Student Contribution (FSC)***

Figure 2 provides us with descriptions of teaching-related equitable thoughts that we are using to guide the development of our survey scenarios. Figure 3 shows the structure of the scenario component of our survey development.



**Figure 3: Structure of the Scenario Component**

Each scenario is followed by a series of three questions, with each question having three to four choices and the option to write one's own response. Question 1, "What first comes to your mind?" is intended to access the extent to which teachers embrace the complex nature of disabilities as being part of students' human identities. Question 2, "What would you do next?" is intended to access the extent to which students with and without disabilities have a voice in the classroom. The response to Question 1 informs the response choices to Question 2. Question 3, "Why did you choose that response?" is designed to access their justifications for their answer in

Question 2 through a view of the student's identity. Our analysis of the set of teachers' responses for each scenario will provide a model of their perceptions towards students with disabilities. The collection of modeled perceptions that emerge from our analysis of all the data will be compared with the equity affirmations and their application in Figure 2 to provide information about the current state of mathematics teachers' perceptions toward disabilities.

### Conclusion

In this paper we have shared our process for developing a survey for accessing teachers' perceptions toward disabilities and students with disabilities using a specific context—the teaching practice of *building*. The structure of our scenario component reflects the interrelationship between both types of perceptions. Our expectation for the survey is that it can be used as a tool for teachers, researchers, and educational administrators to access teachers' perceptions toward disabilities and use what they find to inform professional development oriented to developing inclusive mathematics classrooms.

### Acknowledgments

This material is based upon work supported by the National Science Foundation under Grant No. DRL-1720613. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the NSF.

### References

- Gutiérrez, R. (2012). Context matters: How should we conceptualize equity in mathematics education? In B. Herbel-Eisenmann, J. Choppin, D. Wagner, & D. Pimm (Eds.), *Equity in discourse for mathematics education* (pp. 17-33). Springer, Dordrecht. [https://link.springer.com/chapter/10.1007/978-94-007-2813-4\\_2](https://link.springer.com/chapter/10.1007/978-94-007-2813-4_2)
- Leatham, K. R., Van Zoest, L. R., Freeburn, B., Peterson, B. E., & Stockero, S. L. (2021). Establishing student mathematical thinking as an object of class discussion. In D. Olanoff, K. Johnson, & S. M. Spitzer (Eds.), *Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1392-1400). Philadelphia, PA: PME-NA.
- National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.
- National Council of Teachers of Mathematics. (2014). *Principles to actions: Ensuring mathematical success for all*. Reston, VA: NCTM.
- Sfard, A., & Cobb, P. (2022). Research in Mathematics Education: What Can It Teach Us about Human Learning? In R. Sawyer (Ed.), *The Cambridge Handbook of the Learning Sciences* (pp. 467-485). Cambridge University Press. doi:10.1017/9781108888295.029
- Tan, P., Padilla, A., Mason, E. N., & Sheldon, J. (2019). *Humanizing disability in mathematics education: Forging new paths*. The National Council of Teachers of Mathematics, Inc. <https://pubs.nctm.org/view/book/9781680540253/9781680540253.xml>
- United Nations and World Health Organization. (2001). *Functioning and Disability: The International Classification of Functioning, Disability and Health*. <https://www.who.int/standards/classifications/international-classification-of-functioning-disability-and-health>
- Van Zoest, L. R., Stockero, S. L., Leatham, K. R., Peterson, B. E., Atanga, N. A., & Ochieng, M. A. (2017). Attributes of instances of student mathematical thinking that are worth building on in whole-class discussion. *Mathematical Thinking and Learning*, 19(1), 33-54. <https://doi.org/10.1080/10986065.2017.1259786>
- Vygotsky, L. S. (1987). Thinking and speech. In R. W. Rieber & A. C. Carton (Eds.), *The collected works of L. S. Vygotsky* (pp. 39-285). New York: Plenum Press.

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

## DO MATH MUSEUMS PROVIDE (E)QUALITY STEM EXPERIENCES?

Kyunghoon Son  
Boston College  
sonky@bc.edu

Lillie R. Albert  
Boston College  
albertli@bc.edu

*The purpose of this study is to analyze the composition of STEM elements in the exhibits of mathematics museums, and to examine whether the museums are delivering a balanced, culturally relevant, and comprehensive mathematics learning experience to students. To overcome an equity issue, South Korea established twenty math museums with purpose of providing equitable access to students in rural areas. Eight mathematics museums' exhibits, including MoMath as representative model of math museums, were classified and compared. MoMath's exhibits overall provide a comprehensive experience to student by considering all integrated STEM elements, whereas it was revealed that the exhibits of Korean math museums are skewed towards a small number of specific elements.*

Keywords: Informal Education; Integrated STEM/STEAM; Social Justice

### Introduction

Despite the worldwide interest in STEM education (Sanders, 2009; Brown et al., 2011), equity concerns exist. Supposed students with STEM traits are expected to play as the leaders in the future society; we cannot help but asking whether all students are accessible to enough as well as quality STEM education. Considering that STEM is currently a vital issue for the future society and that many higher-income workforces are concentrated within the STEM-based professions, it is worrisome, in terms of hindering social mobility, that students from marginalized groups tend to distance themselves from STEM majors. According to Ireland et al. (2018), students of color exhibit a significantly higher aversion to STEM-related majors in post-secondary education than white students.

Equity issues vary for each country. For South Korea, regional marginalization is one of the highly concerned equity issues. More than half of the population resides in Seoul and its surrounding areas; therefore, the majority of governmental, social, cultural, and educational accesses are concentrated in the capital city. Students living in areas far from Seoul, inevitably, deprived of opportunities to access education and culture (Han, 2022). The Korean Ministry of Education (2015) acknowledged that the change towards a technology-based society would start with STEM and emphasized mathematics which serves as the foundation for the overall concept of STEM. For this reason, the Ministry of Education announced the Second Comprehensive Plan for Mathematics Education as well as began establishing public mathematics museums nationwide to provide students in rural areas with easy access to mathematics and mathematics culture. As of 2022, twenty mathematics museums have been established and operating in South Korea (Kim, 2022).

The policy considers the mathematical experience of students who are regionally marginalized, which is commendable. However, can the establishment of mathematics museums itself be considered as an equal opportunity to provide all learners with quality mathematical culture experience? Are the museum exhibitions providing students with new and balanced STEM engagement? Are the museums providing students with culturally relevant and responsive mathematics learning experience, with respect to their geographical locations?

This study aims to analyze mathematics museums in South Korea from the perspective of STEM education. Through this analysis, we aim to determine whether museum exhibitions provide equal and quality mathematics and STEM experiences. In doing so, we compare mathematics museums in different regions in South Korea. This comparison will help to provide insights for the construction of future mathematics museums in many other countries, especially when there is a need to ensure students' equitable access to mathematics culture.

### **Theoretical Background**

#### **Integrated STEM Education**

The theory of integrated learning provides the theoretical framework for this paper. This theory is rooted in the progressivism, which emphasizes the connection between learning and real-life situations. John Dewey stated, "Relate the school to life, and all studies are of necessity correlated" (Dewey, 1910, p.32). This statement highlights the notion that learning is more meaningful when students related their experiences and learning. The goal of integrated STEM education is to teach subjects as one cohesive entity, diffusing knowledge, skills, values, and language differences (Corlu et al., 2014). By doing so, teachers and students can collaboratively construct new knowledge, skills, and beliefs at the intersection of multiple STEM subject areas.

Corlu et al. (2014) also argues that STEM teaching can happen at the intersection of multiple STEM subjects, where the interactions draw on the content and processes of both mathematics and science, such as problem-solving and quantitative reasoning. By using mathematics in science or mathematically rigorous science education, education can create an interdisciplinary understanding of STEM while preserving subject-specific knowledge, skills, and attitudes.

### **Methods**

#### **Samples**

The study's sample consists of one model mathematics museum in the United States and seven targeted museums in South Korea. The MoMath, which is located in Manhattan, New York, and is renowned worldwide as a place where visitors of all ages can engage in various mathematical activities. MoMath offers well-balanced programs that involve mathematicians, math educators, teachers, parents, and students from diverse backgrounds. For these reasons, MoMath was used as a benchmark museum to compare the math museums in South Korea and to confirm the validity of the analysis framework. The math museums in South Korea are listed according to their respective locations, including Nowon, Daegu, Daejeon, Wonju, Jeju, and Chungbuk.

#### **Procedure**

The procedure of this study involved two analyses: 1) the development of an analysis framework to identify the STEM integration elements necessary for implementing integrated education in mathematics museums; and 2) an analysis of differences between the experiences and exhibits of mathematics museums. In the development of the analysis framework, literature relevant to the theoretical background was utilized to create a framework for analyzing the STEM integrated elements of the exhibits in mathematics museums. The framework was then applied to the exhibits at MoMath, with each sub-element coded, and refined through several rounds of discussion with three mathematics education experts. In the data collection and analysis stage, data on exhibits from eight mathematics museums were collected and analyzed using the final analysis framework. The results were used to assess the coherence and appropriateness of the integrated elements in the exhibits and to conduct an analysis of each museum's distinctive characteristics.

## Data Collection

The data collection focused on the permanent exhibitions of the research targets, excluding special exhibitions or educational programs offered for a limited period of time, as well as structures and exhibits in the rest areas. We recorded video footage, photographs, and interviews during the visits to these museums in 2020. In addition, research materials developed by each museum, brochures, and previous studies on the museums were used as supplementary data.

For the Jeju, which was also difficult to visit during the study period, we utilized the museum's video guide for the mathematics experience exhibitions. For the Nowon, Daegu, Daejeon, Wonju, and Chungbuk, we visited and collected data by taking photographs of all the exhibits in the exhibition halls, filming the exhibits following the route, and collecting materials through interviews with museum operators.

## Analysis

Using the analysis framework developed in this study, three math education experts applied the framework to code the math museum exhibits at the eight math museums. In cases where disagreements arose during the coding process, the code selected was the one supported by the majority, or a new code was produced through re-analysis and discussion to ensure the validity and reliability of the analysis. Once all the exhibits were coded, the proportion of each museum's exhibits were calculated and mapped based on the result. Later, the similarity of exhibit distribution will be statistically validated using a chi-square test. For now, mapping was used to tentatively compare the characteristics of museums and conduct in-depth analysis of exhibits in museums that show distinctive distribution. The analyzing frameworks are shown in Table 1.

**Table 1: Integration Units for the Math Museum Exhibits**

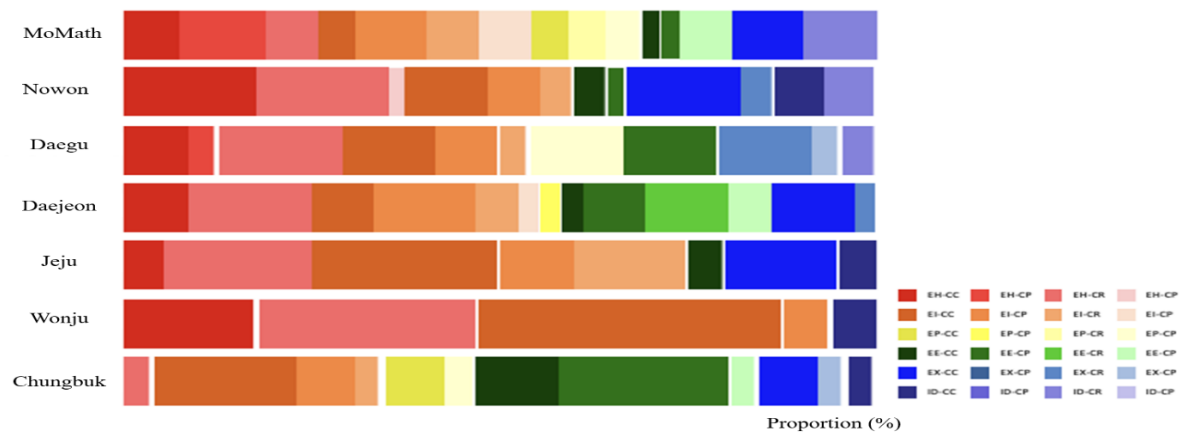
Dimension 1: Integration Units for Math Museum Exhibits		
Unit	Code	Definition
Concept/Principle	CC	Integration of mathematics with other disciplines based on specific math concepts or principles
Problem	CP	Integration of mathematics with other subjects based on problems to be solved
Real-life	CR	Integration of mathematics or other subjects with real-life phenomena or materials
Activity	CA	Integration of mathematics with other subjects based on specific activities
Dimension 2: Evaluation Units for Math Museum Exhibits Experiences.		
Hand-on	EH	A method where the visitor touches or manipulates the exhibit item with body parts
Interactive	EI	An interactive method where there is an exchange between the exhibit and the visitor, and the results can be evaluated based on the visitor's response
Performance	EP	A method of making various products such as math structures and drawings using exhibits
Entertaining	EE	A method of experiencing exhibits through games, quizzes, activities, and other forms of play
Experiment	EX	A method of acquiring knowledge of principles through experimentation
Indirect Exhibition	ID	A passive method of viewing self-operating items, video materials, and special effects exhibits fixed in one location

## Results

The most prominent result is the balanced distribution of integrated elements in MoMath's exhibits, while other museums did not exhibit such balance. There are several reasons why MoMath could maintain such balance from the foundation to the maintenance and improvement processes. For MoMath, experts such as mathematicians, math educators, and math teachers set the museum's operational aims and constructed exhibits based on sufficient theoretical

background. In contrast, Korean math museums rely on a small number of experts or educational authorities to develop problems without sufficient expertise. Therefore, it can be expected that they exhibit a distribution biased towards one side.

Wonju is located in the most rural area among Korean museums in this study, and its exhibit composition was revealed as the simplest. In addition, Wonju and Jeju accounted for nearly half of the EI-CC in all museums. According to interviews with museum officials, both museums' construction was outsourced by the same private company. The distribution map of museum exhibitions is shown in Figure 1.



**Figure 1: Distribution Mapping of Museum Exhibitions**

### Conclusion and Limitations

The purpose of this study is to analyze the STEM elements of mathematics museums established to provide equal access to regionally disadvantaged students. The analysis framework developed by the authors confirmed that MoMath's exhibits were structurally balanced, while many Korean museums were not. In order to deliver a meaningful STEM experience to all students, a balanced composition of museum exhibits should be prioritized. Therefore, compared to their large number of museums, Korean math museums are concerning in terms of their exhibit composition. However, this study is still under development and has limitations since statistical comparison methods or in-depth analysis have not yet been conducted.

### References

Brown, R., Brown J., Reardon, K. & Merrill C. (2011). Understanding STEM: current perceptions. *Technology and Engineering Teacher*, 70(6), 5-9.

Corlu, M. S., Capraro, R. M., & Capraro, M. M. (2014). Introducing STEM education: Implications for educating our teachers in the age of innovation. *Eğitim ve Bilim*, 39(171), 74-85.

Dewey, J. (1910). *How we think*. Boston, MA: Heath.

Han, S. (2022). Spatial stratification and socio-spatial inequalities: the case of Seoul and Busan in South Korea. *Humanities and Social Sciences Communications*, 9(1), 1-14.

Kim, K. (2022, December 13). The biggest math museum is about to open in Busan. Hankyoreh. Retrieved from <https://www.hani.co.kr/arti/area/yeongnam/1071410.html>

Korea Ministry of Education. (2015). The 2<sup>nd</sup> mathematics comprehensive plan.

Ireland, D. T., Freeman, K. E., Winston-Proctor, C. E., DeLaine, K. D., McDonald Lowe, S., & Woodson, K. M. (2018). (Un) hidden figures: A synthesis of research examining the intersectional experiences of Black women and girls in STEM education. *Review of Research in Education*, 42(1), 226-254.

Pearson, J., Giacomo, L. A., Farid, A., & Sadegh, M. (2022). A systematic multiple studies review of low-income, first-generation, and underrepresented, STEM-degree support programs: Emerging evidence-based models and recommendations. *Education Sciences, 12*(5), 333. <https://doi.org/10.3390/educsci12050333>

Sanders, M. (2009). STEM, STEM education, STEMmania. *The Technology Teacher, 68*(4), 20-26.

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.



## ENHANCING OUR THEORETICAL LENS: SECOND-ORDER MODELS AS ACTS OF EQUITY

Amy J. Hackenberg  
Indiana University  
ahackenb@iu.edu

Erik S. Tillema  
Indiana University  
etillema@iu.edu

Andrew M. Gatza  
Ball State University  
amgatza@bsu.edu

*In this theoretical paper, we respond to a call for all Mathematics Education researchers to become equity researchers (Aguirre et al., 2017) by articulating how equity is foundational to making second-order models of students' mathematics. First, based on prior research, we view equity to be about power and respect. We define an act of equity as acting on social boundaries with the intent of changing them in order to address known inequities. Second, we explain why making second-order models is an act of equity, showing how it respects students and can affect power in research settings. Third, we demonstrate how attention to social identity categories and social identities can enhance current second-order models to better support acts of equity.*

Keywords: Equity, Inclusion, and Diversity; Learning Theory; Research Methods

In 2017, the Research Committee of the National Council of Teachers of Mathematics called for all Mathematics Education researchers to engage in the intentional, collective, and professional responsibility of becoming equity researchers (Aguirre et al., 2017). To take up this call, we consider two conceptions of equity in relation to doing research in Mathematics Education. In the first, researchers articulate how their research addresses equity-related issues. In the second, researchers articulate how equity is foundational to their research. With the first conception, equity issues may be added on to a research program without considering the principles, assumptions, and aims of the program (Barajas-Lopez & Larnell, 2019). In this paper, we make an account grounded in the second conception of equity.

We are researchers who make second-order models of students' mathematics (Steffe, et al., 1983) that are rooted in radical constructivism (von Glasersfeld, 1995). These models are explanatory accounts of students' learning that can be used to orchestrate future mathematical interactions with other students. They can also be used as a basis for curricular design and work with teachers. We view the making and using of second-order models to be acts of equity in research settings and to be a basis for engaging in acts of equity both in curricular design and work with teachers. However, this position has seldom been articulated (cf. Ellis, 2022).

We propose that researchers who make second-order models should define what they mean by an act of equity (Gutierrez, 2002), explain how these models can support acts of equity in Mathematics Education, and identify how these models could be enhanced to better support acts of equity. We take on these three tasks in this theoretically-focused paper. In addition, we view our paper as one example of how researchers working within a specific research tradition might respond to the Research Committee's (Aguirre et al., 2017) call. Our aim is to engage other researchers—those who use a radical constructivist framework and those who do not—in conversations that make explicit how equity informs their research.

We make two notes before proceeding. First, since we are researchers of learning, when we articulate how building second-order models can support equity, we focus primarily on the outcomes of supporting learning and opportunities for learning. Second, we view the call to become equity researchers as an active, incomplete process of *becoming*.

### Defining an Act of Equity

In order to respond to the Research Committee's (Aguirre et al., 2017) call, we organized themes in research on equity issues in the field, highlighting those relevant for learning. These themes include: identifying structural and institutional inequities in the field (e.g., Battey, 2013; Berry, 2015); disrupting deficit narratives in order to focus on asset-based framings of knowing (e.g., Adiredja, 2019; Gholson & Martin, 2014; Maloney & Matthews, 2020; Stinson, 2013); understanding the impact of people's identities on mathematical learning (e.g., Walshaw, 2013; Shah, 2017); supporting students to learn mathematics as a tool for inquiry into issues of social justice (e.g., Gutstein, 2016; Kokka, 2022); and reconsidering power and access by broadening what counts as mathematical knowledge and who generates it (e.g., Gutiérrez, 2018).

Based on this literature, we conceive of equity to be about how power in society affects people (Ball, 2013; Foucault, 1980; Hayward, 1998) combined with respect for people in all of their varied ways of being and identifying in the world (Gutiérrez, 2018; Walshaw, 2013). We conceive of *power* as the "social boundaries that, together, define fields of action for all actors" (Hayward, 1998, p. 12). *Social boundaries* include "laws, rules, norms, institutional arrangements, social identities and exclusions" (Hayward, 1998, p. 2). Our definition of power leads to a particular view of *freedom*: "the capacity to act upon the boundaries that constrain and enable social action, for example, by changing their shape or direction" (Hayward, 1998, p. 12).

We define *respect* as "an act of giving particular attention, including high or special regard" (Merriam-Webster dictionary online). We view respect for people as an important component of what the literature on re-humanizing mathematics education (Gutiérrez, 2018) calls for: respect for students in math classrooms. For example, Gutiérrez has called for students to be regarded as thinkers and as whole people, stating, "not until we seek to stand in the shoes of our students, to understand their conceptions, will we be on the path toward recognizing and embracing their humanity" (p. 2). We concur with this view and seek to show in this paper what we mean by it.

The focus on respect for each and every person means taking an interest in how people identify themselves and how others identify them. A person's *identity* refers to the person's ways of associating themselves with (or outside of) a racial group, gender, etc., as well as with or against mathematics (Cobb et al., 2009; Martin, 2000). Walshaw (2013) characterized identities as changeable, even across contexts, and as partly shaped by the expectations of others and social boundaries. People have partial agency over their identities within these social boundaries (Cobb et al., 2009). Furthermore, one's individual sense of agency may be linked to the extent to which a person experiences freedom as defined above.

With these definitions in mind, we define an *act of equity* as acting on social boundaries with the intent of changing their shape or direction in order to address known inequities. This definition is very close to our definition of freedom, but we view freedom as a capacity or potential, whereas an act of equity is an action. To complete an act of equity requires identifying how the changes produce more equitable outcomes (i.e., intent alone is insufficient). For example, in math classrooms, students are often trained to learn the thinking of their teacher, as we will explain. The first part of an act of equity (i.e., the intent) would be changing that norm so that students' mathematical ways of thinking, and especially the thinking of students at the margins, are a central and sustained subject matter of the class. We would consider these changes to be theoretical until an actor (e.g., a teacher-researcher) establishes markers that the changes produce more equitable outcomes. Because we are focused on mathematical learning, this documentation would focus on how the change in social boundaries positively affected the

learning of the students or the opportunities for students' learning.

To be able to engage in acts of equity as a researcher requires, at the very least, knowledge of historical inequities, knowledge of how one has participated in and benefited from the institutions of one's society, and a willingness to question and continue to learn about both. Within the U.S., we consider these institutions to have been infused with racial, gender, linguistic, and other biases, and so we offer brief information about our positionalities (D'Ambrosio et al., 2013). We are former public middle and high school teachers who are white, cisgender, and monolingual. We have all participated in groups working for equity in schools that have been led by both scholars of color and white scholars. All of us have taught racially, socio-economically, linguistically, geographically, gender, and cognitively diverse students, both as former classroom teachers and in our research. All of us studied issues of equity during our academic training, and one of us has a Ph.D. in Urban Education Studies, with an explicit focus on racial equity, as well as a Ph.D. in Mathematics Education. All of us are makers and users of second-order models of students, two of us experienced. All of us consider it our professional responsibility to inquire into and disrupt inequities, to shape our research and teaching to support equity, and to continue to learn. Although we have space for only a few details, we hope this paragraph helps readers see why we are compelled by the issues in this paper.

### **How Second-Order Model Building is an Act of Equity**

In this section we explain how building second-order models of students' mathematics is an act of equity that demonstrates respect for students and that can influence power. We write about ourselves as teacher-researchers in research settings and classrooms.

#### **First- and Second-Order Knowledge**

We view rational human knowing as an active process of perceiving and conceiving (von Glasersfeld, 1995). Humans can never get outside of their ways of perceiving and conceiving in order to check whether their ways of perceiving and conceiving are the way things "really" are in the world. Humans can check with sources outside themselves, certainly! But the check a person makes is performed via their ways of perceiving and conceiving. Therefore, the function of knowing is to organize one's experiential world rather than to discover an objective ontological reality. Yet, this position on knowing is not solipsistic, as humans repeatedly encounter constraints through interaction that help them shape their constructions. These constraints include social boundaries, as well as actions or attempted actions upon these boundaries.

Within this framing of knowing, we differentiate between first-order and second-order knowledge. First-order knowledge is the knowledge that a person constructs "to order, comprehend, and control his or her experience" (Steffe, et al., 1983, p. xvi). In contrast, second-order knowledge is the knowledge that a person constructs about another person's knowledge "in order to explain their observations (i.e., their experience) of the subject's [another person's] states and activities (Steffe et al., p. xvi)." A *second-order model* is a constellation of constructs that a teacher-researcher makes to describe and account for another person's mathematical reasoning and learning (Steffe et al., 1983; Ulrich et al., 2014). So, a second-order model is a *scientific* model of someone else's first-order knowledge.

#### **Why is Making a Second-Order Model an Act of Equity?**

Making a second-order model is an act of equity because it is an act of respect for the thinking of the students whose mathematical thinking is being modeled. The message of making a second-order model is: "I see your way of thinking, and I am going to hold it out as important and unique." Making second-order models is a time-intensive endeavor, on the order of months

and years; it is not something to engage in lightly. In our view, the time-intensive nature of model-making deepens it as an act of respect for the students whose thinking is being modeled. The making of a second-order model is never fully complete due to the complexity of each individual student's thinking. However, teacher-researchers can get to the point where they have a model that holds up across all of their interactions with the student (Steffe & Thompson, 2000).

And yet, if the making of second-order models was only about highlighting the individual thinking of particular students, it would be an honorable pursuit with limited impact. A central reason making second-order models is an act of equity is because making them can contribute to the construction of epistemic subjects (Beth & Piaget, 1966), although we use the term *epistemic students*.

### **What is an Epistemic Student?**

An *epistemic student* is “that which is common to all subjects [students] at the same level of development, whose cognitive structures derive from the most general mechanisms of the coordination of actions” (Beth & Piaget, 1966, p. 308). We view an epistemic student as a generalization that teacher-researchers make of their second-order models in the context of using them with future students. That is, creating an epistemic student comes from a researchers' iterative process of making and refining second-order models of student thinking, using these models in future interactions, and engaging with other researchers' second-order models.

For example, in a first-grade classroom with 25 students, a teacher has 25 individual thinkers. Yet, in terms of how the students think about number, second-order model building has shown that there are similarities among students that can help teacher-researchers organize interactions with them (Steffe & Cobb, 1988; Steffe et al., 1983). Teacher-researchers have found that children construct four different number sequences as they construct whole number knowledge. These number sequences are epistemic students, and they are one support in managing the complexities of interacting with a multitude of students; see Hackenberg et al. (2022) for more.

However, epistemic students should be orienting for a teacher-researcher, not deterministic. What this means is that a teacher-researcher uses their knowledge about epistemic students as a guide to support teaching interactions, but they do not use that knowledge to create a list of forbidden topics. In fact, students operating with different number sequences are likely to work on some topics or problems in different ways, but they are not barred from working on certain topics or problems. As Steffe (1992) has explained, he lets the students be the guide regarding what topics to work on, as he adapts to their ways of thinking.

### **How is Using Epistemic Students an Act of Equity?**

Now we turn to how using epistemic students in interaction with particular students is an act of equity. In our view, using epistemic students in interaction is a higher level of respect for particular students than simply making second-order models. It is a higher level of respect because it involves taking action in order to interact supportively with these students in the moment and with students who have similar ways of thinking in the future. A teacher-researcher can use their epistemic students to inform their learning goals, task design, and plans for interaction (e.g., questioning). In so doing, a teacher-researcher is trying to open possibilities for students to engage and learn. A teacher-researcher is also simultaneously building their knowledge of how to interact with students who think similarly in the future. We view using epistemic students in interactions with particular students as “respect for students in action.”

In addition, using epistemic students in interactions in research settings and classrooms can allow teacher-researchers and students to act upon social boundaries to change their shape or

direction in order to rectify past inequities. For example, a norm that is typical in secondary math classrooms is that students are to learn the first-order knowledge of the teacher (Teuscher et al., 2016). In other words, in typical secondary math classrooms, teachers present mathematical ideas neutrally, as “the mathematics” to be learned. However, the mathematics to be learned in any given classroom is determined by the teacher’s first-order knowledge. Most of the time, teachers attempt to convey their first-order knowledge to students. And, students often tacitly agree to this norm and try to learn these meanings (Liljedahl, 2021; Thompson, 2016). And yet, students cannot learn these meanings directly; they can only interpret the teacher’s activity with the meanings that they have constructed at that point—with their own first-order knowledge. Nevertheless, many of them try. And thus, the norm of students being tasked with learning the teacher’s knowledge is generated and reinforced. This norm becomes one of the central boundaries that shapes the fields of action for teachers and students.

We view this norm as a fundamental inequity because of the heavy burden it places on students and because of the way in which it keeps student thinking hidden. First, trying to learn the first-order knowledge of the teacher means that students are tasked with trying to learn what they often do not have the ways of thinking to learn. Second, student thinking stays hidden. It may surface at times, but when it does, it usually is as a curiosity or anomaly. There is no sense that students’ thinking could be a body of knowledge that is different from the teacher’s first-order knowledge, and yet is still knowledge, with coherence and depth, and with ways that it could be modified and advanced under supportive pedagogical environments.

A teacher-researcher who is using epistemic students in interaction with particular students can subvert this larger norm of students being tasked with learning the teacher’s first-order knowledge. In a research setting, the typical norm may surface by students waiting for the teacher-researcher to tell them what to do, or by not sharing their ideas. A teacher-researcher who is using epistemic students in interaction must learn the mathematical ways of thinking of the particular students. So, this teacher-researcher will have goals of bringing student thinking to the forefront of discussion and using that thinking to support the progress of all students involved. This orientation and these goals, if enacted, can disrupt the typical norm of students being tasked with learning the first-order knowledge of the teacher. This orientation and goals can instead work to establish the norm that the teacher-researcher must build second-order knowledge of the students, and that the students’ mathematics should be the subject matter of the research setting—even in whole classrooms. Thus, using epistemic students in interaction with particular students can influence how power affects students and the teacher-researcher in research settings by shifting whose mathematical knowledge is valued and centered.

### **How Can We Enhance Second-Order Models?**

We now turn to our third task, to identify how second-order models could be enhanced to better support acts of equity. To do so, we focus on the inclusion of social identities in the creation of second-order models. In addition, we adopt two terms from van Es et al. (2022), *expanse* and *stretch*. *Expanse* is “the breadth and range of what teachers [-researchers] identify as noteworthy” (p. 115) in a research interaction. *Stretch* is the ways in which a teacher-researcher “reaches back historically and forward futuristically” (p. 117) to consider their own and their participants’ pasts and futures. We apply these terms to using social identities in the creation of second-order models because researchers making second-order models have typically focused on making accounts of students’ mathematical knowing without including how teacher-researchers’ and participants’ social identities may impact these interactions. To situate our discussion of

expanding and stretching current second order models, we first define social identity categories and differentiate them from social identities.

### **Social Identity Categories and Social Identities**

We use the term *social identity categories* to refer to the range of identifiers commonly used to categorize people, including in the reporting of research, such as race, gender, socioeconomic status, etc. (Langer-Osuna & Esmonde, 2017; Martin, et al., 2017). Researchers have argued that social identity categories are deeply ingrained in society in that they are encoded in law (e.g., Feagin, 2010; Lipsitz, 1995) and used to shape institutions like schools (Battey & Leyva, 2016; Ladson-Billings, 2006; Tyack, 1974). Therefore, they have material consequences for all individuals, and these material consequences are differential, depending on how an individual is categorized by others and how the individual categorizes themselves within the broader system (Battey, 2013; Leonardo, 2009). Moreover, individual members of a society have highly differential access to creating or making changes to these categories or to the consequences of the system of categorization (Gimenez, 2014). And, the categories can be used by those who have more access as one way to define the social boundaries of action for those who have less access (Freire, 1993). Thus, the extent to which individuals experience a sense of agency to change the shape or direction of these social boundaries depends on the access they have to mechanisms that can effect change, often requires collective action over time, and is related to the material consequences an individual may experience for working to make such changes.

With these observations in place, and commensurate with radical constructivist epistemology, we position social identity categories as constraints, which function similarly to physical or conceptual constraints (von Glasersfeld, 1995), in that they are the reality that “kicks back” as individuals’ construct their *social identities* in interactions with each other. We note that *social identity categories are themselves constructions*, but because they have a legal basis and history, the categories have a different level of durability and social sanctioning than, for example, individuals’ construction of their social identities in a mathematical interaction. Moreover, the durability of these categories, and the way that they constrain individuals’ interactions, depends on the context in which they are invoked, and on how they have been encoded and used within that context. We further note that the constraints that individuals experience in constructing their social identities are differential across social identity categories. That is, laws that result in differential benefits across different social identity categories (e.g., access to well-funded schools) form the basis for individuals to experience differential constraints in their construction of their social identities within those contexts.

With social identity categories positioned as constraints that individuals experience in their construction of their social identity, we return to an epistemological point that von Glasersfeld (1995) frequently made: An individual’s first order knowledge is constructed in relation to, but not determined by, their experience of constraints. That is, we propose that social identity categories form one set of constraints in individuals’ constructions of their social identities, but that there are multiple *viable* responses to these constraints. We see this theoretical point as an important way to understand differences in the way members of the same social identity categories construct their social identities: There are multiple viable responses to any set of constraints. We note, too, that the intersection of multiple of these social identity categories are often at play as individuals construct and enact their social identities within a particular context (Collins & Bilge, 2020).

These considerations allow us to differentiate between first- and second-order knowledge within the arena of teacher-researchers' and students' social identities. We have described how social identity categories can form constraints in a person's construction of their first-order knowledge of their social identities. Second-order knowledge of social identities, then, entails a teacher-researcher making second-order models of the ways that their own and their participants' social identities impact interactions, and in particular impact interactions aimed at mathematical learning. We now turn to two examples of stretching and expanding second-order models by attending to the teacher-researchers' and participants' social identities. Due to space limitations, we can only outline them briefly but will elaborate on them more fully in the presentation.

### **Example 1: Addressing Gender Equity in Interactions**

During a recent teaching experiment that the second author conducted with two pre-service secondary mathematics teachers, a graduate student interested in gendered patterns of interaction pointed out that the two participants were engaged in some gender normative patterns of interaction related to their expressions of confidence in their mathematical thinking (Lubienski & Ganley, 2017; Bench, et al., 2015) and how they each attended to the other persons' thinking. Specifically, the female participant was quite attentive to her male partners' mathematical language and reasoning. Although she was a powerful reasoner herself, she also often doubted her mathematical thinking in these interactions. Her male partner, on the other hand, rarely doubted his mathematical thinking and at the same time had difficulty making sense of his female partners' thinking (see Ippolito, et al., 2021). These gender normative patterns of interaction, which were linked to the social identities of the participants and researcher, were one component of the mathematical interactions in the teaching experiment.

As part of the experiment, the teacher-researcher formed the goal of disrupting these gender normative patterns of interaction by framing each participant as possessing a strength—the female participant being a powerful mathematical reasoner who could make strong interpretations of her partner's mathematical thinking, and the male participant having a high level of confidence in his mathematical thinking. He asked each participant to work on the other person's strength relative to themselves—the female participant to work on being more confident in her mathematical thinking, and the male participant to work on understanding his partners' thinking. The pre-service teachers appeared to appreciate the teacher-researcher's suggestion at the time he made it, and they willingly took it up in future teaching episodes.

We see the teacher-researcher and participants' actions as an example of an act of equity in that the researcher and participants acted on a social boundary with the intent of changing its shape or direction in order to address known inequities in patterns of participation related to the gender identities of the participants and researchers. In presenting the example, we will show evidence that the teaching action produced a more equitable outcome regarding the gender normative patterns of interaction. However, that by itself does not tie the act of equity to learning or opportunities for learning. In the presentation, we will identify how making second-order models that attend to the teaching action aimed at disrupting gendered patterns of interaction could be linked to participants' mathematical learning.

### **Example 2: Designing to Address Equity**

A second way to expand second-order models is by designing mathematical work on an issue of social justice into a study in order to make models of both students' mathematical reasoning and students' understanding of an equity-related issue. For example, the third author designed a study in which he asked middle grades students to explore racial bias in jury selection (Gatza,

2021). There were five participants in his study, most of whom were students in his eighth-grade algebra class. Of the five participants, three self-identified as Black/African-American, one self-identified as Hispanic, and one self-identified as multi-racial, Black and White.

Gatza (2021) conducted the design study in an after-school setting, and his analysis focused on two students, one African-American male and one Hispanic male, who participated in 18 teaching episodes. Gatza's study began with an initial interview in which he identified key features of students' multiplicative reasoning coupled with key features of their understanding of race, racial identity, racism, and racial bias. This initial interview was designed so that the researcher could share information about how he thought about his own racial identity, and to allow the students to share their understandings of their racial identity.

The broad intent of the design study was to understand how students' mathematical reasoning impacted their understanding of racial bias, and how their understanding of racial bias impacted their mathematical reasoning. As part of his study, Gatza (2021) also worked with students on differentiating conscious from unconscious bias as one way to support explanations of how an actual outcome could differ significantly from an expected outcome. He elicited initial ideas about conscious and unconscious bias by having students examine a flyer from the American Red Cross about pool safety that showed children phenotypically presenting as white labeled as acting good and children phenotypically presenting as children of color as acting bad. The students initially interpreted the flyer as not involving bias, but simply as happenstance. Through multiple discussions of the meaning of both conscious and unconscious bias that occurred over time, the students were able to interpret what each form of bias would "look like" when they analyzed artifacts like the flyer. They were, then, able to apply these ideas to the jury selection process to reason about factors that could produce differences between actual and expected outcomes. Thus, their development of schemes related to racial bias supported the development of mathematical understandings.

Within the interactions, Gatza's (2021) design study aimed to cultivate respect for the individual participants by basing the teaching episodes on the students' evolving mathematical and racial bias schemes. Moreover, the design was tied to a social and historical inequity, racial bias. Researchers have found that this particular inequity impacts many facets of Black and Latinx students' lives, including their experiences in schools (Gutiérrez, 2018; Kozol, 2005; Ladson-Billings, 2006; Martin, 2009), and that Black and Latinx students who have strong understandings of race, racism, and racial bias are more likely to thrive in schools (Carter, 2008; Oyserman, Brickman, & Rhodes, 2007). Thus, the teacher-researcher explicitly designed contexts to open opportunities for students to develop capacity to act upon the social boundaries that constrain and enable fields of action by changing their shape or direction. Moreover, throughout the study, there was indication that the participants willingly engaged, working to develop their ideas about race, racism, and racial bias. In the presentation, we give details about the relationship between the students' mathematical schemes and their schemes of racial bias.

### **Conclusion**

Our basic stance in writing this paper is that radical constructivism, as a theory of knowing, has to be paired with first- and second-order models in many domains in order to develop the theory and its usefulness. For example, radical constructivism has already been augmented by a model of communication (Thompson, 1999, 2013), as well as second-order models of specific mathematical domains like students' construction of whole numbers (Steffe & Cobb, 1988; Steffe et al., 1983). We consider this paper to be a step in contributing to a model of equity. We



invite discussion to produce refinements of constructs and continued enhancements of second-order models and their uses. The goals are to advance equity and disrupt inequities.

## References

- Adiredja, A. P. (2019). Anti-deficit narratives: Engaging the politics of research on mathematical sense-making. *Journal for Research in Mathematics Education*, 50(4), 401-435.
- Aguirre, J., Herbel-Eisenmann, B., Celedón-Pattichis, S., Civil, M., Wilkerson, T., Stephan, M., Pape, S., & Clements, D. H. (2017). Equity within mathematics education research as a political act: Moving from choice to intentional collective professional responsibility. *Journal for Research in Mathematics Education*, 48(2), 124-147.
- Ball, S. J. (2013). *Foucault, power, and education*. New York: Routledge.
- Barajas-López, F., & Larnell, G. V. (2019). Unpacking the links between equitable teaching practices and standards for mathematical practice: Equity for whom and under what conditions? *Journal for Research in Mathematics Education*, 50(4), 349-361. <https://doi.org/10.5951/jresmetheduc.50.4.0349>
- Battey, D. (2013). Access to mathematics: A possessive investment in Whiteness. *Curriculum Inquiry*, 43(3). <https://doi.org/10.1111/curi.12015>
- Battey, D. & Leyva, L. A. (2016). A framework for understanding whiteness in mathematics education. *Journal of Urban Mathematics Education*, 9(2), 49-80.
- Bench, S. W., Lench, H. C., Liew, J., Miner, K., & Flores, S. A. (2015). Gender gaps in overestimation of math performance. *Sex roles*, 72(11), 536-546
- Berry, R. Q. (2015). Addressing the needs of the marginalized students in school mathematics: A review of policies and reforms. In T. G. Bartell, K. N. Bieda, R. T. Putnam, K. Bradfield, & H. Dominguez, (Eds.). *Proceedings of the 37th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, (pp. 19-32). East Lansing, MI: Michigan State University.
- Beth, E. W., & Piaget, J. (1966). *Mathematical epistemology and psychology* (W. Mays, trans.). Dordrecht: D. Riedel.
- Carter, D. J. (2008). Cultivating a critical race consciousness for African American school success. *The Journal of Educational Foundations*, 11-28.
- Cobb, P., Gresalfi, M., & Hodge, L. L. (2009). An interpretive scheme for analyzing the identities that students develop in mathematics classrooms. *Journal for Research in Mathematics Education*, 40(1), 40-68.
- Collins, P. H., & Bilge, S. (2020). *Intersectionality*. John Wiley & Sons.
- D'Ambrosio, B., Martin, D. B., Frankenstein, M., Moschkovich, J., Gutierrez, R., Taylor, E., Kastberg, S., & Barnes, D. (2013c). Positioning oneself in mathematics education research. *Journal for Research in Mathematics Education*, 44(1), 11-22.
- Ellis, A. B., (2022). Decentering to build asset-based learning trajectories. In A. E. Lischka, E. B. Dyer, R. S. Jones, J. N. Lovett, J. Strayer, & S. Drown (Eds), *Proceedings of the forty-fourth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 15-29). Middle Tennessee State University.
- Feagin, J. R. (2010). *Racist American: Roots, current realities, and future reparations, second edition*. New York: Routledge.
- Foucault, (1980). *Power/knowledge: Selected interviews and other writings, 1972-1977*. (C. Gordon, trans.). New York: Pantheon Books.
- Friere, P. (1993). *Pedagogy of the oppressed*. New York: Bloomsbury Academic
- Gatza, A. M. (2021). Not just mathematics, “just” mathematics: Investigating mathematical learning and critical race consciousness. [Doctoral dissertation, Indiana University, Bloomington]. ProQuest.
- Gholson, M., & Martin, D. B. (2014). Smart girls, Black girls, mean girls, and Bullies: At the intersection of identities and the mediating role of young girls' social network in mathematical communities of practice. *Journal of Education*, 194(1), 19-33.
- Gimenez, M. E. (2014). Latino/”Hispanic”—Who needs a name? The case against a standardized terminology. In A. Darder and R. Torres (Eds.), *Latinos and education: A critical reader* (pp. 93-104). New York: Routledge.
- Gutiérrez, R. (2002). Enabling the practice of mathematics teachers in context: toward a new equity research agenda. *Mathematical Thinking & Learning*, 4(2/3), 145-187. [https://doi.org/10.1207/S15327833MTL04023\\_4](https://doi.org/10.1207/S15327833MTL04023_4)
- Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

- Gutiérrez, R. (2018). The need to rehumanize mathematics. In I. Goffney & R. Gutierrez (Eds.), *Rehumanizing mathematics for Black, Indigenous, and Latinx students* (pp. 1-10). Reston, VA: National Council of Teachers of Mathematics.
- Gutstein, E. (2016). “Our Issues, Our People—Math as Our Weapon”: Critical Mathematics in a Chicago neighborhood high school. *Journal for Research in Mathematics Education*, 47(5), 454-504.
- Hackenberg, A. J., Aydeniz, F., & Borowski, R. (2022). *Differentiating instruction on speed for middle school students* [Manuscript under revision].
- Hayward, C. R. (1998, 10/01/). De-facing power. *Polity*, 31(1), 1-22. <https://doi.org/10.2307/3235365>
- Ippolito, D., Ataide Pinheiro, W., Liu, J. (2021, July 11-18). *Gender differences in student-student interaction* [Conference presentation]. The 14th International Congress on Mathematical Education, Shanghai, China.
- Kokka, K. (2022). Toward a theory of affective pedagogical goals for social justice mathematics. *Journal for Research in Mathematics Education*, 53(2), 133-153. <https://doi.org/10.5951/jresmetheduc-2020-0270>
- Kozol, J. (2005). *The shame of a nation: The restoration of the apartheid schooling in America*. New York: Three Rivers Press.
- Ladson-Billings, G. (2006). From the achievement gap to the education debt: Understanding achievement in U.S. schools. *Educational Researcher*, 35(7), 3-12.
- Langer-Osuna, J. M., & Esmonde, I. (2017). Identity in research on mathematics education. *Compendium for research in mathematics education*, 637-648.
- Leonardo, Z. (2009). Reading whiteness: Anti-racist pedagogy against white racial knowledge. In W. Ayers, T. Quinn, & D. Stovall (Eds.), *Handbook of Social Justice in Education* (pp. 231-248). Routledge.
- Liljedahl, P. (2021). *Building thinking classrooms in mathematics, grades K-12: 14 teaching practices for enhancing learning*. Corwin Press, Inc.
- Lipsitz, G. (1995). The possessive investment in whiteness: Racialized social democracy and the ‘white’ problem in American studies. *American Quarterly*, 47(3), 369-387.
- Lubienski, S. T., & Ganley, C. M. (2017). Research on gender and mathematics. *Compendium for research in mathematics education*, 649-666.
- Maloney, T., & Matthews, J. S. (2020). Teacher care and students’ sense of connectedness in the urban mathematics classroom. *Journal for Research in Mathematics Education*, 51(4), 399-432.
- Martin, D. B. (2000). *Mathematics success and failure among African- American youth*. Erlbaum.
- Martin, D. B. (2009). Researching race in mathematics education. *The Teachers College Record*, 111(2), 295-338.
- Martin, D. B., Anderson, C. R., & Shah, N. (2017). Race and mathematics education. In J. Cai (Ed.), *Compendium for research in mathematics education*, pp. 607-636. Reston, VA: National Council of Teachers of Mathematics.
- Oyserman, D., Brickman, D., & M. Rhodes (2007). Racial-ethnic identity in adolescence: Content and consequences for African American and Latino youth. In A. Fuligni (Ed.) *Social Categories, Identities and Educational Participation*, pp. 91-114. NY: Russell-Sage.
- Shah, N. (2017). Race, ideology, and academic ability: A relational analysis of racial narratives in mathematics. *Teachers College Record*, 119(7), 1-42.
- Steffe, L. P. (1992). Schemes of action and operation involving composite units. *Learning and Individual Differences*, 4(3), 259-309. doi:10.1016/1041-6080(92)90005-Y
- Steffe, L. P., & Cobb, P. (1988). *Construction of arithmetical meanings and strategies*. New York: Springer-Verlag.
- Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In A. E. Kelly & R. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 267-306). Erlbaum.
- Steffe, L. P., von Glasersfeld, E., Richards, J., & Cobb, P. (1983). *Children’s counting types: Philosophy, theory, and application*. New York, NY: Praeger Scientific.
- Stinson, D. (2013). Negotiating the “White male math myth”: African American male students and success in school mathematics. *Journal of Research in Mathematics Education* 44(1), 69-99.
- Teuscher, D., Moore, K., & Carlson, M. (2016). Decentering: A construct to analyze and explain teacher actions as they relate to student thinking. *Journal of Mathematics Teacher Education*, 19(5), 433-456. <https://doi.org/10.1007/s10857-015-9304-0>
- Thompson, P. W. (1999). Some remarks on conventions and representations. In F. Hitt & M. Santos (Eds.), *Proceedings of the 21st Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.
- Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

- Thompson, P. W. (2013). In the absence of meaning... . In Leatham, K. (Ed.), *Vital directions for research in mathematics education* (pp. 57-93). New York, NY: Springer.
- Thompson, P. W. (2016). Researching mathematical meanings for teaching. In L. D. English & D. Kirshner (Eds.), *Handbook of international research in mathematics education* (pp. 435-461). Taylor & Francis.
- Tyack, D. (1974). *The one best system: A history of American urban education*. Cambridge, MA: Harvard University Press.
- Ulrich, C., Tillema, E. S., Hackenberg, A. J., & Norton, A. (2014). Constructivist model building: Empirical examples from mathematics education. *Constructivist Foundations*, 9(3), 328–359.
- van Es, E. A., Hand, V., Agarwal, P., & Sandoval, C. (2022). Multidimensional noticing for equity: Theorizing mathematics teachers' systems of noticing to disrupt inequities [Article]. *Journal for Research in Mathematics Education*, 53(2), 114-132. <https://doi.org/10.5951/jresmetheduc-2019-0018>
- von Glasersfeld, E. (1995). *Radical constructivism: A way of knowing and learning*. London, England: Routledge Falmer. doi:10.4324/9780203454220
- Walshaw, M. (2013). Post-structuralism and ethical practical action: Issues of identity and power. *Journal for Research in Mathematics Education*, 44(1), 100-118.

## ENVISIONING THE ROLE OF THE TEACHER IN EQUITABLE MATHEMATICS INSTRUCTION

Cara Haines  
University of  
Nevada-Las Vegas  
cara.haines@unlv.edu

Charles Munter  
a public  
research university in Missouri  
munterc@missouri.edu

Erica N. Mason  
University of  
Illinois Urbana-Champaign  
enmason@illinois.edu

*We investigated the development of teachers' and other practitioners' visions of equitable mathematics instruction. This involved analyzing data from annual interviews conducted across three school years with 25 practitioners who were working in an urban school district that was trying to decrease a racial opportunity gap in secondary mathematics. During this time, participating teachers attended two five-week professional learning workshops that were designed to support them in confronting sources of racial inequity in school mathematics. Our results include rubrics that model developing visions of equitable mathematics instruction along five dimensions: role of the teacher, mathematical tasks, grouping, evidence of equity, and race consciousness. In this paper, we focus on our findings regarding the role of the teacher.*

Keywords: Instructional Vision; Equity, Inclusion, and Diversity.

There is general agreement within the mathematics education community that all K–12 students should be afforded opportunities to make sense of important mathematical concepts, engage in argumentation, and develop a positive disposition toward the discipline (Hiebert & Grouws, 2007; National Council of Teachers of Mathematics [NCTM], 2014). The community has also converged on a set of instructional practices that can afford such opportunities. Often referred to collectively as *ambitious mathematics instruction* (e.g., Lampert et al., 2010), those practices include posing cognitively demanding tasks that can be solved in multiple ways, providing opportunities for students to reason about those tasks with peers, and facilitating whole-class discussions of students' strategies and solutions and connections between them.

As some researchers have pointed out, ambitious mathematics instruction is limited in terms of its potential to address issues of *equity* that have long permeated schools and classrooms in the United States (e.g., Gutiérrez, 2012; Martin, 2015). For example, while NCTM's *Principles to Actions* (2014) framed teachers' enactment of ambitious instruction as a means of ensuring that all students including those who have been historically marginalized in mathematics (e.g., Black, Latina/o, Indigenous, and poor students) can experience success, the document provided little guidance for working to change the system that is largely responsible for the marginalization of those students in the first place (Martin, 2015). Additionally, the instructional practices promoted therein, such as implementing tasks with “multiple entry points and varied solution strategies” (NCTM, 2014, p. 10), are aimed more at addressing dominant dimensions of equity—*access* and *achievement*—than critical ones—*identity* and *power*—and thus are unlikely to disrupt the status quo in mathematics education (Gutiérrez, 2012).

More recently, however, NCTM (2018; 2020) acknowledged ambitious mathematics instruction as, alone, insufficient for working toward equity and identified the enactment of *equity-based* ideas and practices as another important goal for teachers' learning. Such ideas and practices focus more explicitly on issues of identity and power and include, for example, culturally relevant teaching (Ladson-Billings, 1995), challenging conventional notions of

mathematical competence (Aguirre et al., 2013), and supporting students in developing a sociopolitical disposition (Bartell et al., 2017).

Although prior research provides insight into how teachers develop ambitious mathematics instruction, fewer studies have focused on teachers' development of *equitable* instruction. As an example of the former, Munter (2014) examined teachers' developing visions of ambitious—or high-quality—mathematics instruction to better understand how teachers learn to enact such instruction. This work resulted in an assessment tool that models developing visions of high-quality mathematics instruction along three dimensions of practice: role of the teacher, classroom discourse, and mathematical tasks.

As he noted, a limitation of Munter's (2014) assessment tool is that it does not explicitly account for the development of instructional visions with respect to *equity*. That is, while the tool models developing visions of ambitious instructional practices that are important for addressing issues of equity (e.g., posing tasks with multiple entry points), it does not characterize visions of the equity-based practices the mathematics education community has, thanks to the persistent work of a few, more recently deemed important (e.g., posing culturally relevant tasks). The work shared in this paper marks our initial attempt to understand the development of these latter visions. The research question that guided this work was: Along what trajectories do teachers' (and other practitioners') visions of equitable mathematics instruction develop?

### **Conceptual Framing**

Our work draws from research indicating that teachers' discourses are correlated with (or predictive of) teachers' enactment of instructional practices. For example, research has shown that mathematics teachers' enactment of ambitious instruction relates to their mathematical knowledge for teaching (Wilhelm, 2014), views of their students' mathematical capabilities (Jackson et al., 2017), and conceptions of what constitutes high-quality instruction (Munter & Correnti, 2017; Wilhelm, 2014). The last of these includes instructional vision: “the discourse that teachers or others currently employ to characterize the kind of ideal classroom practice to which they aspire but have not yet necessarily mastered” (Munter & Wilhelm, 2021, p. 343). Teachers' instructional visions are important because they can provide a kind of “leading indicator” of where teachers' practice may be likely to grow.

### **Methods**

Our study was conducted in the context of a research-practice partnership that was aiming to decrease a racial opportunity gap in secondary mathematics in an urban school district in the Northeastern United States serving a student population that was approximately 55% Black, 30% white, and 60% designated economically disadvantaged. The project involved five-week professional learning workshops in two consecutive summers that supported participating teachers in confronting sources of racial inequity in school mathematics, with follow-up monthly sessions during the following school years. Workshop leaders worked to support teachers in reconceptualizing conventional notions of knowing and doing mathematics, interrogating instantiations of racism in the district and in school mathematics, considering the role of identity in mathematics teaching and learning, and pursuing ambitious and equitable instructional practices such as the five practices (Stein et al., 2008) and culturally relevant teaching (Ladson-Billings, 1995). Given its context and foci, the project provided opportunity to observe changes in teachers' visions of equitable mathematics instruction.

### **Data Collection**

The project's research team interviewed district leaders, principals, and teachers to understand how the project and other district improvement initiatives were playing out in schools

and classrooms. Of interest for this analysis were sections of interviews in which participants described equitable mathematics instruction. To capture change over time, we focused on participants with whom we conducted repeated interviews. Our sample included 25 practitioners, six of whom were people of color, and yielded a total of 62 interviews, which were audio recorded and transcribed. The 25 practitioners included 20 teachers, three principals, and two district leaders.

### **Data Analysis**

Our analysis focused on responses to the interview prompt *If you were asked to observe another math teacher's classroom for one or more lessons, what would you look for to determine whether the instruction is equitable?* and to any follow-up questions. First, we identified and coded *dimensions of practice* along which individuals characterized equitable mathematics instruction. When appropriate, we collapsed codes to characterize participants' visions along as few dimensions as possible. For example, we collapsed a preliminary "teacher questions" code under our "role of the teacher" code because participants' talk about the kinds of questions they would expect to hear the teacher asking was always situated within descriptions of what they would expect to see the teacher doing to work toward equity.

Next, we modeled the development of instructional visions by creating *trajectories* for each dimension. Following Munter (2014), this involved creating *levels* for each trajectory by making qualitative distinctions among participants' responses, with the top levels representing aspects of equitable instruction described in relevant research literature (e.g., Aguirre et al., 2013; Cabana et al., 2014; Horn, 2012; Wilson et al., 2019). To create levels, we differentiated between participants' visions in terms of their degree of alignment with the top levels. This involved making distinctions in participants' descriptions of different forms of practice and their intended functions (Saxe et al., 1999), diversity (e.g., in students' mathematical ability), and dimensions of equity (Gutiérrez, 2012). For example, when participants described the teacher as tailoring support to students' abilities, we differentiated between more and less productive framings of ability (Jackson et al., 2017). Finally, we named the levels and identified examples from interview data to illustrate the discourse associated with them.

### **Findings**

Our analysis produced five rubrics that model developmental trajectories of visions of equitable instruction along five dimensions: role of the teacher, mathematical tasks, grouping, evidence of equity, and race consciousness. While Munter's (2014) framework for characterizing visions of high-quality mathematics instruction includes trajectories specific to two of those five dimensions (role of the teacher and mathematical tasks), the trajectories that resulted from this analysis pertain to facets of those dimensions beyond what the original work specified. Due to space limitations, we discuss only the role of the teacher rubric.

#### **The Role of the Teacher Rubric**

In what follows, we elaborate on distinctions between the four levels of the rubric. While reading about the levels, we encourage readers to refer to Figure 1, which includes example quotes from interview data.

Level 1, the lowest level, describes the teacher *ensuring fairness* so that all students have an equal chance at success. Such descriptions include portrayals of the teacher distributing opportunities to all students at the same frequency or tailoring instructional supports to individual students' abilities and learning needs. However, Level 1 lacks attention to the *quality* of the teacher's treatment of students, such as the rigor of the learning opportunities the teacher affords; depicts scenarios in which the teacher holds different students to different standards;

and/or includes *unproductive* framings of ability, implying that ability is hierarchical (Louie, 2017) or attributable to characteristics of some students or their families (Jackson et al., 2017).

**Table 1: Quotes Illustrating Levels of the Role of the Teacher Rubric**

Level	Example Quotes
4	<ul style="list-style-type: none"> <li>• “When the teacher is picking ideas and sharing them, they shouldn’t just be picking from a certain group of students. [I would look to see that] a wide variety of students’ ideas are being shared and discussed, so that there isn’t a type of student that seems like they know what they’re doing more than the other ones.”</li> <li>• “[The teacher should give students] autonomy to be able to decide what direction they were going to take a problem. Maybe they didn’t get the right answer, but they were looking at it in a different way, and the teacher [is] showing that what they did was just as good as what everybody else did, even though they didn’t come to the final conclusion that everybody else did.”</li> </ul>
3	<ul style="list-style-type: none"> <li>• “I would make sure that the teacher is being conscious of who they’re calling on...because I think there are a lot of students who want to talk and have a lot of faith in their own math ability, but if you always let those students talk, there are other students who haven’t had math experiences that set them up to feel like they’re gonna be successful in the classroom. So giving them opportunities to talk too will help their confidence and how they’re doing.”</li> <li>• “So if it’s equitable, I would like to think that the teacher did not see certain members of the class as disadvantaged and treat them like they were coming from a deficit. And maybe having some of those kids bring something from their lives, their hobbies, whatever. ‘What do you bring from your life that maybe could enrich all of us?’”</li> </ul>
2	<ul style="list-style-type: none"> <li>• “I would see that the teacher is talking to all groups. Questioning the same way, not giving one group answers and the other group questions.”</li> <li>• “[The teacher should be] understanding that kids are coming from a certain perspective...so when you’re interacting with them, you’re not expecting them to be doing the problem all the same way... [but] if students are starting to head towards a misconception, [the teacher should make sure] they see why it’s a misconception...or think about another pathway.”</li> </ul>
1	<ul style="list-style-type: none"> <li>• “[I would look to see that] every student is talked to, not just like the smart kids, or the ones that are struggling. Everyone gets equal attention.”</li> <li>• “I always have fast finisher work on the board, so those students that are more advanced can get to that.”</li> </ul>

Level 2 depicts the teacher *having high expectations for all students* by ensuring that their mathematics learning opportunities and participation, while different perhaps, are of a certain *quality*. This includes suggestions that the teacher minimize differences in how the quality of learning opportunities is distributed among students or provide instructional supports that are tailored to students’ abilities and learning needs but also aimed at supporting students in attaining *similar* achievement outcomes over time. Unlike Level 1, Level 2 describes scenarios in which students are supported to participate in ways that are equally rigorous and includes *productive* framings of ability, which suggest that students’ abilities are multifaceted or likely to change.

Level 3 describes the teacher's role in *attending to students' classroom experiences* based on (intersections) of their various social identities, potentially at the expense of affording high-quality learning opportunities. While the teacher's actions are described as serving the purpose of affirming students' identities (Ladson-Billings, 1995) or supporting students in feeling or seeing themselves and each other in positive ways (Boaler, 2008), they are also conservative (Apple, 2004) in that they lack intention and are unlikely to counter power structures that uphold the status quo in school mathematics (Martin, 2015).

Finally, Level 4 focuses on the teacher's role in *challenging spaces of marginality* (Aguirre et al., 2013)—countering practices, systems, and structures that have contributed to marginalizing students in school mathematics. Instructional recommendations depict the teacher being *explicit* and *intentional* in their efforts to “change the game” (Gutierrez, 2012) of school mathematics to prevent the reproduction of inequities, including centering the perspectives and experiences of students of marginalized backgrounds; challenging conventional notions of mathematics, who it is for, and what doing it entails; and making clear connections between mathematics and power.

To help illustrate the difference between Levels 3 and 4, consider the example quotes at each level in Table 1. The first example at each level describes a scenario in which the teacher is inviting students to share with the class. The response at Level 3 focuses on the experience of the imagined student, to “set them up to feel like they're gonna be successful” and “help their confidence.” This is crucial, to be sure, but at Level 4 the response captures more about the intentionality of teachers' positioning of students, where it is not just about the experience of the student who shares, but also about attending to the narratives that can be built up over time across multiple students, working to resist there being “a type of student that seems like they know what they're doing more than the other ones.”

### Significance

Recent years have seen an increased interest among mathematics education researchers in instructional vision and the roles that such discourses play in teachers' learning. Our findings contribute to what has been a missing piece of that work—equity-specific aspects of instructional vision. The trajectory presented in this paper—along with the other four trajectories—could be helpful to anyone who is studying or working to support mathematics teachers in learning about and enacting more equitable instruction. Though each individual's learning may follow a unique path, our trajectory models one that is likely for teachers' instructional visions, particularly those who are engaged in equity-focused professional learning. In this sense, our results provide a kind of roadmap to those working to support teachers' and others' learning. With interviews like those we conducted, users can characterize individuals' instructional visions with respect to equity and use that information to inform their next support efforts and anticipate where the individuals' visions might be headed.

Before concluding we wish to acknowledge that equity is a continual pursuit. Although our paper may seem to frame “Level 4” instructional visions as goals for teachers' perceptions of equity in the mathematics classroom, the presence of such visions, while encouraging, does not indicate that any real resolution has been reached. In fact, others have questioned whether it is even possible to achieve equity within the current system (e.g., Martin, 2019) or whether inequity is an inherent by-product of teaching school mathematics in the first place (e.g., Pais, 2014). It is our hope that at least some in the field will continue to ask such questions and that the work presented in this paper might, in some way, help to answer them.



## Acknowledgments

This material is based upon work supported by the National Science Foundation under Grant No. 1321216. Many thanks to the teachers and leaders we interviewed.

## References

- Aguirre, J. M., Mayfield-Ingram, K., & Martin, D. B. (2013). *The impact of identity in K–8 mathematics: Rethinking equity-based practices*. Reston, VA: National Council of Teachers of Mathematics.
- Apple, M.W. (2004). *Ideology and curriculum*. (3rd ed.) New York, NY: Routledge Falmer.
- Bartell, T., Wager, A., Edwards, A., Battey, D., Foote, M., & Spencer, J. (2017). Toward a framework for research linking equitable teaching with the standards for mathematical practice. *Journal for Research in Mathematics Education*, 48(1), 7–21.
- Boaler, J. (2008). Promoting ‘relational equity’ and high mathematics achievement through an innovative mixed-ability approach. *British Educational Research Journal*, 34(2), 167–194.
- Cabana, C., Shreve, B., Woodbury, E., & Louie, N. (2014). *Mathematics for equity: A framework for successful practice*. Teachers College Press.
- Gutiérrez, R. (2012). Context matters: How should we conceptualize equity in mathematics education? In B. Herbel-Eisenmann, J. Choppin, D. Wagner, & D. Pimm (Eds.), *Equity in discourse for mathematics education: Theories, practices, and policies*. New York, NY: Springer.
- Hiebert, J. & Grouws, D.A. (2007). The effects of classroom mathematics teaching on students’ learning. In F.K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 371–404). Charlotte, NC: Information Age Publishing.
- Horn, I. S. (2012). *Strength in numbers: Collaborative learning in secondary mathematics*. National Council of Teachers of Mathematics.
- Jackson, K., Gibbons, L., & Sharpe, C. J. (2017). Teachers’ views of students’ mathematical capabilities: Challenges and possibilities for ambitious reform. *Teachers College Record*, 119(7), 1–43.
- Ladson-Billings, G. (1995). Toward a theory of culturally relevant pedagogy. *American Educational Research Journal*, 32(3), 465–491.
- Lampert, M., Beasley, H., Ghouseini, H., Kazemi, E., & Franke, M. (2010). Using designed instructional activities to enable novices to manage ambitious mathematics teaching. In M. K. Stein & L. Kucan (Eds.), *Instructional explanations in the disciplines* (pp. 129–141). New York, NY: Springer.
- Louie, N. L. (2017). The culture of exclusion in mathematics education and its persistence in equity-oriented teaching. *Journal for Research in Mathematics Education*, 48(5), 488–519.
- Martin, D. B. (2019). Equity, inclusion, and antiblackness in mathematics education. *Race Ethnicity and Education*, 22(4), 459–478.
- Martin, D. B. (2015). The collective Black and *Principles to Actions*. *Journal of Urban Mathematics Education*, 8(1), 17–23.
- Munter, C. (2014). Developing visions of high-quality mathematics instruction. *Journal for Research in Mathematics Education*, 45(5), 585–636.
- Munter, C., & Correnti, R. (2017). Examining relations between mathematics teachers’ instructional vision and knowledge and change in practice. *American Journal of Education*, 123(2), 171–202.
- Munter, C., & Wilhelm, A. G. (2021). Mathematics teachers’ knowledge, networks, practice, and change in instructional visions. *Journal of Teacher Education*, 72(3), 342–354.
- National Council of Teachers of Mathematics. (2020). *Catalyzing change in middle school mathematics: Initiating critical conversations*. Reston, VA: Author
- National Council of Teachers of Mathematics. (2018) *Catalyzing change in high school mathematics: Initiating critical conversations*. Reston, VA: Author.
- National Council of Teachers of Mathematics. (2014). *Principles to actions: Ensuring mathematical success for all*. Reston, VA: Author.
- Pais, A. (2014). Economy: The absent centre of mathematics education. *ZDM*, 46(7), 1085–1093.
- Saxe, G. B., Gearhart, M., Franke, M. L., Howard, S., & Crockett, M. (1999). Teachers’ shifting assessment practices in the context of educational reform in mathematics. *Teaching and Teacher Education*, 15(1), 85–105.
- Stein, M., Engle, R., Smith, M., & Hughes, E. (2008). Orchestrating productive mathematical discussions: Five practices for helping teachers move beyond show and tell. *Mathematical Thinking and Learning*, 10, 313–340.
- Wilhelm, A. G. (2014). Mathematics teachers’ enactment of cognitively demanding tasks: Investigating links to teachers’ knowledge and conceptions. *Journal for Research in Mathematics Education*, 45(5), 636–674.
- Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

Wilson, J., Nazemi, M., Jackson, K., & Wilhelm, A. G. (2019). Investigating teaching in conceptually oriented mathematics classrooms characterized by African American student success. *Journal for Research in Mathematics Education*, 50(4), 362–400.

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

# EPISTEMIC VIOLENCE AS DEATH DECIMALS: ADVANCING BLACK LOGIC IN MATHEMATICS EDUCATION

Rolanda Payne  
University of Maryland  
rpayne18@umd.edu

Balke o'Neal Turner  
University of Maryland College  
bclark37@umd.edu

Keywords: Social Justice, Systemic Change, Research Methods

Drawing on the work of critical scholars, we assert that mathematics education—teaching, research, praxis, and policy—has proven to be a particularly violent space for Black students (Cedillo, 2018; Gutiérrez, 2008; Gutiérrez & Dixon-Román, 2011; Larnell & Martin, 2021a; Madden et al., 2019; Martin, 2019; Martin et al., 2019). Antiblack epistemic violence (Teo, 2010) in theoretical interpretations creates spatial imaginaries through “discourses, images, and texts that tell a story of Blackness as a problem, non-human, and placeless” (Jenkins, 2021, p. 119). Particularly in mathematics education research, scholars have been complicit in (re)producing antiblack spatial imaginaries of Black doers and learners by using and producing research to invalidate Black students’ ability to learn and do mathematics. While critical scholars have written about how research functions as an antiblack spatial imaginary (e.g., Jenkins, 2021; Morrison et al., 2017), few studies have challenged how quantitative researchers engage in antiblackness and epistemic violence in mathematics education. Thus, we ask, *what antiblack spatial imaginaries about Black doers and learners are sustained within quantitative mathematics education research?*

This study employs two complementary but distinct theoretical frameworks. Black Critical Theory (BlackCrit) (Dumas & Ross, 2016) serves as an analytical framework to illuminate, critique, and dismantle antiblackness in education. As a framework, BlackCrit serves as a prism to capture antiblack epistemic violence in mathematics education research. The second theoretical framework used for this study is Quantitative Critical Theory (QuantCrit) (Gillborn et al., 2018). QuantCrit is a toolkit that allows researchers and practitioners to interrogate quantitative research for manifestations of whiteness. In tandem, both theoretical frameworks allow us to counter antiblack violence in quantitative research by leaning on Black logic (Larnell & Martin, 2021b).

To identify relevant literature, we conducted a systematic literature review of 434 quantitative research articles in mathematics education, including interdisciplinary studies, in the last thirty years. Our inclusion criteria yielded 253 articles that were included in our final analysis. Our holistic view of each article included the literature review, methods, frameworks, results, conclusions, and citations. For each article, we identified the following information: 1) how, if at all, race was defined and used, 2) the ways Black doers and learners were positioned, and 3) how the authors built on or deviated from deficit antiblack logic of Black doers and learners. The current analysis reveals that researchers use white methodologies that perpetuate antiblackness, which positions Black students as problematic or needing repair.

This study extends scholarship problematizing gap gazing in mathematics education (e.g. Gutiérrez, 2008; Gutiérrez & Dixon-Román, 2011) by providing tangible examples of the frameworks, methods, and language employed by scholars whose work is centered in or tangential to mathematics education.

## References

- Cedillo, S. (2018). Beyond Inquiry: Towards the Specificity of Anti-Blackness Studies in STEM Education. *Canadian Journal of Science, Mathematics and Technology Education*, 18(3), 242–256.
- Dumas, M. J., & Ross, K. M. (2016). “Be Real Black for Me”: Imagining BlackCrit in Education. *Urban Education*, 51(4), 415–442.
- Gillborn, D., Warmington, P., & Demack, S. (2018). QuantCrit: education, policy, “Big Data” and principles for a critical race theory of statistics. *Race, Ethnicity and Education*, 21(2), 158–179.
- Gutiérrez, R. (2008). A “Gap-Gazing” Fetish in Mathematics Education? Problematizing Research on the Achievement Gap. *Journal for Research in Mathematics Education*, 39(4), 357–364.
- Gutiérrez, R., & Dixon-Román, E. (2011). Beyond Gap Gazing: How Can Thinking About Education Comprehensively Help Us (Re)envision Mathematics Education? In B. Atweh, M. Graven, W. Secada, & P. Valero (Eds.), *Mapping Equity and Quality in Mathematics Education* (pp. 21–34). Springer Netherlands.
- Jenkins, D. A. (2021). Unspoken Grammar of Place: Anti-Blackness as a Spatial Imaginary in Education. *Journal of School Leadership*, 31(1-2), 107–126.
- Larnell, G. V., & Martin, D. B. (2021a). Urban mathematics education as a political and personal project. In *Handbook of Urban Education* (pp. 355–368). Routledge.
- Larnell, G. V., & Martin, D. B. (2021b). Home as the Quintessential Anti-Racist School: Reflections on Black Logics of Place and Opportunity, Parenting and Learning, Being and Striving. *Multicultural Perspectives*, 23(3), 173–180.
- Madden, K., Pereira, P., Rezvi, S., Trinder, V. F., & Martin, D. B. (2019). Cartographies of race, gender, and class in the white (male settler) spaces of science and mathematics: Navigations by black, Afro- Brazilian, and Pakistani/American womxn. In E. O. McGee & W. Robinson (Eds.), *Diversifying STEM : Multidisciplinary Perspectives on Race and Gender* (pp. 69–106). Rutgers University Press.
- Martin, D. B. (2019). Equity, inclusion, and antiblackness in mathematics education. *Race Ethnicity and Education*, 22(4), 459–478.
- Martin, D. B., Price, P. G., & Moore, R. (2019). Refusing systemic violence against Black children: Toward a Black liberatory mathematics education. In *Critical race theory in mathematics education* (pp. 32–55). Routledge.
- Morrison, D., Annamma, S. A., & Jackson, D. D. (Eds.). (2017). *Critical Race Spatial Analysis: Mapping to Understand and Address Educational Inequity*. Stylus Publishing, LLC.
- Teo, T. (2010). What is epistemological violence in the empirical social sciences? *Social and Personality Psychology Compass*, 4(5), 295–303.

# EXPLORING THE MEDIATING ROLE OF TEACHER EXPECTANCY ON PARTICIPATION IN WHOLE CLASS MATHEMATICS DISCUSSION

Mehmet Kirmizi  
Texas State University  
m\_k307@txstate.edu

*Teacher expectancy (TE) refers to the inferences that teachers make about academic achievement, and future career choice of their students, it is a teacher level variable, and it mediates the teacher-student interaction. I hypothesize that the TE impacts the distribution of learning opportunities in 8th grade mathematics classes. So, I investigate the distribution of learning opportunities in terms of TE. An 8th grade class interaction is recorded (n =16), and investigated by using the EQUIP. The result of this study reveals that seven high expected students despite being the 35% of the class have 76.71% of entire student talk. In other words, a big chunk of the class interaction is only between teacher and high expected students.*

Keywords: Equity & Justice, Middle School Education, Teacher Beliefs

## Introduction

The mathematics education community has given explicit attention to issues of equity over the last 20 years. Gutiérrez defines equity as “the goal of being unable to predict student patterns (e.g., achievement, participation, the ability to critically analyze data or society) based solely on characteristics such as race, class, ethnicity, sex, beliefs, and creeds, and proficiency in the dominant language.” (Gutierrez, 2002, p., 153). Regardless of the characteristics of students, all students should have the fair, not necessarily the same, chance to participate and learn.

Opportunities to learn mathematics are supposed to be accessible to all students rather than to a privileged few.

Students who have more opportunities to participate will learn more whereas students who participate less will learn less (Shah et al., 2021). The distribution of opportunities to participate in classroom discussions is an issue of equity. Equity in participation concerns equal and fair distribution of participation and opportunities to participate to the ongoing discussion (Ernest et al., 2019). Teachers, as the leaders of the classroom community, have power to distribute learning opportunities through how they structure participation. Yet, teachers, like all people, are not free of biases. Many studies report that pre-service and in-service teachers have implicit biases and attitudes toward their students from ethnic minorities and migrant families (Glock & Klapproth, 2017; Glock, et al., 2013). Teachers’ evaluation of student potential is not necessarily accurate (Copur-Gencturk et al. 2020; Soto-Ardila et al., 2022). Teachers’ implicit biases is one possible explanation for the occurrence of inequalities in our classroom. Some empirical evidence suggests that implicit biases impact students’ academic achievement via Teacher Expectation (TE) (Van den Bergh et al., 2010). Teacher expectancy (TE) refers “inferences that teachers make about the present and future academic achievement and general classroom behavior of their students” (Good & Brophy, 1997, p. 79). Overestimated students will have more opportunities to participate and then likely learn more while, underestimated students will have less opportunities to participate and will likely learn less. Differential interaction occurs when some students receive different opportunities to interact or participate in class (Wang et al. 2018). This difference is mediated by the expectation the teacher has for the student. Therefore, students have different experiences based on what is expected from them.

Current studies on inequalities and biases in mathematics classrooms investigate the pattern and the distribution of learning opportunities among students from different communities and backgrounds-- who participates and how are two important research questions that researchers want to understand. Researchers, commonly, compare participation and contributions of students with different identity markers such as race/ethnicity, class, and gender (Reinholz & Shah, 2018; Reinholz et al., 2020; Shah et al., 2021). The mathematics education community has learned a lot about ways inequalities occur in our classrooms. This work has been incredibly valuable in helping us better understand how markers such as race and gender can play a role in biases and thus in opportunities for students to participate in discussion. Yet, there may be other factors also playing a role in how opportunities are distributed by teachers. For example, Cohen and Lotan (2014) note that even in groups of people where the students are largely homogeneous regarding gender and race, for example, participation is not equally distributed. Inequalities and status orders quickly emerged within a group of four to seven white male Harvard sophomore students. Despite the initial similarities and homogeneity of the group, the status order emerged quickly, and the contributions of members were not distributed equally (Bales, 1950). According to Bales (1950), group members tended to consider the most talkative members as the one who contributes most and significantly; whereas they tended to consider quiet group members as the one who had the least significant ideas. Following the logic of that example, two students from a similar set of identity and demographic markers might still have very different characteristics in other regards and thus each student might have very different interactions with the teacher. That is, teachers might be picking up on characteristics not related to race and gender. This is not to suggest that race, for example, is not important; merely that in the absence of race, for example, as a distinguishing characteristic, teachers may be forming expectations using other features. In other terms, TE mediates the interaction between students and TE can be formed in the absence of obvious demographic distinctions. If we want to understand the inequalities within our classroom, we should take into consideration TE and the dyadic interactions between students and teacher. In this study, I aim to explore TE in a classroom environment where the students are largely homogenous with regard to race, gender, and language and attempt to better understand how TE develops and mediates classroom participation.

### **The Mediating Role of Teacher Expectancy**

Teacher expectancy is a teacher level variable, and its impacts on students' academic achievement is well reported (De Boer et al., 2018; McKown & Weinstein, 2008). It mediates the interaction between students and teacher where teachers interact more favorably with students whom they believe have the potential to succeed and less favorable/less frequent interactions with the students they believe have less potential for success and is well supported in teacher expectancy literature (İnan-Kaya & Rubie-Davies, 2021; Wang et al., 2018). Once teachers form their expectations for their students, they start interacting with their students differently. High-expected students tend to have more positive interactions (more frequent eye-contact, smile, etc.) with teachers, whereas the low-expected students have negative or less positive interactions (less frequent eye-contact, smile, etc.). Teachers tend to give high expected students more opportunities to demonstrate their thinking (being called to explain their idea), and low-expected students have less opportunities to do so (Good & Brophy, 1978).

Based on the aforementioned literature, I hypothesize that TE influences the distribution of learning opportunities in an 8<sup>th</sup> grade mathematics where racial, gender, and language characteristics were stable. This study aims to explore TE's role in mediating participation

opportunities in a setting where demographic features are more homogenous across the group of students. Therefore, the RQ of this study is the following:

How are the learning opportunities distributed across individual students based on teacher expectation?

## Method

### Framework

This study adopts the Motivation and Opportunity as Determinants model (MODE; Fazio & Olson, 2003). This model says we usually make decisions in two separate but related ways: (1) deliberate and (2) spontaneous.

The deliberate decision-making process refers to the idea that when there are sufficient time and resources available people make their decisions based on careful thought process and evaluations. The deliberate process is data driven decision-making. An individual considers the potential advantages or disadvantages of the situation with all the resources available to them, then they choose a best course of action. On the other hand, most of the social actions and decisions that people make in their lives happen in the absence of time and necessary resources- which is where spontaneous decisions occur. People make this kind of decision when they must respond immediately and do not have time and resources to consider the consequences of the choice that they make. In this kind of decision-making process, which happens spontaneously, their beliefs and attitudes toward the subject influence the decision that is being made. Once beliefs are activated in the decision-making process it will serve the role of a “filter”. The positive attitudes usually help individuals to attend and notice the positive aspects of the subject. Whereas the negative attitudes will influence individuals to attend the negative side of the subject (Fazio, 1990; Fazio & Olson, 2003).

### Data

The participants of this study include a Turkish mathematics teacher, and her 8<sup>th</sup> grade female students ( $n = 20$ ). The teacher is teaching in a religious-oriented school in one of the southern provinces of Turkey. The teacher has more than 15 years of teaching experience. She is well respected among her colleagues, and she is pursuing her doctoral degree in mathematics education. The 40% percent of the students ( $n = 8$ ) are Turkish, and the rest of the classroom ( $n = 12$ ) have migrated to Turkey from Syria. The class was a fairly homogeneous environment with regard to the parent education, income, and other variables. All students are fluent in Turkish which is the language of the instruction. After obtaining the ethics board approval the teacher rated each student on a scale of 1 (very low) to 5 (very high). Each category represents the potential track that the students might have to go for high school. *High expected students* ( $n = 7$ ) implies that they might go to elective high schools, *neutral expected students* ( $n = 3$ ) means that they might go to unselective academic high school, and the *low expected students* ( $n = 10$ ) might go to vocational schools, or they just stop having a formal education.

In order to understand the potential influence of TE on classroom participation, the class lessons were recorded ( $n = 16$ ) during Fall 2022. The recordings are being investigated by using the classroom observation tool, EQUIP (Reinholz & Shah, 2018). EQUIP captures and analyzes seven dimensions of classroom talk. (1) Discourse type, (2) the length of student talk, (3) the type of student talk, (4) the method of teacher solicitation, (5) wait time, (6) type of teacher solicitation, and (7) explicit evaluation. The table-1 shows the seven dimensions, and subcategories of EQUIP.

The key idea in this study is to find the participation ratio which compares the actual distribution of learning opportunities for high expected and low expected students and the

predicted distribution which assumes equal opportunity regardless of TE. For instance, the teacher identifies 35% of the class is considered high TE, 10% of neutral expected and the rest which is 50% of the class is considered low TE, then the predicted distribution would be that the high, neutral and low TE students, respectively, 35%, 15%, and 50% of the learning opportunities. If the actual distribution differs statistically significant (i.e., if the high TE students get 50% of the questions) from the predicted, there is reason to suspect that the learning opportunities are not distributed equally.

**The influence of Teacher Expectancy to the classroom**

Kruskal-Wallis which is non-parametric alternatives of one-way ANOVA is conducted to test the distribution of students’ total number of participations is influenced by the TE. The results of the Kruskal-Wallis test reveals that there is a significant difference between the total number of participation and TE ( $X^2 = 14.384$ ,  $df = 2$ ,  $p\text{-value} = .0007527 < .05$ ). In order to establish the relationship between the TE and the classroom participation the simple linear regression with the Heteroskedasticity-Consistent Standard Error. The model is significant and it explains the nearly 83.36% data ( $F_{(2,16)} = 40.07$ ,  $p\text{-value} = 5.884e-07 < .05$ ). The model is as follows:

$$Participation = 4 + 65.667 * NeutralExpectation + 124.429 * HighExpectation$$

According to the model being a neutral expected students means having a 65.667 more learning opportunity than the low expected students. Similarly, being a high expected students means on average having 124.429 more learning opportunities than the lower expected students.

**Table 1: Dimensions of EQUIP**

Dimension	Level
Discourse type	Contextual Logical
Length of Student Talk	1-4 words 5 – 20 words 21 or more words
Type of student talk	What How Why Other
Method of Teacher Solicitation	Random selection Called on Not called on
Wait time	More than 3 seconds Less than 3 seconds NA



Type of Teacher Solicitation	Why
Explicit evaluation	How Why that O t h e r Yes No

## Results

### Distribution of Length of the Student Talk

During the recordings students had 1172 opportunities to participate in the ongoing mathematical discussion. The teacher has high expectations from seven of her students which is 35% of the total number of students, and these seven students have contributed 76.71% of the total student contributions (n = 899) to the ongoing mathematical discussion. The neutral expected students (n= 3) are responsible for 17.83% (n = 209) contributions, and the low expected students (n = 10) had only 5.46% (n = 64) of the opportunities that occurs during the recordings. The table 2 summarizes the distribution of the student talk with respect to the TE and the migration status.

**Table 2: Distribution of Student Talk**

	High Exp. (35%)		Neutral Exp. (15%)		Low Exp. (50%)		Native (40%)		Migrant (60%)	
	Exp. Distr.	Obs. Distr.	Exp. Distr.	Obs. Distr.	Exp. Distr.	Obs. Distr.	Exp. Distr.	Obs. Distr.	Exp. Distr.	Obs. Distr.
Student Talk	35%	76.71%	15%	17.83%	50%	5.46 %	40%	58%	60%	42%

As it can be seen in the table 2, native students have contributed 58% of the entire student talk whereas the migrant students have only 42% of student talk. The native students despite being a smaller group, they have more opportunity to participate than the migrant students.

However, a bigger difference lies between the distribution based on the expectation. The high expected students despite being only 35% of the entire class they are responsible 76.71% of total student talk, and neutral student have 17.83% and the low expected student have only 5.46% of the student talk, despite being half of the classroom.

By default, EQUIP captures three levels of student talk: 1-4 words, 5-20 words, and 21+ words. *1-4 words* is a strong indicator of traditional Initiation-Response-Evaluation (IRE) pattern, and the *5-20 words* is roughly a sentence, and it is beyond the IRE pattern, but it is not considered as a high level of participation. *21+ words* is a very strong indicator of student mathematical thinking. A big portion of the length of the student talk is between *1-4 words* which is a strong indicator of IRE. 82.76% (n = 970) of students' participation were low level participation, and it was between *1-4 words*. Of student talk that were *1-4 words* the high expected students were responsible for the 77.94%, the neutral expected students are responsible for 18.87% and the low expected student, despite being the half of the class, only have only 3.20% of the low-level participation. The distribution of high-level learning opportunities is also similar to the low-level participation. The high level-participation typically occurred when the teacher gave her authority to a student to solve a problem at the board individually. Of those, high expected students have 68.64% (n = 81), neutral expected students have 11.02% (n = 13), and low expected students have the rest 20.34% (n = 24) high level participation opportunities that occurs during recordings.

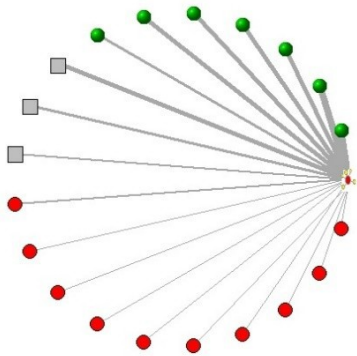
### **Distribution of Teacher Solicitation Method**

In this particular classroom, the high expected students (n = 7) have more learning opportunities than the neutral (n = 3), and low expected (n = 10) students. The high expected students have 76.70% (n = 899) of total learning opportunities that are identified and recorded. They have more academic interaction with their teacher not just because they are given more opportunities, they also seek opportunities to participate.

A little over 68% (n = 798) of the total learning opportunities that were recorded in this class were not called on explicitly by the teacher. The teacher frequently uses a modified version of IRE where she asks a question to the whole class to make sure her students/ or a group of students are following her. In this case, she just wants to hear a response. Usually, the response comes from high expected students. Moreover, high expected students share their thinking and knowledge without any teacher solicitation. They want to show their knowledge and mathematical thinking to the teacher. The distribution of *not called on* is skewed toward the *high* (n = 7) and *neutral* (n = 3) expected students. Together the neutral and high expected students have almost 97% total *not called on* learning opportunities (n = 798). The neutral expected students (n = 3) have 17.17% (n = 137), and finally the high expected students (n = 7) are responsible for the 79.82% (n = 637) of the classroom participation without being called by the teacher. Low expected students (n = 10), despite being half of the class, have only 24 opportunities to participate without being called on which is equivalent to a little over 3%.

### **The interaction diagram**

The results of this study indicate that the high expected students have more opportunities than the neutral and low expected students. They have more opportunity to participate than the low and neutral expected students. They have more interaction with the teacher. In order to illustrate the distribution of the learning opportunities based on TE. The following graph summarizes the student teacher interaction. In the graph each node represents a student, and the shape of the nodes represents the expectation. The teacher's solicitation is used as an edge between nodes, and the width of the edge is the average number of teacher's solicitation in a class time. The Figure 1 represents the teacher student academic interaction in total 16 class time.



**Figure 1: The teacher-student interaction pattern**

As it is clearly seen that spheres nodes (high expected students), have more interaction with the teacher (raster) than the squares nodes (neutral), and circle nodes (low) expectations. In the graph the width of edges gets a steady increase if you start from the raster (teacher) and follow it from the clockwise all the way back to the raster. In other words, TE is almost perfectly aligned with the students' participation to the ongoing mathematical discussion.

### **Conclusion**

First, the result of this study confirms the hypothesis that the distribution of the learning opportunities is influenced by the teacher's expectations. In this study high expected students have more opportunities than the low or neutral expected students. However, high expected students are not just given more opportunities they also ask more questions and seek every opportunity to participate more than others. The result of this study indicates that the dyadic teacher-student interaction is an important variable; however, most of the scholarship on the issue is coming from the late 70s and early 80s (Brophy & Good, 1970; Good, 1980).

Considering how much the population and our classroom have changed we need to update the knowledge on this issue. This area is also important because it tells us so much about the inequalities within our classroom, and it suggests a new perspective to explore the inequalities occurs in a classroom.

### **References**

- Bales, R. F. (1950). A Set of Categories for the Analysis of Small Group Interaction. *American Sociological Review*, 15(2), 257-263. doi:10.2307/2086790
- Brophy, J. E., & Good, T. L. (1970). Teachers' communication of differential expectations for children's classroom performance: Some behavioral data. *Journal of educational psychology*, 61(5), 365.
- De Boer, H., Timmermans, A. C., & Van Der Werf, M. P. (2018). The effects of teacher expectation interventions on teachers' expectations and student achievement: Narrative review and meta-analysis. *Educational Research Evaluation*, 24(3-5), 180-200.
- Copur-Gencturk, Y., Cimpian, J. R., Lubienski, S. T., & Thacker, I. J. E. R. (2020). Teachers' bias against the mathematical ability of female, Black, and Hispanic students. *Educational Researcher*, 49(1), 30-43
- Ernest, J. B., Reinholz, D. L., & Shah, N. (2019). Hidden competence: Women's mathematical participation in public and private classroom spaces. *Educational Studies in Mathematics*, 102(2), 153-172.
- Fazio, R. H., & Olson, M. A. (2003). Implicit measures in social cognition research: Their meaning and use. *Annual Review of Psychology*, 54(1), 297-327.
- Good, T. L. (1987). Two decades of research on teacher expectations: Findings and future directions. *Journal of Teacher Education*, 38(4), 32-47.
- Good, T. L., & Brophy, J. E. (1997). *Looking In Classrooms*. New York: Longman.
- Gutiérrez, R. (2002). Enabling the practice of mathematics teachers in context: Toward a new equity research agenda. *Mathematical Thinking and Learning*, 4(2-3), 145-187.

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (Vol. 2)*. University of Nevada, Reno.

- Gutiérrez, R. (2012). Context matters: How should we conceptualize equity in mathematics education? In *Equity in discourse for mathematics education* (pp. 17-33): Springer.
- İnan-Kaya, G., & Rubie-Davies, C. M. (2021). Teacher classroom interactions and behaviours: Indications of bias. *Learning and Instruction*, 101516.
- McKown, C., & Weinstein, R. S. (2008). Teacher expectations, classroom context, and the achievement gap. *Journal of school psychology*, 46(3), 235-261.
- Reinholz, D. L., Bradfield, K., & Apkarian, N. (2019). Using analytics to support instructor reflection on student participation in a discourse-focused undergraduate mathematics classroom. *International Journal of Research in Undergraduate Mathematics Education*, 5(1), 56-74. doi:10.1007/s40753-019-00084-7
- Reinholz, D. L., & Shah, N. (2018). Equity analytics: A methodological approach for quantifying participation patterns in mathematics classroom discourse. *Journal for Research in Mathematics Education*, 49(2), 140- 177.
- Reinholz, D. L., Stone-Johnstone, A., & Shah, N. (2020). Walking the walk: Using classroom analytics to support instructors to address implicit bias in teaching. *International Journal for Academic Development*, 25(3), 259-272.
- Rosenthal, R., & Rubie-Davies, C. M. (2015). How I spent my last 50-year vacation: Bob Rosenthal's lifetime of research into interpersonal expectancy effects. In *Routledge International Handbook of Social Psychology of the Classroom* (pp. 313-323): Routledge
- Soto-Ardila, L. M., Caballero-Carrasco, A., & Casas-García, L. M. (2022). Teacher expectations and students' achievement in solving elementary arithmetic problems. *Heliyon*, 8(5), e09447.
- Van den Bergh, L., Denessen, E., Hornstra, L., Voeten, M., & Holland, R. W. (2010). The implicit prejudiced attitudes of teachers: Relations to teacher expectations and the ethnic achievement gap. *American educational research journal*, 47(2), 497-527.
- Wang, S., Rubie-Davies, C. M., & Meissel, K. (2018). A systematic review of the teacher expectation literature over the past 30 years. *Educational Research and Evaluation*, 24(3-5), 124-179.

## FIGURED WORLDS OF WOMEN MATHEMATICS EDUCATION SCHOLARS

Lili Zhou  
California State University, Los Angeles  
Lzhou18@calstatela.edu

Ricki Geller McKee  
Purdue University  
gellerr@purdue.edu

Brooke Max  
Purdue University  
bmax@purdue.edu

Hyunyi Jung  
University of Florida  
hyunyi.jung@coe.ufl.edu

Bima Sapkota  
Ball State University  
bima.sapkota@bsu.edu

Jill Newton  
Purdue University  
janewton@purdue.edu

Lindsay M. Keazer  
Sacred Heart University  
keazerl@sacredheart.edu

*Drawing on the concept of figured worlds (Holland et al., 1998), this project focuses on addressing, responding to, and understanding the self within the figured world of the mathematics education community. Specifically, we examine a group of women with diverse backgrounds in terms of race, class, and cultural contexts, who are engaged in various roles as mathematics education scholars, including teachers, teacher educators, and researchers. Using a dialogical self approach, we facilitate both internal and external discourses, exploring personal histories, narratives, and the development of evolving identities. Our findings reveal that culture and social positions, such as gender, class, and race, inform our positionalities within the mathematics education community. The understanding of the roots of our identities serves as a foundation for constructing a more inclusive figured world of mathematics education.*

Keywords: Dialogical self approach, Figured worlds, Identity, Women

### Figured Worlds Theoretical Perspective

With the emergence of feminist theories in the 1990s, mathematics education researchers began to advocate for a new epistemology that promotes gender justice in the field (e.g., Boaler, 1997; Kaiser & Rogers, 1995; Walkerdine, 1998). Subsequently, discussions shifted towards amplifying voices of girls and women in mathematics by delving into the intricate relationships among identity and sociocultural contexts (e.g., Boaler, 1997; Boaler & Greeno, 2000).

Holland and colleagues (1998) illustrated the theory of identity in the context of “as if” realms known as *figured worlds*. These worlds are characterized by the enactment of activities, discourses, performances, and artifacts, shaping individuals’ identities (Urrieta Jr, 2007). According to Holland and colleagues, a figured world is “a socially and culturally constructed realm of interpretation in which particular characters and actors are recognized, significance is assigned to certain acts, and particular outcomes are valued over others” (p. 52).

Drawing on the concept of figured worlds, Holland et al. (1998) further elaborated on two aspects of identity, positional and figurative identity. Positional identity encompasses an individual’s perception of their social position within a lived world while figurative identity involves the abstraction of significant everyday activities and events, focusing on signs that evoke storylines or plots involving generic characters. These two aspects of identity are interconnected, intertwined, sometimes completely coincident and other times one dominates another.

Following Bakhtin's (1981) concept of dialogism, Holland et al. (1998) further expanded that individuals' identity and agency are manifested through the act of authoring the self. According to Bakhtin, the self is a position that is "addressed" by and "answers" others within the context of figured worlds. Through the process of answering, the self "authors" the figured worlds, and in turn actively shapes and constructs the figured worlds. Whether and how individuals enter specific figured worlds depends on their personal history, referred to as history-in-person (Holland & Lave, 2001).

In this study, we bring our own personal histories into the figured world of the mathematics education community, where we encounter/co-construct a range of practices, discourses, interactions, and power relations that shape and inform our new collective identity as women mathematics educators. Through the process of authoring the self, we, within this community, continuously co-construct and reconceptualize the figured world and meanwhile make sense of the evolving self. The following two questions guided our study: *What is our own history-in-person that drives us to be drawn or recruited into the figured world of the mathematics education community? How do we author ourselves within the figured world of the mathematics education community, and how do the narratives we create reflect and shape our professional identities and social identities, including those related to class, race, gender, and nationality?*

### **A Dialogical Self Approach**

Building on Bakhtin's concept of dialogism, Holland et al. (1998) emphasized the self and the "I" are pivotal in the process of authoring figured worlds. To create a space for authoring the figured world and narrating our identities and affordable positions, we adopt a dialogical self approach that promotes internal and external discourse (Hermans & Gieser, 2011).

Researchers have developed dialogical self-theory that establishes relationships focusing on collective voices, social relations, complex positions, and embodied forms of dialogue (e.g., Hermans, 2001). The dialogical self approach has been used as a method in mathematics education and teacher education to understand the complexity of identity development. For example, Solomon (2012) explored how two women authored selves in the figured world of mathematics, where the field is perceived as masculine and gendered. Akkerman and Meijer (2011) used a dialogical self approach to re-conceptualize teacher identity as both unitary and multiple, continuous and discontinuous, and individual and social.

Previous research has substantiated that the dialogical self approach contributes to personal and cultural positioning, enabling a deeper comprehension of the multi-voiced and dialogical nature of the self. Our group consists of seven women mathematics educators. We have been meeting monthly for the past year, and each session involves a member bringing several prompts that create a space for answering the world and authoring the self, as described by Holland et al. (1998). The following were some examples of prompts we used: *Think about a significant episode or a memory that you remember from a stage of your life. What kind of a person were you during this stage? Who were significant people for you during this stage, and why? Where are the places in your journey where you have felt or still feel the most resistance? How have you/do you respond in those contexts of resistance? How are these areas of resistance influenced by working within the field of mathematics/mathematics education?*

During the meetings, we first discuss the prompts, and then we spend 30-45 minutes writing our own narratives or stories to respond to them. Individual writing provides a space for us to construct our internal discourse. Finally, we come back together to share our stories, which leads to an evolving inquiry in the group. The sharing often inspires external discourse on present domination and asymmetry social relations within the figured world of mathematics education.

## Findings

Through a dialogical self approach, we have delved into our personal histories, interrogating past experiences and embracing inner voices for personal and professional growth. The diverse nature of our group has inspired curiosity, fostering a collective sharing of personal narratives and attentive listening. Due to page limitations and the preliminary nature of our findings, we share only some examples from our own narratives below.

### **Personal Histories towards the Figured World of Mathematics Education Community**

Each member of our group possesses a distinct personal history that has driven us to pursue careers in the field of mathematics education. Specifically, our decision to pursue a Ph.D. degree in mathematics education stems from a convergence of social influences, inner reflections, and individual past experiences. Across our personal narratives, two themes consistently emerged: mathematics and teaching. Despite the steep learning curves associated with mathematics, a shared passion for the subject has remained prominent. In fact, from a young age, many of us held strong aspirations to become teachers. Brooke stated, *“I wanted to be a teacher since I was a young child of 4 or 5 years old.”* Hyunyi said, *“I wanted to be a math teacher since I was 13 years old.”* Lili said, *“in fifth grade, I started to think about becoming a math teacher.”*

Some members identify that, despite their passion for teaching, their choice to pursue a career in education may have been influenced by gendered norms that traditionally associate teaching with a profession deemed suitable for women. Ricki shared,

I was lucky to receive an interview for graduate school for mathematics. Upon entering the room for my interview, I was greeted with “Oh, you’re female.” The discovery of my femininity cast a shadow on the interview, disregarding the qualifications that I held. I didn’t get into graduate school for mathematics; it left me feeling “not good enough” yet again in my mathematics journey. I decided to get my masters in administration to better understand the complexities of teaching.

As we navigate our positions in the figured world of mathematics education, we also recognize other people who hold significant roles as actors in our lives, to whom we may have developed strong emotional attachments or who have exerted influence on our personal histories.

As a first-generation college student, my family always wanted me to go to college, but couldn’t afford college. My mother sacrificed and took a secretary position at Purdue University so that I could attend the institution for a discounted tuition (Ricki).

My father gave me advice by asking me how many music teachers were in an average school, and how many math teachers were in an average school. He wanted me to see that there would be more opportunities to be a math teacher and reminded me I could still pursue music at church or in other avenues (Brooke).

### **Identities in the Figured World of Mathematics Education Community**

Our exploration delves into the intricate relationship between multiple identities and our roles as mathematics education scholars. Through this inquiry, we uncover the multifaceted nature of our identities as sources of tension, challenges, and transformative experiences. Here, we highlight the significant roles of being a woman and a mother, which inform our professional identities within the figured world of the mathematics education community.

### ***Identity as a Woman***

We share a collective identity as women. Our knowledge construction is rooted in the lived experiences of women, and reciprocally, within the dynamic activity of the Figured World, our identity as women undergoes continuous formation and transformation. The following narratives, carefully chosen, illuminate the profound influence of our gender identity on how our professional roles are perceived by others.

Even now, as I try to get my foot in the door teaching mathematics at the college level, I'm met with gender inequality. Rarely getting past the initial phase of applying, my qualifications are minimalized compared to male counterparts. My experience and success are overlooked. On the rare chance that I make it to the interview phase, I find myself sitting in a room of all males, being judged and compared to male counterparts, again being told that I'm "not good enough" (Ricki).

Being a mathematics teacher in a school with excellent academic performance, I was recognized as a good teacher because I was patient, diligent, and cared for my students. Even though I took a leadership role as Grade Department Chair, people appreciated my dispositional character more than my intellectual contribution as I was succeeding in the position (Lili).

### ***Identity as a Mother***

Motherhood holds great significance in a woman's life, encompassing various roles and responsibilities. As teachers, educators, and researchers, we also undertake a vital role in educating our own children. This unique experience of being a mother has a profound impact on our professional identities and how we navigate our positionality within the mathematics education community (e.g., Vomvoridi-Ivanovic & Ward, 2021). The selected narratives shed light on how our identity as mothers influences and guides our roles within the figured world of mathematics education.

Prior to having a child, I believed that all children needed in the classroom to learn was encouragement and a teacher to explain concepts to them. After having a child with ADHD, I learned that my prior belief was not enough. Children all learn differently, and some, especially those with ADHD, need activities, movement, patience, repetition, and clear, concise instructions given multiple times. I now feel that I'm much more equipped to assist and help students with ADHD in my classroom. (Ricki).

### **Discussion and Conclusion**

In the figured world of mathematics education community, we, a group of women mathematics education scholars, actively engage in (re)constructing our personal "as if" realms using a dialogical self approach. We examined our individual histories, including significant events and influential actors, in order to shed light on the formation and transformation of our identities. The findings of our inquiry highlight the crucial role of culture and social positions, such as gender, class, and race, in shaping our positionalities within the mathematics education community. The figured world then becomes both personal and professional, embodied over time, through individuals' continual participation (Holland et al., 1998). We call for attention on mathematics educators' ordinary lived experiences, particularly those from marginalized communities. By exploring the social, cultural, and institutional dynamics that shape individuals' identities, we can work towards establishing a more equitable figured world in the mathematics education community. This understanding of our identities and positions within this community



serves as a foundation for constructing a more just and inclusive figured world of mathematics education.

### References

- Akkerman, S. F., & Meijer, P. C. (2011). A dialogical approach to conceptualizing teacher identity. *Teaching and Teacher Education, 27*(2), 308–319.
- Bakhtin, M. (1981). Discourse in the novel (C. Emerson & M. Holquist, Trans.). In M. Holquist (Ed.), *The dialogical imagination* (pp. 259–422). University of Texas Press.
- Boaler, J. (1997). Reclaiming school mathematics: The girls fight back. *Gender and Education, 9*(3), 285-305.
- Boaler, J., & Greeno, G. G. (2000). Identity, agency, and knowing in mathematics worlds. In J. Boaler (Ed.), *Multiple perspectives on mathematics teaching and learning* (pp. 171–200). Ablex Publishing.
- Hermans, H. J., & Gieser, T. (Eds.). (2011). *Handbook of dialogical self theory*. Cambridge University Press.
- Holland D., Lachicotte W. Jr., Skinner D., & Cain C. (1998). *Identity and agency in cultural Worlds*. Harvard University.
- Holland, D., & Lave, J. (2001). History in person. History in person. Enduring struggles, contentious practice, intimate identities, 3-33.
- Kaiser, G., & Rogers, P. (1995). Introduction: Equity in mathematics education. In P. Rogers & G. Kaiser (Eds.), *Equity in mathematics education: Influences of feminism and culture* (pp. 1–10). Falmer.
- Kamberelis, G., Dimitriadis, G., & Welker, A. (2018). Focus group research and/in figured worlds. In N. K. Denzin & Y. S. Lincoln (Eds.), *The SAGE handbook of qualitative research* (pp. 692–716). SAGE.
- Solomon, Y. (2012). Finding a voice? Narrating the female self in mathematics. *Educational studies in mathematics, 80*, 171-183.
- Urrieta Jr, L. (2007). Figured worlds and education: An introduction to the special issue. *The urban review, 39*(2), 107-116.
- Vomvoridi-Ivanovic, E., & Ward, J. K. (2021). Academic Motherhood in Mathematics Teacher Education during COVID-19: Breaking the Silence and Shifting the Discourse. *REDIMAT-Journal of Research in Mathematics Education, 10*(1), 41-61.
- Walkerdine, V. (1989). *Counting girls out*. Virago.

# HUMANIZING MATHEMATICS THROUGH BOUNDARY CROSSING COLLABORATION

Lili Zhou  
California State University, Los Angeles  
Lzhou18@calstatela.edu

*What mathematics is and how to teach it are questions that mathematics educators constantly confront. A challenge identified in mathematics education is supporting students to see mathematics as normal human activity. Mathematics educators' viewpoints of mathematics determine whether they can recognize and exploit learning opportunities for student. This study presents the boundary crossing collaboration between two female educators within informal learning settings. Drawing on a boundary crossing learning perspective, I explored two female educators' mathematics experiences and interrogated their boundary crossing experiences as sites for humanizing mathematics. Based on this exploration, I address the possibilities for humanizing mathematics by facilitating boundary crossing between mathematics education and informal education.*

Keywords: boundary crossing, informal mathematics, humanized mathematics

## Introduction

In a *Journal of Research in Mathematics Education* commentary, Stephan et al. (2015) reported a challenge and an opportunity for mathematics education to support the public to understand the role of mathematics in society. To achieve the goal, the mathematics education community needs to receive widespread support to energize the public toward “changing the image of or public perception about mathematics” (p. 139). Previous studies have recognized that facilitating crossing boundary collaboration between mathematics education and other practices outside mathematics education can stimulate this initiative (e.g., Bakker et al., 2021; Stephan et al., 2015).

The current study is a boundary crossing between two female educators engaging in a collaborative relationship. Laura (pseudonym), the primary participant in the study, started the first *Girls Excelling in Math and Science (GEMS)* club for in 1994 and has continued her contribution to informal education to the present. I, a researcher from the mathematics education community, work alongside Laura and collaborate with her to further develop the *GEMS* program. At first, I doubted that collaboration was related to mathematics or mathematics education. Laura who has taught mathematics activities in informal spaces for more than two decades, always claims, “I am not good at math” and calls me “math person.” As we continued engaging in boundary crossing collaboration, our view of mathematics and mathematics teaching began evolving. This study tends to provide a reference providing opportunities to humanize mathematics through building boundary crossing collaboration between the mathematics education community and communities outside of mathematics education.

## Literature Review and Theoretical Framework

Two sets of relevant literature inform the theoretical perspective of this study. First, the literature on humanizing mathematics helps me reconceptualize mathematics and mathematics education. Following that, I describe a boundary crossing perspective which provides an analytic lens to understand Laura's and my boundary crossing experiences. In particular, four learning mechanisms were unitized to understand different stages of boundary crossing collaboration.

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

## Humanizing Mathematics

In public, mathematics is perceived to be "difficult, cold, abstract, and in many cultures, largely masculine" (Ernest, 1996, p.802). Sam and Ernest (2000) surveyed British 548 adults, including people not indirectly involved in mathematics education and interviewed a sub-sample of 62 adults. They found one's school learning experience was one of the major influential factors for forming their image of mathematics. The public image of mathematics is also widely dispersed in popular culture. For instance, Darragh (2018) examined young adult fiction and found school mathematics was portrayed as tense, terrible, difficult, and less useful. In addition, mathematicians in many movies were depicted as exceptional people who were geniuses and born in that way that marked them as out of the ordinary (Mendick, 2005). Given the growth of future careers in science, technology, engineering, and mathematics (STEM) and the significant role of mathematics in STEM, reframing mathematics as normal, ordinary, and relevant to people is an urgent task for the mathematics education community (e.g., Darragh, 2018; Sealey & Noyes, 2010).

Researchers attempted to use the concept of *humanizing mathematics* to highlight the human practice characteristic of mathematics, shifting the public image of doing mathematics as special people to do extraordinary work. Though substantial literature uses the term *humanizing mathematics*, still, there are no agreed-upon descriptions of what mathematics is humanized and how to humanize mathematics. As early as the 1970s, researchers started to seek ways to humanize mathematics. Wheeler (1975) described humanizing mathematics is to foster students' awareness of self and the world through inner sights on mathematics, which he explicates as "the act of attention that preserves the significant parts of experience, that pegs and holds them in the self so that they are available for future use" (p. 8). Students use mathematics to understand at least part of their world, to see this part of their world through the eyes of mathematical relationships (Falkenberg, 2006).

D'Ambrosio (1985) introduced *ethnomathematics*, which refers to mathematical practices in identifiable cultural groups. Ethnomathematics recognizes that the development of mathematics is embedded in a cultural context. That is, people within various cultural groups develop unique methods and techniques to understand their realities in response to the problems they encounter (Rosa & Orey, 2008). D'Ambrosio (2015) suggested that teachers need to help students realize their full mathematical potential by acknowledging the importance of culture and bringing culturally relevant mathematics into teaching.

Rochelle Gutiérrez (2018) brought up the term *rehumanizing mathematics* to advocate for developing practices for those historically marginalized communities (i.e., Black, Indigenous and Latinx). She claimed, "a student should be able to feel whole as a person—to draw upon all of their cultural and linguistic resources—while participating in school mathematics" (p. 1). Similarly, D'Ambrosio (2001) cautioned that without a culturally relevant curriculum, most students leave school thinking that mathematics is something to be done only at school and has no relevance to their lives.

Nevertheless, school mathematics often focuses on the mastery of skills, accumulation of facts, rules, and algorithms necessary for standardized tests (Cooper, 2011). Conversely, mathematics in other contexts (i.e., designed informal learning environment) is not typically accompanied by traditional academic assessment allowing students to explore relevant mathematics and experience satisfaction in learning (Zhou et al., 2021; Nemirovsky et al., 2017). Different from the literature that tends to bring cultural knowledge into school mathematics (i.e., Gutiérrez, 2018), I conceptualize *humanizing mathematics* as a process that blurs the boundary


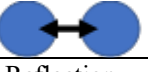


between mathematics in school and other practice, which allow teachers and students to see mathematics learning as ordinary human activity rather than special work.

### Boundary Crossing Learning

Boundary crossing has become an explicit concept in communities of practice (Wenger, 1998) and the theory of expansive learning (Engestrom, 1987; 2015). Though these two theories are rooted in different learning theories, situated learning theory and sociocultural learning theory, respectively, their descriptions of boundary crossing are compatible and coincide with my literature review. Wenger et al. (2002) argued that the “learning potential of an organization lies in this balancing act between well-developed communities and active boundary management” (Wenger p. 154). Engestrom (2001) pointed out that learning is also a horizontal movement in which a learner should be approached as a whole person who participates in school and many other practices. In this study, the horizontal aspect of learning provides a powerful lens for analyzing collaboration between Laura and me from different activity systems, which is conceptualized as boundary crossing (Engestrom, 1987/2015).

Akkerman and Bakker (2011) identified four learning mechanisms at the boundaries, identification, coordination, reflection, and transformation. Akkerman and Bruining (2016) further developed the four learning mechanisms. They suggested that boundary crossing in each form can occur at institutional, interpersonal, and intrapersonal levels. For this study, I only focus on boundary crossing at the interpersonal and intrapersonal levels (i.e., interpersonal experiences between researcher and participant and intrapersonal experiences of individual researcher or participant) (see Table 1).

**Table 1: Boundary Crossing Framework**

Learning Mechanism	At the Interpersonal Level (Action and Interaction Between Actors from Different Practices)	At the Intrapersonal Level (Participation of a Person in Two or More Practices)
 Identification	People come to (re)define their different and complementary roles and tasks.	A person comes to define [ <i>their</i> ] own simultaneous but distinctive participatory positions.
 Coordination	People seek shared means or procedures for exchange and cooperative work.	A person seeks means or procedures to distribute or align [ <i>their</i> ] own participatory positions in multiple practices.
 Reflection	People come to value and take up another’s perspective.	A person comes to look differently at their own participatory position because of the other participatory position.
 Transformation	People face a shared problem, start collaborative work, and may build group identity.	A person develops a hybridized position that integrates previously distinctive ways of thinking, doing, communicating, and feeling.

Adapted from Akkerman and Bruining (2016)

At an interpersonal level, boundary crossing is about actions and interactions between specific groups of people from different practices. I come from a mathematics education community in this study, and Laura comes from the informal education community. We establish collaborative relationships to work together on a certain project. Both Laura and I entailed multiple roles to participate in more than one separate practice at an intrapersonal level.

## Narrative Inquiry Methodology

With the goal of building a reciprocal and collaborative relationship with Laura, narrative inquiry enables me to co-construct the inquiry with the participant. Clandinin and Connelly (2000) described narrative inquiry as “a way of understanding experience. It is a collaboration between researcher and participants, over time, in a place or series of places, and social interaction with milieu” (p. 20). Clandinin and Connelly’s suggestion, I engage in narrative inquiry research by embracing narratives as both the method and phenomena of study and entailing the dual role as researcher and participant in the study.

### Data Source

Because of the nature of the narrative inquiry, this study’s data are co-constructed between Laura and me. The narrative data consist of two forms, field texts and interviews. In this study, field texts include various forms: informal conversations, autobiographical writing, reflections, emails, etc. (Clandinin & Connelly, 2000). The field texts record ongoing interactions between Laura and me and my reflections, thoughts, and feelings in particular moments. Two interviews have been conducted to understand Laura’s experiences. The first interview focused on her past experiences with mathematics and *GEMS*. The second interview focused on her reflections on the collaborative experiences with people from mathematics education, including me.

### Data Analysis

The narrative data are dynamic and influenced by contextual factors and the interaction of Laura and me. Moreover, data interpretation is personal and dynamic, which requires change and recharge, often including further reading. The first data analysis stage is to transfer field texts to research texts (Clandinin & Connelly, 2000). During the transfer, I stepped back for a while from fields and shifted back to a researcher position and through the theoretical lens to view data.

I adopted Polkinghorne’s (1995) analytical approaches: *analysis of narrative and narrative analysis*. The *analysis of narrative* method employs paradigmatic reasoning to collect stories as data and analyze them in descriptions of themes that hold across the stories or in taxonomies of types of stories or characters. Using *analysis of narrative* approach, I first applied boundary crossing theory. I identified learning mechanisms from the collaborative experience of Laura and me in both interpersonal level and intrapersonal level. Then, themes were identified directly from data cross Laura’s and my experiences, such as becoming a teacher and being a female educator were common themes identified.

I also employed the *narrative analysis* method, which uses narrative reasoning to collect descriptions of events and synthesize or configure them into a plot in a story or stories. This method allows me to recognize plots of the narratives by reading data repeatedly. I have identified two major plots of the study: women’s experiences and experiences with mathematics, which are also interwoven, sometimes entangled. The plots also determine how the stories are told: what emotions are displayed or the certain phrases or words the narrator uses, which manifests deeper layers of the narrator’s identity.

### Findings

Due to the page limitation, I share part of the finding from the study. First, I present Laura’s experiences with mathematics in *GEMS*. Then, through a boundary crossing lens, I describe an emerging collaborative experience between Laura and me with a mathematical task. The collaboration starts from recognizing mathematics and ends to reframing mathematics, within which Laura and I coordinate with each other on the task design and reflect on our own experience.

## **Mathematics Was Dropped Out in GEMS**

Laura recognizes that mathematics in *GEMS* presents a dilemma; on the one hand, the goal of *GEMS* is to increase girls' interest in mathematics; on the other hand, Laura has discovered that engaging girls in mathematics can create tension with another goal, presenting *GEMS* as different from school. Unlike other subjects, such as engineering or science, that are easily integrated into hands-on activities and experiments, Laura sees mathematics as built on mental strategies and memorization, which could cause girls to lose interest very quickly.

Laura has struggled with the certainty of mathematics that may diminish the passion for exploration. She has expressed concern that, as *GEMS* girls seek a correct answer, they might overlook the opportunity to discover other possibilities in the process. If a problem is challenging, it can be perceived as difficult and frustrating. Students who are not confident in mathematics thus fear failure and may lose interest and the confidence to explore. Laura said:

I think with math—what was always so hard for me is that it's very tricky to make it not be the right answer—you see, to experiment. I don't even know how to begin. With the technology, when we would do computer stuff with kids, it's something that they could do at home. It's not just school stuff. It's something fun, and so that's what I mean—not like school. It's not intended to put math down. (Laura, Interview, August 2019)

On the *GEMS* website, we provide sample activities for leaders to use. Some activities are labeled as science or engineering on the website, and mathematics components can be identified in the activities. However, mathematics is not emphasized in the activities. I expressed my concern that mathematics is often hidden in *GEMS* activities. Laura proposes that leaders do not label activities at the beginning but later guide students to reflect on the subject knowledge. She said:

There's a lot of math in there. I think we're going to have to help the leaders, not label it or maybe make it a mystery. At the end after you do these really fun activities—let's think about what kinds of categories we were using today. Were we using science or [math]? The teachers are really going to have to rethink how they bring math into *GEMS*. (Laura, Interview, August 2019)

After noticing Laura's hesitancy to communicate directly with *GEMS* girls about mathematics, I asked if she was concerned that by emphasizing mathematics, girls would lose interest in *GEMS*. Laura answered:

No, because again, as long as we always have to keep at the front that it's not school and it's fun, I don't think there's going to be a problem. Because there's going to be hands-on activities. I'm not worried about that. I think as long as we keep those things at the top. (Laura, Interview, August 2019)

Presenting mathematics in a different way from school—fun and hands-on—is Laura's goal for promoting mathematics in *GEMS*. Laura expects we can continue to offer mathematics activities and even highlight the math and said, "I'm really interested in what you guys come up with for fun math in *GEMS*, because this could make a huge difference."

## **Evolving Mathematics in Collaboration**

In this section, I share evolving mathematics in our collaboration. I describe a mathematics task called Quilt Problem, to illustrate informal educators as curriculum developers who capture emerging learning opportunities from everyday practice and teaching practice. Specifically, I

describe the development of the task through a boundary crossing lens (i.e., identification, coordination, reflection, and transformation).

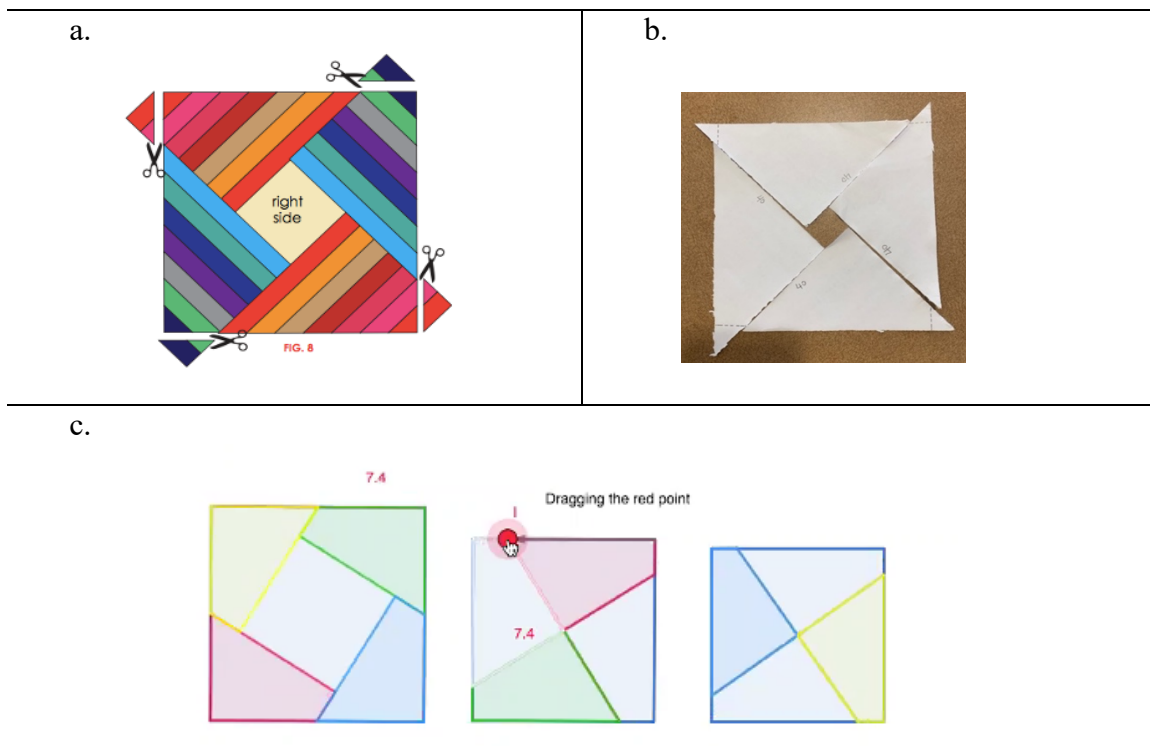
**Identification.** The task started from an email from Laura to me, which was titled *I need help with a math problem*. In the email (See Figure 1.a), she said,

Here is the quilt pattern. I am trying to figure out how to make it a little bigger because I have gorgeous fabric, and I want to expand the portions marked 28 inches wide to about 38-40 inches wide. Unfortunately, they will stay the same length. The problem I am having is many-faceted: 1) Do I make the square bigger? If so, what size? 2) Do I change the location of the angled cut to make it more proportional? 3) What will the finished size be? (Laura, Email, April 2021)

It took me quite a while to make sense of the context. Then I grabbed a piece of paper to manipulate the model, see Figure 1.b, and responded to Laura,

If you have 40\*40 fabric, you can decide the size of the quilt. It also depends on how big the square in the center you want. See the pic below. You can move the triangles to make the square inside bigger; then, the quilt will be bigger too. (Lili, Email, April 2021)

Laura replied that she did not get it. Then we set up a virtual meeting to discuss the fabric and model. Later, using geometry software, I made a model for this problem and demonstrate the relations between fabric and the final quilt (See Figure 1. c). Laura thought the visual model is very helpful for her to understand the spatial relations.



**Figure 1: Quilt Problem**

**Coordination.** Laura and I recognized mathematics from this real-life problem and felt interesting to make this problem a mathematics task. We have worked together to develop a problem called *Ms. Jones's Quilt Problem*.

Ms. Jones has a question. She is making a quilt for her sister. The pattern she used for the quilt is below. She has two fabulous squares of fabrics 40''x40.'' She wants the finished quilt to be a square with sides between 46''-56.'' Could you help her figure out where the cuts are and how you do you make the quilt?

I brought this task to my pre-service teachers and mathematics education colleagues to let them try and collect feedback from them.

**Reflection.** Laura and I both reflected on this task. Laura said,

I'm clueless about spatial stuff. That is an example of my bad spatial skills. I also see my daughters' and other girls' struggles in spatial. They need to understand [spatial]; if they cannot figure out how to do it, the boys take it over. So, the only thing I want to do for GEMS is developing a spatial curriculum.

My reflection as:

This problem is from real life and developed as a contextualized mathematics problem. However, as mathematical relationships are recognized, the context is not important anymore and is even removed in the solution process. It changes back to a pure math problem. We need to think about making mathematics tangible and engaging throughout the problem-solving process.

Laura and I were not very satisfied with the problem in the reflection. Laura thought it was *too school*, and I believed it reflects the idea of optimization, which is a goal of mathematics but not all I wanted to communicate.

**Transformation.** Building on our reflection, Laura and I decided to expand this task to a project. We integrate art, cultural, real-life elements, and geometry, technology, and algebra into the project. See Table 2.

**Table 2: Quilt Project**

Session	1	2	3	4	5	6
Topic	Knowing patterns	Appreciate quilt designs and share stories about quilts	Use Geogebra to draw your quilt design	List materials you need	Make your quilt	Exhibition
Element	Art, real life, and mathematics	Cultural, community, and personal assets	Mathematics, technology, and art	Mathematics, real-life	Hands-on	Involving parents



The integration characteristic of the project involves creating a task that does not connect to one specific subject but instead transfers from a mathematics task to an integrated STEM task. Though less focused on mathematics, this provides multiple entry points for students, allowing them to bring their cultural and life experiences to the project. Laura and I both like the project and see this project as not “too school” but fun, which aligns with the mission of *GEMS*.

### **Discussion and Conclusion**

The boundary crossing efforts in this study build a bridge between mathematics practice in school and mathematics practice in informal learning environments. Informal learning environments like *GEMS* encourage excitement, where having fun is a primary consideration for selecting and implementing activities. Such an emphasis can be potentially disruptive to the seriousness assumed to be essential in learning mathematics. Educators from the mathematics education community support identifying and making mathematics explicit in STEM integration, whereas educators from the informal education community contribute their expertise to bring excitement and engagement to mathematics learning. Learning across boundaries leads to discontinuities (Akkerman & Bruining, 2016), which can be understood as learning mechanisms to generate new practices that create a space for reframing mathematics. The new collaborative practices emerged as crossing boundary between mathematics education and informal education create a space that blurs the border of mathematics in different contexts which leads to a new understanding of mathematics as normal human practices rather than mandatory.

Humanizing mathematics requires a broader approach incorporating culturally relevant methods, objectives, and content in solidarity to reframe mathematics (Yeh & Otis, 2019). In this study, narrative inquiry provides a method that value subjective resource as a form of knowledge that sets a stage for humanizing mathematics. Moreover, the goal of the collaboration between Laura and me is to foster alternative mathematics practice. The shared objective of the collaboration is to work towards a broad understanding of mathematics that makes sense for people from inside the mathematics education community and outside of the community. Humanized mathematics consists of an accessible curriculum and culturally relevant pedagogy, which satisfies learners in working with mathematics. To humanize mathematics and make mathematics “normal and ordinary” (Darragh, 2018), mathematics education researchers need to build connections between mathematics and other practices. Efforts crossing boundaries (Akkerman & Bruining, 2016) between the mathematics education community and the informal education community bridges in-school mathematics and out-of-school mathematics, providing opportunities to humanize mathematics.

This study provides an example of building boundary crossing collaborations between the mathematics education community and other disciplines or communities outside of mathematics education to reconceptualize the understanding of mathematics. As such, the study contributes to changing the traditional image of mathematics by reframing and humanizing mathematics as a normal practice that encourages a wide range of participation in mathematics (Darragh, 2018). In particular, the effort of humanizing mathematics in this study has the potential to inspire historically underserved students in mathematics to actively participate in mathematics/STEM (e.g., Joseph et al., 2019).

Through this boundary crossing collaboration, Laura and I both have been involved in developing non-traditional mathematics curriculum and have developed a new understanding of mathematics. The unique experiences allow me to reframe and humanize mathematics by viewing mathematics experience as a human practice that positions learners and teachers as subjects interacting with mathematics. Using narrative inquiry methodology entailed the role of

researcher and participant in collaborative research, narrowing the gap between school mathematics and informal mathematics (Clandinin & Connelly, 2000). Reflecting on my own experience with mathematics learning and teaching, the image of mathematics I held likely aligns with the public image of mathematics. This study demonstrates the possibilities to change the public image of mathematics by promoting cross-disciplinary collaboration.

### References

- Zhou, L., Suazo-Flores, E., Sapkota, B., Mbeve, R. & Newton, J., (2021). Image of mathematics in- and out-of-school: A case study of two original participants in an afterschool STEM club—Girls Excelling in Math and Science (GEMS). *Proceedings of the 43rd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp.1210-1219). Philadelphia, PA.
- Akkerman, S. F., & Bakker, A. (2011). Boundary crossing and boundary objects. *Review of Educational Research, 81*(2), 132–169. <https://doi.org/10.3102/0034654311404435>
- Akkerman, S. F., & Bakker, A. (2019). Persons pursuing multiple objects of interest in multiple contexts. *European Journal of Psychology of Education, 34*(1), 1-24. <https://doi.org/10.1007/s10212-018-0400-2>
- Bakker, A., Cai, J., & Zenger, L. (2021). Future themes of mathematics education research: an international survey before and during the pandemic. *Educational Studies in Mathematics, 107*(1), 1-24.
- Bronkhorst, L. H., & Akkerman, S. F. (2016). At the boundary of school: Continuity and discontinuity in learning across contexts. *Educational Research Review, 19*, 18–35. <https://doi.org/10.1016/j.edurev.2016.04.001>
- Civil, M. (2007). Building on community knowledge: An avenue to equity in mathematics education. In N. S. Nasir & P. Cobb (Eds.), *Improving access to mathematics: Diversity and equity in the classroom* (pp. 105-117). Teachers College.
- Clandinin, J. & Connelly, M. (2000). *Narrative inquiry: Experience and story in qualitative research*. Jossey-Bass.
- Cooper, S. (2011). An exploration of the potential for mathematical experiences in informal learning environments. *Visitor Studies, 14*(1), 48-65. <https://doi.org/10.1080/10645578.2011.557628>
- Darragh, L. (2018). Loving and loathing: Portrayals of school mathematics in young adult fiction. *Journal for Research in Mathematics Education, 49*(2), 178- 209. <https://doi.org/10.5951/jresmetheduc.49.2.0178>
- D'Ambrosio, U. (1985). Ethnomathematics and its place in the history and pedagogy of mathematics. *For the Learning of Mathematics, 5*(1), 44-48. <https://www.jstor.org/stable/40247876>
- D'Ambrosio, U. (2001). What is ethnomathematics and how can it help children in schools? *Teaching Children Mathematics, 7*(6), 308-310.
- D'Ambrosio, U. (2015). From mathematics education and society to mathematics education and a sustainable civilization: A threat, an appeal, and a proposal. In *Proceedings of the Eighth International Mathematics Education and Society Conference* (Vol. 8, pp. 19-30).
- Engeström, Y. (2001). Expansive learning at work: Toward an activity theoretical reconceptualization. *Journal of Education and Work, 14*(1), 133-156. <https://doi.org/10.1080/13639080020028747>
- Engeström, Y. (1987/2015). *Learning by expanding*. Cambridge University.
- Ernest, P. (1991). *The philosophy of mathematics education*. Routledge Falmer.
- Ernest, P. (1995). Values, gender and images of mathematics: A philosophical perspective. *International Journal of Mathematical Education in Science and Technology, 26*(3), 449-462. <https://doi.org/10.1080/0020739950260313>
- Ernest, P. (1996). Popularization: Myths, massmedia and modernism. In A. Bishop, K. Clements, K. Christine, K. Jeremy, & L. Colette. (Eds.), *International Handbook of Mathematical Education* (pp. 785-817). Kluwer Academic.
- Ernest, P. (2012). What is our first philosophy in mathematics education?. *For the Learning of Mathematics, 32*(3), 8-14. <https://www.jstor.org/stable/23391968>
- Falkenberg, T. (2006, July). Moral education in the teaching of mathematics. In *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 49-56).
- Gutiérrez, R. (2018). Introduction: The need to rehumanize mathematics. In *Rehumanizing mathematics for Black, Indigenous, and Latinx students*. National Council of Teachers of Mathematics.
- Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. Cambridge University.
- Mendick, H. (2005). A beautiful myth? The gendering of being/doing 'good at maths.' *Gender and Education, 17*(2), 203-219. <https://doi.org/10.1080/0954025042000301465>
- Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

- Nemirovsky, R., Kelton, M. L., & Civil, M. (2017). Toward a vibrant and socially significant informal mathematics education. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 90-101). National Council of Teachers of Mathematics.
- Polkinghorne, D. E. (1995). Narrative configuration in qualitative analysis. *International Journal of Qualitative Studies in Education*, 8(1), 5-23. <http://doi.org/10.1080/0951839950080103>
- Rosa, M., & Orey, D. C. (2008). Ethnomathematics and cultural representations: Teaching in highly diverse contexts. *Acta Scientiae-ULBRA*, 10, 27-46.
- Sam, L. C., & Ernest, P. (2000). A survey of public images of mathematics. *Research in Mathematics Education*, 2(1), 193-206. <https://doi.org/10.1080/14794800008520076>
- Sfard, A. (1998). On two metaphors for learning and the dangers of choosing just one. *Educational Researcher*, 27(2), 4-13. <https://doi.org/10.3102/0013189X027002004>
- Sealey, P., & Noyes, A. (2010). On the relevance of the mathematics curriculum to young people. *The Curriculum Journal*, 21(3), 239-253.
- Stephan, M. L., Chval, K. B., Wanko, J. J., Civil, M., Fish, M. C., Herbel-Eisenmann, B., Konold, C., & Wilkerson, T. L. (2015). Grand challenges and opportunities in mathematics education research. *Journal for Research in Mathematics Education*, 46(2), 134-146. <https://doi.org/10.5951/jresmetheduc.46.2.0134>
- Wenger, E. (1998). *Communities of practice: Learning, meaning, and identity*. Cambridge University.
- Wenger, E., McDermott, R. A., & Snyder, W. (2002). *Cultivating communities of practice: A guide to managing knowledge*. Harvard Business School.
- Wheeler, D. (1975). Humanizing mathematics education. *Mathematics Teaching*, 71, 4-9.
- Yeh, C., & Otis, B. M. (2019). Mathematics for whom: Reframing and humanizing mathematics. *Occasional Paper Series*, 2019 (41). Retrieved from <https://educate.bankstreet.edu/occasional-paper-series/vol2019/iss41/8>

## INSERVICE TEACHERS' ATTRIBUTIONS FOR MATHEMATICAL SUCCESS

Dionne Cross Francis  
University of North Carolina  
dicross@unc.edu

Jeffrey Franklin  
IUPIU  
frankljj@iu.edu

Eric Jacobson  
Indiana University  
erdajaco@indiana.edu

Naomi A. Jessup  
Georgia State University  
njessup@gsu.edu

Sheniqua Pierce  
Georgia State University  
spierce8@student.gsu.edu

Craig Willey  
IUPUI  
cjwilley@iupui.edu

Selim Yavuz  
Indiana University  
syavuz@iu.edu

Anna Gustaveson  
University of North Carolina at  
Chapel Hill  
annagus@unc.edu

*Beliefs teachers hold influence the judgments they make about their students, and opportunities they provide for engaging them in rigorous mathematics. While math-related beliefs have been widely studied, less is known about teachers' attributional beliefs (i.e., beliefs about people's actions or behaviors) for mathematical success. In this study we investigated in-service elementary teachers' stated beliefs about mathematical success. Findings show that teachers attribute mathematical success to factors that are both internal and external to the student. Although teachers explicitly stated that race and gender were not factors, many used descriptors that served as proxies for students' demographic markers.*

Keywords: Elementary School Education, Teacher Beliefs, Equity

Teachers play a pivotal role in creating equitable mathematical learning experiences, yet teachers' beliefs about their students' capabilities and what attributes to that success varies, particularly for historically marginalized student groups (Jackson et al., 2017; Wickstrom, 2015). More specifically, beliefs that teachers hold about their students influence the judgments they make about their students (Ernest, 1989; Hoy et al., 2009), how they organize their classrooms (Cross Francis, 2015; Pajares, 1992), and opportunities for engaging in rigorous mathematics (Jackson et al., 2017). Attribution beliefs are individuals' thoughts about the causes of actions or behaviors, and attribution theory assumes that individuals try to determine why people do what they do by attributing behavior to its underlying cause, with either internal or external attributions (Fiske & Taylor, 1991; Graham, 2020). For example, Jackson et al. (2017) found that in-service middle school teachers attributed students' lack of mathematical success to various kinds of deficit-oriented beliefs connected to innate traits and families' funds of knowledge, resulting in lowering the cognitive demand of the mathematics activity.

Cognitive bias assumes that what a person says or does is dependent on the "kind" of person they are (Ross, 1977). Research suggests teachers often hold attributional biases and attribute student failure to factors internal to students (e.g., lack of effort) and external to themselves (Gosling, 1994; Jackson et al., 2017; Kulinna, 2007). Teachers' attributions relate to student demographic markers with gender and race being the most prominent (Espinoza, 2014; Fennema and Leder, 1990; Teidemann, 2002). Research has shown the strong relationship between teachers' math-related beliefs and their practices, suggesting that teachers' attributions about students' mathematical success or failure may influence how they architect and support students' opportunities to learn. Interestingly, despite the abundance of research on teachers' math-related beliefs, very few studies focus on

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

teachers' attribution beliefs about mathematical success in elementary contexts. In this study, we investigated teachers' stated beliefs about reasons for mathematical success.

### Attributional Beliefs

Three kinds of attribution beliefs closely inform our work: belief in genetic determinism (Keller, 2005), belief in social determinism (Rangel & Keller, 2011), and belief in school meritocracy (Wiederkehr et al., 2015). Belief in Genetic Determinism (BGD) holds that innate biological or genetically determined traits play the largest role in molding an individual (Keller, 2005). Statements aligned to BGD communicate underlying beliefs such as, "People can do things differently, but the important parts of who they are cannot really be changed" (Keller, 2005, p. 691). In relation to mathematical success, BGD is expressed by the belief that some people are naturally good at math. This physiological essentialism is correlated with race, gender, stereotyping, and prejudice (Hoffman & Hurst, 1990; Keller, 2005; Martin & Parker, 1995). The foundation these beliefs is connected to an individual's understanding, and contribute to the formation and endorsement of stereotypes (Keller, 2005) that perpetuate deficit narratives about who is considered good at math (Adiredja, 2019).

Belief in Social Determinism (BSD) is the understanding that an individual's fundamental essence is shaped permanently by social factors (e.g., upbringing, social background) (Rangel & Keller, 2011). Statements grounded in BSD communicate underlying beliefs like, "...[T]he social background a person comes from is strongly reflected in the person's character" (ibid, p. 8). An example of BSD in mathematics contexts communicate faultiness in Black students' capabilities based on their family's background and socioeconomic status (Martin, 2012). BSD is strongly linked with negative stereotyping, prejudice, and discriminatory tendencies, along with in-group favoritism (ibid), which can contribute to out-group bias and racism (Brewer, 2001).

Belief in School Meritocracy (BSM) describes an individual's belief that school success can be explained in terms of effort (Wiederkehr et al., 2015). It specifies the general meritocratic belief that social institutions reward individual ability and effort (Young, 1961; Jost et al., 2003). Individuals expressing this belief may state, "To succeed at school, one only has to work hard." For mathematics teachers, a student's motivation and growth mindset toward learning are direct indicators of their mathematical outcomes. However, Zavala and Hand (2019) warn against the guise of growth-mindset beliefs because those do "not take into account institutional forces and historical patterns of marginalization and trauma" (p.850). Thus, belief in meritocracy is associated with out-group favoritism for members of low status groups, and in-group favoritism for members of high-status groups (Jost & Hunyady, 2005). In this study we attempt to develop a better understanding of teachers' attributional beliefs by answering the following questions: (i) *What do elementary, in-service teachers believe about students' success in mathematics?* (ii) *Are there commonalities across teachers' demographics in relation to their beliefs about students' success in mathematics?*

### Methods

#### Participants

This study is situated within a larger project called *Attributions of Mathematical Excellence in Teaching and Learning* focused on understanding teachers' attributions for mathematical success. The participants included ten, female, elementary in-service teachers who teach math in grades K-5 in the United States. Of these teachers, five held Master's degrees, four held Bachelor's degrees, and one teacher indicated their education level as "other". Years of experience ranged from one year to twenty or more years. Four teachers were still in their first five years of teaching, five teachers in the 6-10 year range, and one teacher had taught for more

than twenty years. Three teachers identified as Black or African American, three identified as White, three identified as Hispanic or Latino, and one teacher identified as Asian.

### **Data Sources and Analysis**

All participants completed a Zoom-recorded, semi-structured interview that lasted approximately 60-minutes. The interview consisted of a range of questions focused on understanding participants' beliefs about the factors that contribute to mathematical success. For this study, we focused on three questions that targeted their beliefs about why some students do well in mathematics and the attributes of students who succeed in mathematics. After the interviews were completed, we listened to the recording and identified the location of the participants' responses to the three questions. We wrote detailed notes that described each participant's beliefs about the factors that contributed to mathematical success and transcribed specific statements that captured these beliefs. All authors read and re-read these descriptions and wrote statements or phrases that reflected the meaning underlying the teachers' statements. Then we met to collectively discuss our observations and identified statements/phrases that cohered. We summarized these statements into themes which are described next.

### **Findings**

Teachers' responses converged around five themes in relation to the reasons for students' success in mathematics. In what follows, we describe teachers' responses under each theme.

#### **Innate abilities and dispositions**

Participants indicated that there was some inborn traits responsible for students' mathematical success. Some alluded to innate cognitive abilities while others described more dispositional traits. Participants stated that some students were born with high mathematical aptitude, or their brains were designed to be proficient at math. Connie and Abby's statements reflected these perspectives, stating respectively "Some people are just born with a high skill in math...[they] think mathematically" and "...that's just how their brain is made up". Others described traits of students who are successful at math in ways that would suggest they were inborn and not developed through experience. Some of these traits included "well-behaved" (Eva), "creative (Elizabeth)", "takes initiative (Kimber)", "has a growth mindset (Pablo)", "resilient" (Joanna). These traits were described as the students' natural inclinations in how to act or behave.

#### **Parental influence**

The majority of teachers attributed mathematical expertise to parental influences to some degree in both positive and negative ways. However, the nature of the influence that teachers described varied. Influences that were regarded as positive included parental support and encouragement to do well, providing opportunities to support brain development, and creating access to resources that support mathematical thinking. One of the three factors supporting mathematical expertise that Connie stated was the student's homelife. In particular, "whether there are clear expectations and encouragement at home". Negative influences included parents projecting their own experiences with math onto their children, and lack of encouragement to do well mathematically. Karen recalled parents making statements like "math was hard for me so it will be hard for them".

#### **Nature of the subject**

Teachers also remarked that developing expertise in math is difficult because understanding concepts in the discipline is generally challenging. Elizabeth stated that developing knowledge of math requires initial interaction with concrete objects and learning how to talk about and describe what you are making sense of. She stated that "students generally struggle with explaining" and that does not bode well for developing expertise. Similarly, Joanna mentioned "Students don't have an opportunity to concretely understand what numbers are, what is happening if you're adding,

subtracting something and have that concrete hands-on experience”. While Joanna and Elizabeth referred to difficulties in developing a strong conceptual foundation, Eva described the role of practice, stating “math is a matter of practice so if you are more active then you will succeed”.

### **Good mathematics teaching**

Four teachers identified high-quality teaching as playing an important role in students’ mathematical success. Joanna described how the role of the educator and teaching became most salient for her when she taught kindergarten after teaching third grade. She reflected “Teaching kindergarten was really humbling. I didn’t realize how much teaching had to happen in kindergarten for my third graders to know what they know”. Similarly, Smith foregrounded the role of the educator in unearthing students’ abilities to engage meaningfully in mathematics. She stated, “It’s the educator’s responsibility to ensure that all students feel heard, that create a space where students feel that they can participate and contribute”. Smith made it clear that distinctions along race and gender lines tend to be imposed by adults and that when students are placed in classrooms with educators who know their value, they will do well mathematically.

### **Race but not race**

Unlike Smith who consistently explained why mathematical expertise should not be attributed to gender and/or race, some teachers would state the race and/or gender were not factors but in their descriptions of reasons would include proxies for race. For example, Abby talked about language being a barrier to success, “Schools in the US, white students (advantaged) do better, and this is due to students being accustomed to English and learning connected to their real-life contexts”. Others ascribed racialized and gendered dispositional traits to students although rejecting this idea in other parts of the interview. Connie described that Black and Latino boys often have a “smart mouth” and that often correlated with success, while girls tended to be more reserved. Further, Abby stated that “lower-level” and Hispanic students do not put forth effort, resulting in negative mathematical outcomes.

With respect to commonalities across teachers’ demographics, there were no distinct patterns by race across the responses that fell within the abovementioned themes, except for *good mathematics teaching*. We noted that both teachers of color and white teachers described innate characteristics (i.e., ability, dispositions) and external factors (i.e., parents, resources) as reasons for students’ mathematical success. However, only teachers of color provided reasons for mathematical success that reflected the belief that high-quality teaching played a role in students’ mathematics success. It may be that while white teachers recognized the factors external to the student support or hinder students’ mathematical outcomes, they may not consider teaching as significant enough to outweigh other factors or they do not position themselves as having a critical role to play in students’ success.

### **Discussion**

The teachers’ responses in the study indicate a wide range of beliefs around mathematical success. Although some teachers highlighted and centered teachers as being partially responsible for inspiring students to engage with mathematics (i.e., contextual factors external to the student), there were other teachers who centered students as being solely responsible for how successfully they relate and connect to mathematics. In this regard, teachers referred to innate characteristics and dispositions as the reason for students’ success, which aligns with the tenets of genetic determinism (Keller, 2005). Thus, success in mathematics is connected to students’ abilities to use their own “creativity”, “focused nature”, “initiative”, and/or “resilience” to deal with challenging mathematics concepts. Teachers also described students’ effort (growth mindset), the role of parents, and resources accessible within the home, as responsible for mathematical success. These reflect attributions aligned social determinism and school meritocracy (cf., Jost et al., 2005; Rangal &

Keller, 2011). Interestingly, some teachers suggested that the mathematics was itself a challenging discipline, thus lack of success was not person-related.

The variety of responses from in-service teachers suggests the need to further research how attributional beliefs about students' mathematical success are developed and sustained. However, the real danger lies in within-service teachers' beliefs in one sole factor that determines success. This leads to the further perpetuation of beliefs such as social determinism, genetic determinism, and school meritocracy which largely emphasize one factor without considering contextual aspects, social identities (e.g., race), stereotyping, and equal access and opportunity. By considering the social, political, racialized and gendered contexts that impact student success, educators consider factors that influence students' mathematical success beyond personal attributions and characteristics. Teachers that consider many different factors can aid in pushing back against both positive and negative stereotypes about students who succeed in math and those who do not while also being more culturally responsive to the needs of all of their students.

### Acknowledgments

This work was supported by the National Science Foundation under Grant DRL-2200990.

### References

- Adiredja, A. P. (2019). Anti-deficit narratives: Engaging the politics of research on mathematical sense making. *Journal for Research in Mathematics Education*, 50(4), 401-435.
- Brewer, M. B. (2001). Ingroup identification and intergroup conflict. *Social identity, intergroup conflict, and conflict reduction*, 3, 17-41.
- Cross Francis, D. (2015). Dispelling the notion of inconsistencies in teachers' mathematics beliefs and practices: A three-year case study. *Journal of Mathematics Teacher Education*, 18(2), 173 - 201.
- Ernest, P. (1989). The knowledge, beliefs and attitudes of the mathematics teacher: A model. *Journal of education for teaching*, 15(1), 13-33. <https://doi.org/10.1080/0260747890150102>
- Espinosa, A. (2014). Teachers' educational beliefs and culturally and linguistically diverse students within response to intervention. University of Miami.
- Fennema, E., & Leder, G. C. (1990). *Mathematics and gender*. Teachers College Press, PO Box 20, Wiliston, VT 05495-0020.
- Fiske, S. T., & Taylor, S. E. (1991). *Social cognition*. Mcgraw-Hill Book Company.
- Gosling, P. (1994). The attribution of success and failure: The subject/object contrast. *European journal of psychology of education*, 9, 69-83. <https://doi.org/10.1007/BF03172886>
- Graham, S. (2020). An attributional theory of motivation. *Contemporary Educational Psychology*, 61, 101861. <https://doi.org/10.1016/j.cedpsych.2020.101861>
- Guskey, T. R. (1982). The effects of change in instructional effectiveness on the relationship of teacher expectations and student achievement. *The Journal of Educational Research*, 75(6), 345-349. <https://doi.org/10.1080/00220671.1982.10885407>
- Hoffman, C., & Hurst, N. (1990). Gender stereotypes: Perception or rationalization? *Journal of personality and social psychology*, 58(2), 197. <https://doi.org/10.1037/0022-3514.58.2.197>
- Hoy, A. W., Hoy, W. K., & Davis, H. A. (2009). Teachers' self-efficacy beliefs. In *Handbook of motivation at school* (pp. 641-668). Routledge.
- Jackson, K., Gibbons, L., & Sharpe, C. J. (2017). Teachers' views of students' mathematical capabilities: Challenges and possibilities for ambitious reform. *Teachers college record*, 119(7), 1-43. <https://doi.org/10.1177/016146811711900708>
- Jost, J. T., Glaser, J., Kruglanski, A. W., & Sulloway, F. J. (2003). Exceptions that prove the rule--Using a theory of motivated social cognition to account for ideological incongruities and political anomalies: Reply to Greenberg and Jonas (2003). <https://doi.org/10.1037/0033-2909.129.3.383>
- Jost, J. T., & Hunyady, O. (2005). Antecedents and consequences of system-justifying ideologies. *Current directions in psychological science*, 14(5), 260-265. <https://doi.org/10.1111/j.0963-7214.2005.00377>
- Keller, J. (2005). In genes we trust: the biological component of psychological essentialism and its relationship to mechanisms of motivated social cognition. *Journal of personality and social psychology*, 88(4), 686.
- Kulinna, P. H. (2007). Teachers' attributions and strategies for student misbehavior. *The Journal of Classroom Interaction*, 21-30. <https://www.jstor.org/stable/23869790>

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.



- Martin, C. L., & Parker, S. (1995). Folk theories about sex and race differences. *Personality and Social Psychology Bulletin*, 21(1), 45-57. <https://doi.org/10.1177/0146167295211006>
- Martin, D. B. (2012). Learning mathematics while Black. *Educational Foundations*, 26, 47-66.
- Pajares, M. F. (1992). Teachers' beliefs and educational research: Cleaning up a messy construct. *Review of educational research*, 62(3), 307-332. <https://doi.org/10.3102/00346543062003307>
- Rangel, U., & Keller, J. (2011). Essentialism goes social: Belief in social determinism as a component of psychological essentialism. *Journal of personality and social psychology*, 100(6), 1056. <https://doi.org/10.1037/a0022401>
- Ross, L. (1977). The intuitive psychologist and his shortcomings: Distortions in the attribution process. In *Advances in experimental social psychology* (Vol. 10, pp. 173-220). Academic Press. [https://doi.org/10.1016/S0065-2601\(08\)60357-3](https://doi.org/10.1016/S0065-2601(08)60357-3)
- Tiedemann, J. (2002). Teachers' gender stereotypes as determinants of teacher perceptions in elementary school mathematics. *Educational Studies in mathematics*, 50(1), 49-62. <https://doi.org/10.1023/A:1020518104346>
- Weiner, B. (1985). An attributional theory of achievement motivation and emotion. *Psychological review*, 92(4), 548.
- Wickstrom, M. H. (2015). Challenging a Teacher's Perceptions of Mathematical Smartness Through Reflections on Students' Thinking. *Equity & Excellence in Education*, 48(4), 589-605.
- Wiederkehr, V., Bonnot, V., Krauth-Gruber, S., & Darnon, C. (2015). Belief in school meritocracy as a system-justifying tool for low status students. *Frontiers in Psychology*, 6, 1053. <https://doi.org/10.3389/fpsyg.2015.01053>
- Young, M. (1961). *The Rise of the Meritocracy, 1870–2033: An Essay on Education and Equality*. Baltimore, MD: Penguin Books.
- Yzerbyt, V. Y., Corneille, O., & Estrada, C. (2001). The interplay of naive theories and entitativity from the outsider and the insider perspectives. *Personality and Social Psychology Review*, 5, 141-155. [https://doi.org/10.1207/S15327957PSPR0502\\_5](https://doi.org/10.1207/S15327957PSPR0502_5)
- Yzerbyt, V., Judd, C. M., & Corneille, O. (2004). Perceived variability, entitativity, and essentialism: Introduction and overview.
- Zavala, M. D. R., & Hand, V. (2019). Conflicting narratives of success in mathematics and science education: challenging the achievement-motivation master narrative. *Race Ethnicity and Education*, 22(6), 802-820.

## “IT WAS MEANINGFUL BECAUSE [THIS] IS NOW MY HOME”: LOCALITY-IDENTITY AND SOCIAL JUSTICE MATHEMATICS

Stephen L. Caviness  
Syracuse University  
Scavines@syr.edu

Nicole L. Fonger  
Syracuse University  
Nfonger@syr.edu

Karley Voyias  
Syracuse University  
Kvvoyias@syr.edu

Emmy Njue  
Syracuse University  
Ewnjue@syr.edu

Brian Odiwuor  
Syracuse University  
Bodiwuor@syr.edu

Waleed Raja  
Syracuse University  
Waraja@syr.edu

*Designing for and enacting social justice mathematics tasks is one approach to supporting students' meaningful mathematics learning. We share our design conjectures that were developed out of a theoretically grounded, empirically based investigation of undergraduate students' experiences of a social justice mathematics lab. We engaged an interdisciplinary lens of historically responsive literacy and identity in both the lab design and analyses of student responses. We have conducted four cycles of action research that contribute to a broader program of design-based research. Based on thematic analyses of qualitative data, we introduce the construct of locality-identity as a main theme in the data. Attending to locality-identity in the design of this lab increased student engagement with mathematics by making mathematics relevant and authentic.*

Keywords: Social Justice, Culturally Relevant Pedagogy, Precalculus, Undergraduate Education

Supporting students' meaningful learning of mathematics through theory-driven instructional design remains a perennial challenge for mathematics educators. Our recent work in this domain has been motivated by a need to think, act, and design differently in mathematics education to address issues of inequity (cf. Anderson, 2014; Gutstein, Fey, Heid, DeLoach-Johnson, et al., 2005; Tate, 2005). Within this agenda, we focus on designing for meaningful mathematics learning at the undergraduate level that may help address the problem of students' perceptions of mathematical (ir)relevance. According to Leyva, Walkington, Perera and Bernacki's (2022) research with undergraduate students, "increasing relevance in mathematics instruction can motivate students to learn and achieve in mathematics" (n.p.).

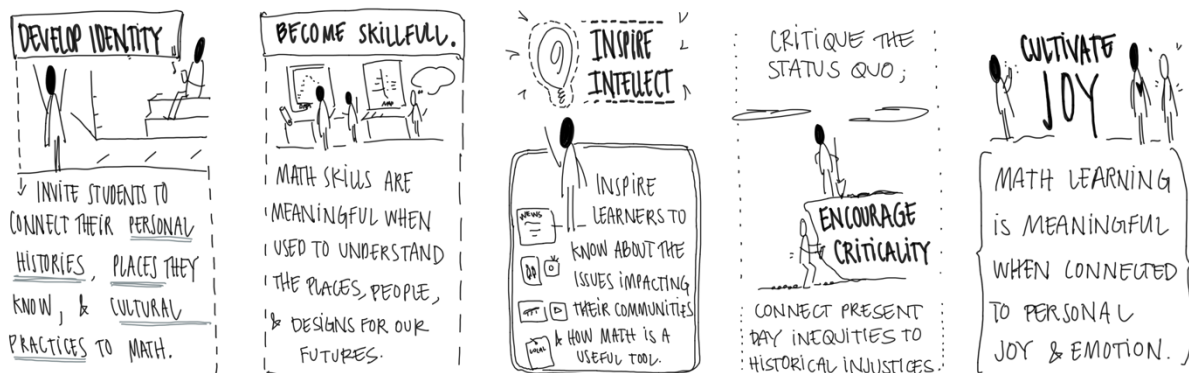
Situated in urban mathematics education, we sought to engage students' identities and utilize mathematics as a problem-solving tool that is relevant not only to where situations are mathematized, but how they are connected to broader social injustices. As Larnell, Bullock, and Jett (2016) elaborate, there is not a one-sized fits all version of the meaning of teaching mathematics for social justice (TMSJ), nor how it is enacted. In this paper, we frame TMSJ from a "social-justice-as-critical-consciousness" lens wherein mathematical skills are interwoven with students' development of an understanding of power and injustices. At a more nuanced level, we focus specifically on identity because starting with who students are is a necessary precursor for teaching skills (Muhammad, 2020). It follows, in our view, that teachers need to design mathematics learning opportunities that connect explicitly with students' identities (see also Larnell, 2016; Valoyes-Chavez, Darragh, 2022).

Based on this background, we conjectured that undergraduate students may identify with the geographical area that the university is situated in, and that eliciting students' sense of belonging to this locality would engage them in meaningful learning. With this prediction (cf. Hiebert et al.

2023), our research was guided by the following research question: *What are theoretically grounded, empirically driven design conjectures that can support teacher-researchers in attending to undergraduate students' identities in place-based social justice mathematics tasks?*

### Theoretical Framework: HRL and Identity

For students to engage in culturally responsive learning, they need to understand their own identities as well as the identities of diverse others (Muhammad, 2020). Muhammad defines identity as “who we are, who others say we are (in both positive and negative ways), and whom we desire to be” (Muhammad, 2020, p. 67). Because the nature of our work incorporates social justice issues into the teaching of mathematics, we also view identity from a post-structural lens that acknowledges the broader power-structures which shape one’s identity (Langer & Osuna, 2017). Lastly, we adopt Sfard and Prusak’s (2005) operationalization of identity as storied in which we define identity as the stories that one shares. These stories are simultaneously shaped at the macro level where structures of power influence the lived experiences of individuals.



**Figure 1: Muhammad’s (2020) culturally and historically responsive literacy framework (visualized by Fonger, 2022).**

Ghody Muhammad’s (2020) Historically Responsive Literacy (HRL) model, visualized in Figure 1, interweaves critical literacy, identity, intellect, and mathematical skill. We not only conceptualize learning through the HRL framework but operationalize it in the design of the social justice mathematics task discussed in this paper. Muhammad argues that to teach skill, you must first connect to students’ identities. Muhammad’s HRL model is an interdisciplinary framework. We have interpreted this framework for mathematics education. From this lens, identity means inviting students to connect their personal histories, places they know, and cultural practices to mathematics. Mathematical skills are meaningful when used to understand the places, people, and designs for our futures. Intellect is cultivated when learners are inspired to know about the issues impacting their communities and how math is a useful tool. Criticality entails critiquing the status quo and connecting present-day inequities to historical injustices. Finally, mathematics learning in the HRL model is meaningful when connected to personal joy.

### Methods

#### Research Context

This research study took place at a private university in the Northeastern United States. The university is a predominantly White institution with 51.3% of the 21,103.5 students identifying as White. Participants in the study were undergraduate students in a precalculus course spread across eight total sections in the spring 2022 and fall 2022 semesters.

## Task Design



Figure 2: Desmos slides eliciting a sense of locality.

The social justice mathematics lab that students participated in during this study was originally designed by Mr. K, a mathematics teacher at a local high school (Keech et al., 2022). Our research team has collaborated with Mr. K and other colleagues at the local high school and have engaged in four cycles of action research (Segal et al., 2009) to iteratively improve the lab. Above, Figure 2 displays three slides from the lab that were intended to elicit students' sense of belonging to the local area and its history. As instructional designers and teacher-researchers, we designed instructional interventions that were originally developed for historically marginalized youth in urban high school classrooms (Keech et al., 2022). These slides depict maps and images of residents in [the area] from the early 1950s before an interstate was built through [the neighborhood] which was a thriving African American community. In this design, we engage Muhammad's (2020) strategy of layering multiple texts (i.e., articles, videos, primary source documents, data) to support students' identity from an HRL lens.

### Data Analysis

We collected qualitative and quantitative data from students' responses to the lab, students' pre and post lab written reflections, and responses to a post lab Qualtrics survey. Consistent with Erskine et al. (2022) and Morse (2009), the quantitative survey responses were meant to underpin the qualitative data from students' responses. This study reports on the theme of locality-identity that we generated during analysis of responses to the problem-solving lab and the Qualtrics survey data. We analyzed student responses following Braun and Clarke's (2006) outline for conducting thematic analysis, a qualitative analytic method for "identifying, analysing and reporting patterns (themes) within data" (Braun & Clarke, 2006, p. 79). For this report, we focused on a specific question from the post-lab survey of data from two cycles of action research, spring 2022 and fall 2022, to assess our conjecture that eliciting a sense of locality would engage students in meaningful mathematics.

### Results

We conjectured that eliciting undergraduate precalculus students' sense of belonging to a specific location would result in meaningful mathematics learning. We asked students if they found the problem-solving lab meaningful. Of the students who responded, 71.43% (30/42) said YES, 21.43% (9/42) said NO, and 7.14% (3/42) said MAYBE. We interpret these results as strongly supporting our design conjecture. Given that our theoretical framework views identity as storied, our findings represent the voices of those who participated in this learning experience. Below, we highlight three themes in Table 1 based on our analysis of locality-identity, which we define as, "one's sense of belonging to and understanding of a specific geographical area." All

data excerpts were responses to the question, “Did you find the problem-solving lab meaningful to you? Please explain.”

**Table 1: Findings from Thematic Analysis of Locality-Identity**

Findings	Excerpts from Data
<p>Theme 1. Many students cited a sense of belonging to the geographical area as a reason for why the lab was meaningful to them regardless of their prior familiarity with the area.</p>	<p>Students who were previously familiar with the area</p>
	<ul style="list-style-type: none"> <li>• <i>“I have lived here my entire life and feel like I have learned more about the place I call home, its’ history, and underlying problems.”</i></li> <li>• <i>“Yes. I’ve lived near [the area] my entire life ([the area]) and I never knew this about [the area]. I am glad I learned about [the interstate] and how it affected the [neighborhood].”</i></li> </ul>
	<p>Students who were not previously familiar with the area</p>
	<ul style="list-style-type: none"> <li>• <i>“Yes, it was cool to do a lab and connect math to a real life problem in the city I now live in.”</i></li> <li>• <i>“It was meaningful because [the area] is now my home and I need to be informed on what goes on around me.”</i></li> </ul>
<p>Theme 2. Attending to locality-identity in the design of this lab increased student engagement with mathematics by making mathematics relevant and authentic.</p>	<ul style="list-style-type: none"> <li>• <i>“Yeah, this is let me realize math can be used on many aspects of our life.”</i></li> <li>• <i>“The problem solving lab was a nice incorporation of social issues into the math course. It gave students a chance to analyze real-life data and not made up numbers from a textbook.”</i></li> <li>• <i>“I thought it was a very helpful connection to how math can be used in a real life situation”</i></li> </ul>
<p>Theme 3. The lab did not generate a sense of belonging to the geographical area for all students</p>	<ul style="list-style-type: none"> <li>• <i>“Slightly, I did find it interesting but it did not affect me in a major way.”</i></li> <li>• <i>“No, It was not meaningful to me. I am not resident here so I don't really care about the issues that's going on but the lab was filled with facts and history I never known about.”</i></li> </ul>

### Discussion and Conclusions

A key contribution of our work is to elaborate on the construct, locality-identity. When designing this social justice mathematics task, we conjectured that eliciting students' locality-identity would result in meaningful learning. Our findings confirm this prediction and exemplify how, as a design feature in curriculum, locality-identity has high potential for engaging learners. In addition, this research corroborates Muhammad's (2020) assertion that attending to students' identities as an essential starting point for supporting students' skill and intellect in literacy practices (in this case, mathematics). In our most recent iteration of implementing this lab, we have attempted to address the finding that attending to locality-identity did not make mathematics meaningful for some students by intentionally asking students to envision how these social justice issues might relate to where they are from or other locations.

As graduate students and faculty, we are newcomers (Lave & Wegner, 1991) to the domain of scholarship on TMSJ (Larnell, Bullock, Jett, 2016), urban mathematics education (Tate 2008; Hoglebe & Tate, 2013; Larnell and Bullock, 2018), and place-based mathematics education (Showalter, 2013). We intend for this brief report to spark conversations, critique, and discussion among colleagues interested in attending to identity and geography when TMSJ.

## References

- Anderson, C. R. (2014). Place matters: Mathematics education reform in urban schools. *Journal of Urban Mathematics Education*, 7(1).
- Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative research in psychology*, 3(2), 77-101.
- Erskine, A., Odiwuor, B., & Fonger, N. L. (2022). An exploratory action research study of social justice mathematics in undergraduate precalculus. In Lischka, A. E., Dyer, E. B., Jones, R. S., Lovett, J. N., Strayer, J., & Drown, S. (Eds). *The 44th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 382-386).
- Fonger, N. L. (2022a). Cultivating math genius. <https://nicolefonger.com/2022/08/16/cultivating-math-genius/>
- Gutstein, E., Fey, J. T., Heid, M. K., DeLoach-Johnson, I., Middleton, J. A., Larson, M., ... & Tunis, H. (2005). Equity in School Mathematics Education: How Can Research Contribute?: NCTM Research Committee. *Journal for Research in Mathematics Education*, 36(2), 92-100.
- Hiebert, J., Cai, J., Hwang, S., Morris, A. K., & Hohensee, C. (2023). *Doing research: A new researcher's guide* (p. 136). Springer Nature.
- Hogrebe, M. C., & Tate, W. F. (2013). Place, Poverty, and Algebra: A Statewide Comparative Spatial Analysis of Variable Relationships. *Journal of Mathematics Education at Teachers College*, 3(2). <https://doi.org/10.7916/jmetc.v3i2.746>
- Keech, K., Routhouska, B., & Fonger, N. L. (2022). People, Place, and Population Predictions. *Mathematics Teacher: Learning and Teaching PK-12*, 115(8), 566-575.
- Langer-Osuna, J. M. & Esmonde, I. (2017). Identity research in mathematics education. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 637-648). National Council of Teachers of Mathematics.
- Larnell, G. V. (2016). More than just skill: Examining mathematics identities, racialized narratives, and remediation among black undergraduates. *Journal for Research in Mathematics Education*, 47(3), 233-269.
- Larnell, G.V., Bullock, E.C. (2018). A Socio-spatial Framework for Urban Mathematics Education: Considering Equity, Social Justice, and the Spatial Turn. In: Bartell, T. (eds) *Toward Equity and Social Justice in Mathematics Education*. Research in Mathematics Education. Springer, Cham. [https://doi.org/10.1007/978-3-319-92907-1\\_3](https://doi.org/10.1007/978-3-319-92907-1_3)
- Larnell, G. V., Bullock, E. C., & Jett, C. C. (2016). Rethinking teaching and learning mathematics for social justice from a critical race perspective. *Journal of Education*, 196(1), 19-29.
- Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. Cambridge university press.
- Leyva, E., Walkington, C., Perera, H., & Bernacki, M. (2022). Making mathematics relevant: An examination of student interest in mathematics, interest in STEM careers, and perceived relevance. *International Journal of Research in Undergraduate Mathematics Education*, 1-30.
- Morse, J. (2009). Procedures and practice of mixed method design: maintaining control, rigor, and complexity. In *Procedures and Practice of Mixed Method Design* (pp. 340-352).
- Muhammad, G. (2020). *Cultivating genius: An equity framework for culturally and historically responsive literacy*. Scholastic Incorporated.
- Segal, S. U. (2009). *Action research in mathematics education: a study of a master's program for teachers*. Montana State University.
- Sfard, A. & Prusak, A. (2005). Telling Identities: In Search of an Analytic Tool for Investigating Learning as a Culturally Shaped Activity. *Educational Researcher*, 34(4), 14-22. <https://doi.org/10.3102/0013189x034004014>
- Showalter, D. A. (2013). Place-based mathematics education: A conflated pedagogy. *Journal of Research in Rural Education*, 28(6), 1-13.
- Tate, W. F. (2008). Putting the "Urban" in mathematics education scholarship. *Journal of Urban Mathematics Education*, 1(1), 5-9.
- Tate, W. F. (2005). Access and opportunities to learn are not accidents: Engineering mathematical progress in your school. ERIC Clearinghouse.
- Tate, W. F. (2008). Putting the "Urban" in mathematics education scholarship. *Journal of Urban Mathematics Education*, 1(1), 5-9.
- Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

Valoyes-Chávez, L., & Darragh, L. (2022). Identity work, racialized emotions, and equity in mathematics education. *Journal for Research in Mathematics Education*, 53(5), 372-378.

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

## LEARNING TO PARENT MATHEMATICALLY: CRITICAL FACTORS IN PARENT-CHILD MATH ENGAGEMENT

Anastasia Betts  
University at Buffalo, SUNY  
albetts@buffalo.edu

Ji-Won Son  
University at Buffalo, SUNY  
jiwonson@buffalo.edu

Hee Jin Bang  
Age of Learning, Inc.  
Heejin.bang@aofl.com

*Dramatic differences in children’s math knowledge at school entry are thought to originate in the Home Math Environment (HME), where parents and caregivers are the primary provider of experiences that influence children’s early math knowledge development. Little is known about what informs parent decision-making around “mathematical parenting” (i.e., parents’ cognitions, motivations, and behaviors that impact and influence child math development in the HME). This study uses the RESET Framework and survey instrument to investigate parents’ mathematical parenting perceptions (n = 847) across the domains of Role, Expectations, Skills, Efficacy, and Time. Parent self-reports of early childhood math knowledge and of shared math activity are also examined to shed light on the factors that influence mathematical parenting of 4-5-year-old children in the home.*

Keywords: early childhood mathematics, home math environment, family engagement, early numeracy, parent involvement

Strong performance in mathematics is associated with higher performance in school and success beyond school (Ritchie & Bates, 2013; Schweinhart et al., 2005), yet standardized assessments of achievement continue to show that more than two-thirds of children struggle to demonstrate grade-level proficiency in mathematics (deBrey et al., 2019). With insufficient exposure to early mathematics knowledge prior to schooling, young children may find it difficult to learn and master key early mathematics competencies required to establish the strong numeracy foundation needed to effectively engage in later mathematics learning in elementary school and beyond (Claessens & Engel, 2013; Nguyen et al., 2016). To engage all students in learning mathematics throughout their K-12 journey and beyond, it is important to examine how learner variability at the onset of schooling can be accounted for and addressed. Learner variability prior to pre-kindergarten or kindergarten is believed to originate with differences in the Home Math Environment (HME), in which parents play a primary role in helping children to develop early math knowledge and skills (Cankaya & LeFevre, 2016; Daucort et al., 2023; Napoli & Purpura, 2018).

Parents are often their children’s first teachers: they provide not only for their child’s basic needs, but as more knowledgeable others, they also stimulate their child’s cognitive development and lead their thinking and learning across many areas, including mathematics (Blevins-Knabe, 2016; Vygotsky, 1986). However, limited research has examined parent motivations and decision-making related to the HME, and little is known about how parents learn to do the work of “mathematical parenting” – i.e., the cognitions, utterances, behaviors, and practices of parents that influence the development of their children’s math understanding, knowledge, and skills. Parents differ greatly in their mathematical parenting ability and knowledge, stemming from their differing perceptions of the role they should play in their children’s mathematics

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.



development, their wide-ranging expectations for their children’s math learning, their perceptions of their own math skills and efficacy in helping their children learn math, and their perceptions about the availability of time and energy to engage mathematically with their children (Betts, 2021; Betts & Son, 2022).

The purpose of present study is to examine parents’ perceptions across the domains of Role, Expectations, Skills, Efficacy, and Time (i.e., the RESET Framework), and how these areas may interact with other areas of mathematical parenting, such as parent knowledge of appropriate early childhood mathematics knowledge and skills, and reports of shared parent-child math activity in the home. Better understanding of the factors that influence mathematical parenting and decision-making in the HME has the potential to not only increase our understanding of the home environment may drive early development of mathematics knowledge, but also help stakeholders better understand how partnerships between schools and communities may be built to support children’s engagement in math learning.

### Theoretical Framework

This study is grounded in theoretical frameworks of Vygotsky, Mowder, and Bronfenbrenner. Vygotsky’s (1986) social-cultural theory of learning and the importance of the *more knowledgeable other* describe the essential role parents play in guiding and leading their child’s learning beyond what the child can learn on their own. To effectively guide and lead the child’s learning, parents must know what they are leading the child toward. They must have some sense of what expectations are appropriate for the child at any given developmental level, in addition to having a sense of their own role in that development, along with the knowledge, confidence, and time to engage in such guidance. Despite the importance of the parent-child learning dynamic, parents often report feeling lost when it comes to supporting their children’s early math development (Betts, 2021; Blevins-Knabe, 2016; Clements & Sarama, 2014).

Mowder’s (2005) parent development theory posits that individuals learn to parent as a result of one’s own experiences, combined with one’s personality, education, and other factors. Similarly, Bronfenbrenner’s (1992) ecological systems theory of development describes the individual at the center of successively expanding layers of influence, both proximal (e.g., family and friends) and distal (e.g., society, government, etc.), which impact the formation of that individual’s identities and beliefs, including mathematical parenting.

Grounded in these theories, the RESET framework was designed to better understand the relationships between parents’ perceptions about math and their mathematical parenting practices (Betts, 2021; Betts & Son, 2022). As shown in Table 1, the dimensions of the framework describe parent or caregiver perceptions of their **Role** in mathematical parenting, their **Expectations** for their child’s math learning, their perception of their own math **Skills**, their sense of self-**Efficacy**, and their perceived **Time** and energy available to support their child’s development of math knowledge and skills.

**Table 2: RESET domains as related to mathematical parenting in early childhood**

---

### RESET Framework Domain Descriptions

---

<b>R</b> (Role)	A parent's construction of their parental role is shaped by their early experiences in learning math, the ways in which they were parented, values and beliefs related to education, and other peer and societal or cultural influences. A parent's role is socially constructed and may change in response to changing social conditions, parent efforts and education (e.g., parent education or intervention programs, or even the child themselves), or the accumulation of life experiences.
<b>E</b> (Expectations)	The parent's expectations for the child's performance or development in mathematics is influenced by the value that the parent places on the learning of mathematics, its perceived role in the life of the parent and child, its perceived impact on the future success of the child, and the parent's knowledge and awareness of the mathematics concepts and skills appropriate for the child's age and developmental level.
<b>S</b> (Skills)	Parents' perceptions of their own mathematics skills and knowledge impact the ways in which they choose to interact with their children through mathematics, the types of skills and concepts they emphasize, and the expectations they have for their children's math development (e.g., if a parent feels like his or her life opportunities were limited because of weak math skills, they may conversely adopt higher expectations for their child's math learning in order to ensure the child is not limited by lack of math knowledge and skills).
<b>E</b> (Efficacy)	Parents' sense of self-efficacy is related to their belief in their ability to successfully support the math development of their child. It is influenced by their perceptions of math skills and knowledge, and influences their expectations for their child's math learning, as well as the ways in which they engage their child in mathematics activities.
<b>T</b> (Time)	Parent engagement in shared math activity is influenced by their perceptions of the time and energy available to participate. Further, parent perceptions of time and energy may be impacted by parents' perceived skills and knowledge and personal sense of self-efficacy. For example, more time and energy may be required from parents with low self-efficacy to engage meaningfully with their children through mathematics (more time to prepare in order to feel confident and comfortable, more anxiety that saps energy, etc.).

### Methodology

The RESET survey instrument (see Betts, 2021 and Betts & Son, 2022) was used to gather data on the perceptions of 847 parents of 4-to-5-year-old children who had not yet started kindergarten. The research questions guiding the study are: (1) how do parents perceive themselves in terms of their Role, Expectations, Skills, Efficacy, and Time (RESET) as they relate to supporting their children's development of early math skills, (2) how do parents' perceptions across RESET domains relate to their knowledge of early childhood math skills, and (3) how do parents' perceptions across RESET domains relate to their reports of shared parent-child math engagement?

### Participants

Participants from across the United States were recruited and screened by Innovate MR, a commercial panel provider, using various recruitment sources (e.g., nationwide database of volunteers for survey research, large-scale advertising networks, and mail and media advertisements, etc.). Participants were mothers (n = 626), fathers (n = 210), and non cis-identifying parents (n = 11), and represented a diverse set of ethnic (Caucasian 70%, African

American 10%, Multiracial, 9%, Latinx 7%, Asian 3%, Indigenous 1%), income (over 100K annually 19%, from 50K-100K annually 46%, less than 50K annually 35%), and educational (Undergrad/Graduate degree 34%, some college 29%, HS grad/trade/tech school 37%) backgrounds.

### Instruments and procedures

The survey instrument included items related to parent perceptions along the RESET domains, parent knowledge of early childhood mathematics, and parent self-reports of the frequency with which they engage in various types of shared math activities in the home environment. Six items per domain were included in the survey, which had been tested previously and found to be reliable with alphas all above .700 (see Table 2). Parents were asked to indicate their agreement with the item (statement) using a 7-point Likert Scale ranging from *Strongly Disagree* to *Strongly Agree*. Parents were also asked to evaluate when certain common early math concepts and skills should be learned (e.g., prior to kindergarten or after, etc.), and to report on how frequently they and their child engaged in a variety of common math activities in the home (e.g., *never, monthly, weekly, daily*).

## Results

### Parent perceptions on RESET

Table 2 shows simple descriptive statistics that describe how parents perceived themselves across the five RESET domains. Parent responses on domain Likert items were combined to create mean scores for each domain (e.g., Role Mean of 5.67, etc.). These domain means acted as proxies for the degree to which parents experienced higher or lower perceptions of themselves across the five RESET domains, with higher means indicating more positive perceptions, and lower means indicating more negative perceptions. In general, parents expressed higher perceptions of Role ( $M = 5.66$ ,  $SD = .87$ ) and Expectations ( $M = 5.76$ ,  $SD = .84$ ), and lower perceptions of Skills ( $M = 4.58$ ,  $SD = 1.41$ ), Efficacy ( $M = 5.09$ ,  $SD = .99$ ), and Time ( $M = 4.76$ ,  $SD = 1.06$ ).

**Table 3: Descriptive statistics for parent perceptions on RESET ( $N = 847$ )**

	Role ( $\alpha = .72$ )	Expectations ( $\alpha = .78$ )	Skills ( $\alpha = .85$ )	Efficacy ( $\alpha = .74$ )	Time ( $\alpha = .73$ )
Mean	5.66	5.76	4.58	5.10	4.67
Median	5.67	5.83	4.67	5.00	4.50
Mode	6.00	6.00	5.00	4.67	4.00
Std. Deviation	.87	.84	1.41	.99	1.06
Variance	.75	.71	2.00	.98	1.13

Separate multiple linear regression analyses were performed to determine relationships between parent Gender, Income, Education level, Ethnicity (independent variables), and parent perceptions along the RESET domain means (dependent variables). Parents of differently identifying gender roles (i.e., mothers, fathers, etc.) demonstrated some differences across the RESET domains, with Mothers expressing significantly higher perceptions of their Role in their children's math learning ( $\beta = 0.07$ ,  $p = 0.03$ ). When compared to Mothers, Fathers expressed significantly higher perceptions of their personal math Skills ( $\beta = 0.11$ ,  $p = 0.001$ ).

Tukey's HSD Test for multiple comparisons revealed a significant difference between Mothers and Fathers in their perceptions of Role ( $p = 0.012$ , 95% CI = [0.03, 0.36]), with Mothers having higher perceptions of their Role in their children's math learning. Conversely, Fathers demonstrated significantly higher perceptions of their math Skills ( $p < 0.001$ , 95% CI = [0.36, 0.88]), and self-Efficacy ( $p = 0.0$ , 95% CI = [-0.01, 0.36]) than Mothers when engaging in mathematical parenting. Non-cisgender identifying parents, while too small a group to perform statistical analysis ( $n = 11$ ) trended more with self-identifying mothers than fathers.

Multiple linear regression analyses showed that Income was a significant predictor of positive parent perceptions along three RESET domains: Expectations ( $\beta = 0.09$ ,  $p = 0.03$ ), Skills ( $\beta = 0.12$ ,  $p = 0.001$ ), and Efficacy ( $\beta = 0.08$ ,  $p = 0.05$ ). However, Income was a significant predictor of *negative* parent perceptions of Role ( $\beta = -0.09$ ,  $p = 0.03$ ). In other words, parents with higher incomes had lower perceptions of their Role. Additionally, Parents' Education level was a significant predictor of positive parent perceptions in Expectations ( $\beta = 0.14$ ,  $p < 0.001$ ), Skills ( $\beta = 0.14$ ,  $p < 0.001$ ), and Efficacy ( $\beta = 0.11$ ,  $p = 0.003$ ) domains. Parents with more education tended to have higher expectations for their child's learning and more positive perceptions of their own math skills and self-efficacy. No significant differences were observed on the RESET domains across Ethnicity; parents did not differ significantly along the Time domain.

### RESET and Parent ECE Math Knowledge

Parent understanding of early math skills may have important implications for their math parenting, since poor understanding of early math learning and how to support it may lead to insufficient support or expectations. To better understand parent knowledge of early math skills, parents were presented with 10 early math skills and asked to indicate when they believed the target skill should be learned (see Table 3). The 10 early math skills presented to parents on the RESET survey all appear in various preschool skills frameworks including those from New Jersey, New York, and California and were considered appropriate for children to learn before beginning kindergarten. During data analysis, parent responses were categorized as (1) before beginning preschool, (2) in preschool/pre-k, (3) in kindergarten or beyond. Additionally, the skills presented were categorized as either Simple Number Skills or as Advanced Number Skills, since early mastery of Advanced Number Skills is more closely associated with later mathematics achievement in school and beyond (Nguyen et al., 2016).

**Table 4: Simple and Advanced Number Skills**

Level of Rigor	Early Childhood Math Items used on the survey
Simple Number Skills (SNS)	• Say the number names in order from 1 to 10
	• Recognize written numbers 1 to 10
	• Say the names of numbers in order when counting a group of up to 10 things (like when counting toys, 1...2...3...4... etc.)
	• Recognize or make simple patterns (e.g., red, yellow, red, yellow)
Advanced Number Skills (ANS)	• Tell how many there are "in all" when counting a group of up to 10 items
	• Recognize / create more advanced patterns (e.g., blue, blue, green, blue, blue, green)
	• Say the number names in order from 10 to 20
	• Count two different groups of things and say how many there are all together ("this group has 3, and this group has 5, that's 8 all together")

- Correctly say the number names backwards from 10 to 1
- Start counting forward from a number that isn't the number one. For example, start at 6... then 7...8....9...10 etc.

Separate multiple linear regression analyses were performed to identify relationships between parent perceptions along the RESET domains (dependent variables) and parent knowledge of early childhood math skills (independent variables). Both Expectations ( $\beta = 0.31$ ,  $p < 0.001$ ) and Time ( $\beta = 0.28$ ,  $p < 0.001$ ) domains were significant predictors of parent knowledge of Simple Number Skills (SNS) as appropriately learned before kindergarten. In other words, parents with higher perceptions of Expectations and more Time available seemed to have more accurate knowledge of SNS as appropriate for children to engage in prior to kindergarten. Regarding parent knowledge of the more Advanced Number Skills (ANS), parent perceptions of Role ( $\beta = 0.14$ ,  $p = 0.003$ ), higher Expectations ( $\beta = 0.15$ ,  $p < 0.001$ ), and higher perceptions of Time ( $\beta = 0.104$ ,  $p < 0.03$ ) available significantly predicted parent knowledge of ANS as appropriately learned prior to kindergarten.

### RESET and Shared Math Activity

For math activity in the home, parents were presented with 12 different items describing common parent-child shared math activities. Parents were asked to indicate the frequency with which they engaged in the presented activities. Responses were categorized as: (1) never, (2) monthly, (3) weekly, or (4) daily. During data analysis, activities were further categorized as Formal/Informal, Explicit/Implicit, or Simple/Advanced (see Table 4).

**Table 5: Types of activity categories based on items presented on the survey**

Category	Description	Sample Survey Item
Formal	Activities consistent with traditional learning approaches found in school such as workbooks, worksheets, flashcards, etc.	<i>"We do flashcards together to help my child learn the names of numbers."</i>
Informal	Spontaneous or impromptu activities involving authentic integration of number concepts and skills within natural life activities.	<i>"We use our everyday activities to do addition problems (example: I have 4 toys and you have 2... how many do we have all together?)."</i>
Explicit	Activities whose specific purpose is to teach, practice, or reinforce math concepts or skills.	<i>"I help my child practice math skills using digital learning apps on our smart phone or tablet."</i>
Implicit	Activities whose specific purpose is something other than to teach, practice or reinforce math concepts or skills and involve math concepts and skills as part of the activity.	<i>"We play board games and count out the spaces one by one when moving the game piece."</i>
Simple	Activities focused on the simple number skills of count sequence, cardinality to ten, and numeral recognition to ten.	<i>"We use our everyday activities to practice counting up to 10 things together (like fingers, toys, snacks, or other things)."</i>

Advanced	Activities focused on more advanced number knowledge (e.g., 1:1 correspondence and cardinality beyond ten, concepts of addition and/or subtraction, comparing quantities, and patterns).	<i>“I help my child notice when one group of things has more (or less) than another group of things (for example, there are 5 trucks and 2 cars - there are more trucks than cars).”</i>
----------	--	--

Once again, separate multiple linear regression analyses were conducted to examine relationships between RESET domain means (dependent variables) and parent self-reports of shared math activity in the home (independent variables). Both the Expectations ( $\beta = 0.33, p < 0.001$ ) and Time ( $\beta = 0.29, p < 0.001$ ) domains were significantly related to overall math activity frequency. Parents with higher perceptions of Expectations for their child’s math learning and Time available to spend on shared math activity reported more frequent overall math activity engagement. Parent perceptions of Expectations and Time were significantly positively related to all activities ( $p < 0.001$ ), regardless of their categorization (e.g., formal vs. informal, explicit vs. implicit, simple vs. advanced). One unexpected result was that parent perceptions in the Role domain were significantly negatively correlated with reports of Formal activity frequency ( $\beta = -0.13, p = 0.003$ ), meaning that parents with higher perceptions of Role in their child’s math learning were more likely to report *less* frequent Formal math activity engagement (e.g., workbooks, flashcards, or digital math apps).

Additionally, regression analyses exploring relationships between parent demographics and activity frequency showed that the only significant relationships existed between Parent Gender, Education Level, and Explicit activities: Fathers engaged in less frequent Explicit activity than Mothers ( $\beta = -0.07, p = 0.03$ ), while parents with higher Education levels ( $\beta = 0.076, p = 0.05$ ) reported engaging in more frequent Explicit activities.

### Discussion & Implications

The RESET framework and its corresponding survey instrument provided effective means for examining parent perceptions around mathematical parenting, and specifically their role in the Home Math Environment. Findings related to Gender Roles pose some interesting considerations for stakeholders, and for teachers and schools who have primarily focused their efforts on working with mothers. Findings suggest that while mothers generally view themselves as having an important Role to play in the early math development of their children, fathers exhibit higher perceptions across the domains of Expectations, Skills, Efficacy, and Time. A potential implication here is that fathers see less of a Role for themselves in the early math learning than mothers do, even though they rate themselves more highly across the other domains. This raises the question of whether educators and schools are conducting adequate outreach to fathers to leverage their high expectations, skills, efficacy, and time available to help when it comes to supporting their children’s early math learning (e.g., “Daddy and Me” programs, etc.). There may be opportunities to help fathers see a role for themselves in their children’s early math learning and to help them better understand what that role might entail.

More complex questions arise around mothers and their significantly lower self-perceptions of their math skills, efficacy, and time. These results are consistent with prior research on math and gender that show lower perceptions of math skills and efficacy among girls and young women, even though measures of mathematics achievement show no significant gender

differences (Else-Quest et al., 2010; Huang, 2013; Perez-Felkner, Nix, & Thomas, 2017). Increasing mothers' self-efficacy presents an enormous challenge that may not be easily met. However, examining ways to better guide the math parenting time and energy of mothers may present a more attainable pathway. Given mothers' perceptions of limited amount of time to engage in active math parenting (e.g., shared math activity), parent engagement programs can do much to ensure that invitations for involvement include accessible activities that parents can easily implement, or that can be conveniently integrated within family routines (Betts, 2021).

The relationships between parents' Income and Education levels on their perceptions along RESET were mostly expected. Research has consistently shown relationships between higher income and education and higher achievement in mathematics (Ritchie & Bates, 2013). Parents of higher income and education that demonstrated significantly higher perceptions of Expectations, Skills, and Efficacy is consistent with that research. However, the somewhat unexpected result that higher parental Income was associated with decreased perceptions of parental Role may suggest that higher income parents—while still maintaining high expectations for their children's early math development—may not view themselves as having a *direct* Role to play in that development (which is how the items are presented). Rather, these parents may use their resources to ensure that their child's math learning needs are met (e.g., through preschool, private daycare, etc.), even if they are not directly or personally meeting them.

The relationship between the domain of Expectations on the RESET framework and parent engagement in math activities is also consistent with prior research (e.g., Missall et al., 2015). Parents' high expectations are associated with their increased involvement in their children's learning (Walker et al., 2005). Additionally, the significant relationships between parents' perceptions of Time available to spend with their child is also not unexpected, as individuals tend to make time for those things they value or for which they have high expectations (Marjoribanks, 1976).

An unexpected finding is the result that parents with higher perceptions of their Role in the math development of their children are less likely to engage in Formal activities such as workbook or flashcard activities. This result, combined with the finding that higher perceptions of Role were also associated with better knowledge of appropriate early childhood math knowledge and skills, and also more strongly associated with parent knowledge of Advanced Number Skills specifically, may indicate that “high Role” parents may be more likely or able to recognize opportunities for informally exploring a variety of math experiences in their everyday activities rather than relying on formal math resources such as workbooks or flashcards. Parents who have less knowledge of which early math skills are appropriate for their child—but recognize that they should be supporting their child's math development in some way—may be more likely to use more formal (i.e., vetted, or trusted) resources to help them.

#### **Limitations and Areas for Future Study**

One of the study limitations is that the sample was limited to only English-speaking/reading parents with access to the Internet. Participants were also limited to those within the communications reach of the commercial panel provider. Furthermore, this study relies solely on parent self-reports of their perceptions along RESET, early math knowledge, and shared math activity in the home, which subject them to bias. Future studies should include different groups of parents *and* children and incorporate observations and measures of child math knowledge, allowing for more concrete connections to be made between factors that influence mathematical parenting and the impact of that parenting on child learning and achievement.

## Conclusion and Contributions

Though this study reports on early work with the RESET Framework and survey instrument, it represents a significant contribution to the study of the Home Math Environment – a field that has been limited by too few standardized, validated tools and methods. As better tools and methods are developed to study and increase our understanding of the Home Math Environment, so too increases the potential for better parent, family, and community engagement programs that will support children’s early development of math knowledge and understanding in the home.

## Acknowledgments

The authors of this paper would like to acknowledge [FUNDER] which funded portions of this study related to generation of the sample.

## References

- Betts, A. (2021). The RESET Framework: Examining critical factors in parent-child math participation. In the *IAFOR International Conference on Education–Hawaii 2021 Official Conference Proceedings*. The International Academic Forum (IAFOR).
- Betts, A., & Son, J-W. (2022). Why parents do what they do: Developing and validating a survey for the mathematical lives of parents and children. *The Paris Conference on Education 2022 Official Conference Proceedings*. Paper presented at the Paris Conference on Education. Paris, France: The International Academic Forum.
- Blevins-Knabe, B. (2016). Early mathematical development: How the home environment matters. In Blevins-Knabe & Berghout Austin, A.M. (Eds.) *Early childhood mathematics skill development in the home environment*, (pp. 7-28). Springer International Publishing.
- Bronfenbrenner, U. (1992). *Ecological systems theory*. Jessica Kingsley Publishers.
- Cankaya, O., & LeFevre, J.-A. (2016). The home numeracy environment: What do cross-cultural comparisons tell us about how to scaffold young children’s mathematical skills? In B. Blevins-Knabe & A.M. Berghout-Austin (Eds.) *Early childhood mathematics skill development in the home environment* (pp. 87-104). Springer International Publishing.
- Claessens, A., & Engel, M. (2013). How important is where you start? Early mathematics knowledge and later school success. *Teachers College Record*, 115(6), 1–29.
- Clements, D. H., & Sarama, J. (2014). *Learning and teaching early math: The learning trajectories approach*. Routledge.
- Daucourt, M. C., Napoli, A. R., Quinn, J. M., Wood, S. G., & Hart, S. A. (2021). The home math environment and math achievement: A meta-analysis. *Psychological bulletin*, 147(6), 565.
- de Brey, C., Musu, L., McFarland, J., Wilkinson-Flicker, S., Diliberti, M., Zhang, A., Branstetter, C., and Wang, X. (2019). *Status and trends in the education of racial and ethnic groups 2018 (NCES 2019-038)*. U.S. Department of Education. Washington, DC: National Center for Education Statistics. Retrieved from <https://nces.ed.gov/pubsearch/>
- Else-Quest, N. M., Hyde, J. S., & Linn, M. C. (2010). Cross-national patterns of gender differences in mathematics: a meta-analysis. *Psychological Bulletin*, 136(1), 103.
- Huang, C. (2013). Gender differences in academic self-efficacy: A meta-analysis. *European Journal of Psychology of Education*, 28, 1-35.
- Marjoribanks, K. (1976). Social learning theory of the family: An analysis. *Psychology in the Schools*, 13(4). 457-462.



- Missall, K., Hojnoski, H., Caskie, G.I.L., & Repasky, P. (2015). Home numeracy environments of preschoolers: Examining relations among mathematical activities, parent mathematical beliefs, and early mathematics skills. *Early Education and Development, 26*(3), 356-376.
- Mowder, B. A. (2005). Parent development theory: Understanding parents, parenting perceptions and parenting behaviors. *Journal of Early Childhood and Infant Psychology, 1*, 45-65.
- Napoli, A. R., & Purpura, D. J. (2018). The home literacy and numeracy environment in preschool: Cross-domain relations of parent–child practices and child outcomes. *Journal of Experimental Child Psychology, 166*, 581-603.
- Nguyen, T., Watts, T.W., Duncan, G.J., Clements, D.H., Sarama, J.S., Wolfe, C., & Spitler, M.E. (2016). Which preschool mathematics competencies are most predictive of fifth grade achievement? *Early Childhood Research Quarterly, 36*, 550-560.
- Perez-Felkner, L., Nix, S., & Thomas, K. (2017). Gendered pathways: How mathematics ability beliefs shape secondary and postsecondary course and degree field choices. *Frontiers in psychology, 386*.
- Ritchie, S. J., & Bates, T. C. (2013). Enduring links from childhood mathematics and reading achievement to adult socioeconomic status. *Psychological Science, 24*(7), 1301-1308.
- Schweinhart, L., Montie, J., Xiang, Z., Barnett, W., Belfield, C., & Nores, M. (2005). *Lifetime Effects: The HighScope Perry Preschool Study Through Age 40*. In Monographs of the HighScope Educational Research Foundation, 14. HighScope Press.
- Vygotsky, L. (1986). *Thought and language: Newly revised and edited*. A. Kozulin (Ed). Cambridge, MA: The Massachusetts Institute of Technology.
- Walker, J.M.T., Wilkins, A.S., Dillaire, J.R., Sandler, H.M., Hoover-Dempsey, K.V. (2005). Parental involvement: Model revision through scale development. *The Elementary School Journal, 106*(2), 85-104

## LIBERATING AGENCY AND TRANSFORMING COMPETENCE THROUGH MATHEMATICAL PLAY

Melissa Gresalfi  
Vanderbilt University  
melissa.gresalfi@vanderbilt.edu

Madison Knows  
Vanderbilt University  
Madison.knowe@vanderbilt.edu

Jamie Vescio  
Vanderbilt University  
Jamie.vescio@vanderbilt.edu

*Many scholars have argued that mathematics classrooms often offer narrow conceptions of mathematical excellence, recognizing only some kinds of thinking and some kinds of people as valuable, and conflating mathematical aptitude with overall intelligence. Play offers the potential to disrupt such classroom mathematical practices, by offering new and broader ways to exercise agency, and, relatedly, more expansive visions of who is seen as mathematically capable. Offering an in-depth analysis of the participation of two students as they engage in mathematics in their Kindergarten class, whole group rug time and small group play centers, we investigate how different activity structures create space for students to exercise agency in ways that demonstrate multiple forms of competence, creating liberating mathematical spaces.*

Keywords: Social Justice, Early Childhood Education, Equity, Inclusion, and Diversity

### Objectives

Many scholars have argued that mathematics classrooms often offer narrow conceptions of mathematical excellence (Ladson-Billings, 1998), recognizing only some kinds of thinking and some kinds of people as valuable (Joseph et al., 2017), and conflating mathematical aptitude with overall intelligence (Martin, 2019). Taken as a whole, such practices serve to establish mathematics classrooms as spaces for white supremacy. This is in part because such classrooms are particularly problematic for students of color who are more likely to have less-qualified teachers (DeMonte & Hanna, 2014), to be removed from school by suspension (Gregory & Roberts, 2017), and to face stereotyped low expectations for their mathematical success (McGee & Martin, 2011; Nasir & Shah, 2011). Play offers the potential to disrupt such classroom mathematical practices, by offering new and broader ways to exercise agency, and, relatedly, more expansive visions of who is seen as mathematically capable. This is in comparison to other more traditional math class activities like whole group instruction which by nature require compliance and thus limit students' agency, choice, and interest, decreasing the multiplicity of ways to demonstrate mathematical competence. Playful mathematics learning allows for students to make choices about their own problem solving, to test and critique their own reasoning and the reasoning of others, and to seek out their own mathematical interests and inquiries.

Consequently, mathematical play has the potential to resist problematic and often racist perpetuations of normalizing homogeneous ways of doing mathematics.

This study seeks to investigate how different types of activity structures create space for students to exercise agency in ways that demonstrate multiple forms of competence, creating liberating mathematical spaces. We offer an in-depth analysis of the participation of two students, Tiana and Quentin, as they engage in mathematics in their Kindergarten class in two contexts, whole group rug time and small group play centers. On the rug, opportunities to exercise agency are relatively limited students can generally either comply or resist. In contrast, small group play centers are designed to allow for mathematical exploration, collaboration,

decision making, and problem solving, thus providing more diverse ways for students to exercise agency. In the more constrained space of the whole group rug activity, Tiana and Quentin typically exercise agency in different ways (one generally complies, and the other often resists). We follow these two students to the play stations to explore the following question: How do playful mathematical learning spaces serve to create new opportunities for students to explore mathematical ideas and be seen as mathematically competent?

### **Theoretical framework**

To make sense of student engagement and the potential of mathematical play to disrupt ingrained classroom practices, we leverage the ideas of *agency* and *competence*, which simultaneously highlight both the individual and collective nature of classroom participation (citations). Agency refers simply to action; students can exercise agency in different ways depending on the organization of the classroom space. In some tightly-controlled classrooms, students can exercise agency primarily by rejecting or complying (Golann, 2018). In such spaces, only a very narrow range of behaviors might be endorsed as competent. Alternatively, other classrooms might create multiple ways for students to exercise agency that are seen to be competent; such classrooms often invite students to share their own ideas or debate ideas with others, rather than practicing or reproducing a set of procedures (Boaler & Staples, 2008). Such classroom organization is not solely under the control of teachers; Yoon and Templeton (2019) note that student agency is often limited in schools, particularly when classroom teachers face pressure to meet curricular or assessment demands. Such climates stifle opportunities for students to participate in ways that are meaningful to them, in exchange for the prioritization of adults' demands, such as standards or assessment goals.

Conversely, Yoon and Templeton (2019) point to “flexible spaces” (p. 58), such as free play, as promising contexts in which students can insert their own voices into curricula or classrooms. Although there is variation amongst the definitions of play, most involve characterizations of spontaneity, interest, choice, and pleasure (Brown, 2009; Burghardt, 2010), thus always, by definition, inviting agency. Although often play in classroom settings is seen as separate from learning and a reward for good behavior, play also offers a context for learning (Parks, 2020; Wager & Parks, 2014), and it is this use of play that is our focus here. Many scholars have highlighted how play allows one to think beyond oneself, to test and explore the limits of ideas (Gadamer, 1975; Vygotsky, 1978), thus opening new spaces to exercise agency and explore nuances of mathematical ideas. In such a context, a wider range of behaviors would be seen as competent—not simply engaging a task as directed, but exploring the boundaries of the task, working with others on the task, and inventing modifications of the task. All these actions are central to play, and thus are likely to be recognized as versions of competent behavior. In the analysis that follows, we examine interactions to explore the ways that students can exercise agency, and how their behavior is recognized and marked as more or less competent.

### **Methods**

This analysis focuses on the contrasting cases (Flyvbjerg, 2006; Stake, 1995) of Quentin and Tiana, two kindergarten students in Ms. Lane's class. We chose these cases for close analysis as we seek to offer in-depth accounts of the interrelation between structure and agency, between individual action and collective recognition. Case study lends itself to such an analysis as we can look in depth across interactions and over time. We have chosen contrasting cases that help

illuminate our question, specifically to explore the ways that different activity structures serve to create new opportunities for students to explore mathematical ideas and be seen as competent.

These cases were identified as part of a larger study of integrating mathematical play into kindergarten classrooms. The study site, a public charter school in the southeastern United States, serves a community of students that is both racially and socio-economically diverse. Approximately half of the student population is White (48%) and half is African American (42%), with 55% of students qualifying for free and reduced lunch. The larger study takes place in partnership with six kindergarten teachers in three classrooms at the school; the data for this paper come from one of the three classrooms. As part of this study, each classroom is observed twice per week, with the observer taking in-depth ethnographic field notes to document student activity, classroom participation structure, teacher questioning, and students' affective experiences. The two focal students for this paper, Quentin and Tiana, were identified early on as contrasting cases through these observations and discussions.

Quentin is a black boy who joined Ms. Lane's class about six weeks into the semester, following some disruption with hiring and classroom reorganization in the entire kindergarten cohort. Quentin had already established a reputation as a child who struggled during whole-group time; in the first classroom he was assigned to he typically was asked to leave the collective mathematical time (that usually took place at the rug) and sit in a chair separate from the group as an effort to calm himself and not distract his peers. He was moved to Ms. Lane's class because of his disruptive behavior. Upon moving to Ms. Lane's classroom, Quentin was able to participate more regularly in whole class instruction, responding well to Ms. Lane's more structured and consistent classroom management style. However, Quentin continued to stand out as a child who needed frequent reminders about where to look and how to control his body.

Tiana is a black girl who was enrolled in Ms. Lane's class from the start of the school year. Tiana is attentive during whole group time, frequently raising her hand, almost always following directions (indeed, anticipating directions), and responding correctly to questions when invited to share. Tiana was often a leader in the class, at times reminding other students about what they should be doing or how they should be behaving during different classroom routines.

After identifying Tiana and Quentin as focal cases, we reviewed video of the students during whole-class mathematical time (almost always on the rug), and at the play centers. Using methods of interaction analysis (Jordan & Henderson, 1995), we examined the videos attending to when and how the students exercised agency, what evidence we had of the ways those behaviors were recognized and interpreted, and ultimately, when the students were seen as being competent either by an adult or peers.

### **Findings**

We begin with an overview of the typical ways the two students exercised agency during whole-class work time, and the resulting ways their behaviors were typically recognized as more or less competent. It is important to note that there was variation across these videos; at times Quentin was recognized as competent for his contributions and Tiana was reminded of the expectations for behavior. Thus, this overview is a representation of typical behaviors rather than a comprehensive account of every interaction. We then follow with examples of each child's interaction in one mathematical play center, offering those examples as contrasts with the typical engagement that was observed during whole class time.

## Whole-Class Instruction

Tiana and Quentin typically experienced mathematical activities in this classroom in very different ways, which was most evident in the whole group activities that took place on the rug at the front of the room. In this environment, each child was assigned to sit on their own square.

The class had established a significant number of procedures about how students should position their bodies, talk in groups, distribute and use materials, respond to questions, and transition between activities. In this context, it was readily apparent who was complying with expectations, and who was resisting or rejecting them. Tiana thrived in this environment, consistently remembering and following these rules and procedures and often instructing others to do the same. Overall, Tiana typically exercised agency in ways that were compliant with the expectation, and was seen as mathematically competent for doing so, either through not being called out for correction, or at times being complimented directly for her behavior. In contrast, Quentin struggled in these constrained conditions and was often dismissed from his spot on the carpet to the “safe space,” a chair that sits up against a wall away from the carpet, for not following these rules or procedures. While sitting in this safe space he was often redirected to continue to engage in the whole group mathematics activities, but was physically separated from his classmates. Such removal and public corrections served to mark Quentin as not behaving competently in the space. Although this participation structure led to different outcomes for different students—Tiana remains on the carpet while Quentin is removed—it reinforced a system of mathematical competence for all students that favored particular forms of behavior over others. Consequently, agency was ultimately limited for both students.

On the other hand, when Tiana and Quentin transitioned to the small-group play stations, they engaged with open-ended tasks that allowed for multiple forms of participation. Although each play station differed in its goals and materials, all play stations created expectations of and opportunities for exploration and imagination. As a result, Tiana and Quentin had more opportunities to exercise agency and to be seen as mathematically competent. Below, we highlight Tiana’s experience from a video from September at the pattern block station, and a video from November of Quentin’s time at the “zoo” station—highlighting the ways in which both learners engaged with mathematics and ultimately asserted their agency throughout each episode.

### Tiana’s Participation in a Play Space

Tiana’s participation in the play space resembled her participation in the whole class episode, in that she was comfortable reminding others of rules and expectations. However, in this space Tiana was also likely to propose her own ideas and invent new games, thus exercising agency in ways that were not simply about compliance. In this episode, Tiana was playing with three other students at the pattern block station: Will, Nicholas, and Nora. Tiana’s playfulness led to the invention of a song and game that all four students contributed to in the same way. For example, when Will stated that he needed a square for his puzzle, Tiana sang, “*Squares, squares, squares! Squares are everywhere!*” Tiana continued this same melody for the entirety of the station, but for different shapes. She even assisted Will, Nicholas, and Nora with shape names so that they could continue to play along. For example, in turn 10 Nicholas asked Tiana to identify a shape.

After Tiana stated that it was a trapezoid, Nicholas returned to the game and said, “*I need a trapsa-toid!*” so that Tiana could sing the song: “*Trapezoids, trapezoids, trapezoids! Trapezoids are everywhere!*” Tiana also took several opportunities to reshape the rules of the game. For instance, she eventually declared herself the “singing robot” and that the other students could only take a pattern block from the bin after Tiana finished singing about their shape. Later, Tiana

changed the rules again so that she would now be the one to hand over the shapes after singing. Tiana's final alteration of the rules occurred when she played with her own leadership role by asking Nora to sing so that she could be the one to ask for shapes.

- 1 Tiana: Only if one of y'all need a square or a different type of shape. I'll sing the shape song.
- 2 Nicholas: I made a fish! *Clears fish puzzle and switches to giraffe puzzle.*
- 3 Nora: I need a triangle. *Grabs triangle from bin.*
- 4 Tiana: Triangles, triangles, triangles! Triangles are everywhere! *Sings.*
- 5 Will: I need a hexagon! *Looks at Tiana and smiles. Grabs hexagon from bin.*
- 6 Tiana: Hexagons, hexagons, hexagons! Hexagons are everywhere. *Sings.*
- 7 Nora: *Smiles and laughs.*
- 8 Will: I need a triangle! *Looks at Tiana and smiles. Grabs triangle from bin.*
- 9 Tiana: Triangles, triangles, triangles! Triangles are everywhere! *Sings.* I'm the singing robot. There's a voice box in me. There's a radio in my mouth.
- 10 Will: What is this called? *Grabs trapezoid from bin and looks at Tiana.*
- 11 Nora: Um...hexagon.
- 12 Tiana: Trapezoid.
- 13 Will: I need a trapsa-toid!
- 14 Tiana: Trapezoid, trapezoid, trapezoid! Trapezoids are everywhere! *Sings.*
- 15 Will: I need a diamond!
- 16 Tiana: Diamonds, diamonds, diamonds! Diamonds are everywhere! *Sings and Hands Will a blue diamond.*
- 17 Will: No, not that diamond, the other diamond. *Grabs white diamond from bin.* Oh, here, here, here!
- 18 Tiana: That's not a diamond. Um that's a rhombus.
- 19 Nicholas: Rhombus, rhombus, rhombus! Rhombuses are everywhere! *Sings.*
- 20 Tiana: When I sing it means you can get it.

Although Tiana was the leader of the game, a role that is typical for her participation in general, her song created a space in which all students could play and engage with mathematics together. This is particularly noteworthy given that pattern block puzzles are not an inherently collaborative task. However, the ill-structured nature of the activity opened opportunities for Tiana to exercise agency in ways that were creative and ultimately extended the original goal of the play station. Furthermore, Tiana's continual shifting of the rules throughout the song and game (turns 1 and 20) were in contrast to her typical compliant nature in mathematical spaces. As Tiana continued to mold and reshape the direction of the game, she played with her own role and position in relation to the other students. Although she was on task and engaging with mathematics throughout the entirety of her time at the play station, her song was a form of personalization and ultimately agency that went against her usual tendency to stick within the bounds of the teacher's exact instructions. What's more, these different ways of exercising agency were collectively recognized as competent, seen in her peers' desire to play the game with her, and in students' requests for shape names (for example, Will's request in turn 10).

#### **Quentin's Participation in a Play Space**

For Quentin, a shift to a playful center created opportunities for him to exercise agency in several ways, all of which kept him in the activity and supported his mathematical thinking.

While playing at a “zoo” center, which involved placing animals in pens to match their capacity (indicated with a number written on a post-it note in the corner of each pen), Quentin appeared to prioritize mutual play, and made many bids to the student across the table, King, to play together. Although King did not take up these bids, he also did not reject them, and the two boys continued to place animals on the same small square without argument, resulting in a kind of parallel play. This form of collegial, conflict-play was completely in-line with the expectations of the classroom; evidence that Quentin (and King’s) play was viewed as competent can be seen when Ms. Lane approached the group, observing them and saying only: “*After you finish, you have to make sure that you count, to make sure that the amount of animals matches this number (tapping on the 13).*” This reminder galvanized King who immediately started to count the animals the boys had placed. At first Quentin simply observed, but after animals began to fall over upon being touched, Quentin made a new bid to play a new game, saying: “*We can knock all of them down, and count up. I’m gonna count while I put ‘em on there.*” What followed was

a cooperative counting task, which was initiated and sustained by Quentin; although King was happy to count, he did not initially coordinate with Quentin to do so.

- 1      Quentin:    So. Onnnneeeee *places an animal.*
- 2      King:        *Places an animal One.*
- 3      Quentin:    No. That hasta be two. *Touches the animal J just put down. So onnnn/eeee, Touching the tiger*
- 4      King:        *touching the animal he put down /onnnneeee (reaches into the bin for another animal)*
- 5      Quentin:    *Touches the animal J just put down Two*
- 6      King:        No, No. *grabs the tiger Patrick had placed, and then picks up all animals on the board, placing each down as he counts one, two*
- 7      Quentin:    *Placing down a third animal Three*
- 8      King:        *Puts down the third animal that was in his hand three*
- 9      Quentin:    *Puts down another animal four*
- 10     King:        *Points at each animal as he counts one two three four five*
- 11     Quentin:    *Places a sixth animal six*
- 12     King:        *Places a seventh animal seven*
- 13     King:        *Both boys put down an animal at the same time*
- 14     Quentin:    Eigggghh/hhtttt
- 15     King:        /Eight
- 16     King:        Nine *placing an animal*
- 17     Quentin:    Ten *placing an animal*
- 18     King:        Both *place another animal*
- 19     Quentin:    Eleven, *places another animal, twelve, places another animal thirteen. There.*
- 20     King:        No, that’s thirteen. There. We’re done! *Tidying the animals. There are 15 animals on the page.*

In this exchange Quentin was again able to exercise agency in ways that appear to be satisfying to him; he made several bids for cooperative play (turns 1 and 3), and then coordinated

his counting with King, seen in turn 11 when, for the first time, his count came after Quentin's last count (instead of his own last count). Here his competence is widely apparent, as he counts to the number that is indicated on the chart, noting his completion with a satisfied "There." Evidence that King viewed this work as mutual and competent can be seen in turn 20, when he said "We're done." Overall, this play space created opportunities for Quentin to exercise agency in ways that were satisfying to him while also allowing him to be seen as competent both by the teacher and by other students. In so doing, he created new opportunities to engage with mathematical ideas in ways that were personally satisfying and creative, fulfilling his own preference for collective and collaborative play.

### **Discussion and Significance**

These brief excerpts offer examples of how these mathematical play stations served to offer new and different forms of participation, inviting students to exercise agency differently and creating space for a wider range of behaviors to be seen as competent. Both Tiana and Quentin were seen as competent in the mathematical play spaces, a shift from how they were generally recognized during more traditional math instruction. Looking across instances, it appears that the more permissive norms around the play stations—anticipating that students will find their own path through the task and are allowed to explore and imagine—was an important aspect of this disruption. What's more, the open-ended nature of the tasks themselves that were designed to invite more than one pathway, additionally served to offer more diverse opportunities for students to exercise mathematical agency (Esmonde, 2009).

Broadening notions of competence is important in any mathematics classroom, particularly when such broadened notions create spaces for new and different ways of engaging with mathematics. In addition, such broadened notions influence students' own perceptions of themselves and their mathematical capabilities. In this context, we wonder whether and how play offers a potential inroad to the question posed by Martin: "What can and should refusal of dehumanizing and violent mathematics education look like in principle and practice?" (2019, p. 461). As Joseph et al., (2019), wrote: "A second area for further research is the idea of humanizing Black girls by creating a space for play—a place for them to be happy, gregarious, social, and "goofy" .... Our emerging analysis illuminates that when the Black girls in this study were afforded the opportunity to be both serious and silly, they were more engaged in mathematics learning" (pp. 149). The examples shared here offer an index of Yoon and Templeton's (2019) notion of "flexible spaces" (p. 58), which allow students to insert their own voices into curricula. At times, student voice may seem irrelevant, like Tiana's song; however, these moments of agency can be conceptualized as "acts of resistance" (p. 80) in an adult-regulated world. When classroom structures enable students to assert their agency, adults send the message that students' "dispositions, discourses, and actions" (p. 80) are in fact valuable forms of knowledge.

However, before making the fundamental error of romanticizing children's activity, we note that play is not a panacea—it characterizes a form of human interaction that is fraught with the same perils, biases, and oppression that characterizes all interactions (Bryan, 2020). We know that children's play can be racist, mean, gendered, and exclusionary. Although it has potential to liberate students from the conventions of the discipline, it by no means transcends the everyday structures that dictate our interactions. Therefore, even as we look for sites of potential disruption, we must stay vigilant at noticing the ways it might fail, lest we contribute to the continued reproduction of mathematics reform that ultimately serves to change nothing for students who are already being oppressed by its structures (Martin, 2019).



## References

- Boaler, J., & Staples, M. (2008). Creating mathematical futures through an equitable teaching approach: The case of Railside school. *Teachers College Record*, 110, 608-645.
- Brown, S. L. (2009). *Play: How it shapes the brain, opens the imagination, and invigorates the soul*. Penguin.
- Burghardt, G. M. (2010). Defining and Recognizing Play. In P. Nathan & A. D. Pellegrini (Eds.), *The Oxford Handbook of the Development of Play* (pp. 0). Oxford University Press.  
<https://doi.org/10.1093/oxfordhb/9780195393002.013.0002>
- DeMonte, J., & Hanna, R. (2014). Looking at the Best Teachers and Who They Teach: Poor Students and Students of Color Are Less Likely to Get Highly Effective Teaching. *Center for American Progress*.
- Esmonde, I. (2009). Ideas and identities: Supporting equity in cooperative mathematics learning. *Review of educational research*, 79(2), 1008-1043.
- Flyvbjerg, B. (2006). Five misunderstandings about case-study research. *Qualitative inquiry*, 12(2), 219-245.
- Gadamer, H.-G. (1975). *Truth and method*. Seabury Press.
- Golann, J. W. (2018). Conformers, adaptors, imitators, and rejecters: How no-excuses teachers' cultural toolkits shape their responses to control. *Sociology of Education*, 91(1), 28-45.
- Gregory, A., & Roberts, G. (2017). Teacher beliefs and the overrepresentation of Black students in classroom discipline. *Theory into Practice*, 56(3), 187-194.
- Jordan, B., & Henderson, A. (1995). Interaction analysis: Foundations and practice. *The Journal of the Learning Sciences*, 4(1), 39-103.
- Joseph, N. M., Hailu, M., & Boston, D. (2017). Black Women's and Girls' Persistence in the P-20 Mathematics Pipeline: Two Decades of Children, Youth, and Adult Education Research. *Review of Research in Education*, 41(1), 203-227.
- Ladson-Billings, G. (1998). It doesn't add up: African American students' mathematics achievement. *Journal for Research in Mathematics Education*, 28, 697-708.
- Martin, D. B. (2019). Equity, inclusion, and antiblackness in mathematics education. *Race Ethnicity and Education*, 22(4), 459-478.
- McGee, E. O., & Martin, D. B. (2011). "You would not believe what I have to go through to prove my intellectual value!" Stereotype management among academically successful Black mathematics and engineering students. *American Educational Research Journal*, 48(6), 1347-1389.
- Nasir, N. i. S., & Shah, N. (2011). On defense: African American males making sense of racialized narratives in mathematics education. *Journal of African American Males in Education*, 2(1), 24-45.
- Parks, A. N. (2020). Creating Joy in PK-Grade 2 Mathematics Classrooms. *Mathematics Teacher: Learning and Teaching PK-12*, 113(1), 61-64.
- Stake, R. E. (1995). *The Art of Case Study Research*. Sage.
- Vygotsky, L. S. (1978). *Mind and society: The development of higher psychological processes*. Harvard University Press.
- Wager, A., & Parks, A. N. (2014). Learning mathematics through play. *Handbook of play and learning in early childhood*, 216-227.
- Yoon, H. S., & Templeton, T. N. (2019). The practice of listening to children: The challenges of hearing children out in an adult-regulated world. *Harvard Educational Review*, 89(1), 55-84.

## HOW SCHOOL PRINCIPALS SUPPORT TEACHERS TO TEACH MATHEMATICS TO DIVERSE LEARNERS

Kellie J. Pop  
University of Nevada, Reno  
kpop@nevada.unr.edu

Lynda R. Wiest  
University of Nevada, Reno  
wiest@unr.edu

*The purpose of this study was to investigate how school principals support culturally sustaining mathematics education. Interviews were conducted with 10 practicing K-12 public school principals in the Western US who report that they implement culturally sustaining leadership and encourage culturally sustaining pedagogy in mathematics teaching in their schools.*

Keywords: culturally relevant pedagogy; instructional leadership; equity, inclusion, and diversity

**School principals are vitally important to fostering equitable learning environments (Ainscow, 2020; Blaik-Hourani & Litz, 2019; Corcoran, 2017; Reid, 2021). With increased awareness of sociopolitical issues in education (Alim et al., 2020; Caraballo et al., 2020; Paris, 2012), it is important to investigate the perspectives of school principals who practice culturally sustaining leadership for their mathematics teachers (Hattori, 2016; Holmes & Young, 2018).**

### Methods

This study investigated how school principals guide their teachers to teach mathematics in a culturally sustaining manner. Data were gathered through semi-structured interviews conducted via Zoom. Participants were recruited using social media sites, regional union newsletters, email listservs, and respondent-driven sampling or word-of-mouth (von der Fehr et al., 2018). Data were analyzed using a non-linear coding system (Fifty et al., 2022; Williams & Moser, 2019).

### Results and Findings

The ten principals in this study stated that they rely on state-issued evaluation frameworks, with a few exceptions. One used a modified state evaluation framework aligned with the Danielson Group Framework, as required by the school district, and two reported that they must use an additional evaluation rubric as an International Baccalaureate program. Two other principals had heard of the IMET and Equip mathematics evaluation rubrics while attending leadership courses in their master's degree programs, but they had not used the rubrics. All noted that the state and district issued evaluation frameworks were not sufficient in providing criteria and indicators that specifically focus on culturally sustaining pedagogical practices, even though that is a criterion on their own evaluations as school principals.

Five principals financially support teachers to attend professional development, and they provide substitute teachers for these days. Two hold professional learning committees daily after school to review student achievement data. These principals also incorporate common teacher preparation times to ensure teachers are teaching the same content. One school principal candidly expressed their desire to do more to preserve and sustain cultures but they “do not have the time to unpack the concept and really take it apart to see how to make it work” in their unique school setting.

### References

- Ainscow, M. (2020). Promoting inclusion and equity in education: Lessons from international experiences. *Nordic Journal of Studies in Educational Policy*, 6(1), 7–16. <https://doi.org/10.1080/20020317.2020.1729587>
- Alim, S., Paris, D., & Wong, C. (2020). Culturally sustaining pedagogy: A critical framework for centering communities. In *Handbook of the Cultural Foundations of Learning* (pp. 261–276). Routledge.
- Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

- Blaik-Hourani, R., & Litz, D. R. (2019). Aligning professional development, school self-evaluation and principals' performance standards: A UAE case study. *School Leadership & Management*, 39(2), 222–249. <https://doi.org/10.1080/13632434.2018.1479848>
- Caraballo, L., Martinez, D. C., Paris, D., & Alim, H. S. (2020). Culturally sustaining pedagogies in the current moment: A conversation with Django Paris and H. Samy Alim. *Journal of Adolescent & Adult Literacy*, 63(6), 697–701. <https://doi.org/10.1002/jaal.1059>
- Corcoran, R. P. (2017). Preparing principals to improve student achievement. *Child & Youth Care Forum*, 46(5), 769–781. <https://doi.org/10.1007/s10566-017-9399-9>
- Fifty, D., Beisiegel, M., & Zimmerman, S. (2022, November 17). Investigating mathematics department leaders' experiences and understanding of equity. *Proceedings of the Forty-Fourth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Critical Dissonance and Resonant Harmony, Nashville TN. <http://www.pmena.org/pmenaproceedings/>
- Hattori, M. (2016). Culturally Sustaining Leadership: A Pacific Islander's Perspective. *Education Sciences*, 6(1), 4. <https://doi.org/10.3390/educsci6010004>
- Holmes, W. T., & Young, S. (2018). Culturally sustaining instructional leadership: Perspectives from Native American public school principals in Montana and Wyoming. *Journal of Educational Leadership, Policy and Practice*, 33(2), 17–31. <https://doi.org/10.21307/jelpp-2018-010>
- Paris, D. (2012). Culturally sustaining pedagogy: A needed change in stance, terminology, and practice. *Educational Researcher*, 41(3), 93–97. <https://doi.org/DOI:10.3102/0013189X12441244>
- Reid, D. B. (2021). US principals' sensemaking of the future roles and responsibilities of school principals. *Educational Management Administration & Leadership*, 49(2), 251–267. <https://doi.org/10.1177/1741143219896072>
- Rossman, G. B., & Rallis, S. F. (2017). *An Introduction to Qualitative Research: Learning in the Field* (4th ed.). Sage.
- von der Fehr, A., Solberg, J., & Bruun, J. (2018). Validation of networks derived from snowball sampling of municipal science education actors. *International Journal of Research & Method in Education*, 41(1), 38–52. <https://doi.org/10.1080/1743727X.2016.1192117>
- Williams, M., & Moser, T. (2019). The art of coding and thematic exploration in qualitative research. *International Management Review*, 15(1), 45–56.

## MATHEMATICS TEACHER QUESTIONING: COGNITIVE AND AFFECTIVE DOMAINS TO NURTURE STUDENTS' MATHEMATICAL IDENTITY FORMATION

Ana-Maria Haiduc  
Purdue University  
ahaiduc@purdue.edu

Elizabeth Suazo-Flores  
University of North Dakota  
elizabeth.suazo@und.edu

Signe E. Kastberg  
Purdue University  
skastber@purdue.edu

Rachael Kenney  
Purdue University  
rhkenney@purdue.edu

Sarah E. Leach  
Purdue University  
leachs@purdue.edu

Andrea F. Barber-Dansby  
Purdue University  
abarberd@purdue.edu

*Mathematics teacher questioning plays a significant role in students' learning. Research has persisted in analyzing teacher questioning from a cognitive perspective. Considering teaching as a relational practice, we explored teacher questioning from cognitive and affective perspectives. Data comprises 14 video-audio recordings, field notes, teacher's reflections, and pictures of a mathematics high school teacher while he taught linear functions. We analyzed the teacher's questioning using Bloom's cognitive and affective domains. We then characterized the type of questions concerning students' mathematical identity formation. We found that the teacher questioning techniques consisted of shifts between cognitive levels domains and added affective connotations with influence on students' mathematical identity. Using questions to promote students' cognition and mathematical identity formation is a form of caring teaching practice.*

Keywords: teachers' questioning, caring teaching, students' identity, cognition, affect.

Teacher questioning during mathematics classwork plays a significant role in guiding students' thinking and gathering information about their ways of knowing mathematics. This study aims to reveal one high school mathematics teacher's questioning, the intentions of his questions, and ways questioning was used to support students' mathematical identity formation. We aimed to answer the question: How does teacher questioning offer insights about teaching practices that promote students' mathematical identity formation?

### Theoretical Framework

Students' mathematical identity is shaped by their stories of learning mathematics (Sfard & Prusak, 2005a, 2005b). These stories include narratives about students' relationships with teachers (Noddings, 1994, 2013, 2017) and mathematics (Ingram, 2015), stories about who they are (actual identities), and what they can become (designated identities) (Sfard & Prusak, 2005a, 2005b), and stories about how their peers perceive them as mathematics doers (Ingram, 2015). When students construct knowledge, their stories change (Sfard & Prusak, 2005a, 2005b; Sfard, 2008). Thus, identities are dynamic. Sfard and Prusak (2005a) draw the concept of identity from human communication, asserting that "learning to think mathematically is tantamount to being initiated into a special form of discourse, known as mathematical." (p. 41 - 42). One form of communication in schools is in the form of questions and answers. Dillon (1981) pointed out that "the teacher typically speaks in questions. Students speak in answer" (p. 51). Teacher questioning maintains the classroom discourse and represents perhaps a central part of students' mathematical identity formation. That is because teachers get to know their students through questioning, which also informs their design of subsequent lessons. In addition, questioning allows students to share their ideas and be exposed to their peers' ideas.

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

In care theory, teaching is a relational practice (Noddings, 2013). Teachers who get to know students as learners and individuals become carers. Teachers as carers are receptive listeners and respond to the students' needs. Students' recognition of their care completes a cycle that defines caring relations (Noddings, 2012). In a caring relationship, "teacher [the one caring] and student [the cared for] become partners in fostering the student's growth" (Noddings, 2017, p. 224). Developing caring relationships facilitates mathematics learning. Knowing the students as learners means understanding their mathematical cognitive processes. Knowing the students as individuals means having a "sympathetic understanding of individuals as individuals which gives [the teacher] an idea of what is actually going on in the minds of those who are learning" (Dewey, 1988, p. 33). In learning mathematics, students use cognitive functions and experience feelings that correspond to the affective domain: anxiety, arousal, attitude, attributions, beliefs and opinions, confidence, the expectancy of success, interests, motivational level, motives, perceived relevance, satisfaction, self-efficacy, and values (Bohlin, 1998).

A caring teacher gives voice to all students regardless of their proficiency, so they feel their contribution is valued (Davis, 1997; Louie, 2017). Teachers who evaluate students' various abilities inform students about the skills they might have. Boaler (2010) called this process *assigning competence*, and it takes place in a classroom where teachers value many dimensions of mathematical work. Boaler (2010) observed that to be able to value multiple skills, teachers adapted the "problems from different curriculum to make them group-worthy" (p. 42). Such problems were "open-ended problems that illustrated important mathematical concepts, allowed for multiple representations, and had several possible solution paths" (Boaler, 2010, p. 42). In addition, these problems facilitated discussions, engaging students in collaboration and sharing ideas. The teacher praised students' ideas publicly, raising the students' statuses which helped to build their mathematical identity.

Additional studies center on teacher questioning as a relational practice that frames students' mathematical ability (Battey, 2012). Battey showed that teachers' questioning and teacher-student micro-interaction facilitate students' engagement in mathematical work. Battey identified four relational practices through which teachers mediate students' access to mathematics: "addressing behavior, framing mathematics ability, acknowledging student contributions, and attending to culture and language" (p.125). Dillon (1981) explained that "discussion is a process of engaging students in some activity other than transmitting or exchanging knowledge of a factual or conventional type. The material for discussion is not factual knowledge itself, but its applications, implications, interpretations, and the like." (p. 53). In addition, Dillon (1981) highlighted that "affective processes are involved either directly or by implication, as when material bears on attitudes, beliefs, opinions, personal experiences and the like." (p. 53-54). These two statements potentially illustrate that cognitive and affective processes evolve simultaneously during relational interactions.

## **Literature Review**

### **Ways of Categorizing Teachers' Questions**

Bloom et al.'s (1956) cognitive domain taxonomy provides a hierarchical classification of students' cognitive processes when learning consisting of knowledge, comprehension, application, analysis, synthesis, and evaluation of learners' cognition. These levels serve as a framework for instructors "to construct and revise assessments so that they are consistent with what has been taught" (Kastberg, 2003, p.1). Besides written assessments, teachers assess students' learning in the classroom through questioning. Bloom et al.'s (1956) cognitive domain taxonomy served many mathematics researchers as a tool to analyze the type and pattern of the

questions. A common classification of teacher questioning focuses on students' cognition and uses two phases, such as lower-order and higher-order (e.g., DeJarnette et al., 2020; Drageset, 2014; Kaya & Ceviz, 2017). Lower-order questions elicit learners' memorized information, and higher-order questions elicit students' critical thinking. Studying teachers' fluctuations and adjustments of questions across the hierarchy of cognitive levels, Nathan and Kim (2009) described four levels of elicitation in teacher questioning: choice (requiring a yes or no decision), product (request for factual knowledge), process (probe for explanations and interpretations), and meta-process (requires a justification for a response earlier provided). *Choice* and *product* are considered lower-order questions, and *process* and *meta-process* are considered higher-order questions. Other types of teacher questioning include: "correcting questions (redirecting), closed progress detail questions to clarify steps of a process or procedure (progressing), to open progress questions with more than one possible answer (focusing)" (DeJarnette et al., 2020, p. 4). In addition, they mention elaboration questions and open-ended questions.

### **The Relationship Between Cognition and Affect**

The affective domain describes ways someone reacts to other people's emotions. Bloom's et al. (1956) original taxonomy of the affective domain was revised by Krathwohl et al., (1964) to comprise: receiving the phenomenon, responding to the phenomenon, valuing, organizing, and internalizing values. Research on the relationship between affect and cognition in mathematics education has focused mostly on anxiety and attitude toward mathematics, concluding with negative results such as "anxiety inhibits cognitive processes, e.g., recall of prior learning, reducing performance" (Zan et al., 2006, p. 113). Zan et al. (2006) stated that "affect has generally been seen as 'other' than mathematical thinking, as just not part of it." (p. 113). That is because there is a belief that reasoning must overcome emotions. However, other research shows that when instruction is based solely on the cognitive domain, students struggle to value the content they learn (Bolin et al., 2005), asking the teacher, " 'Why do we have to learn this?' 'When will we ever need this?' and 'Will this information be on the test?' " (p. 154). A reconciliation between affective and cognitive domains is emphasized by Leder (2005), who pointed out a series of affective factors which pair up with the cognitive domain enabling the understanding of teachers' instructional practices. Students need the teacher's affective support to increase their motivation and interest. Affective support helps students develop a positive attitude and behavior that shapes their mathematical identity formation.

### **Methods**

Data was comprised of 14 video-audio recordings, field notes, teacher's reflections, and pictures of slides and student work from a mathematics high school teacher while he taught the concept of linear functions. The study took place in a Midwestern high school over a period of five weeks during the fall of 2021. The school focused on project-based learning. The teacher we observed worked with students in mathematics-specific sessions to strengthen their skills and their identities as learners of mathematics. The teacher taught with student-centered activities that used estimations, science contexts, and real-life applications. Initially, 20 days of instruction were planned. However, due to the COVID-19 pandemic, a few lessons were canceled, thus instruction was reduced to 14 days. All sessions were video-audio recorded and transcribed using a machine transcription. We analyzed all the teachers' questions in two rounds. In the first round, we categorized the questions based on how they linked to the cognitive domain of Bloom's taxonomy. Because some of the questions had an affective connotation where we observed the teacher verifying students' feelings when doing mathematics, we found it useful to use the affective domain of Bloom's taxonomy for an additional interpretation of the questions.

Consequently, we used the different levels of Bloom's cognitive and affective domain to interpret the teacher's questioning practice.

## Results

The teacher in this research was flexible when applying questioning techniques during instruction. The analysis showed that the teacher's questions had an affective dimension in relation to the learning event. These learning events corresponded to the cognitive phases described by Bloom et al. (1956), namely, knowledge, comprehension, application, analysis, synthesis, and evaluation. These are considered hierarchical levels. However, the teacher's questions did not follow this hierarchical order because the teacher shifted strategically between different levels during his interactions with the students. We situated the teacher's questioning within Bloom's taxonomy, characterizing the teacher's questions according to how he responded to students' reasoning. We thus considered both the cognitive and affective domains of Bloom's taxonomy. The affective dimension in the teacher's questions encouraged all students to speak, promoted students' mathematical identity formation, and demonstrated the teacher's openness to students' struggle: "Why is this [problem] throwing you off?" "You feel good about what's being asked?" "What information could you get from me that I might have that could help you?" This resulted in our organization of findings into three themes related to the teacher's questioning: Identity Forming, Strategy-Sharing, and Unpacking Students' Mathematics.

### Identity Forming

Questions grouped under this theme illustrated the teacher's practice of giving all students an opportunity to freely communicate their struggle with mathematics reasoning. The teacher identified himself with the students by using the pronoun "we." The researchers identified four categories of questions that supported students' mathematical identity formation. In most of the lessons, the teacher posed questions that we categorized as *caring questions*. Through these questions, the teacher tested the students' stimulation or depletion level (Hackenberg, 2005), linked to their work or to their comprehension of the context. Examples of such questions were: "You feel good about what's being asked?" or "Were you starting to stumble on something?" The teacher broke the students' silence with questions when the students could not recall knowledge. Since the teacher was checking students' understanding of and feelings towards ideas using questioning, we linked the questions to the comprehension level of the cognitive domain and to the receiving level of the affective domain of Bloom's taxonomy. Another category of questions that we included under this theme was *Checking for mathematical confidence*. These questions tested students' self-confidence in their own way of reasoning; for example, "Who knows that their estimate is closer to correct?" The teacher invited students to judge the value of their work and decide the best-suited solution. Thus, we linked the synthesis cognitive domain and the evaluation affective domain. The *Revoicing with questions about the students' reasoning* category was comprised of questions that helped students see the value of their contributions, such as "X was saying we need to know like, how long does it last?" Through such questions, the teacher invited students to analyze a problem and to value the knowledge that they acquired from others, thus valuing others' thinking. This links the analysis level of the cognitive domain with valuing level of the affective domain in Bloom's taxonomy. We included the category *Questions to test students' attention/motivation* under this theme because such questions encourage students to participate actively in the lesson. For example, questions like "What is our task right now?" encourage students to provide responses even when they are unsure of a mathematics approach to take to the posed task. Student responses inform

the teacher about their attentiveness. Knowledge in the cognitive domain and responding in the affective domain are thus linked.

**Table 1: Analyzing Questions in Relation to Identity-Forming Theme**

Categories	Definition of the Category	Example Questions	Bloom’s Taxonomy Cognitive/Affective
Caring questions	The teacher feels the students’ struggle (he is putting himself in the students’ shoes).	“You feel good about what’s being asked?” “Were you starting to stumble on something?”	Comprehension/Receiving
Checking for mathematical confidence	The teacher is testing students’ self-esteem in their ways of thinking.	“Who knows that their estimate is closer to correct?”	Evaluation/Valuing
Revoicing with questions about the students’ reasoning	The teacher uses students’ words/responses to explain the solution.	“X was saying we need to know like, how long does it last?”	Analysis/Organizing
Questions to test students’ attention/motivation	Checking students’ interest in being active (including being prepared for the lesson)	“What is our task right now?” “Anyone need something to write with?”	Knowledge/Receiving

### Strategy-Sharing

This theme includes four groups of questions that invited students to justify their work or to explain what they understood when listening to others. We named one group of questions *Asking about the mathematical process*. These questions aimed at eliciting students’ reasoning. For example, questions like “What’s that based on?” or “[say] why you drew it the way you drew it?” invited students to analyze and make inferences while responding to a stimulus that required students to react with an explanation. Therefore, these questions linked analysis in the cognitive domain with responses in the affective domain. The category of questions, *Reminding what needs to be found*, asked students to make connections between the part of the work done and the learning goal. An example of such questions was “So now am I able to answer the question, is it a good deal?” The students demonstrated that they organized their ideas and thus comprehended and gave value to the knowledge they acquired. The third category of questions, *Guidance with hints/extra examples*, provided support by diverting students’ thinking away from struggling and towards obtaining clarification of ideas. Examples of such question include: “What about this starting point?” and “If I said it’s negative seven degrees, and it gets three degrees colder, so it’s like more negative, right?” The hints and extra examples provided a bridge between students’ thinking about their prior knowledge and the stimulus supplied by the teacher. Hence,



application in the cognitive domain and responding in the affective domain were linked to these questions. The fourth category of questions under this theme was *Guidance to think/to find strategies for solutions*. The teacher collaborated with the students to find solutions. Questions such as “How much [have] you spent in a week? How long is it going to be worth until it’s going to be worth it? How much would 100 trips cost? How many weeks would it take them before buying the \$139 mug as a better deal?” involved the teacher and students in collaboration through sharing problem-solving strategies to make sense and decide. Thus, such questions link the synthesis cognitive domain and organizing affective domain.

**Table 2: Analyzing Questions in Relation to Strategy-Sharing Theme**

Categories	Definition of the Category	Example Questions	Bloom’s Taxonomy Cognitive/Affective
Asking about the mathematical process	Eliciting student’s reasoning	“What’s that based on?” “Why you drew it the way you drew it?”	Analysis/Responding
Reminding what needs to be found.	Linking part of the work done with the goal	“So now, am I able to answer the question, is it a good deal?”	Comprehension/Valuing
Guidance with hints/extra example	Questioning in parallel with other examples less sophisticated or simplified	“What about this starting point?” “If I said it’s negative seven degrees, and it gets three degrees colder, so it’s like more negative, right?”	Application/Responding
Guidance to think/ find strategies for solutions.	Teacher-student collaboration to find solutions.	“How much you spent in a week? How long is it going to be worth until it’s going to be worth it? How much would 100 trips cost? How many weeks would it take them before buying the \$139 mug is a better deal?”	Create/Receiving

### **Unpacking Students’ Mathematics: Linking to Students’ Ways of Knowing**

Questions under this theme focused on students’ ways of making sense and confidence in their knowledge development processes. This theme included 7 categories of questions. The first category, *Checking mathematical noticing and its use in solutions*, includes questions that ask students to intuit a response. Intuition is the most subtle form of intelligence. Here students analyze contexts and try to locate or identify helpful information to obtain a logical statement quickly. Since there might be a difference in what every student observes, these kinds of questions invite students to share and listen respectfully to their peers, which demonstrates the

connection between the analysis cognitive domain and receiving affective domain. An example of such a question is: “How many movies if you’re just thinking about it, and just looking at that...?” The second category, *Initiating guidance*, guides students in identifying one helpful piece of information to start solving a problem: “Which picture do you want to use?” Such questions also show the connection between the analysis and receiving domains. The third category, *Searching questions*, helps arouse students’ curiosity to explore ideas: “What would you need to be able to figure it out?” Searching questions are important because they help students to be creative when solving problems, bringing to bear facts that they consider useful. Such questions connect the application and valuing domains. The fourth category, *Checking mathematical concepts/notations*, requires students to recall facts that they have learnt; students are passive in such a situation: “Do you remember, the rate for slope ever being talked about as rise over run?” A question like this connects knowledge and receiving. The fifth category, *Checking for understanding/ validity/ sense-making*, requires students to analyze their work and make connections, and look for what is reasonable. Therefore, the link between comprehension and organizing is evident. An example of such questions is: “Do you think you could have done something wrong?” The sixth category, *Questions for different methods/to verify the results*, requires students to apply/analyze and respond accordingly. Examples of such questions are: “Can somebody improve that?” and “Who’s got other strategies?” The seventh category, *Sense-making in real life*, encourages students to make connections with real life, for example, “Does that affect how much goes into the landfill?” A reasonable student response would be to evaluate their own work.

**Table 3: Analyzing Questions in Relation to Unpacking Students’ Mathematics**

Categories	Definition of the Category	Example Questions	Bloom’s Taxonomy Cognitive/Affective
Checking mathematical noticing and its use in solutions	What students intuit logically	“How many movies if you just think about it, and just looking at that you don’t have to calculate anything necessarily, how many movies it’s going to take to make it worth it?”	Analysis/Receiving
Initiating guidance	Identifying what can be used to start the solution	“Which picture do you want to use?” “So what’s the thing that I don’t know?”	Analysis/Receiving
Searches questions	Instilling curiosity for the exploration of ideas	“What would you need to be able to figure it out?”	Application/Valuing
	The teacher checks students’		Knowledge/

Checking mathematical concepts/ notations	mathematical foundation	“Do you remember, the rate for slope ever being talked about as rise over run?”	Receiving
Checking for understanding/ validity/ sense-making	The teacher asks to analyze the work, make connections, and look for what is reasonable.	“Do you think you could have done something wrong?”	Comprehension/ Organizing
Questions for different methods/verify the results	The teacher looks for different ways of thinking, asking students to observe and interpret the results.	“Can somebody improve that?” “Who’s got other strategies?”	Application/ Responding
Sense-making in real life.	Connection with real life.	“Does that affect how much goes into the landfill? “	Evaluation/ Responding

### Discussions and Limitations

We aimed to answer the question: How does teacher questioning offer insights about teaching practices that promote students’ mathematical identity formation? Our analysis of one teacher’s questioning illuminates how questions can have cognitive intentions, focusing on students’ mathematics, and affective intentions, focusing on students’ feelings when doing mathematics. The teacher’s use of questions promoted students’ cognition and affect while positioning them as thinking persons in mathematics class. The teacher asked questions we described as *identity formation*, *strategy-sharing*, and *unpacking students’ ways of knowing mathematics*.

This teacher’s questions contributed to his intentional goal of building students’ mathematical identities and skills as doers of mathematics. He provided opportunities for students to share strategies, which created space for students to feel that they were listened to, their ideas were valued, and encouraged them to value others’ ideas. There was evidence in our analysis of the questions that the teacher demonstrated great care for the students and wished to support confidence building in their mathematical identity. One limitation of our study is the absence of sufficient student data or reflections on their perceived mathematical identity. Future work is needed in this area. However, by attending to the teacher’s intended actions, we recognize his use of questions to promote students’ cognition and mathematical identity formation as an effective, caring teaching practice that can be modeled and understood by others.

### References

- Batthey, D. (2013). “Good” mathematics teaching for students of color and those in poverty: The importance of relational interactions within instruction. *Educational Studies in Mathematics*, 82(1), 125-144.  
<https://doi.org/10.1007/s10649-012-9412-z>
- Boaler, J. (2008). Promoting ‘relational equity’ and high mathematics achievement through an innovative mixed--

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

- ability approach. *British Educational Research Journal*, 34(2), 167-194.
- Bloom, B. S., Engelhart, M.D., Hill, W.L., Furst, E.J., and Krathwohl, D.R. (1956). *Taxonomy of educational objectives: Cognitive domain*. Longman.
- Bohlin R. M. (1998). The affective domain: A model of learner-instruction interactions. In M.R. Simonson & D.J. Seepersaud (Eds), *Proceedings of Selected Research and Development Presentations at the National Convention of the Association for Educational Communications and Technology* (pp. 38–44).  
<https://files.eric.ed.gov/fulltext/ED423824.pdf>
- Bolin, A. U., Khramtsova, I., & Saarnio, D. (2005). Using student journals to stimulate authentic learning: Balancing Bloom’s cognitive and affective domains. *Teaching of Psychology*, 32(3), 154–159.  
[https://doi.org/10.1207/s15328023top3203\\_3](https://doi.org/10.1207/s15328023top3203_3)
- DeJarnette, A. F., Wilke, E., & Hord, C. (2020). Categorizing mathematics teachers’ questioning: The demands and contributions of teachers’ questions. *International Journal of Educational Research*, 104, 101690..  
<https://doi.org/10.1016/j.ijer.2020.101690>
- Dewey, J. (1998). *Experience and education* (60th-anniversary ed.). Kappa Delta Pi.
- Dillon, J. T. (1981). To question and not to question during discussion: 1. Questioning and discussion. *Journal of Teacher Education*, 32(5), 51–55. <https://doi.org/10.1177/002248718103200512>
- Drageset, O. (2014). Redirecting, progressing, and focusing actions—A framework for describing how teachers use students’ comments to work with mathematics. *Educational Studies in Mathematics*, 85(2), 281–304.  
<https://doi.org/10.1007/s10649-013-9515-1>
- Ingram, N. (2015). Students’ relationships with mathematics: Affect and identity. In M. Marshman, V. Geiger, & A. Bennison (Eds.), *Mathematics education in the margins. Proceedings of the 38th Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 301–308). MERGA.
- Kastberg, S. E. (2003). Using Bloom’s Taxonomy as a Framework for Classroom Assessment. *The Mathematics Teacher*, 96(6), 402–405. <https://doi.org/10.5951/MT.96.6.0402>
- Kaya, S., & Ceviz, A. E. (2017). Pre-service teachers’ use of dynamic discourse variables during classroom teaching. *Journal of Education and Practice*, 8(12), 82–89. <https://eric.ed.gov/?id=EJ1140633>
- Krathwohl, D. R., Bloom, B. S., & Masia, B. B. (1964). *Taxonomy of educational objectives: The classification of educational goals. Handbook II: Affective domain*. David McKay Company.
- Leder, G. C. (1993). Reconciling affective and cognitive aspects of mathematical learning: Reality or a pious hope? In I. Hirabayashi, N. Nohda, K. Shigemathu, & F.-L. Lin (Eds.), *Proceedings of the Seventeenth PME Conference* (pp. 46–65). Tsukuba, Ibaraki, Japan: University of Tsukuba.
- Nathan, M. J., & Kim, S. (2009). Regulation of teacher elicitations in the mathematics classroom. *Cognition and Instruction*, 27(2), 91-120. <https://doi.org/10.1080/07370000902797304>
- Noddings, N. (2013). *Caring: A relational approach to ethics and moral education*. University of California Press.
- Noddings, N. (2017). Care ethics and education. In N. Aloni & L. Weintrob (Eds.), *Beyond bystanders: Moral development and citizenship education* (pp. 183–190). Sense Publishers. doi:10.1007/978-94-6351-026-4\_14.
- Sfard, A. (2008). *Thinking as communicating: Human development, and the growth of discourses, and mathematizing*. Cambridge University Press.
- Sfard, A., & Prusak, A. (2005a). Identity that makes a difference: Substantial learning as closing the gap between actual and designated identities. In Chick, H. L., & Vincent, J. L. (Eds.). (2005). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*. Vol. 1, (pp. 37-52). Melbourne: PME.
- Sfard, A., & Prusak, A. (2005b). Telling identities: In search of an analytic tool for investigating learning as a culturally shaped activity. *Educational Researcher*, 34(4), 14-22. <https://doi.org/10.3102/0013189X03400401>
- Zan, R., Brown, L., Evans, J., & Hannula, M. S. (2006). Affect in mathematics education: An introduction. *Educational Studies in Mathematics*, 63(2), 113–121. <https://doi.org/10.1007/s10649-006-9028-2>

## MENTOR AND NOVICE TEACHER CO-LEARNING TO PROMOTE STUDENT ENGAGEMENT AND PARTICIPATION

Torry Kulow  
Portland State University  
kulow@pdx.edu

Imani Goffney  
University of Maryland  
igoffney@umd.edu

Heather Fink  
Portland State University  
hfink@pdx.edu

Ruth Heaton  
Teacher Development Group  
ruth.heaton@teachersdg.org

Melinda Knapp  
Oregon State University-  
Cascades  
Melinda.Knapp@osucascades  
.edu

Taylor Stafford  
University of Washington  
taylores@uw.edu

Manqing Gao  
Portland State University  
manqgao@pdx.edu

*This design research study describes how one mentor teacher-teacher candidate dyad co-learned to promote student engagement and participation through using a “Collaborative Learning Structure” (CLS) tool that we are developing. We share how the dyad used the CLS, with support from a professional development facilitator and fellow teachers, to better identify student assets (a critical component of promoting student engagement and participation). Our analysis demonstrates that (1) co-noticing is a productive means of supporting dyad co-learning to promote student engagement and participation and (2) teachers need to co-learn across time while using tools supportive of their learning. We discuss implications for how to support teachers in promoting student engagement and participation.*

Keywords: Preservice Teacher Education, Equity, Inclusion, and Diversity, Teacher Noticing

Preparing and supporting teachers to promote meaningful student engagement and participation is an enduring and critical challenge in math education. Our project team is currently in Year 3 of a four-year, NSF-funded, design research (Cobb et al., 2017) study to iteratively design tools to support teacher dyads (teacher candidates and mentor teachers) in collaboratively learning (i.e., co-learning) about ways to promote student engagement and participation. We aim to help dyads develop a vision for math teaching and learning where:

All students, in light of their humanity – personal experiences, backgrounds, histories, languages, and physical and emotional well-being – must have the opportunity and support to learn rich mathematics that fosters meaning making, empowers decision making, and critiques, challenges, and transforms inequities and injustices ... equity demands that responsive accommodations be made as needed to promote equitable access, attainment, and advancement in mathematics education for each student. (Aguirre, Mayfield-Ingram, & Martin, 2013, p. 9)

In addition, we aim to help dyads continually seek to understand their students’ identities and assets as well as their own implicit biases and deficit perspectives (e.g., Featherstone et al., 2011; İnan-Kaya & Rubie-Davies, 2021; Moll et al., 1992; Paris, 2012, 2016). We also aim to help

dyads enact practices that celebrate and draw on students' identities and assets, elicit and validate non-dominant forms of math competence, establish equitable participation norms that position all students as capable, and critique and dismantle classroom structures of power and privilege that suppress their students' success (Aguirre, Mayfield-Ingram & Martin, 2013; Aguirre, Turner, & Bartell, et al., 2013; Bartell et al., 2017; Celedón-Pattichis et al., 2018; Gutiérrez, 2013; Ukpokodu, 2011).

We believe that teachers' learning how to promote student engagement and participation is an adaptive, ongoing, deliberate, and collaborative process since teachers must continually respond to the varied needs of their diverse students across their teaching careers as well as to the ever-shifting contexts in which they work. Engaging with people with diverse perspectives and experiences (notably colleagues, students, parents, community members) regularly across time helps teachers notice and analyze parts of their practice that may be "invisible" to them. These interactions support teachers' development by making their intentions explicit and the impacts of their actions (whether they are equitable or not) open to collective inquiry. This aligns with the benefits of collaborative professional learning opportunities within communities of practice (Wenger, 1999). We are investigating how the clinical experience can be a productive site for dyads' co-learning about promoting student engagement and participation. Using mentoring models that position teachers as "critical friends," "co-enquirers," and "partners" (Furlong & Maynard, 1995; Males et al., 2010), our tools aim to minimize power hierarchies between teachers, draw on assets of both teachers, and support teachers in developing the humble and vulnerable stance required of lifelong teacher learners (Feiman-Nemser, 2012).

The tools we are designing, referred to as Collaborative Learning Structures (CLSs), have three key features that support dyads in working in adaptive, ongoing, deliberate, and collaborative ways. First, our CLSs prompt dyads to investigate "enduring questions" (Cochran-Smith et al., 2008) related to student engagement and participation (e.g., recognizing, understanding, and disrupting inequitable patterns of student participation; honoring and making sense of students' diverse ideas; creating opportunities for students to learn collaboratively).

Second, our CLSs prompt dyads to notice (van Es & Sherin, 2008) collaboratively, or "co-notice", salient aspects of their instruction and students' experiences in the classroom that advance or constrain the students' participation and engagement. When co-noticing, teachers identify elements of their practice that might otherwise go *unnoticed* as well as collaboratively generate and experiment with more equitable practices that aim to disrupt features of the classroom and instruction that lead to inequitable participation (Louie et al., 2021; van Es et al., 2017). Third, our CLSs guide dyads' interactions when collaboratively planning, enacting, and reflecting on lessons over time. The CLSs frame these teaching activities (lesson planning, enactment, reflection) as interconnected so that the mentor teachers and teacher candidates continue making sense of and having structured conversations about student engagement and participation during all phases of lesson development and across time.

The research question guiding this study is "How do CLSs support dyads in co-learning about student engagement and participation?" This paper shares one example of teachers' co-learning to promote student engagement and participation while using a CLS.

### Methods

The CLS used in this study, shown in Figure 1, is our initial draft of this tool (to be revised in summer 2023 and again after data collection during the 2023-2024 academic year). This tool includes three parts. Part 1 is a protocol for planning to co-notice during a lesson. Here the *lead teacher* (i.e., teacher leading instruction) first *gives a high-level summary of the lesson*<sup>1</sup>, then the

dyad selects a *focus to co-notice* (i.e., a co-learning goal related to an enduring question) for their lesson planning-enactment-debriefing cycle and highlights places in the lesson they feel curious or uncertain about related to their focus. Then they decide when/how they will check in with one another during the lesson and what each teacher will notice/pay attention to during the lesson related to the curiosity/uncertainty shared. Part 2 is a protocol for co-noticing and checking in during the lesson. Here teachers circulate and observe students and the lead teacher while capturing noticings related to their goal/enduring question. They check in with each other at the moments they planned and share something they noticed related to their goal in order to work together to think about how to proceed given what they noticed. Part 3 is a protocol for co-debriefing after the lesson. Here teachers share reflections about their noticing, identify a particular moment when they responded to what they noticed, and summarize their learning and planning for future co-learning.

Co-Noticing to Co-Learn Protocol	
<b>Directions and Purpose</b>	
The purpose of this protocol is to support teachers' in co-learning about how to enact more equitable mathematics instruction through opportunities to share their noticings and collaboratively make instructional decisions during instruction. The protocol below walks you through this process and offers prompts/ideas to support your conversation.	
<b>Part 1: Planning to Co-Notice</b>	
High level summary of the lesson (~5 min): "Lead teacher" (whoever is teaching the lesson) gives a quick summary of the lesson and the lesson's goal. <i>Note: It may be helpful to share the lesson with your colleague prior to the day you're planning to try the co-noticing so that both members of the dyad are oriented to the lesson.</i>	
Select a focus to co-notice about today:	
<ul style="list-style-type: none"> <li>Recognize, understand, and disrupt inequitable participation patterns: <i>Who is talking/expressing their ideas and what are they talking about/expressing? Who is not talking or expressing ideas? Why might that be? How might we create more equitable opportunities for engaging specific students deeply in the content?</i></li> <li>Honor and make sense of students' diverse ideas: <i>What ideas and partial understandings are students offering? In what ways are these ideas mathematically rich and relevant? What might "buds" or beginnings of justifications and generalizations sound like given the content we are working on? How can we leverage these "buds"?</i></li> <li>Create opportunities for learning collectively: <i>How are students oriented to the brilliance and ideas of their peers? How are they learning from one another? Who is making sense of other people's ideas and how do I know?</i></li> </ul>	
Lead teacher: Highlight places in the lesson that feel important to your focus. Share what you feel curious or uncertain about.	
Decide when/how you'll check in with one another during the lesson: Together, anticipate some times during the lesson where you think it would be useful to discuss what you're noticing and collaborate about what to do next. This might look like pausing while teaching to ask a colleague for ideas, or a quick huddle during a turn and talk or small group/partner work.	
Decide what each of you will notice/pay attention to: Sketch out some questions to guide your noticing, related to your goal(s). (Sample questions are provided on the next page.)	
<b>Part 2: Co-Noticing and Checking in during the lesson</b>	
Circulate and observe students and the lead teacher. Capture noticings related to your goal/enduring question (using a notebook).	
Sample Questions to Guide Your Noticing <ul style="list-style-type: none"> <li>What mathematical ideas have you heard?</li> <li>What student strengths or assets are you noticing?</li> <li>How are students talking/expressing ideas?</li> <li>Whose voices are carrying the conversation? (Paying attention to issues of gender, race, and language proficiency.)</li> <li>What do you notice about the kinds of opportunities students have to demonstrate their mathematical knowledge?</li> <li>OR, write your own!</li> </ul>	
Check in with your colleague at the moments you planned (as students are working or talking together) and share something you noticed related to your goal for today. <ul style="list-style-type: none"> <li>Something interesting I saw...</li> <li>Something confusing I saw...</li> <li>An idea I have...</li> <li>A question I have...</li> <li>What do you think we should/could do next?</li> </ul>	
Then, work together to think about how you might proceed in the lesson given what you've noticed.	
<b>Part 3: Co-Debriefing</b>	
After the lesson, share some reflections about your noticings with your colleague. <ul style="list-style-type: none"> <li>What did you/your partner notice that surprised you?</li> <li>Is there anything else you noticed that you'd like to share?</li> <li>(If applicable) What did we notice that was different/similar from last time?</li> </ul>	
Identify a particular moment when you responded to what you noticed. <ul style="list-style-type: none"> <li>Think together about a time when you and your partner made a decision about what to do next with students? How did it go? What might we try differently next time?</li> <li>How do you think the instructional decisions you made together related to your focus: <ul style="list-style-type: none"> <li>Pressing on existing participation patterns</li> <li>Honoring and making sense of students' diverse ideas</li> <li>Creating opportunities for learning collectively</li> </ul> </li> </ul>	
Summarize learning and plan for future co-learning <ul style="list-style-type: none"> <li>What new ideas, understandings or questions do you have related to your focus today?</li> <li>What stands out to you as most important about today?</li> <li>What questions about teaching do you have now?</li> <li>How did it feel to try this protocol with your colleague?</li> <li>When would it be helpful to try this again? What might we want to focus on or try out?</li> </ul>	

<sup>1</sup> In this paper, italicized text indicates words and prompts that are taken directly from the CLS (see Figure 1).

### Figure 1: Initial Design of a Collaborative Learning Structure (CLS)

#### Context, Data Collection, and Analysis

Data were collected in fall 2022 during a two-day professional development (PD) experience for elementary teachers. Day two of the PD included a lesson enactment when participants observed a lesson taught in a focal dyad's classroom. On day one of the PD, the PD facilitator, Aurora, met with the focal dyad to prepare for the upcoming lesson enactment. The focal dyad consisted of Maggie (mentor teacher, 20+ years teaching) and Amy (teacher candidate, graduate student). On day one, Aurora, Maggie and Amy used CLS Parts 1 and 2 to learn about students in Maggie and Amy's 5<sup>th</sup> grade classroom, refine the plan for lesson enactment, and plan to co-

notice during the lesson. On day two, Aurora and five dyads (including the focal dyad) used CLS Parts 1 and 2 *before* the lesson enactment to plan for co-noticing while the lesson was happening, then used CLS Part 2 *during* the lesson enactment to guide their notetaking and co-noticing check-in discussions, and finally used CLS Parts 2 and 3 *after* the lesson enactment to share observations and reflect on their collective learning. In addition, all teacher candidates used the CLS in a math methods course and dyads used the CLS outside of the PD while teaching in their own classrooms. The five dyads taught varying grade-levels at the same elementary school.

A project researcher observed and took field notes during the entire PD (including lesson enactment), and video recorded all of the PD except the 35-minute lesson enactment. Data for this study is drawn from the field notes and video-recordings collected during the PD. We analyzed data from all conversations/interactions using the CLS to characterize how the CLS supported the dyads in co-learning about student engagement and participation. We narrowed in on teachers' discussions related to student assets, building on the belief that meaningful student engagement and participation can be promoted by identifying and leveraging the assets students bring to classrooms. Our analytic questions included: "In what ways did the CLS seem to support (or not) productive discussions, particularly about student assets?", "What student assets (if any) did teachers identify?", and "How might these findings inform future revisions of the CLS tool?"

### Results

The following vignettes demonstrate how the CLS supported Maggie and Amy, with support from Aurora and other PD participants, in co-learning how to identify student assets that were previously unnoticed by them.

#### **Vignette 1: Facilitator (Aurora) and Focal Dyad (Maggie and Amy) Use CLS Parts 1 and 2 to Plan to Co-notice During the Lesson Enactment**

In preparation for the PD, Aurora used CLS Part 1 to select a co-learning focus for the entire PD experience. She selected the focus to "Recognize, understand, and disrupt inequitable participation patterns" since it aligned with the teacher candidates' math methods course and with initiatives identified by the elementary school administrators. At the start of day one, Aurora shared this co-learning focus with Maggie and Amy, and the three discussed the roles they each would play as co-learners during the lesson enactment. Using CLS Part 1, Aurora then *gave a high level summary of the lesson* she planned to teach the next day in Maggie and Amy's class and *highlighted places in the lesson that she had curiosities or uncertainties about related to their focus*, specifically students' participation and math thinking since she had spent little time in the focal classroom. Next, the trio refined the lesson plan while keeping in mind the participation of individual students. To do this, Aurora asked the dyad to identify students for focused attention during the lesson. They identified two students they had participation concerns about: Carly, who "thinks her partner is so much smarter", and Lila, who "won't write anything down." Prompted by a question in CLS Part 2, Aurora then asked the dyad what they knew about these students' strengths. Amy described Carly as "super artist and very giving." Maggie added that Carly "is trying... she hasn't shut down" and "she's social." Amy then said, "[Lila] is so capable, but she won't - I don't know what it is." Maggie offered that Lila is "super sweet, very friendly" and "gets along with her tablemates well." In sharing these strengths, the teachers described characteristics of each student that are not typically considered to be mathematical strengths. The trio concluded by making a few final revisions to the lesson plans, incorporating visual models and partner discourse prompts. In doing so, they deferred *deciding when/how they will check in with one another during the lesson* and *deciding what each teacher will notice/pay attention to during the lesson* to day two of the PD session.



## **Vignette 2: All Dyads and Aurora Use CLS Parts 1 and 2 to Plan to Co-notice During the Lesson Enactment**

At the start of day two, the five dyads and Aurora used CLS Parts 1 and 2 to plan to co-notice during the lesson enactment. First, Aurora distributed copies of the CLS to dyads, saying “think about what you’re hoping to co-notice and learn.” She suggested dyads look at the list of *Sample Questions to Guide Your Noticing* in Part 2 then *decide when/how you’ll check in with one another during the lesson* in Part 1. During this time, Maggie and Amy talked about several guiding questions, ultimately deciding to focus on the question *What student strengths or assets are you noticing?* Their conversation, transcribed below, shows how both Amy and Maggie offered ideas during this exchange and seemed to reach consensus together.

Amy: I think whose voices are carrying [the conversation] will be really interesting because- at least with these two- because we’re not expecting them to have necessarily equal exchange.

Maggie: Uh-hmm.

Amy: The next one too [What do you notice about the kinds of opportunities students have to demonstrate their mathematical knowledge?]. Like, with- well, with all of them, but specifically with [Lila], like, what if she can represent? You know, instead of just having a blank piece of paper in front of her. I’m really curious if she’s gonna actually-.

Maggie: Start using the visual model?

Amy: Yeah. I think that Carly probably will.

Maggie: I think so too... But you know what might be interesting for us to look for too is their student strengths even though we’re not really- it’s hard. It’s hard to tell with Lila in particular and for our kids who are similar to her. Like, they have strengths. They’re just not clear to us because they’re so quiet. I mean, it might be like a combination of what strengths do they have and what opportunities are they using to show what they know.

Amy: Uh-hmm. Because maybe that is their strength.

Maggie: Right.

Amy: Maybe the visual is their strength or something like that.

Maggie: Yeah. I’d be interested in trying to find more strengths for those two, for Lila for sure.

## **Vignette 3: All Dyads and Aurora Use CLS Part 2 to Co-Notice During Lesson Enactment**

During the lesson enactment on day two, Aurora told dyads to take a few minutes to check in, as directed in CLS Part 2 (*Check in with your colleague at the moments you planned*). Amy and Maggie stepped into the hallway. They started their conversation by talking about how the two focal students were not talking and were not writing anything on their papers. Amy then asked, “But what about their strengths?” Maggie replied, “Yeah, their strengths.” They both paused to think, then noted how Carly had participated verbally during the choral response in the beginning of lesson. They were unable to identify any specific strengths for Lila. The initial focus of the dyad conversation centered around what their focal students were NOT doing. Amy then used a question in Part 2 (*What student strengths or assets are you noticing?*) to reorient the dyad. She asked, “But what about their strengths?” Maggie accepted the shift in conversation.

#### **Vignette 4: All Dyads and Aurora Use CLS Parts 2 and 3 to Co-Debrief Their Co-Noticing**

After the lesson enactment on day two, Aurora and the five dyads used CLS Parts 2 and 3 to *share something they noticed related to their goal and summarize learning* as a group. Aurora prompted the group to think about leveraging student strengths by asking, “Did you see evidence of students engaging in habits that are strengths as mathematicians? We can start to notice ourselves and have other students notice them and use them to engage students.” Teachers shared strengths they noticed about the class as a whole, including “students did a really good job of persevering and seeking more,” “kids were engaged and excited,” and “they were going back and forth, working through tough questions.” Aurora then asked Maggie and Amy to share strengths they noticed about Carly and Lila. Maggie shared that Carly’s strength was counting orally; Aurora connected that observation to using mathematical structures, patterns, and regularity. The dyad could not initially identify any strengths for Lila, so they invited input from the other teachers, as they examined Lila’s work. As demonstrated in the following transcript, through collaborative noticing and reflecting with support from Aurora and another teacher named Nia, Maggie and Amy identified Lila’s strength of justifying and explaining her mathematical reasoning, a crucial step in leveraging students’ assets to promote meaningful student engagement and participation.

Maggie: [Lila] would look at her partner’s work and then start working herself. But again, that’s the one who isn’t confident.

Aurora: But she’s resourceful.

Maggie: But resourceful. Right? So, I don’t know, what would you call that one?

Aurora: Hmm. I don’t know what she was looking at. Do you know what she was looking at?

Maggie: Well, when you were giving them private reasoning time to write down their initial thoughts about who had more, I mean it was very blatant, she was like, “Ok, what’s my neighbor doing?” And then it was like, you know, doing the work.

Nia: Is it cuz she doesn’t know how to do it or she lacks the confidence to do it? I’m just curious.

Maggie: Yeah, I think she *could* do it.

Nia: So, do you think maybe she’s trying to compare her own thoughts to others?

Maggie: Ummmm ...

Nia: Maybe she’s looking to see if she’s thinking about it correctly?

Maggie: Yeah. I think Amy is looking for her partner’s work right now to compare. But it could be.

Aurora: And we don’t know until we ask ... what does this mean? What else could it mean? Maggie: Oh, and they have different work, now that we’re looking at it too. And she’s actual-

She’s showing more. But she- she- her explanation is using more-.

Amy: Justification.

Maggie: She has more justifi- more mathematical reasoning in her answer than her partner did. But she just had to know- because they both started out the same way. Lance had less, so they both started out that same way, but then she goes on and is more specific, whereas the partner is not. Or they go different routes. The partner said, “because in the tenths place Lance has 4 and Angel has 5.” And [Lila’s] says, “Lance has less and Angel has more because in Angel’s work she shows that 5 tenths

is equal to 500 thousandths and 500 thousandths is bigger than 485 thousandths.” So, they both - they branch off. So maybe she was just like comparing in her head first. And then like, “Ok, we both started off knowing that Lance has less, but now I can explain it in my own way.”

### **Discussion**

These vignettes suggest the CLS was useful in helping Maggie and Amy co-learn new ways of promoting student engagement and participation by expanding their conceptions of their students’ mathematical assets. The CLS prompts were useful in both orienting teachers to look for student assets instead of deficits and in re-focusing their attention when they shifted toward deficit orientations. Based on these vignettes, we have identified two central claims. Claim 1: teachers co-noticing is a productive means of supporting co-learning since working in partnership provides opportunities for teachers to broaden what they see in their students’ thinking and work. Claim 2: learning to promote student engagement and participation is not fast or easy; rather, it requires teachers to engage in goal-oriented activities together across time.

#### **Claim 1: Co-noticing is a Productive Means of Supporting Dyad Co-learning to Promote Student Engagement and Participation**

The CLS is designed to support dyads in *co-noticing* aspects of their instruction and students’ experiences in the classroom in order to collaboratively make instructional decisions. While initially struggling to identify the focal students’ mathematical assets (as demonstrated in vignettes 1 and 3), teachers articulated curiosities about the focal students’ participation (as demonstrated in vignette 2) and began noticing the different ways the focal students were participating in the lesson and the diverse assets they offered (as demonstrated in vignettes 3 and 4), with the help of the CLS, Aurora, and other teachers in the PD. A rich body of research in math education suggests that teacher noticing is a central dimension of teaching expertise and has important implications for how teachers attend to students’ mathematical reasoning and sensemaking (e.g., van Es & Sherin, 2008) and equity in their classrooms (e.g., van Es et al., 2017). The vignettes show how co-noticing about focal students’ participation was an important and necessary precursor for helping the dyad make their unconscious bias and deficit perspectives more visible so that they had the opportunity to disrupt aspects of their instruction that led to inequitable student participation. Further, when focusing their shared work on co-noticing (as demonstrated in all vignettes) both Amy and Maggie had the opportunity to share their individual perspective about their students and salient moments of instruction, thereby drawing on their collective (teacher) assets. Thus, in contrast to collaborative work that elicits and privileges one perspective or interpretation of classroom events, co-noticing invites multiple perspectives and interpretations and presses teachers to understand in new ways how features of the classroom and instruction advance or constrain student participation and engagement.

#### **Claim 2: Teachers need to Co-Learn Across Time While Using Supportive Tools**

These vignettes indicate that the work of co-learning and co-noticing is challenging for teachers (ideally a productive struggle) and unlikely to happen without deliberate tools and support across time. As they were continually prompted to focus on student assets while using all three parts of the CLS, the focal dyad appeared to find this work challenging, as evidenced by the struggle they had when identifying and articulating the mathematical strengths of each focal student in vignettes 1 and 3. The teachers used a positive disposition toward the two students, but had limited knowledge of the range of strengths that students bring to and share in the

mathematical work (beyond “traditional” strengths such as getting the right answer quickly) or were perhaps limited by their constrained views of the students and what the students were capable of doing in their class. The CLS prompts were useful in inspiring the dyad’s discussion and focus on co-noticing, yet in their initial collaborative discussion (in vignette 1) the focal dyad quickly reverted to identifying and naming what the focal students were not doing instead of focusing on what the students were doing in math. However, after Amy refocused the discussion by asking “But what about their strengths?” (in vignette 3), both teachers began to look for and notice student assets. It is not likely that this re-orienting would be possible, meaningful or productive had the status roles of the mentor teacher and teacher candidate not been flattened in the PD session and CLS use. This reorienting was useful in eliminating and shifting the deficit narrative that the teachers had used on day one, although the focal dyad continued to grapple with noticing authentic mathematical strengths and assets of their focus students. The larger discussion with multiple dyads (in vignette 4) helped the dyad identify more authentic mathematical assets of the focal students. The other dyads posed questions that prompted the dyad to more deeply examine and broaden their conception of their students’ participation and assets in ways that helped Maggie and Amy see Carly and Lila in new ways.

Thus, looking across the vignettes shows how teachers must engage in sustained conversations about how to promote student engagement and participation across multiple teaching activities (including lesson planning, enactment, and reflection) with guidance from supportive tools.

### **Conclusions and Implications**

One of our explicit goals in developing the CLSs is to re-frame teacher orientations about equitable math teaching with a specific focus on broadening conceptions of what participation in math classrooms could and should look like. Implicitly, we aim to develop a tool that will a) disrupt deficit perspectives and narratives of students, and b) support teachers with developing new skills by offering replacement practices that leverage an assets/strengths-based perspective. In this way, we anticipate that our CLS will support teachers with both interrupting deficit language and perspectives as well as expanding their skills for noticing the assets and strengths students, especially Black and Latiné students, bring to learning and doing math. We operate from the assumption that teachers have good intentions and are deeply invested in their students’ academic success, are committed to their students developing positive learning identities, and have a desire to improve their own practice. We also leverage our own asset-based perspective about teachers in developing the CLS. We believe if teachers have better tools, then they will be better positioned to operationalize their curiosity about students’ thinking, their commitments to students’ developing positive identities, and their work as equity-oriented math teachers.

Preliminary data from our work suggests that the CLS is promising as a powerful tool for supporting math teachers in recognizing, understanding, and disrupting inequitable patterns of student participation, honoring and making sense of students’ diverse ideas, and creating opportunities for students to learn collaboratively as they co-plan, co-notice, and co-debrief in service of continually becoming equity-oriented teachers. The CLS appears useful to dyads by providing a structure for co-noticing and co-debriefing. It also appears useful for flattening power hierarchies between teacher candidates and mentor teachers in ways that created new spaces for co-learning and new opportunities for multiple perspectives and strengths to be used in service of equity-oriented math teaching. Despite early evidence about the usefulness of the CLS, we remain curious about how we can continue to revise the CLS through design research cycles to better accomplish our goals and create a theory of co-learning equity-oriented

instruction. We are also curious about what other types of supports might be needed by dyads for co-learning, co-noticing and co-debriefing, especially as related to disrupting inequitable patterns of student participation and students' opportunities for learning.

### Acknowledgments

This paper is based in part on work supported by the National Science Foundation under grant No.2010634. Any opinions, findings, conclusions, or recommendations expressed in this paper are those of the authors and do not necessarily reflect the views of the National Science Foundation. We thank Kara Jackson at the University of Washington and Mary Alice Carlson at Montana State University for their ongoing collaboration on this project.

### References

- Aguirre, J. M., Mayfield-Ingram, K., & Martin, D. B. (2013). *The Impact of Identity in K-8 Mathematics: Rethinking Equity-based Practices*. Reston, VA: National Council of Teachers of Mathematics
- Aguirre, J. M., Turner, E. E., Bartell, T. G., Kalinec-Craig, C., Foote, M. Q., Roth McDuffie, A., & Drake, C. (2013). Making connections in practice: How prospective elementary teachers connect to children's mathematical thinking and community funds of knowledge in mathematics instruction. *Journal of Teacher Education*, 64(2), 178–192. <https://doi.org/10.1177/0022487112466900>
- Bartell, T., Wager, A., Edwards, A., Battey, D., Foote, M., & Spencer, J. (2017). Toward a framework for research linking equitable teaching with the standards for mathematical practice. *Journal for Research in Mathematics Education*, 48(1), 7-21. <https://doi.org/10.5951/jresmetheduc.48.1.0007>
- Celedón-Pattichis, S., Borden, L. L., Pape, S. J., Clements, D. H., Peters, S. A., Males, J. R., ... & Leonard, J. (2018). Asset-based approaches to equitable mathematics education research and practice. *Journal for Research in Mathematics Education*, 49(4), 373-389. <https://doi.org/10.5951/jresmetheduc.49.4.0373>
- Cobb, P., Jackson, K., & Dunlap, C. (2017). Conducting design studies to investigate and support mathematics students' and teachers' learning. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 208- 236). National Council of Teachers of Mathematics.
- Cochran-Smith, M., Feiman-Nemser, S., McIntyre, J., & Demers K (Eds.). (2008). *Handbook of Research on Teacher Education: Enduring Questions in Changing Contexts*. Routledge.
- Featherstone, H., Crespo, S., Jilk, L. M., Oslund, J. A., Parks, A. N., & Wood, M. B. (2011). *Smarter Together! Collaboration and Equity in the Elementary Math Classroom*. National Council of Teachers of Mathematics.
- Feiman-Nemser, S. (2012). *Teachers as Learners*. Harvard University Press.
- Furlong, J., & Maynard, T. (1995). *Mentoring student teachers: The growth of professional knowledge*. Routledge.
- Gutiérrez, R. (2013). Why (Urban) Mathematics Teachers Need Political Knowledge. *Journal of Urban Mathematics Education*, 6(2), 7–19. <https://doi.org/10.21423/jume-v6i2a223>
- İnan-Kaya, G., & Rubie-Davies, C. M. (2021). Teacher classroom interactions and behaviours: Indications of bias. *Learning and Instruction*, 78, 1-13. <https://doi.org/10.1016/j.learninstruc.2021.101516>
- Louie, N., Adiredja, A. P., & Jessup, N. (2021). Teacher noticing from a sociopolitical perspective: The FAIR framework for anti-deficit noticing. *ZDM–Mathematics Education*, 53(1), 95-107. <https://doi.org/10.1007/s11858-021-01229-2>
- Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

- Males, L. M., Otten, S., & Herbel-Eisenmann, B. A. (2010). Challenges of critical collegueship: Examining and reflecting on mathematics teacher study group interactions. *Journal of Mathematics Teacher Education*, 13(6), 459-471. <https://doi.org/10.1007/s10857-010-9156-6>
- Moll, L. C., Amanti, C., Neff, D., & Gonzalez, N. (1992). Funds of knowledge for teaching: Using a qualitative approach to connect homes and classrooms. *Theory Into Practice*, 31(2), 132–141. <https://doi.org/10.1080/00405849209543534>
- Paris, D. (2012). Culturally sustaining pedagogy: A needed change in stance, terminology, and practice. *Educational Researcher*, 41(3), 93-97. <https://doi.org/10.3102/0013189X12441244>
- Paris, D. (2016). *On Educating Culturally Sustaining Teachers*. Teaching Works, University of Michigan.
- Ukpokodu, O. N. (2011). How do I teach mathematics in a culturally responsive way?: Identifying empowering teaching practices. *Multicultural Education*, 18(3), 47–56.
- van Es, E. A., Hand, V., & Mercado, J. (2017). Making visible the relationship between teachers’ noticing for equity and equitable teaching practice. In E. O. Schack, M. H. Fisher & J. A. Wilhelm (Eds.). *Teacher noticing: Bridging and broadening perspectives, contexts, and frameworks*. Springer International Publishing.
- van Es, E. A., & Sherin, M. G. (2008). Mathematics teachers’ “learning to notice” in the context of a video club. *Teaching and Teacher Education*, 24(2), 244-276. <https://doi.org/10.1016/j.tate.2006.11.005>
- Wenger, E. (1999). *Communities of practice: Learning, meaning and identity*. Cambridge University Press.

## MODULES THAT INTERSECT NOTICING AND EQUITY: PASSING AN INFLECTION POINT WITH ELEMENTARY PRESERVICE TEACHERS

## MÓDULOS QUE INTERSECAN CON LA MIRADA Y EQUIDAD: PASANDO UN PUNTO DE INFLEXIÓN CON MAESTROS PRESERVICIO DE PRIMARIA

Molly Fisher  
University of Kentucky  
Molly.fisher@uky.edu

Walker Mask  
University of Kentucky  
Walker.mask@uky.edu

Cindy Jong  
University of Kentucky  
Cindy.jong@uky.edu

Jonathan Thmoas  
University of Kentucky  
Jonathan.thmoas1@uky.edu

*Recent developments in mathematics education research seek to better incorporate knowledge of equity into teacher education, in particular practical ways that teachers can promote equity in the mathematics classroom. This paper discusses the results of using a comparison, partial, and full implementation of a set of microlearning modules designed to introduce preservice mathematics teachers to equitable noticing in elementary mathematics teaching. The results of ANOVA analyses on the scores of a video-based assessment instrument relating to professional noticing and themes of equity in a classroom moment indicate that for these modules, there are gains when analyzing the comparison to partial implementations, indicating that using more of the modules leads to a deeper understanding of the introductory equity and professional noticing frameworks used to frame the modules and this study.*

Keywords: Preservice Teacher Education; Teacher Noticing; Equity, Inclusion, and Diversity

### Introduction

Over the past decade, the research that intersects equity and professional noticing has surged (Schack et al., 2017; Louie et al., 2021). When we prepare teachers to actively notice student learning, participation, and engagement in the classroom, we impose onto our students our own ideas of what is deemed important to observe and respond to. In addition, each teacher will bring their own experiences and perspectives into what they tend to notice or ignore in the classroom, some of which may have influences based on social structures such as race, gender, class, language, or ability. Recent work in this area has pushed the notion of how teachers can learn to consider inequities within the mathematics classroom, and by extension, inequities in their teaching behaviors around noticing and responding to students' learning (Van es et al., 2017).

Learning how to promote equity, as well the action of doing so, is a continuous process that does not end when a preservice teacher leaves their teacher education program. But how much time and theoretical space should be devoted to introducing these concepts in order to establish a strong foundation that can continue growing without taking away from the other important aspects of teacher education?

The purpose of this study is to analyze data from a video-based assessment instrument to determine the impact that exposure to micromodule curriculum focused on introducing equitable noticing in mathematics has on preservice elementary teachers' (PSETs) knowledge and ideas of what it means to professionally notice with equity in mind in the elementary mathematics classroom. We focus on comparing two levels of exposure to these micromodule lessons for their

differences in PSET knowledge of the components of noticing and equity frameworks with relation to a mathematics classroom scenario.

### **Theoretical Framework**

The Modules that Intersect Noticing and Equity (MINE) Project was informed by Gutiérrez's (2009) four dimensions of equity framework which are categorized into two axes: the dominant axis (access and achievement) and critical axis (identity and power). Access covers both access to mathematical learning and the resources needed to support said learning. Achievement corresponds to notions of academic performance, correctness in work, assessment, and fitting standard benchmarks for evaluating learning. Together, access achievement comprises a dominant axis that is largely formed by the dominant social-cultural factors that are reproduced in classrooms. While this framework does not capture the complexity of how social inequities are reproduced in mathematics learning, it serves as a useful entry point into this area of teacher knowledge for preservice teachers. The critical axis includes dimensions that seek to contrast the dominant axis. Identity refers to how a student's identity impacts their participation, opportunities, and learning in the mathematics classroom, while the power dimension incorporates student participation in challenging the status quo that is upheld within dominant systems of education. The critical axis serves to combat the norms and systems supported by the dominant axis, but in this way both are necessary for broad equity in mathematics education (Gutiérrez, 2002). While this framework does not reflect the complexity that discourse around equity in mathematics education often reaches, we find that it is a useful entry point into these discussions and perspectives, especially for our population of students, which is primarily white and female.

To bridge this introduction of equity in mathematics education into authentic teacher preparation, we combine the axes and dimensions of equity with Jacobs, Lamb, and Philipp's (2010) professional noticing framework for children's mathematical thinking. This framework divides the in-the-moment internalized actions behind decision-making to support mathematical learning into three interconnected components: attending, interpreting, and deciding. Attending relates to the observation of the moment, interpretation is the analysis of what was attended to, and then deciding is the action that is taken as a result of attending and interpreting. These three actions are connected and also largely internalized. Even when a teacher can vocalize what they are attending to, how they interpret the situation, and the decision they make, there are still subconscious processes that make studying how teachers participate in professional noticing complex. These frameworks come together to produce a framework for equitable noticing in mathematics education, which we use as a central scaffold for the curricular modules we developed and a structure for our assessment instrument used to investigate the modules' effectiveness.

### **Methodology**

This study utilized a descriptive research design to determine the extent that an intervention could develop equitable noticing practices and impact attitudes and beliefs towards mathematics. The MINE instructional modules were used at six institutions with varying levels of implementation within elementary mathematics methods courses. There were eight instructional modules developed that were designed to be taught in any order after the introductory module and in any quantity. Those instructors who taught seven to eight modules were considered a "full implementation" site and those that taught three to four modules were considered a "partial implementation" site, other comparison sites were used that did not teach any of the modules. It should be noted, however, that these modules were piloted in the fall semester of 2020, then



taught in all of 2021, which occurred during a global pandemic where many institutions and instructors were teaching online, hybrid, or in classrooms without the capacity to collaborate in small groups. Despite these challenges, the modules were transferred to the modality required.

Prior studies by this research team to study professional noticing used a cumulative approach to instructional modules on teacher noticing (Schack et al., 2013; Fisher et al., 2018). The instructional modules from these prior studies were intended to take approximately one hour (or one class session) and there were only three to five modules to be taught. This current project is unique in that the modules were not designed in a linear approach, rather each module focused on a mathematical topic to allow flexibility in planning for the instructor. All instructors started with Module 1 as an introductory module, and then chose the order based on how they aligned with their general pacing of the course. Each module was designed to be taught in 30 minutes or less so they could be easily integrated into the instructor's regular lesson. Each module focused on at least one component of professional noticing and at least two dimensions of equity. The mathematical focus of each ranged from topics such as ratio and proportions, functions, and fractions.

### **Participants**

Six institutions across two states in the south, Appalachian, and Midwest regions participated in the study. Three of the institutions are in urban regions and the other three are in rural areas that serve the Appalachian region. Following the demographics typically found in elementary education majors, the population predominantly identified as white and female. PSETs completed the Video-based Assessment of Noticing Equity (VANE) instrument before and after participating in the MINE modules. The data analyzed in this report comes from three semesters of implementation. Total participation included  $n = 158$  in the full implementation group,  $n = 168$  in partial implementation, and  $n = 66$  in the comparison group.

### **Instrumentation**

Building upon frameworks from prior video-based noticing assessments (Schack et al., 2013; Fisher et al., 2018), the VANE was designed as a media-anchored measure of PSETs' noticing quality with respect to dimensions of equity. For this assessment, PSETs viewed a 74-second video segment featuring racially diverse students participating in a number-talk focused on the open number sentence,  $10+10 = \underline{\quad} + 5$ . PSETs responded to five prompts that focused on observing and analyzing the mathematical moment in the video. After an introductory question, three prompts relate directly to the component processes of noticing (Questions 2-4). For example, the first prompt asks, "What mathematical thinking and actions did you observe" which corresponds directly with the component process of attending. The equity-based prompts (Questions 5 and 6), however, allow for a more synthetic approach to noticing via the coordination of multiple component processes. They ask PSETs to "describe how equity relates to this classroom scenario in different ways" (Q5) and "how might you strengthen equity in this classroom scenario if you were the teacher" (Q6). Responding to these prompts requires PSETs to create some conception of the term "equity" and connect that conception to elements of the scenario including children's mathematical thinking and their own actions as hypothetical teachers in that space. Moreover, these prompts allow for the blending of component processes (e.g. attending and interpreting in a single response) that reflects a more coherent vision of noticing as a practice (Thomas et al., 2021). Given the explicit grounding of equity in questions 5 and 6 as well as opportunities for more synthetic responses regarding the component processes of noticing, we ground our equity analyses in these prompts. Professional Noticing (PN) items were scored using previously developed flow charts for analyzing PSET short responses on a

scale of 0 to 3 that focus on student positioning and mathematical context. The equity items were scored on a scale of 0 to 4 that followed the inclusion and interpretation of the different dimensions of equity. All items were scored independently by at least two of the authors at random, and any discrepancies in scoring were discussed until an agreed upon score was obtained.

### Results and Discussion

To understand the effects of differing exposure to the MINE modules on PSETs' change in knowledge of professional noticing and equity, we analyzed the PN and equity items data of the VANE instrument using a linear mixed effects ANOVA. Timepoints (pre-module vs. post-module) were nested inside individuals which in turn were nested inside instructors. Timepoint, treatment condition, and their interaction were entered as predictors. The treatment effect was tested using an omnibus test of the timepoint  $\times$  condition effect. Separate models were fit for each item. The omnibus tests for each item are included in Table 1. Post-hoc analyses were conducted on the items that had statistically significant ANOVA tests. Mean attending scores increased by 0.114 ( $d = 0.145$ ) in the full implementation group and decreased by 0.146 ( $d = -0.186$ ) in the partial implementation group and decreased by 0.176 ( $d = -0.225$ ) in the control group.

**Table 1: ANOVA F-values and significance for VANE items**

	<i>F</i> (2, 386)	<i>p</i>		<i>F</i> (2, 386)	<i>p</i>
Attending item (Q2)	3.368	.035*	Q5 Dominant Axis score	0.543	.581
Interpreting item (Q3)	1.998	.137	Q5 Critical Axis score	1.008	.366
Deciding item (Q4)	0.505	.604	Q6 Dominant Axis score	1.169	.312
			Q6 Critical Axis score	1.719	.181

These results align with prior research that shows that some professional noticing components can be taught in the course of a semester; however, in the previous research, the interpreting component generally showed a significant increase as well, while deciding remains a more difficult construct to measure (Schack et al., 2013; Fisher et al., 2018). More interesting is that the partial implementation scores for attending decreased, thus it appears that in order to find a measurable increase, more sustained focus on professional noticing should occur. When reviewing the equity components, while there were no significant increases, there was a glimmer of hope in the area of the critical axis in that while the partial implementation and control group decreased, there was an increase in the score for the full implementation group for Q5.

Additionally, all three implementation groups increased in Q6. It should be noted that, this is a first attempt at measuring equitable thinking using Gutierrez's (2002) dimensions of equity.

Many could argue that you cannot measure equitable thinking, but we feel this study is a beginning of potential numeration of this theoretical construct.

### References

Fisher, M. H., Thomas, J., Schack, E. O., Jong, C., & Tassell, J. (2018). Noticing Numeracy Now! Examining Changes in Preservice Teachers' Noticing, Knowledge, and Attitudes. *Mathematics Education Research Journal*, (30)2, 209-232.

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

- Gutiérrez, R. (2002). Enabling the practice of mathematics teachers in context: Toward a new equity research agenda. *Mathematical Thinking and Learning*, 4, 145-187.
- Gutiérrez, R. (2009). Framing equity: Helping students “play the game” and “change the game.”. *Teaching for Excellence and Equity in Mathematics*, 1(1), 4-8
- Jacobs, V. R., Lamb, L. L. C., & Philipp, R. A. (2010). Professional noticing of children’s mathematical thinking. *Journal for Research in Mathematics Education*, 41, 169-202.
- Louie, N., Adiredja, A. P., & Jessup, N. (2021). Teacher noticing from a sociopolitical perspective: the FAIR framework for anti-deficit noticing. *ZDM–Mathematics Education*, 1-13
- Schack, E. O, Fisher, M. H., Thomas, J., Eisenhardt, S., Tassell, J., & Yoder, M. (2013). Prospective Elementary School Teachers’ Professional Noticing of Children’s Early Numeracy. *Journal of Mathematics Teacher Education*, (16), 379-397.
- Schack, E. O, Fisher, M. H., Wilhelm, J. (Eds.) (2017). *Teacher noticing – bridging and broadening perspectives, contexts, and frameworks*. New York, NY: Springer.
- Thomas, J., Dueber, D., Fisher, M.H., Jong, C., & Schack, E.O. (2021). *Professional Noticing Coherence: Exploring Relationships between Component Processes*. *Mathematical Thinking and Learning*.
- van Es, E., Hand, V., & Mercado, J. (2017). Making visible the relationship between teachers’ noticing for equity and equitable teaching practice. In E. O. Schack, M. H. Fisher, & J. A. Wilhelm (Eds.), *Teacher noticing: Bridging and broadening perspectives, contexts, and frameworks*(pp. 251-272). New York, NY: Springer

## PARENTS AND TEACHERS DOING MATH TOGETHER

Erin Turner  
University of Arizona  
eturner@arizona.edu

Pilar Ester Mariñoso  
Univ. Camilo José Cela  
pester@ucjc.edu

Marta Civil  
University of Arizona  
civil@math.arizona.edu

Beatriz Quintos  
University of Maryland  
bquintos@umd.edu

Fany Salazar  
University of Arizona  
fanysalazar@arizona.edu

Maura Varley Gutiérrez  
University of Arizona  
marav@arizona.edu

*Collaborative partnerships between families and teachers have the potential to support and transform students' mathematics learning experiences. This study focused on interactions among mothers and teachers of multilingual elementary grade students who participated together in workshops focused on teaching and learning mathematics. We analyzed participants' engagement in open-ended, culturally responsive mathematics tasks designed to foster collaboration and equitable participation. We describe several findings related to the potential of these tasks, including how they supported opportunities a) to recognize one another's strengths; b) to challenge traditional power differentials between parents and teachers, and c) to collaboratively generate mathematical ideas. We also discuss challenges that arose, and implications for the design of collaborative learning experiences for teachers and parents.*

Key Words: equity, inclusion, diversity; problem solving; culturally relevant pedagogy; professional development

This study is part of a broader research project focused on an innovative mathematical partnership that engages teachers, parents, and multilingual children in elementary schools in underserved, predominantly minoritized communities. A central aim of this work is to develop parent-teacher partnerships that enhance mathematics learning for multilingual students.

Traditional school-based approaches to parental engagement can be particularly problematic for multilingual families from underserved communities, as they are often school-centric and adhere to deficit-based views about families' support of their children's learning (Olivos, 2006). Needed are innovative, transformative models for parent-teacher partnerships that build on the strengths and resources that participants bring, and that recognize families' deep commitment to education (González et al., 2005; Jiménez-Castellanos et al., 2016).

To address these challenges, our partnership model integrates two approaches to advancing equity in education- funds of knowledge (González et al., 2005) and positioning theory (Davies & Harré, 1990; Van Langenhove & Harré, 1994). While both theories have been applied in mathematics education, they have not been integrated in a parent-teacher partnership program with the aim of enhancing mathematics learning for multilingual students.

Funds of knowledge refers to the historical accumulation of skills, knowledge, culture, and ways of interacting in communities and households (Vélez-Ibáñez & Greenberg, 1992). Applied to school-family partnerships, a funds of knowledge perspective explicitly contests deficit views of culturally and linguistically diverse communities through a focus on families' knowledge and experiences as intellectual resources that support learning (Civil & Andrade, 2003; González et al., 2005). Research guided by a funds of knowledge approach emphasizes the importance of two-way dialogues between families and teachers, and opportunities to build relationships and

trust (Delgado-Gaitán, 2012; Jiménez-Castellanos et al., 2016). This work also emphasizes the complexity of the transformation of household knowledge into mathematical knowledge for school practice (González et al., 2001). Other challenges include the power issues that may arise when parents and teachers work together on mathematics (Civil & Bernier, 2006), and that parents' mathematical resources are often unnoticed by teachers, often due to language differences or the informal nature of the ideas (Quintos, et al., 2019). Our work builds on the promising directions highlighted in prior research guided by funds of knowledge approaches, while also considering documented challenges (Civil & Quintos, 2009; Quintos, et al., 2019).

To deepen our understanding of these challenges, we draw on positioning theory as it focuses on how social interactions influence opportunities to participate, which in turn impact identity development and learning. Positioning theory (Davies & Harré, 1990; Van Langenhove & Harré, 1994), explores the narratives that people use to position themselves and others in social interactions. Positioning theory attends to how participants use multiple forms of discourse (oral language, gestures, written text), in moment-to-moment interactions, to position themselves and others in ways that over time, lead to storylines about each participant's rights or duties to act in particular ways. In mathematics education, positioning theory has been used to understand how teachers position students in ways that affirm their mathematical competence, or how students position themselves, or one another, in ways that support or undermine their opportunities to contribute ideas (Chval et al., 2015; Esmonde & Langer-Osuna, 2013; Turner et al., 2013). While positioning theory has been applied to analyze interactions within mathematics classrooms, in our partnership model we use positioning theory to understand interactions among mothers and teachers in project workshop sessions, with particular attention to distribution of power and opportunities to learn from one another.

## Methods

### Context and Participants

This research is part of a larger NSF-funded, collaborative research project that involves three universities, and their partner school districts in diverse geographic regions of the country. This analysis focuses on workshops with mothers and teachers at one site, a large public school district in the southwest. Participants included 9 2<sup>nd</sup> through 5<sup>th</sup> grade teachers, from 4 partner schools, who range in teaching experience from 3 to 25 years. The partner schools serve a predominantly Mexican American, Mexican immigrant student population, many of whom have Spanish as home language. While three of the schools use English as the language of instruction for all subjects, one of the partner schools has a dual language model and mathematics is taught in Spanish. Four teachers spoke both Spanish and English, three spoke only English, and two spoke English and other non-Spanish languages. Participants also included 11 mothers (1 was a grandmother) of 2<sup>nd</sup> through 5<sup>th</sup> grade multilingual students. All mothers spoke Spanish as a home language, with varying levels of English/Spanish proficiency. Five of the mothers had attended school in Mexico and had been living in the United States for over 9 years. Mothers were invited to participate by the participating teachers.

### Workshop Activities

This study focuses on three workshop sessions in Fall 2022. Each session included opportunities for mothers and teachers to work together on an open-ended mathematics task. Other activities included time to reflect together on mathematics learning experiences, to discuss classroom videos, and to build relationships by sharing personal stories and learning about one another's background. In this analysis, we focused specifically on mothers' and teachers' interactions during the joint mathematics tasks. We designed these tasks with several principles

in mind. First, the tasks were open-ended, with multiple possible answers depending on decisions made by participants. Second, the tasks reflected mathematics content relevant to our target grade levels (e.g., fractions, multiplication, and division), and connected to familiar practices or contexts in the local community. We specifically aimed to connect to contexts that might be familiar to mothers, to position their knowledge and experiences as resources.

During each workshop, a facilitator (a member of the research team) presented the mathematics task, in both Spanish and English, often using images or video clips to build a shared understanding of the context. The facilitator then invited participants to share experiences relevant to the task with the group, and to brainstorm possible strategies or problem-solving approaches. Next, parents and teachers worked on the task in collaborative groups (typically one or two teachers with two or three mothers). The facilitators circulated among groups, supporting participants with questions, and encouraging collaboration. Following small group work, groups shared solution strategies and compared the different approaches.

### **Collaborative Mathematics Tasks**

In the first workshop, the mathematics task focused on making paper flowers, like those used in Day of Dead (*Día de los Muertos*) celebrations. The problem posed was: How many flowers can you make from a package of 24 sheets of tissue paper? To solve the task, participants made decisions about the size of the flowers and the number of layers of paper for each flower. In the second workshop, the mathematics task focused on a photograph of a sugar skulls display. The display included skulls of different sizes, arranged in multiple layers, and in rows and columns. (Sugar skulls are a common addition to Day of the Dead altars). The problem posed was: How much sugar would you need to make the sugar skulls in the display? Participants estimated the number of skulls of different sizes, and then used a sugar skull recipe to figure out how much sugar was needed. The mathematics task in the third workshop focused on the rows of brightly-colored ribbons that adorn ballet folklórico skirts, which are used for traditional dances in various regions of Mexico. The problem posed was: How much ribbon do you need for one skirt? Participants used sample skirts as well as information about the length of the edge of the skirt compared to the length around the waistline or middle to estimate the ribbon needed.

### **Data Sources and Analysis**

Data sources included video recordings of parents and teachers as they worked on the three mathematics tasks and shared their solutions. Recordings were summarized and transcribed for analysis. Transcripts were imported into Atlas-TI, a qualitative research program, for coding and analysis. Our analysis was guided by the following research questions: As parents and teachers engaged in collaborative mathematics tasks, to what extent did they have opportunities to:

- learn about one another's experiences, and to recognize the strengths and intellectual resources that each participant brings?
- challenge and reconstruct traditional power differentials between parents and teachers?
- collaboratively generate ideas and build knowledge?

Codes attended to the ideas and experiences shared by participants (e.g., personal stories, knowledge related to task context, mathematical ideas), and how participants positioned themselves and one another as they contributed ideas, including how power was distributed equitably or inequitably through these positionings. Differences in codes and interpretation were

resolved through discussion between researchers, and emerging themes were triangulated across data sources (across small groups, and across workshop sessions). In the next section, we describe our findings, organized by salient themes related to each research question. While the themes reported were evident across the three workshop sessions, for the purpose of this report, we focus on examples from the first mathematics task, making paper flowers.

### Findings

#### Facilitator prompts and task contexts supported telling stories and sharing strengths

Across each of the three math problem solving sessions, we identified instances when both teachers and parents shared personal stories relevant to the task context to push forward the work of their group. These instances seemed to be supported by moves facilitators made as they launched each task. For example, in the paper flowers task, the facilitator asked participants, in Spanish and English, to raise their hand if they had made paper flowers in the past. When few participants responded, another facilitator offered further encouragement, (“*No sean tímidas, yo sé que varias de Uds. sí. Venga, venga.*” (Don’t be shy. I know some of you do. Come on, come on. (Said with humor in a jovial manner))). This prompted several mothers and teachers to raise their hands and share related experiences. Brooke, a mother noted that she knew how to make flowers with a different kind of paper. Rocio, a teacher shared that when she was young her mother taught her to make flowers with kleenex. Several other participants indicated similar experiences. In response, the facilitator positioned the mothers and teachers as experts who had valuable experiences relevant to the mathematical task (“We have Brooke (mother) as an expert and Leonor (grandmother), and Rocio (teacher) [proceeds to name other mothers and teachers].. *y aunque la experiencia sea diferente, ellas tienen esas experiencias*” (and even though the experiences may be different, they still have these experiences).

This positioning by the facilitators seemed to prompt participants to continue to draw on their prior experiences as they worked on the task in small groups. For example, in one small group that included three teachers, a mother and a grandmother, the group was trying to decide how to cut the large sheet of tissue paper to make smaller rectangles that could serve as layers. They had multiple ideas about the size of the paper and how many layers they should use. Darla (one of the teachers) paused the conversation and explicitly asked Leonor (the grandmother who had previously indicated her experience making flowers), to share how she made flowers at home.

Leonor initially described an informal process based on visually cutting paper into different sizes, but when one of the teachers (Kevin) suggested partitioning the paper into 8 smaller parts, Leonor intervened and shared more mathematical details about her method. While most of the participants in this group were bilingual (Spanish/English) and moved between languages as they worked on the task this portion of the group conversation occurred mostly in English.

Darla (teacher): (to Leonor) I know you make them at home, how big do you make them?

Leonor (grandmother): different sizes, I just get the paper and I make them different sizes.

... Darla (teacher): You just cut them, you don’t think about the math? (smiles)

Leonor (nods her head, and laughs): it’s like *un puñito de esto* (it’s like a pinch of this), a little bit of this, no measurements ...

Kevin (teacher): We could probably just do it like this [points to a drawing on Darla’s paper, where she has used lines to partition the sheet of paper into 8 smaller sections]

Leonor (grandmother): But how many inches is that one? That’s too small.

Darla: This paper (the whole sheet), they said we could cut it into fourths or sixths or eighths.

Leonor: I think sixths, look, it's already folded in 6 parts (opens a sheet of tissue paper that was previously folded to show that the paper was folded in 6 parts), and it [the size] is perfect. ... What I do is if I get big pieces I get more paper [layers] to make a big flower. If you don't put like 5 [layers] it's not going to look good.

The group then discussed different numbers of layers and ultimately decided (likely based on Leonor's suggestion) to make small-medium size flowers that have 6 layers each. We found similar patterns across groups and across tasks, where participants - sometimes teachers, but often mothers, shared experiences relevant to the context that helped the group make decisions to move the problem solving forward. In some instances, these experiences were explicitly invited by another member of the group, as when Darla asked Leonor to share. But in other instances, participants spontaneously shared stories as they began to explore the task. For instance, Rocío (teacher) shared that her mother, who taught her to make flowers, always said that the more layers, the fuller the flower ("*entre más hojas, se va a ver más lleno*"). In both cases, this sharing of stories not only supported the groups' problem solving, but also helped participants learn about one another's knowledge and experiences -which over time, we conjecture, may support teachers to recognize parents' intellectual resources that support their children's learning.

### **Open-ended tasks created the potential to restructure power dynamics, though this potential was not always maintained**

Our goal in designing open-ended, culturally relevant mathematics tasks was to foster equitable problem-solving spaces where both mothers and teachers contributed ideas, made decisions, and shaped solutions. Since none of the tasks included all the necessary information, but rather required decisions, assumptions, and consideration of multiple options, groups often began with a reciprocal dialogue where mothers and teachers worked together to make sense of the task. For example, the small group highlighted above began by re-reading the task, and then posing initial thoughts or questions. Leticia (mother) suggested that they could use rulers to measure the tissue paper sheets. Kevin (teacher) noted that they would need to determine how to cut the sheets, and Valeria (teacher) and Leonor (grandmother) suggested possible sizes (e.g., 6-inch by 4-inch rectangles). This initial sharing was followed by the invitation for Leonor to share how she made flowers at home. However, this reciprocal dialogue was short-lived, as two of the teachers (Kevin and Darla) began to guide the group's process, positioning themselves in a "teacher-like" role, and positioning the mother and grandmother to "follow along" with the mathematics. Kevin began this shift by naming "the math we need to figure out" and Darla continued, inviting the group to "do some math" as she stated and recorded the first problem they needed to figure out (If we cut 24 sheets into 6 pieces each, how many total pieces do we have?).

Kevin: So do we want to have an equal number of sheets in each flower, so there is nothing left over? (Other participants nod to affirm) Ok, so that is the math we need to figure out.

Darla: (taking a marker, to write on the group solution paper). So, we are deciding that we are going to cut it in 6? (writes "cut in 6" on group paper)

Kevin: 6, uh-huh.

Darla: But we have to keep in mind we're trying to figure out how many flowers would you make out of 24 sheets? ...

Leonor: 24 sheets. So we have to get like a pair of numbers, *verdad?* (right?) (Looking at Darla, waiting for Darla to answer her question).

Darla: We might have to get even numbers to do the math, that's a good idea. Do you want to do some math?

Leonor: Yes.



Darla: So we have 24 sheets, right? And we're going to cut each sheet into 6 pieces.

(Writes, "Decisions: Each sheet will be cut into sixths")

Darla and Kevin proceed to do the computation without input from the other participants. While Leonor previously positioned herself as an expert as she shared her flower-making knowledge, in this excerpt she accepted the role of student, asking Darla to verify her ideas (*verdad?*), and accepting Darla's invitation to follow along as she "does some math." This shift in power dynamics seemed prompted by the teachers' desire for the group to work together towards a single solution. While the open-ended task created the potential for participants to explore different decisions which could have led to different solutions, the teachers responded by narrowing the task and restructuring traditional teacher/parent power differentials.

In other instances, groups maintained the initial, reciprocal dialogue throughout their problem-solving work. In contrast to the prior example, participants strove to keep the task open to allow for diverse decisions among group members. When suggestions to agree on a common decision or solution arose, they were rejected in favor of allowing group members to explore different possibilities. This maintained a more open space for mothers and teachers to contribute and evaluate ideas. For example, Rocio and Nishaan (teachers) and Bárbara and Veronica (mothers) began by acknowledging that many solutions were possible. Next, they brainstormed several possibilities (large flowers with many layers, or tiny flowers with smaller rectangles).

When one of the teachers (Nishaan) suggested that they make a group decision about the number of layers, other participants pushed back, insisting that "We don't have to do the same size."

Rocio (teacher): *No hay una sola forma* (There is not just one way)

Verónica (mother): *Si las vamos a hacer de 5 (capas) cada una, o si vamos a usar las 24 hojas para hacer una sola flor grande?* (Are we going to make them 5 layers each, or are we going to use the 24 sheets to make one large flower?) ... *Eso es lo que ellos quieren, que no hay sólo una respuesta.* (That is what they want, that there is not just one answer)

Rocio: *Que no hay una sola no más.* (That there is not just one) ...

Bárbara (mother): (agrees) We can make [them] different because it is not 1 correct answer. We can make 1, we can make [it with] 24 layers, or we can make tiny little flowers and make 100 *flowercitas* (little flowers).

Nishaan (teacher): So you guys want to decide how many layers we are going to do? Or not? Rocio: Well, she (Bárbara) wants to make her own...

Bárbara: We don't have to do the same size

Rocio: No, we don't have to do the same size. You can do your size; however you want and just explain however many layers you want.

Bárbara: I don't like that question, because like... ? (Gestures, holding up both hands and raising shoulders to indicate that the math task is lacking direction)

Verónica: There is so much left out. There are so many right answers. So, what we are trying to say is we either put all our papers together and make one flower, or we just cut them in different sizes and make a bunch of different flowers?

Bárbara: *Me intriga más resolver el problema matemático que hacer la flor* (I am more interested in solving the math problem than making the flower)

While both mothers expressed puzzlement, or even frustration towards the task (i.e., "I don't like that question" and "There is so much left out"), they opted to keep the task open. Following

this exchange, participants explored multiple solutions, sometimes working independently, and other times coming together to share their discoveries. They made tables to organize possible solutions (Nishaan, teacher, and Veronica, mother), and folded the paper to compare different flower sizes (Rocio, teacher, and Bárbara, mother). Their efforts to maintain the openness of the task created space to pursue diverse ideas, which seemed to support a more equitable distribution of power.

This led to collaborative generation of knowledge, which we explore next.

### **Maintaining space for diverse ideas supported co-construction of knowledge**

Each math problem solving session ended with a whole group discussion where participants shared their thinking. While small groups often came together to share a group solution, we found that it was not a push for consensus that facilitated the co-construction of knowledge, but rather interactions that created space to explore multiple ideas. For example, in the group highlighted above, Bárbara (mother) and Rocio (teacher) folded sheets of tissue paper to explore different sized paper flowers. They worked in parallel, each experimenting with different folds and occasionally exchanging ideas. At one point, Bárbara noticed that while they both folded the paper in eighths, the results were flowers that appeared to be different sizes. This generated a rich dialogue between Bárbara, Veronica, and Rocio about how the direction of the folds into eighths impacts the shape of the resulting rectangles, but not the area.

Bárbara (mother): *Ella (Rocio, teacher) la dividió en 8 (rectángulo) y yo lo dividí en 8 (cuadrado), Nuestros ojos fueron diferentes.* (She divided it in 8 (rectangle shape) and I divided it in 8 (square shape). Our eights were different.)

Verónica (mother): *Me está saliendo un nudo en el cerebro.* (I am getting a headache) ... Rocio (teacher): *¿Y qué hay de la otra medida? Tienes que tener dos medidas.* (starts to

gesture the length and the width) (And what about the other measurement? You must have two measurements.)

Bárbara: *El mío no le he medido, pero yo le calculo 6x5 y el de ella también lo dividió en 8.* (I did not measure mine, but it looks like 6 by 5, and she also divided it in 8).

Facilitator1: *Se supone que todos los papeles son del mismo tamaño, si lo dividieron en 8 las dos se supone...* (All the sheets are supposedly the same size, so if you both divided it in 8, then it should be. ...)

Bárbara: *Porque nuestros doblados son distintos. Yo lo doblé así, y ella lo dobló así* (Because our folds were different. I folded it like this, and she folded it like this)

Verónica: *A lo largo.* (The long way) Like the area. The area is the same, just different shape. Rocio: (nods to agree)

Verónica: So you know, if you have 5 rows of blocks, you know, the way some of the kids are learning, you have 5 blocks going this way and you have 5 blocks going this way, you have your area, right? 25, so it's perfectly square. (Gestures to show rows and columns). But then and you make a rectangular, you are still using the same amount of squares, so it means you still have the same area, just a different shape.

We found similar patterns across groups and tasks – when mothers and teachers kept tasks open and explored diverse ideas, their interactions were often characterized by a collaborative generation of knowledge, as in the previous exchange. This co-construction of mathematical ideas was also evident later in the group's discussion, as Bárbara, Verónica, and Nishaan became increasingly interested in systematically representing a range of possible solutions. Verónica

continued to record possible solutions, repeatedly dividing the tissue paper in half to make smaller and smaller flowers, noting, “You go from one big flower, which is the least you can do, and the more you keep going, the more you have.” She also described the difference between what is possible mathematically “you could even do 128” (cutting the paper into 128 parts), and what is realistic “obviously it is going to be super small, but in the math I mean, it could be done.” Nishaan pursued a similar strategy, starting with a single 6-layer flower, and then cutting the tissue paper in half to make 2 smaller 6-layer flowers, and then in half again, to make 4 smaller flowers 6-layer flowers. After listening to Nishaan describe her process, Verónica noted her agreement, and the similarities in their approaches. Meanwhile, Bárbara attempted to write a general formula for the total number of flowers, given variation in how the tissue paper was divided (number of parts), and the number of layers per flower. She defined different variables and explored various ways to combine them in an equation. She insisted that a general equation to represent all the solutions of the group had to be possible, “*No sé si me ecuación es correcta pero he estado buscando porque tiene que haber, tiene que ....* I don’t know if my equation is correct, but I have been searching because there has to be an equation, there has to be).

We find this interaction particularly important because as the mothers and teachers continued to explore multiple ideas – without a focus on consensus – this led to the co-construction of rich mathematical ideas, such as ideas about the maximum and minimum number of solutions, the difference between mathematical and real-world solutions, and generalized equations. We conjecture that these collaborative exchanges can foster partnerships between parents and teachers that could enhance ongoing work to support children’s mathematics learning.

### **Discussion**

Parents and teachers are often called to be partners in children's education, yet the school system often positions them in asymmetrical relations, particularly in marginalized communities. Furthermore, efforts to disrupt these historical power asymmetries are often superficially successful (Quintos et al., 2019). The analysis of the interactions of mothers and teachers as they participate in mathematical tasks shows the possibilities and challenges of creating spaces to support their collaboration. Their participation in open-ended, culturally relevant tasks allowed parents and teachers to learn about one another’s experiences and strengths. Also, the individual disposition of teachers or parents to share and listen to each other is another key factor, which can be reinforced by the facilitation of the tasks.

We described several contrasting interactions as mothers and teachers solved mathematical tasks. In one group, teachers adopted a “teacher-like” role, positioning parents as students or listeners. The teachers defined the question and led the problem solving. The discourse was mostly controlled by the teachers and consequently, the task was interpreted more narrowly. In the second group, mothers and teachers maintained the openness of the task, which allowed them to explore and construct diverse mathematical ideas. This openness of the task seemed key to disrupting hierarchical power relations, and counters narrow views of school mathematics that exclude different processes for sense making. Mothers were positioned as knowledgeable contributors of ideas, countering deficit views of families in underserved communities.

While the interactions noted above were supported by multiple features of the workshops, including the privileging of both Spanish and English, time to build relationships, and to discuss mathematics learning experiences, we conjecture that the open-ended tasks and connections to relevant contexts offered support. Participants had opportunities to see beyond their teacher or parent role, and to engage with one another in a more informal, personal manner. These types of interactions have the potential to nurture connections between parents and teachers (Civil &

Bernier, 2006; Civil & Quintos, 2009), and support teachers to perceive families as holders of mathematical knowledge (Civil & Andrade, 2003; González et al., 2001).

### Acknowledgments

This research was supported by the National Science Foundation grants (DRL-2010417, DRL-2010260, DRL-2010230).

### References

- Chval, K.B., Pinnow, R.J. & Thomas, A. (2015). Learning how to focus on language while teaching mathematics to English language learners: a case study of Courtney. *Mathematics Education Research Journal*, 27, 103–127. <https://doi.org/10.1007/s13394-013-0101-8>
- Civil, M., Andrade, R. (2003). Collaborative Practice with Parents. In: Peter-Koop, A., Santos-Wagner, V., Breen, C., Begg, A. (Eds) *Collaboration in Teacher Education. Mathematics Teacher Education, vol 1*. Springer. [https://doi.org/10.1007/978-94-017-1072-5\\_11](https://doi.org/10.1007/978-94-017-1072-5_11)
- Civil M., Bernier, E. (2006) Exploring Images of Parental Participation in Mathematics Education: Challenges and Possibilities. *Mathematical Thinking and Learning* 8(3), 309-330 doi: 10.1207/s15327833mtl0803\_6
- Civil, M., Quintos, B. (2009) Latina mothers' perceptions about the teaching and learning of mathematics: Implications for parental participation- Culturally responsive mathematics education. Routledge.
- Civil, M. (2002). Culture and mathematics: A community approach. *Journal of International Studies*, 23(2), 133-148
- Davies, B., & Harré, R. (1990). Positioning: The discursive production of selves. *Journal for the Theory of Social Behaviour*, 20(1), 43–63. <https://doi.org/10.1111/j.1468-5914.1990.tb00174.x>
- De Freitas, K.A., Alves, A., (2010) Reflexiones sobre el papel de la contextualización en la enseñanza de ciencias. *Enseñanza de las Ciencias*. 28(2), 275
- Delgado Gaitan, C. (2012). Culture, Literacy, and Power In Family–Community–School Relationships, *Theory Into Practice*, 51, 4, 305-311. Retrieved from: <http://dx.doi.org/10.1080/00405841.2012.726060>
- Ensign, J. (2005). Helping teachers use students' home cultures in mathematics lessons: Developmental stages of becoming effective teachers of diverse students. In A. Rodriguez, & R.S. Kitchen (Eds.), *Preparing mathematics and science teachers for diverse classrooms: Promising strategies for transformative pedagogy*, 225-242. Lawrence Erlbaum.
- Esmonde, I., & Langer-Osuna, J. M. (2013). Power in Numbers: Student Participation in Mathematical Discussions in Heterogeneous Spaces. *Journal for Research in Mathematics Education*, 44(1), 288–315.
- González, N., Andrade, R., Civil, M., & Moll, L. (2001). Bridging funds of distributed knowledge: Creating zones of practices in mathematics. *Journal of Education for Students Placed at Risk*, 6(1-2), 115–132. [https://doi.org/10.1207/S15327671ESPR0601-2\\_7](https://doi.org/10.1207/S15327671ESPR0601-2_7)

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

- González, N., Moll, L. C., Amanti, C., & Neff, D. (2005). *Funds of Knowledge: Theorizing Practices in Households, Communities, and Classrooms*. Routledge.  
<https://doi.org/10.4324/9781410613462>
- Greenberg, J.B. (1989, April). Funds of knowledge: Historical constitution, social distribution, and transmission. Paper presented at the annual meetings of the Society for Applied Anthropology, Santa Fe, NM.
- Harré, R., Moghaddam, F.M., Cairnie, T.P., Rothbart, D., Sabat, S.R. (2009). Recent Advances in Positioning Theory. *Theory and Psychology*, 19, 31 - 5.
- Jiménez-Castellanos, O., Ochoa, A. M., & Olivos, E. M. (2016). Operationalizing transformative parent engagement in Latino school communities: A case study. *Journal of Latino/Latin American Studies*, 8(1), 93–107. <https://doi.org/10.18085/1549-9502-8.1.93>
- Moll, L., Amanti, C., Neff, D., Gonzalez, N.(1992) Funds of knowledge for teaching: Using a qualitative approach to connect homes and classrooms, *Theory Into Practice*, 31(2), 132-141, DOI:10.1080/00405849209543534
- Moll, L.C., Vélez-Ibáñez, C., Greenberg, J., Whitmore, K., Saavedra, E., Dworin, J., & Andrade, R. (1990). *Community knowledge and classroom practice: Combining resources for literacy instruction*. OBEMLA
- Nasir, N. S., Hand, V. M. (2006). Exploring Sociocultural Perspectives on Race, Culture, and Learning. *Review of Educational Research*, 76(4), 449–475.  
<https://doi.org/10.3102/00346543076004449>
- Olivos, E. M. (2006). *The power of parents: A critical perspective of bicultural parent involvement in public schools*. Peter Lang.
- Quintos, B., Civil, M., & Bratton, J. (2019). Promoting change through a formative intervention: Contradictions in mathematics education parental engagement. *Mind, Culture, and Activity*, 26(2), 171–186. <https://doi.org/10.1080/10749039.2019.1602656>
- Ramirez, A. Y. (2003). Dismay and disappointment: Parental involvement of Latino immigrant parents. *The Urban Review*, 35(2), 93–110.
- Turner, E., Dominguez, H., Empson, S., & Maldonado, L. (2013). Latino/a bilinguals and their teachers developing a shared communicative space. *Educational Studies in Mathematics*, 84(3), 349–370. doi:10.1007/s10649-013- 9486-2
- Van Langenhove, L. & Harré, R. (1994). Cultural stereotypes and positioning theory. *Journal for the Theory of Social Behaviour* 24(4), 359-372.
- Vélez-Ibáñez, C. G., & Greenberg, J. B. (1992). Formation and Transformation of Funds of Knowledge among U.S.-Mexican Households. *Anthropology & Education Quarterly*, 23(4), 313–335. <http://www.jstor.org/stable/3195869>

## REHUMANIZING MATHEMATICS THROUGH EMBODIMENT-FOCUSED NOTICING

Liza Bondurant  
Mississippi State University  
lb2206@msstate.edu

Diana Moss  
University of Nevada, Reno  
dmoss@unr.edu

Claudia Bertolone-Smith  
CSU-Chico  
cmbertolone-  
smith@csuchico.edu

Lisa Poling  
Appalachian State University  
polingll@appstate.edu

Hortensia Soto  
Colorado State University  
Hortensia.soto@colostate.edu

Jonathan David Troup  
CSU-Bakersfield  
jtroup@csub.edu

*In this manuscript we, six math teacher educators (MTEs), share a collaborative self-study investigation into increasing our embodiment-focused noticing. Initially, we individually viewed cognitive interview videos and recorded our embodiment-focused noticings. Next, we met as a group to share, compare, and merge our noticings. We used the merged noticings to develop an exemplar and a rubric. We will discuss how layering an embodiment lens over our noticings can contribute to rehumanizing mathematics.*

Keywords: Teacher Noticing, Equity, Inclusion, and Diversity

### Background and Theoretical Perspectives

We are six math teacher educators (MTEs) with different lived experiences and areas of expertise. As critically conscious MTEs, we acknowledge that social, cultural, and political factors impact our noticings (Louie et al., 2021; Seda & Brown, 2021). We believe our differences strengthened this collaborative self-study because our diverse perspectives pushed us to broaden our noticings. We embarked on this collaboration due to a common problem of practice. The preservice mathematics teachers (PSTs) we work with often notice from a narrow or deficit perspective, exclusively attending to only what students say or write, often ignoring valuable embodiment evidence. Our collaborative self-study draws upon the literature on teacher noticing, rehumanizing mathematics, and embodiment. The conference theme of “Engaging All Learners” is embedded throughout this theoretical brief report in that we discuss that developing a broad assets-based perspective on noticing that considers student embodiment can promote rehumanizing mathematics for all levels of mathematics educators.

### Noticing

Noticing is multifaceted, but one essential aspect is what it means to acknowledge a critical event in the noticing model. A critical event, as defined by Rotem and Ayalon (2023), is the ability, through attending and interpreting, to fully see a myriad of evidence that represents an individual’s mathematical thinking and provides an opportunity to facilitate instructional strategies that support deeper understanding. It is the crucial moment in which an observer can fully see the way in which a task is being evaluated and performed, allowing instructional strategies to match the needs of the students and push their thinking forward (van Es, 2011). Noticing mathematical understanding can never be isolated to the physical act of solving a task, because individuals bring past histories and social structures that operate within their schema with them as they engage in mathematics. The triad of noticing (attend, interpret, respond, or AIR) requires the observer to pay full attention to significant mathematical details along with the personal effects demonstrated by the student when approaching a task (Jacobs et al., 2010).

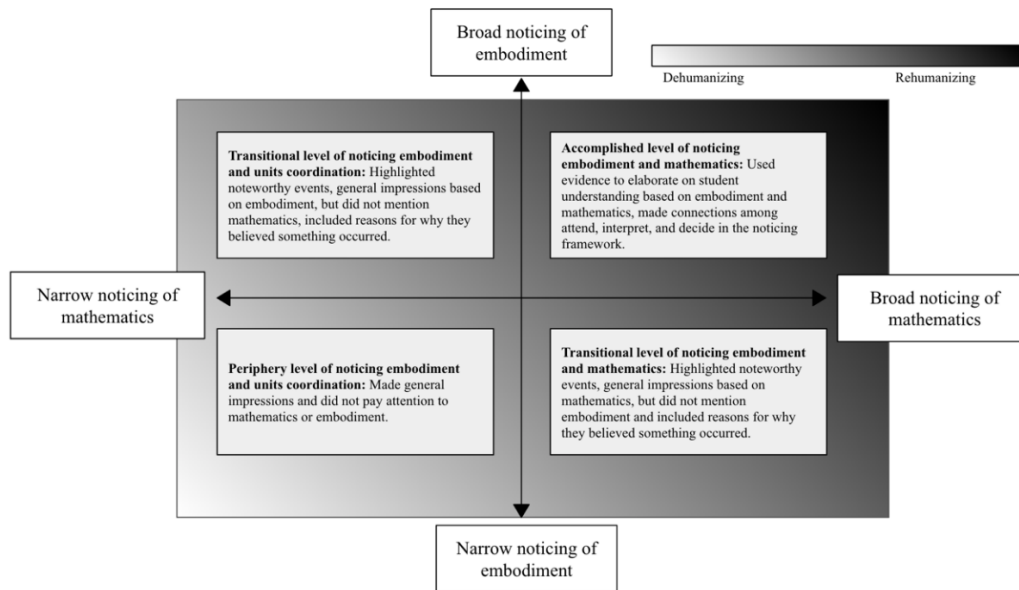
Critical events, also identified as pivotal teaching moments (Stockero & Van Zoest, 2013) occur when teaching episodes provide a transition that allows instructors to change the trajectory of student thinking and consider all dimensions found within the structure, which include cognitive, affective, and social aspects of the teaching and learning process.

### **Rehumanizing Mathematics**

To promote more equitable noticing, we layered Gutiérrez's (2018) rehumanizing mathematics (RM) framework as a lens over the AIR framework. There are eight dimensions in the RM framework. The three dimensions most relevant to our self-study include: (1) emotions and body, (2) participation and positioning, and (3) broadening mathematics. Attending to students' body movements can promote rehumanizing mathematics by broadening what counts as mathematical understanding and who is considered mathematically competent. This is because children often show their mathematical thinking in physical movements (e.g., finger counting), facial expressions, or utterances of thought before they can cognitively process and express a verbal explanation. According to the RM framework, teachers can promote equity by holding high expectations for all students, avoiding deficit views of diverse students, focusing on sensemaking instead of answer-getting, valuing diverse contributions, leveraging early conceptions, and acknowledging and positioning all learners as brilliant mathematicians.

### **Embodiment**

Embodiment refers to the use of "body-based resources to make meaning and to connect new ideas and representations to prior experiences" (Nathan, 2022, p. 4), including gestures, body forms, simulations, and the use of materials (e.g., manipulatives). The literature suggests that attending to embodiment may promote equity by expanding educators' ideologies and pedagogies. Noticing embodied actions can serve as micro and macro affirmations that promote equity and access (Abrahamson, et al., 2020). Attending to embodiment may expand educators' views of evidence of learners' mastery of the counting principles (Gelman & Gallistel, 1978). Embodied cognition has become prevalent in math education literature (e.g., Abrahamson et al., 2020; Alberto et al., 2022; Alibali & Nathan, 2018; Nemirovsky & Ferrara, 2009; Nemirovsky et al., 2012; Tall, 2008), though it remains a broadly interpreted term. Noticing how students coordinate units is related to embodiment because students initially rely on concrete objects, their fingers, and visual representations to explain their mathematical reasoning. Therefore, an observer who uses embodiment as evidence is capturing a more accurate assessment of students' understanding of mathematics. Below is a figure of the relationships among noticing of embodiment and RM (adapted from Moss & Poling, 2019; Thomas et al., 2017).



**Figure 1: Rehumanizing Noticing through Embodiment Methodology**

We adopted a collaborative self-study methodology for this research (Butler & Bullock, 2022). According to Kosnik and colleagues (2009), the characteristics of self-study methodology include openness, multiple perspectives, collaboration, paradoxical nature, reframing, and postmodernism. Throughout this self-study, we pushed each other to keep our minds open to perspectives that differed from our initial stances. Additionally, collaboration with each other played a critical role in helping us consider concepts from multiple perspectives. The nature of this self-study was paradoxical because as “critical friends” we pushed each other to question and reframe our initial beliefs. Finally, we adopted postmodern assumptions, recognizing how our life experiences influence our beliefs and our beliefs are not necessarily the “best practices.”

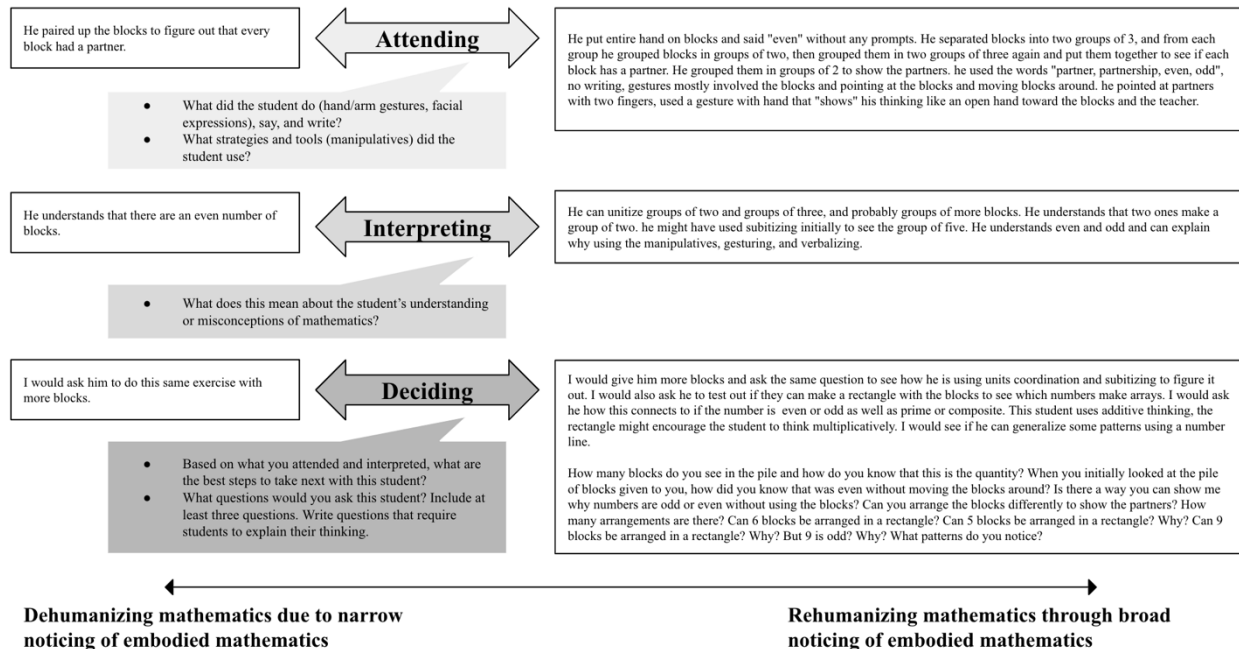
### Data Collection and Analysis

Initially, we individually viewed a cognitive interview video from the course resources of *Mathematics for Elementary Teachers: A Contemporary Approach* by Musser and colleagues (2013) and recorded our embodied noticings. In the video, a second-grade boy is given six blocks and asked, “Is the number of blocks even or odd?” We chose this video because it addresses a common early conception and the video frame includes the student’s face and body, providing us with opportunities to attend to embodiment in our noticings. In the video, the student initially declared that six is even without any interaction with the blocks, suggesting a unitizing strategy. When asked how he knew, he initially organized the six blocks into two rows of three, stating they were “even.” When asked to explain, the student used a different method, likening even number to having “partners” and combining the blocks into three groups of twos. Ball and Bass (2003) reported that third-grade students might perceive numbers like six as both even and odd because of their reasoning that the number of groups of two is odd. Further exploration showed that third-grade children may use fair sharing, groups of two, or the alternation of even and odd numbers on the number line strategies (Bass, 2005). Next, we met as a group to share, compare, and merge our noticings. We used open coding to analyze our individual noticings and develop a merged embodiment exemplar (Corbin & Strauss, 2014).



## Findings

Figure 2 illustrates examples of narrow versus broad attending, interpreting, and deciding of embodiment (structure of figure adapted from Thomas et al., 2017). The rehumanizing of embodied noticing of student thinking could help advance equity and access by broadening what counts as evidence of mathematical understanding.



**Figure 2: Examples of Noticing of Embodiment and Mathematics Discussion**

In this collaborative self-study we set out to illustrate what rehumanizing mathematics through broad noticing of embodied mathematics looks like. Based on our review of the literature, layering an embodiment lens over our noticing may promote equity and access by providing opportunities to highlight and leverage every student as a brilliant mathematician (Abrahamson, et al., 2020; Gutiérrez, 2018). Layering an embodiment lens over our noticing of mathematics provided an "aha" moment for us. We realized the limitations of our prior noticing work that had not explicitly or intentionally focused on embodiment. Through this collaborative self-study, we feel that we progressed from the transitional level (Quadrant IV) to the accomplished level of noticing embodiment and mathematics (Quadrant I). We attribute the broadening and rehumanizing of our noticing skills to our learning about the field of embodiment and our layering of embodiment lenses over our noticings of mathematics.

A limitation of our work is that we only viewed one cognitive interview video of one student completing one mathematics task. Moreover, having diverse MTEs view a representative collection of videos would enhance this study. MTEs could use the merged noticings we share in Figure 2 as an exemplar for preservice teachers (PSTs) or to develop criteria and gradations of quality for evaluating PSTs' noticings. This provocative collaborative self-study pushes the field to consider how layering an embodiment lens over noticing may rehumanize mathematics teaching and learning and promote equity and access. Additionally, it challenges the field to consider the nuanced complexity of potential critical events that may occur during the teaching and learning of mathematics.

## References

- Abrahamson, D., Nathan, M. J., Williams–Pierce, C., Walkington, C., Ottmar, E. R., Soto, H., & Alibali, M. W. (2020). The future of embodied design for mathematics teaching and learning. In S. Ramanathan & I. A. C. Mok (Guest Eds.), *Future of STEM education: Multiple perspectives from researchers* [Special issue]. *Frontiers in Education*, 5(147), 1-29. <http://doi.org/10.3389/educ.2020.00147>
- Alberto, R., Shvarts, A., Drijvers, P., & Bakker, A. (2022). Action-based embodied design for mathematics learning: A decade of variations on a theme. *International Journal of Child-Computer Interaction*, 32, 100419. <https://doi.org/10.1016/j.ijcci.2021.100419>
- Alibali, M. W., & Nathan, M. J. (2018). Embodied cognition in learning and teaching: Action, observation, and imagination. In F. Fischer, C. E. Hmelo-Silver, S. R. Goldman, & P. Reimann (Eds.), *International handbook of the learning sciences* (pp. 75-85). Routledge.
- Ball, D. L., & Bass, H. (2003). Making mathematics reasonable in school. *A research companion to principles and standards for school mathematics*, 27-44.
- Bass, H. (2005). Mathematics, mathematicians, and mathematics education. *Bulletin of the American Mathematical Society*, 42(4), 417-430.
- Corbin, J., & Strauss, A. (2014). *Basics of qualitative research: Techniques and procedures for developing grounded theory* (4th ed.). Sage.
- Gelman, R., & Gallistel, C. R. (1978). *The Child's Understanding of Number*. Harvard University Press.
- Gutiérrez, R. (2018). Introduction: The need to rehumanize mathematics. In I. Goffney & R. Gutiérrez (Eds.), *Rehumanizing mathematics for Black, Indigenous, and Latinx Students* (pp. 1–10). NCTM.
- Jacobs, V. R., Lamb, L. L., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. *Journal for Research in Mathematics Education*, 41(2), 169-202. <https://doi.org/10.5951/jresmetheduc.41.2.0169>
- Kosnik, C., Lasonde, C., & Galman, S. (2009). What does self-study research offer teacher educators? In C. Lasonde & S. Galman (Eds), *Self-study research methodologies for teacher educators* (pp. 225-239). Brill.
- Louie, N., Adiredja, A. P., & Jessup, N. (2021). Teacher noticing from a sociopolitical perspective: the FAIR framework for anti deficit noticing. *ZDM–Mathematics Education*, 1-13. <https://doi.org/10.1007/s11858-021-01229-2>
- Moss, D., & Poling, L. (2019). Using a noticing framework in a mathematics methods course. *Journal on Empowering Teaching Excellence*, 3(1), 1-16. <https://doi.org/10.26077/yjw6-6s37>
- Musser, G. L., Peterson, B. E., & Burger, W. F. (2013). *Mathematics for elementary teachers: A contemporary approach*. John Wiley & Sons.
- Nathan, M. J. (2022). *Foundations of embodied learning: A paradigm for education*. Routledge.
- Nemirovsky, R., & Ferrara, F. (2009). Mathematical imagination and embodied cognition. *Educational Studies in Mathematics*, 70(2), 159-174. <https://doi.org/10.1007/s10649-008-9150-4>
- Nemirovsky, R., Rasmussen, C., Sweeney, G., & Wawro, M. (2012). When the classroom floor becomes the complex plane: Addition and multiplication as ways of bodily navigation. *Journal of the Learning Sciences*, 21(2), 287-323. <https://doi.org/10.1080/10508406.2011.611445>
- Rotem, S. H., & Ayalon, M. (2023). Changes in noticing multiple dimensions in classroom situations among pre-service mathematics teachers. *Teaching and Teacher Education*, 121, 103932. <https://doi.org/10.1016/j.tate.2022.103932>
- Seda, P., & Brown, K. (2021). *Choosing to see: A framework for equity in the math classroom*. Dave Burgess Consulting, Incorporated.
- Stockero, S. L., & Van Zoest, L. R. (2013). Characterizing pivotal teaching moments in beginning mathematics teachers' practice. *Journal of Mathematics Teacher Education*, 16, 125-147. <https://doi.org/10.1007/s10857-012-9222-3>
- Tall, D. (2008). The transition to formal thinking in mathematics. *Mathematics Education Research Journal*, 20(2), 5-24. <https://doi.org/10.1007/BF03217474>
- Thomas, J., Jong, C., Fisher, M. H., & Schack, E. O. (2017). Noticing and knowledge: Exploring theoretical connections between professional noticing and mathematical knowledge for teaching. *The Mathematics Educator*, 26(2), 3-25. <https://openjournals.libs.uga.edu/tme/article/view/2030>
- van Es, E. A. (2011). A framework for learning to notice student thinking. In M. Sherin, V. Jacobs, & R. Philipp (Eds.), *Mathematics Teacher Noticing* (pp. 164-181). Routledge.

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.



## QUEER HIGH SCHOOL STUDENTS' TAKEAWAYS FROM THE TEACHING OF MATHEMATICS FOR SOCIAL JUSTICE

Weverton Ataíde Pinheiro  
Texas Tech University  
[wataidep@ttu.edu](mailto:wataidep@ttu.edu)

Rosa Chávez  
Texas Tech University  
[Rosa.d.chavez@ttu.edu](mailto:Rosa.d.chavez@ttu.edu)

*This study investigated Queer high school students' participation in the teaching of mathematics for social justice. The goal of this research was to introduce the TMSJ to Queer students and understand their experiences with this type of teaching. This study is centered on the transformative paradigm using a poststructuralist approach. Therefore, knowledge about oneself and the world is the product of discursive constructions. Queer, Critical, and poststructuralist theories guided the design of this study; however, their discussion is omitted in this paper due to space limitations. In addition, a TMSJ framework was implemented in a three-lesson experiment about the injustices Queer high school students go through in schools. Through qualitative analysis using inductive coding and thematic analysis, findings show that Queer high school students see a great value in the TMSJGS and had a positive experience with the learning of math. However, efforts are still needed to make this teaching successful in current schools.*

Keywords: Gender, Social Justice, LGBTQIA+; Equity, Inclusion, and Diversity

Mathematics education scholarship to support all learners has expanded to attend to gender and sexuality identities (Neto & Ataíde Pinheiro, 2021; Ataíde Pinheiro, 2022; McGraw et al, 2019). Most current work argues for the importance of either the inclusion of Queer students in the mathematics curricula (Dubbs, 2016; Rands, 2013; Rubel, 2016; Waid, 2020) or the queerization of mathematics education through dismantling gender and sexuality normativity in the curriculum (Kersey & Voigt, 2021; Rands, 2009; Yeh & Rubel, 2020). The Teaching of Mathematics for Social Justice (TMSJ) is one approach that has recently gained momentum in Mathematics Education and has taken up the recommendations above regarding the inclusion of Queer students in the curriculum and the queerization of ways to conceptualize mathematics teaching (Berry III et al., 2020; Rands, 2013). In this study, we refer to the TMSJ that examines issues of gender and sexuality as TMSJGS-- Teaching Mathematics for Social Justice of Gender and Sexuality. Few studies that examine TMSJGS have addressed the experiences of Queer students in the mathematics classroom or the challenges the TMSJGS might cause for Queer students (see Ataíde Pinheiro, 2022). One underlying assumption of the recommendation for the TMSJGS is that Queer students will always benefit from the process of queerizing mathematics education through lessons with social justice aims. But these students may have serious concerns or reservations about such lessons. In order to address this issue, this study investigated: (a) how Midwestate (pseudonym for the significant U.S. Midwestern state where the study was conducted) Queer students experience TMSJGS lessons; (b) under what conditions they experience them as beneficial; and (c) what supports may need to be in place for teachers who are interested in teaching TMSJGS lessons. This study intended to respond to the following research question: What are the takeaways of Midwestate Queer High School students during the TMSJGS?

### Review of Literature and Theoretical Perspectives

Researchers (Martin & Larnell, 2013; Tate, 2008; Stinson & Bullock, 2012) have indicated that the current methods for teaching and learning mathematics in K-12 classrooms have resulted

in inequitable outcomes that do not support all students. Many (e.g., Leyva, 2017; Solomon, 2007; Waid, 2020) have argued that mathematics has historically been taught to propagate a belief system that favors white men and encourages heteronormativity (Ataide Pinheiro, 2021). This becomes increasingly true as students progress to middle and high school (Ataide Pinheiro, 2022). The current instructional systems discourage and may, in fact, prevent the mathematical advancement of people of color, women, LGBTQ+ groups, people with disabilities, and other groups targeted for oppression.

Because of the concerns above and many others, a top priority of the field of mathematics education has been to create equitable classrooms that serve all students (NCTM, 1989; 2000). As theorized by Berry et al. (2020), the TMSJ might be an ideal place to start. A critical component of TMSJ is for students to develop a conscious awareness of the historical injustices that plague society. Below I present the six elements that composed the TMSJ framework:

1. Equitable Mathematics Teaching Practices – 1) Build on social, cultural, family, and community knowledge; 2) Challenge spaces of marginality; and 3) Develop positive social, cultural, and mathematical identities.
2. Authentic, challenging social and mathematical questions or concerns – Local and authentic contexts can increase student engagement and motivation to learn mathematics.
3. Social and mathematical understanding – 1) Mathematics content: what we want students to know; 2) Mathematics practice: how we want students to show what they know; and 3) Social justice standards: how we want students to demonstrate understanding and a response to an issue.
4. Social and Mathematical Investigation – Lessons need to be grounded in the mathematically driven investigation of the social context.
5. Social and Mathematical Reflection – High cognitive demand tasks require students to reflect on mathematics, the social issue, and how one informs the other.
6. Action and Public Product – Take action and develop a public product.

Having the six elements above, the TMSJ is proposed as a method of teaching where students are (a) exposed to a social injustice in society, (b) encouraged to model the social injustice with mathematics, and (c) encouraged to produce a public product or act towards the injustice in order to dismantle it. These three divisions of the TMSJ were used in this study to teach Queer students TMSJGS. It is important to mention that what makes TMSJGS distinct from TMSJ is just the fact that the social injustice being investigated and modeled through mathematics specifically pertains to injustices concerning Gender and Sexuality.

### **Methods and Methodologies**

We approach this study through the transformative paradigm. In this paradigm, researchers work to transform society, recognizing that power and knowledge are situated politically and structurally to maintain oppression among historically marginalized groups of people (Mertens, 2010). Like others using the transformative paradigm, this research sought to bring social transformation to the lives of Queer students through collaborative efforts among the researchers and research participants (Mertens, 2014). Queer high school students' voices foreground their analyses of the TMSJGS, co-participating with the researchers in the production of knowledge regarding the TMSJGS.

### **Positionality**

The first author is an immigrant Queer of color man. He has lived in the United States since

2017 and in the US, he has undergone training that supported his development of conscientização, or critical consciousness (Freire, 1970/2017). He shares multiple experiences with this study students' participants because of his Queerness. However, he also shares differences since he is not originally from the US. Therefore, he sought support during the data analysis to a critical peer to fully understand students' experience regarding unfamiliar social context of the US to him.

The second author is a cis-gender Latina woman born and raised in the borderlands of South Texas. While she does not identify as a Queer woman, there are other marginalized identities that provide a solidarity lens to make meaning of the participants' experiences. From a young age, she was often positioned as less capable of doing mathematics due to her gender. Additionally, her experiences growing up in poverty and emergent bilingual practices inform her critical consciousness and commitment to support historically marginalized students engaging TMSJ.

### **Participants**

Ten Queer high school students from Midweststate participated in this study. These students came from all around Midweststate and were recruited by Gender and Sexuality Alliance (GSA) club advisors. All advisors were contacted through email requesting them to forward a survey for Queer students inquiring their interest to participate in the study. The emails of the advisors were obtained on school's websites and through a major organization in Midweststate that oversees all GSA clubs. Participants came from a variety of school types (e.g, urban, suburban, rural, etc.), identified with a variety of gender and sexual identities, as well as racial identities, and were at different grades (8-12). It is important to note that when the study started in Spring 2021, all students were in high school. However, one student (Cameron) graduated prior to interview data being collected, but they were still included in the study since they started participating as a high school student. Because of space, a full description of the participants will be provided at the conference.

### **Data Collection**

Primarily, the first author taught ten Midweststate Queer high school students three TMSJGS lessons (named in this study as unity study). The unit study used adapted lesson plans published in the book *High School Mathematics Lessons to explore, understand, and respond to social injustice* (Berry III et al., 2020) [Lesson 5.3]. The TMSJGS lessons explored national data on the bullying and harassment Queer high school students go through in society. On these lessons, Queer high school students needed to use the data provided to model with mathematics how many Queer students in their own schools might be going through bullying and harassment due to their Queerness. Finally, Queer students were encouraged to reflect how to act towards ending bullying and harassment in their schools. After being exposed to the unit study, the ten students participated in a 60-minute semi-structured individual interview. After individual interviews, all students were invited to participate in a focus group interview. Five of the ten students returned to participate in a 60-minute focus group.

### **Data Analysis**

In order to answer our research question, we analyzed the interview data through inductive coding (Strauss & Corbin, 1990) followed by *thematic analysis* (Braun & Clarke, 2006). All data was transferred to a software that can be used for qualitative analysis (MAXQDA) and coded to exhaustion (until authors could no longer find new codes). Then, authors met to discuss the coding and the themes found to ensure they were in alignment and agreed 100% to the codes and the themes. Finally, Queer students were presented with a write up of the analysis for member-checking (Koelsch, 2013) to ensure reliability of the findings.

## Findings

Through thematic analysis, we found four themes for Queer students' takeaways in the TMSJGS. These themes can be summarized as (a) comfort in being around the Queer community; (b) the importance of the TMSJGS; (c) perspectives in the TMSJGS in schools; and (d) the intersections of mathematics and social identities.

### **Comfort in being around the Queer community**

Being part of a minoritized group, such as LGBTQ+, makes Queer students feel alienated from those who are not part of the Queer community. Students discussed how feeling safe plays a significant role in how they experience school in general and mathematics in particular. Therefore, many times, Queer students need to negotiate their identities (not be who they are). However, in the TMSJGS, all the participants identified as Queer. Thus, the sense of comfort during the lessons became apparent, and Queer students were satisfied with the TMSJGS because it was a safe space for them. When Eric was asked what he liked about the TMSJGS lessons, he said:

Hmmm, well, working together with people that I don't know, but I know have experienced something similar to what we were even studying. It felt comforting to know that we can you know, work together, maybe even power through this discriminatory stuff. Like see the numbers, but we're trying to change the numbers, to be lower percentage of people that are getting discriminated against, hopefully, one day, there will be no discrimination. And it's comforting to know that we might be able to do that one day.

Eric not only expressed how much he appreciated being able to work with people who also experienced bullying and harassment because of their historically marginalized gender and/or sexual identities (Queer students), but he also expressed how comforting that experience was for him. He said how pleasant he felt to know that there is hope that someday, they, Queer students, might even be able to lower the percentage of Queer students that go through discrimination. He felt comforted knowing that with the Queer community's efforts, they might succeed "one day" in lowering discrimination. Eric's feelings of comfort can be connected to his feelings of hope and agency. The lack of agency Queer students feel at school is being turned into a sense that working together as a community can bring about change (transformation). Closely related to the fact that by being around Queer students, the experience with the TMSJGS was positive, Queer high school students also demonstrated the willingness and hope for change, as emphasized in Eric's last quote above.

### **The importance of the TMSJGS**

Amid the TMSJGS lessons applied in this study to Queer high school students, they had the opportunity to find out, through matrix multiplication, how many students in their school were experiencing harassment and assault due to having historically marginalized gender and/or sexual identities. As Queer students reflected on the experiences during the TMSJGS, they brought back the importance they saw in this type of teaching. To better understand the second theme, we analyze Tom's quote below. When asked what he liked the most about the lessons, Tom said:

If I'm honest, looking at what I did like the most was the discussions. I liked how we were able to reflect on the data, and I liked how when you guys would ask specifically: what did this data mean to you? And, what is this data trying to say? I liked how the lessons were

about more than just getting the numbers, but actually understanding what they meant and understanding what they reflected in the real world.

Above, Tom expressed the power of discussions during the TMSJGS. He appreciated being given the opportunity to reflect on what the data meant. Being engaged in this activity helped Tom shift his perspectives on the doing of mathematics. The TMSJGS went beyond “just getting the numbers,” it was an opportunity to understand what the numbers meant and the impact the data has in the real world. This was so important to Tom that later in the interview, when trying to understand his mathematical identity and experiences with the TMSJGS, he said, “I don't care about you know finding  $x$ , but I do care about the social issues.” This statement shows that the social issues being discussed, bullying and harassment, were more important than doing “procedural mathematics.” Therefore, by implementing the TMSJGS in the classroom, we might be able to help Queer students develop more powerful mathematical identities since this type of teaching can shift the perspectives of what mathematics is and why it is crucial to learn it (mathematics can be used to combat social injustices).

Queer students who had experienced bullying and harassment at their schools because of their Queer identity also expressed how change could be accounted for through the TMSJGS. When we asked Olive what stood out the most about the lessons, she said:

Olive: It was just really how calm and just collected everyone was. There was no, like, craziness and we actually felt like we were doing something at the end. It wasn't just like useless numbers, I felt like these numbers could actually do something. And help people at my school.

Interviewer: Nice, you said like we felt like we're doing something, can you tell me more about that?

Olive: So, finding out these potential numbers could help us identify people at, like, my school. And then identifying people that can help bring them together, have a community, and then ultimately try to heal and learn from everyone's experiences and make it better.

Olive argued that through the TMSJGS, it felt like “we were doing something.” She compared the TMSJGS to her experiences with other mathematics classes where the numbers seemed “useless” because of a lack of connection to problems that could actually change people's lives. The numbers that Olive was referring to are the ones she found using matrix multiplication that represented the number of students at her school who were going through bullying and harassment because of their Queer identities. Olive also saw the possibility of using these numbers to help students in her school going through these experiences. She continued to discuss how the numbers could be used to find people going through bullying and harassment and create a pro Queer community in her school. She argued for the possibility of helping students heal, as we infer she recognized the traumas Queer students go through as they experience bullying and harassment. In sum, Queer high school students saw an opportunity to reflect on the consequences that bullying and harassment have on Queer high school students' lives through engaging in TMSJGS. Using mathematics to model discrimination and understand how discrimination looks in their schools, Queer students showed sentiments toward the community and argued the importance of advocating for changes.

### **Perspectives in the TMSJGS in schools**

Multiple times throughout their interviews, Queer students shared their perspectives of what would happen in their regular classroom if the TMSJGS was introduced. When asked about her



impression of the TMSJGS, Olive said:

Um, I mean, personally if this was to be taught at my school, I'm sure there would be like, if it was not just a bunch of LGBTQ teens, it would go a lot differently, and there might have been some like retaliation, and like read this, I don't know, I just appreciated it for once that when we're talking about these topics that there was no like jeering or just like general put downs and stuff like that.

Olive's experiences in school showed that the TMSJGS could lead to retaliation rather than a productive conversation in her high school. It is possible to infer that when things such as Queerness were to be brought up in her school spaces, students would not be open to it. She said she appreciated the TMSJGS study because there was peace. No students were jeering and disturbing the lessons. She expressed that the TMSJGS would only be possible if it were to happen among many LGBTQ teens; otherwise, there could be retaliation by other students. To follow up with Olive, I asked her to go deeper and share her school experiences with me. She said:

Okay, so probably about half of my school wouldn't really enjoy these lessons very much. They wouldn't try to listen or try to do anything because they simply don't like LGBTQ people, and that's sad and horrible, but it's also a reality that we have to face, you know. So if it were to be taught at a real school um, I don't know, maybe, just like for GSAs or I don't know, but I feel like this might help people become more acquainted and understand what LGBT people go through, so I'm not sure.

Olive complemented the above with her personal experiences in school and how they reflect on her perspectives of the TMSJGS happening in her school. When she mentioned, "half of my school... don't like LGBTQ people," it made it evident that she knew LGBTQ were disliked by half of the people in her school. The consequences would be at least half of non-Queer students not listening or enjoying the TMSJGS. It seems stronger for Olive to see the possibility of the TMSJGS happening in schools through GSAs. However, she also expressed the possible benefits the TMSJGS would have if it happened in the regular classroom. She emphasized the importance that the TMSJGS would have in helping those that are not LGBTQ understand the everyday lives of an LGBTQ person, which are framed by negative experiences due to their Queer identity.

Like Olive, Mackenzie thought that the TMSJGS would not be taken seriously if taught in her school. She expressed these perspectives saying:

Um, I don't know, I think the lesson would probably end up sounding, it probably, end up being a bit more like bottomless, like a joke lesson, I guess. Like it would just be like at the butt end of a lot of jokes, you know.

Mackenzie clearly expressed how the TMSJGS would not be taken seriously in her school. She mentioned the large number of jokes that would happen if this type of teaching was happening in her classroom and how others would not respect the TMSJGS. Her strong statements as "sounding," "joke lesson," and "butt end of a lot of jokes" showed how much her experiences in her school demonstrated that Queer issues are not taken seriously, even when there are discussions of discrimination taking place in the classroom. It is important to mention that the way students discussed the possibilities of the TMSJGS happening in their schools varied significantly according to the experiences and the type of schools these students attended. More nuances on these findings will be presented at the conference.

## **The intersections of mathematics and social identities**

It is not new that mathematics educators have argued that the context of mathematics teaching matters in students' mathematical learning (see Ataide Pinheiro, 2022). The TMSJGS did not prove to be different. Queer high school students had a very different experience with the TMSJGS because it was something relevant to all Queer students in the study. As we see in Lin's quote, "I was a lot more engaged with the lessons because this applied to me. I really wanted to figure out what the numbers meant and how we could improve things." In this quote, it became clear that because the TMSJGS was something that applied to Lin, she could get herself more engaged with the lessons. Consequently, it also made her want to figure out the numbers to improve the experiences of LGBTQ+ students in schools. In alignment with Lin, Olive expressed her experiences with the TMSJGS, she said:

Oh, you know what, yeah, being LGBTQ myself definitely played a role of you know, personal accountability and just interest you know, since I am LGBTQ, I can relate to these numbers and look at them and be like oh there's people like me finally. And like, how can we help people like me, you know. So, I think that definitely plays a role.

For Olive, being able to relate because she is also part of the LGBTQ+ community played a role of accountability and interest. She not only saw herself in the numbers, but she also saw that there are people out there just like her that have gone through similar experiences with bullying and harassment due to their Queer identities. So, the context of the lessons played a role in making her more interested, holding her accountable, and making her want to help make a change in the lives of those going through negative experiences due to their gender and/or sexuality.

## **Discussion and Conclusion**

Queer students specifically liked how the TMSJGS opened up spaces for discussions of how other Queer students were being oppressed in schools, how that oppression related to the study participants, and what those specific oppressions looked like in the world around them. Queer students were fascinated that the TMSJGS lessons were specifically about people like them. They could see themselves reflected in the data, and they could compare the data and their lived experiences as Queer students in high school in the United States. And because in the proposed lessons within the framework for the TMSJ, we specifically aim to use mathematics to model the injustices happening at students' community and/or schools; through the TMSJGS, Queer students were able to find the concrete numbers that reflected the injustices Queer students go through in their own schools. They were able to map the number of students that were probably going through oppression in their schools due to their Queer identities. The numbers became even more important to those coming from schools where Queer identities were not welcomed, such as Olive, Tom, Eric, and Mackenzie. Queer students discussed that during the TMSJGS, they were actually "doing something," which can be interpreted as Queer students contrasting the TMSJGS with their everyday mathematics, where in their mathematics classes they learn little that can be used to change the world around them. And with this new "doing - something," they could bring the changes needed to their schools. One of the Queer students, Olive, discussed the implications of the TMSJGS in her school. She saw the possibility of mapping out Queer students in her school and bringing the students together to create community, heal the traumas, learn from shared experiences, and move towards making changes, as Olive expressed "finding out these potential numbers could help us identify people at like my school and then identifying people that can help bring them together, have a community, and then ultimately try to heal and

learn from everyone's experiences and make it better.” Therefore, I propose that having such a concrete number helped Queer students see that they are not alone and that other Queer students go through bullying and harassment in schools due to their Queer identities. Since numbers are concrete, these numbers also created in Queer students the safety to see and believe that others are going through these experiences, and they need to move towards “doing something about it,” which can be translated as action towards the creation of communities where Queer students are safe, can also work towards healing traumas, and can continue to be organized to brainstorm ways to continue increasing acceptance of Queerness in schools.

TMSJGS was directly connected to Queer students’ gender and/or sexual identities. The lessons Queer students were exposed to specifically discussed the ways Queer students experience harassment and bullying in their everyday lives in school in the U.S.; therefore, TMSJGS lessons hit close to home. Consequently, Queer students demonstrated that they were more interested in learning mathematics that way. There are many reasons why that possibly happened, but we specifically think there are several inferences that can be made from the findings in this study. First, Queer students demonstrated many experiences with mathematics learning through memorization and procedures, which could have impacted the ways these students understand the discipline conceptually, which can also influence the ways that Queer students might not understand the usefulness of mathematics. In the TMSJGS specifically students could develop what it meant to multiply two matrices through a contextualized approach. Throughout the study, Queer students demonstrated that they learned better through the TMSJGS. Perhaps that was one of the first times in high school that they worked with a procedure that maintained meaningful connection to a context Queer students cared about, rather than just memorizing procedures for matrix multiplication. Extending to other mathematics topics, Queer high school students could have thought about how other areas of mathematics would be if they were learning it the way we did through the TMSJGS. Secondly, Queer students saw mathematics closely connected to something relevant to the world (injustices to Queer students) and something relevant to them since they are part of the Queer community. Many of the mathematics Queer student participants had experienced with were not connected to the real world or were not connected to contextualized life issues. But the TMSJGS was a lesson that used real-life data to calculate the number of Queer students in their schools who would be experiencing oppression due to their Queer identities. That made the teaching “real” and meaningful. It was more than just playing with numbers; as many students said, “it felt like they were really doing something important.” Finally, TMSJGS was relevant to Queer students, it was connected to Queer students’ community, and it was about understanding Queer students’ experiences. It helped them see a way to make positive changes towards dismantling the oppression they face in schools.

### References

- Ataide Pinheiro, W. (2021). Dismantling the ‘all-boys club’ a narrative of contradictions women experience in PhD mathematics programs: A Freirean approach. *International Electronic Journal of Mathematics Education*, 16(3), em0652. <https://doi.org/10.29333/iejme/11090>
- Ataide Pinheiro, W. (2022). *At the intersections: Queer high school students’ experiences with the teaching of mathematics for social justice* (Publication No. 29320623) [Doctoral dissertation, Indiana University]. ProQuest Dissertations & Theses Global.
- Berry, R. Q., III, Conway, B. M., IV, Lawler, B. R., & Staley, J. W. (2020). *High school mathematics lessons to explore, understand, and respond to social injustice*. Thousand Oaks: Corwin Press.
- Braun, V. & Clarke, V. (2006) Using thematic analysis in psychology. *Qualitative Research in Psychology*, 3, 77-101.

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

- Dubbs, C. (2016). A queer turn in mathematics education research: Centering the experience of marginalized queer students. In M. B. Wood, E. E. Turner, M. Civil, & J. A. Eli (Eds.), *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1041–1048). The University of Arizona.
- Freire, P. (1970/2017). *Pedagogia do oprimido* (64th ed.). Paz & Terra.
- Kersey, E., & Voigt, M. (2021). Finding community and overcoming barriers: experiences of queer and transgender postsecondary students in mathematics and other STEM fields. *Mathematics Education Research Journal*, 33, 733-756. <https://doi.org/10.1007/s13394-020-00356-5>
- Koelsch, L. E. (2013). Reconceptualizing the member check interview. *International journal of qualitative methods*, 12(1), 168-179.
- Leyva, L. A. (2017). Unpacking the male superiority myth and masculinization of mathematics at the intersections: A review of research on gender in mathematics education. *Journal for Research in Mathematics Education*, 48(4), 397-433. <https://doi.org/10.5951/jresmetheduc.48.4.0397>.
- Martin, D. B., & Larnell, G. V. (2013). Urban mathematics education. In H.R. Milner & K. Lomotey (Eds.), *Handbook of urban education* (pp. 373-393). New York: Routledge.
- McGraw, R., Piatek-Jimenez, K., Wiest, L., Dias, A., Gonçalves, H. J. L., Hall, J., Hodge, A., Kersey, B., & Rubel, L. (2019). Working group on gender and sexuality in mathematics education: experience of people across cultures. In S. Otten, A. G. Candela, Z. Araujo, C. Haines, & C. Munter (Eds.), *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1940–1953). University of Missouri. <https://www.pmena.org/pmenaproceedings/PMENA%2041%202019%20Proceedings.pdf>
- Mertens, D. M. (2010). Philosophy in mixed methods teaching: The transformative paradigm as illustration. *International Journal of Multiple Research Approaches*, 4, 9-18.
- Mertens, D. M. 2014. *Research and Evaluation in Education and Psychology: Integrating Diversity with Quantitative, Qualitative, and Mixed Methods*, Thousand Oaks, CA: Sage Publications.
- National Council of Teachers of Mathematics (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author.
- National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics*. Reston, VA: Author.
- Neto, V., & Ataíde Pinheiro, W. (2021). Análise comparativa entre Brasil e os Estados Unidos: O problema de gênero em livros didáticos de matemática. *Revista De Investigação E Divulgação Em Educação Matemática*, 5(1). <https://doi.org/10.34019/2594-4673.2021.v5.33216>
- Rands, K. (2009). Mathematical inqueery: Beyond “add-queers-and-stir” elementary mathematics education. *Sex Education*, 9(2), 181–191. <https://doi.org/10.1080/14681810902829646>
- Rands, K. (2013). Supporting transgender and gender-nonconforming youth through teaching mathematics for social justice. *Journal of LGBT Youth*, 10(1–2), 106–126. <https://doi.org/10.1080/19361653.2012.717813>.
- Rubel, L. (2016). Speaking up and speaking out about gender in mathematics. *The Mathematics Teacher*, 109(6), 434–439. <https://doi.org/10.5951/mathteacher.109.6.0434>
- Solomon, Y. (2007). Not belonging? What makes a functional learner identity in undergraduate mathematics?. *Studies in Higher Education*, 32(1), 79-96.
- Stinson, D. W., & Bullock, E. C. (2012). Critical postmodern theory in mathematics education research: A praxis of uncertainty. *Educational Studies in Mathematics*, 80(1), 41–55.
- Strauss, A. L. and Corbin, J. (1990). *Basics of Qualitative Research*, 1st edn, Thousand Oaks, CA: Sage.
- Tate, W. F. (2008). “Geography of opportunity”: Poverty, place, and educational outcomes. *Educational Researcher*, 37(7), 397–411.
- Waid, B. E. (2020). Supporting LGBTQ+ students in K-12 mathematics. *Mathematics Teacher: Learning and Teaching PK–12*, 113(11), 874–884.
- Yeh, C., & Rubel, L. (2020). Queering mathematics: Disrupting binary oppositions in mathematics pre-service teacher education. In N. Radakovic & L. Jao (Eds.), *Borders in mathematics pre-service teacher education* (pp. 227-243). Springer. [https://doi.org/10.1007/978-3-030-44292-7\\_11](https://doi.org/10.1007/978-3-030-44292-7_11)

## QUEERING THE MATH GONE WRONG

Katryne Dubeau

University of Saskatchewan  
katryne.dubeau@usask.ca

Sandra Hall

University of Saskatchewan  
sandra.hall@usask.ca

Egan J Chernoff

University of Saskatchewan  
egan.chernoff@usask.ca

Keywords: LGBTQIA+, Probability, Gender

There has been a call to action for including diversity and equity in mathematics education (NCTM, 2023; Rubel, 2016) and the emphasis has primarily been “framed around categories of race, cultural background, language, disability status, sex, or socioeconomic status [while] sexual orientation and gender identity remain largely absent” (Rubel, 2016, p. 435). This call to action asks educators to attempt to integrate gender and sexual diversity into their mathematics courses as mathematics can be used “as a window that looks out onto social justice issues around gender or sexuality” (Yeh & Rubel, 2020, p. 238). While not disturbing the mathematical concepts, educators could use heteronormative and cis-gendered textbook questions (Richard et al., 2022) or extend famous mathematical problems (Dubeau et al., 2023a, 2023b) to reflect and include LGBTQIA+ notions. However, there are social and mathematical pitfalls that educators need awareness of to ensure correct integration.

We argue that it behooves mathematics educators to be alert to social and political meanings behind the integration of LGBTQIA+ culture in mathematical problems so as not to unintentionally marginalize the experiences of queer youth. This poster highlights how the (mis)representation of queer students can occur when mathematics educators try to ‘plug in’ notions from the queer community into established mathematical problems like Martin Gardner’s Two-Child Problem and the Monty Hall Problem, respectively.

As we have presented, our extension of the Two-Child Problem (Dubeau et al., 2023a, 2023b) allows students to see their gender identity represented, making way for discussions of cisnormativity in society and mathematics; it can lead to further classroom conversations surrounding the assumption of gender identity, the prevalence of diverse gender identities, and much more. However, applied to the Monty Hall Problem, we found that the social implications around using the problem could be misused to reinforce oppressive ideas of gender identity (Rubel, 2016) and should not be used to represent any minority group. Several issues arose in our musings, such as gender identity being a prize to be won if the student chooses the correct door; the host, an omniscient figure, already aware of the contestant's gender identity before the game begins; and that the problem uses a door to hold information about the gender identity of a person, which is a closet-like stand-in. Thus, we posit that there is no conceivable way to present this problem in an acceptable and respectful manner regarding a person's gender identity.

Though mathematics educators have the best intentions when rewriting heteronormative and cisnormative questions to better reflect students who do not fit into those categories, these discussions of inclusivity need to represent those minority groups accurately and appropriately (Richard et al., 2022). Including queer identities, experiences, and history into mathematics should be used to enhance the voices of our LGBTQIA+ students and give them a better sense of belonging in the mathematics community (Rands, 2009). Following the success with the Two-Child Problem, we aimed to display that, while a question can be rewritten using 21st century social norms without compromising the mathematics involved, the issue's complexity is

considering the social implications that potentially surround it, as is seen when comparing the sociopolitical connotations of Two-Child Problem to the Monty Hall Problem.

### References

- Dubeau, K., Hall, S., & Chernoff, E. J. (2023a, May 27 – June 1). A queer extension of Martin Gardner's Two-Child Problem [Paper presentation]. Canadian Society for the Study of Education, Congress 2023, Toronto, ON, Canada
- Dubeau, K., Hall, S., & Chernoff, E. J. (2023b, July 11 – 13). Recalculating for the real world [Paper presentation]. The International Association for Statistical Education, Satellite Conference, Toronto, ON, Canada
- Hall, S., Dubeau, K., & Chernoff, E.J. (2023, July 11 – 13). Another queer extension of Martin Gardner's Two-Child Problem: The Facebook edition [Poster presentation]. The International Association for Statistical Education, Satellite Conference, Toronto, ON, Canada
- NCTM. Retrieved February 1, 2023, from [https://www.nctm.org/conferences-and-professional-development/Tips-for-Teachers/Tips-on-Supporting-All-Students\\_-Equity-and-Diversity/](https://www.nctm.org/conferences-and-professional-development/Tips-for-Teachers/Tips-on-Supporting-All-Students_-Equity-and-Diversity/)
- Rands, K. (2009). Mathematical inquiry: Beyond 'add-queers-and-stir' elementary mathematics education. *Sex Education*, 9(2), 181–191. <https://doi.org/10.1080/14681810902829646>
- Richard, T. S., Wiese, E. S., & Rakamarić, Z. (2022, March, 3– 5). An LGBTQ-Inclusive problem set in discrete mathematics [Session]. SIGCSE '22, March 3–5, 2022, Providence, RI, United States of America. <https://doi.org/10.1145/3478431.3499330>
- Rubel, L. H. (2016). Speaking up and speaking out about gender in Mathematics. *The Mathematics Teacher*, 109(6), 434–439. <https://doi.org/10.5951/mathteacher.109.6.0434>
- Yeh, C., & Rubel, L. (2020). Queering mathematics: Disrupting binary oppositions in mathematics pre-service teacher education. *Borders in Mathematics Pre-Service Teacher Education*, 227–243. [https://doi.org/10.1007/978-3-030-44292-7\\_11](https://doi.org/10.1007/978-3-030-44292-7_11)

# RAISING CRITICAL MATHEMATICS: GOING BEYOND NUMBERS

Courtney Harris  
The Ohio State University  
Harris.1050@buckeyemail.osu.edu

Keywords: Critical Mathematics, Middle School Education, Social Justice

Critical Mathematics capitalizes on learning of mathematics through social justice issues (Ernest, 2016, p. 3). Critical mathematics defies neutrality of mathematics (Greer & Mukhopadhyay, 2015) and as Andersson and Barwell (2022) detailed it: *is driven by urgent, complex questions; is interdisciplinary; is politically active and engaged; is democratic; involves critique; and is reflexive and self-aware* (pg. 1). Such approach to curriculum and instruction centers around interpreting, analyzing phenomena drawing on data which document inequities, and injustices relative to access to opportunities (Tutak et al. 2011). In order to do so, learners need to develop skills in reading and interpreting data. In this work we examined how middle school learners viewed available data on undernourishment, drug usage, and adult literacy around the world and the type of teacher interventions which became central in deepening their analysis. The report is a part of a larger project in which we aim to develop curriculum materials around critical mathematics for middle school learners.

## Methodology and Task

Task based clinical interviews (Goldin, 2000) were conducted with each of the five middle school learners. Data for the tasks used came from ourworldindata.org. The participants were asked to examine the data, make inferences about factors that could have contributed to specific trends, and consequences of these conditions for citizens of respective countries. Student interviews were analyzed individually to identify conceptual themes present in each case. A cross analysis of all cases the followed. An example task is shown below:



FIGURE 1: GRAPHICAL TASK

Country	2000	2001	2002	2003	2004	2005	2006	2007
Cambodia	1212	1212	1212	1212	1212	1212	1212	1212
China	162	162	162	162	162	162	162	162
India	242	255	269	283	297	311	325	339
USA	543	532	521	510	500	489	478	467
World	622	612	602	592	582	572	562	552
UN (average)	27	27	27	27	27	27	27	27

FIGURE 2: TABULAR TASK

## Results

All participants were able to identify inequalities regarding human nourishment in various regions across the world based on numerical data. They highlighted lack of access to resources and technology as factors contributing to undernourishment and delineated consequences to include demise of life and increase in rate of crime (people stealing food). Drawing from their knowledge of conditions in their own local communities, they offered that preventative measures could include planting vegetable gardens for family sustenance. These sheltered perspectives were routed in an absence of cultural and political life conditions beyond their own surrounding. Hence, we argue that while mathematics serves a viable tool for highlighting inequities, it may not be sufficient in engaging learners in deep analysis of consequences of data. Supporting curricular materials in social sciences may be needed in raising learners' political awareness.

### References

- Ernest, P., Sriraman, B. & Ernest, N. Eds. (2016) *Critical Mathematics Education: Theory, Praxis and Reality*, Charlotte, NC, USA: Information Age Publishing.
- Goldin, G. A. (n.d.). A Scientific Perspective on Structured, Task-Based Interviews in Mathematics Education Research. 29.
- Greer, B., & Mukhopadhyay, S. (2015). Honoring diversity in intercultural mathematics education. *Intercultural Education*, 26(4), 261–265.  
<https://doi.org/10.1080/14675986.2015.1073876>
- National Research Council 1996. *National Science Education Standards*. Washington, DC: The National Academies Press. <https://doi.org/10.17226/4962>.
- Roth, W.-M. (2022). Inscriptions: Toward a Theory of Representing as Social Practice. *Review of Educational Research*, 68(1), 35-59.  
<https://www.tandfonline.com/doi/full/10.1080/0020739X.2010.510221>



## REAL-WORLD CRITICAL MATHEMATICS LESSON: A WAY TO LEVERAGE STUDENTS' CONCEPTUAL UNDERSTANDING AND SOCIOPOLITICAL AWARENESS

Autumn Pham  
Portland State University  
autpham@pdx.edu

Kathryn E. Roman  
Portland State University  
kroman@pdx.edu

Eva Thanheiser  
Portland State University  
evat@pdx.edu

Molly L. Robinson  
Portland State University  
molrob2@pdx.edu

*We investigated the development of students' sociopolitical consciousness via engagement in a mathematical activity purposefully designed around understanding the unemployment rate in the US. Qualitative data was collected from 12 university seniors participating in an online capstone class. Results showed that students: (1) further developed interpreting data visualizations, (2) understood the nature of the unemployment rate in the United States (i.e. how data is collected, how it is calculated, and how it behaves over time), (3) developed an understanding of the inequality in the US social and political structure, and (4) developed a more critical lens or more critical consciousness by questioning and critiquing the status quo. This study shows that the Unemployment Rate task can be implemented to bridge the disconnect between the mathematics classroom and real-world applications, and may eventually affect a “liberatory social change”.*

Keywords: Data Analysis and Statistics, Design Experiments; Equity, Inclusion, and Diversity; Social Justice

As we move beyond the Equity principle, which provides access to high-quality mathematics instruction to all students from different backgrounds (National Council of Teachers of Mathematics (NCTM), 2000) and move towards “recognizing that inequitable learning opportunities can exist in any setting, diverse or homogenous” (NCTM, 2000, p. 60), we no longer only strive for “mathematics lessons that only focus on the important mathematics” (NCTM, 2000, p. 15). Brantlinger (2013) critiqued the NTCM standards ten years ago that they were not enough to provide equal access to marginalized groups of students. As we move toward the recent years, NTCM in their Access and Equity series of books, they are not only including ensuring that all students should have equal to high quality access (Berlin & Berry III, 2018) but also attend to students’ cultural background, knowledge and experiences (Musgrove & Willey, 2018). Furthermore, Gutstein (2018) leveraged NTCM’s *Principles to Actions* to make a shift toward using mathematics to examine societal structure and empower students to take actions against these power dynamics. Coinciding with the North American Chapter of the International Group for the Psychology of Mathematics Education’s Equity Statement (2019), they recognized that mathematics is not a neutral subject but rather subjective to a human's perspective. Thus, a mathematics lesson that disregards the students’ perspective, experience and cultural background does not equally support all students (Melhuish et al., 2022). Hence, we demand that mathematics lessons not only support mathematical concepts and provide an equitable and inclusive environment but should also promote practical, relevant knowledge and, importantly, develop students’ critical mathematics consciousness.

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

This study is contained in a larger study of 12 university seniors participating in an online capstone class, Mathematics and Society. We aim to examine:

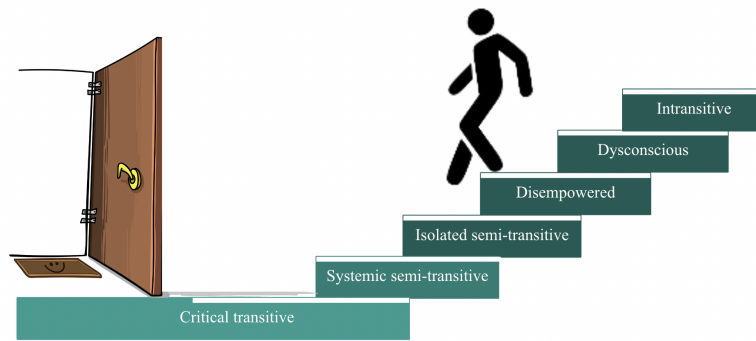
1. How do students' critical mathematics consciousness and sociopolitical awareness develop through a real-world critical mathematics lesson exploring the unemployment rate in the United States?
2. How do students use mathematics and the topic of this lesson to deepen their understanding of the world?

### Theoretical Orientation

#### From Critical Consciousness to Critical Mathematics Consciousness

In Paulo Freire's (1970) *Pedagogy of the Oppressed*, he defined critical consciousness as the ability to be aware of the oppression within the political, social, and economic system and to therefore take action against any forms of oppression. Researchers further narrowed it down to critical mathematics consciousness. For this study, we positioned ourselves in line with Register and colleagues' (2021) Critical Mathematics Consciousness Theoretical framework. Register and colleagues (2021) noticed that students' critical mathematics agency is the triangulation between Ethical Mathematics Awareness, Sociopolitical Mathematics Awareness, and Communicative Mathematics Awareness. This framework attends to three aspects: (1) that mathematics is a tool to represent real-world information by itself and can disparage or liberate certain groups in society, (2) that mathematics carries biases since it is a human activity, and (3) that mathematics can be used to inform or misinform society (Register et al., 2021). We also used Stephan et al.'s (2021) Critical Mathematics Consciousness Growth framework. Stephan and colleagues' (2021) framework boiled down to six levels of mathematical consciousness. First, *intransitive*, where students believe that the cause of the disposition is due to luck or higher being; hence, the situation cannot be changed. Second, *dysconscious*, where students blame the victims for fault and perceive that it is the victim's own responsibility to act. Third, *disempowered*, where students notice either an individual or dominant group is the cause for the oppression, but have no ability to change the situation. Fourth, *isolated semi-transitive*, where students believe the inequality takes place in some isolated incidents and can only act within these situations. Fifth, *systemic semi-transitive*, where students recognize the pervasiveness within the system of oppression, but are not yet ready to act. Lastly and sixth, *critical transitive*, where students acknowledge the pervasiveness of the oppression within our social structure and acquire the critical agency to emancipate the power of oppression (Stephan et al., 2021).

Figure 1 is our reenvisioning of Stephan et al.'s (2021) Critical Mathematics Consciousness Growth framework, where the highest staircase corresponds with the first level of their framework and it steps downwards to the bottom, sixth level - which then connects to the open door and exits to the level of critical mathematics consciousness. Similar to Freire (1973), where he noticed that the critical consciousness is flexible within a situated environment, Stephan and colleagues (2021) also acknowledged that these Critical Mathematics Consciousness levels are context-dependent, and are not a strictly hierarchical order that students can achieve. Thus, these stairsteps would occur depending on the context of the situation and only if students choose to review it during their reflection or conversation. Some stairsteps may be skipped over.



**Figure 1: Reenvisioning of Stephan et al.'s (2021) Critical Mathematics Consciousness Growth Framework [Photo credit: Creozavr from Shutterstock.com]**

### **From the Meaning of Knowledge to the Definition of Mathematics**

Here, we take up our view of knowledge as the relativist whose view, particularly for mathematics, is a justified belief, is context dependent, and is socially-situated within human language (Ernest, 1991). Coinciding with social constructivist view and subjective knowledge, we believe that mathematics knowledge is co-constructed through a community and it is constantly shared and negotiated among members of the community (Ernest, 1991). Thus, instead of breaking our data down to individual students, we used Stephan et al.'s (2021) Critical Mathematics Consciousness Growth framework to analyze the community's knowledge through group discussion and interaction.

Furthermore, our meaning of mathematics has moved beyond the traditional definition of mathematics as numbers, quantities and logic (Heid, 2010) and has shifted toward mathematics as social community interactions within a sociopolitical context (Wagner, 2017). Therefore, we explored our data utilizing Thanheiser's (2023) three-part framework for the meaning of mathematics: (1) logical connections between abstract concepts and quantities, (2) a language to represent the natural order of reality and (3) human interactions that are flexible to one's sociopolitical position.

### **Methods**

We drew data from design-based research (Bakker & van Eerde, 2015) where a US Unemployment Rate lesson was introduced to a group of 12 university seniors who participated in an online capstone class, Mathematics and Society. The lesson was covered over a two-session (110 min/session) span including pre-homework with a survey and post-homework with a survey, consisting of two main tasks which required students to (1) investigate unemployment rate data across racial groups from March 2019 to March 2022 and (2) establish through research the causes for the gaps in the unemployment rates among racial groups. It can be found at <https://tinyurl.com/UErLesson>. The Unemployment Rate lesson was developed and modified to serve students' sociopolitical and traditional mathematics goals. The sociopolitical mathematics goal was to aid students' knowledge of the unemployment rate, especially during the spike of the Covid-19 pandemic, and students' discovery of the inequality in the unemployment rate among racial groups. At the same time, our traditional mathematics goal was for students to understand how the unemployment rate was calculated, how to visually represent the given data, compare the data across time, interpret the visual representation, and connect it to current events. Videos

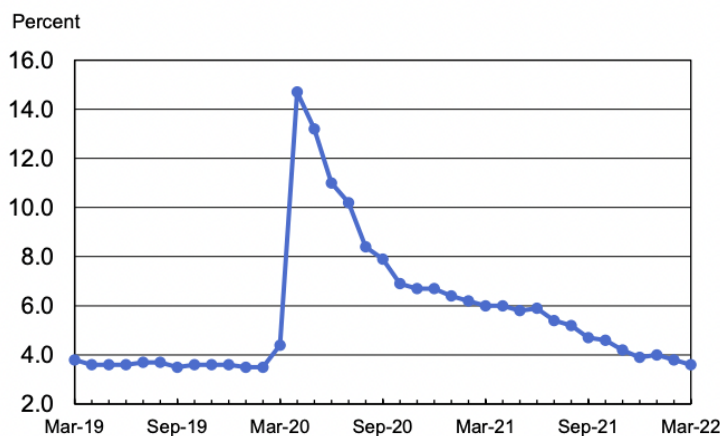
and transcripts were collected and transcribed throughout the two sessions with one main coder and one advisory coder. Through thematic analysis, we read students’ responses during group discussion, reduced them to codes, then later collapsed them into common themes (Creswell, 2014) and let the story arise naturally through those themes. We later applied Stephan and colleagues’ (2021) Critical Mathematics Consciousness Growth framework to students’ responses through an interpretive method. We categorized each statement using the six levels of the framework. If the statements did not fall in any of the categories, we tagged them as inconclusive. Counting each time a student made a statement during group discussions as 1, we were able to quantify the number of responses during discussions in two tasks and create percentages for each of the six levels.

### Findings

Below we will answer (1) how students’ critical mathematics consciousness and sociopolitical awareness developed through a real-world critical mathematics lesson exploring the unemployment rate in the United States and (2) how students used mathematics and the topic of this lesson to deepen their understanding of the world.

#### How Students’ Critical Mathematics Consciousness and Sociopolitical Awareness Developed Through a Real-World Critical Mathematics Lesson Exploring the Unemployment Rate in the United States

**Task 1.** As part of task 1, raw data from the Bureau of Labor Statistics website (at <https://tinyurl.com/UERdata>) were presented to students with informational videos of how the data was collected (see <https://tinyurl.com/UERvideo> for an example of one of these videos). Along with Figure 2, students were asked to examine the graph of the unemployment rate over a period of three years during the pandemic. Students expressed concern about the integrity of the data by questioning the motivations of the data-collectors and the uncounted population. For



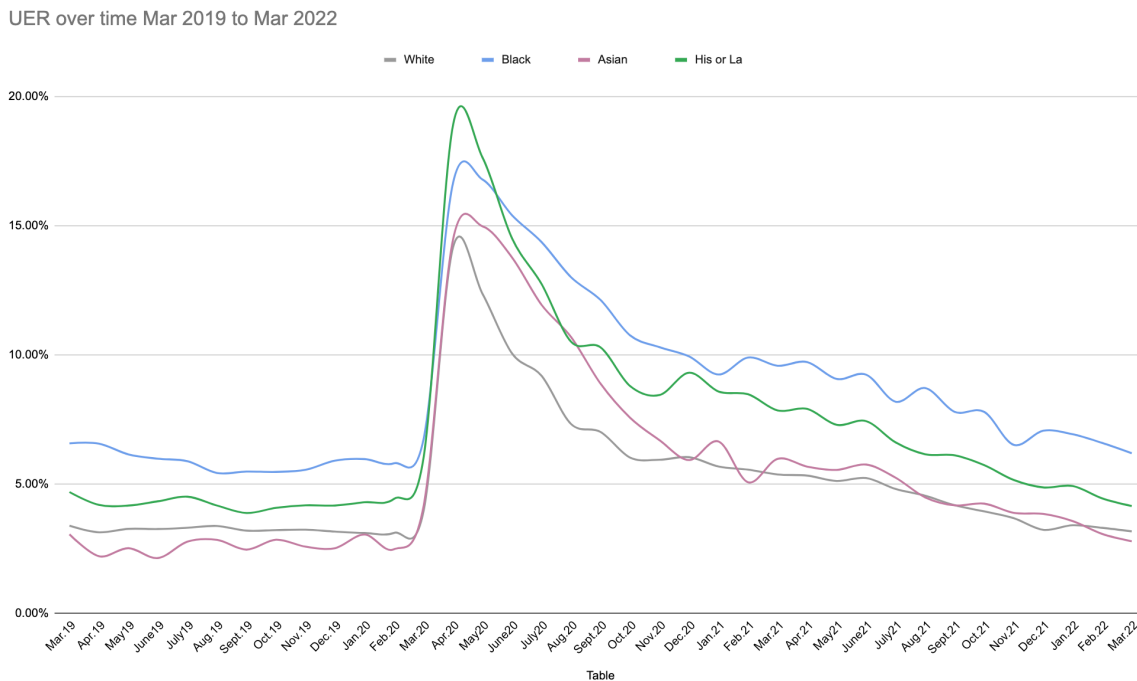
**Figure 2: The US unemployment rate from March 2019 to March 2022**

example, a student commented, “I notice that the system for collecting unemployment numbers is vague and lacks robustness. I wonder about the mechanisms and motivations for the people who conduct unemployment statistics.” Here we coded this response as *concerning the data* and put it under *questioning-motivation of data collectors*. Another student added, “How does the BLS (Bureau of Labor Statistics) account for individuals working under-the-table, and also

underemployment?” We labeled this response as *questioning-the uncounted population* under the same category as *concerning the data* code.

Students were then asked to compute the unemployment rates for White, Black, Asian, and Hispanic groups of population. For example, in the Asian population, students used the number of unemployed Asian workers divided by the number of Asian civilian labor force. As students moved further into modeling these rates using graphs and charts, they focused much of their discussion on making connections between the data (graphs, unemployment rates, etc.) and the events that happened during those times. Comments such as “... the fact that everything changes good or bad pretty much at the same time, you know the event wasn't racist, but the systemic racism is how it affected how high or low they went up at the ground...” or “significant disparity between racial groups, Black unemployment rate is almost double the White unemployment rate” occurred more often. For the two previous statements, we coded them as *noticing the disparity* with a subcode as *systematic racism*. They noticed the disparity between racial groups which negatively impacted the Black, Indigenous, and People-of-Color populations; recognized the existence of systemic racism in our social structure; and made an implication that systemic racism might be a causative factor for this disparity.

Next, students worked on populating the unemployment rate by racial groups across time and used visual graphs to represent the information. Figure 3 was one example of the graphs on the unemployment rate from Mar. 2019 to March 2022. When students examined the data and the graph, especially observing the spike of the unemployment rate during the full



**Figure 3: The US unemployment rate from March 2019 to March 2022 according to racial groups**

government shutdown period of 2020, they noticed, "... we wanted to see the comparison between before the pandemic hit and where we were looking at and that's exactly what we just saw is the pre-pandemic unemployment rates. So this everything is higher." Students not only reasoned how real-world events impacted the behavior of the data (graphs),

Massive spike after Covid hit. Asian population most heavily affected relatively, not particularly surprising considering the prevalence anti-Asian rhetoric and the sentiment it caused. Asian unemployment rate slower to recover to pre-pandemic levels relative to other groups. Hispanic/Latino hit the hardest at the start of the pandemic, but it recovered faster relative to other groups, quickly falling below black unemployment rate.

They also explored which factors they needed to gain further insight on using the data, "I would like industry chart numbers...like you could even see movement between industries, and I think that would be really helpful to give further insight." The three previous statements were coded as *connection between world events and mathematics*.

**Task 2.** As part of task 2, students did research based on their interest topic to investigate the reason for the disparity. The three topics such as how discriminatory hiring practices, the inequity in education opportunities, and the underemployment impacted the racial differences in the unemployment, were chosen. When doing their research, one group noticed:

We found that there's a negative correlation between the level of education at school and the unemployment number. The more education, the less unemployment. The dropout rates over a period of like 93 to 2014 so see that includes astronomically high at 33% and the Black is at 50%, which is also really high and then over time they came down to where Black is within 2% of White and Hispanic is still like double everyone else, they have much, much lower. So that there's been improvement over time in those in those categories.

We noticed that students used traditional mathematics such as comparing the percentages of high school dropout rate among racial groups to justify their sociopolitical mathematics such as the higher the level of education resulted in the lower unemployment rate. With the statement above, we labeled it as *justification for socio-awareness* code. As students moved further into their research, investigating the causes for the disparities among racial groups, their conversations became more reflective and they started to discuss possible practices to combat forms of oppression and criticized practices that perpetuate oppression. One student discussed the hiring practice of removing applicants' names from resumes to reduce bias based on name:

This is not like this is just kind of like individual thing by individual companies, so there isn't like a you know, like a like a joint effort from like a group of companies so. I guess there's two trying to do something to kind of get rid of the discrimination, you know, like the bias to were like Black job applicants, but for the most part, I think the effort is really limited.

For this response, we coded as *socio-action*.

**Applying the Critical Mathematics Consciousness Growth Framework to students' responses in task 1 and task 2 (Percentage Breakdown).** Through the Critical Mathematics Consciousness Growth framework (see Figure 1), during task 1 group discussion, 35% of students' statements were coded at *disempowered* level. For example, students made statements such as, "The unemployment rate among non-white population is higher than that of White population, especially for Black and Hispanic populations" or "It definitely seems like it hit different groups differently, which is really interesting." These responses let us know that the inequity happened to certain groups of the population, but we had yet to know if students had any action plan to combat this disparity. As we looked further, around 4% of the statements were at *systemic semi-transitive* level. For example, students commented, "we've talked about how

like people's labor and the ability to work is related to their skill and their ability and not their race, so these. These numbers would kind of highlight systemic racism like we've been talking about..." Students recognized the pervasiveness of the inequality; however, they had not yet expressed the desire to act. No other level was recorded during this discussion. We noticed students' *disempowered* and *systemic semi-transitive* level statements happened more at the beginning of the discussion, however, they diminished as students dug deeper into traditional mathematics and used more technical language to analyze the behavior of the graphs.

During the task 2 group discussion, 58% of the students' statements were at *semi-transitive* level, 33% were at *critical transitive* level, 8% were at *disempowered* level and the rest were inconclusive. As students moved toward the end of task 2, the *critical transitive* level statements started to show up more in the discussion. Students demonstrated their knowledge of the pervasiveness of the systemic racism in our society and they acquired the agency for understanding how they could act against the displaced oppression. For example, one student reflected:

We have data from the government, we could totally use this data to to regulate these corporations that have no regulation ... every administration, independent of the party, keeps giving them more lax and more rope that we get hung on.

In this case, students acknowledged that they had the power to influence the government to impose regulations on the corporations in question and they were willing to do so.

### **How Students Use Mathematics to Deepen Their Understanding of the World**

Lastly, in order to investigate how students use mathematics to deepen their understanding of the world, we identified students' mathematics actions, such as concretely using numeric data and visually interpreting graph behavior by correlating it with events that occurred in real life. Furthermore, they expanded their knowledge by effectively utilizing visual graphs to represent information. Table 1 lists some of the students' mathematics actions and correlates them with their responses.

**Table 1: Sample of Students' Mathematical Actions and Corresponding Responses**

Sample of the students' mathematical actions throughout the tasks	Sample of students' responses
Concretely using evidence through data to justify how Covid 19 pandemic causes the high unemployment rate	..we wanted to see the comparison between before the pandemic it and where we were looking at and that's exactly what we just saw is the pre pandemic unemployment rates so this everything is higher.
Comparing data/math overtime or across timeline	The string a few others is big bend on the downhill slope until I realized that that was one of our data points...
Predicting the causes/reasons for the behavior of the graph	And how more affluent employees, whether they're white or Asian or more skilled workers might have had more savings and more ability to

	take the less desirable job or live off of the out of the workforce off of their savings when everybody was sent home and so.
Understand the strength and weakness of visual graphs	...our initial graph of the representation of employment was a scatter plot and we concluded that it like puts an emphasis on each individual month and each individual discrete case and there's less of an emphasis on trends....

### Discussion

As we implemented the tasks, students were initially showing the *disempowered* level in which they recognized the disparity between unemployment rates among the racial groups. However, they did not offer any concrete social action to promote equality. As we moved further through the lessons, students displaced the *systemic semi-transitive* level in which they identified the systemic racism in our society as one of the causing factors for this discrepancy, but a practical plan was not yet formed. It was during task 2 that students provided some solutions, such as influencing government regulation. In addition, students noticed that they have the power to make an impact on the current situation. Finally, through our observation of students' mathematics actions, they used mathematics to expand their understanding of the world by analyzing both numeric and visual data and further correlating them with real-life situations.

Through this lesson of real-world critical mathematics, we could observe students' critical consciousness as it surfaced. Although it took quite a strenuous process and arduous time for collecting data and planning out the sequence of the tasks, we successfully contributed a real-world critical mathematics lesson that contained practical mathematics knowledge and promoted students' critical consciousness through sociopolitical mathematics awareness. When mathematics tasks are purposefully designed around a current critical issue that is affecting students' lives and highlighting the oppression in our current socio-political structure, we support students in developing a meaningful understanding of their world and creating a space for them to have meaningful sociopolitical *and* mathematical discussions. As Gutiérrez (2013) argued, "In that sense, taking the sociopolitical turn is a necessary chapter in mathematics education," (p 60).

### Acknowledgment

I would like to acknowledge my colleague Diana Salter, who supported me through modifying and implementing the tasks.

### References

- Bakker, A., & van Eerde, D. (2015). An introduction to design-based research with an example from statistics education. In *Approaches to Qualitative Research in Mathematics Education: Examples of Methodology and Methods*. Springer.
- Berlin, R., & Berry III, R. Q. (2018). Confronting the lies I tell myself. In S. Crespo, S. Celedón-Pattichis, & M. Civil (Eds.), *Access and Equity: Promoting High-Quality Mathematics in Grades 3-5*. National Council of Teachers of Mathematics.



- Brantlinger, A. (2013). Between politics and equations: Teaching critical mathematics in a remedial secondary classroom. *American Educational Research Journal*, 50(5), 1050–1080. <https://doi.org/10.3102/0002831213487195>
- Creswell, J. W. (2014). *Educational research: Planning, conducting, and evaluating quantitative and qualitative research*. Pearson.
- Ernest, P. (1991). *Philosophy mathematics education*. Routledge.
- Freire, P. (1970). *Pedagogy of the oppressed* (pp. 71–86). Bloomsbury Academic.
- Freire, P. (1973). *Education for critical consciousness*. Sheed and Ward.
- Gutiérrez, R. (2013). The sociopolitical turn in mathematics education. *Journal for Research in Mathematics Education*, 44(1), 37. <https://doi.org/10.5951/jresmetheduc.44.1.0037>
- Gutstein, E. (2018). Reading and writing the world with mathematics: Enacting solidarity with students, families, and communities. In D. Y. White, A. Fernandes, & M. Civil (Eds.), *Access and Equity: Promoting high-quality Mathematics in Grades 9-12*. National Council of Teachers of Mathematics.
- Heid, M. K. (2010). Editorial: Where’s the math (in mathematics education research)? *Journal for Research in Mathematics Education*, 41(2), 102–103. <https://doi.org/10.5951/jresmetheduc.41.2.0102>
- Melhuish, K., Thanheiser, E., White, A., Rosencrans, B., Shaughnessy, J. M., Foreman, L., Riffel, A., & Guyot, L. (2022). The efficacy of research-based “mathematics for all” professional development. *Journal for Research in Mathematics Education*, 53(4), 307–333. <https://doi.org/10.5951/jresmetheduc-2019-0053>
- Musgrove, J., & Willey, C. (2018). Problem-Based mathematics learning in urban spaces: Exploring mathematical circles in youth’s community circles. In D. Y. White, A. Fernandes, & M. Civil (Eds.), *Access and Equity: Promoting high-quality Mathematics in Grades 9-12*. National Council of Teachers of Mathematics.
- National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Authors.
- National Council of Teachers of Mathematics. (2014). *Principles to Actions: Ensuring Mathematical Success for All*. Reston, VA: Authors.
- North American Chapter of the International Group for the Psychology of Mathematics Education. (2019). *PME-NA Equity Statement*.
- Register, J., Stephan, M., & Pugalee, D. (2021). Ethical reasoning in mathematics: New directions for didactics in U.S. mathematics education. *Mathematics*, 9(8), 799. <https://doi.org/10.3390/math9080799>
- Stephan, M., Register, J., Reinke, L., Robinson, C., Pugalenti, P., & Pugalee, D. (2021). People use math as a weapon: critical mathematics consciousness in the time of COVID-19. *Educational Studies in Mathematics*, 108, 513–532. <https://doi.org/10.1007/s10649-021-10062-z>
- Thanheiser, E. (2023). What is the mathematics in mathematics education? *Journal of Mathematical Behavior*, 70, 101033. <https://doi.org/10.1016/j.jmathb.2023.101033>
- Wagner, D. (2017). Reflections on research positioning: Where the math is and where the people are. In *The Disorder of Mathematics Education: Challenging the Sociopolitical Dimensions of Research* (pp. 291–306). Springer International Publishing.
- REDESIGNING MATH INTERVENTION TOWARDS AGENCY AND ENGAGEMENT: GAMES AND UNFINISHED LEARNING IN MULTIPLICATION**

Rachel Lambert  
UC Santa Barbara  
rlambert@ucsb.edu

Tomy Nguyen  
UC Santa Barbara  
tomynguyen@ucsb.edu

*While many students have lost opportunity in mathematics to the pandemic, those effects have been compounded for students with low-income backgrounds, multilingual learners, and students with disabilities. We seek math intervention that promotes mathematical agency and engagement to support student identities and equity more broadly. In collaboration with our local school district, we are currently piloting a 2-times a week Games Time where all students play non-digital mathematical games (grades 3 and 4 in 2 schools). We present analysis of the surprisingly limited research on non-digital games in math education, including emerging work on supporting multilingual learners through games. During the pilot, we will research the efficacy on math knowledge in the area of multiplication using interviews and assessments. In a collective Design Team, we will redesign based on the experiences of students and teachers.*

Keywords: Affect, Emotion, Beliefs, and Attitudes; Instructional Activities and Practices; Number Concepts and Operations; Equity, Inclusion, and Diversity

This poster centers on the problem of engaging students in intervention on unfinished learning. We seek to address the issue of unfinished learning in ways that engage, even delight, our students. We know that they respond particularly positively to games. We do so to make sure that students feel agentic in their learning, and empowered rather than passive, knowing that this is a central tenet of providing equity in math for racially and culturally diverse youth (Seda & Brown, 2021). We also believe that this agentic approach can support student social and emotional development (Myung et al., 2021). We were surprised at the limited amount of research in math education on non-digital games (both board and card games). Our poster will discuss the existing research and present initial findings.

Our research questions are as follows:

RQ1: What are the effects of playing mathematical games 3 times a week for 30 minutes on student learning in addition, subtraction, multiplication and division, particularly for students who are currently below grade level expectations in these areas? How does it affect understanding of conceptual basis of multiplication, including understanding of arrays and area model?

RQ2: What are the experiences of students and teachers in this project? What implementation barriers do they identify? What supports students' engagement and teachers' work? How do they innovate within the structure?

### References

- Myung, J., Kimner, H., Cottingham, B. W., Diaz Luna, S., Shiels, S., & Hough, H. J. (2021, May). *Restorative restart: The path towards reimagining and rebuilding schools* [Report]. Policy Analysis for California Education. <https://edpolicyinca.org/publications/restorative-restart/>
- Seda, P., & Brown, K. (2021). *Choosing to See: A Framework for Equity in the Math Classroom*. Dave Burgess Consulting, Incorporated.

## RETURNING TO DISRUPTING DEFICIT NARRATIVES: FUNDS OF KNOWLEDGE BEYOND CURRICULUM

Aditya P. Adiredja  
The University of Arizona  
adiredja@math.arizona.edu

Marta Civil  
The University of Arizona  
civil@math.arizona.edu

Keywords: Equity, Inclusion, and Diversity; Professional Development; Undergraduate Education, Research Methods

We examine the potential for instructor-led student interview in disrupting narratives about minoritized university students and their families. This approach was inspired by a critical component of the original Funds of Knowledge for Teaching project (FoK), the home visit, which involved ethnographic interviews with students and their family to learn about their lives. Study group reflections with project teachers and researchers led to the teachers' examination of biases and disruption of deficit stories about students and their families (Tenery, 2005).

Two frameworks guided the design and the analysis of the study. The *anti-deficit framework* for students from minoritized groups (Adiredja, 2019) prioritizes unpacking students' existing understandings and resources for learning, and explicitly constructing counter-stories about students and their mathematics. The study also interprets the results of the study from a *socio-ecological perspective* (Louie & Zhan, 2022). Story construction is attributed to the individual, the learning community they are in, and the broader societal context.

The data comes from a larger project focusing on engaging university instructors in community learning project focusing on race, gender, and mathematics (NSF DUE 2021313 IUSE). The project began in the summer and the interview occurred in the following Spring. Six participants interviewed a student of their choice, and four attended a follow-up debrief meeting.

Results showed that participants gained novel insights into their students' lives. The focal case, Zaynah shows that the interview can also change an instructor's deficit story about a student and their mathematics into an anti-deficit story. Zaynah's post-interview description of her student, Maria, went from someone who had "weak algebra skills" and was "just kind of there," to a student who had a well-researched and articulated academic and life goals, and a "passion" that Zaynah "knew nothing about." Zaynah accepted that math "wasn't [Maria's] thing," and that "she can and will do the math" to achieve those goals.

The anti-deficit framework and the socio-ecological perspective rejected a simple narrative of a deficit-oriented instructor who changed after the interview experience. Zaynah is not a deficit-oriented instructor. Her description of her other students in the workshop—all women of color—were positive, with no mention of deficits. Instead, it is a more complex story of the interplay of narratives about mathematics and students of color, and the impact of the interview with a student and the role of the community in shifting deficit stories. It is showing promise of applying FoK project's core principles in undergraduate mathematics beyond curriculum design.

### References

- Adiredja, A. P. (2019). Anti-deficit narratives: Engaging the politics of mathematical sense making. *Journal for Research in Mathematics Education*, 50(4), 401-435. <https://doi.org/10.5951/jresmetheduc.50.4.0401>
- Louie, N., & Zhan, W. Y. (2022). A socio-ecological framework for research in mathematics education. *Journal for Research in Mathematics Education*, 53(5), 365-371. <https://doi.org/10.5951/jresmetheduc-2022-0003>
- Tenery, M. F. (2005). La Visita In N. Gonzáles, L.C. Moll and C. Amanti (Eds.), *Funds of knowledge: Theorizing practices in households, communities, and classrooms* (pp. 119-130). Mahwah, NJ: Lawrence Erlbaum.

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

## SITUATING PRESERVICE TEACHERS' EMERGING PROFESSIONAL IDENTITIES IN THE GEOSPATIAL FRAMEWORK

Rose Mbewe  
Purdue University  
rmbewe@purdue.edu

Hillary Ongoyo Omoze  
Purdue University  
homoze@purdue.edu

Keywords: Culturally Relevant Pedagogy, Social Justice, Teacher knowledge

Adjusting to new learning institutional environments, student beliefs and actions, and mentor pressure has been daunting for novice teachers (Losano et al., 2019). Losano et al. argue that newcomers face the contradiction between their theoretical knowledge from educator programs and the reality in the field. The situation contrasts novice teachers' evolving professional identities, professional practices, and 21st-century ways of teaching and learning. While educator curricula are explicit about content knowledge and pedagogical content knowledge (Shulman, 1978), novice teachers' personal and professional identities are often ignored or assumed to be developing as they go through the system. Teacher education curricula must, therefore, deliberately seek to ground novice teachers in theory and practices that sufficiently help consolidate their professional identities (Shulman, 1986).

Novice teachers committed to social justice may be inducted into a profession where standardization and accountability are the norms (Costigan et al., 2004). When new teachers enter the field of education, their beliefs about what social justice looks like and how it should be enacted in the classroom are far from uniform. Instead, such allegiances emerge from a complex set of circumstances and subjectivities that include teachers' prior experiences, their preservice programs, and their particular social localities. As teacher education programs emphasize social justice, equity, and diversity as central concerns in the professional preparation of teachers, social justice teacher education programs view preparing teachers "with the knowledge, dispositions, and practices to work with students from diverse backgrounds as a fundamental responsibility of teacher education" (McDonald, 2007, p. 2048).

In an effort to improve student learning, teacher education programs may ask preservice teachers to draw on their students' culture, family, and interests to create a pedagogy that is culturally relevant to students' experiences within and outside of the school setting. Such pedagogy makes constructive use of the wealth of knowledge in and outside of the classroom. Culturally responsive teaching or pedagogy builds curriculum and pedagogy on the cultural frames of reference and linguistic strengths of historically oppressed communities (Barnes, 2006; Ladson-Billings, 1990; Sleeter, 2005; Zeichner, 2003). Pedagogy constructed through understanding students' culture, background, and interests has been shown to improve student learning in the classroom (Sleeter, 2005).

The study used a qualitative design. Data was collected from preservice teachers in classes both researchers instruct who are on their teaching attachment. We employed the narrative inquiry framework as an analytical tool (Riessman 2002, 2005) to answer the question, *in what ways do teacher educator programs prepare novice teachers for the challenges in varying geospatial locations?* Preliminary findings reveal that by engaging in critical self-reflection and acquiring theoretical tools, teacher education programs may prepare teachers to teach using culturally relevant pedagogies (Lane et al., 2003), be agents of social change (Sleeter

& Grant, 1999), and teach to change the world (Oakes & Lipton, 2003) in different geospatial spaces.

### References

- Barnes, C. (2006). Preparing preservice teachers to teach in a culturally responsive way. *The Negro Educational Review*, 57(1/2), 85-100.
- Ladson-Billings, G. (1990). Like lightning in a bottle: Attempting to capture the pedagogical excellence of successful teachers of black students. *Qualitative Studies in Education*, 34(4), 335-344.
- Losano, L., Fiorentini, D., & Villarreal, M. (2018). The development of a mathematics teacher's professional identity during her first-year teaching. *Journal of Mathematics Teacher Education*, 21(3), 287-315.
- McDonald, M. (2007). The joint enterprise of social justice teacher education. *Teachers College Record*, 109(8), 2047-2081.
- Oakes, J., & Lipton, M. (2003). *Teaching to change the world* (2nd ed.). Boston: McGraw Hill.
- Shulman, L. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard educational review*, 57(1), 1-23.
- Sleeter, C., & Grant, C. (1999). *Making choices for multicultural education: Five approaches to race, class, and gender* (3rd ed.). Upper Saddle River: Prentice-Hall, Inc.
- Sleeter, C. (2005). *Un-standardizing curriculum: Multicultural teaching in the standards-based classroom*. New York: Teachers College Press.
- Zeichner, K. (2003). The adequacies and inadequacies of three current strategies to recruit, prepare, and retain the best teachers for all students. *Teachers College Record*, 105(3), 490-519.
- Riessman, C. K. (2002). Positioning gender identity in narratives of infertility. *Infertility around the globe: New thinking on childlessness, gender, and reproductive technologies*, 152-170.
- Riessman, C. K. (2005). Exporting ethics: A narrative about narrative research in South India. *Health*: 9(4), 473-490.
- Lane, K.L., Wehby, J. H., Little, A. M., and Cooley, C. (2005). Students educated in self-contained classrooms and self-contained schools: Part II—how do they progress over time? *Behavioral Disorders*, 30(4), 363-274.
- Costigan, A. T., Zumwalt, K. K., & Crocco, M. S. (2004). *Learning to teach in an age of accountability*. Routledge.

## THE DEHUMANIZATION OF HEARING “I HAVE ALWAYS HATED MATH”

Becca Jarnutowski  
University of Arizona  
jarnutowski@math.arizona.edu

Aditya Adiredja  
University of Arizona  
adiredja@math.arizona.edu

*We share our theorizing about a common statement regarding a math person that is often perceived as innocuous in society. More specifically, we are referring to people’s responses when an individual shares that they are studying mathematics, such as “I have always hated math” and “Oh, you must be so smart.” We draw on the notions of marked category, narratives, dehumanization, and microaggression. We use these theoretical constructs to argue that people’s responses are an instantiation of mathematics as a marked category and that they function as microaggressions, especially for minoritized students who are multiply marked. Moreover, due to their prevalence, they can contribute to students’ active choice of not doing mathematics in order to prioritize their humanity. Our report further highlights the importance of mathematical microaffirmations and the development of sub-communities within mathematics.*

Keywords: equity, inclusion, and diversity, undergraduate education, LGBTQIA+

“Statisticians may not play football or date cheerleaders, but they can provide statistical guidance for football coaches.” (Wheelan, 2013, p. 76). This quote adds another layer to the stereotype of a statistician (or a mathematician) as an old white heterosexual man. Rhetorically the author is arguing for the worth of a statistician despite their lack of athleticism and social ineptitude. Other non-human characteristics, like being cold or robot-like are also common in describing a mathematician. This has led scholars and mathematicians to disrupt these stereotypes and highlight the humanity behind mathematics (e.g., MathematicallyGiftedandBlack.com, Lathism.org, *Mathematics for Human Flourishing*, 2020). In this paper, we further analyze the idea of being a mathematician or having mathematical ability as deviant and its potential contribution to people actively choosing *not* to study mathematics. We examine how such an idea gets circulated in everyday interactions and discuss potential ways to mitigate it.

There has been a focus on how to get more people into mathematics and to stay in mathematics with particular attention on students with minoritized backgrounds. Having a supportive community and a sense of belonging meaningfully contribute to people’s persistence and success in mathematics. Unfortunately, many of the findings in the literature documented the struggle for finding such a sense of belonging. We are becoming increasingly aware of microaggressions (e.g., Yang & Carroll, 2018; Kim & Meister, 2022) and other dehumanizing experiences, like the burden of managing the threat of stereotypes for racially minoritized students (e.g., Leyva et al., 2021; McGee & Martin, 2011). A welcoming and affirming community, mathematics is not, at least not yet.

This makes us wonder about opportunities for students to affirm their choice to study mathematics or their enjoyment of mathematics. Research interviews with students about their choice of major can end up becoming such an opportunity where they get to affirm their choice of studying mathematics or other STEM fields (e.g., Leyva, 2016; Leyva et al., 2022). These interviews often remind students of their motivation for choosing their field of study. We are interested in another event that can function as an affirming event for students. This event typically starts with a question, “What are you studying at the university?” or “What do you do?”

and after some calculation, the person might respond with “I study mathematics” or “I am a graduate student in mathematics.” Then comes the reply.

These replies take many forms including statements such as “I have always been bad at math,” “I have always hated math,” “you must be really smart,” or “you must have a little bit of Asian in you” (Shah, 2017). The statement of “I do math” is a (rare) opportunity when a student explicitly affirms their membership in the mathematics community. Yet, we argue that the typical responses do not affirm the individual’s choice, even the ones that attribute intelligence to the person. Instead, it distances the individual from the responder. In other words, these responses end up becoming microaggressions that dehumanize the individual. The prevalence of these dehumanizing responses does not normalize or minimize the harm, instead, it heightens the threat, especially for individuals whose identities have historically been excluded in mathematics.

### Theoretical Framework

We draw on several theoretical constructs to make sense of the phenomenon. Specifically, we draw on notions of deficit discourses in mathematics education (Adiredja & Louie, 2020) and dehumanization in the context of Damarin’s (2000) idea of mathematical ability as a *marked category*. We further these ideas by incorporating microaggression theory (Sue, 2020) as a means of operationalizing these constructs in everyday interactions. All these constructs and theories focus on the interrelationship between identity, power, and discourse, and as such we are examining mathematics learning as a sociopolitical activity (Valero, 2004; Gutiérrez, 2013). And thereby, we view race and gender not just as a social marker of identity, but as a dynamic social construct that gets negotiated in interactions (Esmonde, 2009).

Marked category, as explained by Damarin (2000), refers to interrelated social processes that produce and reproduce groups as either normal or deviant in the context of a larger functional group. Thus, marked categories refer to groups of individuals who share marks of deviance based upon larger popular culture. For example, women, people of color, individuals in the LGBTQ+ community, and people with disabilities are commonly constructed as marked categories (Damarin, 2000). Individuals who exist in multiple marked categories, being doubly or multiply marked, end up at the margins of each of their marked communities since their membership in one category marks them as deviant in the other. We can see this double marking being called upon in Black Feminist work such as *All the Women are White, All the Blacks are Men, but Some of Us are Brave* (Hull et al., 1981 as cited in Damarin, 2000) which speaks to the invisibility of Black womanhood.

Damarin establishes the mathematically able as a marked category due to the popular representation of mathematicians as socially inept, hopelessly heterosexual individuals separate from general society, along with the construction of mathematical ability as something that is genetically determined within humans. The quote at the start of our paper is an example of this marked category at play in literature. Moreover, research surrounding popular conceptions of mathematicians’ identities construct them as excessively intelligent (e.g., Piatek-Jimenez, 2008; Shah, 2019), obsessively passionate (e.g., Bartholomew et al., 2011, Mendick, 2005), socially inept (e.g., Piatek-Jimenez, 2008; Hall & Suurtamm, 2020), and innately able (e.g., Bartholomew et al., 2011; Jaremus et al., 2020). Together these narratives work to stereotype mathematicians at the extremes of excess and deficit, with consequences such as establishing mathematicians as incompetent in basic life skills (Picker & Berry, 2000), unemotional and cold (Ernest 1995), and uncreative (Mendick, 2005). These narratives have disparate effects and can be utilized in a large variety of settings.

Shah (2019) extended the application of math ability as a marked category by showing that a narrative of excess can also be a dehumanizing racial narrative. Shah's (2019) analysis of the statement "Asians are good at math" demonstrates its dehumanizing impacts on Asian personhood by positioning them as racial subjects. Shah argued that mathematical ability is used as a proxy for excessive intelligence at the expense of core humanistic traits. This type of narrative of excess reduces Asian people to a calculator or a tool, which strips the person of their autonomy and their ability to reason. Shah explained the connections of this intelligence with the conception of a hyper-evolved android further marks Asians as subhuman and deviant. We extend this narrative of excess and the ways it threatens personhood to other dominant discourses about gender and sexuality in mathematics.

In contrast to narratives of excess, narratives of deficits are more commonly discussed in mathematics education research. Narratives about the underperformance of different racial groups and women are quite common in mathematics education (e.g., Leyva, 2016, Jaremus et al., 2020, Mendick, 2005). As a narrative, deficit narratives are also socially constructed and reified. However, deficit narratives about minoritized groups in mathematics play an important role in broader deficit discourses in mathematics education (Adiredja & Louie, 2020). Supported also by dominant narratives about mathematics (as hierarchical, apolitical, etc.), and dominant institutional practices (e.g., standardized testing as the main form of assessment), deficit discourses systematically function to create differences between groups and utilize these differences as substantiation of the inferiority of marginalized groups (see also the racial hierarchy of mathematics ability, Martin, 2009). The positioning comes as a result of over-emphasis of assumed intellectual shortcomings of students from marginalized groups and attributing these shortcomings to their families, communities, and cultures.

Narratives of both excess and deficit play a role in dehumanizing people by positioning them to have too much or too little of what makes someone human. As Shah (2019) has argued, as part of the racial contract in the US, Asian people have an excess of mathematical ability that renders them not human, and Black, Latinx, and Indigenous people have too little mathematical ability that makes them subhuman, leaving only white people as the only group having the right amount of mathematics ability as humans. More generally, dehumanization occurs when "...individuals are necessarily deprived of human dignity at the same time that they are deprived of the possibility and opportunity to shape their own lives, to be seen and heard, in the community of others" (Ericson, 1991, p. 34). At the core of humanity is personal autonomy and being recognized as a member of a community. This dehumanization and threatening of someone's human dignity happens both on the individual level and within entire systems.

Microaggressions operationalize dehumanizing narratives in everyday interactions. They are defined as brief and subtle verbal, behavioral, and environmental indignities toward members of minoritized groups (Sue, 2020). Microaggressive statements may appear innocuous and harmless, but they have detrimental impacts on the psychological health of recipients due to their continual and everyday nature. Solórzano and Perez Huber (2020) draw connections between racial microaggressions to larger systems of racism through a tree model of the white supremacist roots of racial microaggressions. Within this model, the roots represent white supremacy, the trunk represents institutional racism, and the leaves represent racial microaggressions. They use this model to demonstrate how microaggressions reify racism in everyday interactions as well as maintain systems of institutionalized racism.



### Sources of Data Illustration

We draw on data from two separate research studies to illustrate our theorizing. One study focuses on the beliefs, knowledge, and practices of prospective and beginning middle and high school teachers related to equity. The other focuses on the impact of professional development on university instructors' understanding of anti-deficit teaching of mathematics. We included two conversations during which an individual shared the impact of people's responses to their studying mathematics. The fact that these individuals shared similar interactions in two separate studies, while also not being prompted, highlights the prevalence and impact of such interaction.

The two excerpts referred to the perceived impact of people's responses to the statement that "I do math." The first excerpt is from Chris, an undergraduate student in mathematics education who identified as white and non-binary. The second excerpt comes from a dialogue between three white individuals, two doctoral students in mathematics and one instructor in a mathematics department. One is the first author, and the others were Cedric and Hannah. Cedric, the center of the excerpt, identified as a cis white gay man.

### Positionality Statement

The first author is a white graduate student in mathematics and mathematics education. They identify as disabled and trans non-binary. They are interested in this topic because of their experience in hearing these types of responses and the impact such responses had on their identification with mathematics. The second author is a queer Asian male mathematics educator. While he often hears similar responses as ones that are the focus of this paper, he was initially skeptical about the impact of such statements. It was through working with the first author and hearing the interactions below that he became interested in the issue.

### Data Illustration and Analysis

The phenomenon came up with Chris when the interviewer was asking about what they had learned about equity in their teacher education program. Chris began the conversation with the use of carceral pedagogy within classrooms (e.g., the disproportionate rate of Black students being asked to leave the classroom) and student ideas being undervalued. They were then prompted to connect back specifically to mathematics. Chris responded with mathematics as a "white cis-gendered heteronormative space" and connected it to people's responses to their being a math major.

*Chris:* Yeah. Math, math is something that I'm really passionate about with this. Because I feel like it kind of goes hand in hand with science and the kind of cold understanding of what it means to do math and be a mathematician. Where it's like, it's a very kind of white cis-gendered heteronormative space where it's like, males are privileged really high. It's there's, there's a, there's a feeling, a cultural kind of feeling of what a good mathematician looks like. And it's, it's important for me to kind of break out of that.

Um, and it's, it's unfortunately something that I see a lot, I've met a lot of adults where I'll be saying that I'm a math major and they'll be like, "Oh, I was never good at math. I hate math." And it's, it's just brutal because it's, these people are in these groups. Like it's never, you know, a cis white guy that I'm talking to. Who's like, oh, I hate math. It's always women most of the time. And it's just like, it just, it's brutal.

Chris shared their awareness of dominant discourses ("cultural kind of feeling") about what a mathematician looks like and who is presumed to be good at mathematics, mainly white cis-gendered heterosexual men. This type of narrative is relational (see also Shah, 2019 for racial

narratives). As much as it defines who can be good at math, it also defines who is not good at math. Interpreted in this way, Chris was aware of this narrative of deficit attributed to non-dominant groups, including women and non-cis-gendered people. During their interview, issues of gender were very present in the discussion. So, it was not surprising that they connected the discussion about mathematics to gender.

We interpreted Chris' use of "brutality" to mark people's responses as a prevalent microaggression. The dominant discourse functioned as a constraint on their mathematical identity, from which they felt the need to "break out" of. However, their attempt to break out of the deficit discourse by sharing their major was frequently met with the very same discourse. Moreover, this constraining discourse also came from women. Thus, what was an opportunity for Chris to affirm their mathematical identity was instead met with a resistance by groups that could have been their co-conspirator in the fight. Being good at math challenged Chris' identification with this non-dominant group, i.e., non-white-cisgendered-heterosexual men.

Cedric shared similar accounts of his membership being challenged in a particular non-dominant group, mainly gay men. However, Cedric highlighted the dehumanization that comes from not being seen as a human being when he shared his mathematical identity. People's responses to Cedric stripped him of his humanity by putting him in the foreign box of being a math person, whose interests are not worth pursuing further in conversations.

*Cedric:* When Weston [another graduate student] was talking about, sort of, this mutual dehumanization, I had a very vivid sense memory. And I'm pretty sure all of us have had this experience of, you know, introducing yourself to somebody in a social setting, and they go, "Oh, what do you do?" you say, "I study math." And they go, "Oh, I hate that sh\*t, you're so smart. How do you do it?" And it's just like this weird, like, you don't feel like you're seen anymore. Just like, okay, cool, I'm off in this weird wilderness that's separate from you. And I don't know, I can't help but feel like—

*Hannah:* Dehumanized as "just like math."

*Cedric:* Well yeah. But yeah, there's something that I just feel like, curiosity in general about things. Is the way that I don't know, in my experience, like, I, if I'm, if somebody is curious about what I do, I find myself really engaged with that person. But if they just sort of go "ahh" and put me in a box, like, oh okay, you don't get it at all /.../

*Becca:* So, you just gave me another proof of concept for my thesis. So, I'm doing a thesis on the way that people respond when you bring up math. Typically, with "I hate math, I could never do that" or "You must be so smart to do that." And how, like, isolating that is, particularly when it's people that are within your same group and like— So if you're a woman and you say that I do math and then other women are like "Oh my gosh, that's the worst thing ever, I could never do it," how isolating that is from this group that you're supposed to identify with and you feel like you can't anymore. And then bring that into the math classroom and you also feel like you can't identify. That's kinda what I'm looking into. Very happy that it [this concept] happened in the wild.

*Cedric:* It happens a lot with gay men as well. So, there you go.

Consistent with Chris' account, Cedric also noted the prevalence of such responses from people ("all of us have had this experience"), especially from gay men, a community with whom Cedric closely identified. Hearing dissociation with mathematics from people "that are within your same group" is isolating, as Becca argued. Becca concluded that this double exclusion from

mathematics and from one's own home community makes it difficult to persist and claim ownership of studying mathematics.

In contrast to Chris, however, Cedric and Becca raised another common positioning in people's responses: "You're so smart." This positioning appears affirming as it attributes a positive characteristic to the individual. Yet it can also be dehumanizing in two ways. This narrative of excess functions similarly with the narrative of "Asians being good at math," in the way that it attributes surplus intelligence to a person. This excess intelligence stops others from relating to the individual while simultaneously stripping them of their humanity, becoming "just math," as Hannah said. The narrative puts Cedric in the foreign box separate from his own community. Returning to Damarin's idea of mathematical ability as a marked category, this form of smartness was not necessarily desirable.

### Discussion

In this theoretical report, we drew on several theoretical ideas to argue that responses like, "I have always hated math" to an individual sharing their math identity are dehumanizing. We highlighted the deviant positioning of excelling in mathematics by interpreting mathematics as a marked category. Narrative of excess allows us to interpret a statement that appears affirming as dehumanizing. The response is dehumanizing in the way that it strips the individual from core traits of being human, such as self-autonomy and relationality. In this report, we also highlight the dehumanizing impacts of these responses particularly for doubly marked individuals, like Cedric and Chris, who are both good at math and members of the LGBTQ+ community. The prevalence and everyday nature of these statements and their impact of othering individuals within particular groups qualify them as microaggressions.

Just like many microaggressions, one can dismiss the impact of these seemingly innocuous slights to an individual's identity. Yet, what might be the long-term impact of hearing negative responses about studying math to individuals? While future research can examine that question, we posit that the prevalence of hearing such responses can slowly chip away at someone's mathematical identity. We recognize this type of interaction as a rare opportunity wherein an individual gets to affirm their mathematical identity outside of the academic setting. If such an opportunity is met with resistance and threatens one's membership in a community, then individuals who are doubly marked can become further isolated in pursuing mathematics.

Microaggression theory also offers some ways to mitigate this potential othering and exclusion. Microaffirmations are "the often-subtle verbal and/or non-verbal strategies (moments of shared cultural intimacy) People of Color consciously engage that acknowledge and affirm each other's value, integrity, and shared humanity" (Pérez Huber, L., & Solórzano, D. G., 2015, as cited in Pérez Huber, L., 2018, p. 1-2). We extrapolate from the theory's original focus on racial microaffirmations to mathematical microaffirmations between people within the same doubly marked categories. This also highlights the importance of subcommunities within mathematics where such microaffirmations can take place.

To get at the long-term impact of these responses on mathematics students, Becca is planning to design and conduct a study focusing on this topic, where they can systematically interview BIPOC women and trans individuals in mathematics (i.e., multiply marked individuals) on their experiences with such responses and how they negotiate their mathematical identities. The study aims to examine the generality of the responses we shared in this report, as well as additional ideas on how to mitigate these dehumanizing microaggressions.

## Acknowledgment

This material is based upon work supported by the National Science Foundation under Grant Nos. 2021313 and Nos. 1758401.

## References

- Adiredja, A., & Louie, N. (2020). Untangling the Web of Deficit Discourses in Mathematics Education, For the Learning of Mathematics, 40(1). 42-46.
- Bartholomew, H., Darragh, L., Ell, F., & Saunders, J. (2011). 'I'm a natural and I do it for love!': exploring students' accounts of studying mathematics, International Journal of Mathematical Education in Science and Technology, 42(7), 915-924. <https://doi.org/10.1080/0020739X.2011.608863>
- Damarin, S. (2000). The Mathematically Able as a Marked Category, Gender and Education, 12(1), 69-85, <https://doi.org/10.1080/09540250020418>
- Ericson, D. P. (1991). Humanization, democracy, and political education. Studies in Philosophy and Education, 11(1), 31-43. <https://doi.org/10.1007/BF00368402>
- Ernest, P. (1995). Values, gender and images of mathematics: a philosophical perspective. International Journal of Mathematical Education in Science and Technology, 26(3), 449-462. <https://doi.org/10.1080/0020739950260313>
- Esmonde, I. (2009). Ideas and identities: Supporting equity in cooperative mathematics learning. Review of Educational Research, 79, 1008-1043. <https://doi.org/10.3102/0034654309332562>
- Gutiérrez, R. (2013). The sociopolitical turns in mathematics education. Journal for Research in Mathematics Education, 44, Equity Special Issue, 37-68. <https://doi.org/10.5951/jresmetheduc.44.1.0037>
- Hall, J., & Suurtamm, C. (2020). Numbers and Nerds: Exploring Portrayals of Mathematics and Mathematicians in Children's Media, International Electronic Journal of Mathematics Education, 15(3), <https://doi.org/10.29333/iejme/8260>
- Hull, G.T., Scott, P.B. & Smith, B. (1981) All the Women are White, All the Blacks are Men, but Some of Us are Brave: black women's studies, Old Westbury, NY, The Feminist Press.
- Jaremus, F., Gore, J., Prieto-Rodriguez, E., & Fray, L. (2020). Girls are still being 'counted out': teacher expectations of high-level mathematics students, Educational Studies in Mathematics, 105, 219-236. <https://doi.org/10.1007/s10649-020-09986-9>
- Kim, J. Y., & Meister, A. (2022). Microaggressions, Interrupted: The Experience and Effects of Gender Microaggressions for Women in STEM. Journal of Business Ethics, 1-19. <https://doi.org/10.1007/s10551-022-05203-0>
- Leyva, L. A. (2016) . An intersectional analysis of Latin@ college women's counter-stories in mathematics. Journal of Urban Mathematics Education, 9(2), 81-121 .
- Leyva, L. A., McNeill, R. T., Balmer, B. R., Marshall, B. L., King, V. E., & Alley, Z. D. (2022). Black Queer Students' Counter-Stories of Invisibility in Undergraduate STEM as a White, Cisheteropatriarchal Space. American Educational Research Journal, 59(5), 863-904. <https://doi.org/10.3102/00028312221096455>
- Leyva, L. A., Quea, R., Weber, K., Battey, D., & López, D. (2021). Detailing racialized and gendered mechanisms of undergraduate precalculus and calculus classroom instruction. Cognition and Instruction, 39(1), 1-34. <https://doi.org/10.1080/07370008.2020.1849218>
- McGee, E., & Martin, D. (2011). "You would not believe what I have to go through to prove my intellectual value!" Stereotype management among academically successful black mathematics and engineering students. American Educational Research Journal, 48(6), 1347-1389. <https://doi.org/10.3102/0002831211423972>
- Mendick, H. (2005). A beautiful myth? The gendering of being/doing 'good at maths'. Gender and Education, 17(2), 203-219. <https://doi.org/10.1080/0954025042000301465>
- Pérez Huber, L. (2018). Racial microaffirmations as a response to microaggressions. Research Briefs (No. 15). Los Angeles: Center for Critical Race Studies at UCLA.
- Pérez Huber, L., & Solórzano, D. G. (2015). Racial microaggressions as a tool for critical race research. Race Ethnicity and Education, 18(3), 297-320. <https://doi.org/10.1080/13613324.2014.994173>
- Piatek-Jimenez, K. (2008). Images of mathematicians: a new perspective on the shortage of women in mathematical careers, ZDM Mathematics Education, 40, 633-646. <https://doi.org/10.1007/s11858-008-0126-8>
- Picker, S. H., & Berry, J. S. (2000). Investigating pupils' images of mathematicians. Educational Studies in Mathematics, 43, 65-94. <https://doi.org/10.1023/A:1017523230758>
- Shah, N. (2017). Race, ideology, and academic ability: A relational analysis of racial narratives in mathematics. Teachers College Record, 119(7), 1-42. <https://doi.org/10.1177/016146811711900705>
- Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

- Shah, N. (2019). "Asians Are Good at Math" Is Not a Compliment: STEM Success as a Threat to Personhood, *Harvard Education Review*, 89(4), 661–686. <https://doi.org/10.17763/1943-5045-89.4.661>
- Solórzano, D. G., & Pérez, H. L. (2020). *Racial microaggressions : Using critical race theory to respond to everyday racism*. Teachers College Press.
- Su, F. (2020). *Mathematics for human flourishing*. Yale University Press. <https://doi.org/10.12987/9780300248814>
- Valero, P. (2004). Sociopolitical perspectives on mathematics education. In P. Valero & R. Zevenbergen (Eds.), *Researching the socio-political dimensions of mathematics education: Issues of power in theory and methodology* (pp. 5–24). Norwell, MA: Kluwer. <https://doi.org/10.1007/b120597>
- Wing Sue, D. (2020). *Microaggressions in everyday life: Race, gender, and sexual orientation*, 2nd Ed. Hoboken, NJ: Wiley.
- Yang, Y., & Carroll, D. W. (2018). Gendered Microaggressions in Science, Technology, Engineering, and Mathematics. *Leadership and research in Education*, 4, 28-45.

## THE IMPACT OF COLLABORATIVE INTERACTIONS ON A MATH TEACHER'S POSITIONING OF EMERGENT BILINGUAL STUDENTS

Ji Yeong I  
Iowa State University  
jiyeong1@iastate.edu

Coskun Erden  
Iowa State University  
cerden@iastate.edu

Jasmine Sourwine  
Iowa State University  
jasmines@iastate.edu

This research study explores how collaborative interactions between researchers and teachers can enhance the positioning of emergent bilingual (EB) students from an asset-based perspective. The research question that drives this study is: "How does a mathematics teacher transform her positioning through collaborative interactions with researchers to foster the mathematical learning and linguistic development of emergent bilinguals?"

### Positioning Framework

Positioning theory, the main framework in this study, explores how individuals are placed in conversations, co-constructing storylines (Davies & Harré, 1999). In education, it analyzes how discourse impacts students' problem-solving skills, reasoning abilities, academic competencies, and interactions with teachers. Reflexive positioning involves a person positioning themselves in a context, while interactive positioning involves others positioning a person or group in a context. In a classroom setting, reflexive and interactive positioning can be shaped by a teacher's implicit power, giving them rights, duties, and responsibilities and the authority to influence students' participation in activities (Zangorri & Pinnow, 2020).

### Methods

This study was conducted in an EB-only Algebra 1 classroom within a large urban school district. We collaborated with the teacher through co-developing and co-teaching mathematics lessons that embedded EBs' real-world contexts and co-analyzing students' data to better understand students' learning and social interactions in the classroom. We observed and videotaped one lesson before the first co-teaching and one lesson after the last co-teaching. Both pre/post lessons were planned and taught by only the classroom teacher to measure the teacher's growth through the year-long collaboration. The videotaped lessons were transcribed and read by three coders (authors) several times. First, each coder identified teacher discourses relevant to the teacher's positioning toward students by coding them as asset-based and deficit-based on Dedoose. Then, we sub-coded the focus of each coded excerpt in mathematics, language, and context. From the analysis, we developed storylines representing teacher positioning changes.

### Findings

The findings demonstrate an increase in asset-based teacher talks between the pre and post lessons in terms of number and percentage, which rose from 58.67% to 83.53%. Deficit-based codes decreased from 41.33% to 16.47%. We found the math teacher shifted her positioning on EBs in Algebra 1 class after having 1-year long collaboration regarding both teacher's authority and students' autonomy. In reflexive positioning, the general positioning pattern of the teacher moved from a conveyor of knowledge who has all authority in the classroom to a facilitator who shares authority with students. As for interactive positioning, the teacher's positioning on EBs shifted from dependent learners to independent learners and contributors. Our preliminary positioning analysis shows that the collaboration influenced the teacher's positioning to increase asset-based and decrease deficit-based discourses and practices.

## References

- Davies, B., & Harré, R. (1999). *Positioning and personhood*. In R. Harré & L. van Lagenhove (Eds.), *Positioning theory: Moral contexts of intentional action* (pp. 32-51). Blackwell: Oxford
- Zangori, L., & Pinnow, R. J. (2020). Positioning participation in the NGSS era: What counts as success? *Journal of Research in Science Teaching*, 57(4), 623-648. doi.org/10.1002/tea.21607

## UNDERGRADUATE MATHEMATICS INSTRUCTORS AS CULTURAL AGENTS

Rachel Tremaine  
Colorado State University  
rachel.tremaine@colostate.edu

Jess Ellis Hagman  
Colorado State University  
jess.ellis@colostate.edu

Enhancing equity, diversity, and inclusion (EDI) at the undergraduate level includes reducing the cultural work required of traditionally marginalized students when they are asked to assimilate to a dominant university culture and, more critically, seeking to align university culture to validate the experiences of those students. Faculty can play a role in this by acting as cultural agents who “seek to reduce the gap between students’ cultures of origin and the campus culture” (Dee & Daly, 2012, p. 169). Actions of cultural agents fall along four interrelated dimensions: decoding the text of higher education, building peer and professional networks, validating students’ cultures, and structuring learning environments which promote academic engagement (Dee & Daly, 2012). When faculty effectively act as cultural agents, students experience greater levels of academic preparedness and support for their well-being (Schademan & Thompson, 2016). In this poster, we qualitatively investigate the following research question: *In what ways do university mathematics instructors incorporate the dimensions of cultural agency when describing their roles?*

Data for this study was collected during the Fall 2019 at two research institutions in the United States. As part of a larger study involving college precalculus and calculus, six hour-long individual interviews were conducted with two professors, one associate professor, two contingent faculty members, and one seminar instructor (Hank, Felipe, Susan, Tara, Edgar, and Kaia, respectively), in which they discussed their role at the university and how that role related to EDI initiatives. The transcripts underwent a thematic analysis with *a priori* themes, using the dimensions of cultural agency as specified by Dee and Daly (2012).

Instances of each of the four dimensions of cultural agency were present in faculty’s narratives. Kaia and Edgar both identified ways in which they aimed to assist students in navigating undergraduate mathematics spaces through decoding, while Hank attended more to decoding the actions of professional mathematicians. Kaia, Hank, Susan, and Edgar all discussed ways in which they support students via building both peer-to-peer and professional within-university networks, and Tara and Hank emphasized that they intentionally structure learning environments to support student engagement via problem-posing and creating group work environments. Kaia was the only instructor who incorporated notions of validating students’ home cultures in her description of her role at the university.

Utilizing a lens of cultural agency enabled attention to how instructors with various department rank and identity markers differentially embody cultural agency when describing their roles on campus. Further, this lens highlighted tensions instructors may face when acting as cultural agents. For example, Edgar’s desire to decode university mathematics contexts for his students mitigated his desire to incorporate more engaging learning environments, as he felt that such environments would be incongruous with their later mathematics experience. In addition, this lens allowed us to explore the concept of *secondary* cultural agent actions; those actions which perhaps do not directly assist undergraduates in culturally navigating university mathematics spaces, but deeply influence the navigability of those spaces. Felipe’s discussions of his role were very oriented toward acting as a secondary cultural agent through using his status



as a full Professor and member of the Faculty Senate to advocate for inclusive hiring practices. We conclude that mathematics instructors are well-positioned to act as cultural agents.

### **References**

- Dee, J. R., & Daly, C. J. (2012). Engaging faculty in the process of cultural change in support of diverse student populations. In S. D. Museus & U. M. Jayakumar (Eds.), *Creating Campus Cultures* (pp. 168-188). Routledge.
- Schademan, A. R., & Thompson, M. R. (2016). Are college faculty and first-generation, low-income students ready for each other?. *Journal of College Student Retention: Research, Theory & Practice*, 18(2), 194-216.

## UNDERSTANDING THE INTERSECTIONS OF STUDENT BELONGING TO CREATE INCLUSIVE ENVIRONMENTS: A NARRATIVE INQUIRY APPROACH

Darolyn A. Flaggs  
Kennesaw State University  
dflaggs@kennesaw.edu

Alison Hedrick  
Kennesaw State University  
ahedric1@kennesaw.edu

Wendy Sanchez  
Kennesaw State University  
wsanchez@kennesaw.edu

Belinda Edwards  
Kennesaw State University  
bedwards@kennesaw.edu

Amy Hillen  
Kennesaw State University  
ahillen@kennesaw.edu

*In this qualitative research study, we used a narrative inquiry approach to reveal the unique experiences of belonging amongst preservice mathematics teachers. We explore the intersections of student belonging within and across participants from marginalized populations. We then discuss four emerging themes from our initial interviews. Reflecting on these stories, we sought to gain a deeper understanding of how mathematics teacher educators could and should cultivate inclusive environments. We conclude with recommendations for faculty to create environments supportive of student belonging and engagement.*

**Keywords:** Affect, Emotion, Beliefs, and Attitudes; Social Justice; Systemic Change; LGBTQIA+; Preservice Teacher Education; Teacher Educators; Equity, Inclusion, and Diversity

Students' sense of belonging has implications towards their academic success (Freeman et al., 2007), engagement in learning environments (Hurtado & Carter, 1997; Strayhorn, 2018), and overall well-being (Moeller et al., 2020). Strayhorn (2018) defines sense of belonging as "a basic human need, a fundamental motivation, sufficient to drive behaviors and perceptions" (p. 9). While Maslow's Hierarchy of Needs (1943) also characterizes belonging as a basic human need for all individuals, students of color and other marginalized populations may have different perceptions of belonging (Strayhorn, 2012; Walton & Brady, 2017). These differences are based on their identities and experiences and often impact the extent to which students feel safe to be their authentic selves on a college campus and in their courses.

Because belonging goes beyond fitting in for marginalized populations (Walton & Brady, 2017), it may also impact how students react and perform in academia. When students do not feel they belong at an institution or they are valued by the campus community, including by their peers and instructors, they do not engage as fully in learning (Freire, 1970), and thus their academic success suffers (Hausmann et al., 2007; Johnson et al., 2007; Strayhorn, 2018; Walton & Cohen, 2007). This is a level of complexity with which not all students must cope. Further, professors may consciously and/or subconsciously lack culturally responsive tenets that are important for supporting students who may lack a sense of belonging in their classes (Jett, 2013). Jett explains how this inaction steals the identity of students, "thereby causing some students years to recover their natural mathematical states, if they return to them at all" (p. 103).

To this end, we pursued research to (a) provide mathematics teacher educators (MTEs) a nuanced understanding of ways sense of belonging impacts preservice teachers' (PSTs) engagement, success, and retention, and (b) bring awareness to barriers, challenges, and systemic forms of inequalities that marginalized students experience, along with possible strategies to address them. Driving this study, we offer the following guiding research questions: How do PSTs navigate interaction in spaces where they lack a belonging in their teacher preparation

programs? How can MTEs cultivate inclusive environments that support students' sense of belonging? In this research brief, we discuss emerging findings from our initial interviews and use student voices to inform equity and inclusivity at the classroom level and broader structural changes at the campus level.

### **Theoretical Framework**

Researchers have shown that “intersectionality provides a useful framework to discuss how mobility and marginalization exist on our college campuses” and “has a long history of calling to attention marginality and oppression in different situations and is not solely housed in the halls of academia” (Brunn-Bevel et al., 2019, p. 5). Critical Theory (Friere, 1970) allowed us to focus our attention on situations “in which some individuals prevent others from engaging in the process of inquiry ... the means used are not important; to alienate human beings from their own decision-making is to change them into objects” (p. 84).

### **Methods**

Our study is a narrative inquiry aimed at developing better support systems for marginalized and/or traditionally underserved students in teacher education programs. The stories in our data are powerful, and a narrative inquiry design helped us focus on the three-dimensional space of temporality, sociality, and place (Clandinin, 2013). This approach provided the flexibility to hear participant's voices situated within the spaces they were “constituted, shaped, expressed, and enacted” (Clandinin, 2007, p. 43).

### **Participants and Study Context**

Part of a larger research, the present study focuses on two women who were enrolled in or graduated from a mathematics education program in the southeast region of the United States. Both women were African American, lesbian, and came from an impoverished background. Participants were asked to select a pseudonym; they chose Batman and Wonder Woman.

**Batman and Wonder Woman.** Before sharing their stories and our findings, it is important to share some of the circumstances that Batman and Wonder Woman faced while trying to navigate college. Batman lost her off-campus housing during her program. She was having difficulty communicating with her professors about her situation. She explained,

I'm not sleeping, I'm working, I'm homeless...I was about to fail, so I had to say something... I felt like I was just fighting to get here, fighting to pass, fighting to communicate with people. And I just kind of was ready to give up.

Wonder Woman came from a large family—8 children and her parents. Her life growing up was filled with uncertainty of food and housing. She shared, “what we ate at school would be it... I know what it's like to get yourself put out on the street.” Wonder Woman's motivation to finish her degree was closely tied to food insecurity. She explained, “I have to finish. I cannot stop. I have to graduate with this degree so I can get a job so I can get a place to live so I can have something to eat.”

### **Data Collection**

For this study, we conducted two semi-structured interviews for each participant, which allowed both breadth and depth of variations (Han & Ellis, 2019). Interview one was completed late Spring/early Fall 2019; interview two was completed Spring 2022. After the interview data were transcribed, the researcher who had taught the students and conducted the interviews wrote narratives about each participant. Some of the researcher's own thoughts were included as part of the narratives. Both participants reviewed the narratives and approved them for use in the study.

## Data Analysis

Before analyzing the data, we generated a list of a-priori codes from the literature. For example, the codes *Cognitive SOB* and *Affective SOB* emerged from Strayhorn's research (2018). Then as we analyzed the data, we used open coding to add additional codes that emerged. Some of the open codes were *lack of family resources*, *food insecurity*, and *trying to hide/not participate*. After organizing the data, four themes emerged: sources of not belonging, coping with not belonging, navigating structures/programs, and campus and community resources.

## Findings

This study sought to understand how PSTs navigate interaction in spaces where they felt a lack a belonging in their teacher preparation programs and how MTEs can cultivate inclusive environments that support students' sense of belonging. During their interviews, both Batman and Wonder Woman touched on all four of the themes; for the purposes of this research brief, selected quotes are extracted that underline the significance of each theme. Their perspectives can help MTEs create more inclusive environments in their classrooms and programs.

### **How do PSTs navigate interaction in spaces where they felt a lack a belonging in their teacher preparation programs?**

Research shows that *sources of not belonging* include academic and social experiences (Strayhorn, 2012) and low representation of one's group can influence students' sense of belonging (Murphy et al., 2007). Both participants felt a lack of belonging during collaborative group work. Wonder Woman shared "I don't belong here. I'm not supposed to be here, because I don't see a lot of people that I can relate to..., but I know I need to be here to do what I want to do in life." She continued by sharing her discomfort with asking for help and mentions being one of two Black women in the class. "Everybody else was Caucasian," she exclaims, "so, I'm not saying I wasn't comfortable because they're white, but I was just in fear of being judged or treated differently, because once again, I'm not supposed to be in a class like that." This sense of not belonging highlights the intersection of Wonder Woman's racial background, gender, and math self-efficacy, and the impact she perceived it had on her ability to fully engage. This is also an example of how she was *coping with not belonging* in the classroom; she hid and did not speak up. Batman, on the other hand, voiced her feelings and shared "I felt suppressed. I felt like I was fighting to speak my case... to my actual group members." She shares how teachers would praise her "but my group members would like not even listen... And I just was like, you know, why are you not listening to me? Why are you always looking away from me when I speak?"

Just as Wonder Woman and Batman had different approaches to coping with not belonging, students' educational experiences may differ vastly depending on their program of study. Both students were faced with *navigating structures/programs* in spaces they did not feel they belonged. Batman, an engineering student turned middle school teacher education student shared "I think just you have to learn different in education programs... It's not straightforward, so it was just a different mental challenge for me." She perceives her STEM content courses as "pretty direct" but found education courses to be different. She explains how "[they talk] around everything... you have to go find information yourself that they explore, to really learn." These uncomfortable spaces for both women did not stop in the classroom. Their interactions, or lack thereof, with *campus and community resources* also played a role in their lack of belonging. While walking to class one day, Wonder Woman came across an LGBT protest on campus. She expressed the importance of universities "making it obvious that this campus is a safe space as far as LGBT" especially since "most LGBT are probably afraid to [voice what they believe]."

She shared "... [my partner] found this one group that was for LGBT. I have no idea how she came across it, because I had no clue it was there."

### **How can MTEs cultivate inclusive environments that support students' sense of belonging?**

It is extremely important for mathematics teacher educators to not only cultivate inclusive environments in all classes (Bature et al., 2016), but to show compassion and flexibility at times, especially considering all students are unique with distinct stories and challenges. When asked what can be done to help you feel like you belong, Batman shared that one of the *sources of not belonging* occurs by professors "not being understanding, you know, just being so rigid... I understand rules and things have to be done, but some people really have extreme circumstances. And [school] wasn't the only thing on their plate, ... It's just some, some discretion". She urged mathematics teacher educators to "just check on me and just just make sure I'm Oh, you know, okay,... Just a "what's going on?" When asked a similar question, Wonder Woman shared that there were professors who helped her in *coping with not belonging*. She shared how they "...affected me more on a personal level and just standing up for who you are. African American woman, lesbian." Wonder Woman discusses how her professors affected her more on "the math side and the education side, as far as getting my education and being a teacher to educate other people. You guys helped me realize some things that took away those insecurities that I had."

In terms of *navigating structures/programs*, Wonder Woman suggested that MTEs provide support for their students in multiple ways. She shared "I wish professors knew that I wasn't the typical student. I needed more than what they were giving, and me being an adult and being in college, of course, I assumed, Okay, I have to go get that myself." While most colleges and universities provide resources to help students succeed and navigate challenges that interfere with their learning, students aren't always aware they exist. In the classroom, the professor can be that liaison, but outside of the classroom, students must be made aware of *campus and community resources*. Batman, who faced housing insecurity, was proud and struggled to ask for help. Not until she was on the cusp of failing did she decide to say something to one of her professors. Once she did "... everybody just started helping me like the [campus resources center] helped me a lot. They gave me a place to stay... meal passes and stuff like that."

### **Discussion and Implications**

Batman and Wonder Woman might seem like outliers, but students like these are present on many campuses. We learn from their stories how peers, instructors, and class and campus structures, all impacted their belonging and engagement in their teacher preparation programs. They remind us that (a) showing care and concern can go a long way in helping cultivate a sense of belonging amongst students, (b) MTEs can take an active role in supporting praising, and provided critical feedback to students, particularly during group work, and (c) MTEs have opportunities to support students' success from beyond the classroom. Research already shows that belonging impacts engagement for marginalized students (Walton & Cohen, 2007), but the unique experiences and powerful voices of these students provide evidence of and implications for how MTEs can further cultivate inclusive environments for marginalized students. Instructors must be aware of ambiguous cues that may heighten belonging uncertainty (Walton & Cohen, 2007) and further threaten how marginalized students interact in mathematics classrooms.

It is equally important that we prepare PSTs to interact with diverse communities. We know that teacher preparation programs are preparing future teachers, those who will have a tremendous impact on future generations of students. Thus, intervention studies with PSTs may be an important step towards creating learning environments that are less exclusionary. This is a

concern for Batman who felt disrespected by her peers, especially during group work, and said “You can’t make somebody respect you and consider your opinion. But my thing is, you know, I’m worried about when they become teachers, because we are becoming a teacher.”

### References

- Bature, I. J., Atweh, B., & Treagust, D. (2016). Inclusivity: An Effective Tool for Achieving Quality Mathematics Classroom Instruction in Nigerian Secondary Schools. *Universal Journal of Educational Research*, 4(1), 173–180.
- Brunn-Bevel, R. J., Ovink, S. M., Byrd, W. C., & Mahoney, A. D. (2019). *Always crossing boundaries, always existing in multiple bubbles: Intersected experiences and positions on college campuses*. In W. C. Byrd, R. J. Brunn-Vevel, & S. M. Ovink (Eds.), *Intersectionality and higher education: Identity and inequality on college campuses* (1<sup>st</sup> ed., pp. 3-24). Rutgers University Press.
- Clandinin, D. J. (Ed.). (2007). *Handbook of narrative inquiry: Mapping a methodology*. Sage Publications, Inc. <https://doi.org/10.4135/9781452226552>
- Clandinin, D. J. (2013). *Engaging in Narrative Inquiry* (1<sup>st</sup> ed.). Routledge. <https://doi.org/10.4324/9781315429618>
- Freire, P. (1970). *Pedagogy of the oppressed*. Bloomsbury.
- Freeman, T. M., Anderman, L. H., & Jensen, J. M. (2007). Sense of Belonging in College Freshmen at the Classroom and Campus Levels. *The Journal of Experimental Education*, 75(3), 203–220.
- Han, F., & Ellis, R. A. (2019). Using Phenomenography to Tackle Key Challenges in Science Education. *Frontiers in Psychology*, 10. <https://doi.org/10.3389/fpsyg.2019.01414>
- Hausmann, L. R. M., Schofield, J. W., & Woods, R. L. (2007). Sense of Belonging as a Predictor of Intentions to Persist among African American and White First-Year College Students. *Research in Higher Education*, 48(7), 803–839.
- Hurtado, S., & Carter, D. F. (1997). Effects of College Transition and Perceptions of the Campus Racial Climate on Latino College Students’ Sense of Belonging. *Sociology of Education*, 70(4), 324–345. <https://doi.org/10.2307/2673270>
- Jett, C. C. (2013). Culturally Responsive Collegiate Mathematics Education: Implications for African American Students. *Interdisciplinary Journal of Teaching & Learning*, 3(2), 102–116.
- Johnson, D. R., Alvarez, P., Soldner, M., Inkelas, K. K., Leonard, J. B., Rowan-Kenyon, H., & Longerbeam, S. (2007). Examining sense of belonging among first-year undergraduates from different racial/ethnic groups. *Journal of College Student Development*, 48(5), 525–542–542. <https://doi.org/10.1353/csd.2007.0054>
- Moeller, R. W., Seehuus, M., & Peisch, V. (2020). Emotional intelligence, belongingness, and mental health in college students. *Frontiers in Psychology*, 11. <https://doi.org/10.3389/fpsyg.2020.00093>
- Murphy, M. C., Steele, C. M., & Gross, J. J. (2007). Signaling Threat: How Situational Cues Affect Women in Math, Science, and Engineering Settings. *Psychological Science*, 18(10), 879–885.
- Strayhorn, T.L. (2012). *College Students’ Sense of Belonging: A Key to Educational Success for All Students* (1<sup>st</sup> ed.). Routledge. <https://doi.org/10.4324/9780203118924>
- Strayhorn, T.L. (2018). *College Students’ Sense of Belonging: A Key to Educational Success for All Students* (2<sup>nd</sup> ed.). Routledge. <https://doi.org/10.4324/9781315297293>
- Walton, G. M., & Brady, S. T. (2017). The many questions of belonging. In A. J. Elliot, C. S. Dweck, & D. S. Yeager (Eds.), *Handbook of competence and motivation: Theory and application* (pp. 272–293). The Guilford Press.
- Walton, G. M., & Cohen, G. L. (2007). A question of belonging: Race, social fit, and achievement. *Journal of Personality and Social Psychology*, 92(1), 82–96. <https://doi.org/10.1037/0022-3514.92.1.82>

## UNPACKING EQUITY-ORIENTED TEACHING DILEMMAS IN ELEMENTARY MATHEMATICAL MODELING LESSONS

Mary Alice Carlson  
Montana State University  
mary.carlson5@montana.edu

Erin Turner  
The University of Arizona  
eturner@arizona.edu

Jennifer Suh  
George Mason University  
jsuh4@gmu.edu

Julia Aguirre  
University of Washington-Tacoma  
jaguirre@u.washington.edu

*Our paper details the ways teachers understand and navigate equity-oriented dilemmas (Berlak & Berlak, 1981) when teaching mathematical modeling and how mathematics teacher educators can support teachers' learning of culturally responsive mathematics teaching. Using Zavala and Aguirre's (in press) framework for culturally responsive mathematics teaching, we explored the ways teachers describe and frame their choices when faced with dilemmas. Findings revealed that teachers identified dilemmas with Rigor and Support most often, followed by Knowledges and Identities. Dilemmas related to Power and Participation occurred far less frequently. Implications for teacher professional development are discussed.*

Keywords: Equity, Inclusion and Diversity; Modeling, Culturally Relevant Pedagogy; Professional Development

Teaching mathematical modeling involves supporting students to pose problems; make assumptions and define variables; and create, validate, and share usable models (COMAP & SIAM, 2016). Mathematical modeling has the potential to advance equity because it gives students opportunities to draw on their own knowledge bases as they answer meaningful questions (Aguirre et al., 2019; Anhalt et al, 2018; Carlson et al., 2016; Cirillo et al, 2016; English & Watters, 2004; Suh et al., 2018; Turner et al., 2022), immerses students in rigorous content (Fulton, 2018), broadens participation, and disrupts traditional classroom power structures (Anhalt, 2014; Featherstone et al, 2012; Lesh & Doerr, 2003). Teaching modeling is pedagogically demanding (Carlson, 2021), as teachers must anticipate, elicit, and interpret students' ideas about problem contexts and mathematical approaches, and then respond in-the-moment (Jacobs, Lamb, & Philipp, 2010). Research has demonstrated that teaching mathematical modeling and developing equity focused pedagogies is challenging, but learnable (Anhalt et al, 2018). Meeting these challenges requires support and more research is needed to understand how teachers navigate the demands of equity-focused mathematical modeling.

### Theoretical Perspectives

We focus on dilemmas that arise when culturally responsive pedagogies are foregrounded in teaching mathematical modeling. The language of *dilemmas* is useful in understanding the complexities of instructional decision-making (e.g. Berlak & Berlak, 1981; Lampert, 1985). Berlak and Berlak (1981) use dilemmas to describe “both the forces which shape teachers' actions...and the capacity of teachers not only to select from alternatives, both to act to create alternatives” (p. 124). Instead of focusing on a “right” or “best” course of action, we focus on understanding dilemmas from teachers' perspectives and on supporting teachers to develop and select from a range of alternatives.

Teaching mathematical modeling is challenging because modeling problems are more open and less predictable than tasks in most lessons (Cai et al., 2014). They require teachers to know about the contexts that motivate modeling problems, potential mathematical solutions, and ways to maintain rigor and support for students as they develop, refine, and communicate their models (Carlson, 2021). Historically, mathematical modeling has been enacted at the secondary and undergraduate level but a growing number of researchers have found that elementary students can successfully engage in mathematical modeling (Albarrucín, 2021; Turner, et al., 2021; English 2012; English & Watters, 2006). We posit that mathematical modeling is also a lever for equity. Modeling empowers teachers to elicit and build on the knowledge and cultural resources that students bring to the classroom and empowers students to draw on their identities and experiences to inform mathematical work and take action (Aguirre, et al., 2019; Turner and Bustillos, 2017). In addition, modeling elicits diverse student contributions and gives teachers opportunities to assign competence and “recognize and reward a broader range of mathematical abilities than those traditionally emphasized” (Lesh & Doerr, 2003, p. 23).

Our perspective on equity-focused pedagogies is informed by Zavala and Aguirre’s (in press) three-strand framework for culturally responsive mathematics teaching. The first strand, *knowledge and identities*, focuses on how teachers draw on students’ funds of knowledge, build on student thinking, and support positive mathematics identities. The second strand, *rigor and support*, attends to how teachers maintain high cognitive demand, support students by scaffolding, and affirm multilingualism. The third strand, *power and participation*, focuses on distributing intellectual authority among students, disrupting status differences and stereotypes, and supporting students to take action. Explicitly and intentionally foregrounding culturally responsive pedagogies turns the potential mathematical modeling holds as a lever for equity into a reality for students. However, there is a gap in the extant literature related to how teachers learn to advance equity through mathematical modeling. The purpose of our study was to explore how teachers understand and navigate equity-oriented dilemmas when teaching modeling lessons.

### **Methods**

Our study focused on the following research questions: (1) What dilemmas do teachers encounter as they teach culturally responsive mathematical modeling? (2) When faced with dilemmas, how do teachers frame their choices?

### **Context and Participants**

This study is part of a collaborative, multi-year, funded project on mathematical modeling in the elementary grades. The project involves four universities and four geographically, racially, and culturally diverse school districts. Teachers engage in seven workshops spanning across the school year with in-person sessions and asynchronous assignments between workshops. There they learn about elementary mathematical modeling and ways to engage in culturally responsive mathematics teaching practices. In between workshops, teachers also implement at least three modeling lessons in their classrooms. This study draws on data from post-lesson enactment interviews with 15 teachers new to modeling lessons at four different sites from across grades K-5. Modeling lessons focused on deciding how a classroom should share snacks fairly. Students created models to answer questions such as “How many snacks do we need?” or “Do we have enough?” or “How long will these snacks last?”

### **Data and Analysis**

Our analysis focused on dilemmas related to the three strands of the culturally responsive mathematics teaching framework described above. Our primary data source was transcriptions of post-lesson teacher interviews. Using an inductive, open-coding process (Saldaña, 2021) we first



coded interview transcripts to identify teacher described dilemmas (i.e., moments when teachers recounted uncertainty about what to do in a lesson, or reflected on decisions made). Next, we identified dilemmas that related to one or more strands of our equity framework and generated a codebook (Table 1). A second round of analysis focused themes within each strand of equity-oriented dilemmas and on how teachers understood their choices in commonly occurring dilemmas. We wrote memos to describe patterns and themes within each strand and created heat maps to analyze the relative frequency of codes within each strand.

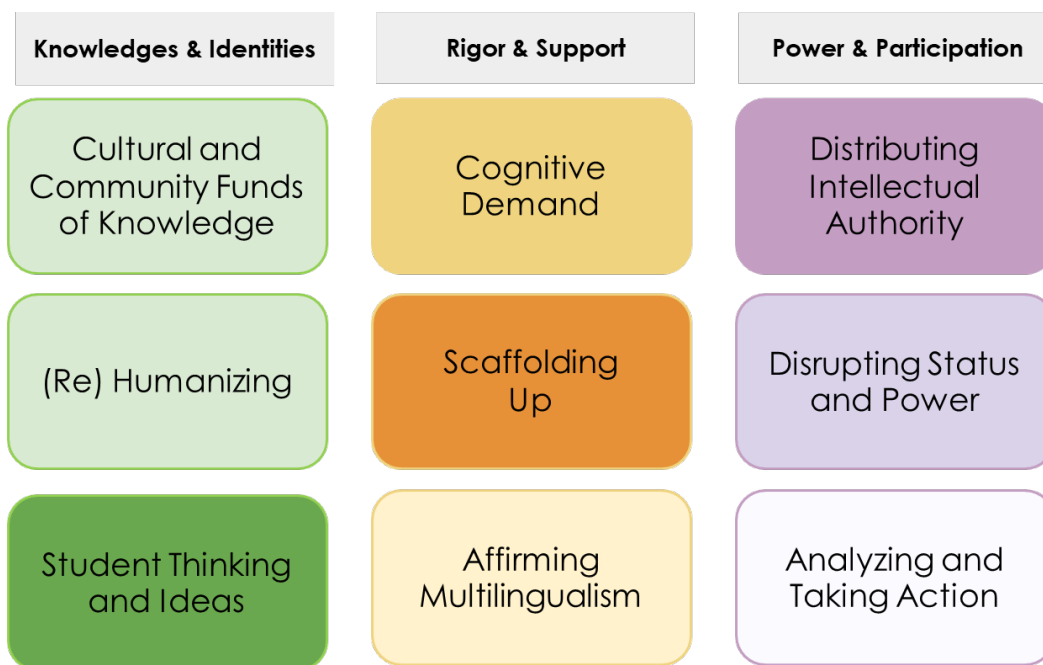
**Table 1: Codebook for Equity-Oriented Dilemmas in Elementary Mathematical Modeling**

Knowledge and Identities	
Cultural/ Community Funds of Knowledge	How to elicit or build on students' funds of knowledge, including connecting students' out-of-school experiences with ideas central to the mathematical demands of the modeling task.
Rehumanizing	Affirming mathematical identities, supporting creativity, broadening what counts as knowledge, and helping students feel confident taking intellectual risks.
Student Thinking and Ideas	Eliciting to and responding to student ideas, getting many student ideas on the table and deciding how to respond to unanticipated/unreasonable ideas.
Rigor and Support	
Cognitive Demand	How to support students to engage in the rigor of the task, especially whether the task seems too easy or too hard.
Scaffolding Up	Actions and interventions teachers take, or could take, to maintain high rigor with high support for students.
Affirming Multilingualism	Giving multilingual learners access to the task, and to centering multilingual learners' perspectives during modeling.
Power and Participation	
Distributing Intellectual Authority	Transferring authority from the teacher to the students, especially deciding what quantities/strategies to use or whether a model is correct.
Disrupting Status and Power	Attending to the ways students are positioning each other or to the ways teachers position students.
Taking Action	The ways the lesson is supporting students to use mathematics to analyze, critique and address power relationships and injustice in their lives.

### Findings

Looking across the three categories, Knowledges and Identities, Rigor and Support, and Power and Participation, teachers described dilemmas related to Rigor and Support most often ( $n = 52$ ), followed by Knowledges and Identities ( $n = 40$ ). Dilemmas related to Power and Participation occurred far less frequently ( $n=17$ ). The figure below shows the relative frequency of the dilemmas teachers described within each category. Darker colors indicate more dilemmas in a particular culturally responsive mathematics teaching dimension. Teachers identified dilemmas related to *student thinking and ideas*, *scaffolding up*, *cognitive demand*, and *distributing intellectual authority* most often. In the section that follows, we share examples of

dilemmas from these categories, following each with a discussion of how teachers framed their options.



**Figure 1: Culturally Responsive Math Teaching Framework Shaded to Indicate the Relative Frequency of Codes**

### Knowledge and Identities: Student Thinking and Ideas

In the Knowledge and Identities category, dilemmas coded as *student thinking and ideas* occurred most often. These dilemmas centered on eliciting and responding to students’ ideas about the modeling context, the quantities and assumptions relevant to the context, and the mathematical strategies they might want to use. The most prominent pattern was that teachers wondered about what to do when students did not generate the ideas, strategies, or approaches teachers anticipated, or when students struggled to come up with an approach at all. For example, M. R., a kindergarten teacher, recounted challenges helping students share and discuss options around what to do with leftover snacks.

And then the other ones, I guess, didn't know what to do with the [extra snacks]. So then one time, I chimed in and said, “I'll eat them,” and so they put me down because they didn't know what to do with the extra. I was hoping that they would say split it, but I couldn't pull that out of them at all. I guess my struggle is trying to get like, I know that they can do it, but I think they need a lot of help getting there.

Mrs. R. anticipated students would want to split any extra snacks and when they struggled to come up with options for the leftovers, she offered to eat the extras. Underlying her dilemma about what to do when students are hesitant to share their thinking were questions around how much to guide and support students in generating ideas to begin with. It seemed Mrs. R.’s belief that students *could* generate ideas was in tension with what she saw as her best option for moving the lesson forward; namely offering an option for the leftovers herself.

### **Rigor and Support: Scaffolding Up**

Dilemmas coded as *scaffolding up* were related to tensions teachers experienced around actions they took, or could take, in order to maintain high rigor and high support for students. Because modeling problems are open and foreground student decision-making, teachers wondered when and how to make use of examples, model mathematical strategies, or guide students' work and they often struggled to decide when and how to provide scaffolds that give students access while maintaining rigor and support. For example, Ms. S. reflected on her efforts to provide appropriate support to her fourth-grade class.

I didn't have them write [the number of fourth grade classes] down because this is the first time we did it, and I didn't want to overwhelm them with too many numbers. So we put it on the board. There were 26 kids, and me, so we had 27. [We had] about 530 pretzels. We figured that out using multiplication because there were 24 servings and 22 pretzels in a serving. That was the problem we had together.  $22 \times 24$  is a little too advanced for them to do without the manipulatives, and then I just kind of let them go. And that was, I think, the mistake. Not talking about it first, like just a little bit at least.

Initially, Ms. S. decided to ease the demands of the task by limiting the number of quantities the students worked with at one time. However, directing students to decide whether or not they had enough pretzels for their class and rather than the entire fourth grade funneled students to use a single approach: find the number of pretzels needed per class, and then find the number of pretzels needed per grade. As her reflection indicates, Ms. S. also felt students needed additional support finding the number of pretzels in a container but chose to “let them go” - a decision she recounted as a “mistake.” Ms. S.'s dilemma highlights teachers' challenges to use teaching strategies that maintain high rigor *with* high support. At times, teachers responded by offering supports that reduced demand and constrained students' choices. At other times, and in an effort to maintain rigor, they withheld support but let students become sidetracked by large numbers and challenging computations.

### **Power and Participation: Disrupting Intellectual Authority**

Teachers discussed dilemmas related to *power and participation* less frequently than the other two categories. When they did discuss power and participation, dilemmas were most often related to *distributing intellectual authority*. Many teachers described these dilemmas as a tension between intervening with explicit guidance and stepping back so that students could drive the discussion or take ownership of ideas. Teachers used terms like “stepping back” or “standing back” to signal a shift in power structures from the teacher as the driver of mathematical ideas to the students. For teachers, shifting from the role of explicit instructor to listener and facilitator was significant. Mrs. B, a fourth-grade teacher, described her dilemma as follows.

So really, for me, just like letting go of what I thought was maybe going to be the incorrect way. Because honestly, when they started wanting to add all those 16s together, I was like, oh my gosh. My brain was like. “No, no, no, no. Let's not do that. Don't do that.” But the math that came out of that was so beneficial that I wouldn't do. I would have let them keep going. Now, if I were ever to do that again, I would have said, “Yeah. Absolutely. Try that.” Right? So that was challenging just to be able to slow down and to just praise what they were doing correctly and listening. And a lot of it was not me talking, but just like letting them listen or let go or letting them share out.

Mrs. B's description of her internal struggle to step back and let students pursue what appeared to be an incorrect strategy brings the dramatic shift from resisting to affirming student ideas into focus. At the same time, it highlights her opportunity to benefit from letting students develop and explore their own strategies, and the significance of the shift she was learning to make.

### Discussion

Teaching culturally responsive mathematical modeling involves ambitious and challenging teaching practices. It requires teachers to learn new ways of working with student ideas and new ways of providing scaffolds and support. Like Mrs. B, many teachers also confront a mismatch between the roles they and their students play in traditional mathematics lessons and the new roles and responsibilities they have to adopt. Investigating dilemmas teachers new to modeling encounter gives the field insights into the nature of the demands teachers are facing, and of the ways teachers understand their options. Such insights can help professional development designers create resources to broaden teachers' perspectives on what aspects of their practice need to be problematized and give teachers expanded options for navigating dilemmas.

Given our explicit focus on culturally responsive mathematical modeling, we believe it is also important to attend to the dilemma categories that did not come up in teacher interviews. Although our current analysis does not focus on why particular dilemmas occurred more often than others, we believe the task context and teachers' prior experiences played an important role. We note that the modeling context was snack sharing and was situated within the teachers' classrooms. Snack sharing is a common classroom routine and is familiar to most teachers and students. Mathematical opportunities embedded in snack sharing are clear and connections to K-5 mathematics content are readily available. However, the *connecting to students' cultural and community funds of knowledge* and *analyzing and taking action* dimensions of the culturally responsive mathematics teaching framework involve explicit connections to contexts and practices outside the classroom. Because snack sharing does not require teachers to make substantive connections to students' out of school experiences, they may have limited opportunities to connect to and affirm students' cultural and community knowledge bases. Likewise, snack sharing may give classes opportunities to discuss, define, and mathematize fairness for the purpose of their model, but it does not come with ready opportunities to analyze, address, and critique power relationships and injustice.

Second, teachers in our project may have had more opportunities to develop language and practice around *student thinking and ideas* and *scaffolding up* outside of our professional development. Working with student ideas and providing scaffolds and supports are common practices in elementary school teaching (e.g., Cohen, 2004; van de Pol, Volman, & Beishuizen, 2010). Thus, teachers may be more adept at noticing and reflecting on their own practices in these areas. We posit that *(re)humanizing* mathematics, *disrupting status and power*, and *analyzing and taking action* introduced new ways to frame mathematics teaching, so teachers may need additional support to learn to notice their own teaching habits in these areas, as well as more opportunities to learn new practices.

### Conclusion

Unpacking and understanding both the named and unnamed dilemmas that arise as teachers learn to implement culturally responsive mathematical modeling lessons has implications for the design and implementation of teacher professional development. In our own project, we have

used our growing understanding dilemmas to design “teacher moves tables” that suggest strategies aligned with specific dimensions of our culturally responsive mathematics teaching framework, developed annotations that foreground teacher decisions to accompany lesson planning tools, and worked to provide time during lesson debriefs and teacher professional development sessions to discuss culturally responsive teaching dilemmas. We are also beginning to investigate the varied potential of different kinds of modeling tasks to address dimensions of culturally responsive mathematics teaching, especially *power and participation*, and the ways modeling tasks give teachers opportunities to notice and reflect on their pedagogical choices.

Although our work focuses on culturally responsive mathematical modeling, we believe a focus on teaching dilemmas could benefit a range of professional development initiatives. The ways teachers understand and frame their choices provides a bridge between the ideals espoused and modeled during professional development and the ways teachers negotiate those ideals during instruction. Explicitly foregrounding equity through tools like Zavala and Aguirre’s (in press) culturally responsive mathematics teaching framework grounds teacher learning in the practices critical to realizing the equitable and just learning experiences each and every child deserves.

### Acknowledgments

The work here is supported by NSF DRL 2008997, 2010202, 2010269, and 2010178.

### References

- Aguirre, J.M., Anhalt, C., Cortez, R., Turner, E.E., & Simi-Muller, K., (2019). Engaging teachers in the powerful combination of mathematical modeling and social justice. *Mathematics Teacher Educator*. 7(2). DOI: 10.5951/mathteaceduc.7.2.0007
- Albarracín, L. (2021). Large number estimation as a vehicle to promote mathematical modeling. *Early Childhood Education Journal*, 49(4), 681-691. DOI:10.1007/s10643-020-01104-x
- Anhalt, C. (2014). Scaffolding in mathematical modeling for ELLs. In Civil, M & Turner E. (Eds.) *Common core state standards in mathematics for english language learners* (pp.111-120). Tesol International Association.
- Anhalt, C. O., Staats, S., Cortez, R., & Civil, M. (2018). *Mathematical Modeling and Culturally Relevant Pedagogy*. In Y.J Dori, Z.R. Mevarech, and D.R. Baker (Eds). *Cognition, Metacognition, and Culture in STEM Education* (pp. 307-330). Springer Publications.
- Association of Mathematics Teacher Educators (AMTE). (2017). *Standards for preparing teachers of mathematics*. Author. amte.net/standards
- Berlak, A. & Berlak, H. (1981). *Dilemmas of schooling: Teaching and social change*. Methuen & Co. Ltd.
- Cirillo, M., Bartell, T.G., & Wager, A. (2016). Teaching mathematics for social justice through mathematical modeling. In C. Hirsch & A. Roth McDuffie (Eds.), *Annual perspectives in mathematics education: Mathematical modeling and modeling with mathematics* (pp. 86-97). National Council of Teachers of Mathematics.
- Cai, J., Cirillo, M., Pelesko, J. A., Borromeo Ferri, R., Borba, M., Geiger, V. Stillman, G., English, L. D., Wake, G., Kaiser, G., & Kwon, O. N. (2014). *Mathematical modeling in school education: Mathematical, cognitive, curricular, instructional and teacher education perspectives*. In P. Liljedahl, C. Nicol, S. Oesterle, & D. Allan (Eds.), *Proceedings of the joint meeting of PME 38 and PME-NA 36* (pp. 145–172). Vancouver, Canada: PME.
- Carlson, M.A., Wickstrom, M.H., Burroughs, B. & Fulton, E. (2016). *A Case for Mathematical Modeling in the Elementary School Classroom*. In C. Hirsch & A. Roth McDuffie (Eds.), *Annual perspectives in mathematics education (APME) 2016: Mathematical modeling and modeling mathematics* (pp. 121-129). 121–29. Reston, Va.: National Council of Teachers of Mathematics.
- Cirillo, M., Bartell, T. G., & Wager, A. (2016). Teaching mathematics for social justice through mathematical modeling. In C. Hirsch & A. Roth McDuffie (Eds.), *Annual perspectives in mathematics education: Mathematical modeling and modeling with mathematics* (pp. 87-96). National Council of Teachers of Mathematics.
- Cohen, S. (2004). *Teachers' professional development and the elementary mathematics classroom: Bringing understandings to light*. Routledge.

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

- Consortium for Mathematics and its Applications (COMAP) and Society for Industrial and Applied Mathematics (SIAM). (2016). Guidelines for assessment & instruction in mathematical modelling education. Authors. <https://www.comap.com/Free/GAIMME/>
- English, L. D. (2006). Mathematical modeling in the primary school: Children's construction of a consumer guide. *Educational studies in mathematics*, 63, 303-323. DOI: 10.1007/s10649-005-9013-1
- English, L. D. (2012). Data modelling with first-grade students. *Educational Studies in Mathematics*, 81, 15-30. DOI:10.1007/S10649-011-9377-3
- English, L. & Watters, J. (2004), Mathematical Modeling in the Early School Years, *Mathematics Education Research Journal*, 16 (3), 59-80. DOI:10.1007/BF03217401
- Featherstone, H., Crespo, S., Jilk, L. M., Oslund, J. A., Parks, A. N., & Wood, M. B. (2011). Smarter together! Collaboration and equity in the elementary math classroom. National Council of Teachers of Mathematics.
- Jacobs, V. R., Lamb, L. L., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. *Journal for research in mathematics education*, 41(2), 169-202. DOI: 10.5951/jresmetheduc.41.2.0169
- Lampert, M. (1985). How do teachers manage to teach? *Harvard Educational Review*, 55(2), 178-194. DOI: <https://doi.org/10.17763/haer.55.2.56142234616x4352>
- Lesh, R. A., & Doerr, H. M. (2003). Beyond constructivism: Models and modeling perspectives on mathematics problem solving, learning, and teaching. Routledge.
- Saldaña, J. (2021). The coding manual for qualitative researchers (4th ed.). Sage Publications Inc.
- Suh, J., Britton, L., Burke, K., Matson, K., Ferguson, L., Jamieson, S., & Seshaiyer, P. (2018). Every penny counts: Promoting community engagement to engage students in mathematical modeling. In Goffney & Gutiérrez (Eds) *Rehumanizing mathematics for Black, Indigenous, and Latinx students*. pp. 63-76. Reston, VA: NCTM.
- Turner, E., Roth McDuffie, A., Aguirre, J., Foote, M. Q., Chapelle, C., Bennett, A., Granillo, M. & Ponnuru, N. (2021). Upcycling Plastic Bags to Make Jump Ropes: Elementary students leverage experiences and funds of knowledge as they engage in a relevant, community-oriented mathematical modeling task (pp. 235-266). In J. Suh, M. Wickstram, L. English (Eds). *Exploring Mathematical Modeling with Young Learners*. The Netherlands: Springer. DOI:10.1007/978-3-030-63900-6\_11.
- Turner, E. & Bustillos, L. (2017). ¿Qué observamos aquí? ¿Qué preguntas tienen? Problem posing in Ms. Bustillos's Second-Grade Bilingual Classroom. In S. Celedón-Pattichis, D. White and M. Civil (Eds.) *Access and equity: Promoting high quality mathematics in grades Pre K-2* (pp. 45-62.). Reston, VA: National Council of Teachers of Mathematics.
- Van de Pol, J., Volman, M., & Beishuizen, J. (2010). Scaffolding in teacher–student interaction: A decade of research. *Educational psychology review*, 22, 271-296. DOI: 10.1007/s10648-010-9127-6
- Zavala, M.R. & Aguirre, J.M. (in press) *Cultivating mathematical hearts: Culturally responsive math teaching in elementary classrooms*. Thousand Oaks, CA: Corwin.

## USING BIPOC STEM PROFESSIONALS' CAREER STORIES TO PROMOTE ENGAGEMENT AND EQUITY

Jill V Hamm  
UNC Chapel Hill  
Jill.Hamm@unc.edu

Robert Martinez  
UNC Chapel Hill  
rrmartin@email.unc.edu

Daniel J Heck  
Horizon Research, Inc  
dheck@horizon-research.com

Keywords: Informal education; Communication; Equity, Inclusion, and Diversity

CASCADE involves simulations set in STEM careers for BIPOC teens to practice, with virtual partners, aspects of tasks, discourse, and social dynamics important to math collaboration (Hamm & Hoffman, 2016; Heck & Hamm, 2016; Smith & Stein, 1998; Sztajn et al., 2021). To enhance student engagement and equity, the simulations provide feedback from BIPOC STEM professionals about using math and collaboration in their careers, and guidance for navigating math-related careers. We partnered with 3 Black and 3 Latinx STEM professionals who shared math-related feedback and career navigation guidance.

### **Designing Simulations with Career-Relevant Feedback and Guidance to Promote Equity**

Career-relevant feedback and guidance is grounded in the Community Cultural Wealth (CCW) framework's (Yosso, 2005) premise that BIPOC communities possess diverse cultural capital (aspirational, familial, linguistic, navigational, resistant, and social) that are assets to their members' adjustment and empower resistance to structural racism and oppression. Personal narratives of BIPOC on these themes, that account for race and racism and their intersectionality with gender and SES, known as counterstories, are a source of and used to share CCW, helping BIPOC feel unity with others and revealing possibilities and strategies for the self (Solorzano & Yosso, 2001). As teens play the simulation, they have repeated chances to try mathematics collaborative skills, experience consequences, and get career-relevant feedback and guidance.

### **Evidence of the Benefit of CCW and Counterstories for Student Engagement and Equity**

Data from 2 focus groups with 9 Latinx high schoolers responding to BIPOC professionals' stories support the value of CCW to promote both engagement and equity and informed which feedback to include in the simulations. Participants identified engaging messages, such as:

Maybe there was two girls of color and me, [my engineering class] was very male dominated and the boys would try to make it seem that I look stupid instead of genuinely trying to explain it to me. It's really hard being a person of color, a woman, in a very male dominated field of work. I really related to her [BIPOC professional's counterstory] a lot in that case.

He [BIPOC professional] saw he wanted to help with the survey results and that made him realize that he wanted to use statistics to help others. And I liked that 'cause he went out of his way to do something else to realize that was his calling.

Black and Latinx teens in college pathway programs are playtesting the simulation; impacts of career guidance and feedback on their dispositions for careers using math will be reported.

### **Acknowledgments**

This research is funded by the National Science Foundation (NSF) to Hamm [2048993] and Heck [2048985]. The findings reported here do not necessarily reflect the opinions of the NSF.



Hamm and Heck are married, and Heck is employed by and co-owns Horizon Research, Inc. UNC reviewed these arrangements, concluding that the possible benefit to Hamm or Heck is not likely to affect participant safety or the scientific quality of the study.

### References

- Hamm, J. V. & Hoffman, A. (2016). Teachers' influence on students' peer relationships and peer ecologies. In Wentzel, K., Ramani, G. (Eds.), *Handbook of Social Influences on Social-Emotional, Motivation, and Cognitive Outcomes in School Contexts*. Taylor & Francis.
- Heck, D. J & Hamm, J. V. (2016). Small group learning environments in middle grades classrooms. In M. B. Wood, E. E. Turner, M. Civil, & J. A. Eli (Eds.), *Proceedings of the 38th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1325–1328). Tucson, AZ: University of Arizona. <https://doi.org/10.51272/pmena.38.2016>
- Solorzano, D.G. & Yosso, T.J. (2001). Critical race and LatCrit storytelling: Counterstories. *International Journal of Qualitative Studies in Education*, 14:4, 471-495. DOI: 10.1080/09518390110063365
- Stein, M. K., & Smith, M. S. (1998). Mathematical tasks as a framework for reflection: From research to practice. *Mathematics Teaching in the Middle School*, 3, 268-275. <https://doi.org/10.5951/mtms.3.4.0268>
- Sztajn, P., Heck, D.J., & Malzahn, K. (2021). *Activating math talk*. Corwin Press.
- Yosso, T.J. (2005). Whose culture has capital? A critical race theory discussion of community cultural wealth. *Race, Ethnicity, and Education*, 8:1, 69-91. DOI: 10.1080/1361332052000341006

## WHAT COMES BEFORE A THEORETICAL FRAMEWORK? THE AESTHETIC DIMENSION OF THEORIZING IN MATHEMATICS EDUCATION RESEARCH

Brady A. Tyburski  
Michigan State University  
tybursk2@msu.edu

*In the mathematics education literature, there is plenty written on the roles that theoretical frameworks should play once they are already crafted; however, there is much less guidance on how scholars—especially emerging scholars—might construct a theoretical framework that serves these roles. When guidance is provided, it focuses on issues of epistemology and ontology. In this brief theoretical report, I emphasize the importance of the oft-ignored aesthetic dimension of theorizing. I first argue that aesthetics cannot be separated from our everyday behaviors and choices and therefore our research practice. I then share my experience about how reflecting on my personal aesthetics allowed me to craft a framework that enables me to bring my whole self—worldviews and all—into my research. By sharing my story, I hope to support other emerging scholars in bringing their ways of being, thinking, and feeling into their research.*

Keywords: Research Methods; Doctoral Education; Affect, Emotion, Beliefs, and Attitudes; Equity, Inclusion, and Diversity.

In conversations I've had with graduate students just getting started with research (e.g., Tyburski et al., 2022), it's not uncommon for some to express that the theoretical framework they're using "just doesn't feel right." Often, the framework they've crafted satisfies the many roles—both epistemological and ontological—that a theoretical framework should according to the mathematics education literature: identifying what counts as evidence for or against a particular claim, what set of constructs should be attended to, etc. (Cai et al., 2019; Spangler & Williams, 2019). But satisfying these utilitarian purposes is not always enough to quell the unrelenting uncertainty some graduate students feel they ought to resolve before proceeding with their research in a way that feels right to them.

In this brief theoretical report, I suggest that a possible cause for this "just doesn't feel right" dilemma is a felt tension between one's implicit worldviews/theories (Spangler & Williams, 2019; Stinson, 2020) and the assumptions of the theoretical framework they are using. Ensuring a harmony between one's personal worldviews and the (implicit) philosophical assumptions of the theoretical framework they choose to employ is an underdiscussed but important aspect of developing a theoretical framework (Stinson, 2020). There is literature in mathematics education on the roles a theoretical framework should play once it is crafted (e.g., Cai et al., 2019; Spangler & Williams, 2019); however, this literature contains very little guidance for how one might go about creating such a theoretical framework in the first place. In sharing my own story of crafting a theoretical framework that jives with my worldviews, my intention is to add guidance in this domain. Specifically, I focus herein on the oft-ignored aesthetic dimension of crafting a theoretical framework. After all, as I will argue, our personal aesthetic cannot be disentangled from our experience as people and therefore from our inquiry as researchers.

By adding such guidance, I hope to lower the bar for other emerging scholars to develop theoretical frameworks idiosyncratic to their worldviews and personal aesthetics, so that they can bring their true selves into their work. This crafting process is no easy task for newcomers to mathematics education (Hass et al., 2022), so if we do not provide explicit guidance, we increase

the chance that emerging scholars will feel the pressure to simply imitate what has come before and already exists (Baucom, 2019), thereby perpetuating a culture of exclusion in mathematics education (Louie, 2017) and gatekeeping in our doctoral programs (Westby, 2021).

### **The Status Quo of Theorizing in Mathematics Education**

I use *theoretical framework* broadly, “to encompass the set of assumptions, theories, hypotheses, and claims (as well as the relationships between them) that guide a researchers’ thinking about the phenomenon being studied” (Cai et al., 2019, p. 219). Crafting a theoretical framework—or *theorizing*—is a dynamic process underpinned by philosophical assumptions (Bikner-Ansbahs et al., 2014). Yet, Stinson (2020) cautions that most existing mathematics education research offers only a limited acknowledgement of these philosophical underpinnings. Accordingly, he calls us to center these philosophical considerations—epistemology, ontology, and axiology (i.e., a researcher’s worldview)—and make them the starting point of theorizing.

### **Personal Aesthetics Influence our Research Practice**

Although much of the literature in the philosophy of mathematics and mathematics education equates aesthetics to beauty (e.g., Ernest, 2016), what I mean by aesthetics expands beyond one dimension to include negative aesthetic judgements and qualities—messy, impure, ugly—alongside positive ones—joy, wonderment, and curiosity. Hermerén, for example, describes five different types of aesthetic qualities: “emotion qualities (e.g., somber, gay); behavior qualities (bold, nervous); gestalt qualities (unified, disorganized); taste qualities (elegant, delightful); and affective qualities (funny, glaring)” (Leddy, 1995, p. 262). As this list of qualities demonstrates, our personal aesthetics are fundamentally entangled with our behaviors as well as our emotional senses and feelings (Sinclair, 2009). Indeed, such aesthetic considerations have been shown to play an important role in mathematicians’ research practices (Burton, 2004), including selecting which research questions to pursue and which solutions to favor (Sinclair, 2006). Aesthetic reactions such as wonderment or surprise have also been shown to motivate mathematics students to pursue extended inquiry (Dietiker & Richman, 2021). Personal aesthetics have even been shown to interact with students’ sense-making (Jasien & Horn, 2022) and the choices they make while problem solving (Fiori & Selling, 2016). In a similar way, our research inquiry in mathematics education is governed by the sway of our personal aesthetic sensibilities.

According to *everyday aesthetics*, a modern branch of aesthetics (Marini, 2021; Saito, 2007), our personal aesthetics are intertwined with our everyday experiences and impact our behaviors, whether we consciously recognize it or not. For example, as I’ve been typing, I’ve had my arm resting on my desk. My unconscious decision *not* to remove my arm for the past half hour was likely influenced by factors such as the smooth texture of the desk as well as the sense of balance I felt as my arm rested there. Whether I was aware of them or not, these aesthetic considerations almost certainly influenced my behavior. Aesthetics cannot be disentangled from our experience. In the same way, our personal aesthetics cannot be disentangled from our inquiry as mathematics education researchers, including the philosophical and practical choices we make when constructing the theoretical frameworks we use in our research.

### **Epistemological and Ontological Choices Cannot be Separated from Our Aesthetics**

Conversations in our field about the role of theoretical frameworks almost exclusively center on utilitarian epistemological and ontological questions (e.g., Cai et al., 2019; Spangler & Williams, 2019). It is exceedingly rare—outside of perhaps a brief comment about the role of personal choice in theory selection (e.g. Lerman, 2019)—for the aesthetics of a theoretical framework to receive explicit attention. Yet, from the perspective of everyday aesthetics, it becomes clear that aesthetic considerations are inseparable from our epistemological and

ontological ones. Our personal values, or axiologies—both our aesthetics and our ethics—are bound up in our epistemological and ontological choices. Indeed, the view that aesthetics and epistemology go hand-in-hand is a commonplace understanding in the world of the humanities (Allen, 1995; Paxman, 1992) as well as from the perspective of many Indigenous philosophies (e.g., Gutiérrez, 2017). As Bowers and Lawler (2021) point out, “Axiology palpably shapes every aspect of research: our sense of how we do good while doing research, our sense of what questions are interesting or worthwhile” (pp. 323–324). As a field, we must transcend the Enlightenment era myth that our minds and our thought processes—including our theorizing processes—can be separated from our bodies, personalities, and feelings. We should strive to acknowledge, interrogate, and embrace the subjectivities in our ways of knowing and being so that we conduct accountable, just, and personally resonant research (Haraway, 1988).

### **Embracing Aesthetic Reflection: Overcoming the “Just Doesn’t Feel Right” Dilemma**

In order to resolve the “just doesn’t feel right” dilemma of theorizing, we must be willing to acknowledge our alexithymia head on (Bowers, 2022). We must create space to acknowledge and reflect on our personal values that lead us to favor or disfavor elements of particular theoretical frameworks. Only then, once we have taken stock of our worldviews and personal philosophies, can we hope to construct theoretical frameworks that *do* feel right—frameworks that align with our epistemological, ontological, *and* axiological beliefs. In this section, I share my story of embracing aesthetic reflection to resolve this dilemma. My goal in sharing my theorizing story is not to prescribe a path that all researchers should or will take in resolving this tension. Rather, I aim to “reveal my work” (Siy, 2019) that has led me to where I am today so that it might serve as a guiding light for others who find themselves grappling with this dilemma.

In the second year of my doctoral program, students are tasked with designing, proposing, and carrying out their own research study. My theoretical framework featured a variant of social semiotic theory (Presmeg et al., 2016) that emphasized students’ meta-representational competence (diSessa, 2004) in an attempt to catalog students’ symbolic representation systems from an anti-deficit perspective (Peck, 2020). Although finding and networking these theories to create a framework for my study had been challenging, I felt good about my choices.

And yet, when it came to analyzing the data from the interviews I’d conducted, I froze. Although I had outlined a clear plan for how to analyze each representation students created in my proposal, each time I sat down to do the analysis, it felt like something was missing. I made some progress. Slowly. But as months passed, I found myself becoming detached from my work. Instead, I gravitated toward reading literature on philosophy and theory, trying to figure out what changes I might make to my theoretical framework to overcome my own “just doesn’t feel right” dilemma. Eventually, I ran out of time without any concrete answers. I scrambled to write up and share the results of my work. I passed, but I didn’t feel good about it.

I was studying topics I was interested in and using theories that (more or less) aligned with my epistemological and ontological beliefs. But this was not enough. Only retrospectively do I recognize that I had not carefully considered the litany of aesthetic questions pertaining to my choice of theoretical framework. For example, in what ways does the framework allow/enable me to engage all my senses and feelings during analysis? In what ways does the framework embrace complexity rather than reducing it? Does the framework highlight open-ended possibilities or favor close-ended answers? All these questions concern, to varying degrees, epistemology and ontology, but whether I found the answer the framework provided agreeable or not depended primarily on my personal aesthetics.

The goal of this research experience had been gaining autonomy and confidence in carrying out research, yet I found myself more lost than ever. I worried that all research would feel like this. On the worst days, I considered quitting my doctoral program.

### **Mood Boards as an Artistic Means of Aesthetic Reflection**

Luckily, a course I was taking at the time was framed with an aesthetic question: What is *your desire* to do research? (Loveless, 2019). By stripping away the utilitarian concerns of theoretical frameworks for a moment, the reflections I engaged in around this question have since allowed me to construct theoretical frameworks that feel consistent with my personal aesthetics. I also allowed myself to create art as a mean of reflection. One particularly impactful reflective exercise was the creation of a “research mood board.”

Mood boards are commonplace in various design industries such as fashion or interior design, as a planning tool to determine the aesthetics of a design. They are typically used to establish a sense of cohesion in the project (or, in my case, theoretical framework) before diving head-first into creating a final product. The boards themselves can consist of various visual elements—pictures, fabrics, textures, etc.—that align with the intended aesthetic of the project. Cassidy (2011) explains that “mood boards provide a ‘space’ to arrange the collected visuals in a meaningful manner to the designer that enables the flow of thoughts, inspirations, and creativity for design outcomes—products” (p. 230). Viewed in this light, mood boards are also a tool for creative thinking and problem solving as well as reflection. It is for this reason that researchers have suggested that mood boards might also be helpful in laying out and designing qualitative research studies (Garner & McDonagh-Philp, 2001). The act of creating this mood board allowed me to get a feel for my own aesthetics in relation to the theoretical framework of my next study. It allowed me to explicitly consider the aesthetic questions I had ignored before.

I continue to refer to this mood board as my study has proceeded. Keeping it in view reminds me of the value in ensuring an alignment between the aesthetics of the theoretical frameworks I use and my own personal aesthetics. Nowadays, I go as far as to actively incorporate everyday aesthetics and the paradigm of arts-based research (Leavy, 2018) into my work on student thinking. By shattering the divide between feeling and thinking, creation and research, I’ve found myself free to develop an aesthetic of mathematics education research that is not limited by what came before and more consistent with who I am as a person in and out of academia.

### **Looking Forward**

The act of constructing a theoretical framework that not only serves basic utilitarian purposes of research but is also in harmony with one’s worldviews and personal aesthetics is no easy task. Yet, there is minimal guidance for how one might go about this process in the mathematics education literature. The guidance that exists centers mainly on epistemological and ontological considerations, but in this report I have suggested that we must explicitly contemplate aesthetic considerations, too. This careful axiological reflection may not look the same for everyone as it has for me, but I sincerely hope that by revealing my work and sharing my painful struggle, that others from across the sub-fields of mathematics education will, in turn, join me in revealing their personal axiological reflections that have shaped their research.

By sharing our stories, we can empower emerging scholars who are just beginning to grapple with theoretical frameworks to bring themselves into their work. If we instead hide behind the guise of objectivity and an artificial separation between thinking and feeling, the risk of our field simply recreating what came before is too great. We must be willing to welcome in a “new aesthetics” of mathematics education (Bowers et al., 2022), one that allows all newcomers the freedom to breathe life into new, not-yet-imagined fields of study (Dubbs, 2021).

## References

- Allen, P. B. (1995). *Art is a way of knowing: A guide to self-knowledge and spiritual fulfillment through creativity*. Shambhala Publications.
- Baucom, L. (2019). Two paths; Tensions of an emerging scholar. *Proceedings of the 41st Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, 22–28. <http://www.pmena.org/pmenaproceedings/PMENA%2041%202019%20Proceedings.pdf>
- Bikner-Ansbahs, A., Prediger, S., Artigue, M., Arzarello, F., Bosch, M., Dreyfus, T., Gascón, J., Halverscheid, S., Haspekian, M., Kidron, I., Corblin-Lenfant, A., Meyer, A., Sabena, C., & Schäfer, I. (2014). Starting points for dealing with the diversity of theories. In A. Bikner-Ansbahs & S. Prediger (Eds.), *Networking of theories as a research practice in mathematics education* (pp. 3–13). Springer. [https://doi.org/10.1007/978-3-319-05389-9\\_1](https://doi.org/10.1007/978-3-319-05389-9_1)
- Bowers, D. M. (2022, March 18). Rage, rebellion, and revolution in mathematics education: Pursuing change with sincerity [Seminar]. Critical Philosophical & Psychoanalytic Institute for Mathematics Education. <https://drive.google.com/file/d/12maRMOOrPeSqJtRaYW5iz9faPqIGNlcyx/view>
- Bowers, D. M., Dubbs, C. H., & Moore, A. S. (2022). A manifesto from the margins: A new epoch for (non)theoretical mathematics education research. *Journal for Theoretical and Marginal Mathematics Education*, 1(1), 1–23. <https://doi.org/10.5281/zenodo.7499269>
- Bowers, D. M., & Lawler, B. R. (2021). Anarchism as a methodological foundation in mathematics education: A portrait of resistance. In D. Kolloche (Ed.), *Proceedings of the 11th International Mathematics Education and Society Conference* (Vol. 1, pp. 321–330). <https://www.mescommunity.info/proceedings/MES11.pdf>
- Burton, L. (2004). *Mathematicians as enquirers: Learning about learning mathematics*. Kluwer Academic Publishers. <https://doi.org/10.1007/978-1-4020-7908-5>
- Cai, J., Morris, A., Hohensee, C., Hwang, S., Robison, V., Cirillo, M., Kramer, S. L., & Hiebert, J. (2019). Theoretical framing as justifying. *Journal for Research in Mathematics Education*, 50(3), 218–224. <https://doi.org/10.5951/jresmetheduc.50.3.0218>
- Cassidy, T. (2011). The mood board process modeled and understood as a qualitative design research tool. *Fashion Practice: The Journal of Design, Creative Process, & the Fashion Industry*, 3(2), 225–251. <https://doi.org/10.2752/175693811X13080607764854>
- Dietiker, L., & Richman, A. S. (2021). How textbooks can promote inquiry: Using a narrative framework to investigate the design of mathematical content in a lesson. *Journal for Research in Mathematics Education*, 52(3), 301–331. <https://doi.org/10.5951/jresmetheduc-2020-0318>
- diSessa, A. A. (2004). Metarepresentation: Native competence and targets for instruction. *Cognition and Instruction*, 22(3), 293–331. [https://doi.org/10.1207/s1532690xci2203\\_2](https://doi.org/10.1207/s1532690xci2203_2)
- Dubbs, C. H. (2021). *Mathematics education atlas: Mapping the field of mathematics education research*. Crave Press.
- Ernest, P. (2016). An overview of the philosophy of mathematics education. *Philosophy of Mathematics Education Journal*, 31. <https://education.exeter.ac.uk/research/centres/stem/publications/pmej/pome31/index.html>
- Fiori, N., & Selling, S. K. (2016). Truth isn't everything: Promoting aesthetically guided choice in mathematical work. *The Journal of Mathematical Behavior*, 41, 219–234. <https://doi.org/10.1016/j.jmathb.2015.10.002>
- Garner, S., & McDonagh-Philp, D. (2001). Problem interpretation and resolution via visual stimuli: The use of mood boards in design education. *International Journal of Art & Design: Design Education*, 20(1), 57–64. <https://doi.org/10.1111/1468-5949.00250>
- Gutiérrez, R. (2017). Living mathematx: Towards a vision for the future. In E. Galindo & J. Newton (Eds.), *Proceedings of the 39th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 2–26). Hoosier Association of Mathematics Teacher Educators. <http://www.pmena.org/pmenaproceedings/PMENA%2039%202017%20Proceedings.pdf>
- Haraway, D. (1988). Situated knowledges: The science question in feminism and the privilege of partial perspective. *Feminist Studies*, 14(3), 575–599. <https://doi.org/10.2307/3178066>
- Hass, C. A. F., El-Adawy, S., Hancock, E., Sayre, E. C., & Savić, M. (2022). Emerging mathematics education researchers' conception of theory in education research. In S. S. Karunakaran & A. Higgins (Eds.), *Proceedings of the 24th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 1009–1014). <http://sigmaa.maa.org/rume/RUME24.pdf>
- Jasien, L., & Horn, I. (2022). Fixing the crooked heart: How aesthetic practices support sense making in mathematical play. *Journal for Research in Mathematics Education*, 53(1), 41–64. <https://doi.org/10.5951/jresmetheduc-2020-0228>
- Leavy, P. (2018). Introduction to arts-based research. In P. Leavy (Ed.), *Handbook of arts-based research* (pp. 3–21).
- Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

The Guildford Press.

- Leddy, T. (1995). Everyday surface aesthetic qualities: “Neat,” “messy,” “clean,” “dirty.” *The Journal of Aesthetics and Art Criticism*, 53(3), 259–268. <https://doi.org/10.2307/431351>
- Lerman, S. (2019). Theoretical aspects of doing research in mathematics education: An argument for coherence. In G. Kaiser & N. Presmeg (Eds.), *Compendium for early career researchers in mathematics education* (pp. 309–324). Springer. [https://doi.org/10.1007/978-3-030-15636-7\\_14](https://doi.org/10.1007/978-3-030-15636-7_14)
- Louie, N. L. (2017). The culture of exclusion in mathematics education and its persistence in equity-oriented teaching. *Journal for Research in Mathematics Education*, 48(5), 488–519. <https://doi.org/10.5951/jresmetheduc.48.5.0488>
- Loveless, N. (2019). Polydisciplinarity. In *How to make art at the end of the world: A manifesto for research-creation* (pp. 59–76). Duke University Press. <https://doi.org/10.1215/9781478004646-004>
- Marini, G. (2021). An introduction to everyday aesthetics in education. *Studies in Philosophy and Education*, 40, 39–50. <https://doi.org/10.1007/s11217-020-09740-x>
- Paxman, D. (1992). Aesthetics as epistemology, or knowledge without certainty. *Eighteenth-Century Studies*, 26(2), 285–206. <https://doi.org/10.2307/2739321>
- Peck, F. (2020). Towards anti-deficit education in undergraduate education: How deficit perspectives work to structure inequality and what can be done about it. *Problems, Resources, and Issues in Mathematics Undergraduate Studies*, 31(9), 940–961. <https://doi.org/10.1080/10511970.2020.1781721>
- Presmeg, N., Radford, L., Roth, W.-M., & Kadunz, G. (2016). *Semiotics in mathematics education*. Springer. <https://doi.org/10.1007/978-3-319-31370-2>
- Saito, Y. (2007). *Everyday aesthetics*. Oxford University Press. <https://doi.org/10.1093/acprof:oso/9780199278350.001.0001>
- Sinclair, N. (2006). The aesthetic sensibilities of mathematicians. In N. Sinclair & W. Higginson (Eds.), *Mathematics and the aesthetic: New approaches to an ancient affinity* (pp. 87–104). Canadian Mathematical Society. [https://doi.org/10.1007/978-0-387-38145-9\\_5](https://doi.org/10.1007/978-0-387-38145-9_5)
- Sinclair, N. (2009). Aesthetics as a liberating force in mathematics education? *ZDM*, 41(1–2), 45–60. <https://doi.org/10.1007/s11858-008-0132-x>
- Siy, E. (2019). Creating a healthy and rigorous culture of research by revealing our work. In S. Otten, A. G. Candela, Z. de Araujo, C. Haines, & C. Munter (Eds.), *Proceedings of the 41st Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 16–21). <http://www.pmena.org/pmenaproceedings/PMENA%2041%202019%20Proceedings.pdf>
- Spangler, D. A., & Williams, S. R. (2019). The role of theoretical frameworks in mathematics education research. In K. R. Leatham (Ed.), *Designing, conducting, and publishing quality research in mathematics education* (pp. 3–16). Springer. [https://doi.org/10.1007/978-3-030-23505-5\\_1](https://doi.org/10.1007/978-3-030-23505-5_1)
- Stinson, D. W. (2020). Scholars before researchers: Philosophical considerations in the preparation of mathematics education researchers. *Philosophy of Mathematics Education Journal*, 36, 1–18. <http://socialsciences.exeter.ac.uk/education/research/centres/stem/publications/pmej/pome36/index.html>
- Tyburski, B. A., Drimalla, J., & Alibek, A. (2022, November 28). Graduate student meeting to discuss doctoral programs in mathematics education. 44th Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Nashville, TN.
- Westby, K. R. (2021). Disability in mathematics education: A call to action. In D. Olanoff, K. Johnson, & S. Spitzer (Eds.), *Proceedings of the 43rd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 232–233). <http://www.pmena.org/pmenaproceedings/PMENA%2043%202021%20Proceedings.pdf>

## WHY AM I SUPPOSED TO LOVE MATH?: DIGITAL MATHEMATICS STORYTELLING IN ASIAN AMERICAN COMMUNITIES

Theodore Chao  
The Ohio State University  
chao.160@osu.edu

Angga Hidayat  
The Ohio State University  
hidayat.8@osu.edu

Ruth Nneoma Oliwe  
The Ohio State University  
oliwe.1@osu.edu

*In this research study, we detail how Digital Mathematics Storytelling, in which youth create video stories detailing the mathematics knowledge existing within their families and communities, can actively create counter-stories to the model minority myth. Through intergenerational video storytelling in historic Asian American communities, the research team and participants used a community participatory action research and narrative inquiry framework to engage elementary and middle-school aged youth in mathematics-based storytelling that not only detailed the painful effects of the model minority myth but also showcased that mathematics identities within Asian American communities can be rich and joyful.*

Keywords: Equity, Inclusion, and Diversity; Informal Education; Social Justice; Technology

### Objectives

Within the realm of mathematics education, Asian Americans youth must actively combat the destructive stereotype known as the “model minority myth,” which historically reinforces the idea that Asian Americans are somehow naturally gifted in mathematics (Chen & Buell, 2017; Lee, 1996; Shah, 2019). This stereotype not only erases the diversity of experiences within the Asian American community, but also positions Asian American youth as only being able to achieve success in mathematics-related enterprises. This harmful stereotype, coupled with the lack of Asian American stories in the mainstream media, paint Asian Americans as a singular monolith, ignoring the vast academic diversity within the Asian American community and leading to generalizations, hate, and even violence against Asian Americans (Gover et al., 2020; Takaki, 1998).

This research proposal explores how Digital Mathematics Storytelling can be used to elicit culturally sustaining mathematics stories within Asian American communities. Specifically, this research focuses on how intergenerational youth-led video stories created within a community participatory action research and narrative inquiry research project frame how Asian American youth position themselves around mathematics. Through these stories, Asian American youth actively craft counter-stories that not only create unique mathematics identities honoring their culture, but also push back against destructive stereotypes such as the model minority myth.

This research project is framed by two research objectives:

1. To examine the ways in which Digital Mathematics Storytelling can be used to elicit intergenerational stories about mathematics within Asian American communities.
2. To analyze counter storytelling by youth and their community and family members that actively combat stereotypes about Asian Americans and mathematics.



## Theoretical Perspectives

The use of Digital Mathematics Storytelling as a means of exploring culturally significant mathematics stories in Asian American communities draws on multiple, intersecting theoretical frameworks, as detailed below.

### Critical Race Theory

A Critical Race Theory perspective positions race as neither a fixed or real construct, but rather a socially constructed phenomenon produced and reinforced by existing systems of power and oppression (Delgado & Stefancic, 2012; Ladson-Billings & Tate, 2006). This perspective problematizes the traditional understandings of race and racism, recognizing that racism is not a series of isolated incidents but rather an enduring part of society and the legal system (Bell, 1995; Crenshaw, 1991). Through its emphasis on narrative and counter-storytelling, Critical Race Theory disrupts dominant narratives, thereby unveiling ways in which existing power and educational structures perpetuate racialized forms of inequality and injustice. By challenging the presumed neutrality and objectivity of legal structures and social institutions, Critical Race Theory reveals that our structures are often inherently imbued with racial biases. Consequently, Critical Race Theory functions not only as a theoretical lens but also as a methodological tool for interrogating the relationships between power, law, and race.

### AsianCrit

AsianCrit is a specific framework of Critical Race Theory that seeks to challenge dominant narratives about Asian Americans and promote a more nuanced understanding of the experiences and perspectives of Asian Americans (Museus & Iftikar, 2013). Drawing from the field of Asian American studies, the key tenets of AsianCrit include the intersection of race, class, gender, and sexuality, and the need for community-led activism and solidarity across diverse Asian American communities. The tenets of AsianCrit are powerful frameworks for recognizing and challenging harmful stereotypes about Asian Americans. Particularly in the realm of mathematics education, an AsianCrit framework helps to challenge the monolithic representations and recognizing the multifaceted realities of these students, an AsianCrit perspective in mathematics education can promote equitable educational outcomes, not only in terms of academic performance, but also in cultivating a truly inclusive, supportive, and representative educational environment.

### The Model Minority Myth

The model minority myth is a stereotype that categorizes Asian Americans as uniformly successful, industrious, and academically outstanding, suggesting that they serve as a “model” minority group within the United States (Chang & Au, 2007; Chow, 2017). However, this simplistic narrative overlooks the rich diversity and multifaceted experiences within the Asian American community, unintentionally perpetuating harmful stereotypes that can lead to further marginalization and discrimination. Moreover, the myth disregards the past and present racial prejudice and violence endured by Asian Americans, including systemic exclusion and hate crimes (Hartlep, 2013).

The model minority myth also incorrectly suggests that other minority groups could achieve similar levels of success if they were to “work hard enough,” thereby negating the impacts of systemic and structural barriers that obstruct upward mobility. This presumption is detrimental not only to Asian Americans but also to other minority groups, as it fosters harmful stereotypes and veils the realities of racial inequality and discrimination in the United States. For instance, the stereotype of Asian Americans as passive and compliant workers is used to unfavorably compare and thus disadvantage other racial and ethnic minority groups in the USA, reinforcing a system of power that incites division among non-white communities (Ng et al., 2007).

In mathematics education, the AsianCrit framework helps to make visible this “model minority myth”. It highlights the absurdity of generalizing that all Asian Americans excel in mathematics and shows how this flawed assumption conceals the academic needs and struggles of various groups within the Asian American community.

### **Culturally Sustaining Pedagogy**

Culturally Sustaining Pedagogy is an educational framework that seeks to promote equitable and inclusive education for students from diverse cultural backgrounds (Paris & Alim, 2014). Drawing on the fields of multicultural education and critical pedagogy, the construct of culturally sustaining pedagogy emphasizes validating and valuing students’ cultural identities and learning experiences. This approach recognizes the need for educators to engage in ongoing critical reflection on their own cultural biases and assumptions, and to actively work towards creating a learning environment that promotes equity, diversity, and social justice. Culturally sustaining pedagogy also emphasizes the importance of engaging with communities and families as active partners in the educational process.

### **Narrative Identity and Counter-Storytelling**

Identity is not only embodied within the stories a person tells about themselves, but also encompasses the actual act of narrating or storytelling (Sfard & Prusak, 2005). Identity is a verb, made and remade through the act of storytelling. The stories are not merely descriptions of a static reality, but rather dynamic constructs that can change over time and context. The narratives serve as constructs that embody someone’s range of experiences, characteristics, and expectations, thereby defining the creation and evolution of one’s personal and social identities. Even more important than telling a story to explore one’s identity is the way that identities are reified and endorsed through the acceptance, validation, and re-telling of these narratives.

Counter-storytelling, therefore, involves sharing stories and experiences that challenge existing dominant narratives and stereotypes, with the goal of promoting a more nuanced and equitable understanding of social issues in order to challenge dominant (and oppressive) narratives (Solórzano & Yosso, 2002). Counter-storytelling is a tool for marginalized communities to highlight their experiences and perspectives, and challenge destructive narratives that perpetuate harmful stereotypes. Through counter-storytelling, individuals and communities reclaim their own narratives and thereby their own identities (Author, 2021).

Together, these theoretical frameworks provide a theoretical foundation for using of digital mathematics storytelling to explore culturally sustaining mathematics stories told by youth within Asian American communities.

### **Modes of Inquiry**

This research proposal uses a community participatory action research and narrative inquiry approach to explore the ways that Asian American youth engage in digital mathematics storytelling.

### **Community Participatory Action Research**

Community participatory action research emphasizes the importance of involving community members in the research process, not just as participants, but as actual decision makers in the research planning and enactment (Kim, 2016; Mirra et al., 2015). This research approach recognizes that knowledge production is not an objective or neutral process, but rather a process shaped by the social, political, and economic context in which it is produced. By using community participatory action research, this research proposal engages with Asian American youth, families, and communities as active participants in the research process.

## **Narrative Inquiry**

Narrative inquiry (Clandinin & Connelly, 2000) is a research methodology that focuses on the stories that people tell about their experiences and how these narratives are their identities (Sfard & Prusak, 2005). This approach recognizes that stories are a powerful mechanism of communicating and understanding complex phenomena that connect to our central humanity. By using digital mathematics storytelling to elicit stories about mathematics, this research project elicits, analyzes, and tells counter-stories crafted by youth and their community and family members that actively combat stereotypes about Asian Americans and mathematics.

## **Research Participants**

The research participants in this proposal were 8 youth who self-identified as Asian American and who lived in a predominantly Asian American community in the Western part of the United States of America. They ranged in age from 9-years old to 13-years old. Each participant engaged in a four-part digital mathematics storytelling workshop focusing on crafting and telling mathematics videos. Because of the use of youth participatory research methodology, each youth storyteller engaged in the research project differently, based on their own agency and ideas about what the research project should be.

Some youth crafted their stories individually over multiple sessions with the research team. Some youth crafted their stories in small groups with each other. Every youth involved elders from their own family and community in crafting the stories. And members of the research team, who identified as Asian American community members, interacted with the youth both as university-based educational researchers and members of the youth participants' communities. Because of this proximity between members of the research team and the Asian American community, all recruitment of participants occurred through already established relationships between the research team members and the Asian American community centers that served as the physical location of the storytelling workshops and screening.

Within the structure of digital mathematics storytelling that the 8 Asian American youth enacted, the research team collected data based on (1) field note observations, (2) informal interviews, and (3) the final stories that the youth created. Additionally, (4) the final showcase community screening of the digital mathematics stories at an Asian American community center was video recorded and analyzed. This final community screening served as a culminating event for family and community members to watch, react, and discuss the feelings and stories surfacing from watching the youth-created stories together.

## **Analysis**

The process of data analysis was carried out in several stages. First, our research team meticulously watched the data and engaged in multiple conversations, fostering a collective understanding of the video stories, conversations, and observations generated throughout the project. These discussions offered diverse perspectives and interpretations, enriching our comprehension of the data.

Next, the team conducted follow-up interviews with the participants to further deepen our understanding of their experiences. This step allowed us to clarify any ambiguities and add layers of depth to our initial interpretations of the data. During this stage, we also engaged in member checking with the participants to ensure that our interpretations resonated with the participants' experiences and perspectives.

Finally, after a rigorous process of data immersion and refinement, the research team engaged in thematic analysis. This entailed a systematic process of coding and categorizing the data to identify themes and patterns. Our primary focus lay on the ways in which the youth

participants actively crafted and framed counter-stories that challenged the model minority myth and their positioning in relation to mathematics.

### **Researcher Positionality**

The first author identifies as a 2.5 generation Asian American, which brings personal experiences of Asian American family and community positioning along different gradients of “Americanness”. The 2nd author identifies as an Asian individual residing in the United States, navigating an ongoing journey of balancing Asian cultural values and ways of thinking with their life in the United States. The 3rd author identifies as a person of color, embodying a spectrum of experiences that revolve around being continually othered seen as deficient in mathematics. Our shared lived experiences fostered a sense of connection with our youth participants, enabling us to establish trust through the recognition and validation of shared experiences, good and bad.

However, we must also acknowledge and be critically aware of the positional power we held, as adult researchers within a university setting. We recognize that this power dynamic influenced the way our participants engage with us. We also recognize that our experiences as not identical to those of our youth participants. We may have experienced our own set of challenges, but we must remain attuned to the ways in which the participants’ experiences may differ from, and at times echo, our own.

### **Results**

The use of digital mathematics storytelling elicited several counter stories around mathematics that challenge the ways that Asian American youth position themselves and their communities. While we do not have space to fully describe our findings here, we hope this glimpse of the stories created within this research project shows the intricacies of Asian American mathematics identities that came out through the digital mathematics stories shared.



**Figure 1: Screenshot from a participant’s Digital Mathematics Story exploring the mathematics of a shared counting sequence within a community Kung Fu practice.**

### **Theme 1: Mathematics in the Culture**

The first theme of *mathematics in the culture* focused on the way that youth positioned mathematics as more than just the numbers and formulas they work with in school. By telling stories in which they detail the very use of mathematics in their own communities, several youths showed that mathematics in their own worlds was an active and constant construct that they were very aware of. For instance, in Figure 1, a youth presented the precise ways that their entire community kung fu studio counted together in English while enacting their routine. This use of

choral counting as a mechanism of unity and cohesion shows the ways that very simple mathematics, such as counting, was a constant within the rich lives of the Asian American community. Additionally, the conversations during the community screening highlighted the ways that these shared experiences, such as kung fu, created historical ties within the community, as the basics sequences of kung fu practice have survived for multiple generations, with a history more than 4,000 years old. One elder shared that the counting sequence that their grandchild enacted in this practice was the same counting sequence that he had enacted as a child more than 70 years ago.



**Figure 2: Screenshot from a participant’s Digital Mathematics Story exploring the crafting of intricate designs in boiled sugar for a traditional Chinese dessert.**

### **Theme 2: Asian American Creativity**

Several participants also shared the ways they saw mathematics in culturally sustaining practices that were actively different than the mathematics they felt they were often positioned within. For example, in Figure 2, a participant detailed the mathematics they noticed during a Lunar New Year celebration featuring an artisan crafting intricate animals out of boiled sugar. In this case, the participant detailed the intricate mathematics involved in the melting of sugar, the timing of how long the sugar will stay malleable, and the precision needed by an artisan in molding this sugar into intricate animal patterns in a traditional dessert. The discussion during the community screening opened up ideas about how Asian Americans engaged in multiple artistic practices that involved creativity, ingenuity, and tradition, which countered the stereotypes of regurgitation and precision often associated with mathematics achievement. Simply put, the myth of Asian Americans as non-creative and non-artistic was one that all youth participants felt and wanted to counter.



**Figure 3: Screenshot from a participant’s Digital Mathematics Story exploring the feeling of loss following the deadly Monterey Park shooting.**

### **Theme 3: Mathematics as a Mechanism for Processing Violence**

Several participants shared poignant narratives of the trauma they had experienced, notably in response to attacks on their Asian American community. These traumas highlighted the impact of Anti-Asian hate onto the youths’ mental health—an area often overlooked in Asian American students. A poignant example of this trauma came when a participant analyzed the devastating effects of the Monterey Park shooting of 2023, detailing the lost lives and opportunities and the survivors’ experiences (Arango et al., 2023). These narratives resonated deeply with participants and their families, giving voice to intergenerational stories of feeling “hunted” and “attacked” in the United States solely due to their Asian American identity.

These stories led to conversations about how the current wave of Anti-Asian hate was not new; this violence was embedded in the fabric of our country since its inception, as evidenced by historical instances such as the Japanese internment camps and the Chinese Exclusion Act (Takaki, 1998). Therefore, while these violent episodes caused significant distress among the youth, this research project offered an opportunity to delve into Asian American history and address violence and trauma head-on, as opposed to disregarding or minimizing it.

While the initial aim of this research project was not specifically to aid participants in processing trauma, the stories and ensuing discussions made it evident that participants required a safe space not only to tell these stories but also to acknowledge the collective trauma resulting from these events.

### **Discussion**

Utilizing digital mathematics storytelling as a medium to delve into culturally sustaining mathematics narratives in Asian American communities via community participatory action research not only fostered a more nuanced understanding of Asian Americans, but also actively disputed detrimental stereotypes associated with mathematics. A scan of existing literature revealed a dearth of similar research, underscoring the importance and novelty of this project in illuminating previously invisible perspectives about Asian American youth and mathematics.

By creating a platform for counter-stories that challenge the model minority stereotype, the youth-produced video stories and subsequent discussions underscored the diversity of experiences and viewpoints within Asian American communities. In essence, these stories served as platforms for youth expression, allowing participants space to reclaim control of harmful

narratives about their identity and to counteract stereotypes. These counter-stories thus dispute prevalent narratives about Asian Americans and their relationship to mathematics, narratives that often serve to bolster white supremacy and oppressive educational practices (Chen & Buell, 2017).

Through this project, the research team witnessed the transformative potential of collaborative and participatory research methodologies. We not only engaged the youth as active participants, but also embraced the ways they involved their families and community members. We found that centering the experiences and perspectives of community members in the research process unveiled tangible ways that youth and their communities were challenging systemic barriers to their success and agency beyond academic settings.

Nonetheless, we recognize the limitations of this research project. The application of digital mathematics storytelling and community participatory action research does not present a universal solution to the multifaceted issues confronting mathematics education in Asian American communities. Furthermore, given that participant recruitment was primarily facilitated by the researchers' own standing as community members, we acknowledge the potential for sample bias.

In conclusion, we posit that digital mathematics storytelling or any narrative mechanism that prioritizes youth creative agency can facilitate a more nuanced understanding of mathematics education in relation to Asian American communities in the USA. These stories and ensuing conversations serve as potent reminders of the oversimplified and harmful nature of the model minority myth, and that Asian American experiences with mathematics diverge substantially from prevalent portrayals in our field.

Interestingly, while many students initially believed their videos needed to focus solely on mathematics, they discovered the potential to tell deeper stories about identity and their emotional connection to mathematics. They shared their mathematics identities by telling and retelling their stories.

## References

- Arango, T., Kim, V., Sandoval, E., Karlamangla, S., & Cowan, J. (2023, January 23). Monterey Park Shooting: Police in California Ask: 'What Drove a Madman to Do This?' The New York Times. <https://www.nytimes.com/live/2023/01/23/us/shooting-monterey-park-california>
- Author. (2021).
- Bell, D. A. (1995). Who's afraid of critical race theory. *University of Illinois Law Review*, 893–910.
- Chang, B., & Au, W. (2007). You're Asian, how could you fail math? *Unmasking the myth of the model minority. Rethinking Schools*, 22(2).
- Chen, G. A., & Buell, J. Y. (2017). Of models and myths: Asian(Americans) in STEM and the neoliberal racial project. *Race Ethnicity and Education*, 0(0), 1–19. <https://doi.org/10.1080/13613324.2017.1377170>
- Chow, K. (2017, April 19). "Model Minority" Myth Again Used As A Racial Wedge Between Asians And Blacks. NPR. <https://www.npr.org/sections/codeswitch/2017/04/19/524571669/model-minority-myth-again-used-as-a-racial-wedge-between-asians-and-blacks>
- Clandinin, D. J., & Connelly, F. M. (2000). *Narrative inquiry: Experience and story in qualitative research*. Jossey-Bass.
- Crenshaw, K. (1991). Mapping the margins: Intersectionality, identity politics, and violence against women of color. *Stanford Law Review*, 1241–1299.
- Delgado, R., & Stefancic, J. (2012). *Critical Race Theory: An Introduction*. NYU Press.
- Gover, A. R., Harper, S. B., & Langton, L. (2020). Anti-Asian Hate Crime During the COVID-19 Pandemic: Exploring the Reproduction of Inequality. *American Journal of Criminal Justice*, 45(4), 647–667. <https://doi.org/10.1007/s12103-020-09545-1>
- Hartlep, N. D. (2013). The model minority stereotype: Demystifying Asian American success. IAP.
- Kim, J. (2016). Youth involvement in participatory action research (PAR): Challenges and barriers. *Critical Social*

Work, 17(1).

- Ladson-Billings, G., & Tate, W. F. (2006). Toward a critical race theory of education. In A. D. Dixson & C. K. Rousseau (Eds.), *Critical race theory in education: All God's children got a song* (pp. 11–30). Routledge.
- Lee, S. J. (1996). *Unraveling the “model minority” stereotype: Listening to Asian American youth*. Teachers College Press.
- Mirra, N., Garcia, A., & Morrell, E. (2015). *Doing youth participatory action research: Transforming inquiry with researchers, educators, and students*. Routledge.
- Museus, S. D., & Iftikar, J. (2013). An Asian Critical Theory (AsianCrit) Framework. In M. Y. Danico & J. G. Golson (Eds.), *Asian American Students in Higher Education* (pp. 18–29). Routledge.
- Ng, J. C., Lee, S. S., & Pak, Y. K. (2007). Contesting the model minority and perpetual foreigner stereotypes: A critical review of literature on Asian Americans in education. *Review of Research in Education*, 31(1), 95–130.
- Paris, D., & Alim, H. S. (2014). What are we seeking to sustain through culturally sustaining pedagogy? A loving critique forward. *Harvard Educational Review*, 84(1), 85–100.
- Sfard, A., & Prusak, A. (2005). Telling identities: In search of an analytic tool for investigating learning as a culturally shaped activity. *Educational Researcher*, 34(4), 14–22.
- Shah, N. (2019). “Asians Are Good at Math” Is Not a Compliment: STEM Success as a Threat to Personhood. *Harvard Educational Review*, 89(4), 661–686. <https://doi.org/10.17763/1943-5045-89.4.661>
- Solórzano, D. G., & Yosso, T. J. (2002). Critical race methodology: Counter-storytelling as an analytical framework for education research. *Qualitative Inquiry*, 8(1), 23–44.
- Takaki, R. (1998). *Strangers from a Different Shore: A History of Asian Americans (Updated and Revised)*. Little, Brown and Company.



---

# EARLY ALGEBRA, ALGEBRAIC THINKING AND FUNCTION

---

## ALGEBRA INTERVENTIONS AT THE ELEMENTARY AND SECONDARY LEVELS: SEARCH FOR A DEFINITION

Sheree T Sharpe  
University of New Hampshire  
Sheree.sharpe@unh.edu

Matthew Mauntel  
University of New Hampshire  
Matthew.mauntel@unh.edu

*This paper is a part of a larger research study that we are conducting to develop a framework consisting of the data-driven best practices around teaching and learning algebra in K-12 classrooms. The purpose of this paper is to develop a usable definition for algebra interventions for the second stage of screening within the larger study, which will be done by reviewing seminal or systematic reviews in the algebra field across grades K-12. Combining the algebra definitions with the type of interventions produces three questions to ask during the second stage of screening to decide if a study is an algebra intervention.*

Keywords: Algebra, Algebraic Thinking, Types of Interventions

It is well known that algebra is essential and a gatekeeper to higher level mathematics and to the pursuit of a career in the Science, Technology, Engineering, and Mathematics (STEM) fields (Hughes et al., 2014; National Mathematics Advisory Panel, 2008). Despite this, many middle and high school students struggle with understanding and achieving in school algebra (Bednarz, 2001; Kieran, 2007). Sharpe (2019) found that difficulties occur when students transition from arithmetic to school algebra. Hence, researchers have argued that students need to gain algebraic thinking in the elementary grades (Carraher, Schliemann, & Schwartz, 2008). This paper is a part of a larger research study that we are conducting to develop a framework consisting of the data-driven best practices around teaching and learning algebra in the K-12 classrooms. The purpose of this paper is to describe what we mean by algebra interventions in grades K-12 classrooms to aid in identifying relevant research.

### Perspective

Meta-research is the quantitative (meta-analysis) and the qualitative (meta-synthesis) analysis of studies (both published and unpublished). The advantage of this type of research is the cumulative power of analyzing many samples on a specific topic, the result of which has much greater explanatory power than one study in isolation (Cooper, 2017). In the overarching study, the authors are in the process of conducting a meta-research study. To conduct this larger research the authors will need to search databases and gray literature, screen studies, extract data from the studies, and analyze the data via meta-analysis and qualitative meta-synthesis. The screening stage consists of three levels of screening: (1) Title/Abstract screening, (2) Full-text screening, and (3) Methods screening. During the title and abstract screening stage of over 11,000 studies, conflicts between raters arose regarding whether or not the studies included an intervention involving algebraic thinking, especially at the elementary level. A screening protocol was developed with seven questions, where the screener needs to answer *yes*, *no*, or *more info is needed*. If all questions received a *yes* or *more info is needed*, then the abstract moved from level 1 to level 2 of screening, however, if any of the questions received a *no*, then the abstract moved to the rejected folder. The two screening questions that the screeners had conflicts with while trying to answer “yes/no/more info” are:

- Does the study involve algebra, an algebra-related topic (equation, function, variable), or algebraic reasoning (patterns, relations, quantitative relationships, change in various contexts)?
- Does the study involve at least one of the following teaching approaches:
  - A strategy or intervention for learning about algebra, functions, or equations?
  - A program or instructional technique for learning about algebra, functions, or equations?
  - A game, technological tool, or curricular materials for learning about algebra, functions, or equations?

Hence, the aim of this paper is to develop a usable definition for algebra interventions for the second stage of screening, which will be done by reviewing seminal or systematic reviews in the algebra field across grades K-12.

### **Modes of Inquiry**

Our process for developing screening questions began by first identifying key studies about early algebra and algebraic thinking. A search of ERIC was conducted using the term “algebraic thinking.” The top search results were inspected, snowballed, and reversed snowballed (Sayers, 2007) to identify key articles that were referenced in the field. Five papers were included from this search based upon their being referenced by multiple publications in algebraic thinking and presenting a definition of algebraic thinking (Carraher and Schliemann, 2007; Kaput, 2008; Blanton et al., 2018; Chimoni et al., 2018; and Kieran, 2021). Kieran (2022) was not cited extensively but was included because several of her algebra papers were referenced frequently and this work was the latest iteration. Also included were the NCTM standards (2000) for algebra since they are the basis for most US mathematics teaching standards.

Next, we investigated systematic reviews in algebra to develop a list of potential interventions. A search of ERIC on EBSCOhost, APA PsycInfo, and Education Source was conducted using the terms: “algebra interventions AND systematic reviews.” This initial search produced three reviews all related to learning disabilities and algebra instructions (Bone et al., 2021; Hughes et al., 2014; Watt et al., 2016). A search of Hughes et al., (2014) produces an additional two reviews (Haas, 2005; Rakes et al., 2010) on algebra instructions. And a search of Bone et al., (2021) produces one additional review (Hwang et al., 2019). This resulted in a total of six reviews to investigate the type of algebra interventions.

### **Results**

#### **Algebra Definition**

The goal of this section is to operationalize a definition of algebra to be utilized in the meta-research. This means producing a set of yes/no questions that screeners can utilize to determine if a study pertains to early algebra. This was accomplished by looking at surveys of literature, systematic reviews, and organizational reports (such as NCTM) that attempt to define algebra and early algebra. The goal of this section is to synthesize the working definitions together to produce a list of characteristics of an algebra study across grades K to 12. The six publications that we review in this section include: the NCTM standards for algebra (2000), Carraher and Schliemann (2007), Kaput (2008), Blanton et al. (2018), Chimoni et al. (2018), and Kieran (2021). After reviewing these works we looked across the definitions to formulate a set of questions that could be used on a screening protocol to detect if a study involves algebraic thinking.

In 2000, NCTM issued standards relating to mathematics across the K-12 curriculum including algebra standards for grades K-2 and 3-5. These standards while enacted differently across the two groups highlighted the importance of four key standards (p.296): “Understand patterns, relations, and functions; Represent and analyze mathematical situations and structures using algebraic symbols; Use mathematical models to represent and understand quantitative relationships; and Analyze change in various contexts.” Examples of the standards include students using a variety of notations and objects including pictures, lists, and equations; students engaging in learning about the properties of numbers and the rules governing them such as the commutative property; and students modeling a variety of real-life scenarios such as growth of plants. As students enter grades 6-8, the same standards remain, but their subjects differ as more emphasis is placed on functions, graphs, equations, tables, algebraic expressions, and linear equations. Finally, as students transition to grades 9-12 greater emphasis is placed on a wide variety of functions including exponential, logarithmic, and period functions and more complex work with functions including composing, combining, and inverting. Students are recommended to study global behavior of graphs including finding the intercepts, asymptotes, and zeros of a graph. Students are encouraged to symbolize the relationships between different expressions.

Carraher and Schliemann (2007) build on the idea of early algebra by posing five questions that will help define early algebra. The core themes behind these questions are the relationship between arithmetic and algebra, the role of procedural mathematics in algebra, the role of modeling in algebra, when formal mathematics notation should be introduced, and the role of non-conventional notations such as student illustrations and informal student language. They highlight the idea that “algebra is latent in the existing early mathematics curriculum,” (p.25) through introducing ideas that could relate to functions, properties of number systems, properties of the number line, and algebraic equations. Carraher and Schliemann make it clear that there are opportunities for learning algebra throughout elementary school and there are activities such as modeling, representing, and generalizing that extract the deeper algebraic structure from lessons.

Kaput (2008) defines algebra as consisting of two core aspects that are embodied by three main strands which view algebra “as a cultural artifact expressed mainly as conventional symbol systems and as a certain kind of human activity” (p.10). The first core aspect involves generalizing and symbolizing patterns or regularities that are noticed. The second core aspect takes these symbolized generalizations and reasons with them to elaborate on structure and this is where the traditional rules of algebra are established. The first core aspect embodies much of early algebra and the second core aspect often applies later in one’s algebraic career, but this is not guaranteed. The three strands are much like topics that are studied in algebra. The first strand concerns computations, arithmetic, and quantitative reasoning. The second strand encompasses relations, joint variations, and functions. The final strand concerns modeling situations both inside and outside of mathematics.

Blanton et al. (2018) builds on the framework of Kaput et al. (2008) to define four key aspects/activities involved in early algebraic thinking. These activities include condensing information through generalizing, representing with pictures and symbols, justifying by developing arguments to explain their reasoning, and expanding their built knowledge to other situations by reasoning with generalizations. In their proposed curriculum they discuss some of the contexts in which students will be using these activities. These include the relational understanding of the equal sign, looking at the fundamental properties of arithmetic, and

modeling with linear equations. In Blanton's work we continue to see the importance of generalizing as well as several other activities including representing, justifying, and relating. The topics include looking at equations, equality, arithmetic, and relations.

Chimoni et al. (2018) looked at algebra for grades 4-7 and developed four different dimensions of algebraic thinking based partially on the work of Kaput. The first dimension was the strands from Kaput (2008). The second dimension involves key concepts/topics which are essential to algebra, and they identify as "equal signs, equality, equations, properties of numbers, properties of operations, variables, unknown quantities, symbols, co-variation, and correspondence" (p. 60). The third dimension consists of processes which expand upon activities from earlier works and include "noticing, conjecturing, representing, generalizing, justifying, and validating" (p. 60). The final dimension involves types of reasoning and includes both inductive and deductive reasoning. Through Chimoni et al.'s (2018) we see common topics that arise and the importance of certain forms of activity such as generalization.

Finally, Kieran (2022) conducted a historical analysis of the development of early algebra and identified three dimensions: analytical, structural, and functional. The analytical dimension looked at how students came to understand unknowns and issues relating to equations such as equivalence. The structural dimension referred to "expressing structure and properties within numbers, operations, and expressions" (p.1144). This involved comparing and relating numbers while representing and symbolizing them with expressions. The functional-thinking dimension involved generalizing especially when it was extracted from covariational and relational reasoning. While generalizing was only referenced by name in the last dimension, Kieran admits that generalizing is "the scarlet thread that runs through all three dimensions" (p.1134).

Looking across these different definitions of algebra there are several things that we noticed. First there is an importance given to certain human activities or processes. These are most often represented by -ing words. The most important example which is prominent across almost every definition is the act of generalizing. Other activities include modeling, representing, and symbolizing, justifying, relating, validating, expressing, solving, and graphing. These human activities extract the structure from observations such as generalizing or relating and then apply that structure to other problems through activities such as solving, justifying, and validating. In this way, we feel the activities communicate the essence of Kaput's (2008) core aspects, Chimoni et al's processes, and the thread that runs through Kieran's (2022) three dimensions.

While human activities are important, it is also important to collect the strands that they explore while engaging in algebraic thinking. These are the topics of their activity and include quantities, generalized arithmetic, co-variation, equations, equality, properties of numbers systems, operations, variables, change, patterns, graphs, models, expressions, or functions. Some of these topics include reifications (Sfard, 1991) of the mentioned activities. The topics allow us to distinguish between other mathematical subjects such as geometry where generalization, validating, and justifying are also important activities. The activities allow us to distinguish between early algebra and middle or high school algebra by looking at the topics of the activity. If we look at the NCTM standards (2000) for example, the standards are the same for grades K-12, but the topics are different. As students progress through early algebra, they begin by looking at patterns and formulating generalizations of these patterns by extending them and drawing early graphs and understanding equivalence. In middle school they are expected to model linear equations and solve equations. Finally, in high school they solve multiple different types of equations, graph inequalities and model a variety of situations using algebraic concepts they have developed along the way.

The goal of this section was to develop a set of questions to distinguish algebra from other topics of mathematics to distinguish early algebra from topics like arithmetic. First, we want to ensure that if a study claims to represent algebra, pre-algebra, algebraic thinking, or early algebra then it is included. If the study does not describe itself as algebraic then we will include it if it includes both an algebraic activity and topic or rather if both of the following questions are yes:

- Does the study involve generalizing, modeling, representing and symbolizing, justifying, relating, validating, noticing, conjecturing, reasoning, expressing, solving, or graphing?
- Does the study involve any of the following topics: quantities, generalized arithmetic, co-variation, equations, equality, properties of numbers systems, operations, variables, change, patterns, graphs, models, expressions, or functions?

### **Types of Interventions**

The goal of an algebra intervention is to intervene in the regular practices of the classroom as a way to improve the teaching or learning of algebra. The goal of this section is to explore the different types of algebra interventions used in a classroom for inclusion in our meta-research. This was accomplished by synthesizing six studies (i.e., meta-analyses, systematic reviews, reviews, and synthesis) related to algebra intervention as a means to capture the different types of algebra interventions. The six studies that were synthesized are Haas (2005); Rakes et al. (2010); Hughes et al. (2014); Watt et al. (2016); Hwang et al. (2019); and Bone et al. (2021).

Haas (2005) conducted a meta-analysis of studies published between 1980 and 2002 at the secondary school level (grades 7 to 12) wherein the authors conducted an experimental investigation with algebra instruction as the focus. Haas' (2015) meta-analysis resulted in the inclusion of 35 independent experimental studies which he used to classify (by effect sizes) six teaching method categories correlated to positive effects on students' achievement in the secondary-algebra classroom. The following are the six teaching method categories as described by Haas (2015): "Cooperative learning, Communication and study skills, Technology-aided instruction, Problem-based learning, Manipulatives, models, and multiple representations, and Direct Instruction" (p. 27-28).

Rakes et al. (2010) conducted a systematic review and meta-analysis of studies published between 1968 to 2008 on algebra instructional improvement strategies that used random experiments and quasi-experimental designs. Rakes et al. (2010) meta-analysis resulted in the inclusion of 82 relevant studies from mostly the secondary level with one study at grade 3. Five categories of improvement strategies emerged: "Technology-based curricula, Non-Technology curricula, Instructional strategies, Manipulatives, and Technology tools" (p.382).

Hughes et al. (2014) conducted a meta-analysis of Algebra Interventions aimed at improving algebra performance of students with disabilities and struggling learners published between 1983 and 2013. Hughes et al. (2014) included studies that were either experimental or quasi-experimental designs which resulted in 12 relevant articles both at the elementary and secondary levels. The type of interventions described by the authors included: Cognitive/Model-based Interventions, Co-teaching (general ed teacher and special ed teacher), CRA - Concrete-Representational-Abstract Instructional Sequence, Graphic Organizers, Single-sex Instruction, and Technology.

Watt et al. (2016) conducted a review and meta-analysis of studies published between 1980 and 2014 for teaching algebra to students with learning disabilities using either experimental, quasi-experimental, or single-subject designs. The review included the results of 15 studies (10

experimental and 5 single-subject designs) at both the elementary and secondary level. Effect sizes were computed for the experimental studies. The interventions were: Concrete-Representational-Abstract Instructional Sequence, Cognitive Strategy instruction, Enhanced Anchored Instruction, Tutoring, and Graphic Organizers.

Hwang et al. (2019) conducted a review of algebra interventions published after 1990 about the cognitive processes involved in problem solving and their instructional strategies. These algebra interventions focused on secondary students (grades 6 to 12) with learning disabilities. The review included the results of 11 effective algebra interventions. Four conceptualizations of the cognitive processes involved in problem solving were identified: (a) sequential concrete–semi-concrete–abstract, (b) sequential virtual-abstract, (c) integrated concrete–semi-concrete–abstract, and (d) abstract only. Hwang et al. (2019) also discussed several instructional strategies incorporated in the interventions: advance organizer, modeling, guided practice, independent practice, post-test, feedback/rewards.

Bone et al. (2021) conducted an evidence-based synthesis on algebra instruction and interventions for secondary students (grades 6 to 12) with learning disabilities published between 1999 and 2019 by critically analyzing the practices for teaching appropriate algebraic concepts. The synthesis included 18 studies across 17 articles, where only 14 met the Council of Exceptional Children standards of high quality. Twelve of the 18 studies used single-case design and the other 6 used group design methodology. Across the 18 studies, 8 instructional categories emerged: (a) concrete-representational-abstract (CRA) framework (b) schema-based instruction, (c) enhanced anchor instruction (EAI), (d) manipulatives, (e) peer-assisted learning strategies, (f) virtual-abstract (VA) framework (g) graphic organizers and diagrams; and, (h) explicit inquiry routine. However, Bone et al. (2021) found that none of the instructional approaches met the criteria for evidence-based, and only 5 were found to be potentially evidence-based (i.e., CRA, manipulatives, EAI, schema based instruction, and peer assisted learning strategy).

Looking across these definitions of interventions, we found six general intervention types for algebra across the grade levels: Instructional, learning, curricula, technology, tutoring, and manipulatives. Although these categories form a general base from which to discuss effective algebra interventions, one type of intervention not covered in the aforementioned reviews was teacher development. Sharpe and Schliemann (2017) provide a good example of the effectiveness of this as an intervention. In their study, Sharpe and Schliemann analyzed the impact of a teacher development program based on a functional approach to algebra on their grade 7 students' understanding of equations. The teacher development intervention was a three semester-long online graduate level courses on algebra, functions, multiple representation, modeling, applications, and student thinking. The teachers were not given a class-level intervention or told what to implement in their classrooms. Sharpe and Schliemann's (2017) found that the students of the teachers who participated in the teacher development intervention showed significantly greater improvement than the students in the control teachers' classrooms.

We have included teacher development as a potential algebra intervention measure to be considered further in this context. The results of our synthesis of the six studies produced the following categories of interventions:

- Instructional Strategies – this consists of different instructional methods such as cooperative learning, focus on communication and study skills, explicit inquiry routine, mastery learning, cognitive strategies, schema-based instruction, enhanced anchored instruction, concrete-representational-abstract instructional sequence, co-teaching, single-

sex instruction, and direct instruction (Bone et al., 2021, Haas, 2005; Hughes et al., 2014; Hwang et al., 2019; Rakes et al., 2010; Watt et al., 2016). Enhanced anchored instruction is an inquiry-based learning where students solve problems or complete tasks using video anchors of real life situations (Bone et al., 2021, Watt et al., 2016). Concrete-Representational-Abstract Instructional Sequence is an instructional sequences which moves students through the concrete (manipulatives), representational (pictures) to the abstract stage of symbols (Bone et al., 2021, Haas, 2005; Hughes et al., 2014; Hwang et al., 2019; Watt et al., 2016).

- Learning Strategies – this includes problem-based learning and model-based learning which is learning to solve real-world or word problems using problem solving strategies (Bone et al., 2021; Haas, 2005; Hughes et al., 2014).
- Curricula – this consisted of reformed-based curricula and researcher-developed curricula (Rakes et al., 2010).
- Technology-aided Instruction/Curricula/Tools – this includes using computers, calculators, software, graphic organizers, or virtual representation as a basis for curricula, instruction, or a tool in the classroom (Bone et al., 2021; Haas, 2005; Hughes et al., 2014; Hwang et al., 2019; Rakes et al., 2010; Watt et al., 2016).
- Tutoring - this can consist of one-on-one tutoring with a peer, tutoring from preservice teachers, special education teachers, or paraprofessionals, or incorporating peer-assisted learning strategies (Bone et al., 2021, Watt et al., 2016).
- Manipulatives – this is when concrete objects are used to help students understand a concept (Bone et al., 2021, Rakes et al., 2010).
- Teacher Development - this is when the intervention is geared towards the teachers with no explicit instructions on how to implement a specific strategy or curricula with the students (Sharpe & Schliemann, 2017).

The goal of this section was to explore the different types of algebra interventions that may be expected when screening studies for inclusion. If a study claims to intervene in the regular practices of the classroom as a way to improve the teaching and/or learning of algebra via instructional strategies, learning strategies, curricula, technology, tutoring, manipulatives, or teacher development, then we will include it into the pool for review in the larger study.

#### **Discussion and/or Conclusions**

Recall that this paper is a part of a larger study that we are conducting to develop a framework consisting of data-driven best practices around teaching and learning algebra in the K-12 classroom, by conducting both a meta-analysis and a qualitative meta-synthesis. The aim of this paper is to develop a usable definition for algebra interventions for the second stage of screening within the context of the larger study. Combining the algebra definitions with the type of interventions produces three questions to ask during the second stage of screening to decide if it's an algebra intervention (Figure 1).



Does the study implement?	Does the study involve?	Does the study address?
<ul style="list-style-type: none"> <li>• Instruction</li> <li>• Learning</li> <li>• Curricula</li> <li>• Technology</li> <li>• Tutoring</li> <li>• Manipulatives</li> <li>• Teacher Development</li> </ul>	<ul style="list-style-type: none"> <li>• Generalizing</li> <li>• Modeling</li> <li>• Representing &amp; Symbolizing</li> <li>• Justifying</li> <li>• Relating</li> <li>• Graphing</li> <li>• Validating</li> <li>• Expressing</li> <li>• Solving</li> <li>• Graphing</li> <li>• Conjecturing</li> <li>• Reasoning</li> <li>• Solving</li> <li>• Noticing</li> </ul>	<ul style="list-style-type: none"> <li>• Quantities</li> <li>• Generalized Arithmetic</li> <li>• Co-variation</li> <li>• Expressions or Equations</li> <li>• Equality</li> <li>• Properties of Number Systems</li> <li>• Operations</li> <li>• Variable/Change</li> <li>• Patterns</li> <li>• Graphs/Models</li> <li>• Functions</li> </ul>

**Figure 1: What is an algebra intervention?**

Next, we will provide an example of an algebraic thinking intervention in elementary school, as it relates to our model in the figure above. Blanton et al. (2019) conducted an intervention study in three school districts across 46 schools with 23 schools randomly assigned to treatment and the other 23 randomly assigned to the control condition. The authors describe the intervention as follows:

“The intervention consisted of 18 one-hour lessons at each of Grades 3 to 5, with lessons taught throughout the school year (approximately September through March). As noted earlier, lessons were designed to engage students in the algebraic thinking practices of generalizing, representing, justifying, and reasoning with mathematical structure and relationships within the Big Ideas of generalized arithmetic; equivalence, expressions, equations, and inequalities; and functional thinking” (Blanton et al., 2019, p. 1942).

As underlined in the above paragraph, the type of intervention is instructional, the type of activities are generalizing, representing, justifying, and reasoning, and the type of topics are generalized arithmetic; equivalence, expressions, equations, and inequalities; and functional thinking.

An additional example elucidates how the information in the figure above can be applied to studies involving algebra interventions in secondary school. Thompson and Senk (2001) examine the performance of 8 pairs of algebra classes from four schools (Atlanta, Chicago, Mississippi, and Philadelphia), where one of the pair uses the University of Chicago Mathematics Project (UCSMP) Advanced Algebra Curriculum and the other their regular textbook. A total of 150 students were in the UCSMP classes and 156 in the comparison classes, all students were either in 10th or 11th grade. Topics across the intervention and comparison schools included “work with linear, quadratic, higher degree polynomial, exponential, logarithmic, and trigonometric expressions, equations, functions, ... powers and roots, sequences and series, systems of equations and inequalities, matrices, and conic sections” (p. 64). The UCSMP textbook “emphasizes four dimensions of understanding: skills, properties, uses, and representations.”

As underlined in the above paragraph, the type of intervention is curricula, the type of activity includes representing, and the type of topics include functions and equations. What is an algebra intervention? It is a study of a practice that intervenes in the regular practices of the classroom as a way to improve the teaching and/or learning of algebra, and if the study claims to address algebra, pre-algebra, algebraic thinking, or early algebra then it’s an algebra intervention. Our analysis produced a protocol for our screeners to decide if a publication included an algebra intervention. The following three new screening questions will replace the initial two screening questions for the full-text screening each being answered with a dichotomous yes vs no:

- Does the study implement any of the following: instructional strategies, learning strategies, curricula, technology, tutoring, manipulatives, or teacher development?
- Does the study involve generalizing, modeling, representing and symbolizing, justifying, relating, validating, noticing, conjecturing, reasoning, expressing, solving, or graphing?
- Does the study involve any of the following topics: quantities, generalized arithmetic, co-variation, equations, equality, properties of numbers systems, operations, variables, change, patterns, graphs, models, expressions, or functions?

### Acknowledgement

This material is supported by the National Science Foundation DRK-12 Grant #2142659.

### References

- Bednarz, N. (2001). A problem-solving approach to algebra: Accounting for the reasoning and notations developed by students. In H. Chick, K. Stacey, J. Vincent, & J. Vincent (Eds.), *The future of the teaching and learning of algebra: Proceedings of the 12th ICMI Study Conference* (Vol. 1, pp. 69-78). Melbourne, Australia: The University of Melbourne.
- Blanton, M., Brizuela, B. M., Stephens, A., Knuth, E., Isler, I., Gardiner, A. M., ... & Stylianou, D. (2018). Implementing a framework for early algebra. *Teaching and learning algebraic thinking with 5-to 12-year-olds: The global evolution of an emerging field of research and practice*, 27-49.
- Bone, E., Bouck, E., & Witmer, S. (2021). Evidence-Based Systematic of Literature on Algebra Instruction and Interventions for Students with Learning Disabilities. *Learning Disabilities: A Contemporary Journal*, 19(1), 1-22.
- Carraher, D. W., Martinez, M. V., & Schliemann, A. D. (2008). Early algebra and mathematical generalization. *ZDM*, 40, 3-22.
- Chimoni, M., Pitta-Pantazi, D., & Christou, C. (2018). Examining early algebraic thinking: insights from empirical data. *Educational Studies in Mathematics*, 98, 57-76.
- Cooper, H. (2017). *Research synthesis and meta-analysis: A step-by-step approach* (5th ed.). Thousand Oaks, CA: SAGE Publications, Inc.
- Haas, M. (2005). Teaching methods for secondary algebra: A meta-analysis of findings. *NASSP Bulletin*, 89 (642), 24-46.
- Hughes, E. M., Witzel, B. S., Riccomini, P.J., Fries, K. M., & Kanyongo, G. Y. (2014). A meta-analysis of Algebra interventions for learners with disabilities and struggling learners. *The Journal of the International Association of Special Education*, 15(1), 36 – 47.
- Hwang, J., Riccomini, P.J., & Morano, S. (2019). Examination of Cognitive Processes in Effective Algebra Problem-Solving Interventions for Secondary Students with learning Disabilities. *Learning Disabilities: A Contemporary Journal*, 17(2), 205-220.
- Kaput, J.J., Carraher, D.W., & Blanton, M.L. (Eds.). (2008). *Algebra in the Early Grades* (1st ed.). Routledge.
- Kieran, C. (2007). Learning and teaching Algebra at the middle school through college levels: Building meaning for symbols and their manipulation. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (Vol. II, pp. 707–762). Charlotte, NC: Information Age Publishing.
- Kieran, C. (2022). The multi-dimensionality of early algebraic thinking: background, overarching dimensions, and new directions. *ZDM–Mathematics Education*, 54(6), 1131-1150.
- National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. National Council of Teachers of Mathematics.
- National Mathematics Advisory Panel. (2008). *Foundations for success: the final report of the National Mathematics Advisory Panel*. U.S. Department of Education.
- Rakes, C.R., Valentine, J.C., McGatha, M.B., & Ronau, R.N. (2010). Methods of Instructional Improvement in Algebra: A Systematic Review and Meta-Analysis. *Review of Educational Research*, 80(3), 372-400.
- Sharpe, S.T. (2019). An algebraic translation task solved by grade 7-9 students. *Mathematical Thinking and Learning*, <https://doi.org/10.1080/10986065.2019.1564970>.
- Sayers A. (2007). Tips and tricks in performing a systematic review. *The British journal of general practice : the journal of the Royal College of General Practitioners*, 57(538), 425.
- Watt, S. J., Watkins, M., & Abbitt, J. (2016). Teaching Algebra to Students with Learning Disabilities: Where Have We Come and Where Should We Go? *Journal of Learning Disabilities*, 49(4), 437-447.
- Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

## COLLEGE STUDENTS' CONCEPTIONS OF SYMBOLIC PROPERTIES IN ALGEBRA

Claire Wladis  
BMCC/CUNY  
CUNY Graduate Center  
cwladis@bmcc.cuny.edu

Benjamin Sencindiver  
CUNY Graduate Center  
bsencindiver@gc.cuny.edu

Kathleen Offenholley  
BMCC/CUNY  
koffenholley@bmcc.cuny.edu

*Here we investigate how college students may conceptualize symbolic algebraic properties. This work uses the theory of Grundvorstellungen (GVs) to analyze how learners' conceptions may or may not align with some desired goals of instruction. Through the analysis of interviews with students across a variety of courses, we describe several categories of conceptions, or descriptive GVs, that emerged in the data. We expect these categorizations to be a helpful first step in understanding learners' thinking and improving instruction on algebraic properties.*

Keywords: algebraic properties; syntactic reasoning; equivalence; algebraic transformation

Mathematical properties justify transformation across mathematical domains, especially those that rely on symbolic representation such as algebra. Despite the central role that algebraic properties play in mathematical transformation, learners often do not use them in mathematically valid ways (e.g., Hoch & Dreyfus, 2004; Mok, 2010) and instruction may not support students in learning about properties explicitly enough (e.g., Barnett & Ding, 2019; Eaves et al., 2021; Larsen, 2010). In this paper we focus on how learners identify parallel syntactic structure between symbolic properties and mathematical objects such as expressions, and we explore how this may relate to their conceptions of symbolic properties in algebra.

### Forms and Symbolic Properties

The framework of student conceptions of symbolic properties presented here is part of a larger structure sense model, which has been presented in more detail in other work (Wladis, 2019; Wladis et al. 2022a, 2023a); here we focus in more detail on conceptions of symbolic properties specifically. As part of this framework, we view a *property* as any mathematical statement that may be used to transform a mathematical object into an equivalent object with a different form. Using this definition, both axiomatic and derived statements are properties.

Consider the examples: 1) a definition of rational exponents, written for example as:  $x^{\frac{1}{n}} = \sqrt[n]{x}$  for all positive integers  $n$ ; and 2) the statement about two equivalent equations:  $A \cdot B = C \leftrightarrow A = \frac{C}{B}$  for all nonzero  $B$ . Under our model, both examples are considered to be mathematical properties. The key characteristic is that the definition of negative exponents is a valid justification for replacing an expression of the form  $x^{\frac{1}{n}}$  with one of the form  $\sqrt[n]{x}$  (or vice versa), and similarly, the statement about equivalent equations is a valid justification for replacing equations with other equivalent equations that have a particular form. Here we are interested in how properties can be used to transform symbolic representations, so we use the term *symbolic properties* to denote symbolic representations of properties. Properties are made up of smaller sub-structures (e.g., each side of a formal property statement can be viewed as a separate object), which are often referred to colloquially during mathematics instruction as *forms* (e.g., the “form”  $x^{\frac{1}{n}}$ ,  $\sqrt[n]{x}$ ,  $A \cdot B = C$ , and  $A = \frac{C}{B}$ , in the properties above). Here with the term “form” we are generalizing a common practice that is often used to refer to particularly common forms used in

computation. For example, it is very common for instructors to ask students if a particular expression has the form  $ax^2 + bx + c$  or whether an equation has the form  $y = mx + b$ .

This work addresses a gap in existing literature on learners’ use of mathematical properties, where much of that literature has focused on classifying the types of errors students make when using properties to simplify expressions or solve problems (e.g., Hoch & Dreyfus, 2004; Mok, 2010), or on learners’ justifications for why properties are true or their ability to derive properties from arithmetic patterns (e.g., Hunter et al., 2022). Some work has focused on student structure sense for specific properties, such as the distributive property (e.g., Schüler-Meyer, 2017). Since it is a critical skill for working with more complex symbolic representations (Kieran, 2011), we focus on learners’ conceptions of *symbolic* properties and forms. This work aims to describe conceptions of properties and forms more generally by building a theory of how learners’ conceptions of properties and syntactic reasoning (Wladis et al, 2022a, 2023b) are connected.

### Theoretical Framework

In this work, we draw on the theory of prescriptive and descriptive *Grundvorstellungen* (GVs) (or “fundamental conceptions”). Prescriptive GVs describe aspirational mental models that we aim for learners to attain during instruction (vom Hofe, 1995); while descriptive GVs describe students’ actual conceptions. Descriptive and prescriptive GVs are not intended to be static, nor to present a monolithic view of what it means to learn a particular idea. Comparing prescriptive and descriptive GVs, however, may be beneficial for instructors and curriculum writers to assess the success of their intended goals for instruction and curriculum, and refining materials to align with their goals for students’ thinking (Greefrath et al., 2016). We begin by describing two related prescriptive GVs for symbolic properties (Table 1).

**Table 1: Two Prescriptive GVs for Symbolic Properties**

<b>Equivalence -Preserving GV</b>	Symbolic properties by definition describe a valid method for replacing one symbolic object (e.g., expression, equation) with another equivalent one, with respect to a particular context-dependent pre-existing definition of equivalence (e.g., insertion equivalence of expressions; Prediger & Zwetzscher, 2013).
<b>Mapping GV</b>	In order for equivalence to be preserved when properties are used for transformation, the following criteria must be met: The form on one side of the symbolic property must be mapped bijectively (one-to-one, so that no symbols in the symbolic object or the form are left out or used more than once) to the symbolic object (e.g., expression, equation) so that: 1) A unified subexpression <sup>9</sup> is mapped to each variable in the form; 2) All other symbols are mapped to notation in the form with the same syntactic meaning (e.g., different notation for multiplication can be mapped to one another).

### Framework for Classifying Descriptive GVs of Symbolic Properties

Our framework for classifying descriptive GVs of symbolic properties conceptualizes students’ conceptions as existing on two axes: operational vs. structural conceptions of properties

<sup>9</sup> By unified subexpression, we mean a substring for which placing parentheses around it would not change the syntactic meaning of the overall object (e.g., in the expression  $-3x^2$ ,  $x^2$  is a subexpression, but  $-3x$  is not).

(Sfard, 1992), and extracted vs. stipulated definitions of properties (Edwards & Ward, 2004). A student with a *structural* conception thinks of properties as abstract objects (e.g., canonical representations of particular algebraic structures), whereas a student with an *operational* conception thinks of mathematical properties as a process of computation. A student with a structural conception sees objects as reified processes (e.g., the form  $a(b + c)$  is seen as an object, and not just as the process of adding  $b$  and  $c$  and then multiplying  $a$  by the result), while a *pseudostructural* conception is when a student views something as an object that is not the reification of any process (Sfard, 1992, p. 75). We see the operational/structural distinction as relating to the prescriptive Mapping GV of Symbolic Properties, which focuses on a learner’s ability to conceptualize forms within a property structurally as an object (although what that reified object is may vary by learner).

*Extracted* definitions are definitions that one creates to describe the observed usage of a term (e.g., a learner may extract a meaning for a property from how their instructor uses the term during in-class computations). *Stipulated* definitions, in contrast, are stated explicitly, allowing for one to consult the definition directly to determine if something fits the definition (Edwards & Ward, 2004). We see this distinction as relating to the Equivalence-Preserving GV of Symbolic Properties, where one key stipulated feature of properties is that they preserve equivalence (the type of equivalence that is preserved is also based on a stipulated definition of equivalence in that context).

**Table 2: Framework to Categorize Descriptive GVs for Symbolic Properties**

	Extracted Definition	Stipulated Definition
Operational Conception of Properties	<b>Pseudo-process view:</b> Learners see properties as a cue to a computational process, and their approaches are extracted from prior experience rather than based on stipulated definitions. They often draw on surface structure rather than syntactic meaning. For example, students may conceptualize the distributive property as an instruction to “take what is on the outside of the parentheses and put it next to each thing on the inside”, regardless of the specific operations involved.	<b>Process view:</b> Learners see properties as a cue to a computational process, but attend to syntactic meanings and/or equivalence as a justification (e.g., checking for appropriate operations in the expression; checking that original and resulting expressions are insertionally equivalent). However, they may struggle to conceptualize properties as objects to which structures in the expression or equation can be mapped one-to-one, and as a result may have difficulty generalizing the use of properties to more syntactically complex symbolic representations.
Structural Conception of Properties	<b>Pseudo-object view:</b> Learners conceptualize a property as something that requires mapping to the specific forms in the property, but the mapping is still somewhat ill-defined and/or	<b>Object view:</b> Learners conceptualize the property as an object, such as a canonical form, to which the specific mathematical object (i.e., expression, equation, etc.) must be mapped one-to-one, in such a way that preserves

---

based on extracted notions, such as what “looks right”. syntactic meaning. They recognize that it is this criterion that preserves equivalence.

---

### Methods

This study is based on 102 cognitive interviews conducted with US college students on items from a concept inventory about Algebra topics (Wladis et al., 2018, 2023c). Interviewees came from 18 different courses, ranging from elementary algebra (similar to Algebra I in high school) to Linear Algebra. In this work, we analyze students’ responses to questions that were focused primarily on their reasoning around properties or forms, using thematic analysis (Braun & Clarke, 2006). Our analysis was influenced by an initial theoretical stance focused on noticing how students’ responses may reflect extracted and stipulated definitions (Edwards & Ward, 2004) or operational and structural (Sfard, 1992) conceptions, as well as the extent to which students appeared to show evidence of Equivalence-Preserving or Mapping GVs. Through iterative refinements, the analysis led to a more nuanced emergent framework of learners’ conceptions, or descriptive GVs, of symbolic properties, which we present here.

### Results and Analysis

We illustrate the framework by presenting a few excerpts from student interviews that demonstrate different ways that students may conceptualize symbolic properties in algebra. These examples were chosen because we felt that they reflected some of the most common types of reasoning observed in the sample.

#### Operational Conceptions

In this section, we present several segments from an interview with a student (whom we call *Iota*) who was enrolled in an introductory statistics course that had a school algebra pre-requisite. In these segments, Iota appeared to be drawing on operational GVs of symbolic properties when given a series of seven related questions, including the item shown below (Figure 1).

- Q6:** Which of the following could result from using the **distributive property** to rewrite the expression  $(x + 2)(3x + 7)$ ?
- $x + 2 \cdot 3x + 7$
  - $x \cdot 3x + 2 \cdot 7$
  - $x + 2 \cdot 3x + 2 \cdot 7$
  - $(x + 2) \cdot 3x + (x + 2) \cdot 7$
  - None of the above.
  - I don’t know the distributive property.

#### Figure 1: One Representative Item from a Series of Seven Related Items

Each of the items asked the result of applying distributive property to a different expression. Expressions used in other versions of this item included: Q1:  $(2x + 1)^2$ ; Q2:  $x - (2x + 1)$ ; Q3:  $2(2x \div 1)$ ; Q4:  $2(x \cdot y)$ ; Q5:  $(2x + 1)^2$ ; and Q7:  $2(xy)$ . For each, Iota stated that the distributive property could be used to rewrite the expressions: They (correctly) chose d for Q6, and an equivalent expression that could be conceptualized as the result of the distributive property for Q1 ( $2x \cdot 2 + 1 \cdot 2$ ) and Q2 ( $x - 2x - 1$ ). But Iota also incorrectly chose “results” of applying the distributive property to Q3 ( $2 \cdot 2x \div 2 \cdot 1$ ), Q4 ( $2x \cdot 2y$ ), Q5 ( $(2x)^2 + 1^2$ ) and Q7 ( $2x2y$ ). The specific answers that Iota chose suggest that Iota may have a purely operational conception of the distributive property akin to the framing “The distributive property is an instruction to take whatever is on the outside of the brackets and apply it to each ‘thing’ inside

the brackets”. At the same time, Iota’s ability to conceptualize  $(x + 2)$  as a unified sub-expression within  $(x + 2)(3x + 7)$  that could then be “distributed” to each term in the subexpression  $3x + 7$  is an unusual and syntactically sophisticated skill, suggesting that Iota is also capable of thinking structurally. When asked to explain their thinking on Q4 ( $2(x \cdot y)$ ), Iota stated “Because obviously two can distribute [makes motion with fingers as though moving the two from right to left twice] with the one in parentheses. So two in the front can distribute to  $2x$  multiply by  $2y$ . So, it's gonna be  $2x$  multiply by  $2y$  [repeats distributive motion with fingers]—that's the result.”

In their explanation, Iota’s focus is on describing computation, and not on verifying or justifying the mathematical validity of that computation. This is consistent with an operational GV. Thus, at this moment, Iota appears to be drawing on a pseudo-process conception. We see more evidence of this later in the interview when the interviewer asked Iota what the distributive property is:

**Interviewer:** What is the distributive property?

**Iota:** Distribute property is like that you can use the main number or main groups to distribute to each of another number or another groups.

**Interviewer:** So, is that like here [highlighting  $(x + 2)$  in Q6], is  $x + 2$  the main number?

**Iota:** It's a main group. Yes.

**Interviewer:** And then you apply that to each of the ones [motions to  $3x$  and  $7$  in Q6]

**Iota:** Yes.

**Interviewer:** Okay. So, I noticed that this one [highlights  $+$  in expression  $(3x + 7)$  in Q6] has a plus sign in between them. Is the distributive property only for the plus sign or could it also be subtraction? Could it be multiplication or division?

**Iota:** So, yeah, it could be subtraction, multiplication... Could be any sign, but when you calculate, when you are doing it, you have to do with that own sign.

Again, Iota appears to conceptualize the distributive property as a process, in which whatever is outside the brackets is multiplied by each “group” inside the brackets, while maintaining the original operation between the multiple “groups” inside the brackets. In this case, Iota appears to be drawing on a pseudo-process GV of the distributive property. In contrast, when Iota was interviewed about Q7 ( $2(xy)$ ), they start to reveal some evidence of a process view:

**Iota:** sometimes when I see these kind of questions, at first I may think its right answer is A ( $2x2y$ ), but what I normally do is I double check the answer. So I create some equations and I double check it, it's incorrect. So for this case, I create like  $x$  is 3. Okay, let me type it now,  $y$  is 2 (Iota types, producing the following).  $x=3, y=2$

$$2(3*2) = 2*6 = 12$$

$$2*3*2*2 = 24$$

I think it's wrong. So, I say no... I don't know why, but this is very tricky question for me... So,  $x$  and  $y$  multiply each other should be do before multiply the one outside. Now I was thinking. I don't know, it's not look like a distributive property for me. It's look like the way to calculate is you do the  $xy$  first because in parentheses, and after you get the result of  $xy$  you do with the number 2. So, I don't think this one is like a distributive property... to be honest, I don't know why. I don't think it's A, but I just feel it's not.

**Interviewer:** So, this strategy that you were doing, replacing  $x$  and  $y$  with numbers and seeing if they were the same—if you did that for number six, for example, would you get the same answers?

**Iota:** Oh, that's a good question. I don't... Yeah. Right. I don't know... I didn't... I didn't try. But... I mean, I'm just, I'm looking at it right now. Yeah, it should be the same. Because it should be only one value. Mm-hmm.

In this excerpt, there is evidence of both process and pseudo-process conceptions. For the first time Iota shows evidence of the prescriptive Equivalence-Preserving GV, when they substitute numbers to check whether the expression resulting from their distributive property transformation in Q7 produces the same output as the original expression, at least for one value. When they observe that the results are not equal for that value, they question their use of the distributive property to replace  $2(xy)$  with  $2x2y$ . Thus, we see evidence of a process GV. However, their approach still draws on extracted meanings and some pseudo-process conceptions: they mention several times “feeling” that the distributive property is not right here or describing whether the expression “looks like” the distributive property should be used. They did not call on their process GV on the other six similar distributive property questions, until the interviewer asked them whether this would be true for those questions as well. At that point, Iota saw the relevance to other questions by drawing on their knowledge of the distributive property as an equivalence-preserving transformation. However, Iota specifically describes how the way the items “looked” cued them not to take time to call on their equivalence preserving GV in this context (and instead cued a pseudo-process approach). It may be that Iota would benefit from instruction, tasks, and assessments that aim to explicitly link their pre-existing equivalence preserving GV about properties to actual calculation procedures. One component of this may be to focus more on checking and justifying calculation than calculation alone.

### Pseudo-Object Conception

We now present an interview with an elementary algebra student, Eta, where they were asked to interpret whether  $(2x + 1)(3x - 5)$  could be viewed as equal to the form  $(a + b)c$ .

Consider  $(2x + 1)(3x - 5)$  in its current form (don't rewrite it or do anything to it). Is there any part of  $(2x + 1)(3x - 5)$  which could be equal to  $(a + b)c$  if we pick the right expressions to represent  $a$ ,  $b$ , and  $c$ ?

- a. No
- b. Yes, if  $c = 3x$
- c. Yes, if  $c = 1$
- d. Yes, if  $c = 3x - 5$
- e. Yes, if  $c = 3$

### Figure 2: Eta's response to whether $(2x + 1)(3x - 5)$ can have the form $(a + b)c$

**Eta:**  $2x$  could be  $a$  then the one would be  $b$ , then the  $c$  would be  $3x$ . So, then I said that if  $c$  is equal to  $3x$  then it would make sense.... I'm just doing it by order by the first number, second number, third number. Maybe that's not the best way, but that's what I was doing.

**Interviewer:** What's being multiplied in each case [pointing to the expression]?

**Eta:** Two is being multiplied by three. Two is also being multiplied by the negative five. The same thing for the one, the one is being multiplied by three and then the one is also being multiplied by the negative five.

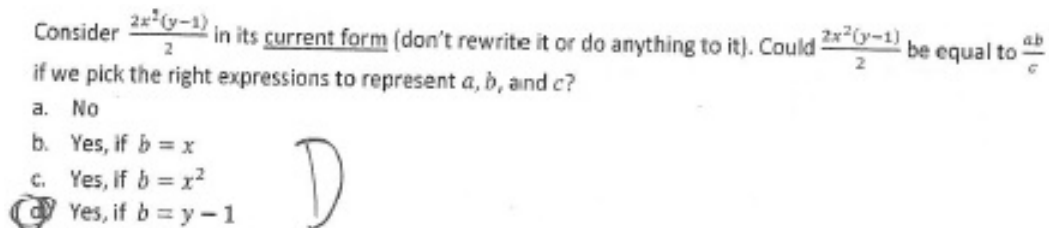
Here Eta appears to be drawing on a pseudo-object conception by mapping sub-expressions to variables in the form “in order”, i.e., mapping the “first subexpression” to the first variable,



the “second subexpression” to the second variable, etc., without attending to the grammatical meaning of syntactic structures in the expression. In  $(2x + 5)(3x - 5)$ , Eta initially does not attend to the second set of brackets while they are mapping subexpressions to the form  $(a + b)c$ ; however, when questioned further, Eta is able to identify that both terms in  $(3x - 5)$  will eventually be multiplied by each term in  $(2x + 1)$ . This suggests that Eta’s pseudo-object GV of properties likely does not stem directly from a failure to recognize the syntactic meaning of the second set of brackets, but that instead, this likely stems from a disconnect between the way that Eta interprets the syntactic meaning of expressions, and what information they focus on when trying to map that syntactic structure to a form. Eta does not identify the current syntactic meaning of  $(2x + 1)(3x - 5)$  as the subexpression  $2x + 1$  being multiplied by the subexpression  $3x - 5$ , but rather conceptualizes this expression as having the syntactic meaning of something like  $2 \cdot 3 \cdot x^2 + 2 \cdot -5 + 1 \cdot 3 \cdot x + 1 \cdot 5$  (which while equivalent to  $(2x + 1)(3x - 5)$ , technically has a different syntactic meaning). By perceiving it as the result of expansion rather than its current literal meaning, Eta is obscuring the structure needed to map this expression to the form  $(a + b)c$ . Thus, this computational view of syntactic structure appears to be negatively impacting Eta’s GV for symbolic properties. Because of this, Eta might benefit from instruction that more explicitly discusses the differences between expressions that have the same syntactic structure vs. expressions that are equivalent, and that explicitly links the syntactic structure of expressions and equations to form mapping. This may better enable Eta to draw on their existing knowledge of syntax, symbolic structures, and forms as objects.

### Object Conception

In this interview with Theta, an elementary algebra student, we asked them to interpret whether  $\frac{2x^2(y-1)}{2}$  could be viewed as equal to the form  $\frac{(ab)}{c}$  (where  $c \neq 0$ ).



**Figure 3: Theta’s work mapping a multi-term expression to a variable in a form**

**Theta:** I felt like D was the best option because looking at the example  $a$  and  $b$  over  $c$  the first equation fit that like  $a$  could be  $2x^2$  squared and  $b$  could be  $y - 1$  and  $c$  could be 2.

**Interviewer:** Did the parentheses impact your decision?

**Theta:** Yes.

**Interviewer:** How?

**Theta:** Because I saw that the  $y - 1$ , I saw it as separate from  $2x^2$ . And I know that looking at the second one that  $a$  and  $b$  in order for them to be multiplied they would most likely have to have parentheses around them. And I saw  $y - 1$  in parentheses so I just... Basically, looking at them all as substitutes like as soon as I saw  $a$  and  $b$  over  $c$  like I was just putting in my head okay,  $2x^2$  squared is  $a$ ,  $y - 1$  is  $b$ , and the two is equal to  $c$ .

In this excerpt, Theta appears to be drawing on an object GV of properties. They identify mathematically valid subexpressions in  $\frac{2x^2(y-1)}{2}$ , and identify which of these should map to each

variable in the form so that the syntactic structure is preserved. The interviewer then asked Theta to identify different syntactic structures in the expression, and Theta was able to do so accurately without further prompting. This is similar to the learners who were able to “treat a compound term as a single entity” when using the distributive property (Schüler-Meyer, 2017). Theta also discusses brackets from an object view (as a grouping mechanism rather than a cue to a procedure [see Wladis, et al, 2022b]) by describing how they “separate”  $2x^2$  from  $y - 1$ . This suggests that Theta has an object view of syntactic structure that they draw on to develop an object view of symbolic properties, because it enables them to identify the subexpression structures that produce a one-to-one mapping from  $\frac{(2x^2)(y-1)}{2}$  to the form  $\frac{ab}{c}$  so that syntactic structure is preserved. In addition, Theta’s conceptions of substitution and substitution equivalence (see Wladis et al, 2020) appear to be related to their conception of properties, because they mention substitution when describing how the subexpressions related to the form. Theta’s explanations are unusually structural here, compared to other students in the sample at all course levels. Theta was part of an intervention that was focused on explicitly teaching students the prescriptive GVs presented here (as well as others related to syntactic structure and equivalence)<sup>10</sup>, so this may have influenced their GV formation. While we can draw no causal conclusions based on this evidence, Theta’s responses indicate that some algebra students are capable of reasoning structurally about symbolic properties.

### Conclusion

In the vignettes presented here, all three students have prior knowledge that may be helpful to leverage when using symbolic properties to transform algebraic expressions or equations. In some cases, the learners drew on that prior knowledge in robust ways. In other cases, that prior knowledge was not cued or viewed as relevant in the moment by the learners as they answered questions about how they make sense of forms and symbolic properties. This may explain some of the results found in existing literature, where students made various computational errors when working with algebraic properties to transform expressions or equations (e.g., Hoch & Dreyfus, 2004; Mok, 2010). Future research is needed to better understand how these conceptions connect to computation and prior knowledge; we continue to investigate these relationships in ongoing research. However, these results shed light on learners’ *reasons* for working with symbolic properties in particular ways, which may be helpful in experimenting with different approaches to tailoring instruction to learners with different conceptions of symbolic algebraic properties. For example, learners who conceptualize a property as an instruction to perform a particular symbolic manipulation without connecting it to the Equivalence-Preserving GV (pseudo-object view) might benefit from tasks that engage them to justify their use of properties by linking transformation back to whether equivalence is preserved. In contrast, learners with a process view might benefit more from tasks that engage them in conceptualizing subexpressions as unified objects and give them opportunities to practice mapping these subexpressions to variables in forms in ways that preserve syntactic structure. The particular types of tasks or instruction that are beneficial to different learners may vary based on their conceptions of properties. This research is just a first step towards understanding student conceptions of symbolic properties in algebra, and significantly more research is needed.

---

<sup>10</sup> We note that this intervention was a separate study, but that some students who were in that intervention group also volunteered to be interviewed as a part of this study.

## Acknowledgements

This research was supported by the National Science Foundation (# 1760491). The opinions expressed here are those of the authors and do not represent those of the granting agency.

## References

- Barnett, E., & Ding, M. (2019). Teaching of the associative property: A natural classroom investigation. *Investigations in Mathematics Learning*, 11(2), 148–166.
- Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative Research in Psychology*.
- Eaves, J., Gilmore, C., & Attridge, N. (2021). Conceptual knowledge of the associativity principle: A review of the literature and an agenda for future research. *Trends in Neuroscience and Education*, 23, 100152.
- Edwards, B., & Ward, M. (2004). Surprises from mathematics education research: Student (mis) use of mathematical definitions. *The American Mathematical Monthly*, 111(5), 411–424.
- Greefrath, G., Oldenburg, R., Siller, H., Ulm, V., & Weigand, H. (2016). Aspects and "Grundvorstellungen" of the Concepts of Derivative and Integral: Subject Matter related Didactical Perspectives of Concept Formation. *Journal Für Mathematik-Didaktik*, 37, 99-129.
- Hoch, M., & Dreyfus, T. (2004). Structure sense in high school algebra: The effect of brackets (M. J. Høines & A. B. Fuglestad, Eds.; 3rd ed., pp. 49–56). *PME*.
- Hunter, J., Miller, J., Bowmar, A., & Jones, I. (2022). “ It Has the Same Numbers, Just in a Different Order”: Middle School Students Noticing Algebraic Structures within Equivalent Equations. *Mathematics Education Research Group of Australasia*.
- Kieran, C. (2011). Overall commentary on early algebraization: Perspectives for research and teaching. *Early Algebraization: A Global Dialogue from Multiple Perspectives*, 579–593.
- Larsen, S. (2010). Struggling to disentangle the associative and commutative properties. *For the Learning of Mathematics*, 30(1), 37–42.
- Mok, I. A. C. (2010). Students’ algebra sense via their understanding of the distributive law. *Pedagogies: An International Journal*, 5(3), 251–263.
- Prediger, S., & Zwetschler, L. (2013). Topic-specific design research with a focus on learning processes: The case of understanding algebraic equivalence in grade 8 (T. Plomp & N. Nieveen, Eds.; pp. 407–424). *SLO*.
- Schüler-Meyer, A. (2017). Students’ development of structure sense for the distributive law. *Educational Studies in Mathematics*, 96(1), 17–32.
- Sfard, A. (1992). Operational origins of mathematical objects and the quandary of reification: The case of function. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy* (Vol. 25, pp. 59–84). *Mathematical Association of America*.
- Vom Hofe, R. (1995). *Grundvorstellungen mathematischer Inhalte*.
- Wladis, C., Offenholley, K., Lee, J. K., Licwinko, S., Dawes, D. (2018). Development of the elementary algebra concept inventory for the college context. In T. Fukawa-Connelly, N. Engelke Infante, M. Wawro, S. Brown (Eds.), *Proceedings of the 21st Annual Conference on Research in Undergraduate Mathematics Education*. (pp. 605-617). San Diego, CA.
- Wladis, C. (2019, October). Symbolic structure sense: A “big idea” connecting arithmetic, algebra, calculus, and other mathematical domains. [Invited keynote talk]. *Mathematical Association of America (MAA) Research in Undergraduate Mathematics Education (RUME-NE) Northeast Regional Conference*, New York, NY.
- Wladis, C., Offenholley, K., Beiting, M., Griffith, S., Jaffee, E., Thakkar, N., and Dawes D. (2020). A proposed framework of student thinking around substitution equivalence: structural versus operational views. In Smith Kavaunakavan, S., Cook, S., Engelke-Infante, N. & Wawro, M. *Proceedings of the 23rd Annual Conference on Research in Undergraduate Mathematics Education*. Boston, MA.
- Wladis, C., Sencindiver, B. and Offenholley, K. (2022a) A model of symbolic structure sense: Meaning-making within abstract symbolic systems. Manuscript under review.
- Wladis, C., Sencindiver, B., & Offenholley, K. H. (2022b). An Exploration of How College Students Think About Parentheses in the Context of Algebraic Syntax. *Proceedings of the 44th Annual Psychology of Mathematics Education-North America (PME-NA) Conference*. (pp. 286-294), Middle Tennessee State University.
- Wladis, C., Sencindiver, B., & Offenholley, K. (2023a). Reconceptualizing algebraic transformation as a process of substitution equivalence. In S. Cook, and N. Infante (Eds.), *Proceedings of the 25th Annual Conference on Research in Undergraduate Mathematics Education (RUME)*, Omaha, NE.

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

Wladis, C., Sencindiver, B., & Offenholley, K. (2023b). Syntactic reasoning and cognitive load in algebra. In S. Cook, and N. Infante (Eds.), *Proceedings of the 25th Annual Conference on Research in Undergraduate Mathematics Education (RUME)*, Omaha, NE.

Wladis, C., Myszkowski, N., Sencindiver, B., Aly, G., Offenholley, K. and Jaffe, E. (2023c) *Creation and Validation of the Algebra Concept Inventory: Measuring the Algebraic Conceptual Understanding of College Students*. Manuscript under review.

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

# CONNECTING UNITS COORDINATION AND COVARIATIONAL REASONING: THE CASE OF DANIEL

Sarah Kerrigan  
George Fox University  
skerrigan@georgefox.edu

*Units Coordination and Covariational Reasoning are powerful frameworks for modeling students' mathematics in arithmetic reasoning and construction of relationships between changing quantities, respectively. This case study of an advanced stage 2, 8th-grade algebra student, Daniel, investigated connections between his units coordination and covariational reasoning on non-graphical covariation tasks. Results show Daniel leveraged his units coordination structures to reason about how two quantities varied together in several distinct ways. From Daniel, new insight was gained into underlying mental structures and actions involved in Carlson and colleagues' (2002) covariational reasoning framework. Implication for engaging a diversity of learners is included.*

Keywords: Number Concepts and Operations; Learning Theory; Middle School Education, Modeling

## Background and Motivation

Building models of mathematical thinking and development has been a powerful tool for mathematics education researchers to better understand the mathematics of students as well as the nature of mathematics (A. Hackenberg, 2014; Steffe & Thompson, 2000; Ulrich et al., 2014). By focusing on the cognitive resources available to the students and how they use them when engaging in mathematical activity, we leverage the students' own mathematics as meaningful in the context of the larger mathematics community rather than approaching understanding students' thinking from a deficit model, which focuses on what resources students lack.

Researchers have built robust models for students' mathematics in a variety of mathematical contexts, including whole numbers (Boyce & Norton, 2017; Olive, 2001; Steffe, 1992; Ulrich, 2015, 2016) and fractions (Boyce & Norton, 2016; A. Hackenberg & Tillema, 2009; Steffe & Olive, 2010). Within both these contexts, the construction of units and units coordination (UC) is essential. UC stage theory identifies what unit structures students assimilate with and how they reason in problems with that unit structure through sensorimotor and mental actions (Steffe, 1992).

The literature has started to expand UC to more advanced mathematics, investigating the role these cognitive structures play in areas such as algebra (A. Hackenberg, 2014; A. Hackenberg et al., 2021; Lee, 2018; Zwanch, 2022), calculus (Boyce et al., 2020; Byerley, 2019), and combinatorics (Tillema, 2013, 2014). Zwanch (2022), focusing on units construction through number sequences, found variations in students' approaches to solving systems of equations were more connected to the number sequence attributed to the student than instruction (math course taken). Hackenberg and Lee (2015) found a similar connection with students' equation writing, fraction knowledge, and multiplicative concepts, where the students' UC structures and available operations (fractions knowledge and multiplicative concepts) directly related to students' engagement in writing algebraic equations. In a more advanced mathematical topic, Tillema (2013) identified new unit structures for students working on combinatorics problems. Tillema

describes higher dimensional unit as pairwise structured unit of units rather than the Steffe (1992) composite unit of units.

This opens questions of what other unit structures students construct in more advanced mathematical contexts. For example, covariational reasoning has emerged as a fundamental component of mathematical development in topics such as functions, graphs, and interpreting quantitative situations (Carlson et al., 2002; Moore et al., 2013; Paoletti & Moore, 2017; Thompson & Carlson, 2017). However, this research largely uses a strictly quantitative reasoning lens and does not connect to the units construction lens. Further Castillo-Garsow (2014) questioned the use of Steffe's iterable units as the foundational unit for covariational and quantitative reasoning, which remains an open question. This report adds to both bodies of literature by addressing the research question: What units, unit transformations, and mental actions are involved with solving covariation tasks with pre-algebra and algebra students?

### **Theoretical Framing**

This research was framed in a radical constructivist perspective that learning happens as an individual interacts with their environment, experiencing novel situations and perturbations that lead to assimilations or accommodations of existing schemes. Here a scheme is a three-part, goal-oriented cognitive structure: a stimulus is received, a corresponding action is enacted, and an expected result of carrying out the action (Glaserfeld, 1995). The focus is on capturing the underlying sensorimotor and mental structures and actions that individuals use and construct during the learning process. From this lens, mathematical cognition and development are framed in terms of an individual's own sensorimotor and mental activity (Beth & Piaget, 1966; Piaget & Szeminska, 1952). This situates the individual's own mathematics in a position of power and allows the generation of fine grain models of mathematical thinking and development based on the students' own cognitive and sensorimotor activity (Tillema & Hackenberg, 2017).

Both UC and covariational frameworks used in this report have constructivist roots and model students' mathematics through the students' mental structures and actions. Both frameworks stem from the broader framing of quantitative reasoning. Within quantitative reasoning, there are several definitions for quantity and different perspectives on the structures of these quantities (Piaget & Szeminska, 1952; Steffe, 1991; Thompson, 1994). Piaget and Szeminska (1952) defined three stages of quantity construction: gross, intensive, and extensive. A gross quantity conception is when an individual relies on perceptual information to determine its size. Next is intensive quantity, which involves coordinating two gross quantities. Lastly, an extensive quantity emerges when an individual assigns units to quantities and quantitative comparisons. Steffe (1991, 1992) approached student construction of number through a quantitative lens but focused on the construction of unit quantities.

### **Units Coordination**

UC emerged from Steffe's (1992) work with young children's construction of number and is a Piagetian-based stage theory. Stages are classified by the number of units an individual has constructed and assimilates with prior to the start of a task. Students are considered to be stage 1 when they are only able to assimilate (take as given) one level of units. For example, through their activity of counting four ones, a stage 1 student can construct the two-level unit structure of a unit of units (composite unit). Whereas a stage 2 student already has a two-level unit structure of a composite unit available for them at the beginning of a task without needing to first construct it. Stage 2 students also construct a third level of units (a unit of units of units) in activity. More recently, an intermediate stage, advanced stage 2, has been identified (Hackenberg

& Sevinc, 2022). Lastly, a stage 3 student has three levels of units available to them and can build a fourth in activity (Norton et al., 2015; Ulrich, 2016).

### Covariational Reasoning

Covariational reasoning merged from the quantitative reasoning literature focused on students understanding of quantitatively rich situations (Thompson & Carlson, 2017). Although there are several definitions of covariational reasoning, a common one used is Carlson and colleagues (2002) definition “the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other” (p. 354). Carlson and colleagues (2002) developed a framework for categorizing students’ covariational reasoning with five mental actions and five levels of reasoning (Table 1).

**Table 2: Recreation of Carlson and Colleagues’ (2002) Covariational Mental Actions**

Mental Action	Description of Mental Action	Behaviors
Mental Action 1 (MA1)	Coordinating the value of one variable with changes in the other	<ul style="list-style-type: none"> <li>• labeling the axes with verbal indications of coordinating the two variables (e.g., y changes with changes in x)</li> </ul>
Mental Action 2 (MA2)	Coordinating the direction of change of one variable with changes in the other variable	<ul style="list-style-type: none"> <li>• constructing an increasing straight line</li> <li>• verbalizing an awareness of the direction of change of the output while considering changes in the input</li> </ul>
Mental Action 3 (MA3)	Coordinating the amount of change of one variable with changes in the other variable	<ul style="list-style-type: none"> <li>• plotting points/constructing secant lines</li> <li>• verbalizing an awareness of the amount of change of the output while considering changes in the input</li> </ul>
Mental Action 4 (MA4)	Coordinating the average rate-of-change of the function with uniform increments of change in the input variable.	<ul style="list-style-type: none"> <li>• constructing continuous secant lines for the domain</li> <li>• verbalizing an awareness of the rate of change of the output (with respect to the input) while considering uniform increments of the input</li> </ul>
Mental Action 5 (MA4)	Coordinating the instantaneous rate-of-change of the function with continuous changes in the independent variable for the entire domain of the function	<ul style="list-style-type: none"> <li>• constructing a smooth curve with clear indications of concavity changes</li> <li>• verbalizing an awareness of the instantaneous changes in the rate-of-change for the entire domain of the function (direction of concavities and inflection points are correct)</li> </ul>

Johnson (2015) extended this to a new framework differentiating between the mental actions associated with different types of quantities (extensive and intensive, Schwartz, 1988) in the context of rates of change. Thompson and Carlson (2017) also extend Carlson and colleagues’ original framework by differentiating between variational and covariational reasoning, and attending to variational reasoning within covariational reasoning and the types of quantities the students are reasoning with. They also focused more on the levels portion of the original

framework rather than the mental actions. Here I use the original Carlson and colleagues (2002) framework because it focuses on mental actions rather than levels and can be applied to a broader context than just rates of change (Johnson, 2015).

### **Methods**

The data collected and analyzed in this report is part of a larger ongoing study examining relations between UC, covariational reasoning, and working memory. This report focuses on one participant's work regarding the UC and covariational reasoning component of the larger study.

#### **Participants**

The larger study consisted of six middle-grade students enrolled in either a pre-algebra or algebra course in the Mid-Atlantic. Middle-grade students were selected for the population of study because this population has the cognitive diversity in both UC and working memory present to generate the desired cognitive diversity. A total of eight students went through three screening interviews to assess UC stage and working memory, from which six were invited to participate in the covariational reasoning part of the study. Daniel, the subject of this report, was an 8th-grade student enrolled in an algebra class and was assessed at a UC stage of advanced stage 2. Daniel was selected for the case study because he had engaging interviews and unique mathematical activity with his advanced stage structures.

#### **Data Collection**

For the covariational reasoning part, all participants completed 12-task-based semi-structured clinical interviews (Goldin, 2000) proctored over Zoom. Each interview was 25-45 minutes long with video and audio recordings. Participants were given an iPad to use to access the web-based covariational tasks designed in GeoGebra. Screen recordings of the iPad were taken. Zoom auto-generated transcripts were used for transcription after being checked for accuracy. Students were allowed to play and pause the animation but were not allowed to write anything down until the third part of the task protocol as part of the working memory component of the study no reported on here.

Each task consisted of a dynamic animation of changing shapes or objects in a GeoGebra applet designed to provide the participants with a medium to actively engage with quantities and help elicit covariational reasoning. Every participant received the tasks in the same and asked questions from the same set of base protocol questions with follow-up questions as needed for clarification. The first part consisted of more general questions about what quantities were present and how the quantities changed. For example, two questions students were asked were, "What quantities are changing in the animation?" and "Are any of the changing quantities changing in the same way?" These were designed to identify what quantities and relationships each student generated without prompting from the researcher. Every participant completed the same set of questions in the first part of the interview.

The second part of each task's protocol consisted of more direct questions between two quantities identified by the researchers based on the task design. The tasks were designed so that students were asked about the relation between time, discrete units (countable dots or squares), lengths, and areas. For example, Task E had a shape that tripled in area for each discrete jump in the animation switching between a square to a rectangle (Figure 1). This was designed for the student to consider the quantities of time and area whereas Tasks I and J was designed for relations between two lengths changing.





**Figure 1: First three stages of Task E**

### **Data Analysis**

Analysis happened both during data collection and after with retrospective analysis upon completion of data collection. The real-time analysis consisted of the researcher taking notes during interviews and doing preliminary analysis to plan for the next interview and build an initial second-order model (Ulrich et al., 2014) of the students' mathematics. A more in-depth conceptual analysis was done after data collection through retrospective analysis using the videos, transcripts, and the researchers' notes for initial models as sources of data. For each round of analysis, the researchers started with Task A and analyzed the tasks in the order given to the students.

There were several rounds of retrospective analysis. The first consisted of an initial viewing of the video data to edit the transcription for accuracy, organize the data with time stamp markers, and make initial coding UC and covariational reasoning. These were given when the student used their units coordinating structures or gave evidence for covariational reasoning (Carlson et al., 2002). The third round consisted of a more in-depth viewing of the videos to build finer-grain models of the student's work identifying the types of units the students constructed, the sensorimotor and mental actions the students used, and coding for covariational reasoning mental actions from the Carlson and colleagues' (2002) framework (Table 1).

After this third round of analysis, the models of the student's work became the data, and the researcher went back through to synthesize the models identifying trends in the types of units and unit transformations the students used when coded for engaging in the various mental actions in the covariational framework.

### **Results**

In this section, I share the unit structures and mental actions identified from Daniel's work that led to an extension of Carlson and colleagues' (2002) framework. Here I focus on the extensions for two of the mental actions, Mental Action 2 (MA2) and Mental Action 3 (MA3). For each mental action, I summarize the extensions and provide evidence with samples of Daniel's work. Note, in the initial framing of the larger research study, the importance of Piagetian quantities was not predicted to be an important factor of the students' covariational reasoning. As such the analysis did not include the distinction between the three types but categorized them into two categories of gross and non-gross (intensive and extensive).

#### **Daniel's Reasoning with MA2**

Recall from Table 1, MA2 involves students coordinating the direction of change (increasing, decreasing) of one quantity with changes in another quantity. In every task, Daniel identified the direction of change of one quantity in terms of another quantity. One typical example is seen in Daniel's response from Task D, a plant growing in height exponentially. In response to the question, "What quantities changed?" Daniel replied, "Well, well it's like getting bigger as time goes on." When asked to explain or justify his claim, Daniel initially said, "I can't really explain it, you probably have to use some kind of tool or something." This response was typical justification for Daniel although more often he would not include the measuring tool idea.

Collectively, this indicates Daniel’s direction of change was primarily rooted in gross quantities and associated mental action of visual sweeps.

Daniel's suggestion of using a tool appeared in a few other tasks in which he described taking measurements of the quantity, length, or area, as the animation played or stopping it to measure, and then comparing the measurements to show either an increase or decrease occurred. Once Daniel evoked a measuring tool, he shifted to reasoning with non-gross quantities. He imagined the process of taking measurements or marking new positions of the quantities to do gross-magnitude comparisons. It was unclear whether those quantities were measured with units or simply generated for the use of comparisons (intensive). Thus, I classified these responses with non-gross quantities and actions of continuous measuring and gross-magnitude comparisons. Continuous measuring is defined as imagining marking or measuring a quantity as time passes or as another quantity changes simultaneously (Table 2).

**Table 2: MA2 Extensions to Carlson and Colleagues’ (2002) Covariational Framework**

Original MA2 Definition	Underlying Structures	Mental Actions
Coordinating the direction of change of one variable with changes in the other variable	Gross Quantity	Perceptual Sweep Continuous Measuring
	Non-Gross Quantity	Gross Magnitude Comparison

### Daniel’s Reasoning with MA3

MA3 consists of coordinating amounts of change in one quantity with changes in another quantity. Daniel's engagement in MA3 was very diverse and varied depending how several factors including whether he worked with lengths or areas and the role time played as a quantity. Here I focus on results for Daniel working with non-gross quantities and use excerpts of Daniel’s work on Task E (Figure 1) because it captures several of the trends represented throughout the tasks. Below is Daniel’s response to the third question of the first part of the task protocol that asks whether the quantities he identified in the proceeding question (size of shape and side length), change in the same way:

D: [watches animation] Um, percentage-wise I think, yes, because it's like changing by like two-thirds when it's going to the right like this [gestures with thumb and index finger as a length and moves it to the right along animation] in the rectangle.

I: Okay.

D: I think it’s going up by two-thirds or tripling the whole thing. [returns to watching animation] And then the square’s like filling up. [stops watching the animation to answer] Yeah it like triples itself like [same sweeping hand gesture] to the right. And then it takes that shape [holds out four fingers flat, then makes it three] and it goes up three times like that [sweeps flat fingers upwards].

I: Okay, and so, when you say like percentage wise what exactly did you mean by that?

D: Um so like a bigger, the bigger square’s changing in like the same like, like amount, but it's just like a different proportion. So, it's just, yeah. [starts to say more but then finishes after hearing interviewer say “Okay”] Daniel: Well, I kind of just looked at how big it was I kind of just estimated that it was going up by three or you know tripling itself.

I: Okay, and so, how did you come to that estimate of what, why do you think it was tripling instead of doubling?

D: Well, a double would be shorter, like if a doubled itself it wouldn't work as long. I think I looked at how many spaces it would take up [makes length measure with thumb and index finger and makes a kind of iterating motion] if they're there were three of them, and if there were two of them.

I: Okay, and so, when you say like taking up space, what do you mean by that?

D: Or okay I looked at how many how big like the square was by itself [points to bottom left corner of animation, starting square position]. And then I looked at um how many [makes bouncing along a line motion with stylus] if like if you added on two more of those same squares [makes length from both index fingers and iterates it twice] how big would it be.

I: Okay, so you're sort of imagining copies of it?

D: Yeah.

I: And it looked like there would be three copies for each step?

D: [nods] mmhmm.

Daniel's initial claim that the quantities change in the same way generated a change unit of two-thirds and then multiplicatively by a change factor of three. This is an immediate change from the answer types we saw for MA2 because Daniel introduces measured quantities and numerical units. Even within Task E, this was the first time that Daniel introduced a numerical value about how the quantities change besides stating there were four sides. This indicates that Daniel no longer used gross quantities.

In his explanation, he constructs several different units and mental actions to transform those units. For example, he constructed two different change units of two-thirds and an iterating unit of change factor with the iterative multiplication by three. These were distinct structures for Daniel because, from his two-thirds example, he seemed to construct 3 as a partitioned unit of 1 and 2 (1,2) in that he decomposed his unit of three into a unit of 1 and a composite unit of 2. His iterative multiplication by three appeared to be an encapsulation of his construction of the partitioned unit, re-unitizing his unit of measure and constructing the next partitioned unit.

In his later explanation of his generation or check of his factor of three, Daniel's verbal responses and hand motions indicate that he iterated his unit to two to get to the next shape. After he constructed the next shape, he re-unitized that new shape as his unit of measure and enacted his tripling scheme again. Thus, Daniel had iterable units of 1 and iterable change units of 3 (iterative multiplication by three). In fact, throughout the 12 tasks, Daniel constructed and used an Iterative Multiplication Scheme (IMS) in which he characterized the change as constant if the change appeared to come from repeated multiplication of a fixed change factor. This structure is different from other change factors that were rate units and a constant rate of change unit (linear).

A few minutes later in the interview, in the first question of the second part of the task, Daniel was asked, "As time passes, how is the shape getting bigger?" He responded:

Um...Okay, so as time goes on, it's increasing it's like tripling itself or actually I think it's... I think it's like... Since it's tripling itself to the right and then doing it three more times, it's technically adding on just nine, nine of that unit, [slight pause] or eight more of that unit until like the one square in this corner. So, it's going to make 1,2,3 [moves finger over to the left with each number spoken] and then six up here [points above previous 1,2,3]. So, it's doing

that three times, but it stops when it does look another 1,2,3 like this because but it's a very big square so.

Here we see a shift in Daniel's characterization of how the shape changes. He still utilized his previous multiplication by three or tripling idea. However, now he has combined two iterations into a single operation or step. Although Daniel still implements his multiplication by three twice, instead of re-unitizing after each step, he maintained this composite unit of 3 constructed from the first step to operate on further in the second multiplication by three. Thus, he now generated a measure for the second square as nine units, keeping his original starting square as the unit of measure.

Once he arrives has his idea of 9, he provides an alternate way to get the nine but adds on eight of the original unit instead. This aligns with Daniel being an advanced stage 2 because when he got to his result, he did not maintain his three-level structure but reverted to measuring with his original unit. Daniel does maintain that he could also do three times three to get the nine, but it is unclear if he is maintaining all the units in the context of the problem or simply relying on number facts once he constructed the three times three as a single object in activity.

**Table 3: MA3 Extensions to Carlson and Colleagues' (2002) Covariational Framework**

Original MA3 Definition	Underlying Structures	Mental Actions
	Gross Quantity Intensity of Change	Perceptual sweep Gross magnitude comparison
Coordinating the amount of change of one variable with changes in the other variable	Non-Gross Quantity Iterating Unit of Measure* IMS* Rate Unit Partitioned Unit*	Gross magnitude comparison Partitioning measured quantity * Unitizing coordinated multiplicative object Re-unitizing*

\* Indicates extensions discussed in this report

### Discussion

In this report, I give evidence of how an 8th-grade advanced stage 2 student leveraged his units coordinating structure to reason covariationally and identify underlying mental structures and actions involved with Carlson and Colleagues' (2002) covariational framework. For example, Daniel's reasoning about amounts of change in two distinct ways in the expert above. He utilized a partitioned unit of (1,2) to describe the amount of change additively and constructed an iterative multiplication amount of change through more multiplicative reasoning. Daniel's characterization of change through iterative multiplication is unsurprising based on Ellis and colleagues' (2016) hypothetical learning trajectory for exponential growth. However, this work did not explicitly account for UC and covariational reasoning.

A new unit structure emerged as important in Daniel's amount of change reasoning, the partitioned unit. As seen in Task E, and others not reported on here, Daniel used a partitioned unit to describe the amount of change when he was reconciling his additive and multiplicative worlds. It is an example of another type of unit of units that is not a Steffe (1992) composite unit.

This adds to the growing base of different structures of units found in higher mathematical contexts than whole number and fraction arithmetic reasoning (Tillema, 2013).

This report also adds to the broader literature in support of building models of epistemic students' mathematics as a way to gain important insight into common cognitive structures in mathematical development to be used to inform pedagogical and curriculum design. The models of epistemic students provide starting recognition templates for researchers and teachers of how students at the various stages might answer different problem types and provide suggestions for the types of activities those students should engage with to continue in their mathematical development (Hackenberg, 2014). By identifying how students with different cognitive structures engage with similar mathematics problems researchers and teachers can highlight how cognitive diversity might show up within a classroom and value all students' mathematical contributions by leveraging the various ideas present in the learning process. For example, marking the progress of the shapes was a common strategy that Daniel employed for both MA2 and MA3, thus adapting the design of the task to include options to see previous iterations for measurement could make the task more accessible to students at either stage in covariational reasoning.

### References

- Beth, E. W., & Piaget, J. (1966). *Mathematical epistemology and psychology* (Trans. W. Mays). Gordon and Breach.
- Boyce, S., Grabhorn, J. A., & Byerley, C. (2020). Relating students' units coordinating and calculus readiness. *Mathematical Thinking and Learning*, 1–22. <https://doi.org/10.1080/10986065.2020.1771651>
- Boyce, S., & Norton, A. (2016). Co-construction of fractions schemes and units coordinating structures. *The Journal of Mathematical Behavior*, 41, 10–25. <https://doi.org/10.1016/j.jmathb.2015.11.003>
- Boyce, S., & Norton, A. (2017). Dylan's units coordinating across contexts. *The Journal of Mathematical Behavior*, 45, 121–136. <https://doi.org/10.1016/j.jmathb.2016.12.009>
- Byerley, C. (2019). Calculus students' fraction and measure schemes and implications for teaching rate of change functions conceptually. *The Journal of Mathematical Behavior*.
- Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying Covariational Reasoning While Modeling Dynamic Events: A Framework and a Study. *Journal for Research in Mathematics Education*, 33(5), 352–378. <https://doi.org/10.2307/4149958>
- Castillo-Garsow, C. (2014). Besides the Iterable Unit: A reply to Steffe et al. *Epistemic Algebraic Students: Emerging Models of Students' Algebraic Knowing*, 3, 157–174.
- Ellis, A., Ozgur, Z., Kulow, T., Dogan, M. F., & Amidon, J. (2016). An Exponential Growth Learning Trajectory: Students' Emerging Understanding of Exponential Growth Through Covariation. *Mathematical Thinking and Learning*, 18(3), 151–181. <https://doi.org/10.1080/10986065.2016.1183090>
- Glaserfeld, E. von. (1995). *Radical Constructivism: A Way of Knowing and Learning*. *Studies in Mathematics Education Series: 6*. Falmer Press, Taylor & Francis Inc. <https://eric.ed.gov/?id=ED381352>
- Hackenberg, A. (2014). Musings on three epistemic algebraic students. *Epistemic Algebraic Students: Emerging Models of Students' Algebraic Knowing*, 4, 81–124.
- Hackenberg, A., Aydeniz, F., & Jones, R. (2021). Middle school students' construction of quantitative unknowns. *The Journal of Mathematical Behavior*, 61, 100832. <https://doi.org/10.1016/j.jmathb.2020.100832>
- Hackenberg, A. J., & Sevinc, S. (2022). A boundary of the second multiplicative concept: The case of Milo. *Educational Studies in Mathematics*, 109(1), 177–193. <https://doi.org/10.1007/s10649-021-10083-8>
- Hackenberg, A., & Lee, M. Y. (2015). Relationships Between Students' Fractional Knowledge and Equation Writing. *Journal for Research in Mathematics Education*, 46(2), 196–243. <https://doi.org/10.5951/jresmetheduc.46.2.0196>
- Hackenberg, A., & Tillema, E. (2009). Students' whole number multiplicative concepts: A critical constructive resource for fraction composition schemes. *The Journal of Mathematical Behavior*, 28(1), 1–18. <https://doi.org/10.1016/j.jmathb.2009.04.004>
- Johnson, H. (2015). Secondary Students' Quantification of Ratio and Rate: A Framework for Reasoning about Change in Covarying Quantities. *Mathematical Thinking and Learning*, 17(1), 64–90. <https://doi.org/10.1080/10986065.2015.981946>
- Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

- Lee, M. Y. (2018). A Case Study Examining Links between Fractional Knowledge and Linear Equation Writing of Seventh-Grade Students and Whether to Introduce Linear Equations in an Earlier Grade. *International Electronic Journal of Mathematics Education*, 14(1). <https://doi.org/10.12973/iejme/3980>
- Moore, K., Paoletti, T., & Musgrave, S. (2013). Covariational reasoning and invariance among coordinate systems. *The Journal of Mathematical Behavior*, 32(3), 461–473. <https://doi.org/10.1016/j.jmathb.2013.05.002>
- Olive, J. (2001). Children's Number Sequences: An Explanation of Steffe's Constructs and an Extrapolation to Rational Numbers of Arithmetic. *The Mathematics Educator*, 11(1).
- Paoletti, T., & Moore, K. (2017). The parametric nature of two students' covariational reasoning. *The Journal of Mathematical Behavior*, 48, 137–151. <https://doi.org/10.1016/j.jmathb.2017.08.003>
- Piaget, J., & Szeminska, A. (1952). *The Child's Conception of Number*. Routledge.
- Schwartz, J. (1988). Intensive Quantity and Referent Transforming Arithmetic Operations. *Research Agenda for Mathematics Education Number Concepts and Operations in the Middle Grades*, 2, 41–52.
- Steffe, L. (1991). Operations that generate quantity. *Learning and Individual Differences*, 3(1), 61–82. [https://doi.org/10.1016/1041-6080\(91\)90004-K](https://doi.org/10.1016/1041-6080(91)90004-K)
- Steffe, L. (1992). Schemes of Action and Operation Involving Composite Units. *Learning and Individual Differences*, 4(3), 259–309. [https://doi.org/10.1016/1041-6080\(92\)90005-Y](https://doi.org/10.1016/1041-6080(92)90005-Y)
- Steffe, L., & Olive, J. (2010). *Children's fractional knowledge*. Springer Science & Business Media.
- Steffe, L., & Thompson, P. (2000). Teaching experiments methodology: Underlying principles and essential characteristics. *Research Design in Mathematics and Science Education, Hillsdale, Nj: Laurence Erlbaum*.
- Thompson, P. (1994). The development of the concept of speed and its relationship of concepts of rate. In *The development of multiplicative reasoning in the learning of mathematics* (pp. 179–234).
- Thompson, P., & Carlson, M. (2017). Variation, Covariation, and Functions: Foundational Ways of Thinking Mathematically. In *Compendium for research in matheamtics education* (pp. 421–456). National Council for Teachers of Mathematics.
- Tillema, E., & Hackenberg, A. (2017). Three Facets of Equity in Steffe's Research Programs. *Conference Papers -- Psychology of Mathematics & Education of North America*, 57–67.
- Tillema, E. S. (2013). A power meaning of multiplication: Three eighth graders' solutions of Cartesian product problems. *The Journal of Mathematical Behavior*, 32(3), 331–352. <https://doi.org/10.1016/j.jmathb.2013.03.006>
- Tillema, E. S. (2014). Students' coordination of lower and higher dimensional units in the context of constructing and evaluating sums of consecutive whole numbers. *The Journal of Mathematical Behavior*, 36, 51–72. <https://doi.org/10.1016/j.jmathb.2014.07.005>
- Ulrich, C. (2015). Stages In Constructing and Coordinating Units Additively and Multiplicatively (Part 1). *For the Learning of Mathematics*, 35(3), 2–7. JSTOR.
- Ulrich, C. (2016). Stages In Constructing and Coordinating Units Additively and Multiplicatively (Part 2). *For the Learning of Mathematics*, 36(1), 34–39.
- Ulrich, C., Tillema, E., Hackenberg, A., & Norton, A. (2014). Constructivist Model Building: Empirical Examples From Mathematics Education. *Constructivist Foundations*, 9(3), 328–339.
- Zwanch, K. (2022). Examining middle grades students' solutions to word problems that can be modeled by systems of equations using the number sequences lens. *The Journal of Mathematical Behavior*, 66, 100960. <https://doi.org/10.1016/j.jmathb.2022.100960>

# DEVELOPING MEANINGS FOR GRAPHS FROM NUMBER LINES: THE CASE OF NAYA

Halil I. Tasova  
California State University San Bernardino  
halil.tasova@csusb.edu

Keywords: Algebra and Algebraic Thinking, Cognition, Mathematical Representations

## Background

The topic of constructing and interpreting graphs in middle school mathematics presents an opportunity for powerful learning, as noted by Leinhardt et al. (1990). However, students encounter various challenges, such as perceiving graphs as representations of situations, events, phenomena, or literal motion of objects, which can negatively impact their understanding and learning of algebra and calculus topics (e.g., Clement, 1989; Hattikudur et al., 2012; Johnson et al., 2020). Despite these challenges, fewer studies have focused on investigating how students create graphs as emergent representations of the relationship between two quantities (Frank, 2016, 2017; Moore, 2021; Tasova, 2021; Tasova & Moore, 2020). My investigation aims to answer the following questions: What ways of thinking do students engage in graphing activities intended to emphasize quantitative reasoning? What ways of reasoning are involved in students developing productive meanings for graphs?

## Methods

This study focuses on Naya's (a sixth grader at a public school in the southeastern United States) interpretation of graphs and the changes in her understanding over the course of the teaching experiment. Naya was involved in a teaching experiment that included 14 one-hour sessions, during which video and transcripts were used to capture her gestures, speech, and drawings. The reader can see all the tasks implemented in the following GeoGebra book: <https://www.geogebra.org/m/c8eajg3b>. I conducted a conceptual analysis to understand the students' explanations and actions (Steffe & Thompson, 2000). I analyzed the data using the generative and axial methods (Corbin & Strauss, 2008). The reader can learn about the other cases in my earlier studies (i.e., Tasova & Moore, 2021; Tasova, 2022a; Tasova, 2022b).

## Findings

Naya initially assimilated the points on the plane in relation to the physical objects that appear in the situation, and her meanings for points were based in iconic or transformed iconic translation (i.e., picture of the situation). As she began to conceptualize the quantities in the situation, Naya's graphical meanings included representing two quantities in non-canonical Cartesian plane. Subsequently, her attention to quantities in the situation, mapping those quantities' magnitudes onto the number lines, and assimilating the axes of the plane as number lines afforded Naya to develop a meaning of points in terms of representing two quantities' magnitudes by reorganizing the space consistent with a Cartesian plane.

## Conclusion

I found four cognitive resources that were critical for developing a productive meaning for points. These resources include (i) conceptualizing quantities and their relationship in the situation, (ii) representing the quantities magnitudes as varying bars on parallel magnitude lines, (iii) assimilating the axes of the coordinate plane in relation to the magnitude lines, and (iv) creating a point on the plane by intersecting the projections of the magnitudes on the axes.

## Acknowledgment

This material is based upon work supported by California State University San Bernardino Mini-Grant.

## References

- Clement, J. (1989). The concept of variation and misconceptions in Cartesian graphing. *Focus on Learning Problems in Mathematics*, 1(1-2), 77-87.
- Corbin, J. M., & Strauss, A. (2008). *Basics of Qualitative Research: Techniques and Procedures for Developing Grounded Theory* (3rd ed.). Thousand Oaks, CA: Sage.
- Frank, K. M. (2016). Students' conceptualizations and representations of how two quantities change together. Paper presented at the 19th Meeting of the MAA Special Interest Group on Research in Undergraduate Mathematics Education, San Diego, CA.
- Frank, K. M. (2017). Examining the development of students' covariational reasoning in the context of graphing. (Unpublished doctoral dissertation), Arizona State University, Tempe, AZ.
- Hattikudur, S., Prather, R. W., Asquith, P., Alibali, M. W., Knuth, E., & Nathan, M. (2012). Constructing graphical representations: middle schoolers' intuitions and developing knowledge about slope and y-intercept. *School Science and Mathematics*, 112(4), 230–240. <https://doi.org/10.1111/j.1949-8594.2012.00138.x>
- Johnson, H. L., McClintock, E., & Gardner, A. (2020). Opportunities for reasoning: Digital task design to promote students' conceptions of graphs as representing relationships between quantities. *Digital Experiences in Mathematics Education*.
- Leinhardt, G., Zaslavsky, O., & Stein, M. K. (1990). Functions, graphs, and graphing: Tasks, learning, and teaching. *Review of Educational Research*, 60(1), 1–64.
- Moore, K. C. (2021). Graphical Shape Thinking and Transfer. In: Hohensee, C., Lobato, J. (eds) *Transfer of Learning. Research in Mathematics Education*. Springer, Cham.
- Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. A. Lesh & A. E. Kelly (Eds.), *Handbook of research design in mathematics and science education* (pp. 267–307). Hillsdale, NJ.
- Tasova, H. I., & Moore, K. C. (2020). Framework for representing a multiplicative object in the context of graphing. In A.I. Sacristán, J.C. Cortés-Zavala & P.M. Ruiz-Arias, (Eds.). *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Mexico* (pp. 210–219). Cinvestav/PME-NA
- Tasova, H. I. (2021). Developing middle school students' meanings for constructing graphs through reasoning quantitatively (Doctoral dissertation). <https://esploro.libs.uga.edu/esploro/outputs/9949391256202959>
- Tasova, H. I., & Moore, K. C. (2021). From number lines to graphs: A middle school student's re-organization of the space. In D. Olanoff, K. Johnson., & S. Spitzer, (Eds.). *Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 598–602). Philadelphia, PA.
- Tasova, H. I. (2022a). Developing meanings for graphs from number lines: The case of Mike. In A. E., Lischka, E. B., Dyer, R. S., Jones, J. N., Lovett, J., Strayer, & S., Drown, (Eds.). *Proceedings of the forty-fourth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 313–322). Nashville, TN. Middle Tennessee State University.
- Tasova, H. I. (2022b, October). Developing meanings for graphs from number lines: The case of Zane [Paper presentation]. The 2022 National Council of Teachers of Mathematics Research Conference, Los Angeles, CA, United States.



## EFFECTIVE INSTRUCTIONAL PRACTICES THAT FOSTER THE DEVELOPMENT OF STUDENTS' EARLY ALGEBRAIC THINKING

Boram Lee  
University of Texas at Austin  
boram\_lee@utexas.edu

Ingrid Ristroph  
University of Texas at Austin  
Ingrid.ristroph@utexas.edu

Despina Stylianou  
City College of New York  
dstylianou@ccny.cuny.edu

Eric Knuth  
National Science Foundation  
eknuth@nsf.gov

Maria Blanton  
TERC  
maria\_blanton@terc.edu

Angela M. Gardiner  
TERC  
angela\_gardiner@terc.edu

Ana Stephens  
University of Wisconsin  
Madison  
acstephens@wisc.edu

Rena Stroud  
Merrimack College  
stroudr@merrimack.edu

### Abstract

*Is it possible to identify instructional practices that have an impact on student learning in mathematics? The study described here is part of ongoing efforts to understand and characterize effective instruction. We drew on the work of several recently developed frameworks for understanding teaching effectiveness to develop a protocol for studying effective instruction that both coordinates and extends existing research in the context of early algebra. Using a large-scale study, we characterized effective instruction in this context and documented the impact of such instruction on students' performance using both qualitative and quantitative analyses. Findings suggest that teachers' abilities to take up curriculum openings are important aspects of teaching. Furthermore, the manner with which teachers react to these moments strongly correlates with gains in student performance.*

Keywords: Early Algebra, Effective Instruction, Teaching and Classroom Practice

### Introduction

There is a convergence of belief in the field of mathematics education that the nature of classroom instruction significantly affects the nature and level of student learning. Ball and Forzani (2011) assert that “student learning depends fundamentally on what happens inside the classroom as teachers and learners interact over the curriculum” (p. 45). Scholars suggest that the nature of classroom teaching is by far the most significant factor in learning, surpassing all other aspects of schooling (e.g., Chetty et al., 2014; Opper, 2019). Policy recommendations echo these positions, presenting several principles describing what constitutes effective instruction that improves mathematics learning for all students (e.g., NCTM, 2000; NGA & CCSO, 2010; RAND, 2003). Yet, research evidence that links particular teaching practices with student outcomes is somewhat lacking. As Blazar (2015) contends, “the nature of effective teaching still largely remains a black box. Given that the effect of teachers on achievement must occur at least in part through instruction, it is critical that researchers identify the types of classroom practices that matter most to student outcomes” (p. 16).

In a rich body of work, several scholars have already *proposed* critical instructional practices that appear to contribute to student learning in mathematics. These practices include selecting rich, cognitively demanding mathematical tasks, and maintaining students' engagement with

these tasks at a high level (Stein et al., 2000), building on students' mathematical thinking (e.g., Cengiz et al., 2011; Fennema et al., 1996), and facilitating discussions that support students in connecting mathematical ideas within the curriculum (Smith & Sherin, 2020).

We are interested in understanding the teacher actions within these practices that are associated with student learning. Hence, we examined this literature on teachers' instructional actions, particularly the frameworks that have been proposed as potentially fruitful for understanding teaching effectiveness, and then drawing upon this literature, we proposed a framework for understanding instructional effectiveness in the context of early algebra. We sought to understand the ways in which teachers take up "curriculum openings" (Remillard & Geist, 2002), that is, opportunities that arise in the moment in response to student reasoning and contributions to classroom mathematical work and how teachers engage with such opportunities. Finally, we used this framework to explore the relationships that exist between teachers' practices around these curricular openings and students' early algebra learning.

## Literature Review

### Teacher Moves and Frameworks

Critical moments within mathematics lessons that afford teachers opportunities to notice and act to further student reasoning and understanding have been well documented (e.g., Leatham et al., 2015; Stockero & Van Zoest, 2013). Ball and Cohen (1999) introduced the notion of teachers "siz[ing] up a situation from moment to moment" (p. 11) and using what they learn to improve their practice. Remillard and Geist (2002) referred to these moment-to-moment situations as "*openings in the curriculum*" (p.13, emphasis in original). These moments might include student questions or solutions that afford teachers the opportunity to engage students more deeply in mathematics. Similarly, van Es and Sherin (2002) used the term "noticing" to describe the ways in which teachers may take up unexpected moments and incorporate them into the lesson. Specifically in the context of algebra, Walkoe et al. (2022) referred to moments that arise as extensions of the curriculum that teachers can take up as "moments of algebraic potential." Additionally, Stockero and van Zoest (2013) defined "pivotal teaching moments" (PTMs) as "instance(s) in a classroom lesson in which an interruption in the flow of the lesson provides the teacher an opportunity to modify instruction in order to extend or change the nature of students' mathematical understanding" (p. 127). Other researchers have described such moments as "teachable moments" (Stockero & van Zoest, 2013), "mathematically significant pedagogical opportunities" (Leatham et al., 2015), "significant mathematical instances" (Davies & Walker, 2005), and "crucial mathematics hinge moments" (Thames & Ball, 2013). Unfortunately, little of this research has been done in the context of early algebra – with the work of Blanton and Kaput (2005), who documented the development of teachers' "algebra eyes and ears"—as one exception.

We chose to examine the specific teaching moves that occur during these opportunities, particularly as they occur during classroom discussions. What actions do teachers take to specifically foster mathematical development and extend students' engagement with rich mathematics in early algebra? To assist our examination, we turned to work on facilitating mathematical discourse.

### Classroom Discourse

Classroom discourse is closely related to responsiveness. As Bakhtin (1986) explains, "every utterance must be regarded as a response to preceding utterances" (p. 91), and, as such, classroom discourse is a responsive act. Studies addressing discourse define responsiveness to students' mathematical thinking as "a characteristic of discourse that reflects the extent to which

students' mathematical ideas are present, valued, attended to, and taken up as the basis of instruction" (Bishop et al., 2022, p. 11).

Several studies have characterized discourse as a responsive act and have examined its function and patterns. Prominent among these are patterns that view discourse as productive versus unproductive. For example, Knuth and Peressini (2001) discussed univocal versus dialogic discourse. Univocal discourse is characterized by an effort to convey information from the speaker (teacher) to the student (listener). The teacher responds to an incorrect student comment with the goal to correct it. Dialogic discourse, on the other hand, is characterized by a dialogue, a back-and-forth where both teacher and students participate equally within which new meaning is created. Wood (1998) described a funneling versus a focusing pattern in classroom discourse. The former involves the teacher leading a series of explicit questions or comments that aim to lead students to a certain correct response while the latter aims to co-develop a mathematical idea while authority is dispersed among all participants (students and a teacher).

Using the concepts of univocal versus dialogic, several studies have identified instructional "moves" that determine the degree of responsiveness or moves that determine the direction of the responsiveness in a univocal-dialogic continuum. At a more fundamental level are those studies that distinguished between the action of acknowledging and taking up a student comment or question, and the action of not taking or setting aside a student contribution (e.g., Ellis et al., 2019). Once a student contribution is taken up, several categories of moves may occur. Truxaw and DeFranco (2008) in particular, detailed moves such as acknowledging student responses (or ideas), revoicing them, responding to them by evaluating them (validating or correcting), eliciting a desired response (asking questions in a funneling pattern to draw a particular response), and extending student thinking (deepening student understanding by inviting generalizations and connections). Extending student thinking often involves moves that invite the participation of several students (e.g., Franke et al., 2015).

### **Situating the Study**

The study draws from a larger longitudinal study that examined the impact and nature of an early algebra intervention in Grades 3–5. A curricular sequence based on an early algebra learning progression was developed that consisted of 18 lessons per year (Blanton et al., 2018). The lessons aimed to develop students' abilities to generalize, represent, justify, and reason with mathematical structure and focused on big ideas of early algebra: (a) mathematical equivalence; (b) generalized arithmetic; (c) functional thinking; and (d) variable (Kaput, 2008). For a detailed discussion of the larger study see Blanton et al. (2018, 2019).

### **Our Framework – Conceptualizing Early Algebra Moments**

In this study, we take up the challenge to examine what constitutes effective instruction of early algebra, using the early algebra data corpus from the aforementioned larger study. We hope to add to the body of early algebra research by increasing our understanding of moments that afford teachers the opportunities to foster students' early algebraic thinking. In particular, we use videotaped lessons across the grades to address the following questions: (1) What types of teacher moves occur spontaneously (outside the curricular materials) in an inquiry-oriented early algebra setting? (2) How can we characterize the instructional moves within these moments? (3) To what extent do these moments impact student learning?

### **Methods**

#### **Data Corpus**

The data corpus for this work is drawn from a large-scale, randomized, longitudinal study of the effectiveness of a Grades 3–5 early algebra intervention. The study involved approximately

3,200 students and 100 teachers at each grade level. It occurred in 46 schools (23 experimental, 23 control) in diverse settings. For this study, we focus on one lesson at each grade level, and the videotaped lessons of 12 teachers at each grade level implementing the respective lessons. Students completed an assessment addressing the intervention's big algebraic ideas that was administered as a pre-test at the beginning of Grade 3 and then at the end of each academic year in Grades 3, 4, and 5.

### **Data Coding and Analysis**

The videotaped lessons were transcribed and coded independently by two team members, and any differences were reconciled through discussion. We used an open coding system (Saldana, 2013) to first identify *anticipated moments* (AMs) - expected instructional actions that might occur based on each lesson and on any accompanying teacher guidance for the lesson. For example, in a task that required students to create a graph, an AM might consist of a teacher inviting students to consider what type of graph would be appropriate or how to label the axes.

We next identified spontaneous early algebra moments (SMs) - unanticipated early algebra moments that arise during the enactment of a lesson. SMs often occur when students introduce unexpected, yet potentially fruitful, mathematical ideas that prompt teachers to take advantage of opportunities to advance students' algebraic thinking, even if this means deviating from the planned lesson. Using the earlier example, a student's suggestion to write an equation for the situation described by the graph, or to find how aspects of that equation relate to the graph were considered SMs.

When these moments were taken up, we subsequently coded for actions within these moments. Earlier work that characterized discourse as univocal versus dialogic (e.g., Knuth & Peressini, 2001) as well as categorizations of moves within responses (e.g., Truxaw & DeFranco, 2008; Ellis et al., 2019) framed our coding: setting aside, acknowledging, responding, eliciting, facilitating, and extending students' algebraic reasoning. These prior frameworks served as the basis for our coding (Syed & Nelson, 2015).

To address our research questions, AM and SM frequencies were noted, and SMs were each assigned one of six hierarchical response categories (i.e., setting aside, acknowledging, responding, eliciting, facilitating, and extending). Each teacher received an SM score by averaging the SM scores across that lesson.

We examined the correlation between the use of AMs and SMs and student overall performance using a simple linear regression (AM or SM frequencies and ratios being the independent variable, and student performance in each teacher's class being the dependent variable). Student performance was calculated by using the proportional difference between pre- and post-test growth on the assessment administered in each grade as part of the broader study. We also examined the correlation between the level of SM scores and student performance.

### **Results**

We first present the frequency of AMs and SMs identified and taken up by teachers in the 36 analyzed videotaped lessons. We then take a closer look at SMs by presenting examples of such moments as well as the teacher actions associated with these SMs. Finally, as the goal of the work is to investigate instructional practices that are effective in fostering the development of early algebra learning, we examine the impact of teachers' interactions with AMs and SMs on students' early algebra learning. We do so by examining both the frequencies of these AMs and SMs as well as the characteristics of SMs and teachers' SM scores.

## Algebra Moment Counts and Relationship with Student Performance

Based on the project team’s identification of anticipated moments in each of the three lessons, there were 19 AMs in Grade 3 (lesson 15), 12 AMs in Grade 4 (lesson 5), and 9 AMs in Grade 5 (lesson 12). However, in the videotaped lessons teachers varied in taking up these opportunities. Table 1 shows the variation in the degree to which teachers took up AMs. Overall, Grade 3 teachers took up an average of 7.58 AMs (min 2, max 16) out of 19 potential AMs, 7.62 were taken up in Grade 4 (min 3, max 12) out of 12 potential AMs, and 6.8 in Grade 5 (min 5, max 8) out of 9 potential AMs. Viewed as a ratio of AMs taken up compared to the total AMs available to each teacher (in other words, the ratio of AMs one could possibly implement within the time that was available to them), Grade 3 teachers took up 86% of AMs (min 60%, max 100%), 88% in Grade 4 (min 55%, max 100%), and 86% in Grade 5 (min 55%, max 100%). (For a more detailed look at Grade 5 AMs, please see Ristroph et al., (2022))

SMs (spontaneous moments) also varied across the implemented lessons we analyzed. Overall, in Grade 3 teachers took up an average of 2.58 SMs (min 1, max 6), 2.00 in Grade 4 (min 1, max 5), and 5.76 in Grade 5 (min 1, max 11). By definition, there was not an expected number of spontaneous moments for each lesson.

**Table 1: AM and SM Frequencies in Grades 3, 4, and 5**

		Minimum	Maximum	Mean	$r^2$
AMs Taken	Grade 3	2	16	7.58	0.088
	Grade 4	3	12	7.58	0.134
	Grade 5	5	8	6.8	0.083
Proportion of AMs taken	Grade 3	0.6	1	0.86	0.001
	Grade 4	0.55	1	0.88	0.035
	Grade 5	0.55	1	0.86	0.185
SMs Taken	Grade 3	1	6	2.58	0
	Grade 4	1	5	2.00	0.098
	Grade 5	1	11	5.76	0.43

A linear regression of the relationship between the frequency of both AMs and SMs in lessons, and student growth as measured by the early algebra assessment (Blanton et al., 2019) did not detect any significant correlations ( $r^2 = 0.088$  and  $0.134$  and  $0.083$  for AMs taken,  $0.001$  and  $0.035$  and  $0.185$  for AM ratio, and  $0$  and  $0.098$  and  $0.43$  for SMs taken for Grades 3, 4 and 5 respectively – see Table 1). In other words, students whose teachers took up more AMs or SMs during these lessons experienced just as much growth in the early algebra assessment as their counterparts whose teachers took fewer AMs/SMs.

### Characterizing SMs

Each SM was assigned one of six “response categories” (i.e., setting aside, acknowledging, responding, eliciting, facilitating, and extending). With the exception of *setting aside*, an indication of teachers not engaging with student reasoning, the remaining five categories occurred as a continuum with an increasing degree of responsiveness and patterns of dialogic discourse. Hence, we considered these categories as “levels”, starting with *setting aside* as Level 0 to “extending” as Level 5.

**Table 2: Categories of Teachers' Responsiveness to Spontaneous Moments**

<b>Response Categories &amp; Teacher Moves</b>	<b>Classroom Examples</b>
<p><b>L1: Acknowledging</b> Acknowledges a response but does not act on it</p>	<p>Task: Fill in the blank: <math>m = \_ \times m</math> St: Or we could put in "0" T: Hmm.. Interesting. Alright, next problem...</p>
<p><b>L2: Responding</b> Validates correct answers or Corrects errors Revoices or highlights a response or contribution</p>	<p>Task: "John is thinking of a number. If he multiplies it by 1 and adds 0, what does he get? What kind of number was he thinking of?" S: Any number. T: Marco says "any number." Yes.</p>
<p><b>L3: Eliciting</b> Elicits procedures, answers &amp; facts, solutions of problems. Bounded by the need to produce an answer to the task</p>	<p>T: What's something you might need to do? S: An equation... we can use variables. T: Okay, what variables would you use? S: <math>t</math> T: Okay but for what? What is our equation? S: <math>t = 2s</math> T: And, what would <math>t</math> stand for? S: Chocolate.</p>
<p><b>L4: Facilitating</b> Facilitating understanding (asking questions) without validating, correcting, or cueing. Facilitating the building of conceptual ideas underlying the task</p>	<p>T: I want you to share your thoughts. S: We have strawberry and chocolate. T: Say more. What other pieces of evidence are there? S: Twice as many strawberry as chocolate. T: What does that mean? S: times 2 T: How are we going to find out how many strawberry? What else can you say? S: We can write two times as many strawberry more than chocolate.</p>
<p><b>L5: Extending</b> Encouraging other students to add to the response</p>	<p>T: Let's start with Anna. S1: You don't know how much so you can use a variable. T: Oh, you can use a variable to represent what you don't know. Anymore or do you want to pass it on to someone else?</p>

Pressing for  
Justification  
Pressing for  
Generalization  
Facilitating  
connections among  
mathematical ideas

S2: I want to add. I agree with Anna. So, like, you can do  $v$  times 2 equals something.  
Ss: You can use  $v$  times 2 equals  $s$  because that's how many strawberry sandwiches.  
T: What does  $v$  mean, Sean?  
S: It means that number of chocolates. You don't know how many.  
T: So we don't know how many, but this is how many chocolate.  
S: You can use  $c$ .  
T: Can we use  $c$ ? And, wait, what does this mean?  
S: Sandwiches.  
T: Do these numbers make sense in these problems?  
S: Because  $s$  is the amount of strawberry and  $c$  is chocolate.  
T: How do you know that these numbers (pointing to function table and then to the equation) work?

---

In this continuum, we identified instructional moves that gradually came to define these categories. As the categories changed from acknowledging, to responding, eliciting, facilitating, and extending, we noted an increase dialogic discourse. To that end, teachers gradually reduced the validation or correction of student responses/reasoning and increased the involvement of students in the discussion and in sharing the mathematical authority. We observed teachers gradually shifting the classroom discourse from focusing on a correct response or procedure to allowing students to negotiate a statement and, ultimately, inviting students to reason algebraically by generalizing, justifying, and connecting algebraic ideas.

As shown in Table 2, instructional engagement and responsiveness increased in each level. In *acknowledging* and *responding* (Levels 1 and 2 respectively), teachers did not build on student responses at all. *Eliciting* (Level 3), represents a relatively dramatic change in which teachers begin to build on student responses by drawing out (cueing, funneling) the desired response. Teachers elicit student participation, but at a very procedural level. The discourse is bounded by the need to produce a (predetermined) answer/solution/strategy/response to a given task. *Facilitating*, Level 4, is an extension of *eliciting* in that, once again, the teacher is drawing responses from students by inviting them to engage in more conceptual reasoning. Teachers orchestrate discourse around conceptual understanding but remain the main authority in the room as they guide the direction of the lesson. The difference between *eliciting* and *facilitating* may be subtle, unless one pays attention to the teacher's cues— "What is something you might *need to do*?" (the teacher's invitation to students in the illustrative eliciting example), as opposed to "I want you to *share your thoughts*." (the teacher's invitation in the illustrative facilitating example). The latter is more open-ended than the former in that it welcomes general student observations and questions. Finally, *extending*, once again, elevates discourse to a new level. The teacher openly shares authority and while pressing students for justification, generalization and for connections among ideas and also invites students to bring their curiosities and understandings to the floor and embraces these new mathematical directions.

### **Relationships Among SMs and Student Performance**

As the last part of our analysis, we examined students' overall performance in early algebra and its relationship with the algebra moments during instruction. To this end, we conducted a

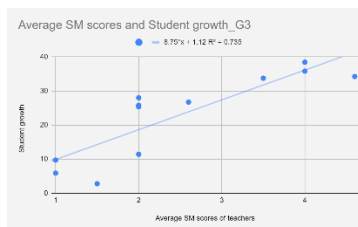
linear regression of the relationship between the frequency of both AMs and SMs in lessons and student growth as measured by the early algebra assessment. As we did not detect any significant correlation between frequencies of either AMs or SMs and student performance, we proceeded to examine the SMs more closely.

Each SM was again assigned one of six characterizations (i.e., setting aside, acknowledging, responding, eliciting, facilitating, and extending). Each teacher received an SM score by averaging the SM levels across that lesson. Table 3 shows the variation in teachers’ SM scores. Overall, in Grade 3 teachers had an average of score of 2.48 (min 1, max 4.6), 2.67 in Grade 4 (min 1, max 5), and 2.47 in Grade 5 (min 1, max 4.5). A simple linear regression between SM score and early algebra growth ( $r^2 = 0.735, 0.623, \text{ and } 0.885$  for Grades 3, 4 and 5 respectively as shown in Table 3) suggests a strong correlation overall between SM-scores and student growth.

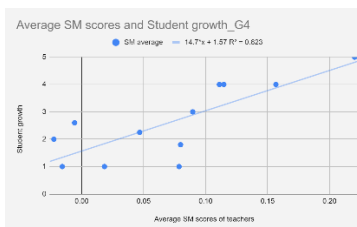
**Table 3:**

		Minimum	Maximum	Mean	$r^2$
SMs Score	Grade 3	1	4.6	2.48	0.735
	Grade 4	1	5	2.67	0.623
	Grade 5	1	4.5	2.47	0.885

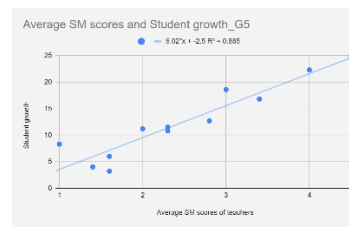
In contrast to the findings regarding *frequency* of either AMs or SMs, the level, or *quality* of SM implementation has a strong correlation with student growth. Students who were in classrooms in which teachers not only chose to take up the algebraic moments that arose spontaneously during classroom discourse, but also chose to respond to these moments by engaging in dialogic discourse in a manner that advances student algebraic reasoning (i.e., justification, generalization and encouraging connections), tended to have higher gains in the algebra assessment. The engagement in dialogic discourse is not a dichotomy, but a continuum of teachers’ moves that increase student engagement with these algebraic practices, and with each other’s reasoning. Figure 1 shows a graphic display of this relationship.



1a. Grade 3  
 $r^2 = 0.735$



1b. Grade 4  
 $r^2 = 0.623$



1c. Grade 5  
 $r^2 = 0.885$

**Figure 1: Student Growth as it Relates to Average SM Teacher Level**

### Discussion and Conclusion

We began this study by asking whether it is possible to identify instructional practices that have an impact on student learning in mathematics. We examined teachers’ implementation of an early algebra intervention and looked closely at curriculum openings – anticipated and spontaneous moments in instruction. Previous work suggests that teachers’ responsiveness to these spontaneous moments are an important characteristic of “good” instruction. Teachers’



abilities to take up openings in the curriculum and to identify important mathematical moments that arise outside of the curriculum have been identified as a *potentially* important aspects of teaching and important to student learning.

Our work corroborates these earlier findings and, in fact, finds a substantial positive relation between teachers' manner of taking up curriculum openings and actual student performance. It also brings us one step closer to understanding aspects of effective instruction regarding early algebra. As discussed earlier, attention to early algebra instructional effectiveness has been sparse, but our work gleans the untapped potential of this area of study. Our examination of several lessons across grades shows a clear pattern that levels of implementation of these spontaneous moments have a strong correlation to gains in student performance. As teachers attended to students' reasoning, be it correct or incorrect, and invited discussions on these issues as they organically arose, student performance on these concepts improved. These results offer promise that this is a fruitful area of research that we can continue to explore, and provide directions for improving teacher preparation to promote more rich early algebra teaching and learning.

### Acknowledgements

The research reported here was supported in part by the National Science Foundation under Award DRL-1721192. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the NSF. We are grateful for the assistance of Jordan Burkland, Hangil Kim, and Bethany Miller who participated in earlier stages of the project, particularly in coding efforts.

### References

- Ball, D. L., & Cohen, D. K. (1999). Developing practice, developing practitioners: Toward a practice-based theory of professional education. In L. Darling-Hammond, & G. Skyes (Eds.), *Teaching as a learning profession: Handbook of policy and practice*. San Francisco: Jossey-Bass.
- Ball, D. & Forzani, F. (2011). Building a Common Core for Learning to Teach and Connecting Professional Learning to Practice, *American Educator*, 35 (2), 17-21
- Bakhtin, M. M. (1986). *Speech genres and other late essays* (V. W. McGee, Trans.; C. Emerson & M. Holquist, Eds.). University of Texas Press.
- Blanton, M. & Kaput, J. J. (2005). Characterizing a classroom practice that promotes algebraic reasoning. *Journal for Research in Mathematics Education*, 36(5), 412–446.
- Blanton, M., Brizuela, B., Stephens, A., Knuth, E., Isler, I., Gardiner, A., Stroud, R., Fonger, N., & Stylianou, D. (2018). Implementing a framework for early algebra. In C. Kieran (Ed.), *Teaching and learning algebraic thinking with 5- to 12-year-olds: The global evolution of an emerging field of research and practice* (pp. 27-49). Cham, Switzerland: Springer International Publishing.
- Blanton, M., Stephens, A., Stroud, R., Knuth, E., Stylianou, D., Gardiner, A. (2019). Does Early Algebra Matter? The Effectiveness of an Early Algebra Intervention in Grades 3–5. *American Education Research Journal*, 56 (5), 1930-1972.
- Bishop, J. P., Hardison, H., Przybyla-Kuchek, J. (2022). Responsiveness to Students' Mathematical Thinking in Middle-Grades Classrooms. *Journal for Research in Mathematics Education*, 53 (1), 10-40
- Blazar, D. (2015). Effective teaching in elementary mathematics: Identifying classroom practices that support student achievement. *Economics of Education Review*, 48, 16-29.
- Cengiz, N., Kline, K., & Grant, T. J. (2011). Extending students' mathematical thinking during whole-group discussions. *Journal of Mathematics Teacher Education*, 14(5), 355–374.
- Chetty, Raj, John N. Friedman, and Jonah E. Rockoff. (2014). "Measuring the Impacts of Teachers II: Teacher Value-Added and Student Outcomes in Adulthood." *American Economic Review*, 104 (9), 2633-79.
- Davies, N., & Walker, K. (2005). Learning to notice: One aspect of teachers' content knowledge in the numeracy classrooms. In P. Clarkson et al. (Eds.), *Building connections: Theory, research and practice—Proceedings of*

- the 28th Annual Conference of the Mathematics Education Research Group of Australasia (pp. 273–280). Sydney, Australia.
- Ellis, A., Özgür, Z. & Reiten, L. (2019). Teacher moves for supporting student reasoning. *Math Ed Res J* 31, 107–132. <https://doi.org/10.1007/s13394-018-0246-6>
- Fennema, E., Carpenter, T. P., Franke, M. L., Levi, L., Jacobs, V. R., & Epton, S. B. (1996). A longitudinal study of learning to use children's thinking in mathematics instruction. *Journal for Research in Mathematics Education*, 27(4), 403–434.
- Franke, M. L., Turrou, A. C., Webb, N. M., Ing, M., Wong, J., Shin, N., & Fernandez, C. (2015). Student engagement with others' mathematical ideas: The role of teacher invitation and support moves. *The Elementary School Journal*, 116(1), 126–148. <https://doi.org/10.1086/683174>
- Kaput, J. (2008). What is algebra? What is algebraic reasoning? In J. Kaput, D. Carraher, & M. Blanton (Eds.), *Algebra in the early grades* (pp. 5–17). Mahwah, NJ: Lawrence Erlbaum Associates/Taylor & Francis Group.
- Knuth, E., & Peressini, D. (2001). A theoretical framework for examining discourse in mathematics classrooms. *Focus on Learning Problems in Mathematics*, 23(2), 5–22.
- Leatham, K., Peterson, B., Stockero, S., & Van Zoest, L. (2015). Conceptualizing mathematically significant pedagogical opportunities to build on student thinking. *Journal for Research in Mathematics Education*, 46(1), 88–124.
- National Center for Teacher Effectiveness. (2014). *Mathematical Quality of Instruction*. Retrieved from [http://sites.harvard.edu/icb/icb.do?keyword=mqi\\_training](http://sites.harvard.edu/icb/icb.do?keyword=mqi_training).
- National Council of Teachers of Mathematics [NCTM]. (2000). *Principles and standards of school mathematics*. Reston, VA: Author.
- National Governors Association Center for Best Practices [NGA] & Council of Chief State School Officers [CCSSO]. 2010. *Common core state standards for mathematics*. Washington, DC: Author. Retrieved from [http://www.corestandards.org/assets/CCSSI\\_Math%20Standards.pdf](http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf)
- Opper, I. (2019). *Teachers Matter: Understanding Teachers' Impact on Student Achievement*. RAND Foundation Report.
- Remillard, J. T., & Geist, P. (2002). Supporting teachers' professional learning by navigating openings in the curriculum. *Journal of Mathematics Teacher Education*, 5, 17–34.
- RAND Mathematics Study Panel Report. (2003). *Mathematical proficiency for all students: A strategic research and development program in mathematics education*. Washington, DC: U.S. Department of Education.
- Ristroph, I. Knuth, E., Boram, L., Stylianou, D., Kim, H., & Miller, B. (2022). Early Algebra Moments: Opportunities to Foster Students' Algebraic Thinking. In Lischka, A. E., Dyer, E. B., Jones, R. S., Lovett, J. N., Strayer, J., & Drown, S. (Eds). *Proceedings of the forty-fourth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 292-280). Middle Tennessee State University.
- Saldana, J. (2013). *The coding manual for qualitative researchers*. Sage
- Smith, M., & Sherin, M. (2020). *The five practices in practice*. NCTM.
- Stein, M. K., Smith, M. S., Henningsen, M. A., & Silver, E. A. (2000). *Implementing standards-based mathematics instruction: A casebook for professional development*. Teachers College Press.
- Stockero, S., & Zoest, L. (2013). Characterizing pivotal teaching moments in beginning mathematics teachers' practice. *Journal of Mathematics Teacher Education*, 16(2), 125–147.
- Syed, M., & Nelson, S. C. (2015). Guidelines for establishing reliability when coding narrative data. *Emerging Adulthood*, 3(6), 375–387. [doi.org/10.1177/2167696815587648](https://doi.org/10.1177/2167696815587648)
- Thames, M. H., & Ball, D. L. (2013). Making progress in U.S. mathematics education: Lessons learned— past, present and future. In K. Leatham (Ed.), *Vital directions for mathematics education research* (pp. 15–44). New York, NY: Springer
- Truxaw, M. P., & DeFranco, T. C. (2008). Mapping mathematics classroom discourse and its implications for models of teaching. *Journal for Research in Mathematics Education*, 39(5), 489–525.
- van Es, E. A., & Sherin, M. G. (2002). Learning to notice: Scaffolding new teachers' interpretations of classroom interactions. *Journal of Technology and Teacher Education*, 10(4), 571–596.
- Walkoe, J., Walton, M., & Levin M. (2022). Supporting teacher noticing of moments of algebraic potential. *Korean Journal of Education Research in Mathematics*, 32(3), 271–286.
- Wood, T. (1998). Alternative patterns of communication in mathematics classes: Funneling or focusing? In H. Steinbring, M. Bartolini Bussi, & A. Sierpiska (Eds.), *Language and communication in the mathematics classroom* (pp. 167–178). Reston: National Council of Teachers of Mathematics.
- Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.



## EMBODIED TEACHER ACTIONS TO PROMOTE STUDENTS' GENERALIZATION IN A VISUAL PATTERN TASK

Lizhen Chen  
Ohio University  
lizchen@ohio.edu

Allyson Hallman-Thrasher  
Ohio University  
hallman@ohio.edu

Keywords: Algebra and Algebraic Thinking, Mathematical Representations, Preservice Teacher Education, Classroom Discourse, Gestures

Teachers often use visual pattern tasks to introduce generalization to young students. Although coordinating both numerical and figural structures in the tasks could be difficult (Radford, 2014), the presence of visuals is beneficial for students to strengthen their numerosity and generalization knowledge (Nathan & Kim, 2007). Teachers translate generalizing ideas between different representational spaces (e.g., visual, numeric, and algebraic spaces) (Hallman-Thrasher et al., 2022). There is little research on how the discursive and embodied perspectives collaborate to make teachers' generalizing promoting actions explicit to students. This study aims to investigate how preservice mathematics teachers enact generalizing promoting actions and translate generalizing thinking between discursive and gestural spaces.

We focus on a mathematics teacher candidate, Mr. Davidson, who enrolled in a year-long master's teacher preparation program. He completed pattern tasks as a learner, analyzed other teachers' videos of teaching pattern tasks and analyzed his own videos. He planned, taught, and reflected on a generalization lesson that focused on a visual pattern task. Our data analysis was guided by Ellis' (2011) seven categories of generalizing promoting actions and Strachota's (2020) three categories of priming actions. Here we only present three categories of teacher actions: 1) priming actions that prompt students to identify mathematical features situated in a specific picture but extendable to other pictures in the pattern task (e.g., a teacher focuses students' attention to numerosity in a picture, expecting them to extend the idea of numerosity to other pictures); 2) publicly revoicing a generalization or idea; 3) building on a previously shared generalization or idea and creating a better or new generalizing idea. We used double coding to capture some teacher actions that share features of more than one category. We divided the teaching video into chunks based on the instructional topics mentioned in their immediate contexts. We coded the teacher's gesture use for each teacher action code. Finally, we discussed and studied how teacher gestures contributed to students' generalizing in teacher actions.

We found that the teacher candidate kept asking students for accurate descriptions of the ways of drawing the fourth picture. He used various representational gestures to re-present students' explanations of what the fourth picture looked like. When the teacher revoiced or built on students' generalizing responses in front of the whole class, his embodiment acted as a resource for all students to comprehend better what was shared in the moment. Through the teacher-student interaction, the teacher candidate created opportunities for students to refine their language of location, space, and relevant geometric features of the boxes and the rows in the picture. As students' generalizing explanations became more accurate in the geometric features in the pattern task, the teacher candidate's embodied generalizing promoting actions became more targeting spatial relationships between boxes and between rows (i.e., next to each other, parallel, below). While facilitating students' spatial construction in a specific picture of the pattern task, the teacher candidate, in effect, supported students' discovery of a generalizable rule

for cases beyond hand afterwards. This support sheds light on a potential of eventually building such a teaching practice of embodied generalizing-promoting teacher actions.

### **Acknowledgments**

The research reported here was supported in part by National Science Foundation Award #1758484. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

### **References**

- Ellis, A. B. (2011). Generalizing-promoting actions: How classroom collaborations can support students' mathematical generalizations. *Journal for Research in Mathematics Education*, 42(4), 308–345. <https://doi.org/10.5951/jresematheduc.42.4.0308>
- Hallman-Thrasher, A., Thompson, J., Heacock, K., & Chen, L. (2022). Relationships between teacher questioning and student generalizing In A. E. Lischka, E. B. Dyer, R. S. Jones, J. N. Lovett, J. Strayer, & S. Drown (Eds.). *Proceedings of the forty-fourth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Middle Tennessee State University.
- Nathan, M. J., & Kim, S. (2007). Pattern generalization with graphs and words: A cross-sectional and longitudinal analysis of middle school students' representational fluency. *Mathematical Thinking and Learning*, 9(3), 193-219. <https://doi.org/10.1080/10986060701360886>
- Radford, L. (2014). The progressive development of early embodied algebraic thinking. *Mathematics Education Research Journal*, 26, 257-277. DOI 10.1007/s13394-013-0087-2
- Strachota, S. (2020). Generalizing in the context of an early algebra intervention (La generalización en el contexto de una intervención algebraica temprana). *Journal for the Study of Education and Development*, 43, 347-394. <https://doi.org/10.1080/02103702.2020.1732611>

## INSERVICE SECONDARY TEACHERS' BELIEFS ABOUT DEDUCTIVE DISCOURSE FOR EQUATION SOLVING

Elizabeth Wrightsman  
Texas State University  
ewrightsman@txstate.edu

Cody L. Patterson  
Texas State University  
codypatterson@txstate.edu

*We investigate teacher beliefs about discourses for equation solving and the challenges these beliefs might pose for the implementation of instructional practices that promote deductive reasoning in algebra. To uncover these beliefs, we recorded three video explanations of solutions to the same linear equation with distinct discursive characteristics and analyzed seven secondary mathematics teachers' small-group critical discussions of these explanations. Three prevalent themes surfaced in our thematic analysis. Teacher beliefs about discourse for equation solving specified different roles and potential benefits of deductive explanations, estimated students' capacity to understand deductive explanations, and hypothesized differences between teachers' and students' potential to understand deductive reasoning. We discuss implications of these beliefs for opportunities to engage all learners in conceptual thinking about equations.*

Keywords: Algebra and Algebraic Thinking, Teacher Beliefs, Classroom Discourse, Reasoning and Proof

The study of algebra serves as an entry point into postsecondary studies and careers in science, technology, engineering, and mathematics. Yet success in algebra remains elusive for many students at the secondary and college levels. Reasons for this are varied and include a lack of equitable access to well-prepared mathematics teachers (Lee, 2012; Sutchter et al., 2019), teaching practices that do not build upon students' knowledge assets (including knowledge from their communities as well as foundational understandings of number and operations; see for example Civil, 2016), and a focus on symbolic manipulation at the expense of opportunities for sensemaking (Chazan, 1996). A key concept within algebra is equation solving, which may be introduced to students using various approaches and choices of language. Our study investigates the language that teachers consider effective in teaching students to solve equations. The Common Core State Standards call for students to "Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution" (NGA & CCSSO, 2010, HSA-REI.A.1). This standard aligns with a view of equation solving as a deductive process: the steps of a solution process can be viewed as steps in an argument which assumes the equality of the values of two expressions and makes successive inferences about the value of a variable. In classroom settings, this deductive process may be modeled using concrete models such as a balance scale (Vlassis, 2002). On the other hand, evidence suggests that in some algebra courses, textbooks and teachers describe novel problem-solving procedures in terms of actions on symbols without attending to underlying algebraic objects and their properties (Patterson & Farmer, 2018).

Although there has been some research conducted to assess how language specific to the algebra of equations is used in mathematics classrooms (e.g., Planas, 2021), work is still needed to advance our understanding of how language may be leveraged to further students' deductive understanding of equation solving. Our research team considers teacher beliefs a driving force behind teaching practices and choices in how mathematical content is communicated. Therefore,

we suggest that a productive first step in understanding how discourse considerations shape teaching and learning about equations is to gain insight into teacher beliefs.

### Theoretical Framework

Our work draws from the commognitive perspective, in which thinking is viewed as a process of communication, and to learn mathematics is to undergo a change in one's participation in a discourse community (Sfard, 2007, 2020). A foundational assumption of our work is that the ways in which teachers communicate when explaining algebraic concepts, and the discourse in which they invite students to participate as they grapple with these concepts, are consequential for students' opportunities to learn to reason flexibly and fluently about algebra. This view is supported by empirical research on connections between language, conceptual understanding, and student achievement (e.g., Bills, 2002; Huntley et al., 2007; Knuth et al., 2006; Van Amerom, 2003). In our study of discourses associated with equation solving in algebra, we draw from the *arithmetical discourse profile* of Ben-Yehuda, Lavy, Linchevsky, and Sfard (2005), which analyzed learners' discourse about concepts and problems in arithmetic along several key dimensions: their uses of *words* and the extent to which these explicitly describe mathematical objects, their uses of *mediators* (symbols and visuals that represent mathematical objects), their *endorsed narratives* and apparent meta-rules for accepting and rejecting narratives, and their uses of routines. In framing our work, we condense the *words* and *mediators* dimensions into a single dimension and use the resulting three dimensions (*words/mediators*, *endorsed narratives*, and *routines*) as a framework for algebraic discourse.

Guided by this discourse framework, we have developed a survey with fourteen open-ended items related to algebraic expressions, equations, functions, and modeling. Each item asks teachers to analyze a hypothetical student solution to an algebra problem, resolve a potentially ambiguous situation (such as what happens when the process of solving a system of linear equations culminates in an equation of the form  $c = c$  where  $c$  is a constant), or explain the conceptual underpinnings of an idea often taken for granted in algebra (such as "combining like terms" or "keeping the sides balanced" when solving an equation). In-service teacher responses to this survey have supported a preliminary finding that teachers' talk about algebraic concepts varies significantly along all three dimensions (Patterson et al., 2021), suggesting that the arithmetical discourse profile can be extended to study the discourse practices of secondary algebra classrooms.

We define a *deductive discourse for equation solving* to be one in which words and mediators frequently serve to make the objects of the discourse (e.g., values of expressions, operations, equality) and their properties explicit, in which narratives about equations and unknowns are endorsed or rejected by deduction from assumptions and other endorsed narratives, rather than by appeal to authority or other communicative rituals lacking an explicit deductive basis; and in which routines are used as flexible tools for generating new narratives about mathematical objects. Our larger study investigates the extent to which teachers engage in deductive discourse when explaining processes for solving equations and how teacher beliefs might support or constrain students' opportunities to engage in deductive discourse for equation solving. This research report focuses on our investigation of the second question.

Our analysis of teachers' beliefs about equation solving is informed by Leatham's *sensible systems* theory, which suggests that rather than focusing on apparent contradictions among beliefs held by an individual teacher, we should view beliefs as occupying an interconnected network in which some beliefs may take precedence over others at specific times (2006).

Leatham encourages mathematics teacher educators to present opportunities for teachers to explore their beliefs in practical contexts. While teacher beliefs may pose challenges for instructional change, beliefs can both inform the design of professional learning experiences and be shaped by collaboration with teachers and teacher educators (Goldsmith et al., 2014).

Guided by this framework, we address the following research questions:

1. What beliefs do teachers have about teaching students a deductive discourse for solving linear equations?
2. What challenges might teacher beliefs pose for teaching students how to reason deductively about equation solving?

### Method

Informed by teachers' responses to survey items dealing with equations and solution processes, we developed an activity titled *Linear Equation Talk-Throughs* that we implemented as part of an 80-hour content-based professional development workshop for seven middle and high school algebra teachers in 2022. In the first stage of this activity, teachers privately recorded "talk-throughs" – video explanations of solution processes – in which they solved the linear equation  $7x - 20 = 3x$ , as if teaching students "who are just learning how to solve this type of equation" (per the written activity instructions). They also privately watched three different video talk-throughs recorded by the second author. These three researcher talk-throughs were designed to exemplify different possible discourse features of explanations of solutions to linear equations. Video 1 exemplified an approach focused on mediators, their spatial arrangement (e.g., which "side" of the equation terms are on), and strategic actions-on-mediators needed to solve the equation. Video 2 exemplified an approach that interprets the given equation as a statement that the values of two expressions are equal and identifies the solution set through a sequence of deductive steps using properties of equality. Video 3 exemplified an approach that the teachers had come to know from a previous workshop activity as "solving by inspection": using the structure of the equation to make successive inferences about values of various terms and factors using number sense. Because the steps taken in Videos 1 and 2 are equivalent in their symbolic representation, these two talk-throughs served to illustrate a contrast between explanations that use words, mediators, and narratives differently. Because the approach taken in Video 3 is noticeably different, assigning values at intermediate steps to terms and factors in the equation, we see this talk-through as illustrating a more flexible approach to the use of narratives and routines during the equation-solving process. Table 1 provides a description and a representative transcript excerpt for each of the three researcher talk-throughs.

After the seven teacher-participants independently watched the researcher's talk-throughs, they divided into small groups (See Table 2) to compare the three talk-throughs and discuss the affordances and drawbacks of each. These discussions were video/audio recorded and transcripts were electronically generated and verified for accuracy. Each author independently coded the discussion transcripts; for each talk turn, we indicated any implicit beliefs about teaching and learning of equation solving that were evident in the teachers' analysis. Subsequently, we came together to discuss the independent themes to arrive at agreement on broad themes related to teacher beliefs. This research report presents the findings from this analysis and discusses possible implications of the teacher beliefs that surfaced.



**Table 1: Researcher Talk-Through Video Samples**

Video	Description	Sample from Researcher Talk-Through (Emphasis ours)
Video 1	Focus on actions-on-symbols Duration: 1:18	“My approach here is I'm going to try to <b>get all my x's on one side</b> of the equation and <b>put all the constants on the other side</b> of the equation. Because there's already a $3x$ on the right side of the equation, <b>I think I want to move the term <math>7x</math>, so it's over on the right side</b> , and I'll leave the $-20$ on the left. I'm going to take this $7x$ here, and I'm going to <b>change sides and change signs</b> , so I'm going to move it over to the right and <b>put a negative sign on it.</b> ”
Video 2	Focus on deductions about equal values Duration: 2:14	“So, I'm going to start by saying <b>because <math>7x</math> minus <math>20</math> has the same value as <math>3x</math>, if I add <math>-7x</math> to both of those values, I should get the same result</b> . So, in other words, $-7x$ plus $7x$ minus $20$ , that should be equal to $-7x$ plus $3x$ , 'cause <b>I took two equal values, <math>7x</math> minus <math>20</math> and <math>3x</math>, and I added the same thing to each</b> . Now, if I look at the left side of the equation, I have $-7x$ plus $7x$ minus $20$ . $-7x$ plus $7x$ , those are additive inverses of each other, so they add to zero. That means I'm left with $0$ minus $20$ equals $-7x$ plus $3x$ .”
Video 3	Using structure and number sense to solve by inspection Duration: 0:58	“ <b>Well, one thing I notice about this equation</b> is I'm starting with $7x$ and I'm subtracting $20$ , and that leaves me with $3x$ . One thing that I know is that if I start with $7x$ and subtract $4x$ , that leaves $3x$ . So that means that if I'm subtracting $7x$ minus $20$ and getting $3x$ , <b>that means that <math>20</math> has to be equal to <math>4x</math></b> . And so now I have an equation that says that $20$ is equal to $4$ times $x$ , $4$ times my number $x$ . <b>So I think what number multiplied by <math>4</math> gives me <math>20</math>? Well, I know that <math>4</math> times <math>5</math> is <math>20</math></b> , so that indicates that $x$ is equal to $5$ .”

**Table 2: Group Composition of Teacher Participants**

Group	Participant	Level(s) Taught
Green	Danielle	High school
	Pablo	High school
	Frances	High school
Pink	Benjamin	High school
	Viola	Middle school (K–8 academy)
Yellow	Denise	High school
	Felipe	Middle school (K–8 academy)

### Summary of Findings: Teacher Beliefs

Three prominent themes surfaced in our analysis of teachers’ discussion of the researcher talk-throughs. Each of the three groups discussed the role or importance of understanding that solving linear equations is a deductive process. Notably, there seemed to be conflicting perspectives on when in the learning process the role of mathematical properties in the equation-solving process should be made explicit to students. A second common theme was estimation of students’ capacity to understand solving linear equations as a deductive process. Two of the groups (Green and Yellow), comprising of five teacher participants, suggested that a deductive approach to solving equations would not be suitable for *all* students. Some teachers drew a

distinction between those students who would be confused by “too many steps” and students who would benefit from an explicit development of the deductive reasoning behind the problem-solving process. The third salient theme was the perception of a difference between those more experienced in algebraic reasoning (teachers) and novices to algebraic reasoning (students) in terms of the potential to understand and engage in deductive algebraic discourse. We present some excerpts from the teachers’ discussions to illustrate the major themes and our interpretations.

### **Role or Importance of Understanding Solving as a Deductive Process**

A common thread among the three group discussions was beliefs related to the role or significance of a deductive discourse for solving linear equations, or of specific features of this discourse. Some teachers believed that it is productive to highlight algebraic properties upon introducing linear equations, thus providing students with the rationale supporting the steps of the problem-solving process, as was the case with Viola and Benjamin in the Pink Group in their exchange about Video 1, which focused on actions-on-symbols rather than deduction.

Viola: It's that part that if I was new to algebra, I would not understand, “Why am I changing sides?” I'm assuming that a student who'd do this is well-versed in why I'm changing sides and why I'm changing signs. That statement assumes understanding is what I'm just saying.

Benjamin: Especially with negative numbers, and that's where they get confused. My experience, they get confused a lot.

Viola: ...I'm going to tell you straight up; sixth grade is where it's introduced. If it's not introduced with concrete [models], they will struggle for a long, long time.

Otherwise, you're going to have to rely on rules and they don't know why it works. So, this is key, right? ...So, the question, “Why does it work?” needs to be happening way down before you... Yeah, because you're too far. You're advanced.

The teachers emphasize the importance of illuminating the “why” behind each step of the solution process, which directly aligns with the call-in standards documents to equip students with the deductive tools necessary to explain and justify each step. As Viola suggests, the language employed within Video 1, “changing sides” and “changing signs,” phrases often used in describing the steps of solving an equation, “assumes understanding” that students may not yet possess.

The Yellow Group made a contrast between the explanations in Video 1 and Video 2 that suggested that Video 2, the deductive explanation, would be appropriate for introducing equation solving, whereas Video 1 is the conventional method to describe the steps of solving equations and would be deemed the “easier” approach.

Felipe: I was going to say, I think using that method [from Video 2] would be a good way to introduce it, which sounds counterintuitive, but I feel like you show it to them, and they're like, "Okay." They can kind of see it, and then you show them the way we usually do it [referring to Video 1's method], and then to them, that seems easier, so they're like, "I like that a lot more."

Denise: Yeah. Okay. I very much would tend to do that with my students. The first time we do it, I make them do it that way, so then when I show them an easier way, it makes sense, and they prefer it, and they're going to do that.

Counter to the belief that students *should* be exposed to the deductive reasoning behind the algebraic steps in equation solving as they learn the steps to solving, Danielle posited that introducing properties too early in the learning process may confuse learners.

Danielle: But that's after they already have learned to solve equations in ninth grade, in algebra one. Then we're doing it in geometry, we're saying, "Okay, these are what these properties are called now to practice those justifications." So, from that standpoint, but again, doing that not on the first time they're learning this. It's like, the second time. So, I love the use of properties, but I agree, I think it would be confusing to the people learning for the first time, and that's what I thought, too.

From Danielle's comment about Video 2, it would seem that she views the introduction of properties as a stepping-stone for inducting students into formal mathematics; she later clarified that she considered this appropriate only for students in advanced-track courses.

Our findings suggest that most teachers recognize benefits of exploring the justifications for steps of the problem-solving process. However, teachers also exhibited beliefs about students that stood in apparent tension with their view of the benefits.

### **Students' Capacity to Understand Solving as a Deductive Process**

Another common theme that arose from our analysis relates to the teachers' beliefs about students' capacity to understand solving as a deductive process. For example, two of the teachers in the Green Group agreed that their students would be confused by the number of steps in Video 2, which explains the algebraic properties underpinning the deductive view of equation solving.

Frances: It's too many steps. And then, I would have simplified the right side instead of taking it to the next step. I would have simplified as I went to the next step on both sides. And he would simplify one side, then bring down to another one, another step, and then simplify on the first side. He wouldn't simplify it as he would go along; he would wait, go to the next step, next step, next step.

Danielle: Yeah.

Frances: Like, step one, step two, step three. My kids would get confused. Danielle: Yeah.

Frances: Yeah, my kids would get confused; too many steps. I already know that, too many steps. Now, the ones that are real bright, they would catch on real easily. But you have to realize you have to accommodate everybody in the class...

Additionally, Frances suggests Video 2, with the explicit reasoning steps, would be appropriate for her "real bright" students, implying that deductive reasoning is for more advanced students. Guided by an imperative to accommodate all students, Frances seemed to consider it important to rely on the equation-solving explanation that she believed would be most accessible.

Felipe and Denise in the Yellow Group reacted similarly to the explanation provided by Video 2, particularly the step where the researcher talk-through included a step to illustrate the reasoning behind combining like terms, which included factoring out the variable  $x$  to first add the coefficients, then redistributing  $x$ .

Felipe: I think right here they were getting confused.

Denise: Yes. They were not really understanding what you were doing there. Felipe: Yeah. This one right there, they'd be like, "What did you do?"

Denise: Yeah. Well, depending on what this is, combining like terms which is something you're going to do before I think you start doing solving, you need you look at that. So,

they would see that combining like term, but doing like that, they would wonder where you got that.

The teacher participants implied that the conceptual reasoning behind “combining like terms” would be something introduced before equation solving, and that if it were integrated into the steps of equation solving, it could be a source of confusion for their students.

### **Differences Between Teachers' and Students' Potential to Understand Deductive Reasoning**

The final notable theme that surfaced through our data analysis is related to teacher beliefs about the difference in potential to understand deductive reasoning between those deemed more knowledgeable about the equation-solving process, and those who are less experienced, as stated by two teacher participants. Regarding the explanation in Video 3, which relies on structure and number sense, Benjamin proclaimed, “For us [teachers] it's no big deal. It's trivial, we understand it,” implying that it would be challenging for students to understand. Similarly, Felipe suggested a discrepancy between his view of Video 3 and the view his students would likely take:

Felipe: That one [Video 3], I think is the more complex of them all. Well, no, not for us. For them to rationalize and understand because to them, when they see  $5x$ , they generally, I think would see it as two units, 5 and  $x$ . Whereas we can see it as one thing that we can manipulate.

### **Discussion**

We wish to acknowledge some limitations of the present study. Most notably, each researcher talk-through, by necessity, contained idiosyncratic features that may have diverted teachers' evaluations from the key discourse features we intended to embed in each video. For example, Video 2 contained steps justifying the process of combining  $3x + -7x$  to obtain  $-4x$ ; while linking this step to the distributive property may be edifying for students, we find that it is typically assumed that students are fluent in combining like terms before they learn to solve linear equations. Several of our teacher-participants, therefore, found it peculiar that the researcher justified this step in such detail. Because teachers understandably focused on critiquing specific choices that the researcher made in each explanation, they did not always discuss broader characteristics of each video, such as the commitment in Video 2 to reasoning deductively from assumptions. (Danielle was a notable exception: she aptly summarized Video 1 as “What do we do to isolate  $x$ ?”, Video 2 as “What keeps both sides equal at all times?”, and Video 3 as “What makes that true?”) We conjecture that adding prompts calling teachers' attention to some of these features in future iterations of the activity might enrich our understanding of teachers' beliefs about the feasibility and benefits of a deductive discourse for equation solving.

Participants' analyses of the researcher talk-throughs suggested that they saw potential benefits in the deductive explanation for the standard solution process given in Video 2 and the structure-oriented approach described in Video 3, though participants did see the role of these alternative explanations differently. For example, Felipe and Denise suggested that they would use an explanation like that in Video 2 to introduce students to the solution process before showing them an “easier” approach, while Danielle suggested that she would defer the in-depth explanation in Video 2 until her students began grappling with deductive reasoning and formal proof in geometry. While Viola and Benjamin stated that they found the “solving by inspection” approach in Video 3 to be “a fabulous tool,” Denise and Felipe hypothesized that this method

would be harder for students to understand and suggested offering it to students only as a “fun challenge.”

### **Directions for Future Research**

If there is strong consensus that deductive explanations and structure-oriented approaches for solving equations are potentially useful for students, why are actions-on-symbols explanations of solution processes so prevalent in teaching, as evidenced by reviews of curricular materials and our own teachers’ recorded talk-throughs? In keeping with a sensible systems view of teacher beliefs, we seek to understand beliefs about instruction and about students that might mediate between teachers’ generally favorable views of deductive discourse for equation solving and their likelihood of modeling this discourse in classroom practice. In this study we have discovered two such families of beliefs: (1) that a deductive perspective on equation solving is likely to prove difficult for students (especially those who have been the target of deficit attributions, such as students in an intervention course), and (2) that explanations that teachers find approachable (and in fact elegant or efficient) might nevertheless be beyond students’ reach. Given that many teachers feel a strong sense of commitment to engaging all learners in successful mathematical practice, it is understandable that an explanation or approach that appears likely to confuse or frustrate learners might be disfavored in instruction. One goal of our project is to persuade teachers that it is feasible and worthwhile to engage all learners in deep and conceptually coherent algebraic reasoning.

Given that beliefs are deeply held and often resistant to change (Conner & Gomez, 2019; Philipp, 2007), we as mathematics teacher educators aim to design and provide professional learning experiences that allow teachers to reflect on, explore, and challenge their own beliefs about algebra teaching, while also helping to lower some of the perceived barriers that might impede students’ access to deductive reasoning. Our teachers’ analyses of the researcher talk-throughs offer some initial suggestions that we plan to incorporate into future iterations of the activity. At one point Frances noted that one factor that contributed to a general sense of “too many steps” in Video 2 was that the researcher rewrote the entire equation each time he wanted to simplify part of an expression; Frances stated that she would instead carry out detailed simplification steps in the margin and incorporate these changes into the solution process once done simplifying. We see this as entirely compatible with a deductive approach to equation solving: a sequence of simplification steps can be viewed as a sub-argument that generates an endorsed narrative about equivalent expressions; this sub-argument can be made separately from the main argument associated with the solution process. Viola pointed out that while she found the structural approach in Video 3 useful, she found it even more important to teach her middle school students to solve equations using concrete models first. Given that concrete models such as algebra tiles can act as mediators for unknown values and can encourage the kind of structure thinking embodied by Video 3, we see this suggestion as a potential bridge between the use of concrete models (which we have found that many teachers embrace enthusiastically) and deductive reasoning about equations. We have attempted this bridging with systems of linear equations with some success in our workshop and look forward to incorporating an explanation involving concrete models into the next iteration of the *Linear Equation Talk-Throughs* activity.

### **Acknowledgments**

This work is supported by the National Science Foundation through the Discovery Research PreK-12 (DRK-12) program (Award #1908825). Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

## References

- Ben-Yehuda, M., Lavy, I., Linchevski, L., & Sfard, A. (2005). Doing wrong with words: What bars students' access to arithmetical discourses. *Journal for Research in Mathematics education*, 36(3), 176–247. <https://doi.org/10.2307/30034835>
- Bills, C. (2002). Linguistic pointers in young childrens' descriptions of mental calculation. In A. Cockburn & E. Nardi (Eds.), *Proceedings of the Twenty-Sixth Annual Meeting of the International Group for the Psychology of Mathematics Education*, (Vol. 2, pp. 97–104). School of Educational and Professional Development, University of East Anglia.
- Chazan, D. (1996). Algebra for all students?: The algebra policy debate. *The Journal of Mathematical Behavior*, 15(4), 455–477. [https://doi.org/10.1016/S0732-3123\(96\)90030-9](https://doi.org/10.1016/S0732-3123(96)90030-9)
- Civil, M. (2016). STEM learning research through a funds of knowledge lens. *Cultural Studies of Science Education*, 11(1), 41–59. <https://doi.org/10.1007/s11422-014-9648-2>
- Conner, A., & Gomez, C. N. (2019). Belief structure as explanation for resistance to change: The case of Robin. *The Journal of Mathematical Behavior*, 53, 196–209. <https://doi.org/10.1016/j.jmathb.2018.08.003>
- Goldsmith, L. T., Doerr, H. M., & Lewis, C. C. (2014). Mathematics teachers' learning: A conceptual framework and synthesis of research. *Journal of mathematics teacher education*, 17, 5–36. <https://doi.org/10.1007/s10857-013-9245-4>
- Huntley, M. A., Marcus, R., Kahan, J., & Miller, J. L. (2007). Investigating high-school students' reasoning strategies when they solve linear equations. *The Journal of Mathematical Behavior*, 26(2), 115–139. <https://doi.org/10.1016/j.jmathb.2007.05.005>
- Leatham, K. R. (2006). Viewing mathematics teachers' beliefs as sensible systems\*. *Journal of Mathematics Teacher Education*, 9, 91–102. <https://doi.org/10.1007/s10857-006-9006-8>
- Lee, J. (2012). Educational equity and adequacy for disadvantaged minority students: School and teacher resource gaps toward national mathematics proficiency standard. *The Journal of Educational Research*, 105(1), 64–75. <https://doi.org/10.1080/00220671.2010.519409>
- Knuth, E. J., Stephens, A. C., McNeil, N. M., & Alibali, M. W. (2006). Does understanding the equal sign matter? Evidence from solving equations. *Journal for Research in Mathematics Education*, 37(4), 297–312. <https://doi.org/10.2307/30034852>
- National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010). *Common Core State Standards*.
- Patterson, C. L., & Farmer, L. C. (2018). Here's what you do: Personalization and ritual in college students' algebraic discourse (pp. 198–212). San Diego, CA, United States: SIGMAA for Research in Undergraduate Mathematics Education. Retrieved from <http://sigmaa.maa.org/rume/Site/Proceedings.html>.
- Patterson, C.L., Wrightsman, E., Kirmizi, M., McGraw R. (2021). Extractive and Inferential Discourses for Equation Solving. In *Proceedings of the Forty-Third Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, (pp. 390-394). Philadelphia, PA.
- Philipp, R. A. (2007). Mathematics teachers' beliefs and affect. *Second handbook of research on mathematics teaching and learning*, 1, 257–315.
- Planas, N. (2021). How specific can language as resource become for the teaching of algebraic concepts?. *ZDM—Mathematics Education*, 53(2), 277–288. <https://doi.org/10.1007/s11858-020-01190-6>
- Sfard, A. (2007). When the rules of discourse change, but nobody tells you: Making sense of mathematics learning from a commognitive standpoint. *The Journal of the Learning Sciences*, 16(4), 565–613. <https://doi.org/10.1080/10508400701525253>
- Sfard, A. (2020). Commognition. *Encyclopedia of mathematics education*, 95–101.
- Sutcher, L., Darling-Hammond, L., & Carver-Thomas, D. (2019). Understanding teacher shortages: An analysis of teacher supply and demand in the United States. *Education policy analysis archives*, 27(35), 1–36.
- Van Amerom, B. A. (2003). Focusing on informal strategies when linking arithmetic to early algebra. *Educational Studies in Mathematics*, 54, 63–75. <https://doi.org/10.1023/B:EDUC.0000005237.72281.bf>
- Vlassis, J. (2002). The balance model: Hindrance or support for the solving of linear equations with one unknown. *Educational Studies in Mathematics*, 49, 341–359. <https://doi.org/10.1023/A:1020229023965>

# LINEAR OR NONLINEAR? STUDENTS' GRAPH REASONING AND SELECTION ON AN ONLINE ASSESSMENT

Robert Knurek  
University of Colorado Denver  
robert.knurek@ucdenver.edu

Heather Lynn Johnson  
University of Colorado Denver  
heather.johnson@ucdenver.edu

*We conducted a collective case study investigating two college algebra students' graph reasoning and selection on an online assessment. Students completed the assessment during individual, semi-structured interviews, as part of a broader validation study. The assessment contained six items; students selected Cartesian graphs to represent relationships between attributes in dynamic situations and explained their reasoning. Each student spontaneously wondered whether to select a piecewise-linear or nonlinear graph. Our qualitative analysis revealed that students' expectations about whether a graph "should" be linear or nonlinear impacted their graph selection. These expectations also influenced how they narrowed down between two graph choices that both represented the same gross covariation in attributes. We conclude with implications for course textbooks to promote students' covariational reasoning.*

Keywords: Assessment, Cognition, Reasoning and Proof, Undergraduate Education

## Introduction

When students sketch a Cartesian graph to represent a relationship between attributes in a dynamic situation, how do they decide if the graph should be linear or nonlinear? What role do their expectations play in this decision? For instance, a student may expect a graph to be curved because they are currently working on a unit on quadratic functions in their college algebra course. Alternatively, they may expect a graph to be linear because physical features of an object in the situation contain straight edges. We investigate how students' graph reasoning relates to their selection of a graph to represent attributes in dynamic situations, in particular when there are linear and nonlinear options.

For decades, researchers have been examining and theorizing secondary and university students' graph reasoning in dynamic situations (Carlson et al., 2002; Kerslake, 1977; Leinhardt et al., 1990). By "dynamic situation", we mean a situation involving change and variation. For example, a dynamic situation could include a cart moving on a turning Ferris wheel. We interpret "graph reasoning" broadly, to encompass students' reasoning about attributes of objects and their relationships, as well as their reasoning about observable aspects of graphs. For our purposes, we focus on Cartesian graphs. For instance, a student could expect that a graph representing the Ferris wheel situation needs to be shaped like the wheel itself. Students' graph reasoning can take on different forms depending on how students interpret attributes of objects represented in dynamic situations (Clement, 1989; Johnson et al., 2020; Moore & Thompson, 2015).

Covariational reasoning (Carlson et al., 2002) can play a role in students' interpretation of graphs as representing relationships between attributes in dynamic situations. Furthermore, covariational reasoning can engender students' differentiation between relationships represented by linear or nonlinear graphs (Paoletti & Vishnubhotla, 2022). For example, a student may choose a nonlinear graph based on their conception of relationships between the attributes of the Ferris wheel car's height and distance. Hence, students' covariational reasoning may impact their selection of a linear or nonlinear graph to represent attributes in a dynamic situation.

The two students in our study are undergraduate students enrolled in college algebra, a course known to be overly congested with content to cover in a short period of time (Gordon, 2008). Because of this, the course tends to focus on procedural applications rather than concept development, and college algebra textbooks reflect that (Mesa et al., 2012). Furthermore, college algebra has higher proportions of underrepresented minority, low income, and first-generation college students (Chen, 2016). Our study, investigating students' graph reasoning and selection, contributes to knowledge about an important, and understudied, population.

In this paper, we present findings from a collective case study (Stake, 2005) of two college algebra students who spontaneously wondered whether graphs should be piecewise-linear or nonlinear during their work on an online assessment. Our research question is: How does students' graph reasoning relate to their graph selection on a fully online assessment?

### Theoretical Framework

#### Thompson's Theory of Quantitative Reasoning

We employ Thompson's theory of quantitative reasoning (1994, 2011) as a lens to frame this study. From Thompson's perspective, students conceive of quantities when they can think of an attribute of an object as being possible to measure. For example, consider a dynamic situation of a cart on a Ferris wheel moving counter-clockwise (Figure 1). A student can conceptualize the height of the cart as a quantity by thinking of it as being possible to measure, regardless of whether they actually measure the height of the cart or assign numerical values to measurements.

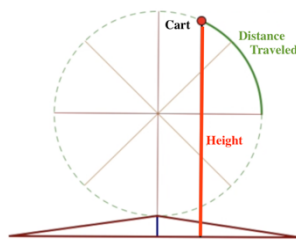


Figure 1: Ferris Wheel Dynamic Situation

When students conceive of two quantities and conceive of a relationship between them in which both quantities are capable of varying simultaneously, they are engaging in covariational reasoning (Thompson & Carlson, 2017). For example, given the Ferris wheel dynamic situation in Figure 1, a student may conceive of both height and distance traveled as quantities. They may engage in covariational reasoning by conceiving of a relationship between these two attributes in which the height fluctuates while the distance traveled increases. Thompson & Carlson put forth “gross coordination of values” as an early level of covariational reasoning. This refers to a loose connection between the direction of change in attributes. For example, a student demonstrating evidence of reasoning at this level may look at the Ferris wheel dynamic situation and say “when the height increases, the diameter decreases.” This level of covariational reasoning does not necessitate identifying specific amounts of change or the rate at which the change happens.

#### A Graph Reasoning Framework

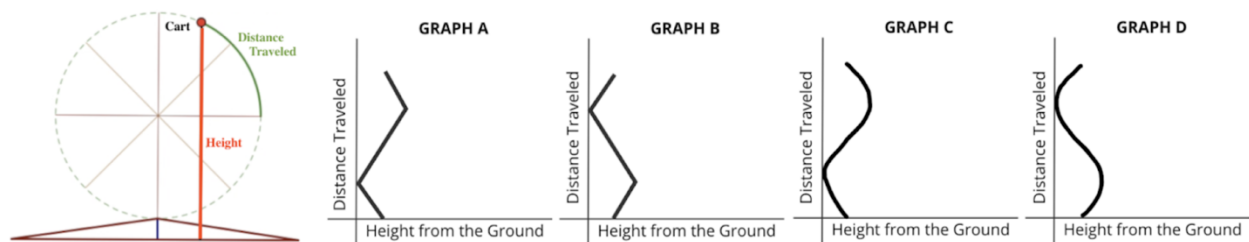
Johnson et al. (2020) conducted interviews with secondary students to investigate their conceptions of what graphs represent and how these conceptions change across digital task sequences. They developed a graph reasoning framework to characterize students' conceptions of what graphs represent. They included four broad forms of reasoning: Covariation, Variation,



Motion, and Iconic. The first two types of reasoning, Covariation and Variation, referred to reasoning compatible with at least Thompson & Carlson’s (2017) gross coordination of values and gross variation, respectively. Motion reasoning refers to students’ reasoning about physical motion of objects in dynamic situations (Kerslake, 1977). Iconic reasoning referred to students’ reasoning about physical features of objects in dynamic situations (Leinhardt et al., 1990). This framework encompassed students’ reasoning about quantities and their relationships (Covariation, Variation), as well as their reasoning about observable aspects of dynamic situations (Motion, Iconic).

To illustrate, consider a dynamic situation involving a turning Ferris wheel, in which students select a graph to represent a relationship between two attributes in the situation: the Ferris wheel cart’s height from the ground and the total distance traveled around one revolution of the wheel. Figure 2 contains a still image of the situation, along with four graph choices. Each graph is unconventional (Moore et al., 2014), with the height from the ground represented on the horizontal axis and distance traveled represented on the vertical axis. Below are examples of the four forms of graph reasoning from Johnson and colleagues’ (2020) framework.

- Covariation: The Ferris wheel’ height increases and decreases while the distance traveled continues to increase.
- Variation: The Ferris wheel’s height increases and decreases.
- Motion: The Ferris wheel moves at a constant speed, so the graph should be straight.
- Iconic: The Ferris wheel is circular, so the graph should be curved.



**Figure 2: Ferris Wheel Dynamic Situation with Graph Choices**

### Students’ Expectations for Graphs

When interpreting dynamic situations, students might wonder whether a linear or nonlinear graph could represent a relationship between attributes in a dynamic situation. Carlson et al. (2002) investigated calculus students’ covariational reasoning on a five-item assessment that tasked students with sketching graphs of dynamic situations, such as a bottle filling with water. One student in their study sketched an appropriate nonlinear graph, and gave this reason for their sketch: “I just know that it must be smooth because this is what these graphs always look like, not disconnected line segments” (p. 364). This response suggested that the student had an expectation about what the graph *should* look like (Johnson et al., 2020), and this opinion influenced their graph sketching. For example, if students were working on a unit about linear function, they might expect a linear graph to represent relationships between attributes in a dynamic situation. Hence, students’ expectations about linear and nonlinear graphs could result from experiences that include only certain kinds of Cartesian graphs.

Graphing conventions (Moore et al., 2014) can influence students' expectations of graph features. For example, if students only encounter graphs that have time as an independent variable on the x-axis, they may think that time is a variable that is always plotted on the x-axis. This can result in students introducing "time" in dynamic situations even when it is only an implicit variable (Kertil et al., 2019; Patterson & McGraw, 2018; Yemen-Karpuzcu et al., 2017). For example, a student aiming to sketch a graph of a relationship between two attributes that vary in their direction of change may instead sketch a graph that has one of these attributes on the y-axis and continuing time on the x-axis. Furthermore, students may exclusively associate dynamic situations involving an object moving at a constant speed with linear graphs and dynamic situations involving an object moving at a non-constant speed with nonlinear graphs (Patterson & McGraw, 2018; Stavley & Vidakovic, 2015). This can be problematic for students' interpretations of dynamic situations that involve linear relationships between attributes of an object moving at a non-constant speed, and vice versa. Hence, students' previous experiences with Cartesian graphs can play a role in their expectations about observable features of graphs.

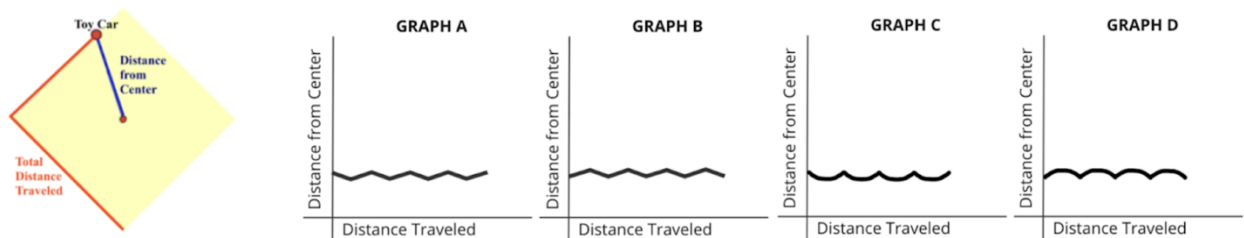
### Methods

This collective case study (Stake, 2005) is part of a larger, National Science Foundation funded project designed to promote mathematical reasoning and instructional transformation in college algebra. This study is a secondary analysis of a larger interview-based validation study (n=31 students) of a fully online assessment, the MGSRDS, designed to measure students' graph reasoning and selection for dynamic situations (Johnson et al., 2021).

#### The MGSRDS

The MGSRDS assessment consists of six items, each appearing in random order. Each item has four parts, split into two screens. On Screen 1, students are to view a short animation of a dynamic situation involving changing attributes. Then, students are to confirm whether they understood the situation. On Screen 2, students are to select a graph, from among four possible graph choices, to best represent a relationship between attributes in the dynamic situation. Then, students are to explain why they chose the graph that they did.

The six items included a cart on a turning Ferris wheel (Ferris Wheel), a person walking forward and back along a path (Nat & Tree), a fishbowl filling with water (Fishbowl), a toy car moving along a square track (Toy Car), a cone changing in size (Changing Cone), and two insects walking back and forth along a path (Ant & Ladybug). Figure 3 contains still images of the Toy Car item, along with the four graphs choices.



**Figure 3: Toy Car Dynamic Situation with Graph Choices**

The first five items contain one attribute that varied in its direction of change and one attribute that did not. For example, in the Toy Car item, the toy car's distance traveled always increases, while the toy car's distance from the center increases and decreases. Graph choices for

the first five items include two piecewise-linear options and two nonlinear options. Each piecewise-linear graph choice has a nonlinear graph choice that represents the same gross covariation in attributes. For example, Graphs B and D for the Ferris Wheel item (see Figure 2) represent the cart's height increasing, then decreasing, then increasing again, while the distance traveled continues to increase. For the Ferris Wheel item, Graph D is correct because it represents the correct direction of change in attributes and it represents the correct (nonlinear) relationship between attributes (i.e., an amount of change in distance traveled does not equal the same amount of change in height). Graph B is partially correct in that it represents the correct direction of change in attributes, but the incorrect relationship between attributes.

The sixth item, Ant & Ladybug, is different from the other five items in two main ways. First, both attributes in the dynamic situation vary in their direction of change (the Ant's distance from home increases and decreases while the Ladybug's distance from home decreases and increases). Second, the four graph choices do not include two piecewise-linear options and two nonlinear options. There is one graph that is correct, and no graph that is partially correct. For this study, we focus only on the first five items, which included both piecewise-linear options and nonlinear options for the graph choices.

### **Data Collection and Participants**

As part of the MGRSRS validation study, Johnson conducted 31 individual interviews with college algebra students, via Zoom video conference. Students volunteered to participate in the interviews, and they received a gift card for their participation. A graduate research assistant (GRA) attended each interview, to observe and take field notes. After each interview, a GRA created a verbatim transcript.

During the interviews, we noticed that there were students who spontaneously wondered about whether they should select a piecewise-linear or nonlinear graph. Thus, we used purposeful sampling (Gliner et al., 2017) to determine participants. Looking at each of the 31 transcripts, we did a keyword search, using the search terms "linear," "curv," and "straight." From this pool, we identified students who wondered about linearity/nonlinearity across five MGRSRS items (Ferris Wheel, Nat & Tree, Fishbowl, Changing Cone, Toy Car). There were six such students. We selected two students, Emma and Maya (pseudonyms), as cases for this study, because they demonstrated evidence of covariational reasoning across the majority of items, in addition to wondering about whether to select a piecewise-linear or nonlinear graph.

### **Data Analysis**

We started by watching Emma and Maya's video recordings and reading their transcript. Then we analyzed in terms of specific and generic properties, to better understand the phenomenon within and across cases (Stake, 2005). In terms of specific properties, we identified and documented any evidence of four forms of students' graph reasoning (Johnson et al., 2020). In terms of generic properties, we noted any excerpts when students spoke about linear and/or nonlinear graphs. To gain a better overall understanding of each individual case, Knurek created a table describing identified instances of Emma and Maya's graph reasoning and selection, in chronological order, for each of the five MGRSRS items (Ferris Wheel, Nat & Tree, Fishbowl, Changing Cone, Toy Car). Then, Knurek looked across these instances to make inferences about each students' graph reasoning and selection process for each item. After creating these tables, Johnson and Knurek met to vet inferences and come to a consensus.

### **Results**

We report on Emma and Maya's work on two MGRSRS items, Ferris Wheel and Toy Car. Their reasoning on these items is representative of the broader set of items. These items include

conventional (Toy Car) and unconventional graphs (Ferris Wheel). Furthermore, these items include graphs having physical features that share resemblances with (Ferris Wheel) and differences from (Toy Car) the dynamic situation. The toy car's track is composed of straight lines, but the correct graph is nonlinear. In contrast, the Ferris wheel is rounded and the correct graph is curved. We organize this section first by each case, outlining how Emma and Maya's graph reasoning relates to their graph selection on each of the two MGSRDS items. Then, we present results from our cross case analysis.

### **Expecting Graph Features Before Seeing Graph Choices: Emma**

**The Ferris Wheel Item.** On Screen 1, Emma watched the video animation and made a connection to her work in her college algebra class:

Emma (Screen 1): I believe we had something like this in my algebra class, and it says, we'll focus on the Ferris wheel cart's height from the ground in the total distance traveled. So, we're going to be analyzing a graph and it's not going to be a straight line due to the increasing distance it gets from the ground and then how it gets closer to the ground. But no matter what, it's always going to be increasing in the total distance traveled.

Before even seeing any of the graph choices, Emma spontaneously shared her expectation of the graph being nonlinear, like the graph that she had seen in her algebra class. Her response on Screen 1 shows evidence of gross covariational reasoning (Thompson & Carlson, 2017) because she talked about changes in the Ferris wheel cart's height from the ground and total distance traveled. When she moved on to Screen 2, Emma looked at the four graph choices and immediately eliminated both piecewise-linear graph choices:

Emma (Screen 2): I would automatically eliminate A and B, because the Ferris wheel doesn't seem to move in a rigidity pattern like that. It is making a circle. So it'd be a curved line. And I'm choosing graph D, because it starts off by going forward, instead of going backwards when you're going, um, vertical. And to me, that just makes more sense because it starts by going up and then it goes down, and then it goes back up.

Emma's justification for eliminating the two piecewise-linear graphs ("the Ferris wheel doesn't seem to move in a rigidity pattern like that") shows evidence of motion reasoning, per the framework from Johnson et al. (2020). When distinguishing between the two remaining graphs, Emma appealed to the direction of change in the height of the Ferris wheel cart, which resulted in her selection of a single graph. In summary, Emma's graph selection process for the Ferris wheel item involved her gross covariational and motion reasoning, along with her prior expectation of the graph being nonlinear.

**The Toy Car Item.** On Screen 1, Emma demonstrated evidence of gross covariational reasoning, similar to how she did on the Ferris wheel item. Again, Emma spontaneously shared her expectation of the graph, saying, "we're going to look at more of, like, a zigzagged line." Upon moving to Screen 2, Emma eliminated two graphs. However, unlike the Ferris wheel item, the two graphs she eliminated represented the same direction of change in attributes. This left her with two graph choices, one piecewise-linear and one nonlinear (Graphs A and C, see Figure 3). To distinguish between these graphs, Emma said, "I'm going to pick Graph A, because to me, it just makes more sense to move on a straight line when we're talking about a square." This response shows evidence of motion and iconic reasoning, per the framework from Johnson et al. (2020), because she talks about motion of the car and the physical shape of the track. Hence,

Emma's graph selection process for the Toy Car item involved her gross covariational, motion, and iconic reasoning, along with her prior expectation of the graph being piecewise-linear.

### **Determining Graph Features After Seeing Graph Choices: Maya**

**The Ferris Wheel Item.** On Screen 1, Maya watched the video animation, described the situation while showing evidence of gross covariational reasoning, and moved on to Screen 2. Upon looking at the four graphs on Screen 2, Maya eliminated two graphs that she didn't think represented the correct direction of change in attributes:

Maya (Screen 2): I'm going to eliminate Graphs A and C because if you look at how the height is going it starts off decreasing but, according to the animation, the Ferris wheel started from the midpoint, the height of the cart was at the midpoint and when it started moving, its increased and didn't decrease so, in Graphs A and C show that the cart starts at a decreasing height so those two are out.

Maya's response shows evidence of gross variational reasoning because she talked about changes in the Ferris wheel cart's height. After eliminating these graphs, Maya spontaneously brought up a challenge with distinguishing between "pointed" and "curved" graphs:

Maya (Screen 2): To be honest, I don't really know the differences between the graphs that are pointed and, and the graphs that are more curved. So, I'm just, I don't know, I'm going to pick Graph D. And this, this isn't because of a really educated guess but since we're dealing with a circular kind of model, I'm assuming that the graph will also look that way.

Maya's response shows evidence of iconic reasoning in her justification for picking a nonlinear graph ("we're dealing with a circular kind of model"). In summary, Maya's graph selection process for the Ferris wheel item involved her covariational, variational, and iconic reasoning.

**The Toy Car Item.** On Screen 1, Maya again demonstrated gross covariational reasoning after watching the video animation, like she did for the Ferris Wheel item. Upon moving to Screen 2, Maya eliminated two graphs. However, unlike the Ferris wheel item, she eliminated both nonlinear graphs. Her justification showed evidence of motion reasoning: "I feel like the toy car, its movement was just, I don't know, linear-like." When distinguishing between the two remaining graphs, Maya appealed to the direction of change in the toy car's distance from the center of the track, saying "the toy car started at an edge and then moved closer to the midpoints of that first edge, meaning that the distance started off with decreasing." Hence, Maya's graph selection process for the Toy Car item involved her gross covariational and motion reasoning.

### **Cross Case Analysis: Graph Selection and Reasoning**

**Physical Graph Features vs. Direction of Change.** The order in which Emma and Maya eliminated graphs reversed between the MGRS items. On the Ferris Wheel item, Emma eliminated two graphs based on linearity, then selected a single graph based on the direction of change in attributes. In contrast, on the Toy Car item, Emma eliminated two graphs based on the direction of change in attributes, then selected a single graph based on linearity. Maya's graph selection followed a similar process, but the order was switched (i.e., direction of change before linearity on the Ferris Wheel item, and linearity before direction of change on the Toy Car item). Notably, Emma and Maya's conceptions of the direction of change in attributes and their expectations for the graph features informed their graph selection. While this was sufficient to select a correct graph for the Ferris Wheel item, it was insufficient to do so for the Toy Car item.

**Physical Features of Situation vs. Physical Features of Graphs.** Emma and Maya both selected the correct graph for the Ferris Wheel item, and the partially correct graph for the Toy Car item. They both chose the same graphs on both items, despite differences in the order of their graph reasoning. Notably, the physical features of the Ferris wheel situation (curved wheel) resembled physical features of the correct graph (curved).

### **Discussion and Conclusion**

We investigated how students' graph reasoning related to their graph selection on the MGSRDS items that included piecewise linear and nonlinear graph choices. Our qualitative analysis revealed two approaches through which students' graph reasoning intertwined with their graph selection. The first approach was to narrow down graph choices to two graphs based on physical features (i.e., both piecewise-linear or both nonlinear), then select a single graph via gross covariational reasoning. The second approach was to narrow down graph choices to two graphs based on the direction of change in attributes (i.e., one piecewise-linear and one nonlinear), then select a single graph via motion and/or iconic reasoning. Notably, Emma and Maya used both approaches. Furthermore, when physical features of the dynamic situation resembled physical features of the graph, either approach resulted in the correct graph choice.

Our findings point to affordances and constraints in students' graph selection when students engage in gross covariational reasoning (Thompson & Carlson, 2017). An affordance is to interpret and select graphs based on direction of change in attributes. Both Emma and Maya selected graphs that represented gross covariation in attributes for the Ferris Wheel and Toy Car items. However, to distinguish between two different looking graphs that both represented the same gross covariation in attributes (e.g., Graphs A and C, see Figure 3), gross covariational reasoning alone was insufficient. Because these interviews were part of the MGSRDS validation, Johnson did not engage in exploratory teaching to investigate whether students might extend beyond gross covariational reasoning. Students' questions about graph curvature could prompt their investigation of more nuanced relationships between quantities (Johnson et al., 2017). In future studies, researchers could investigate conditions under which students might shift beyond gross covariational reasoning, to more advanced forms of covariational reasoning.

The framework from Johnson et al. (2020) distinguishes between motion/iconic reasoning and variational/covariational reasoning. This distinction is compatible with Moore and Thompson's (2015) distinction between static and emergent shape thinking. Static shape thinking involves conceiving of graphs as physical objects, while emergent shape thinking involves conceiving of graphs as in-progress traces of relationships between covarying quantities. Our results suggest that students use both physical feature-based (e.g., motion, iconic) and quantitative-based (e.g., variation, covariation) reasoning to select graphs representing relationships between attributes in dynamic situations. In future studies of students' graphing, researchers can examine how students' static and emergent shape thinking may intertwine.

Our findings pointed to the viability of including unconventional graphs (e.g., Moore et al., 2014) on assessment items. When Maya and Emma encountered unconventional graphs (e.g., on the Ferris wheel item), they engaged in gross covariational reasoning to make sense of relationships between attributes. This supported an earlier claim from Moore et al. (2014), that breaking conventions (e.g., On a Cartesian graph, a variable varying consistent with elapsing time is to be represented on the horizontal axis.) could promote students' quantitative reasoning.

In conclusion, it is valuable for students to question whether a graph should be linear or curved. While we are encouraged that Emma and Maya spontaneously raised this question during an interview, we contend that the organization of function topics in many U.S. College

Algebra textbooks (e.g., Sullivan, 2020) can limit such questioning. If students only encounter particular function types in separate units, there is little room for such exploration.

**Acknowledgement.** This research was supported by U.S. National Science Foundation award 2013186. Opinions, findings, and conclusions are those of the authors.

### References

- Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education*, 33(5), 352-378. <https://doi.org/10.2307/4149958>
- Chen, X. (2016). Remedial Coursetaking at US Public 2-and 4-Year Institutions: Scope, Experiences, and Outcomes. Statistical Analysis Report. NCEES 2016-405. National Center for Education Statistics.
- Clement, J. (1989). The concept of variation and misconceptions in cartesian graphing. *Focus on Learning Problems in Mathematics*, 11(1-2), 77-87.
- Gliner, J. A., Morgan, G. A., & Leech, N. L. (2017). *Research methods in applied settings: An integrated approach to design and analysis* (3rd ed.). Routledge.
- Gordon, S. P. (2008). What's wrong with college algebra? *PRIMUS*, 18(6), 516-541. <https://doi.org/10.1080/10511970701598752>
- Johnson, H. L., McClintock, E., & Hornbein, P. (2017). Ferris wheels and filling bottles: a case of a student's transfer of covariational reasoning across tasks with different backgrounds and features. *ZDM: The International Journal on Mathematics Education*, 49(6), 851-864. <https://doi.org/10.1007/s11858-017-0866-4>
- Johnson, H. L., McClintock, E., & Gardner, A. (2020). Opportunities for reasoning: Digital task design to promote students' conceptions of graphs as representing relationships between quantities. *Digital Experiences in Mathematics Education*, 6(3), 340-366. <https://doi.org/10.1007/s40751-020-00061-9>
- Johnson, H. L., Olson, G., Smith, A., Gardner, A., Wang, X., & Donovan, C. (2021). Validating an assessment of students' covariational reasoning. In Olanoff, D., Smith, K., and Spitzer, S (Eds.), *Proceedings of the 43rd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 63-67). Philadelphia, PA.
- Kerslake, D. (1977). The understanding of graphs. *Mathematics in School*, 6(2), 22-25.
- Kertil, M., Erbas, A. K., & Cetinkaya, B. (2019). Developing prospective teachers' covariational reasoning through a model development sequence. *Mathematical Thinking and Learning*, 21(3), 207-233. <https://doi.org/10.1080/10986065.2019.1576001>
- Leinhardt, G., Zaslavsky, O., & Stein, M. K. (1990). Functions, graphs, and graphing: Tasks, learning, and teaching. *Review of Educational Research*, 60(1), 1-64. <https://doi.org/10.3102/00346543060001001>
- Mesa, V., Suh, H., Blake, T., & Whittemore, T. (2012). Examples in college algebra textbooks: Opportunities for students' learning. *PRIMUS*, 23(1), 76-105. <https://doi.org/10.1080/10511970.2012.667515>
- Moore, K. C., Silverman, J., Paoletti, T., & LaForest, K. (2014). Breaking conventions to support quantitative reasoning. *Mathematics Teacher Educator*, 2(2), 141-157. <https://doi.org/10.5951/mathteduc.2.2.0141>
- Moore, K. C., & Thompson, P. W. (2015). Shape thinking and students' graphing activity. *Proceedings of the 18th Meeting of the MAA Special Interest Group on Research in Undergraduate Mathematics Education*, 782-789.
- Paoletti, T., & Vishnubhotla, M. (2022). Constructing covariational relationships and distinguishing nonlinear and linear relationships. In Karagöz Akar, G., Zembat, İ.Ö., Arslan, S., Thompson, P.W. (eds) *Quantitative Reasoning in Mathematics and Science Education. Mathematics Education in the Digital Era*, vol 21. Springer, Cham. [https://doi.org/10.1007/978-3-031-14553-7\\_6](https://doi.org/10.1007/978-3-031-14553-7_6)
- Patterson, C. L., & McGraw, R. (2018). When time is an implicit variable: An investigation of students' ways of understanding graphing tasks. *Mathematical Thinking and Learning*, 20(4), 295-323. <https://doi.org/10.1080/10986065.2018.1509421>
- Stake, R. E. (2005). Qualitative case studies. In N. K. Denzin & Y. S. Lincoln (Eds.), *The Sage handbook of qualitative research* (3rd ed., pp. 443-466). Sage.
- Stalvey, H. E., & Vidakovic, D. (2015). Students' reasoning about relationships between variables in a real-world problem. *The Journal of Mathematical Behavior*, 40, 192-210. <https://doi.org/10.1016/j.jmathb.2015.08.002>
- Sullivan, M., III. (2020). *Algebra & Trigonometry Enhanced with Graphing Utilities* (8th ed.). Pearson Education, Inc.
- Thompson, P. W. (1994). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 181-234). Albany, NY: State University of New York Press.

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

- Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In L. L. Hatfield, S. Chamberlain & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education. WISDOMe Monographs* (Vol. 1, pp. 33-57). Laramie, WY: University of Wyoming.
- Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 421-456). Reston, VA: National Council of Teachers of Mathematics.
- Yemen-Karpuzcu, S., Ulusoy, F., & İşksal-Bostan, M. (2017). Prospective middle school mathematics teachers' covariational reasoning for interpreting dynamic events during peer interactions. *International Journal of Science and Mathematics Education*, 15, 89-108. <https://doi.org/10.1007/s10763-015-9668-8>



## PRE-SERVICE TEACHERS' ALGEBRAIC THINKING STRATEGIES TO SUPPORT THE MEANING OF THE EQUAL SIGN

Alysia Goyer  
Stockton University  
alysia.goyer@stockton.edu

Teruni Lamberg  
University of Nevada, Reno  
terunil@unr.edu

*Pre-service teachers often rely on their formalized algebra skills to make sense of early algebraic concepts, including the meaning of the equal sign. This study documents pre-service students' conceptions of early algebra as it relates to the equal sign. Specifically, we document how preservice teachers thought about the meaning of the equal sign and the strategies they used to solve arithmetic problems that can be solved using relational thinking.*

Keywords: Algebra and Algebraic Thinking, Preservice Teacher Education,

In pre-service methods courses, the knowledge development of Pedagogical Content Knowledge (Shulman, 1986) is crucial for pre-service teachers to apply teaching strategies to their content knowledge. It can also be said that pre-service teachers need to develop a more nuanced approach which can be viewed in the six domains of Mathematical Knowledge for Teaching (Ball et al., 2008). This is true particularly for algebraic thinking, which is foundational for the developing numeric fluency and flexibility, noticing patterns and generalizations, creating models, and connecting mathematics representations (Blanton & Kaput, 2005; Kaput, 2008; Kieran et al., 2016; Kieran, 2018). One example of the specialized knowledge that pre-service teachers needs to develop is distinguishing between arithmetic and algebraic meanings of the equal sign (Carpenter et al., 2000; Carpenter et al., 2003). This study investigated: What algebraic strategies do pre-service teachers use when asked to solve two-digit and subtraction problems that require relational thinking.

### Review of Literature

Algebraic thinking is a fundamental component in K-16 education. Kaput (2008) defines algebraic thinking in three strands: Early Algebra, Formal Algebra, and Modeling of Algebra. Early Algebraic thinking is defined as generalized arithmetic and focuses on the noticing of patterns and generalizations regarding the overall numeric structure (Blanton, 2008; Kieran et al. 2016; Kieran, 2018). Formal Algebra is building towards the generalization of a function utilizing variables and unknowns. In this context, algebra can be viewed as expressions in the form of patterns that are generalized or equations where the properties of equality are used for solving. It is important for pre-service teachers to distinguish between these two strands, because there is the misconception that doing Early Algebra is the same as doing Algebra Early (Carraher & Schliemann, 2018). Currently, there is little research on pre-service teachers' development of Mathematical Knowledge of Teaching for algebraic thinking (Hohensee, 2017). Suppa & Hohensee (2021) state that PST's have difficulty breaking the "apprenticeship of observation" while in their methods coursework. Specifically, they state PST's struggle with multiple strategy solutions and teaching in a different way than they were taught. Often when pre-service teachers approach ideas of early algebra, they rely on formalized algebraic techniques which are the most familiar as an entry point into building their understanding of early algebra.

In the Mathematical Knowledge of Teaching (Ball et al., 2008) is developing specialized content knowledge. One area of specialized content knowledge pre-service teachers must

develop is in differentiating between the two meanings of the equal sign (Carpenter et al., 2000; 2003). The first meaning is an arithmetic operation denoting to find the solution. The second meaning is an algebraic way of thinking about the equal sign, meaning a relationship of equality between each side, or relational thinking (Carpenter et al. 2000; 2003; 2005; 2015; Kieran et al., 2016).

Example:

Arithmetic:  $5 + 7 = \square$

Algebraic:  $5 + 7 = \square + 8$

The difference between these two equations is that in the arithmetic equation the equal sign signifies the ending of the addition sentence and works as an indicator for the answer of 12. In the algebraic equation, there is a relationship between the two sides that must be balanced or equal.

This study builds on the research done by Hohensee (2017) that found the pre-service teachers were able to distinguish between the differences in arithmetic and algebraic meaning of the equal sign by providing insight into the strategies pre-service teachers use to think relationally about the equal sign.

### Methodology

The research study took place in an undergraduate pre-service elementary mathematics methods course (n=19) with a content focus in algebraic thinking lesson planning with the use of technology. This study focuses on one instructional activity with a specified learning objective of supporting generalizations and algebraic meaning of the equal sign. As part of the whole study, qualitative data collection techniques were used including field notes, audio transcripts, video, and participant work were collected and used to provide and support triangulation (Saldaña & Omasta, 2018). The participant work that was collected and analyzed was a handout of three two-digit addition and three two-digit subtraction problems that used relational thinking to promote the algebraic meaning of the equal sign.

Example:  $10 + \quad = 11 + 15$       and       $73 - 27 = 80 -$

For these problems, participants were asked to use metacognitive strategies and were instructed to record their thinking and explain their solution strategies as they worked through each problem (Schoenbach et al. 2012). After participants worked through the worksheet independently, participants were asked to discuss their strategies in small groups, and then as a whole class.

### Data Analysis

During the whole class discussion, three strategies were shared (See Table 1), serving as the basis for the coding scheme of participant work. Participant work was then coded deductively and analyzed (Saldaña & Omasta, 2018). Upon further examination of the participant explanations, the researchers were able to construct a visualization of the direction of the relational thinking taking place (See Table 1). Upon further analysis, it was also determined that

some participants used more than one strategy, so primary and secondary strategies were coded as well (See Table 2).

**Table 1: Description of Pre-Service Teachers' Algebraic Thinking Strategies**

Description of Pre-Service Teachers' Directional Algebraic Thinking	Example of Pre-Service Teacher Explanation	Relational Thinking Across the Equal Sign
Single-Direction Formal Algebraic Thinking	<p>“ I would first add 11 +15 (combine like terms) insert a variable into blank space so I can visually see what I needed to solve for”</p> <p>“Added 11+ 15, then subtracted 10 from 26”</p>	$10 + \underline{\quad} = 11 + 15$ $10 + x = 26 \quad \text{or} \quad 10 + \underline{\quad} = 26$ $x = 16 \quad \text{or} \quad \underline{\quad} = 16$
Single-Direction Early Algebraic Thinking	<p>“ Add one side together first then think of 10 plus what equal that”</p>	$10 + \underline{\quad} = 11 + 15$ $10 + \underline{\quad} = 26$ $10 + 16 = 26$
Bi-Directional Early Algebraic Thinking	<p>“10 + 1 = 11, 15 + 1 = 16”</p> <p>So answer is 16”</p>	$10 + \underline{\quad} = 11 + 15$ $10 + \underline{\quad} = 10 + 1 + 15$ $10 + 16 = 11 + 15$

**Table 2: Description of Strategy Types**

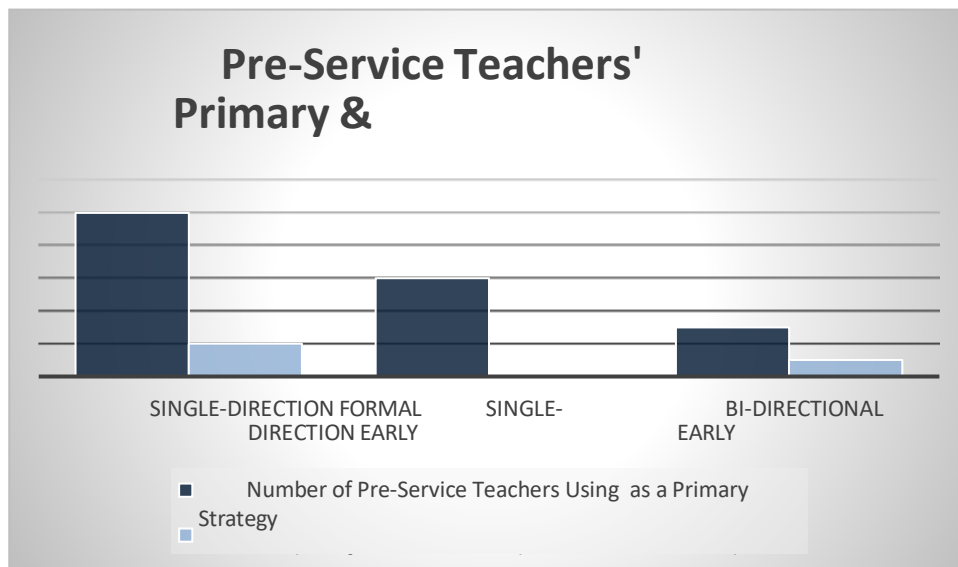
Strategy Type	Description
Primary Strategy	Strategy was used in majority of problem types
Secondary Strategy	Strategy was used only once or twice

### Results

The data highlights the strategies that pre-service teachers will utilize when encountering a relational use of the equal sign (See Figure 1). The primary strategy used by participants (52.6%) was Single Direction Formal Algebraic Thinking, in which participants combined the quantity on one side and then applied formal algebraic properties of equality in order to solve.

The second most used strategy was Single Direction Early Algebraic Thinking (31.6%), where participants combined the quantity on one side, but related the number statement to the quantity. This strategy utilizes Early Algebra in applying addition or subtraction problems where the start, addend, or difference are unknown. Finally, the least utilized strategy was Bi-Directional Early Algebraic Thinking (15.8%), where participants could relate more than one relation across the quantities across the equal sign, creating a bi-directional relational strategy.

In the analysis of primary and secondary strategies it was found that only 15.8% of participants utilized a secondary strategy (See Figure 1). This suggests that for the majority of participants (84.2%), when they found a strategy that worked, they will continue to utilize the same strategy.



**Figure 1: Pre-Service Teachers Primary & Secondary Strategies Discussion**

This study provides insight into how pre-service teachers think relationally when encountering the algebraic meaning of the equal sign. The results are consistent with Hohensee (2017) in that pre-service teachers will rely on traditional, formalized algebra techniques to communicate early algebra concepts. This highlights the importance of making early algebra, as generalized arithmetic, generalizations, and number fluency explicit in pre-service content and methods courses if shifts in pre-service teachers' thinking is to be made. More research is needed to determine ways in which to support pre-service teachers' shift from applying their content knowledge of formalized algebra to specialized content knowledge to support their future students' learning of early algebraic ideas.

### References

- Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special?. Blanton, M. L. (2008). Algebra and the elementary classroom: Transforming thinking, transforming practice. Greenwood International.
- Blanton, M. L., & Kaput, J. J. (2005). Characterizing a classroom practice that promotes algebraic reasoning. *Journal for research in mathematics education*, 36(5), 412-446.
- Carpenter, T. P., Fennema, E., Franke, M. L., Levi, L., & Empson, S. B. (2000). *Cognitively Guided Instruction: A Research-Based Teacher Professional Development Program for Elementary School Mathematics*. Research Report.

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

- Carpenter, T. P., Franke, M. L., & Levi, L. (2003). *Thinking mathematically*. Portsmouth, NH: Heinemann.
- Carpenter, T. P., Levi, L., Franke, M. L., & Zeringue, J. K. (2005). Algebra in elementary school: Developing relational thinking. *Zentralblatt für Didaktik der Mathematik*, 37(1), 53-59.
- Carpenter, T. P., Fennema, E., Franke, M. L., Levi, L., & Empson, S. B. (2015). *Children's mathematics*, second edition: Cognitively guided instruction. Portsmouth, NH: Heinemann.
- Carraher, D. W., & Schliemann, A. D. (2018). Cultivating early algebraic thinking. In *Teaching and Learning Algebraic Thinking with 5-to 12-Year-Olds* (pp. 107-138). Springer, Cham.
- Hohensee, C. (2017). Preparing elementary prospective teachers to teach early algebra. *Journal of Mathematics Teacher Education*, 20(3), 231-257.
- Kaput, J. (2008). What is algebra? What is algebraic reasoning?. *Algebra in the early grades*, 2008, 5-17.
- Kieran, C. (2018). The early learning of algebra: A structural perspective. In *Research issues in the learning and teaching of algebra* (pp. 33-56). Routledge. (Original work published 1989).
- Kieran, C., Pang, J., Schifter, D., & Ng, S. F. (2016). *Early algebra: Research into its nature, its learning, its teaching*. Springer Nature.
- Saldaña, J. & Omasta, M. (2018). *Qualitative research: Analyzing life*. Thousand Oaks, CA: SAGE Publications.
- Schoenbach, R., Greenleaf, C., & Murphy, L. (2012). *Reading for understanding: How reading apprenticeship improves disciplinary learning in secondary and college classrooms*. John Wiley & Sons.
- Suppa, S., & Hohensee, C. (2021). Struggles Pre-Service Teachers Experience When Taking a Pre-Symbolic Algebra Content Course. *Mathematics Teacher Education & Development*, 23(4).

## SHAPE THINKING AND STUDENTS' ACTIVITY WITH SIMULATIONS AND TABLES

Toni York  
Montclair State University  
yorka1@montclair.edu

Nicole Panorkou  
Montclair State University  
panorkoun@montclair.edu

*The construct of static and emergent shape thinking (Moore & Thompson, 2015) characterizes differences in students' reasoning about graphs. In our previous work with middle school students, we found that this construct may also be useful in characterizing students' reasoning about other representations such as simulations and tables. In this paper, we present data from six students' reasoning to initiate a discussion around the possible nature of static and emergent shape thinking in the context of simulations and tables that would contribute to an expansion of the current framework to include these representations.*

Keywords: Design Experiments, Mathematical Representations, Middle School Education, Technology

Covariational reasoning involves the coordination of simultaneous changes in two related quantities that are varying together (Thompson & Carlson, 2017). A quantity is defined as “someone’s conceptualization of an object such that it has an attribute that could be measured” (Thompson & Carlson, 2017, p. 425). These quantities need not be numerical. For example, a student may reason covariationally only about the direction of change (Carlson et al., 2002), such as an observation that “as  $X$  increases,  $Y$  decreases.” Many researchers have also characterized students’ forms of covariational reasoning in different ways. For instance, reasoning in which a student shows a chunky rather than a smooth image of change (Castillo-Garsow et al., 2013). These forms of reasoning are crucial for students’ development of several mathematical concepts, including the understanding of functions (Thompson & Carlson, 2017). In addition to mathematics, students’ covariational reasoning has also been studied in science education since relationships between two or more covarying quantities are also prevalent in the study of many science topics. For example, we have found that middle school students can reason covariationally in the context of various scientific phenomena such as gravity (Basu et al., 2020; Panorkou & Germia, 2021), weather (York et al., 2021), and the greenhouse effect (Basu & Panorkou, 2019).

To support students’ covariational reasoning in these scientific contexts, we iteratively developed and tested module designs that integrate mathematics and science content through interactive simulations as well as accompanying graphing tasks and questioning (Panorkou & York, 2020). As we examined the data from these design experiment iterations, one of the frameworks we used to characterize students’ reasoning in the graphing tasks was Moore and Thompson’s (2015) idea of static versus emergent shape thinking. A student engages in static shape thinking when they reason about a graph as an object in and of itself, such as a wire or a hill whose properties are not connected to an underlying covariational meaning. In contrast, emergent shape thinking involves reasoning about a graph as a meaningful record or emerging trace of the relationship between covarying quantities. Furthermore, emergent reasoning also includes recognizing the same underlying mathematical structures across different graphical representations (Moore, 2021) such as different coordinate systems (Paoletti et al., 2018).

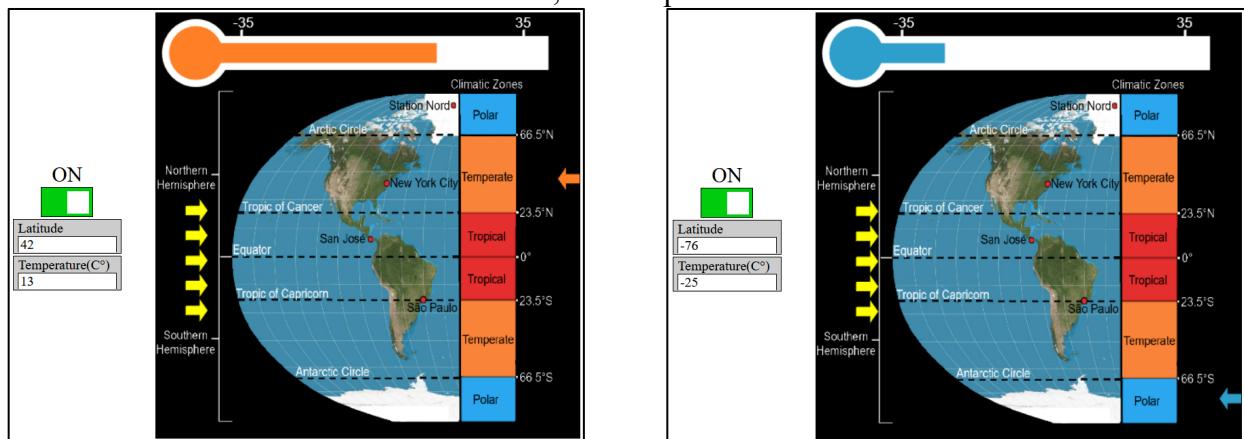
While Moore and Thompson (2015) used the *shape thinking* construct to characterize students' thinking in graphs, as we analyzed our data, we noticed that this construct also seemed relevant to describe students' reasoning in other contexts. Specifically, in our previous work with our Climate Module (Germia et al., 2022; Panorkou et al., 2022), we used the distinction between static and emergent shape thinking to show progressions in two pairs of students' reasoning as they transitioned between working with different artifacts including a simulation, data tables, and graphs. In doing so, we found that these students reasoned both about the simulation and tables as well as the graphs in ways that could be characterized as static or emergent. We therefore conjectured that this framework might be expanded to include these other types of artifacts.

To explore this possible expansion of the framework of static and emergent shape thinking, we conducted a further analysis of our Climate Module data. Specifically, we explored the research question: *What forms of reasoning that can be characterized as shape thinking do students exhibit as they interact with different artifacts?*

### Methods

To explore this research question, we analyzed data collected from the first iteration of a whole-class design experiment (DE) (Cobb et al., 2003) conducted in a sixth-grade classroom in the Northeast of the U.S. The DE focused on an instructional module we designed that aimed to support students in reasoning covariationally about the relationship between latitude and temperature in the Earth's different climatic zones.

First, students were asked to explore the Climatic Zones simulation (Figure 1). The simulation shows how latitude and temperature covary by displaying these quantities in visual and textual readouts. These readouts change as the student moves the arrow on the right side of the screen to different locations with their mouse. The locations are spread in the three main climatic zones: tropical, temperate, and polar. Latitude is displayed using both positive and negative numbers, where positive latitudes are located in the north and negative latitudes are located in the south. As the distance away from the Equator (the quantity measured by latitude) increases either to the north or to the south, the temperature tends to decrease.

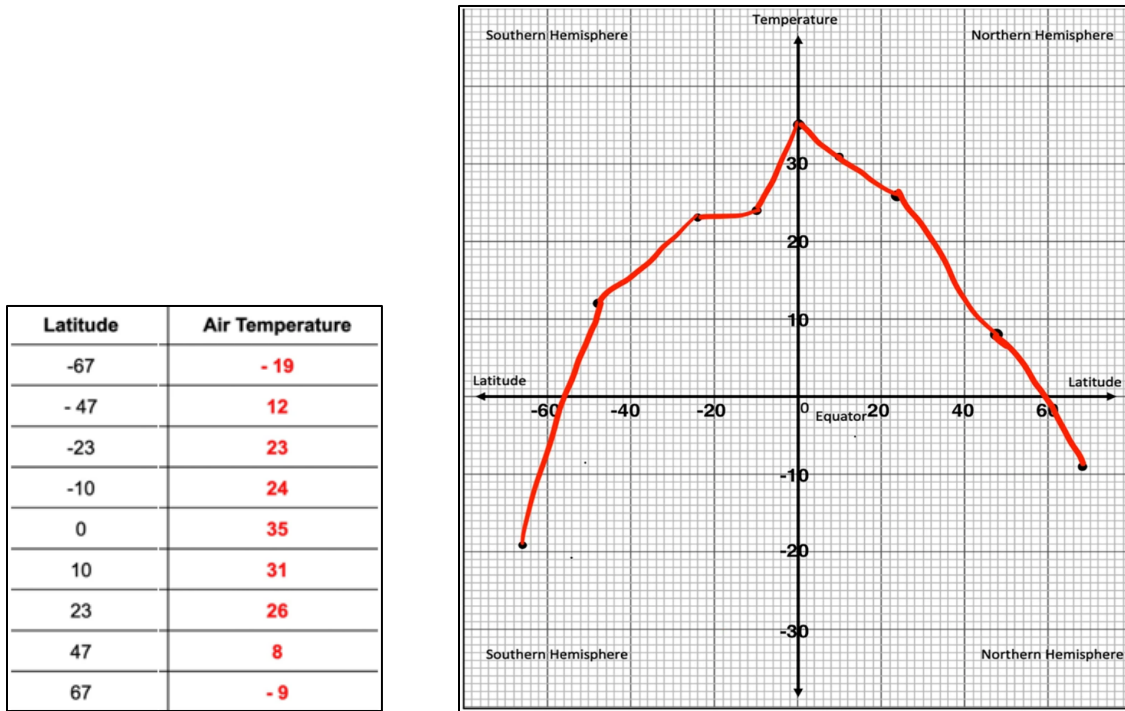


**Figure 1: The Climatic Zones Simulation**

The accompanying task involved creating tables and graphs of the latitude and temperature data found in this simulation (Figure 2). We also designed questioning to elicit students' reasoning about these quantities and the ways in which the different artifacts represent the

relationship between them. For example, we asked questions such as, “What did you notice in the simulation?”, “What patterns do you see in your table?”, and “What does your graph show?”

The class met for two 25-minute sessions via Google Classroom, during which the students were placed in pairs in breakout rooms by the teacher. Three of these pairs were selected by the teacher to be interviewed by the researchers. As each of the selected pairs worked on the module in their breakout room, a researcher joined them to observe their work and follow their reasoning. The six students’ video, text chat, and shared screens during these breakout room sessions were recorded, and these recordings were transcribed for analysis. The transcriptions include speech, gestures, and screen/mouse actions.



**Figure 2: Recreation of a Student Table and Graph**

In this paper we present the retrospective analysis of the activity of all six students: Ali, Jaden, Mikhail, Tasif, Jami, and Gaelyn. This analysis proceeded in two different stages. In the first stage, we identified and coded relevant excerpts of the transcripts in which the students reasoned about either the simulation, a table, or a graph. We use the term ‘relevant’ here to mean any reference to the features of an artifact, conversations about what an artifact shows, and any mention of the quantities or relationships represented in an artifact. As a counterexample, logistical conversations about opening the simulation or the task documents were not coded as relevant to an artifact. Then, in the second stage, we analyzed the identified excerpts again to code them as examples of the students displaying either static or emergent shape thinking.

### Findings

In this section we organize our findings by the type of artifact the students were reasoning about and describe the excerpts that we characterized as showing static or emergent shape thinking in each case. Although the students did reason both statically and emergently with their



graphs (Germia et al., 2022; Panorkou et al., 2022), we decided to omit the familiar context of graphs here for brevity and focus instead on the simulation and table.

### **Static Shape Thinking with a Simulation**

After exploring the Climatic Zones simulation freely for a few minutes, the students were asked what they had noticed. In response, Ali, Jami, and Tasif talked about the three zones. For instance, Ali responded, “The earth is divided into three different zones. The tropical zone, the temperate zone, and the polar zone.” Similarly, Jami stated, “The three major zones are polar, temperate, and tropic.” Tasif also commented that, “What I noticed in the simulation is that the arrow on the right, it points to a certain latitude. So you could change it and there are zones like polar, temperate, and tropical.” In these excerpts, all three students reasoned about visual features of the simulation, thinking of it as showing a map of the Earth and its different regions.

While Ali and Jami’s reasoning illustrated a static image of that map, Tasif was able to identify latitude as a quantity that varied. However, at this stage he did not reason about how both latitude and temperature covary. Like Tasif, Gaelyn also focused on the variation of latitude. She described the Equator as a separator between regions, saying, “From the Equator up is positive [latitude] and from the Equator down is negative [latitude].” She also explained that “the Equator is at the 0-degree [latitude] line between the northern and southern hemisphere.”

Mikhail described the simulation’s arrow and the different zones as well. However, he focused on the changing temperature in these different locations:

Mikhail: This simulation, every time you move the arrow, it shows you the polar, the temperate, or the tropical zones and the things that separate them [moving mouse to point to these different locations on the simulation]. And it tells you every time you move to a certain area, it tells you the heat.

Similar to Tasif and Gaelyn, Mikhail focused on the variation of one quantity. Likewise, Jaden observed that the Arctic and Antarctic Circles are “what separates the polar regions from the temperate regions.” He also noted changes in temperature similarly to Mikhail, stating where it was negative or positive: “In the polar regions, the temperatures are negative. And in the temperate and tropical regions the temperatures are positive.” Mikhail and Jaden’s discussions of the changing temperature note only that this quantity is different in different places, not what the relationship is between the latitudes of those places and their temperatures.

All six students illustrated a simulation-as-map reasoning, which did not involve connections to an underlying covariational meaning of these features, even when the students made note of changes in latitude or temperature as they controlled the simulation’s arrow. We therefore argue that these examples can be characterized as showing the students’ static shape thinking as they reasoned about the simulation.

### **Emergent Shape Thinking with a Simulation**

As they were asked questions about what was changing in the simulation and what patterns they saw in those changes, the students also spoke about the two varying quantities together. When Gaelyn was asked to order the cities in the simulation from warmest to coldest, she stated:

Gaelyn: Because San Jose is right in the middle of the tropical, and then São Paulo is right on the line of the tropical and the temperate, and then New York City is in the temperate, and then Station Nord is in the polar. So that would be the order of it.

Gaelyn’s statement shows that she was able to describe how temperature changes as you move from zone to zone. Jami also illustrated similar reasoning:

Jami: The polar zone is actually very cold because it is up north. And the temperate zone, since it is right between the Equator where it is most hottest and the polar zone, it is pretty warm, like the average temperature. And the tropical zones are very hot because they are more close to the Equator.

Both Gaelyn and Jami's statements may be interpreted as illustrating covariational reasoning because they reasoned that both quantities change in relation to each other. While one might describe Gaelyn's and Jami's reasoning as illustrating change in chunks (zones), Jami's last statement about tropical zones being hot because they are "more close" to the Equator shows evidence of possibly having constructed a smooth image of change.

Mikhail showed a smooth image of change in his reasoning by describing the changes in temperature as a continuous movement from zone to zone:

Mikhail: The closer you go to the tropical, the hotter it is [moving mouse slowly from the northern polar zone to the tropical zone]. The closer you go to the polar, the colder it is [moving mouse from the tropical zone to the southern polar zone]. And the temperate is more in the middle. And, and could go each way [moving mouse within the southern temperate zone].

Mikhail identified that this relationship "could go each way" although he described this continuous change as occurring zone by zone. In contrast, Jaden and Ali showed that they constructed a smooth image of change in latitude and temperature. For example, Jaden noted that "The higher the latitude the colder it is going to be." Similarly, Ali stated, "The closer you are to the Equator the hotter it is."

Of the six students, Tasif illustrated the most sophisticated form of covariational reasoning. Looking at the two hemispheres separately, he stated the relationship he saw between movement towards the Equator and an increase in temperature, saying, "In the northern hemisphere, the more south you go, [it] gets hotter. But if in the southern hemisphere, the more north you go, it'll be a little hotter until we reach 0 latitude." Furthermore, Tasif also reasoned about a pattern he noticed in the rate of this change in different zones: "When you're at the temperate zone, it slowly decreases as you can see in simulation [moving mouse within temperate zones in both hemispheres], but once you reach the polar zones [moving mouse within polar zones], it starts decreasing rapidly."

In sum, all six students made a connection between changes in latitude (distance from the Equator) and changes in temperature in the simulation. In contrast to the static shape thinking examples in the previous section, each of these examples show a student reasoning about the simulation as simultaneously representing the changes in both quantities. While students' forms of covariational reasoning might be different, they involved simulation-as-relationship thinking rather than simulation-as-map thinking. Because of this difference, we thus interpret these excerpts as showing the students' emergent shape thinking as they reasoned about the simulation.

### **Static Shape Thinking with a Table**

Only three of the six students provided data on their reasoning about their tables. Two students focused more on graphing during their interviews and no audio was recorded for one student during this part of the interview. Of the remaining three students, Ali and Mikhail both answered questions about patterns in their tables with descriptions of the numbers that they could see. Ali stated, "I think it's like the same exact numbers, just negative [moving mouse up and down the latitude column]." He then further described this visual pattern he had found in the

latitude column by adding, “So here’s 0 [placing mouse on the entry for 0 latitude]. Then you’ve got -10 and 10, 23 and -23, -47, 47, 67, and -67 [moving mouse up and down to each latitude value as he reads them out].” Similarly, Mikhail noted, “there is some type of pattern repeating. [...] when the latitude is -23, the air temperature is 23. And when the latitude was 23, it [temperature] was 26. It is kind of similar.” He later also described how he saw the temperature changing in his table by discussing these changes as a rising and falling shape.

Mikhail: The temperature, I say it would rise up from negative [latitude], I mean, it would become very low for negative [latitude] since -19 [°C]. [...] But then when it reaches -47 latitude, the temperatures start rising. And for the next, it’s like a slowly curved rising [tracing a hill-like motion with his hand]. And then it slopes down again [moves from the peak to the original starting height].

Ali’s description of the symmetry in the numbers of the latitude column and Mikhail’s description of the repeated appearance of 23 and a “kind of similar” 26 in the table are both examples of visual patterns in the appearance of the table’s numbers (Figure 3).

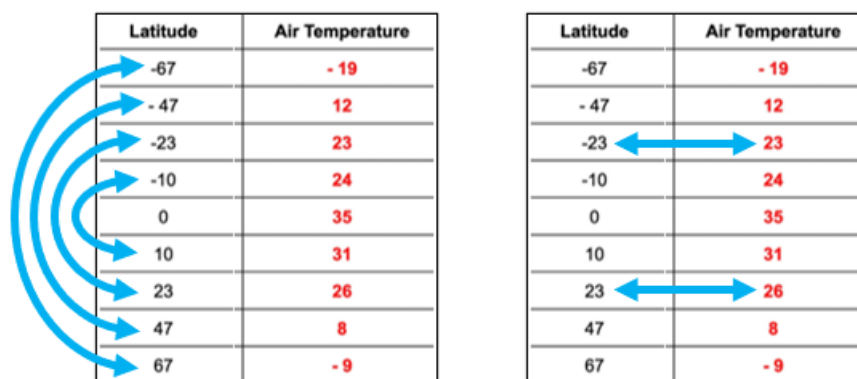


Figure 3: Ali’s (left) and Mikhail’s (right) Visual Patterns

Neither of these patterns carry any meaning associated with the covariational relationship between latitude and temperature, but are rather a result of how the data was chosen in the design of the simulation and how the given latitude values were chosen for the table creation task. Similarly, although Mikhail mentioned the temperature changing for different values of latitude in the table, he reasoned about the visual up-and-down shape of these changing values without connecting this to the underlying covariational relationship that it represents. These excerpts thus show examples of the students’ static shape thinking as they reasoned about the table.

### Emergent Shape Thinking with a Table

Similar to the section before, we focus here on the analysis of the data of only the three students. When asked what his table told him, Ali responded:

Ali: Yeah, so the Equator’s at 0 [pointing to the row with 0 latitude on table]. And then like, the temperature’s dropping faster. Like the negatives [moving mouse from 0 latitude to the top of the table] are the south of it and they’re dropping faster [moving mouse along the temperature values for the south in the top half of the table] than the north of the Equator [moving mouse along the bottom half of the table].

Ali’s reasoning shows a description of how the rate of change of the temperature was different in the southern and northern latitudes. His reasoning about the differing rate of change

of temperature across different regions of latitude implies that he saw the table as representing the simultaneous covariation of these quantities in order to make an observation about how this covariation itself varies. He also later directly stated the relationship he saw in his table between latitude and temperature, saying,

Ali: “The farther from 0 [moving mouse from 0 latitude to the bottom of the table], like, the colder. And even that, I think this [moving mouse from 0 latitude to the top of the table] is just the same.”

This later statement about how the temperature is colder the farther from 0 in both directions on the table makes his reasoning more explicit. For Ali, in these excerpts, the table records how the quantities are changing in relationship with each other.

Similarly, Tasif found a relationship between distance from the Equator and temperature in his table, noting, “the farther you go from the Equator, the colder it will be.” Mikhail then replied to Tasif’s observation by adding, “because there’s less direct sunlight [farther from the Equator].” Tasif’s statement of this relationship also shows that he imagined the table as a record of distance from the Equator (latitude) and temperature varying simultaneously. Mikhail’s reply to Tasif’s statement also implies a similar way of thinking about the table, offering a causal relationship between the three simultaneously varying quantities of latitude, amount of direct sunlight, and temperature.

Each of these excerpts involve a student reasoning about what the numbers in the table meaningfully represent in terms of the underlying covarying quantities, seeing the table as a record of a covariational relationship. We therefore interpret these excerpts to exemplify students’ emergent shape thinking as they reasoned about the table.

### Concluding Remarks

The analysis showed that as the students worked with the simulation and the table, their reasoning varied in ways that can be characterized in terms of shape thinking. We interpret their reasoning to illustrate evidence of both static and emergent thinking at different points during the DE. Specifically, all six students reasoned both statically and emergently with the simulation and two of them, Ali and Mikhail, also reasoned in both ways with the table. Table 1 presents the resulting expansion of the Moore and Thompson (2015) framework with definitions and examples of each type of shape thinking taken from the data presented in this paper.

**Table 1: Expansion of the Moore and Thompson (2015) Shape Thinking Framework**

	<b>Static Shape Thinking</b>	<b>Emergent Shape Thinking</b>
	<i>Reasoning about simulation features without connecting them to an underlying covariational meaning.</i>	<i>Reasoning about how simultaneously varying quantities in the simulation are related to each other.</i>
<b>Simulation</b>	“The earth is divided into three different zones. The tropical zone, the temperate zone, and the polar zone.”	“The closer you are to the Equator the hotter it is.”
<b>Table</b>	<i>Reasoning about a visual</i>	<i>Reasoning about numerical</i>

---

*pattern of numbers without connecting this to an underlying covariational meaning.*

“I think it’s like the same exact numbers, just negative [moving mouse up and down the latitude column].”

*patterns that describe a covariational relationship.*

“The farther from 0 [moving mouse from 0 latitude to the bottom of the table], like, the colder. And even that, I think this [moving mouse from 0 latitude to the top of the table] is just the same.”

---

Furthermore, we found that all six students recognized the same underlying covariational structure in their emergent reasoning with both the simulation and graph (graph data reported in Germia et al. (2022) and Panorkou et al. (2022)). Of the three students for whom we also have table data (Ali, Mikhail, and Tasif), all three of them also reasoned emergently with the table and therefore recognized the same structure across all three artifacts. This implies that Moore’s (2021) discussion of transfer in graphical shape thinking may also be expanded to include different types of artifacts other than graphs.

This expansion of the shape thinking construct allows researchers to analyze students’ reasoning with this lens across multiple representations, broadening its use and offering a way to discuss progressions in students’ reasoning as they transition between artifacts. In practice, this expansion could also contribute to teachers’ understanding of their students’ reasoning, allowing them to more closely tailor their instructional moves to a given student’s current thinking. For example, it is important to note that students may reason both statically and emergently at different times with the same artifact. Reasoning statically in one instance does not mean the student does not see the emergent relationship, while reasoning emergently in one instance also does not mean they are necessarily thinking about that relationship at a different time or in a different context.

Further study is needed into the type of questioning that leads to specific expression of shape thinking, acknowledging that students may be reasoning emergently but not showing evidence of this because the researcher’s questioning does not elicit it. Future work may also consider data from other DEs to see if other students reasoned similarly and explore patterns in the ways that students use shape thinking when working with representations other than graphs. Finally, it would also be interesting to explore what other artifacts or representations of covariation the shape thinking framework might be expanded to include.

### **Acknowledgments**

This research was supported by a grant from the National Science Foundation (#1742125). The data presented, statements made, and views expressed are solely the responsibility of the authors. We would like to thank Erell Germia for her help in the analysis of the data and Jay Singh, Debasmita Basu, Sowmith Etikyala, and Anthony Cuvillo for their help in the design of the simulation and activities.

### **References**

- Basu, D. & Panorkou, N. (2019). Integrating covariational reasoning and technology into the teaching and learning of the greenhouse effect. *Journal of Mathematics Education*, 12(1), 6–23.
- Basu, D., Panorkou, N., Zhu, M., Lal, P., & Samanthula, B. K. (2020). Exploring the Mathematics of Gravity, *Mathematics Teacher: Learning and Teaching PK-12 MTLT*, 113(1), 39–46.

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

- Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education*, 33(5), 352–378.
- Castillo-Garsow, C., Johnson, H. L., & Moore, K. C. (2013). Chunky and smooth images of change. *For the Learning of Mathematics*, 33(3), 31–37.
- Cobb, P., Confrey, J., DiSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. *Educational Researcher*, 32(1), 9–13.
- Germia, E., York, T., & Panorkou, N. (2022). How transitions between related artifacts support students' covariational reasoning. In A. E. Lischka, E. B. Dyer, R. S. Jones, J. N. Lovett, J. Strayer, & S. Drown (Eds.), *Proceedings of the 44<sup>th</sup> Annual Meeting of PME-NA* (pp. 1982–1991).
- Moore, K. C. (2021). Graphical shape thinking and transfer. *Transfer of learning: Progressive perspectives for mathematics education and related fields* (pp. 145–171).
- Moore, K.C., & Thompson, P.W. (2015). Shape thinking and students' graphing activity. In Fukawa-Connelly, T., Infante, N., Keene, K., & Zandieh, M. (Eds.), *Proceedings of the Eighteenth Annual Conference on RUME* (pp. 782–789).
- Panorkou, N., & Germia, E. (2021). Integrating math and science content through covariational reasoning: the case of gravity. *Mathematical Thinking and Learning*, 23(4), 318–343.
- Panorkou, N., & York, T. (2020). Designing for an integrated STEM+ C experience. In A. Sacristán, J. Cortés-Zavala, & P. Ruiz-Arias (Eds.), *Proceedings of the 42<sup>nd</sup> Annual Meeting of PME-NA* (pp. 2233–2237).
- Panorkou, N., York, T., & Germia, E. (2022). Examining the “messiness” of transitions between related artifacts. *Digital Experiences in Mathematics Education*, 1–32.
- Paoletti, T., Lee, H. Y., & Hardison, H. L. (2018). Static and emergent thinking in spatial and quantitative coordinate systems. In T. E. Hodges, G. J. Roy, & A. M. Tyminski (Eds.), *Proceedings of the 40<sup>th</sup> Annual Meeting of PME-NA* (pp. 1315–1322).
- Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), *Compendium for Research in Mathematics Education* (pp. 421–456). National Council of Teachers of Mathematics.
- York, T., Germia, E., Kim, Y., & Panorkou, N. (2021). Students' reorganizations of variational, covariational, and multivariational reasoning. In D. Olanoff, K. Johnson, & S. M. Spitzer (Eds.), *Proceedings of the 43<sup>rd</sup> Annual Meeting of PME-NA* (pp. 308–312).

## STUDENTS' STRUCTURAL REASONING ABOUT RATIONAL EXPRESSIONS

Daniel K. Siebert  
Brigham Young University  
dsiebert@mathed.byu.edu

Dana Steinhorst  
Karl G. Maeser Preparatory Academy  
dana.steinhorst@maeserprep.org

Keywords: Algebra and Algebraic Reasoning, High School Education.

Many students struggle to make sense of and reason about algebraic expressions. Their lack of sense making often results in errors as they manipulate expressions and equations involving variables. Researchers believe these errors are due at least in part to students' lack of structural reasoning (Hoch & Dreyfus, 2004, 2006; Kirshner & Awtry, 2004; Ruede, 2013; Vega-Castro et al., 2012). While studies show that students seldom demonstrate expert structural reasoning when engaged in symbol manipulation, little research has been done to examine what structures students actually perceive and how they decide which manipulations to perform. We recognized that structure is not contained in the written symbols themselves, but rather is constructed by the reader or writer. We defined structure as the hierarchy of parts and the relationships between them that are perceived by an individual (Harel & Soto, 2017; Kieran, 1989; Ruede, 2013). We selected three components of structural reasoning that we thought would be particularly useful in understanding and manipulating algebraic expressions and equations: identifying structure, matching structure to rules, and evaluating matches for correctness and progress toward solving the problem (Harel & Soto, 2017; Hoch & Dreyfus, 2004). Our research question was, how do high school students reason structurally as they simplify and solve equations involving rational expressions?

We recruited six volunteers from a high school AP calculus class. Each student engaged in an hour-long interview in which they explained their reasoning as they solved problems involving rational expressions. To analyze the data, we constructed "solution maps" for each problem solved by the students. These maps captured the structures students identified, the rules they attempted to match, and their evaluations of their matches. We then compared maps to develop a theory for how students were reasoning structurally.

Contrary to past studies, we found that all six students regularly engaged in structural reasoning. Students identified two types of structures: those that were the result of breaking expressions into smaller parts based on the highest level of operation (HLO), and those that were not. Only HLO structures led to correct solutions. Once students identified structures, they tried to match them to rules. Each structure seemed to elicit a collection of possible rules for manipulation, which we referred to as rule banks, that differed across students. Matching consisted of attempts to map parts of the structure onto the rule using a combination of one-to-one and many-to-one correspondences. Some students' rule banks were missing rules that would have matched the structure and led to a correct solution. When students failed to make a match, some would return to the expression and select non-HLO structures. Others would conclude that the expression was in simplest form. Students who selected non-HLO structures either mapped the entire structure to an invented, invalid rule or a part of the non-HLO structure inappropriately to valid rules; both approaches led to incorrect solutions. Evaluation occurred when students assessed whether a match could be made between structure and rule, and when students tried to determine if the match led to progress in solving the problem. Our findings suggest that students

need support in distinguishing between HLO and non-HLO structures, creating robust rule banks, and identifying incorrect, invented rules.

### References

- Harel, G., & Soto, O. (2017). Structural reasoning. *International Journal of Research in Undergraduate Mathematics Education*, 3(1), 225-242.
- Hoch, M., & Dreyfus, T. (2004). Structure sense in high school algebra: The effect of brackets. In M. Hoines & A. Fuglestad (Eds.), *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education* (pp. 49-56).
- Hoch, M., & Dreyfus, T. (2006). Structure sense versus manipulation skills: An unexpected result. In J. Novotna, H. Moraova, M. Kratka, & N. Stehlikova (Eds.), *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education* (pp. 305-312).
- Kieran, C. (1989). The early learning of algebra: A structural perspective. In S. Wagner & C. Kieran (Eds.), *Research issues in the learning and teaching of algebra* (pp. 33-56). NCTM.
- Kirshner, D., & Awtry, T. (2004). Visual salience of algebraic transformations. *Journal for Research in Mathematics Education*, 35(4), 224-257.
- Rüede, C. (2013). How secondary level teachers and students impose personal structure on fractional expressions and equations—An expert-novice study. *Educational Studies in Mathematics*, 83(3), 387-408.



# STUDENTS' UNDERSTANDINGS OF THE DEFINITION OF A FUNCTION

Camille S. Burnett  
Prairie View A&M University  
csburnett@pvamu.edu

*The study discussed here aims to describe students' understandings of the definition of a mathematical function, which was achieved through a pilot case study of clinical interviews with four participants – two ninth graders and two twelfth graders. The participants were recruited from the same urban public high school in the northeast of the United States, which serves a diverse racial, ethnic, and cultural community. All four participants were selected by their mathematics teachers because of their high grades and skill level. The participants were interviewed individually about questions pertaining to the definition of a mathematical function. Analyses of the interview responses revealed that the twelfth graders understand the definition of a mathematical function differently than do ninth graders.*

Keywords: Algebra and Algebraic Thinking

## Purpose of Study

Functions are one of the fundamental objects of mathematics (Doorman & Drijvers, 2011; Dreyfus & Eisenberg, 1982; Eisenberg, 1992; Gagatsis & Shiakalli, 2004; Hitt, 1998; Schwartz, 1999; Schwartz & Yerushalmy, 1992; Zandieh et al., 2017). They are present in (1) elementary school mathematics courses, where students are required to identify patterns or interpret data tables and graphs; (2) middle and high school mathematics courses, ranging from Algebra 1 to Pre-Calculus; and (3) college or university mathematics courses, including Calculus at all levels and Real Analysis. Therefore, functions are a core concept in mathematics, appearing at all mathematics curriculum levels. There is much research within the field of functions. An example of one of the most highly cited pieces of literature regarding functions is Dubinsky and Harel (1992), whose work examines: the development of the function concept; teaching experiments; students' and teachers' conceptions of functions; and the use of technology. The research presented here is a pilot case study that focuses on students' conceptions of functions and aims to further contribute to this line of research, particularly students' *understandings* of the *definition* of a *mathematical function*, given its presence throughout mathematics. Previous studies have shown that (a) junior high school teachers were better able than college students to determine if a given relation was a function based on their concept image of a function (Dreyfus & Vinner, 1982); (b) most prospective teachers knew of the univalence requirement for a relation to be a function, but many did not know why this was a requirement (Even, 1993); (c) there is initial evidence to suggest that secondary students' conceptualization of functions may be connected to the curriculum to which they were exposed (Ayalon & Wilkie, 2019); and (d) many undergraduate mathematics students are unable to define a function or determine if a given graph or rule represents a function (Bardini et al., 2014). The research presented here will highlight differences in ninth and twelfth graders' understanding of a function's definition.

## Theoretical Framework

### Definition of a Function

Several definitions of a function exist, each with a slight variation depending on the audience (e.g., high school vs. college student) receiving the definition. A quick search on the internet suggests that one possible explanation of a function is "A special relationship where each input

has a single output” (Math is Fun, 2018). A high school mathematics textbook suggests the following definition: “Suppose A and B are two sets of objects. A function from A to B is a pairing between A and B such that each element in A pairs with exactly one element of B” (CME Project Development Team, 2009, p. 104). A college-level textbook suggests that the definition of a function is, “A function  $f$  from a set A to a set B is a relation that assigns to each element  $x$  in the set A exactly one element  $y$  in the set B. The set A is the domain (or set of inputs) of the function  $f$ , and the set B contains the range (the set of outputs)” (Larson, 2014, p. 173). The latter two definitions of a function are known as the Dirichlet-Bourbaki definition of a function. They include the set-theoretic notions of Bourbaki and the rule-based notions of Dirichlet. They are also considered a *modern* definition of function for these same reasons. The former definition of a function does not explicitly mention the set-theoretic notions of Bourbaki. Thus, for this paper, the latter two definitions of a function will be the accepted definitions.

### **Characteristics of a Function**

In order to get to this modern definition of a function, the concept of function has undergone a curious evolutionary process due to the change in knowledge of mathematics over time (Kleiner, 1989; Malik, 1980; Markovits et al., 1986; O’Connor & Robertson, 2005; Sfard, 1992; Sierpinska, 1992). Functions may have first appeared in tabular representational form and as trigonometric functions (2000 B.C.E. – 1299 C.E.), then successively as a relationship of dependence (1300 C.E. – 1499 C.E.), a relationship between varying quantities (1500 C.E. – 1599 C.E.), and in the algebraic and graphical representational form (1600 C.E. – 1699 C.E.). Next, functions were defined from an algebraic perspective by Euler (1700 C.E. – 1799 C.E.), then based on an arbitrary correspondence by Dirichlet (1800 C.E. – 1899 C.E.), and finally considered as an arbitrary correspondence between two sets, which followed the emergence of set theory (1900 C.E. – present). Thus, this *modern* definition of a function has two distinct characteristics – *arbitrariness* and *univalence* (Freudenthal, 1983) – which are used in this paper to form the framework for analyzing the data in this study.

**Arbitrariness.** The arbitrariness of a function refers to “both the relationship between the two sets on which the function is defined and the sets themselves” (Even, 1990, p. 528; 1993, p. 96). In terms of the relationship between the two sets, there does not need to be any specific rule of correspondence, i.e., there does not need to be a specific algebraic expression, a set pattern in a table of values, or a graph with a specific shape. In terms of the sets themselves, the sets do not need to be defined on any specific set of objects, i.e., the sets do not necessarily contain numbers. In other words, the sets (which are referred to as variables by Freudenthal [1983]) can consist of “numbers, number tuples, points, curves, functions, permutands, elements of arbitrary sets” (Freudenthal, 1983, p. 528).

**Univalence.** The univalence characteristic of a function refers to the part of the definition that states that for each element in the domain, there is only one element (image) in the range (Even, 1990, 1993). Thus, in terms of a relation between two sets, a relation in which every single element in the domain is mapped to its own single element in the range (i.e., a one-to-one relation) or a relation in which more than one element in the domain is mapped to the same single element in the range (i.e., a many-to-one relation) could be a function. While a relation in which every single element in the domain is mapped to more than one element in the range (i.e., a one-to-many relation) or a relation in which more than one element in the domain is mapped to more than one element in the range (i.e., a many-to-many relation) could not be a function.

## Methods

### Participants

The participants included two ninth graders (Student 9-1 and Student 9-2) and two twelfth graders (Student 12-1 and Student 12-2). These students were from the same urban public high school in the northeast of the United States of America, which serves a diverse racial, ethnic, and cultural community. The ninth graders were learning basic topics in algebra (from an equational perspective), geometry, and probability and statistics in their current mathematics class, and the twelfth graders were learning a combination of topics from pre-calculus and calculus with a focus on various types of functions, as reported by their mathematics teachers. The teachers at the school designed the curriculum for both grades, and it was not supplemented with any specific textbook. Their mathematics teachers selected the participating students to participate in the study based on their performance (grades and skill level) in their current mathematics class.

### Individual Interviews

The interviews consisted of seven questions focused on the definition of function and the transformations and comparisons of functions in various representations. This paper will focus only on the responses to the two questions regarding the definition of function (see Figure 1).

<p><b>Question 1</b></p> <p>(a) Give me an example of a function. Explain.</p> <p>(b) Give me an example of something that is not a function. Explain.</p> <p>(c) What is the definition of a function? Explain.</p>																																								
<p><b>Question 7</b></p> <p>(a) Which of the following seems to you to be the best definition of a function?</p> <ol style="list-style-type: none"> <li>i. A function is an algebraic expression in which you can substitute various values for an unknown.</li> <li>ii. A function is a computational process that produces an output (<math>y</math>) from an input (<math>x</math>).</li> <li>iii. A function consists of two sets <math>S</math> and <math>T</math> together with a “rule” that assigns to each element of <math>S</math> a specific element of <math>T</math>.</li> </ol> <p>(b) The following table of values were computed from a function, but the person who made the table forgot to label one column <math>x</math> and the other <math>f(x)</math>.</p> <table border="1" style="margin-left: auto; margin-right: auto; border-collapse: collapse; text-align: center;"> <tr><td>1</td><td>9</td></tr> <tr><td>4</td><td>2</td></tr> <tr><td>4</td><td>-2</td></tr> <tr><td>5</td><td>25</td></tr> <tr><td>9</td><td>1</td></tr> <tr><td>16</td><td>-4</td></tr> </table> <p>Which of the following labeling is correct? How do you know?</p> <table border="1" style="margin-left: auto; margin-right: auto; border-collapse: collapse; text-align: center;"> <tr> <td style="padding: 5px;"><math>x</math></td> <td style="padding: 5px;"><math>f(x)</math></td> <td style="padding: 5px;"><math>x</math></td> <td style="padding: 5px;"><math>f(x)</math></td> </tr> <tr> <td style="padding: 5px;">1</td> <td style="padding: 5px;">9</td> <td style="padding: 5px;">9</td> <td style="padding: 5px;">1</td> </tr> <tr> <td style="padding: 5px;">4</td> <td style="padding: 5px;">2</td> <td style="padding: 5px;">2</td> <td style="padding: 5px;">4</td> </tr> <tr> <td style="padding: 5px;">4</td> <td style="padding: 5px;">-2</td> <td style="padding: 5px;">-2</td> <td style="padding: 5px;">4</td> </tr> <tr> <td style="padding: 5px;">5</td> <td style="padding: 5px;">25</td> <td style="padding: 5px;">25</td> <td style="padding: 5px;">5</td> </tr> <tr> <td style="padding: 5px;">9</td> <td style="padding: 5px;">1</td> <td style="padding: 5px;">1</td> <td style="padding: 5px;">9</td> </tr> <tr> <td style="padding: 5px;">16</td> <td style="padding: 5px;">-4</td> <td style="padding: 5px;">-4</td> <td style="padding: 5px;">16</td> </tr> </table>	1	9	4	2	4	-2	5	25	9	1	16	-4	$x$	$f(x)$	$x$	$f(x)$	1	9	9	1	4	2	2	4	4	-2	-2	4	5	25	25	5	9	1	1	9	16	-4	-4	16
1	9																																							
4	2																																							
4	-2																																							
5	25																																							
9	1																																							
16	-4																																							
$x$	$f(x)$	$x$	$f(x)$																																					
1	9	9	1																																					
4	2	2	4																																					
4	-2	-2	4																																					
5	25	25	5																																					
9	1	1	9																																					
16	-4	-4	16																																					

### Figure 1: Interview Questions

Question 1 was open-ended and consisted of three parts. The participants were asked to give an example of a function, an example of a non-function, and the definition of a function, with an accompanying explanation for each. Question 7 consisted of two parts. The participants were asked to determine which of three options defined a function correctly. They were also asked to determine which of two given relations (presented in the tabular representational form) were functions. The order of the format of the two questions was intentional. Participants could provide an initial spontaneous and untainted response to the definition of a function in Question 1 before being presented with possible meanings of a function in Question 7. The clinical interviews (Ginsburg, 1997; Piaget, 1929/1976) were videotaped and lasted between 15 and 60 minutes, depending on the participants' length of time to process and answer each question. All interviews were conducted within a month to reduce the likelihood of any student being exposed to more mathematics instruction than their grade-level counterparts.

#### Analysis

The participants' interview responses to Questions 1 and 7 were transcribed. Their written responses and oral responses were combined and summarized using direct quotes and paraphrasing their answers. The general idea of their responses was noted from this summary (see Tables 1, 2, 3, and 4) by identifying mathematical terms related to functions. The participants' responses to Questions 1 and 7 were then coded based on the essential characteristics of a function – *arbitrariness* and *univalence* (see Table 5). Each explanation was scored with a “1” or a “0” depending on whether it met the following criteria:

- Arbitrariness (A): The *relationship* between the two sets comprising the function is arbitrary, and/or the members of the *sets* themselves are arbitrary.
- Univalence (U): The relation between the two sets comprising the function is either one-to-one or many-to-one.

#### Results

For each participant, the results are presented here; they include a narrative summary and a table of their responses to Questions 1 and 7 (see Tables 1, 2, 3, and 4). Brief comparisons of grade-level counterparts and a coded summary of their responses (see Table 5) are also presented. Note that even though the emphasis is placed on the participants being able to define a function, it is even more critical for them to understand the concept of function. Thus, it is more important for a participant to suggest an example of a function or a non-function and determine if a given relationship is a function, indicating that they understand the concept of function. It is also hoped that if the participants can this, then stating the definition of a function will be less challenging for them and will become less critical in describing their understanding of function.

#### Student 9-1

Student 9-1 was consistent in their responses to Questions 1 and 7 (see Table 1). This student believed that a function is an equation such that there would be a specific rule and that there should be unknowns instead of variables involved in a function. This understanding is consistently evidenced in the responses given to the questions posed. It should be noted that this student had never been formally exposed to the *modern* definition of a function per their mathematics teacher and that the exposure to examples of functions was limited to linear functions. Therefore, it is understandable that the student was not fully aware of the need for two arbitrary sets, an arbitrary rule of correspondence, and univalence in a function. It also indicates

that instruction within this area is necessary for understanding the definition of a function and understanding various examples of functions, even linear functions.

**Table 1: Student 9-1 Responses**

Question	Response	Explanation	General Idea
1(a) Function	$a^2 + b^2 = c^2$	“it is an equation, a way to find something out”	equation, unknown
1(b) Non- function	PEMDAS <sup>11</sup>	“an order in which you do something, and it’s not a function, so it doesn’t give you a way to find something out”	unknown
1(c) Definition	“Something that is used to get an answer; a function has a purpose, and the purpose is the answer.”	“a function usually like has a purpose, so the purpose would be the answer. So the function is like an equation that you can ultimately get the answer of”	equation, unknown
7(a) Definition	i	“because like what we were doing before, we were substituting variables, and we were trying to find out what the unknown was.”	variables and unknowns
7(b) Function?	neither	no consistent pattern in either of them	rule

### Student 9-2

Student 9-2 consistently stated or implied that a function is an equation in their responses to Questions 1 and 7 (see Table 2). This implies a belief in the need for a specific rule of correspondence. In addition, since there was constant emphasis on using a function to find something out or solve a problem, the student likely believed that there are unknowns, not variables, in a function. However, the response to Question 7(b) was surprising because the student thought that the lack of a rule of correspondence in the relations made them functions, which, unbeknown to the student, contradicted the definition repeatedly given by the student. The question probably caused a state of disequilibrium (Beth & Piaget, 1966) in the student’s understanding of the definition of a function, as evidenced by the discrepancy in the student’s responses to the questions about the definition of a function. It is possible the student was never asked questions of this nature before, which caused the student to respond with such contradictory answers.

Finally, it should be noted that this student, much like the first ninth grade student, had never been formally exposed to the *modern* definition of a function per their mathematics teacher. The exposure to examples of functions was limited to linear functions, as is expected for ninth grade

<sup>11</sup> PEMDAS – Parentheses, Exponents, Multiplication, Division, Addition, Subtraction, also known as Please Excuse My Dear Aunt Sally, is the acronym used to help students remember the order in which mathematical operations should be applied to an algebraic expression.

students. Therefore, it is understandable why the student was not fully aware of the need for two arbitrary sets, an arbitrary rule of correspondence, and univalence in a function. Again, it seems that instruction within this area and exposure to various functions is necessary for understanding the definition of a function and understanding of multiple examples of functions, even linear functions, that are relatively simple.


**Table 2: Student 9-2 Responses**

Question	Response	Explanation	General Idea												
1(a) Function	$y = mx + b$	“it’s an equation to help you figure out something else”	equation, unknown												
1(b) Non-function	<table border="1"> <tr> <td>People</td> <td>1</td> <td>2</td> <td>3</td> <td>4</td> <td>5</td> </tr> <tr> <td>Donuts</td> <td>2</td> <td>4</td> <td>6</td> <td>8</td> <td>10</td> </tr> </table>	People	1	2	3	4	5	Donuts	2	4	6	8	10	does not match definition of a function, which is “a function is like an equation to help you solve...to help you solve a problem”	equation, unknown
People	1	2	3	4	5										
Donuts	2	4	6	8	10										
1(c) Definition	“An equation or problem used to solve something.”	functions are like equations that are used to find something out	equation, unknown												
7(a) Definition	i, ii, iii	all three options are indicative of different types of functions	types of functions												
7(b) Function?	either	no consistent pattern in either of them	rule												

### Student 12-1

In response to every question posed regarding the definition of a function, Student 12-1 identified the two characteristics of a function (see Table 3). In fact, the student emphasized the need for univalence more than the need for arbitrariness, but the variability in emphasis was consistent throughout the interview. The student also showed a solid awareness of the need for two arbitrary sets but was inconsistent in indicating that the rule of correspondence could be arbitrary.

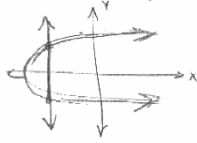
**Table 3: Student 12-1 Responses**

Question	Response	Explanation	General Idea
1(a) Function	$f(x) = x^3 + 3x^2 - 4x + 3$	there is an input, being referred to as $x$ , and an output, and the input “will give you one and only one outcome in the output which makes it a function”	input/output, mapping
1(b) Non-function		“Something that’s not a function is, basically if you put in something for the input and you get two different answers or multiple answers for the output” ... fails the vertical line test	input/output, mapping, vertical line test
1(c) Definition	“A function is an equation that will give only one output for one specific input.”	if there is more than one output for each input, then the graph would be inaccurate	input/output, mapping
7(a) Definition	iii	“two sets ... $S$ and $T$ are ... combined with ... with a rule, that basically says that for each assignment of $S$ you can only get one specific ... element of $T$ , or like one specific output which would be $T$ ”	arbitrariness, univalence
7(b) Function?	second table	“you are giving one input, and it's giving two different outputs, so this [the second table] would be the correct one”	equation, univalence

### Student 12-2

For Student 12-2, there is continuous awareness of the need for two arbitrary sets to define a function (see Table 4). The student also continually expected that the rule of correspondence should be specific. Regarding the characteristic of univalence, the student could only identify and explain it through the vertical line test, which is a graphical test. This suggests that the representation of the function may limit the student’s understanding of this characteristic.

**Table 4: Student 12-2 Responses**

Question	Response	Explanation	General Idea
1(a) Function	$f(x) = x^2 + 9x + 3$	it has roots, factors, and it is a parabola ... passes the vertical line test	vertical line test
1(b) Non-function		has an x-axis and y-axis ..., and "fails the vertical line test"	input/output, vertical line test
1(c) Definition	"If you have two different inputs and have two similar outputs, then it is not a function. If you have two inputs and get two different outputs then it is a function."	it does not fail the vertical line test	vertical line test
7(a) Definition	i and ii	one needs to plug in an input in order to obtain an output	input/output, rule
7(b) Function?	second table	created graph, then noted that it failed the vertical line test	vertical line test

**Comparison of Student 9-1 and Student 9-2**

In comparing Student 9-1 and Student 9-2, both students have a relatively unsophisticated understanding of function. This is evidenced in their examples and their definitions. There is a consistent emphasis on the presence of unknowns and solving, as opposed to the presence of a variable. This unsophisticated understanding is further evidenced in Table 5, where we can see that both students have a similar non-understanding of a function's characteristics. Both students could not fully articulate the need for arbitrariness in a function.

Additionally, both students are similarly unaware of the idea of univalence. These results parallel the work of Ayalon and Wilkie (2019), Bardini et al. (2014), Dreyfus and Vinner (1982), Even (1993), Vinner (1983), and Vinner and Dreyfus (1989), whose research shows that some students are less aware of the univalence characteristic of function than other students. These ninth-grade students' responses also clearly revealed the limited variety of functions they might have been exposed to in their mathematics education, further reinforcing their understanding of function yet indicating the potential for a shift in their understanding.

**Comparison of Student 12-1 and Student 12-2**

In comparing the responses of Student 12-1 and Student 12-2, the first of the two students has a slightly more sophisticated understanding of the concept of function than the second twelfth grade student. This is evidenced in their examples being reasonably similar but somewhat different explanations. In their answers, Student 12-1 consistently refers to the arbitrary nature of a function, and Student 12-2 does not. Their responses to Question 7a are also different as Student 12-1 chose the option in which the univalent nature of function was prominent, and Student 12-2 did not. This difference in understanding is further emphasized in Table 5, where



the slight difference in their understandings is more clearly seen. Again, these results parallel the work of Ayalon and Wilkie (2019), Bardini et al. (2014), Dreyfus and Vinner (1982), Even (1993), Vinner (1983), and Vinner and Dreyfus (1989), whose research shows that some students are more aware of the univalence characteristic of function than other students.

**Table 5: Coded Summary of Responses to Questions 1 and 7**

Participant	1(a) Function		1(b) Non- function		1(c) Definition		7(a) Definition		7(b) Function?	
	A	U	A	U	A	U	A	U	A	U
1 Student 9-	1	0	1	0	1	0	1	0	1	0
2 Student 9-	1	0	1	0	1	0	1	0	1	0
12-1 Student	1	1	1	1	1	1	1	1	1	1
12-2 Student	0	1	1	1	1	1	1	0	1	1

## Discussion

### Overall Findings

This paper aimed to describe students' understandings of the definition of function through a case study analysis. The results and discussion lead to three main findings about the participants:

- the ninth-grade students are slightly more aware of the arbitrariness than of the univalence of functions, while the twelfth grade students are more mindful of both the arbitrariness and univalence of functions;
- the differences in understandings of the definition of a function coincide with a student's grade level and, therefore, a student's exposure to mathematics, which implies that progression from one level of understanding to the next level of understanding can only be achieved through direct learning experiences of the concept (Duckworth, 1973, 1996; Piaget 1975/1985; Vygotsky, 1978); and
- the differences in understanding a function's definition parallel the historical development of the concept of function (Piaget & Garcia, 1983/1989).

### Implications for Teachers and Researchers

This study is essential to both mathematics teachers and mathematics education researchers. For mathematics teachers, it describes students' responses to various questions about functions and highlights their misconceptions. For mathematics education researchers, it implies possible levels of understandings of the concept of function. These descriptions and implications are necessary, given that function is a core concept in the teaching and learning of mathematics. The more we know about how a student understands a concept, the better we will teach it and develop appropriate curricula, as evidenced in the work of Dubinsky and Wilson (2013) and Sherman et al. (2019), and therefore, engage all students in learning.

### Future Research

The research presented here is a pilot case study and is merely descriptive. Thus, more research needs to be done with a larger sample size to make truly conclusive statements regarding students' understanding of functions. Despite these limitations, the research discussed

can form the basis of such a research endeavor, for example, a study focused on one particular grade level. Also, the types of questions asked should be modified based on the results presented here. For instance, the responses to questions about examples of functions were quite varied, implying that a question asking one to determine whether a given relation is a function might provide more streamlined and valuable data and, therefore, more substantiated conclusions.

### References

- Ayalon, M., & Wilkie, K. J. (2019). Exploring secondary students' conceptualization of functions in three curriculum contexts. *Journal of Mathematical Behavior*, 56, 1-17. <https://doi-org.pvamu.idm.oclc.org/10.1016/j.jmathb.2019.100718>
- Beth, E. W. & Piaget, J. (1966). *Mathematical Epistemology and Psychology* (W. Mays, Trans.). Dordrecht-Holland: D. Reidel Publishing Company.
- Bardini, C., Pierce, R., Vincent, J., & King, D. (2014). Undergraduate mathematics students' understanding of the concept of function. *Indonesian Mathematical Society Journal on Mathematics Education*, 5(2), 85-107.
- CME Project Development Team. (2009). *Algebra 2: Common Core*. Pearson.
- Doorman, M., & Drijvers, P. (2011). Algebra in function. In P. Drijvers (Ed.), *Secondary Algebra Education* (pp. 119-136). Sense Publishers.
- Dreyfus, T. & Eisenberg, T. (1982). Intuitive function concepts: A baseline study on intuitions. *Journal for Research in Mathematics Education*, 13(5), 360-380.
- Dreyfus, T. & Vinner, S. (1982). Some aspects of the function concept in college students and junior high school teachers. *Proceedings of the Sixth International Conference for the Psychology of Mathematics Education*, Antwerp, Belgium.
- Dubinsky, E. & Harel, G. (Eds.). (1992). *The concept of function: Aspects of epistemology and pedagogy*. Mathematical Association of America.
- Dubinsky, E., & Wilson, R. T. (2013). High school students' understanding of the function concept. *The Journal of Mathematical Behavior*, 32(1), 83-101. <https://doi.org/10.1016/j.jmathb.2012.12.001>
- Duckworth, E. (1973, October). Piaget takes a teacher's look. *Learning*, 22-27.
- Duckworth, E. (1996). Either we're too early and they can't learn it, or we're too late and they know it already: The dilemma of "applying Piaget." In E. Duckworth, *The having of wonderful ideas and other essays on teaching and learning* (2nd ed., pp. 31-49). Teachers College Press.
- Eisenberg, T. (1992). On the development of a sense for functions. In E. Dubinsky & G. Harel (Eds.), *The concept of function: Aspects of epistemology and pedagogy* (pp. 153-174). Mathematical Association of America.
- Even, R. (1990). Subject matter knowledge for teaching and the case of functions. *Educational Studies in Mathematics*, 21(6), 521-544.
- Even, R. (1993). Subject-matter knowledge and pedagogical content knowledge: prospective secondary teachers and the function concept. *Journal for Research in Mathematics Education*, 24(2), 91-116.
- Freudenthal, H. (1983). *Didactical phenomenology of mathematical structures*. D. Reidel Publishing Company.
- Gagatsis, A., & Shiakalli, M. (2004). Ability to translate from one representation of the concept of function to another and mathematical problem solving. *Educational Psychology*, 24(5), 645-657.
- Ginsburg, H. P. (1997). *Entering the child's mind: The clinical interview in psychological research and practice*. Cambridge, UK: Cambridge University Press.
- Hitt, F. (1998). Difficulties in the articulation of different representations linked to the concept of function. *Journal of Mathematical Behavior*, 17(1), 123-134.
- Kleiner, I. (1989). Evolution of the function concept: A brief survey. *The College Mathematics Journal*, 20(4), 282-300.
- Larson, R. (2014). *College Algebra* (9th ed.). Cengage Learning.
- Malik, M. A. (1980). Historical and pedagogical aspects of the definition of function. *International Journal of Mathematics Education in Science and Technology*, 11(4), 489-492.
- Markovits, Z., Eylon, B., & Bruckheimer, M. (1986). Functions today and yesterday. *For the Learning of Mathematics*, 6(2), 18-24.
- Math is Fun. (2018). Definition of a function. <https://www.mathsisfun.com/definitions/function.html>
- O'Connor, J. J., & Robertson, E. F. (2005). *The Function Concept*. MacTutor History of Mathematics. <http://mathshistory.st-andrews.ac.uk/HistTopics/Functions.html>
- Piaget, J. (1976). *A child's conception of the world* (J. and A. Tomlinson, Trans.). Maryland: Littlefield Adams Quality Paperbacks. (Original work published in 1929)

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

- Piaget, J. (1985). *The equilibration of cognitive structures* (T. Brown and K. J. Thampy, Trans.). The University of Chicago Press. (Original work published in 1975).
- Piaget, J. & Garcia, R. (1989). *Psychogenesis and the history of science* (H. Feider, Trans.). Columbia University Press. (Original work published in 1983).
- Schwartz, J. L. (1999). Can technology help us make the mathematics curriculum intellectually stimulating and socially responsible? *International Journal of Computers for Mathematical Learning*, 4(2-3), 99-119.
- Schwartz, J. & Yerushalmy, M. (1992). Getting students to function on and with algebra. In E. Dubinsky & G. Harel (Eds.), *The concept of function: Aspects of epistemology and pedagogy* (pp. 261-289). Mathematical Association of America.
- Sfard, A. (1992). Operational origins of mathematical objects and the quandary of reification – the case of function. In E. Dubinsky & G. Harel (Eds.), *The concept of function: Aspects of epistemology and pedagogy* (pp. 59-84). Mathematical Association of America.
- Sherman, M. F., Meagher, M. S., Lovett, J., & McCulloch, A. (2019). Transforming pre-service teachers definition of function. *North American Chapter of the International Group for the Psychology of Mathematics Education*.
- Sierpinska, A. (1992). On understanding the notion of function. In E. Dubinsky & G. Harel (Eds.), *The concept of function: Aspects of epistemology and pedagogy* (pp. 25-58).
- Vinner, S. (1983). Concept definition, concept image and the notion of function. *International Journal of Mathematics Education in Science and Technology*, 14(3), 293-305.
- Vinner, S., & Dreyfus, T. (1989). Images and definitions for the concept of function. *Journal for Research in Mathematics Education*, 20(4), 356-366.
- Vygotsky, L. (1978). *Mind In Society*. Harvard University Press.
- Zandieh, M., Ellis, J., & Rasmussen, C. (2017). A characterization of a unified notion of mathematical function: The case of high school function and linear transformation. *Educational Studies in Mathematics* 95(1), 21-38. <https://doi.org/10.1007/s10649-016-9737-0>

## UNDERSTANDING SLOPE THRU TACTILE AND WHOLE-BODY EXPERIENCES

Joy A. Oslund  
Grand Valley State University  
oslundj@gvsu.edu

Paul Woo Dong Yu  
Grand Valley State University  
yupaul@gvsu.edu

*This case study explores theories and enactments for using whole-body movement in helping a student with visual impairment understand slope. While technology has made learning algebraic concepts easier for many students, much of it relies on sight. To accommodate the student's learning, tactile and whole-body experiences were used to give meaning to the concept of slope. The results show that the learner constructed deep, profound, and novel notions of slope. In this paper, we illustrate the way mathematical knowledge was understood and communicated through motion and language and raise theoretical questions about the relationship between the body, the environment, and mathematical ideas. We propose these pedagogies be used with all students reflective of universal design principles.*

Keywords: Algebra and Algebraic Thinking, Cognition, Students with Disabilities.

Mathematics teacher educators have been interested in incorporating multiple representations for mathematical ideas into our lessons to make learning richer and more connected, as well as to include students with disabilities. A key element of learning algebra has been the importance of ‘visualizing’ mathematical representations and making connections between them, including graphs, patterns tables, and symbols. Using multiple representations is part of Universal Design for Learning, which is a model for teaching lessons that meet diverse needs in a classroom (CAST, 2018). However, not all bodies perceive all representations—for example, students with visual impairment may not be able to see diagrams. In recent years, mathematics educators have begun to understand that mathematical experience is embodied, that is, that mathematical ideas, physical objects, the environment, and our bodies are entangled in learning mathematics (de Freitas & Sinclair, 2014). This case study of a visually impaired student provided us an opportunity to think about what this looks like during learning. In this paper, we report two learning activities with which the student engaged, describe the student's changing understandings, and then connect these with theories of embodied learning in mathematics to build on and raise questions about what it means that mathematics is embodied.

### **Theoretical Framework**

For years, math was considered abstract and disembodied (Gerofsky, 2016), but in recent years, mathematical knowing and learning has been theorized as embodied. This change has come about as math educators seek to democratize mathematics learning and understand ways different bodies are societally positioned (Gerofsky, 2016; Sinclair & de Freitas, 2019). Developing an understanding of learning as embodied is important to engaging all learners.

Research and theorizing on embodied cognition are currently being taken up across a wide range of fields and approaches, including linguistics, cognitive science, semiotics, and more (Gerofsky, 2016; Sinclair & de Freitas, 2019). Work has been done on the relationship between language and motion (e.g. Alibali & Nathan, 2012; Lakoff & Nuñez, 2000) as well as ways to use motion and other multi-sensory approaches to support students' learning of mathematics. For example, Gerofsky (2011) has elicited students' motions in response to images of graphs. This research has started to extend to understand the mathematical learning of students with visual

impairments. For example, Gerofsky and Zebehazy (2022) used motion to help elementary students with visual impairments notice mathematically important features of graphs.

Work on embodied mathematics learning represents various relationships between bodies and mathematics and different ontologies of mathematical objects. For example, Lakoff and Nuñez's (2000) work described mathematics as arising from bodily experience, while Alibali and Nathan (2012) described math ideas as communicated by or manifested in gestures. Seeking to critique the dichotomy between abstract and physical, the New Materialism is an approach to embodiment that views bodies, mathematical ideas, physical objects, and the environment as entangled and presses on the boundaries between them (de Freitas & Sinclair, 2014).

As we analyzed data for this project, we began to wonder about the nature of the mathematical learning we were seeing. Was it being communicated by or arising from the student's motions and interactions with objects? Neither of those ways of thinking felt sufficient. We started to question the boundary we had originally assumed existed between motion and mathematical knowing and started to think about entanglements of the student's body, the environment, the objects he was manipulating, and the mathematical ideas he was constructing. In this paper, we attend to the question, how can we make sense of these entanglements as the student worked to understand the concept of slope?

### **Methodology**

To investigate these phenomena, a naturalistic (Moschkovich & Brenner, 2000), multitiered teaching experiment (Lesh & Kelly, 2000) was conducted in an undergraduate intermediate algebra course at a medium sized Midwestern university. Two researchers provided two tiers or perspectives--the researcher level and the teacher level. One assumed the role of teacher-researcher (Ball, 2000), planning, teaching, and videorecording lessons. The second researcher analyzed video data and interviewed the teacher-researcher.

The participant was a legally blind, male international student from Central Asia in his first year at the university. He was totally blind in one eye and had very limited vision in the other. English was not his first language; he had learned English, Braille, Nemeth Braille math script, and UEB Braille math script in the two years prior to his university enrollment. He was assigned to a special section of the algebra course in which the teacher-researcher and student met one-on-one for instructional sessions. Lessons were designed to use whole-body or tactile experiences with stairs and 3-D shapes and facilitate connections to the concept of slope. Lessons were videorecorded and analyzed using iterative refinement cycles for video analysis (Lesh & Lehrer, 2000) in which each video was reviewed by both researchers. The goal was to have multiple perspectives from the researcher and teacher-researcher as they converge their analysis on individual episodes.

### **The Focal Episodes**

In this section we present two short episodes from lessons on slope and connect them to theories of embodied mathematics learning. Prior to this course the student had no recollection of learning slope and said he did not know what the  $x$  or  $y$  axes were. The first lesson was conducted in and near the building in which the class was held on three sets of stairs, each with different slopes, and a ramp adjacent to one of the sets of stairs. The purpose of the lesson was to get the student to feel different degrees of steepness and articulate what those difference feel like. At each set of stairs, instructor and student walked up and down the stairs discussing the experience. The instructor elicited the students' ideas and sometimes provided informal and formal English language for the experience. For example, after the student spent time walking up and down the three sets of stairs and the ramp, the instructor asked him to compare how the ramp

felt compared to the sets of stairs. The student said, “So, [the ramp goes up] but it takes too much space.” When asked to clarify what ‘too much space’ meant, the student replied, “The first stairs go up really high, like a high degree.”

“Like steep,” replied the teacher.

The student continued, “Yes, but here [with the ramp] it’s kind of like laid down...”

“How do you think [the ramp] compares to the second stairs?” asked the teacher.

“It’s the same [overall] height, definitely... [the ramp] is longer than the stairs.”

The teacher said, “What you are experiencing then is this idea of slope. It’s the relationship between the vertical height and the horizontal distance. The vertical, which we’re going to call the Y-axis, that’s called the rise.” As he said this, the student moved his cane up and down. “And the horizontal is the forward and back one, that’s called the run.” The lesson ended with the instructor asking the student to estimate the rise and run of the stairs in centimeters, which the student did. The instructor then said, “We’ve been talking about the rise and run, in terms of two numbers, the horizontal and vertical. What we can do is take those two numbers and set it up as a fraction, and it’s called the slope. The slope is a single number, and it’s when you take the rise, and you divide it by the run.” As they discussed this formal notion of slope, the student would also gesture with his hands the movement of the horizontal, vertical, and slope of the stairs.

In the second episode, the next day’s lesson, the instructor provided the student with a set of 3D solids consisting of various prisms, pyramids, a cylinder, a cone, and a hemisphere. The shapes had heights of 5cm, and widths between 2 to 5cm. The instructor prompted, the student to identify objects with slanted edges, which he did within a few minutes, using his fingers to feel the 3D solids. The teacher then gave the student a textured cm ruler and asked him to “measure the slopes of the various structures.”

Holding the ruler vertically next to a square pyramid, adjacent to the base of the slant height, the student said, “The rise is 5cm,” as he ran his index finger up the textured ruler. Then using his index finger to trace the horizontal space from the tip of the pyramid to the ruler (see Figure 1) he said, “The run is 2[cm], five divided by two is two and a half?” The student then physically compared the slant heights of the square pyramid with slope 2.5 to a triangular pyramid by holding the two pyramids together saying, “And I got this [triangular pyramid], and I count [or measure] this way...” holding the two pyramids adjacent to each other, “...because [the pyramids] have the same height.” Then while holding the triangular pyramid in his right hand, he set the square pyramid down that was in his left hand, and picked up the ruler saying, “but I still need to measure from here to here [the horizontal distance from the tip of the pyramid to the ruler] which is one centimeter,” again tracing the horizontal space from tip to ruler. “Five divided by one, which is [a slope of] five,” he concluded.



**Figure 1: The Horizontal Space Traced by the Student’s Fingertip**

### **What Does it Mean that Mathematical Learning is Embodied?**

These episodes raised theoretical questions for us about mathematical knowledge and the nature of learning. Where is the math? What is the evidence of learning? What exactly is the motion—is it something to learn from, something that communicates, or something more?

Slope is thought of as the relationship between the rise and the run, two concepts that are grounded in a metaphor of motion (Lakoff & Nuñez, 2000) that has vertical and horizontal characteristics. In the first episode, the student was able to informally describe these and their relationship as he participated in that motion—walking forward up or down on the stairs and ramp. His body was experiencing slope as a whole concept and was starting to verbalize informal understandings for rise and run. When the instructor used formal language to ask the student about the rise and run with 3D objects, he seemed to translate his understandings to the new context and use his fingers to show his understanding.

It seems that for the stairs the embodied experiences of movement (Gerofsky, 2011; Gerofsky & Zebehazi, 2022) became tools for which the student to think with. For example, in comparing the two steeper staircases, the student could feel and articulated the difference in steepness through describing the subtle differences between the rise and run of the steps. He was able to physically sense that the steepest stairs had a rise 1 inch greater than the other set of stairs and verified it by measuring the difference using his cane as tool for direct comparison. He then used these physical experiences with the stairs to reason about the ramp, describing the steepness of the ramp through the relationship between its long horizontal run relative to its shorter overall rise.

When asked to reason about slope given a set of 3D solids and textured ruler, he used his fingers to feel not only the concept of slope, but also used his fingertips to estimate small distances. The gesture of using his fingertip across the span of the pyramid tip and vertical ruler was not only communicative, “from here to here,” but served as way for him to estimate the measure thru feeling the distance that his fingertip moved (Alibali & Nathan, 2012). Similar to how a sighted person might be able to estimate a centimeter gap thru sight, he would feel the relative measurements.

The episodes allowed us to think about how the concept of slope can be instantiated through embodied experiences, which may be whole-body experiences, like walking on stairs, or fine motor experiences, like touching small objects. They illustrate the entangled nature of the environment, objects, motion, language, and mathematical concepts in developing understanding. The math learning and knowing seemed not to be located only in the environment or only the student, but in the interplay between them.

### **Conclusions**

Work on mathematics and embodied ways of knowing has happened across many approaches but has not yet produced agreement on what it means that math learning and knowing are embodied (Gerofsky, 2016; Sinclair & de Freitas, 2019). We posit that these cases help to illustrate an embodied way of knowing mathematics. Future scholarship should continue to develop language for describing the entangled elements across which learning happens.

The instructor considered ideas from embodied cognition when designing the instructional sequences to connect with the student’s physical perception. The cases highlight the student’s keen physical intuition (Burton, 1999) supporting the construction of formal mathematical concepts. Universal design for learning raises the possibility that these lessons could support rich math learning for all students by providing multiple ways of perceiving and making connections (CAST, 2018).

## References

- Alibali, M.W. & Nathan, M.J. (2012). Embodiment in mathematics teaching and learning: Evidence from learners' and teachers' gestures. *Journal of the Learning Sciences* 21(2), 247-286.  
<https://doi.org/10.1080/10508406.2011.611446>
- Ball, D. L. (2000). Working on the inside: Using one's own practice as a site for studying teaching and learning. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education*. Mahwah, NM: Lawrence Erlbaum Associates.
- Burton, L. (1999). Why is intuition so important to mathematicians but missing from mathematics education? *For the Learning of Mathematics*, 19(3), 27-32. <http://www.jstor.org/stable/40248307>.
- CAST (2018). *Universal Design for Learning Guidelines version 2.2*. Retrieved from <http://udlguidelines.cast.org>.
- de Freitas, E. & Sinclair, N. (2014). *Mathematics and the body: Material entanglements in the classroom*. NYC: Cambridge.
- Gerofsky, S. (2011). Seeing the graph vs. being the graph: Gesture, engagement and awareness in school mathematics. In G. Stam & M. Ishino (Eds.) *Integrating gestures: The interdisciplinary nature of gesture*. Amsterdam: John Benjamins Publishing.
- Gerofsky, S. (2016). Approaches to embodied learning in mathematics. *Handbook of international research in mathematics education*, 3rd edition. New York: Routledge.
- Gerofsky, S. & Zebehazy, K.T. (2022). Enhancing mathematical noticing of graphs through movement, voice, and metaphor: An intervention with two students with visual impairment. *British Journal of Visual Impairment*, 40(2), 209-221. <http://doi.org/10.1177/0264619620963516>
- Lakoff, G. & Nuñez, R. E. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. NYC: Basic books.
- Lesh, R., & Kelly, A.E. (2000). Multitiered teaching experiments. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education*. Mahwah, NJ: Lawrence Erlbaum Associates
- Lesh, R., & Lehrer, R. (2000). Iterative refinement cycles for videotape analysis of conceptual change. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education*. Mahwah, NJ: Lawrence Erlbaum Associates
- Moschkovich, J. N., & Brenner, M. E. (2000). Integrating naturalistic paradigm into research on mathematics and science cognition and learning. In A. Kelly & R. Lesh (Eds.), *Handbook of research and design in mathematics and science education*. Mahwah, NJ: Lawrence Erlbaum Associates.
- Sinclair, N. & de Freitas, E. (2019). Body studies in mathematics education: Diverse scales of mattering. *ZDM* 51, 227-237. <https://doi.org/10.1007/s11858-019-01052-w>



## USING DYNAGRAPHS TO INVESTIGATE UNDERSTANDING OF SLOPE

Courtney Nagle  
Penn State Erie, The Behrend College  
crt12@psu.edu

Deborah Moore-Russo  
University of Oklahoma  
dmr@ou.edu

*This paper provides a case study account of a preservice secondary mathematics teacher's thinking while engaging in slope tasks using dynagraphs. The data included audio recordings and screen captures of a small group of preservice teachers engaging with these tasks, with our analysis focusing on the case of Robin. Despite familiarity using slope to measure steepness and determine relationships in a Cartesian plane setting, results indicate Robin struggled to reason about these same uses of slope when presented with a dynagraph. Using the APOS-Slope framework as a lens, the analysis suggests that Robin is limited to thinking of slope as an Action when using slope to measure steepness and determine relationships, relying heavily on shape thinking in the Cartesian plane. Implications for further research are provided.*

Keywords: Algebra and Algebraic Thinking; Learning Trajectories and Progressions; Mathematical Representations

Typically introduced in the middle grades (Stanton & Moore-Russo, 2012; Nagle & Moore-Russo, 2014; Nagle et al., 2022), slope serves as a foundational idea for more advanced topics. Slope is used to contrast covariational relationships in linear and nonlinear functions in algebra (Teuscher & Reys, 2010) and serves as a measure of steepness relating to the tangent of an angle in trigonometry (Nagle & Moore-Russo, 2013). Slope is used with regression lines to describe the nature of a data set in statistics (Nagle et al., 2017) and is central to a number of topics in calculus (Asiala et al., 1997; Moore-Russo & Nagle, in press; Zandieh & Knapp, 2006).

One way that slope can be represented is with dynagraphs. Dynagraphs are dynamic representations of functional relationships that allow students to manipulate the value of the input to see the corresponding value of the output, supporting covariational thinking (Lisarelli 2017; Ozen et al. 2021). Dynagraphs use parallel number lines for input and output axes, rather than the traditional perpendicular axes in the Cartesian coordinate plane. Because inputs and outputs are depicted separately, rather than combined in a coordinate pair (Bailey et al. 2020), dynagraphs help direct students' attention to how the two change in relation to each other.

### Literature Review

Students need a robust understanding of slope built on covariational reasoning to understand many topics, but even university-level students often have a limited understanding of linear functions and are unable to transition between their different representations (Adu-Gyamfi & Bossé, 2014; Newton, 2018). Some students rely on shape thinking, where a linear graph is treated as a static object (Moore & Thompson, 2015), and slope is perceived to be its visual tilt. These students may incorrectly compute or compare slopes when non-homogenous axes are used (Zaslavsky et al., 2002). Other students reduce slope to mnemonics like “change in  $y$  over change in  $x$ ” (Walter & Gerson, 2007) and rely on symbolic procedures to calculate a value for slope.

Research provides evidence that students and teachers can fail to make connections between various interpretations of slope (Frank & Thompson, 2021; Hattikudur et al., 2011; Hoban, 2021; Planinic et al., 2012). Students without a robust understanding may think differently about slope depending on the representations (De Bock et al., 2015; Tanışlı & Bike Kalkan, 2018) or the problem contexts (Byerley & Thompson, 2017) within the task, failing to reason conceptually on

application tasks (Lingefjärd & Farahani, 2017) involving slope. Some suggest that this may be attributed to teachers reducing slope instruction to rote procedures (Stump, 1999), and others have reported that pre- and in-service teachers can have limited understandings of slope (Mudaly & Moore-Russo, 2011; Avcu & Türker Biber, 2022; Coe, 2007; Stump, 2001).

### Theoretical Framing

APOS theory serves as framing for this paper in terms of a student's development of mathematical knowledge through four stages: Action, Process, Object, and Schema (Dubinsky, 2014). How students engage with mathematical tasks depends on their stage of understanding of the topic involved. We now describe the four APOS stages.

An *Action* is not connected to other mathematical knowledge but instead involves applying a rote procedure or a memorized fact. Actions are typically associated with specific mathematical representations. Once a student repeats and reflects on an Action, it may be internalized so that the student can perform the same transformation as the Action in his or her mind without the need of external stimuli. A *Process* involves meaningful links to other knowledge that allow the student to imagine the transformation, omitting steps that were necessary in the Action stage. These connecting links allow the student to work with different mathematical representations. As the student is able to extend past the Process to deal with new situations, then the Process has been encapsulated into a mental Object. At the *Object* stage, students are able to extend across, and even past, the different representations of a topic to consider how it may apply in novel contexts. Once a mental framework is constructed so that these Actions, Processes, and Objects form an organized, coherent collection, the individual has constructed a *Schema* for the topic.

One may consider the Transition level from the Action to the Process stage (Arnon et al., 2014). On one slope task a student might exhibit behavior in the Process stage but on a slightly different slope task this same student might appear to be at the Action stage. Such is the case at the *Transition* level when students often seem to be moving back and forth between both the Action and Process stages. The APOS-slope framework (Nagle et al., 2016, 2019) is illustrated in Figure 1 below. It focuses on the Action stage, the Transition level, the Process stage, and the Object stage for students who are developing an understanding of slope.

At the Action stage, The APOS-slope framework involves three distinct, isolated notions of slope as a geometric ratio (G), as an algebraic ratio (A), and as a functional property (F). At the Action stage students consider slope to be an intra-representational value that is calculated or identified. The geometric ratio at the Action stage ( $A_G$ ) involves students who are limited to thinking of slope as a rise-over-run calculation. The algebraic ratio at the Action stage ( $A_A$ ) represents students who are limited to using the formula  $(y_2 - y_1)/(x_2 - x_1)$  as a memorized fact. The functional property at the Action stage ( $A_F$ ) involves using the phrase "slope is the rate of change of a function" or identifying the coefficient of the  $x$ -term in a linear function to be the slope but without any grounding of the underlying covariational relationship between inputs and outputs.

Transitioning from the Action to the Process stage requires moving beyond blindly following a procedure. Students transitioning to the Process stage for geometric ratio ( $T_G$ ) are realizing that slope is independent of the location or size of the right triangle used to determine the vertical displacement ( $\Delta V$ ) and horizontal displacement ( $\Delta H$ ) of a line. Students transitioning to the Process stage for algebraic ratio ( $T_A$ ) are realizing that the slope formula is independent of which two points on the line are used, and it may be applied for a general point  $(x, y)$  on the line to obtain a calculation involving symbols rather than a number. Students transitioning to the Process stage for functional property ( $T_F$ ) are realizing that the format of a linear function impacts if the coefficient of the  $x$  term can be used to identify the slope. They are starting to

recognize that  $y = (-a/b)x + c/b$  and  $kax + kby = kc$  (for  $k$ , a nonzero constant) have the same slope. In the three transition scenarios, students are moving beyond a single action and starting to repeat the action in order to describe linearity. Students transitioning from the Action to Process stage may also be starting to make connections between  $A_G$ ,  $A_A$ , and  $A_F$  as well as  $T_G$ ,  $T_A$ , and  $T_F$ .

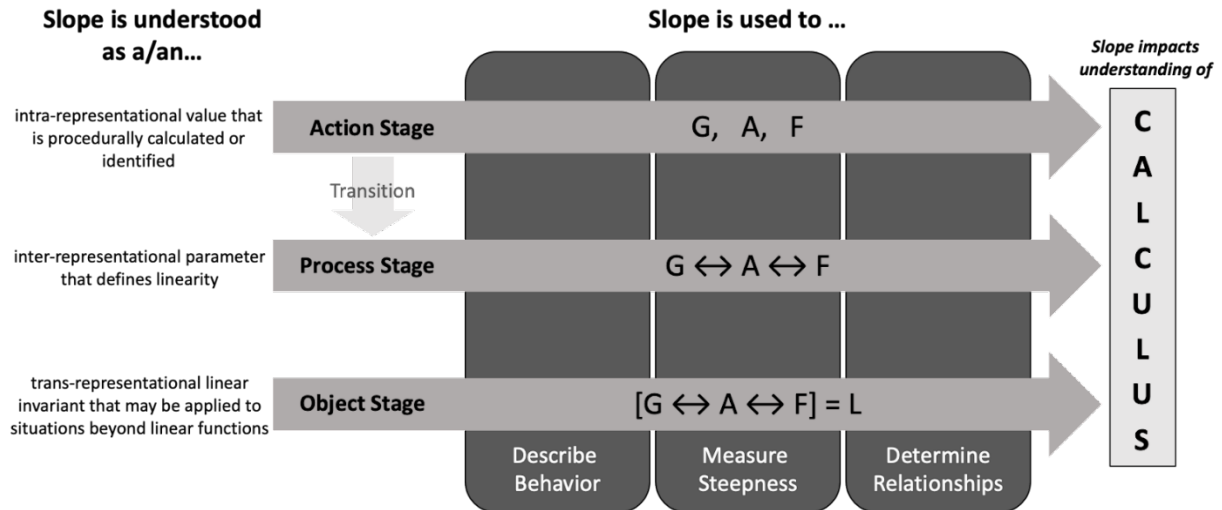


Figure 1: APOS-Slope Framework (Nagle et al., 2019)

At the Process stage, students have an intra-representational understanding of slope that allows them to move easily between visual and analytic representations of slope. They recognize that certain ways of thinking about slope might be more efficient in different contexts, and their understanding of slope is grounded in covariational reasoning (Carlson et al., 2002). Students at the Object stage understand all the Process stage connections. They also are able to view slope as a linear invariant within an equivalence class of ratios that transcends contexts and representations. They can deal with new scenarios that involve unfamiliar units or contexts.

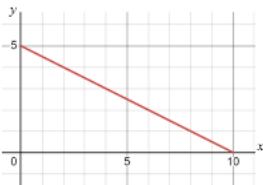
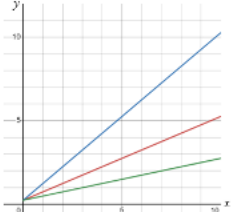
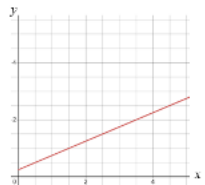
The vertical columns in the APOS-slope framework (see Figure 1) distinguish between the common uses of slope. To *describe behavior* involves using slope to determine if a line's graph or its outputs are increasing, decreasing, or horizontal/constant as corresponding input values increase. To *measure steepness* involves using slope to determine a linear graph's angle of inclination that impacts the severity of its tilt or the outputs' rate of increase when given equal input increments. To *determine relationships* involves using slope to determine if graphs of lines intersect or if systems of linear functions have one, none, or infinitely many solutions. It can also involve using negative reciprocal slopes to determine if a pair of lines intersect in a right angle.

This study leverages dynagraph tasks to evaluate understanding of slope with a preservice teacher. In particular, we consider the following **research question**: *What understanding of slope as a functional property (F) does a preservice teacher evidence while using slope to describe behavior, measure steepness, and determine relationships?*

### Methodology

The participants were a convenience sample of a methods class of three preservice teachers at a regional university in the Northeastern US (pseudonyms: Erin, Jessie, Robin). All three were upper-level undergraduate, preservice teachers preparing for student teaching. Our study focuses only on Robin due to space limitations. The researchers modified a traditional slope problem to

develop a series of tasks to capture the different uses of slope (see Figure 2). Each task was accompanied by an interactive dynagraph. One researcher administered the three slope tasks to the group, who worked collaboratively, with accompanying dynagraphs students could manipulate while solving the tasks. The researchers watched interactions independently and completed a line-by-line analysis of each preservice teacher’s transcript to determine which slope reasoning in the APOS-Slope Framework was implied and how the dynagraph was leveraged. The two researchers jointly discussed Robin’s reasoning, focusing on key shifts in reasoning.

Original Task	Describe Behavior Task	Measure Steepness Task	Determine Relationships Task
<p>It takes Maddy 2.25 hours to mow four lawns and 3.75 hours to mow seven lawns. Find a linear equation that will tell how you how long (<math>y</math>) it takes Maddy to mow <math>x</math> lawns.</p>	<p>1. Explore the relationship between number of lawns and total time spent using the dynagraph at: <a href="https://www.geogebra.org/m/yvheyemg">https://www.geogebra.org/m/yvheyemg</a></p> <p>As <math>x</math> increases, what happens to <math>y</math>?</p> <p>Explain why this makes sense given that <math>x</math> is the number of lawns and <math>y</math> is the number of hours it takes Maddy to mow them.</p> <p>As <math>x</math> decreases, what happens to <math>y</math>? Why does this make sense?</p> <p>2. Now, let’s look at a different situation. Suppose that <math>x</math> still represents the number of lawns Maddy mows. Think about a possible scenario where <math>y</math> could represent something other than the number of hours so the relationship between <math>x</math> and <math>y</math> could be represented by the graph below.</p> 	<p>Pokey Paul and Speedy Stef also mow lawns. It takes them the same amount of time as Maddy to get set up. However, Paul takes twice as long as Maddy to mow one lawn while Stef can mow a lawn in half the time it takes Maddy.</p> <p>Explore Maddy, Paul, and Stef’s lawn versus time relationships at <a href="https://www.geogebra.org/calculator/dkzrsht6">https://www.geogebra.org/calculator/dkzrsht6</a></p> <p>As <math>x</math> increases, how does <math>y</math> change for each person? For whom does <math>y</math> change <i>fastest</i>? For whom does <math>y</math> change <i>slowest</i>? Explain why.</p> <p>Three graphs are provided on the same coordinate plane, where <math>x</math> represents number of lawns and <math>y</math> represents time in hours. Please label each one according to whether it represents Maddy, Paul, or Stef. Explain how you know.</p> 	<p>It usually takes Maddy 15 minutes to set-up. Occasionally, small repairs are needed, or gas and oil must be added to the lawn mowers before she starts mowing. This can increase her set-up time.</p> <p>Explore the relationship between the number of lawns mowed and total time spent mowing for Week 1 and Week 2, where you can alter the set-up time for Week 2. <a href="https://www.geogebra.org/calculator/pbchffoxe">https://www.geogebra.org/calculator/pbchffoxe</a></p> <p>Once Maddy is set-up, does the set-up time impact how quickly she is able to mow lawns? Justify your answer using the dynagraph.</p> <p>The graph representing the amount of time (<math>y</math>) that it takes Maddy to mow <math>x</math> lawns in Week 1 is provided. Sketch two possible graphs for Week 2, one where set-up takes longer than Week 1 and one where set-up is shorter than Week 1. Explain how the graphs are related. Why does that make sense?</p> 

**Figure 2: Series of Slope Tasks Given to Participants on Three Uses of Slope Robin’s Results on Three Slope Tasks**

We describe Robin’s thinking and use of dynagraphs with a number line for number of lawns mowed (input) and parallel number lines for time (outputs) for each of the three tasks. Key elements, including transitions in her thinking, are summarized in Table 1.

### Describe Behavior Task

When presented with this task, Robin and her peers began engaging with the dynagraph by dragging the number of lawns mowed to see corresponding changes in the amount of time it takes Maddy to mow. Robin was the first within her group to note that “As  $x$  increases,  $y$  also increases.” After attending to this directional change relationship, the group’s discussion turned to the rate at which the time increases. Initially, all three students believed the dynagraph showed that Maddy’s time to mow a lawn decreases as the number of lawns increases. Robin used personal experiences, explaining that she “mowed lawns for a summer. It takes practice.” She added, “It’s taking less time. She’s working out how to do it faster or something.”

The students manipulated the dynagraph in silence for six seconds, adjusting the number of lawns mowed from smaller to larger values, and back again. Robin suddenly asked, “Oh wait, is it actually taking less time?” to which a peer responded, “Or is it taking the same time?” While it appears, an important cognitive shift may have occurred, Robin then elaborated that she did not “think the line is changing too drastically.” She added, “I don’t know where it changes” and after

a few seconds of dragging the dynagraph, “It isn’t really getting straight.” These explanations suggest Robin was looking for cues in the visual representation of the connecting segment linking the input and output in the dynagraph, which might also explain why initially she and her peers thought the rate of change was decreasing. When later prompted by the researcher to support the conclusion about the same-directional relationship between number of lawns and elapsed time, Robin returned to the context of the problem by stating, “Because if you’re doing more of something, it can’t take you less time. If you mow more lawns, it wouldn’t take you less time. It would take you more.” Asked to summarize the relationship between lawns mowed and time spent mowing, Robin responded, “They’re dependent.” This response suggests Robin is coordinating change in inputs and outputs (Carlson et. al., 1996), but her previous discussion also suggests she is coordinating direction of change between inputs and outputs.

When presented with the Cartesian plane graph of a decreasing relationship, one of Robin’s peers suggested that the graph could represent the amount of gasoline in the lawn mower, and Robin agreed. At the conclusion of this discussion, Robin reflected on what she learned from the activity, stating, “The relationship between the  $x$  and  $y$  axis. The labels on the axis and what they represent. How positive and negative slope have different meanings.”

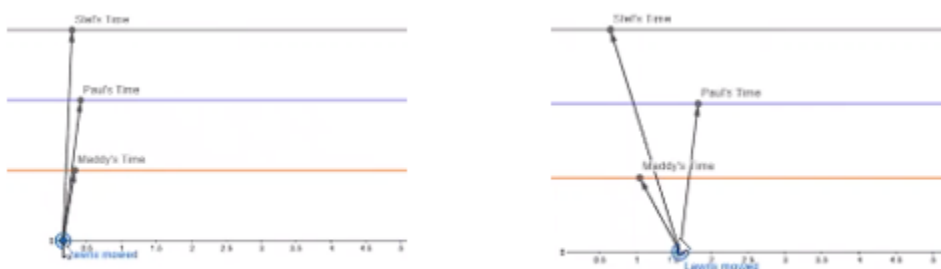
**Table 1: Key Elements of Robin’s Slope Thinking by Task**

	Key Elements of Reasoning	Supporting Quotes	Uses of Dynagraphs	Shifts in Reasoning
<i>Describe Behavior</i> Task	The more lawns mowed; the more time passes.	<i>As <math>x</math> increases, <math>y</math> also increases.</i>	Slides inputs back and forth rapidly.	N/A
	The more lawns mowed, the faster she mows each lawn.	<i>It’s taking less time. She’s working out how to do it faster or something.</i>	Continues to slide inputs back and forth.	The rate at which Maddy mows lawns stays the same
<i>Measure Steepness</i> Task	Paul’s speed is constant; Stef and Maddy get faster.	<i>Paul isn’t really that slow at the beginning. He’s kind of a constant rate, but they [Maddy and Stef] get faster throughout.</i>	Moves input to 0 where behavior is “consistent” and then to larger values (see Figure 3)	Agrees with peers after discussing the context that <i>Paul’s [output] changes the most.</i>
<i>Determine Relationships</i> Task	Week 2 takes more time initially, but eventually the time output will match Week 1.	<i>...as she does more lawns, the setup time doesn’t impact as much.</i>  <i>It catches up eventually.</i>	Adjusts the set-up time and moves the input to large values (see Figure 4)	<i>They’re never going to pass each other. So, week two’s dot will always be further to the right than week one.</i>

### **Measure Steepness Task**

When this task was posed, Robin was initially quiet as her peers discussed whether Pokey Paul’s time changed faster or slower than Speedy Stef’s. While the others engaged in this debate,

Robin appeared to be thinking about the relationship she saw, announcing at the first break in conversation that “It’s gonna change somewhere.” She added, “Paul isn’t really that slow at the beginning. He’s kind of at a constant rate. But they [Maddy and Stef] get faster throughout.” Robin’s classmates continued to discuss the relationship between Pokey Paul and Maddy, but Robin appears stuck in her own reasoning. She reiterated, “Paul’s isn’t changing” adding “his slope is like, look at his line throughout the whole thing. It doesn’t really change.” Robin was attending to the slope of the connecting segment between the lawns mowed (input) and Paul’s time spent mowing (output) as shown in Figure 3. Because the input-output connecting segment does not appear to be changing steepness, Robin believed that Paul’s rate was staying the same. On the other hand, Stef’s and Maddy’s connecting segments changed more drastically as the number of lawns mowed increases, leading Robin to explain, “...Paul’s doesn’t change at all, and Maddy’s is changing the slowest, and Stef’s is changing the fastest.”



**Figure 3: Dynagraph Screenshots that Supported Robin’s Reasoning about Rates**

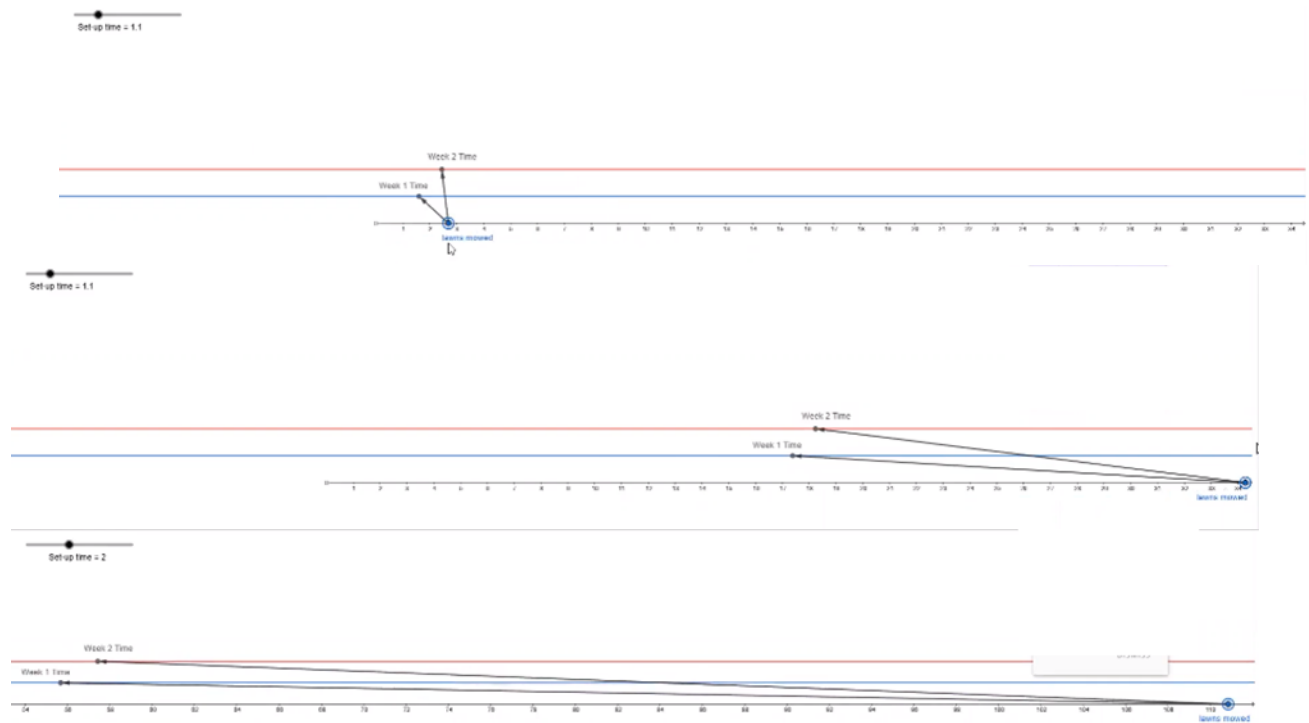
Eventually, one of Robin’s peers applied covariational reasoning about each mowers’ speed and related it to the dynagraph to suggest that Paul’s elapsed time would change the most since it takes him more time to mow the same number of lawns. Although Robin did not acknowledge the dynagraph, she did relate to this contextual explanation, replying, “Oh, I had it backwards. So, Paul’s changes the most.” She and her peers labeled the three Cartesian plane graphs correctly based on steepness and the rate at which each person mows lawns. As the task wrapped up, Robin voiced her frustration with the dynagraph, “I understand the input-output, but it would be easier to look at it, like, as an equation.” In the task reflection she stated, “The relationships matter. For example, just because Paul is slow does not mean his slope takes longer. It is actually a smaller slope.” Her response raises questions about what Robin actually attended to on the dynagraph and how she related it to slope. She knew Paul is the slowest, and that means “Paul’s changes the most.” However, she did not appear to attend to that change via the dynagraph.

### ***Determine Relationships Task***

When the task was posed, Robin immediately recognized that set-up time is important stating, “So, if there’s more setup time, then they’re not going to be the same.” But again, the representation in the dynagraph was at odds with her intuition. Robin moved the set-up time to 1.1 hours (“close to 1”) and then discussed what happens as she changed the number of lawns mowed. For a small number of lawns (see Figure 4a), Robin described that “It’s going to obviously be not the same amount,” but as she dragged the number of lawns to larger values (see Figure 4b) she stated that “They almost catch up.” She continued saying, “At the beginning it makes a difference...but as she does more lawns, the setup time doesn’t impact as much.” This belief persisted. A peer explained that since the rate at which Maddy mows the lawns is the

same, Week 2 time will always stay the same amount ahead of Week 1 time. Robin countered, arguing, “It catches up eventually.” She adjusted the set-up time and moved the dynagraph to much larger values of lawns, seemingly looking for a value at which the “catch up” occurred. These interactions illustrate her focus on the connecting segment’s steepness rather than on the rates at which outputs change relative to inputs. Eventually, the researcher posed the question, “Does the set-up time impact the amount of time it takes Maddy to mow lawns?” This prompted more discussion in the group, and Robin adjusted the set-up time to 2 hours once again moving the number of lawns to large values. She conceded, “I guess she probably always is 2 minutes behind, but it is hard to tell when you do the small numbers.” See Figure 4c.

When the group received the Cartesian plane extension problem, Robin very quickly stated, “Yeah, it's going to be parallel because the time it took to mow didn't change.” Suddenly, Robin recognized that the familiar Cartesian representation should relate to the dynagraph, saying, “Why the heck are these [dynagraphs] so hard to look at? Because you don't realize that these are technically parallel.” The researcher asked how the parallel relationship was represented in the dynagraph. Robin stated, “The dots. They're never going to pass each other. So, Week two's dot will always be further to the right than Week one's. And then the distance between the two jobs will always stay the same.” Robin verbalized frustration with the dynagraph again, stating, “It would be easier to see on a graph. The lines kind of distract from what you're actually trying to work out.” In reflection to the activity, Robin reported that the activity helped her to recognize “The idea of having the same slope. Then lines will be parallel. The overall slope won't change.”



**Figures 4a-4c: Series of Screenshots of Dynagraph Manipulations Discussion**

Despite an extensive mathematical background with both constant and variable rates of change, Robin demonstrated limitations in reasoning about slope as a functional property when

presented with a new representation (i.e., dynagraphs). Robin was successful applying functional property thinking to *describe behavior*, “seeing” the corresponding direction of change in inputs and outputs in the dynagraph almost immediately. It was difficult for her to see that the rate of change was constant, but eventually she led her classmates away from the notion that the rate was decreasing while using the dynagraph to observe the input-output relationship. Although other shifts in Robin’s reasoning did occur, this is the only shift that was prompted by the dynagraph and supported by covariational reasoning. This suggests that Robin’s applied at least the  $T_F$  stage of reasoning when using slope to *describe behavior*.

Robin had much less success applying functional property reasoning of slope to *measure steepness* and *determine relationships*. The dynagraph representation challenged her familiar visual cues from the Cartesian plane. Robin looked for visual cues in the steepness of the connecting segment linking the inputs and outputs, suggesting that she may be limited to  $A_G$  thinking when using slope to *measure steepness*. Her “shape thinking” led her to conclude that Paul’s rate was constant while Stef’s and Maddy’s were speeding up and that eventually the time spent mowing in Week 1 would catch up to the time in Week 2. Despite these misconceptions when working with dynagraphs, Robin was very comfortable using slope when the linear graph was on a Cartesian plane, and she even voiced her frustration at how hard it was to “see” the steepness and parallel relationships in the dynagraph representation. Once the Cartesian plane representation was revealed, Robin made a connection between parallel lines and the distance between Week 1 and Week 2 time values (outputs) staying the same on the dynagraph. Using the two representations in conjunction helped her reason covariationally, an indicator of moving toward  $T_F$  thinking while using slope to *determine relationships*. There is no evidence that she was able to reason about slope as a functional property used to *measure steepness* even after engaging with the Cartesian plane representation of the three graphs. One can deduce that Robin’s prior experiences using slope to *measure steepness* and *determine relationships* have been focused primarily on  $A_G$ . She may need additional opportunities to think about slope as an algebraic ratio and functional property while using it to *measure steepness* and *determine relationships* in order to develop an understanding of slope as an inter-representational parameter of linearity.

### Conclusions

As an unfamiliar representation to Robin, dynagraphs highlighted that she was able to apply functional property reasoning for some, but not all, uses of slope. Despite demonstrating at least  $T_F$  reasoning about slope when *describing behavior*, she did not initially apply even  $A_F$  reasoning when using slope to *determine relationships* and never evidenced  $A_F$  reasoning to *measure steepness*. These results highlight how students’ slope reasoning might vary based on the purpose for which slope is used. The results in this study align with Styers and colleagues’ (2022) findings that teachers need experiences interacting with tasks to build robust notions of steepness through explicit connections to a variety of physical contexts, so they develop imagery and mathematical terminology in rich and meaningful ways. Future slope research should consider which uses of slope are being studied and consider how using multiple representations (e.g., dynagraphs, Cartesian graphs) might provide additional insight into students’ thinking.

### References

- Adu-Gyamfi, K., & Bossé, M. J. (2014). Processes and reasoning in representations of linear functions. *International Journal of Science and Mathematics Education*, 12(1), 167-192. <https://doi.org/10.1007/s10763-013-9416-x>
- Arnon, I., Cottrill, J., Dubinsky, E., Oktaç, A., Roa Fuentes, S., Trigueros, M., & Weller, K. (2014). *APOS Theory: A framework for research and curriculum development in mathematics education*. Springer Verlag: New York.

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.



- Asiala, M., Cottrill, J., Dubinsky, E., & Schwingendorf, K. E. (1997). The development of students' graphical understanding of the derivative. *The Journal of Mathematical Behavior*, 16(4), 399-431. [https://doi.org/10.1016/S0732-3123\(97\)90015-8](https://doi.org/10.1016/S0732-3123(97)90015-8)
- Avcu, S., & Türker Biber, B., (2022). Prospective middle school mathematics teachers' conceptualizations of slope. *International Journal of Mathematical Education in Science and Technology*, 53(11), 3115-3117.
- Bailey, N. G., Reed, S., Fye, K., McCulloch, A., & Lovett, J. (2020). “#WODB: The Power of Dynamic Representations.” *Mathematics Teacher: Learning and Teaching PK-12* 113, 10, 845-850.
- Byerley, C., & Thompson, P. W. (2017). Secondary mathematics teachers' meanings for measure, slope, and rate of change. *The Journal of Mathematical Behavior*, 48, 168-193. <https://doi.org/10.1016/j.jmathb.2017.09.003>
- Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education*, 33, 352-378. <https://doi.org/10.2307/4149958>
- Coe, E. E. (2007). Modeling teachers' ways of thinking about rate of change. Unpublished doctoral dissertation, Arizona State University, Tempe, AZ. Retrieved from <http://pat-thompson.net/PDFversions/Theses/2007Ted.pdf>
- De Bock, D., Van Dooren, W., & Verschaffel, L. (2015). Students' understanding of proportional, inverse proportional, and affine functions: Two studies on the role of external representations. *International Journal of Science and Mathematics Education*, 13(1), 47-69. <https://doi.org/10.1007/s10763-013-9475-z>
- Dubinsky, E. (2014). Actions, processes, objects, schemas (APOS) in mathematics education. In S. Lerman (Ed.), *Encyclopedia of Mathematics Education* (pp. 8-11). Dordrecht: The Netherlands: Springer.
- Frank, K., & Thompson, P. W. (2021). School students' preparation for calculus in the United States. *ZDM—Mathematics Education*, 53(3), 549-562. <https://doi.org/10.1007/s11858-021-01231-8>
- Hattikudur, S., Prather, R. W., Asquith, P., Alibali, M. W., Knuth, E. J., & Nathan, M. (2012). Constructing graphical representations: Middle schoolers' intuitions and developing knowledge about slope and y-intercept. *School Science and Mathematics*, 112(4), 230-240. <https://doi.org/10.1111/j.1949-8594.2012.00138.x>
- Hoban, R. A. (2021). Introducing the slope concept. *International Journal of Mathematical Education in Science and Technology*, 52(6), 879-895. <https://doi.org/10.1080/0020739X.2020.1723726>
- Lingefjärd, T. & Farahani, D. (2017). The elusive slope. *International Journal of Science and Mathematics Education*, 16(6), 1187-1206. <https://doi.org/10.1007/s10763-017-9811-9>
- Lisarelli, G. (2017). Exploiting potentials of dynamic representations of functions with parallel axes.” In G. Aldon & J. Trgalova (Eds.). *Proceedings of the 13th International Conference on Technology in Mathematics Teaching* (pp. 144–150). Lyon, France: ICTMT.
- Moore, K. C., & Thompson, P. W. (2015). Shape thinking and students' graphing activity. In T. Fukawa-Connelly, N. E. Infante, K. Keene, & M. Zandieh (Eds.). *Proceedings of the 18th Meeting of the MAA Special Interest Group on Research in Undergraduate Mathematics Education* (pp. 782–789). Pittsburgh, PA: RUME.
- Moore-Russo, D. & Nagle, C. (in press). Connecting representations and ways of thinking about slope from algebra to calculus. *The Mathematics Enthusiast*.
- Mudaly, V., & Moore-Russo, D. (2011). South African teachers' conceptualisations of gradient: A study of historically disadvantaged teachers in an Advanced Certificate in Education programme. *Pythagoras*. 32(1), 27-33.
- Nagle, C. R., Martínez-Planell, R., & Moore-Russo, D. (2019). Using APOS theory as a framework for considering slope understanding. *Journal of Mathematical Behavior*, 54, 100684. <https://doi.org/10.1016/j.jmathb.2018.12.003>.
- Nagle, C., & Moore-Russo, D. (2013). Connecting slope, steepness, and angles. *Mathematics Teacher*, 107(4), 272-279.
- Nagle, C., & Moore-Russo, D. (2014). Slope across the curriculum: Principles and Standards for School Mathematics and Common Core State Standards. *The Mathematics Educator*, 23(2), 40-59.
- Nagle, C., Moore-Russo, D., & Martinez-Planell, R. (2016). A framework for describing conceptions of slope. In M. B. Wood, E. E. Turner, M. Civil, & J. A. Eli (Eds.) *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Ch. 3, pp. 195-198). Tucson, Arizona: The University of Arizona.
- Nagle, C., Moore-Russo, D., & Fisher, T. (2022). A study of slope explanations across a standards-based textbook series. *International Journal for Mathematics Teaching and Learning*, 23(1), 25-45.
- Nagle, C. R., Casey, S., & Moore-Russo, D. (2017). Slope and line of best fit: A transfer of knowledge case study. *School Science and Mathematics*, 117(1-2), 13-26.

- Newton, X. A. (2018). Undergraduate STEM majors' understanding of slope. In *Improving Teacher Knowledge in K-12 Schooling* (pp. 75-100). Cham, Switzerland: Palgrave Macmillan. [https://doi.org/10.1007/978-3-319-71207-9\\_4](https://doi.org/10.1007/978-3-319-71207-9_4)
- Ozen, D. Y., Bailey, N., Fletcher, S. Sanei, H., McCulloch, A., Lovett, J. & Charity Cayton, C. (2022). Preservice secondary teachers' reasoning about static and dynamic representations of function. In D. Olanoff, K. Johnson, & S. Spitzer (Eds.) *Proceedings of the 43rd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, (pp.1639-1648). Philadelphia, PA: PME-NA.
- Planinic, M., Milin-Sipus, Z., Katic, H., Susac, A. & Ivanjek, L. (2012). Comparison of student understanding of line graph slope in physics and mathematics. *International Journal of Science and Mathematics Education*, 10, 1393–1414. <https://doi.org/10.1007/s10763-012-9344-1>
- Stanton, M., & Moore-Russo, D. (2012). Conceptualizations of slope: A look at state standards. *School Science and Mathematics*, 112(5), 270-277.
- Stump, S. L. (1999). Secondary mathematics teachers' knowledge of slope. *Mathematics Education Research Journal*, 11(2), 124-144. <https://doi.org/10.1007/BF03217065>
- Stump, S. L. (2001). Developing preservice teachers' pedagogical content knowledge of slope. *The Journal of Mathematical Behavior*, 20, 207-227.
- Styers, J. L., Nagle, C. R., & Moore-Russo, D. (2020) Teachers' interpretations of students' slope statements: Implications for professional noticing and mathematical knowledge for teaching. *Canadian Journal of Science, Mathematics, and Technology Education*, 20(3), 504-520.
- Tanışlı, D., & Bike Kalkan, D. (2018). Linear functions and slope: How do students understand these concepts and how does reasoning support their understanding? *Croatian Journal of Education: Hrvatski časopis za odgoj i obrazovanje*, 20(4), 1193-1260. <https://doi.org/10.15516/cje.v20i4.2746>
- Teuscher, D., & Reys, R. E. (2010). Slope, rate of change, and steepness: Do students understand these concepts? *Mathematics Teacher*, 103(7), 519-524.
- Walter, J. G., & Gerson, H. (2007). Teachers' personal agency: Making sense of slope through additive structures. *Educational Studies in Mathematics*, 65, 205-233. <https://doi.org/10.1007/s10649-006-9048-y>
- Zandieh, M. J., & Knapp, J. (2006). Exploring the role of metonymy in mathematical understanding and reasoning: The concept of derivative as an example. *The Journal of Mathematical Behavior*, 25(1), 1-17. <https://doi.org/10.1016/j.jmathb.2005.11.002>
- Zaslavsky, O., Sela, H. & Leron, U. (2002). Being sloppy about slope: The effect of changing the scale. *Educational Studies in Mathematics*, 49(1), 119–140. <https://doi.org/10.1023/A:1016093305002>

## USING EXIT TICKETS TO LEARN FROM MISTAKES AND DEVELOPING GROWTH MINDSET IN AN ALGEBRA CLASS

Veena Paliwal  
University of West Georgia  
vpaliwal@westga.edu

*This study was designed to examine the use of mistakes to promote students' performance in undergraduate Algebra classes by developing a growth mindset. Participants were seventy-four students from three Algebra classes and received one of the three interventions along with regular instruction: (a) growth mindset feedback on mistakes (growth-feedback, n=27), (b) regular feedback on mistakes (feedback-only, n=23), and (c) watching video presentation (control, n=24). Participants from the growth-feedback and the feedback-only groups performed significantly better than the control group in the exams conducted after the intervention. Also, the growth-feedback participants outperformed the other two classes. Findings highlight the importance of valuing mistakes, providing feedback and fostering a growth mindset in developing students' math skills and promoting their academic achievement.*

Keywords: Algebra, Growth-mindset, Constructive feedback.

Growth mindset is the belief that an individual's most basic abilities can be developed through dedication and hard work—brains and talent are just the starting point (Dweck, 2008). A growth mindset identifies a direct correlation between one's efforts and increased knowledge. Mistakes are inevitable in math and are also a source of frustration for both educators and students, as they reflect on potential misunderstandings and lack of learning (Lischka, et al., 2018; Mueller & Dweck, 1998). Growth mindset unleashes the potential of viewing mistakes as opportunities and using them as constructive tools for learning (Boaler, 2016; Mueller & Dweck, 1998; Sun, 2015). Growth mindset promotes individuals to engage with materials and skills with an understanding that failures and setbacks can ultimately bring success.

Although researchers (Eggleton, & Moldavan, 2001; Kaur, 2008) emphasize on the importance of instructional approach based on providing constructive feedback on students' mistakes, there are studies (Moldavan, 1986) that did not find this approach effective in promoting students' learning. This study examines the efficacy of constructive immediate feedback using growth mindset messages about mistakes in developing math skills and maximizing the academic achievement of students.

### **Growth Mindset and Mistakes**

In teaching mathematics, instructors and students often come across mistakes that are a part of the learning process. Mistakes provide the first step in learning by highlighting an opportunity to construct new knowledge or work on misconceptions. An understanding of mistakes can facilitate both instructors and students in the strengthening of their knowledge. Instructors can use mistakes to incorporate teaching strategies that can facilitate students' deepening their learning by providing them opportunities to discover and construct knowledge. Students—using constructive feedback on mistakes—can use the learning opportunity from their mistakes to develop the knowledge that is lacking.

### **Importance of Constructive Formative Feedback**

Instructors need to have a thorough understanding of the concepts that students are lacking and collecting timely feedback from students can help in getting acquainted with students' mistakes and in providing instructors insight to students' understanding (Doabler & Fien, 2013; TÜRKDOĞAN & Adnan, 2021). While it is imperative that instructors provide feedback on students' understanding, the quality and time of feedback plays a crucial role in fostering students' learning.

### **Using Growth Mindset View of Mistakes as a Learning Tool**

A growth mindset is known to foster a notable change in students' attitude towards seeing mistakes as opportunities to learn and improve (Dweck, 2008; Boaler, 2016, 2022; Moser et al., 2011), and the results support the findings that the individuals with a growth mindset had superior accuracy after getting acquainted with mistakes compared to individuals with fixed mindset (Fixed mindset beliefs are assumptions that the brain cannot grow; individuals are either smart or not, and there is nothing that can change their innate ability). Learning from the examination of mistakes allows the brain to form productive pathways for new concepts while focusing on learning concepts that lead to mistakes/misunderstandings (Boaler, 2016; Borasi, 1996). Moreover, instructional experiences based on focusing on mistakes are recommended by the Standards for Mathematical Practice (Common Core State Standards Initiative, 2011): make sense of problems and persevere in solving them (SMP 1) and reason abstractly and quantitatively (SMP 2). Mistakes can help students in recognizing their own strengths and weaknesses and allow a window for instructors into students' thinking.

Available research (An & Wu, 2012; Kingsdorf & Krawec, 2014; Schnepfer & McCoy, 2013) supports using mistakes to address students' misconceptions and learning process and using them to design curriculum. The use of mistakes that go beyond diagnostic tools and as a medium to combat misconceptions and promote learning has not been explored in the available research. It is also not clear how to effectively address mistakes in an instructional approach. In other words, following research questions are not explored by the available research: Can mistakes be used as a pedagogical tool to promote students' learning? Does the effectiveness of feedback on mistakes depend on the messages that they deliver?

### **Present Study**

Present study was designed to understand the role of formative feedback on mistakes and developing a growth mindset towards mistakes in students' academic success in a mathematics classroom. Present study had two goals: (a) investigating the efficacy of capitalizing on mistakes for promoting students' academic success by integrating them in the instruction; (b) effectively engaging students in the instruction to take charge of their own learning by developing a growth mindset towards mistakes. The first step in addressing whether building on mistakes facilitates learning mathematics is to evaluate all participants' performance before the intervention (exam 1). The three classes: control group (class 1), feedback-only (class 2), and growth-feedback (class 3) received identical instructional material. To evaluate the efficacy of building knowledge on mistakes, the feedback-only and the growth-feedback classes received formative feedback on mistakes using exit tickets. Growth-feedback class received growth mindset intervention about mistakes in addition to the feedback on the exit tickets. Follow up exams (exam 2 and exam 3) evaluated the impact of the intervention on promoting learning.

## Hypotheses

The study addressed following hypotheses to examine the efficacy of an intervention designed to investigate the use mistakes for promoting student success in mathematics classrooms:

H1. Does focusing on mistakes promote students' performance? Participants' performance between the experimental groups (the feedback-only and growth-feedback groups) and the control group were compared on the exams conducted before and after the intervention. If focusing on mistakes facilitates students' performance, then feedback-only and growth-feedback groups should perform significantly better than control group on exam 2 and exam 3, but there should be no difference in their performances in exam 1.

H2. Does the growth mindset intervention facilitate students' academic success? Performance of the participants in control group, feedback-only, and growth-feedback classes was compared in the exams conducted before and after the intervention. If growth mindset intervention is successful, then participants in growth-feedback group should outperform participants from control group and feedback-only on exams 2 and exam 3. There should be no difference in their performances in exam 1.

## Method

### Participants

Participants were seventy-four students from three sections of algebra class: class 1: control group (n=24), class 2: feedback-only (n=23), and class 3: growth-feedback (n=27). Participation was voluntary, and all participants signed a consent form before the intervention. Design The study had an experimental design where participants from the three classes were compared on their performances in exams conducted before and after the intervention. All classes received identical instructional material in the classrooms. Students from feedback-only and growth-feedback groups received formative feedback on mistakes and students from growth feedback group received growth mindset intervention on learning from mistakes (see Table 1 for the project timeline).

**Table 1: Project Timeline**

Activities conducted	Control Group	Experimental Group: Feedback-only	Experimental Group: Growth-feedback
Week 1-15: Regular classroom instruction	Yes	Yes	Yes
Week 5: Exam 1	Yes	Yes	Yes
Week 6-15: working on exit tickets after each class and receiving feedback	No	Yes	Yes
Week 6-15: watching and working on problems from the video presentation on the topics covered	Yes	No	No
Week 6-15: growth-mindset based feedback on exit tickets after each class	No	No	Yes
Week 10: Exam 2	Yes	Yes	Yes
Week 15: Exam 3	Yes	Yes	Yes

## Intervention

The three classes met two times a week for 75 minutes and received identical instructional material in the class for 45 minutes. Last 30 minutes of each class were devoted to the problems based on concepts covered that day and approach to solving those problems was different in each class based on the intervention: control group watched a video presentation on solving some problems on concepts discussed in the class that day, feedback-only and growth-feedback worked on exit tickets and received feedback on their performances.

**Formative Feedback.** Feedback-only and growth-feedback groups worked on exit tickets that were based on problems covered in the class that day. It was mandatory for each student to work on the exit ticket and receive feedback from the instructor before exiting the classroom. Exit tickets were used to identify students' misconceptions and lack of knowledge and were used as a tool to improve/modify instruction. Exit tickets were used in two ways: providing immediate feedback to students and analyzing mistakes to restructure instruction in the upcoming classes for both groups.

Instructor provided feedback to each student by grading their exit tickets that included classifying their responses as correct or still needing work, detailing mistakes that lead to a wrong answer, and asking students (if time permits) to correct them before leaving the classroom. Instructor used mistakes to diagnose error patterns and adjusting instruction for next classes in feedback-only and growth-feedback groups by incorporating instructional techniques involving working on correcting possible mistakes on the new topics that were introduced in the class. This involved: revisiting the mistakes based on the analysis of students mistakes from the exit tickets, engaging students in collaborative work for identifying the source of wrong solutions and working on a strategy to correct them, and re-phrasing steps involved in computations by providing algebraic justification and connecting them with students' existing knowledge.

**Growth Mindset Feedback.** Participants from the growth-feedback group also received intervention after exam 1 on growth mindset and on the importance of mistakes in learning. Intervention involved providing students from growth-feedback literature about the importance of mistakes (Boaler, 2022) in mathematics and discussions in the classroom focused on these questions: What are mistakes? How do mistakes help the brain in strengthening existing connections? What role do mistakes play in learning? In addition to this, growth-feedback students received feedback on the exit tickets that included messages about learning from the mistakes in their work on exit tickets and using them to develop conceptual understanding of the concepts covered. Students were asked to rework on the problems from exit tickets (if time permits) by identifying their own mistakes (by comparing with the solution) and devising a strategy to remediate them. Instructor, then, provided feedback on exit tickets detailing how a particular mistake can be corrected by developing conceptual understanding of a particular concept. Both feedback-only and growth-feedback groups received immediate feedback on their work on the exit tickets, but only the growth-feedback students were prompted using the growth mindset approach. Students from growth-feedback were facilitated to take charge of their own learning and work on the mistakes by using their existing knowledge.

## Measures

### Exams

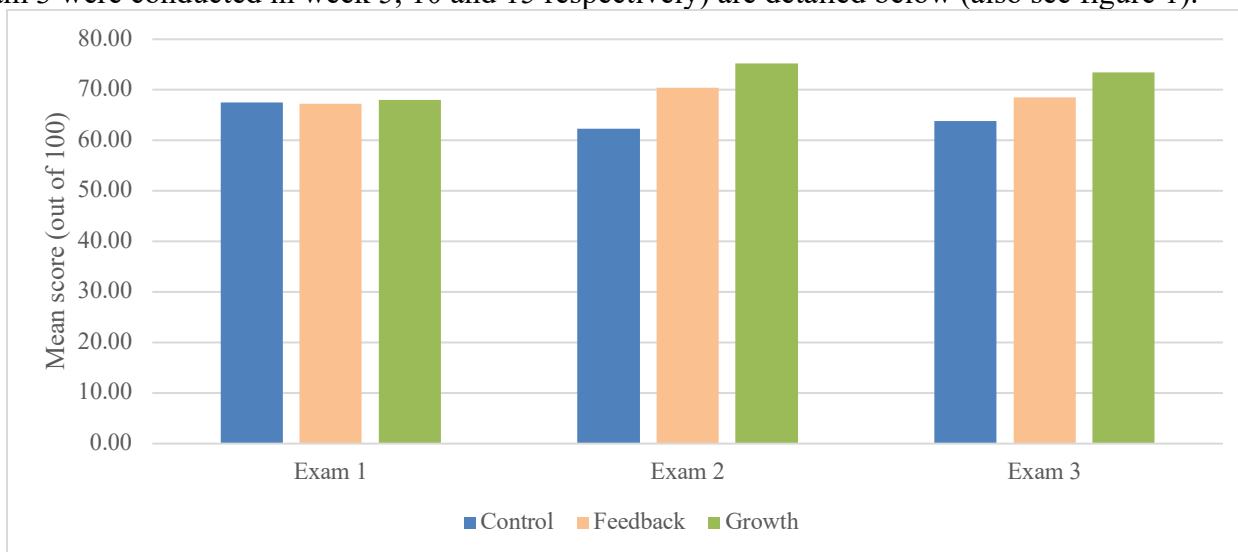
Participants' performance in the three groups were compared in the three exams, exam 1 was conducted before the intervention, and exam 2 and exam 3 were conducted after the intervention. The three exams covered different topics from the syllabus (exam 1-functions, inequalities, linear

functions, equations; exam 2-polynomials and rational functions, and exam 3 exponential and logarithmic functions, systems of equations and inequalities). An exam was scheduled every fifth week of the semester. Research has shown that mathematical concepts are interrelated (Batanero, & Díaz, 2007; Bourbaki, 1950) and build on the understanding of previous concepts (for example, addition and subtraction are interrelated and a thorough understanding of addition facilitate in understanding subtraction  $8-3=?$  can be solved by knowing  $3+?=8$ ). It is crucial for students to have a thorough understanding of topics they learn as a new concept builds on the knowledge of the previous one. Therefore, an instructional approach based on using students' mistakes on everyday topics covered in the class as a learning tool was a step in the right direction.

## Results

### Students' performance (H1)

Students' average score in the three exams was compared between the three classes using ANOVA and Tukey HSD. Students' performances in the three exams (exam 1, exam 2, and exam 3 were conducted in week 5, 10 and 15 respectively) are detailed below (also see figure 1).



**Figure 1: Comparison of participants' mean scores in the three exams**

**Exam 1.** One way ANOVA revealed that there was no significant difference between the performances of participants in the three groups with  $F(2,71)=0.06$ ,  $p=0.93$ . A post hoc Tukey HSD showed that there was no significant difference in the performances of participants in the control group ( $M=67.5$ ,  $SD=6.3$ ), the feedback-only group ( $M=67.3$ ,  $SD= 5.8$ ) and the growth feedback group ( $M=68$ ,  $SD=7.0$ ) with  $p>0.05$  in all three cases.

**Exam 2.** One way ANOVA revealed that there was significant difference between the three groups with  $F(2,71)=23.3$ ,  $p<0.01$ . A post hoc Tukey HSD showed that participants from the feedback-only ( $M=70.35$ ,  $SD=6.1$ ) and the growth-feedback ( $M=75.19$ ,  $SD=7.4$ ) groups performed significantly and substantively better than participants from the control group ( $M=62.29$ ,  $SD=6.4$ ) with  $p<0.01$ ; the growth-feedback group performed significantly better ( $p<0.03$ ) than the feedback-only group.

**Exam 3.** One way ANOVA revealed that there was significant difference between the three groups with  $F(2,71)=14.9$ ,  $p<0.01$ . A post hoc Tukey HSD showed that participants from the feedback-only ( $M=68.56$ ,  $SD=6.4$ ) and the growth-feedback ( $M=73.51$ ,  $SD=6.0$ ) groups

performed significantly and substantively better than participants from the control group ( $M=63.87$ ,  $SD=6.5$ ) with  $p<0.03$  and  $p<0.01$  respectively; the growth-feedback group performed significantly better ( $p<0.02$ ) than the feedback-only group.

### Significance

The results of this study highlight the importance of constructive formative feedback in promoting students' academic achievement. Only feedback-only and growth-feedback groups received immediate feedback on their work on the exit tickets and results revealed that these two groups outperformed the control group that received only usual instruction and watched video on the topics covered. All three classes were taught by the same instructor so the difference in students' performance in the three groups can be exclusively attributed to the intervention they received. Students in the experimental groups were provided immediate feedback on their mistakes which could have resulted in promoting conceptual understanding of the concepts. A timely feedback puts both instructor and students in the same boat of learning where they work collaboratively towards academic success of students. The results further support the significance of active learning that was facilitated with the use of exit tickets in the classroom. Closed monitoring of students' progress in learning and detailed feedback on their work allows instructors to build on the mistakes and use them as an effective pedagogical tool.

The other significant contribution of this research comes from the better performance of growth-feedback group that received growth mindset intervention along with feedback on the mistakes—revealing that students' perception of mistakes makes a difference in their learning. When students develop a growth mindset attitude towards their mistakes and welcome these mistakes as learning opportunities rather than hindrance to their success, they initiate a process of self-assessment and work towards the knowledge that is required and is lacking. In this study, the instructor also noticed that the growth-mindset group was more inclined to ask questions when a concept is taught and requesting justification towards any misconceptions in their learning. Students with a growth mindset tend to make the most from their experiences. The research also pinpoints the instructors' messages on mistakes that have more impact on students' learning. The difference in growth-mindset and feedback-only group can be accounted for by the messages that instructor delivered to the students. Growth mindset messages to students that rely on not only resilience but also commitment to learning, develop an attitude of taking responsibility for their own learning. A study by Canning et al. (2022) also supported this finding by revealing that an instructor's fixed mindset has a negative impact on students. The study calls on all educators to make their growth mindset more evident to their students. These findings corroborate with the results that instructors can instill their growth mindset on their students to promote learning. Growth-mindset messages convey an instructor's positive outlook towards students' potential to succeed and enables students to change their perspective towards mistakes from lack of understanding to a learning experience that they can build on. The consistent better performance of the growth-mindset group in exam 2 and exam 3 indicate the long-term positive impact of developing a growth mindset in the classrooms.

### References

- An, S., & Wu, Z. (2012). Enhancing mathematics teachers' knowledge of students' thinking from assessing and analyzing misconceptions in homework. *International Journal of Science and Mathematics Education*, 10, 717-753.
- Batanero, C., & Díaz, C. (2007). The meaning and understanding of mathematics. In *Philosophical dimensions in mathematics education* (pp. 107-127). Springer, Boston, MA.

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.



- Boaler, J. (2016). Mistakes grow your brain. Youcubed at Stanford University Graduate School of Education. Accessed April, 14, 2016.
- Boaler, J. (2022). *Mathematical mindsets: Unleashing students' potential through creative mathematics, inspiring messages and innovative teaching*. John Wiley & Sons.
- Borasi, R. (1996). *Reconceiving mathematics instruction: A focus on errors*. Greenwood Publishing Group.
- Bourbaki, N. (1950). The architecture of mathematics. *The American Mathematical Monthly*, 57(4), 221-232.
- Canning, E. A., Ozier, E., Williams, H. E., AlRasheed, R., & Murphy, M. C. (2022). Professors who signal a fixed mindset about ability undermine women's performance in STEM. *Social Psychological and Personality Science*, 13(5), 927-937.
- Common Core State Standards Initiative, 2011
- Doabler, C. T., & Fien, H. (2013). Explicit mathematics instruction: What teachers can do for teaching students with mathematics difficulties. *Intervention in School and Clinic*, 48(5), 276-285.
- Dweck, C. S. (2008). Can personality be changed? The role of beliefs in personality and change. *Current directions in psychological science*, 17(6), 391-394.
- Eggleton, P. J., & Moldavan, C. C. (2001). The value of mistakes. *Mathematics Teaching in the Middle School*, 7(1), 42-47.
- Kaur, B. (2008). Teaching and learning of mathematics: What really matters to teachers and students?. *ZDM*, 40, 951-962.
- Kingsdorf, S., & Krawec, J. (2014). Error analysis of mathematical word problem solving across students with and without learning disabilities. *Learning Disabilities Research & Practice*, 29(2), 66-74.
- Lischka, A. E., Gerstenschlager, N. E., Stephens, D. C., Strayer, J. F., & Barlow, A. T. (2018). Making room for inspecting mistakes. *The Mathematics Teacher*, 111(6), 432-439.
- Moldavan, C. C. (1986). *The effect of warning intermediate algebra students about typical errors* (Doctoral dissertation, University of Georgia).
- Moser, J. S., Schroder, H. S., Heeter, C., Moran, T. P., & Lee, Y. H. (2011). Mind your errors: Evidence for a neural mechanism linking growth mind-set to adaptive posterror adjustments. *Psychological science*, 22(12), 1484-1489.
- Mueller, C. M., & Dweck, C. S. (1998). Praise for intelligence can undermine children's motivation and performance. *Journal of personality and social psychology*, 75(1), 33.
- National Governors Association. (2010). *Common core state standards*. Washington, DC.
- Schnepper, L. C., & McCoy, L. P. (2013). Analysis of misconceptions in high school mathematics. *Networks: An Online Journal for Teacher Research*, 15(1), 625-625.
- Stevenson, H. W., Chen, C., & Lee, S. Y. (1993). Mathematics achievement of Chinese, Japanese, and American children: Ten years later. *Science*, 259(5091), 53-58.
- Sun, K. L. (2015). *There's no limit: Mathematics teaching for a growth mindset*. Stanford University.
- Türkdoğan, A., & Adnan, B. A. K. İ. (2021). The Relationship between Mistakes and Feedbacks Encountered in Mathematics Course in the 7th Grade. *Journal of Computer and Education Research*, 9(17), 480-496.

---

# GEOMETRY AND MEASUREMENT

---

# COORDINATION OF GEOMETRY REPRESENTATIONS IN A TEXTBOOK

Waldemar Stepnowski  
Temple University  
wstepnowski@temple.edu

*The study describes how the common representations are coordinated. I analyzed a popular geometry textbook using semiotics and a pragmatic approach to capture the variety of representations into categories and to use descriptive statistics to narrow the focus to the most common representations and coordinations. The major findings are: (1) exposing which representations are most often coordinated like written language; (2) some of the mechanisms in coordination use numbers, point names, and textbook gestures, which include color, arrows, font changes, etc.*

Keywords: Geometry and Spatial Reasoning, High School Education

## Introduction

Some representations are ideal in certain situations, while some are not. Teachers, students, and seasoned mathematicians must make choices in how they represent a mathematical object so that it helps them attain their goal. To be able to make choices in the use of geometry representations, teachers, students, and mathematicians need to know which representations are available and how to use them. That is precisely the goal of my research, i.e., to identify how they are presented and coordinated in a high school geometry textbook, Larson and Boswell's (2015) *Geometry: A Common Core Curriculum*. I do that by analyzing the mechanisms that aid in the most common coordinations.

Besides the above, there are a few more reasons why studying representations is valuable. First, we all want students to solve problems with a high degree of accuracy and efficiency, and representations help us manage mathematical information. Students who coordinate different mathematical representations well solve mathematical problem better than those who do not (Gagatsis & Shiakalli, 2004). Second, there is a difference between changing within one representation and changing or coordinating between different representations. While students have difficulty changing representations (conversion), they have less difficulty in manipulating (treatment) a given representation (Duval, 2006). Therefore, studying which representations appear to be coordinated most often and what are some ways that they are coordinated may help geometry teachers, publishers, and researchers devise strategies that aid students with those conversions. Third, to solve problems students need to use aids or tools like representations to reduce the complexity of a problem. We use representations as a kind of external memory, where we can offload some of the information from our working memory (Scaife & Rogers, 1996). Chandler and Sweller's (1991) seminal work on cognitive load theory shows via experiments that integrated diagrams with text, in contrast to a diagram with text below it, reduced the cognitive load, and reducing cognitive load improves learning. Fourth, we are beginning to develop digital textbooks, online environments, etc., where the choice of representations is important (Presmeg et al., 2016).

## Framework

I take the formalist tradition as to the ontology of what mathematics is, and the pragmatist tradition as to the epistemology of this study. Cognitive Load theory was the learning theory that guided how I analyzed how the signs of geometry could be learned. According to cognitive load

theory, students make use of all three components of working memory, namely central executive, visual-spatial sketchpad and an auditory loop (Sweller, Ayres, & Kalyga, 2011). Our working memory is limited to a sentence, maybe two, an equation or two, a small diagram, a few ordered pairs, etc. As we will see in the methods, a sentence, or more specifically an independent clause, was my unit of analysis.

Semiotics also provides us some tools to analyze representations of mathematical objects. In Peirce's semiotics, the object is what is signified, the representation is the signifier or representamen, and a student's understanding of the object is the interpretant (Sáenz-Ludlow, 2002; Schreiber, 2013). Sáenz-Ludlow (2002) points out that representations are a process (semiosis) among the object, the representamen, and the interpretant. In the case of mathematics, there is little or no direct contact with the object, and therefore a student coordinates representamen ('symbols') and interpretants (students' understanding of the object) to understand the properties of the object or even the object itself.

Converting from one representation to another is difficult for students. Students studying geometry transformations, for example, preferred to use algebraic transformation rules that they memorized rather than think visually (Bansilal & Naidoo, 2012). When students do convert, they are better at it in one direction than the other. Expert teachers in China recognize the importance of converting representation by having students change language to geometrical symbol language, text or oral statements to diagrams, diagram and/or geometrical symbol language to verbal description (Ding, Jones, & Zhang, 2013). The difference between expert and novice may be best explained with Cognitive Load Theory (CLT). Sweller, Ayres, Kalyuga (2011) explain that the difference between a novice and an expert is that an expert has biologically secondary information (geometry representations, algorithms, etc.) stored in his or her long-term memory.

### Methods

I studied a high school geometry textbook and focused on the most common coordinations to expose any common mechanisms that aid in coordination. The textbook in this study is Larson and Boswell's (2015) *Geometry: A Common Core Curriculum*. I chose to study the chapter one (Basics of Geometry), chapter 2 (Reasoning and Proofs), chapter 3 (parallel and perpendicular lines), chapter 4 (Transformations), and most of chapter 11 (Circumference, Area, and Volume), because all the representations were either introduced in those chapters or used extensively enough to study. To determine that selection I had to analyze representations thoroughly in the first chapter, where most representations were introduced.

From multiple initial passes through the textbook, I found many possible distinctions of how objects and concepts are represented in geometry. I also noticed that many chapters used the same representations with little variation (mostly written language and diagram). I excluded some chapters and focused on deeper analysis of coordination and the mechanisms that aid that coordination in chapters 1, 2, 3, 4, and 11. I did not analyze the exercises at the end of the sections and chapters, the preface, index, appendix, glossary, etc. Instead, I analyzed the expository text, postulates/theorems, worked examples, explorations, constructions, etc., that are meant to introduce students to new mathematical content.

I coded almost 4,000 coordinations into a spreadsheet, giving me the ability to look for common types of coordinations like written language to diagram (WL→D). Having those coordinations, I looked for common mechanisms that could aid in the coordination. This was not a linear process, meaning I had to return to previous stages to refine data, definitions, coordinations, etc. I printed each page of the textbook sometimes multiple times and cut out each independent clause and any associated representations like graphs, diagrams, algebraic

expressions, etc. I labeled each snippet with the representations present, and I analyzed in which order they could be read by a student. Some aspects had to be operationalized. For example, I had to decide that for a code to be real-world written language (WDR), the meaning of the sentence and most of the words had to reflect the physical real-world. I used the following codes: written language (WL) with sub-codes of real (R), pure (P), meta (M), declarative (D) and imperative/interrogative (I); diagrams (D) with sub-codes of 2D, 3D, construction (C), dynamic geometry environment (E), and graphs (G); algebra (A); short geometry symbols (Sy), numbers (N); physical objects (P); table (T).

Finally, I focused on the mechanisms. O'Halloran's (2008) linguistic approach, along with Peirce's semiotics, aided to understand some of the mechanism that occur during coordination of representations. By mechanism, as I explained earlier, I mean the method through which we change within one representation or convert one representation to another.

### Results

I found that there are a few common mechanisms to aid coordination like names of points, numbers, algebra, and textbook gestures like color, font change, arrows, etc. The descriptive statistics of the most common representations or combinations of representations along with some examples are shown on the table below. Some representations like diagrams were counted multiple times because other representations referred to them multiple times. Initial representation means that I considered that representation leading to another, the subsequent representation. Table 1 lists combinations of representations that appeared more than 100 times in the coding. The statistics of the combinations of codes helped me choose which coordination of representation combinations to analyze in more detail; it is more pragmatic to study those combinations of representations that students are most likely to come across.

**Table 1: Descriptive statistics of the frequency of representations during coordination**

Representation Combination	Initial	Subsequent	Total	Example (frequency)
WL	1521	797	2318	WL → WL (471), WL → D (257)
D	258	844	1102	NOP → D (131), D → WL (86)
WLSy	471	154	625	WLSy → D (231), WL → WLSy (59)
N	141	243	384	NA → N (55), N → N (51)
WLN	181	182	363	WL → WLN (46), N → WLN (36)
NOP	204	123	327	NOP → NOP (42), OPA → NOP (28)
DN	78	121	199	DN → NSy (21), DN → N (16)
NSy	98	101	199	D → NSy (15), DNP → NSy (12)
A	118	78	196	A → WLA (27), A → A (21)
DP		179	179	WL → DP (89), WLSy → DP (49)
NA	115	61	176	NA → WLN (15), NA → A (10)
WLA	76	98	174	WL → WLA (44), A → WLA (27)
Sy	90	55	145	Sy → D (30), Sy → WL (27)
WLD		118	118	WL → WLD (95)
ASy	67	40	107	ASy → WL (25), ASy → ASy (13)

I have chosen to discuss the following coordinations because they appear often in the textbook. It is obviously only a small selection chosen for brevity. The following examples are not exactly what I have as the most common coordinations, but I tried to choose a representative sample that was not too large and, at the same time, covers most of the common coordinations.

**WL → WL**

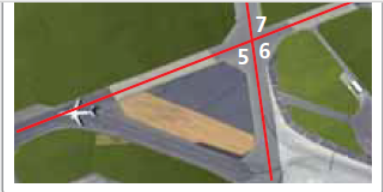
The two sentences below demonstrate two concepts in language that are in proximity to each other.

*A straightedge* is a tool that you can use to draw a straight line. An example of a straightedge is a ruler. (p. 11, Larson & Boswell, 2015)

Both clauses are declarative written language (WDP), and ‘*straightedge*’ is emphasized with italics, a textbook gesture (Gst). The gesture seems to underscore the word ‘straightedge,’ in a similar way a teacher would point to it with her finger as gesture emphasizing the word’s importance. Other important vocabulary words have a yellow background, and the font is in bold. With the italics, a textbook gesture (Gst), we coordinate these two sentences focusing not only on *straightedge* because it is the subject of the first sentence and part of the noun phrase of the second one, but also because it is emphasized through a gesture. Gestures, repeated use of words, and sentence structure (like definitions) aid in connecting representations through emphasis and an easy way to refer to an object when scanning for it.

**WDP+Sy → P+2D**

The picture of airport runways with the red lines and angles (2D) is an example of how physical objects (P) can be used to represent mathematical objects. The diagram overlaps the runways, which are clearly wider than a line, and for that reason a thinner line must be drawn somewhere on the broad runway; the authors chose to draw it in the middle of the runway. This runway example shows the problem with using physical objects (P) to represent mathematical ones; it is more difficult to discuss the abstract features of geometry when coordinating with physical objects than diagrams alone.

<p><b>STUDY TIP</b></p> <p>In paragraph proofs, <i>transitional words</i> such as <i>so</i>, <i>then</i>, and <i>therefore</i> help make the logic clear.</p>	<p><b>Given</b> <math>\angle 5</math> and <math>\angle 7</math> are vertical angles.  <b>Prove</b> <math>\angle 5 \cong \angle 7</math></p> <p><b>Paragraph Proof</b></p> <p><math>\angle 5</math> and <math>\angle 7</math> are vertical angles formed by intersecting lines. As shown in the diagram, <math>\angle 5</math> and <math>\angle 6</math> are a linear pair, and <math>\angle 6</math> and <math>\angle 7</math> are a linear pair. Then, by the Linear Pair Postulate, <math>\angle 5</math> and <math>\angle 6</math> are supplementary and <math>\angle 6</math> and <math>\angle 7</math> are supplementary. So, by the Congruent Supplements Theorem, <math>\angle 5 \cong \angle 7</math>.</p>	
---	--	---

**Figure 1. WMP, syllogism, Sy, Picture of Physical object (PP). (p. 110, Larson & Boswell, 2015).**

In Figure 2, a graph (G) is used to display some of the information in a worked example. To remind the reader, the number of representations for table 3 was reduced; graph (G), construction (C), 2D, and 3D are grouped under one representation diagrams (D).

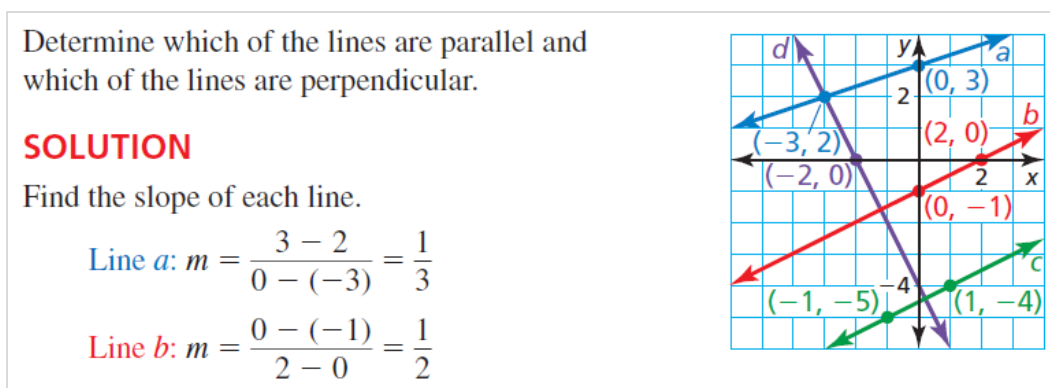


Figure 2. Example of WIP, G, OP, and N combination (p. 157, Larson & Boswell, 2015).

We can notice many features in figure 2: use of color for different objects, ordered pairs, points, grid, short segment (gesture) that points to a point (see point at  $(-3, 2)$ ). Other graphs have other aspects: equations, variables in ordered pairs, angle names and measures. Figure 2 and the next two paragraphs explain how ordered pair with numbers (NOP) were coordinated with written language and arithmetic of numbers.

**WDP+N→G+N+OP**

As in figure 2, numbers (N) and ordered pairs (OP) are often coordinated. In this case, ‘Line  $a$ ’ (WL) and the substituted x- and y-values (N) would be coordinated with the graph (G) and the ordered pairs (NOP). Line  $a$  is not only labeled with the letter ‘ $a$ ’ on the graph but it is also colored blue for easier coordination. Again, such mechanisms are used often in the book, especially when a concept is being explained for the first or second time. A student would need to have the schema of slope (change of y divided by change in x) well established to follow the worked example, or he or she would need to look back at the formula. In this figure, the student would look back and forth possibly a few times to see that the y-values are 3 and 2 (numerator), and that the x-values (denominator) are 0 and -3. Because we are comparing slopes of various lines to see if they are parallel the lines are differentiated by color; when slope was reviewed earlier in the textbook (p. 123) the change in the x- and y-values was color-coded in a worked example.

The most common mechanism in coordinating objects is using the point. Segments, lines, rays, circles, intersections, polygons, etc., are displayed on diagrams (D) containing named points. They are then referred to in written language (WL) or in short geometry statements (Sy) using those points, that is a student needs to look at specific points on a diagram (D) to see where an object is. For example, a student would look for the three points A, B, and C of  $\triangle ABC$  (Sy) on the diagram.

Other mechanism between representations are variables, color, measurement (N), numerals/letters for angles, gestures, and other symbols in statements. Variables or even algebraic expressions (A) often appear as measures of angles or length and then in a paragraph (WL) or statement (Sy, and maybe Sy) near the diagram. Students can then use those variables to coordinate between the two representations. Numbers (N) are also used in a similar manner, often appearing as length or angle measure both on the diagram and in written language and/or statements. Numbers are also crucial in coordination between graphs, points as ordered pairs (NOP), and written language. Students can not only refer to numbers on the axes, but they can also count the squares on the graph to determine how many units a point, line, center, etc., is

away from the x- or y-axis. Numerals and cursive lower-case letters also help with coordination between representations for angles and lines, respectively. Finally, other symbols (Sy) like  $\Delta$ ,  $\angle$ ,  $\parallel$ ,  $\overline{AB}$  (the bar above AB),  $\cong$ , etc., aid in coordination. Many are iconic, visually close to the objects, but others like the congruence symbol ( $\cong$ ) are arbitrary symbols.

Many words convey meaning in geometry through being connected to physical reality, a mechanism that aids in coordination. That connections may aid in students remembering the word. For example, a word like ray is usually thought in connection to light. In common language, a base is usually defined as the bottom part of some structure like bases in triangles; they appear at the bottom of a corresponding altitude. There are other words from the textbook that mean something in the physical world that is different in varying degrees to what they mean in geometry, e.g., line, compass, construction, equal, side, vertical, adjacent, reflect, intersection, slope, etc.

Movement is not often discussed in geometry class; geometry is usually static, but many of its concepts involve motion. As I analyzed the language, diagrams, constructions, and other representations, movement seems to pervade them. There are many concepts that movement appears in. Dynamic geometry and construction obviously involve motion to construct, and with DGE to drag even after construction. Slope is often described as “rise over run,” which implies moving up and then over; even the purer mathematical language of “change in y” implies upward or downward movement. On graphs arrows often show that to move from some point A to some point B, you need to go up a certain number of units and over a certain number of units. Roads or airplane paths, which are superimposed on diagrams, often contain cars or planes (see fig. 1) that are implied to be moving from one point to another. So, when a student thinks of a point or an intersection, he or she might think of a car or a plane passing through it. As another example, term ‘bisect’ is defined as “cut into ... parts,” which implies a kind of slicing of a segment or an angle.

Addressing these depictions of movement in geometry concepts, we must consider that formal definitions, concepts, properties, etc. are static sets of points. For example, a line contains a point, not passes through it; a reflection is the set of points the same distance away on the other side of a line, not a flipping of a figure; a line bisects a segment when one of line’s points is the midpoint of the segment, not a cutting or a slicing of a segment like a knife. These metaphors may aid students to remember and to connect concepts, or they may confuse them. The point is that representations may complement, contradict, mirror each other, and that is part of the mechanism that helps us coordinate between representations. For example, if a student understands bisecting a segment as cutting into two equal parts, then he or she may think that the segment like a piece of rope is not whole but rather two separate smaller segments because when he or she cuts a segment of rope in half, the piece of rope is no longer whole. On the other hand, from similar experiences, the student’s idea of equal parts of a piece of rope may help him or her to remember that bisect means exactly in half.

### Discussion

Analysis of geometry representations is very complex in many ways. First, there are a variety of representations. Second, there are a great number of mathematical objects throughout a typical geometry textbook and course; the objects can be typical like points and segments, they can be aspects of those objects like length, they can be relationships like the ratio between two lengths, they can be more general like explanations of the logic of a mathematical system, etc. Third, there are a great number of combinations of the various representations. My research tried to



clarify this specific field of study on the one hand by making specific distinctions among representations; that was accomplished by studying how and which representations were introduced in the textbook (first research question). Then, I tried to simplify the complexity by focusing on the most common coordinations (second research question). Finally, while analyzing those coordinations, I looked for common mechanisms (third research question) that aided in the coordination between or among representations.

The analysis of the introduction of representations, especially the introduction of their affordances, shows that certain representations like written language and graphs are not introduced with detail and with explicitness, while others like dynamic geometry environments (DGE) have more sections devoted to explanations of use and affordances. Yet, the textbook very rarely provides the constraints of specific representations. Having the descriptive statistics and knowing what coordinations of representations that students are exposed to in a geometry course most often, I was able to focus on some mechanisms in high school geometry. For example, coordination of language, diagrams, etc. in many different combinations is very useful, but coordination of a table of values and a graph is not because it does not appear often in the textbook.

### **Interpretations and Implications**

This study explored the introduction of representations in a textbook, but to do that I needed to make distinctions among those representations and go beyond those found in the extant literature. Therefore, while the representations were introduced, I was searching for specific differences between the ones that appeared earlier in the textbook with those that followed, and also with those in the extant literature. If they were similar the ones that came before, there was no introduction, if they were different, I analyzed the introduction. In this section, I would like to compare the representations introduced in the textbook with those in the extant literature, describe how they were introduced, and describe possible implications of both.

It is helpful to discuss coordination with Duval's distinction of treatment and conversion, and with Peirce's triad of object, representamen, and interpretant. In this textbook examination, most coordination involved conversion, where different representations were compared. Some coordinations (about 10%) were treatments, or coordinations within a representation, but most of it was within written language and a small amount of algebra. It was possible to read an implied reference to Peirce's triad where the authors discuss "interpreting a diagram," and list assumptions that a student can and cannot conclude. For example, we cannot conclude that two segments are congruent because they appear to be. This is the closest that the authors come to making clear that the reader has a role to play in the interpretation of representations. There is a representation (representamen) of two segments that appear to be the same length. There is an interpretant, where a student might think that they are or are not congruent. There is the object that whoever designed the diagram for an exercise or a worked example either planned them to be congruent or not, and that the math community has a specific idea of a congruent segments. Neither idea was discussed explicitly even with more teenage-friendly vocabulary in the textbook. The closest explanation of conversion is the use of 'words' and 'symbols' next to those different representations of an object. Again, the implication for future research is to study whether such descriptions of Peirce's triad or Duval's treatment/conversion distinction, or similar ideas more suitable for adolescents in high school geometry, improves student learning. The distinction might provide students with meta-language to process their learning and understanding, which in turn might improve their self-regulating techniques.

From my perspective as a teacher, I found some mechanisms that aided coordination (the third research question). One was using colors to compare relative parts like x- and y-values in ordered pairs and in equations or on a graph, new arcs on construction, highlighting specific words, etc. Another was sentence structure like conditional (if/then) statements or definitions to aid students connect ideas in a familiar way. A third was using textbook gesture like arrows to point to specific locations or parts or to show a transformation or other movement. Fourth, algebra has its refined mechanisms. Fifth, many of the short geometry symbols like parallel (  $\parallel$  ) are iconic so students can see a smaller version in the flow of text or as separate statements. Sixth, the names of points are the most traditional way of coordinating among representations.

Textbook gestures seem to overlap with mechanisms and studying them more may change how we use them in the classroom, in future research, and in textbooks. First, studying whether they increase cognitive load may warrant using them less or more. For example, having multiple lines on the same diagram as figure 2, color as a textbook gesture may reduce the cognitive load, but there is probably a point when the number of lines overload a student working memory. What is the best number when introducing a new topic like slope of parallel lines? Two? Three? More? For which students, more advanced less? Where is the most practical balance? Another important question is whether the color itself is distracting. It may be that displaying those three or four lines would reduce the load or displaying them in black may reduce load. With digital textbooks being more and more common a third way is possible, being able to turn the color off or on, or displaying it before or after displaying the figures in black. A study using pre- and post-tests could help answer some of these questions.

As with the coordinations, this is not an exhaustive list, and it certainly needs to be studied through the eyes of actual high school students seeing some of these representations for the first time or in new light. Many of the mechanisms are what I termed gestures, e.g., color, arrows, type of font, and they appear in both representations, or join them in some way.

### **Recommendations**

Besides the recommendations for further research, teacher training and textbook authors could benefit from this research. Teacher colleges could train teachers to fill the gaps of textbooks that do not address explicit introduction and coordination of representations. Researchers can learn more about whether explicit introduction to the various representations and their affordances and constraints, explanations of conversions among representations, and teacher knowledge about these topics, leads to better student outcomes. We may discover that students learn these concepts through example, or that teachers fill in the gaps in the textbook well enough that explicit explanations may not be necessary in a textbook. With digital textbooks and websites becoming more abundant, the use of animation, sound, and possibly other representations are possible. Not only is it worthwhile to study digital textbooks and websites to find more dynamic mechanisms than the ones in a printed textbook, but I hope that researchers and textbook and website designers attempt to add more and improved mechanisms that make learning coordination clearer and more engaging.

### **References**

- Ding, L., Jones, K., & Zhang, D. (2013). Teaching geometrical theorems in Grade 8 using the ‘Shen Tou’ method: A case study in Shanghai. In L. Fan et al. (Eds), *How Chinese Teach Mathematics*. Singapore: World Scientific.
- Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational studies in mathematics*, 61(1/2), 103-131. doi:10.1007/s10649-006-0400-z
- Gagatsis, A., & Shiakalli, M. (2004). Ability to translate from one representation of the concept of function to another and mathematical problem solving. *Educational Psychology*, 24(5), 645-657. doi:10.1080/0144341042000262953

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

- Larson, R., & Boswell, L. (2015). *Geometry: A Common Core Curriculum*. Erie, PA: Big Ideas Learning.
- Naidoo, J., & Bansilal, S. (2010). Strategies used by Grade 12 Mathematics learners in Transformation Geometry. *South African Journal of Education*, 32(1), 26-39.
- O'Halloran, K. (2008). *Mathematical discourse: Language, symbolism and visual images*. London, UK: Continuum.
- Presmeg, N., Radford, L., Roth, W. M., & Kadunz, G. (2016). Semiotics in Theory and Practice in Mathematics Education. In *Semiotics in Mathematics Education* (pp. 5-29). Springer International Publishing.
- Sáenz-Ludlow, A. (2002). Sign as a process of representation: A Peircean perspective. *Representations and mathematics visualization*, 277-296.
- Scaife, M., & Rogers, Y. (1996). External cognition: how do graphical representations work? *International Journal of Human-Computer Studies*, 45(2), 185-213. doi:10.1006/ijhc.1996.0048
- Schreiber, C. (2012). Semiotic processes in chat-based problem-solving situations. *Educational Studies in Mathematics*, 82(1), 51-73. doi:10.1007/s10649-012-9417-7
- Sweller, J., Ayres, P. & Kalyuga, S. (2011). *Cognitive load theory*. New York, NY: Springer.

# STEM IDENTIFICATION THROUGH MATHEMATICS-BASED COMPUTATIONAL AND CODING ACTIVITIE

Mina Sedaghatjou  
Rowan University  
Sedaghatjou@rowan.edu

Anton Puvirajah  
University Western  
Ontario  
apuvira@uwo.ca

Ashley Gomez  
Rowan University  
Gomez94@students.rowan.edu

Ali Kooshyar  
Rowan University  
kooshy47@students.rowan.edu

Keywords: Computing and Coding; Integrated STEM / STEAM, Geometry and Spatial Reasoning; Elementary School Education

There is a growing consensus among scholars, educators, and policymakers about the importance of including Computer Science (CS), Computational Thinking (CT) curricular activities for youth to foster creativity, innovation, building foundational skill and identity, and therefore reducing inequality (Century, Ferris & Zuo, 2020; Garcia, et al., 2018). While several studies explored the multi-faceted linkage of CT and K-12 education (Barcelos et al., 2018; Tang et al., 2020), there is a significant gap about the role of mathematics- based CT experience in Science, Technology, Engineering, and Mathematics (STEM) identity (Ye, et al., 2023). To address this purpose, we investigated underserved students' interest and perceptions of themselves as future STEM professionals. We also examined youth's STEM self-identification as they participate in mathematical-based CT activities in Collaborative Engagement learning environment (Sedaghatjou, Rodney, 2018).

## Methods

In this exploratory mixed-method study, we offered a two-week coding and robotics summer camp for newly immigrant students. We designed several mathematical-based (geometry and spatial reasoning) coding activities using scratch, micro-bits, and Lego Mindstorms. Data was collected from a total of 16 English Language learners from newly immigrated students, 10 boys and 6 girls, aged 10 to 13 who recently ( $16 \pm 9$  months) immigrated from different countries, with the highest frequency from the Korea and UAE, followed by the KSA, Jordan and Iraq, and Egypt, China, and Russia.

## Results

Our findings showed effectiveness of the CT experience on youth attitude, interest, and affinity as well as motivations towards STEM career. A Wilcoxon singled-ranks test indicated that median post-STEM affinity test scores, were statistically significantly higher than the median pre- STEM affinity test scores,  $Z= 84.5$ ,  $p<.006$ .



Figure 1: Underserved student Collaborative Engagement in the mathematical-based

### Computational Thinking activities.

Drawing on the theory of Acts of authentication (Verma, Puvirajah, Webb, 2015; 2020); we found that students learn mathematics, engineering practices, and coding, and develop habits of mind, build a friendship network, and develop a STEM attitude and identity even without recognizing the mathematical underlying concepts. The key practices for enacting acts of authentication included: student talk, productive disciplinary engagement, and participation in their communities of practice.

### References

- Barcelos, T. S., Munoz, R., Villarroel, R., Merino, E., & Silveira, I. F. (2018). Mathematics learning through computational thinking activities: A systematic literature review. *Journal of Universal Computer Science*, 24(7), 815–845. <https://www.scopus.com/record/display.uri?eid=2-s2.0-85051735640&origin=inward&txGid=232bb8b5c24cea136a356b7fda1a2bdb>
- Century, J., Ferris, K. A., & Zuo, H. (2020). Finding time for computer science in the elementary school day: a quasi-experimental study of a transdisciplinary problem-based learning approach. *International Journal of STEM Education*, 7(1), 1-16. <https://doi.org/10.1186/s40594-020-00218-3>
- Verma, G., Puvirajah, A., & Webb, H. (2015). Enacting acts of authentication in a robotics competition: An interpretivist study. *Journal of Research in Science Teaching*, 52(3), 268-295. <https://doi.org/10.1002/tea.21195>
- Puvirajah, A., Verma, G., & Campbell, T. (2020). Advancing Minoritized Learners' STEM Oriented Communication Competency Through a Science Center-Based Summer Program. *Journal of Museum Education*, 45(4), 437-449. <https://doi.org/10.1080/10598650.2020.1820812>
- Garcia, A, Ross, M, Hazari, Z, Weiss, M, Christensen, K, & Georgiopoulos, M. Examining the Computing Identity of High-Achieving Underserved Computing Students on the Basis of Gender, Field, and Year in School. Collaborative Network for Engineering and Computing Diversity (CoNECD). Retrieved from <https://par.nsf.gov/biblio/10075621>.
- Hazari, Z., Dou, R., Sonnert, G., & Sadler, P. M. (2022). Examining the relationship between informal science experiences and physics identity: Unrealized possibilities. *Physical Review Physics Education Research*, 18(1), <https://doi.org/10.1103/PhysRevPhysEducRes.18.010107>
- Lunn, S., Ross, M., Hazari, Z., Weiss, M. A., Georgiopoulos, M., & Christensen, K. (2021). How do educational experiences predict computing identity?. *ACM Transactions on Computing Education (TOCE)*, 22(2), 1-28. <https://eric.ed.gov/?id=EJ1358822>
- Sedaghatjou, M., Rodney, S. (2018). Collaborative Engagement Through Mobile Technology in Mathematics Learning. In: Calder, N., Larkin, K., Sinclair, N. (eds) *Using Mobile Technologies in the Teaching and Learning of Mathematics*. Mathematics Education in the Digital Era, vol 12. Springer, Cham. [https://doi.org/10.1007/978-3-319-90179-4\\_7](https://doi.org/10.1007/978-3-319-90179-4_7)
- Ye, H., Liang, B., Ng, O. L., Wong, R. K. W., & Xie, Q. (2023). Integration of computational thinking in K-12 mathematics education: A systematic review on CT-based mathematics instruction and student learning. *International Journal of STEM Education*, 10(1), 3. <https://doi.org/10.1186/s40594-023-00396-w>
- Tang, X., Yin, Y., Lin, Q., Hadad, R., & Zhai, X. (2020). Assessing computational thinking: A systematic review of empirical studies. *Computers & Education*, 148, 103798. <https://doi.org/10.1016/j.compedu.2019.103798>

## ENGAGING WITH A GEOMETRIC TRANSLATIONS TASK USING CODING

Karmen Williams  
Louisiana State University  
kwil419@lsu.edu

Zuhal Yilmaz  
Louisiana State University  
zyilmaz@lsu.edu

Fernando Alegre  
Louisiana State University  
falegre@lsu.edu

Juana Moreno  
Louisiana State University  
moreno@lsu.edu

Rose Kendrick  
Louisiana State University  
rkendr3@lsu.edu

Keywords: Computation, Geometric Translations, Functions, Coding

Portnoy et al. (2006) suggested students need to use patterns, generalizations and spatial reasoning as they learn geometric transformations. Computation tasks are an excellent medium for identifying patterns and exploring the results of manipulations (Schanzer et al., 2015). This study explores how 9th grade students engage with a computation task focused on rigid geometric translations and their teacher's insights on how the task could support their students' understanding of geometric translations. Geometric translations can be interpreted as motion and mapping (Hollebrands, 2003; Yanık, 2011). Motion refers to "the mental or physical manipulation of geometric figures to new positions or orientations" (Boulter & Kirby, 1994, p. 298). In mapping, "translations are considered special functions that map all points in the plane to other points in the plane based on a specific direction and magnitude" (Yanık, 2013, p.273).

This poster presents ongoing research on synergies between computational and mathematical thinking. We observed a one-hour class in which 14 ninth grade students concurrently enrolled in Algebra I engaged with a coding activity where students created a translation of a bird silhouette. Students were given prewritten code that translates the bird to the top of the output panel and were asked to simplify the code, by creating a function. At the end, we debriefed with the teacher on how she used this bird task in supporting her students' understanding and use of geometric translations and how computation tasks could be helpful in her future teaching. Researchers used the Framework for Engagement with Mathematics (Attard, 2012) to create questions for the debrief. Also, students' codes were stored in our platform (Alegre, 2020). Field notes of the observation, debrief with the teacher, and students' codes were used in the data analysis.

Initial analysis showed the majority of the students interpreted translations as motion. They described the translation as "it is like sliding". One student stated "You can move  $x$  and  $y$ . We can move the shape 5 units or 3 units left and right with coordinates". The teacher commented on whether the translation should be applied to all points on the plane. But, there was no check for understanding on mapping interpretation of translations in the lesson. But the analysis of the codes showed that students defined a function in their codes that applies translation on all points.

Preliminary results suggest that computation activities let students externalize and manipulate their thought process, making mathematical thinking explicit while engaging with the task. Also, it allowed students to observe the motion interpretation of the translation through manipulation. Students had a chance to understand the mapping interpretation of translation by defining a function, and they used important mathematical concepts, such as functions and congruence, during the task. Future research will explore rotations, dilations, and composite transformations.

### Acknowledgments

This research project is funded by NSF award CNS-1923573 and the U.S. Department of Education award U411C190287.

### References

- Attard, C. (2012). Engagement with mathematics: What does it mean and what does it look like?. *Australian Primary Mathematics Classroom*, 17(1), 9-13.
- Boulter, D. R., & Kirby, J. R. (1994). Identification of strategies used in solving transformational geometry problems. *The Journal of Educational Research*, 87(5), 298-303.
- Alegre, F., Underwood, J., Moreno, J., & Alegre, M. (2020, February). Introduction to computational thinking: a new high school curriculum using Codeworld. In Proceedings of the 51st ACM Technical Symposium on Computer Science Education (pp. 992-998).
- Hollebrands, K. F. (2003). High school students' understandings of geometric transformations in the context of a technological environment. *The Journal of Mathematical Behavior*, 22(1), 55-72.
- Portnoy, N., Grundmeier, T. A., & Graham, K. J. (2006). Students' understanding of mathematical objects in the context of transformational geometry: Implications for constructing and understanding proofs. *The Journal of Mathematical Behavior*, 25(3), 196-207.
- Schanzer, E., Fidler, K., Krishnamurthi, S., & Felleisen, M. (2015, February). Transferring skills at solving word problems from computing to algebra through Bootstrap. In Proceedings of the 46th ACM Technical symposium on computer science education (pp. 616-621).
- Yanik, H. B. (2011). Prospective middle school mathematics teachers' preconceptions of geometric translations. *Educational Studies in Mathematics*, 78, 231-260.
- Yanik, H. B. (2013). Learning Geometric Translations in a Dynamic Geometry Environment. *Education & Science/Egitim ve Bilim*, 38(168).

## FRACTION AS A QUANTITY: DESCRIBING STUDENTS' REASONING

Jennifer Talbot  
Illinois State University  
jkane@ilstu.edu

Amanda Cullen  
Illinois State University  
almille@ilstu.edu

Cheryl Lizano  
Southern Illinois University  
Edwardsville  
ceames@siue.edu

*Understanding fraction as a quantity has been identified as a key developmental understanding. In this study, students in Grades 5, 8, and 11 were asked to compare the areas of two halves of the same square—a rectangle and a right triangle. Findings from this study suggest that students who understand fraction as a quantity use reasoning related to a generalization, whereas students who understand fraction as an arrangement use reasoning related to visualization, computation, or characteristics of the specific shapes involved. Knowing the reasoning exhibited by students can inform both teachers and mathematics curriculum writers in the creation of and planning for instructional tasks.*

Keywords: Middle school education, High school education, Geometry and spatial reasoning

According to Simon (2006), other researchers (e.g., Skemp's (1976) relational understanding and Hiebert & Lefevre's (1986) conceptual knowledge) have used the term "understanding" in different and productive ways, mainly in outlining valuable types of mathematical knowledge. However, Simon (2006) argued that none have described it in such a way as to "help in identifying critical transitions that are essential for students' mathematical development" (p. 360). To bridge the gap, Simon (2006) introduced a construct known as *key developmental understanding* (KDU). A KDU has two characteristics. The first of which is that a student makes a "conceptual advance" (Simon, 2006, p. 362). Simon (2006) defined "conceptual advance" as "a change in students' ability to think about and/or perceive particular mathematical relationships" (p. 362). The second characteristic of a KDU is that the transition between a student not holding the desired knowledge and holding it requires an accumulation of activity and reflection over the course of multiple experiences. In other words, the desired knowledge cannot be obtained purely through explanation or demonstration.

To further illustrate the construct of a KDU, Simon (2006) provided an episode from a fourth-grade classroom. Students were given geoboards and asked to use a blue rubber band to make a square and a red rubber band around half of their square. Simon (2006) stated that most students demonstrated half, with the red rubber band, by making two congruent rectangles. However, one student showed half by splitting the square into two congruent right triangles. The students' consensus was that both representations of half were correct. The justification involved reasoning that "each of the parts was one of two equal parts and that the two parts made up the whole" (p. 361). When Simon asked students to compare the rectangle half and the right triangle half, their responses were split between three comparisons: the rectangle was larger, the right triangle was larger, and the two shapes were the same size.

Simon (2006) asserted that the students who believed that the rectangular or triangular half was larger viewed a fraction, one-half in this case, as an arrangement. In other words, the whole was separated into identical parts. Students who believed that the rectangle and right triangle were the same size viewed fraction as a quantity. These students demonstrated a conceptual advance, or KDU, in recognizing that the shapes were congruent and that "equal partitioning creates specific units of quantity" (p. 362). According to Simon (2006), researchers must observe



students participating in the same mathematical task and examine the differences in their actions and responses to identify or better understand a KDU. The geoboard task was utilized as a context to highlight an important aspect in understanding fractions.

Similar contexts have been used in other research studies, although with different goals. In Simon et al. (2004), researchers posed a similar task, in which the original squares were cookies, was posed to a third-grade student. The student, Micki, thought that the right triangle was larger than the rectangle. This example was used to exemplify the progression of students' learning processes. In Simon et al. (2018), three fourth-grade participants were given a similar task as part of a pre-assessment, in which the original shapes were rectangular brownies. The pre-assessment results informed an instructional intervention with a goal to promote fraction as measure understanding. None of the students recognized that the rectangular brownie and right triangular brownie were the same size even though they were half of the same whole brownie. Instead, they chose the right triangular piece, claiming it was larger than the rectangular piece. Similarly, in a longitudinal study by Barrett et al. (2017), a similar task was presented to students to understand area conservation. Researchers observed significant inconsistency in student responses, which prompted revisions to a developmental progression involving measurement. However, the reasoning students used to compare shapes in these tasks was not discussed or analyzed in depth. In Barrett et al. (2017), fourth-grade students were given pairs of pre-cut paper shapes in order to compare areas: a square and a right triangle formed by cutting a congruent square along the diagonal, the same square and a rectangle formed by cutting a congruent square in half vertically, and the right triangle and the rectangle, both made from cutting the square. Participants' responses indicated that some students believed the rectangle had a larger area, while others stated that the triangle had a larger area.

Within the contexts described in the current literature (i.e., Barrett et al., 2017; Simon et al., 2004; Simon, 2006; Simon et al., 2018), detailed information about students' reasoning, as it relates to their understanding of fractions (as an arrangement versus as a quantity) has not been communicated. This study describes the reasoning of students as they compared areas of squares, rectangles, and right triangles, and in doing so, extends the existing literature by examining students' reasoning related to each type of understanding, fraction as an arrangement to reasoning about fraction as a quantity, and the transition between.

### **Research Questions**

In this study, we investigated fifth-, eighth-, and eleventh-grade students' reasoning as it related to comparing the areas of rectangles and right triangles. The two research questions addressed were:

1. What reasoning do students use to compare the areas of squares, rectangles, and right triangles?
2. What does it look like for a student to demonstrate understanding of fraction as an arrangement when comparing areas of squares, rectangles, and right triangles? Fraction as a quantity?

### **Theoretical Perspective**

Constructivism is the theoretical perspective that informs this study. Constructivism is based on the idea that learners build their own knowledge, understanding, and ways of reasoning over time by interacting with their environment (von Glaserfeld, 1995). In other words, knowledge cannot be merely transferred, by explanation or demonstration, to the student. Students'

accumulation of experiences, reflection, and abstraction provides structure for developing their conceptual understanding. In addition, the role of the researcher or teacher is to listen to students and make sense of their conceptual understanding. Listening to students is what leads to a continually updated hypothesis or anticipation of student learning. Then, that interpretation can facilitate the way in which misconceptions are addressed and the type of intervention to attempt to further develop students' understandings (von Glaserfeld, 1995).

### Methods

As part of a larger study investigating how students reasoned about areas of triangles, the research team conducted semi-structured individual interviews with fifth-, eighth-, and eleventh-grade students during the summer of 2020. Participants were recruited from a public school district in the Midwestern region of the United States. The district is located in a suburban community and has 1,000 students in attendance of which 74.1% identified as White, 4% as Black, 7.9% as Hispanic, 7.2% as Asian, and 6.8% as multi-racial. In this paper, we examine participants' responses to the first task of the first interview.

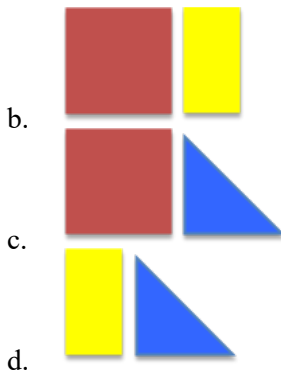
### Procedure

Participants completed at least one zoom-recorded interview. Considering the interviews were conducted through zoom, participants were in places where other people could have been present; however, researchers asked that they not interact with the participant during the interview. All materials associated with the interview (i.e., ruler, blank paper, printed tasks) were mailed to or dropped off at the residences of the participants.

The first interview lasted on average 20 minutes. During each interview, two of the three researchers were present. One researcher acted as the main interviewer, posing each of the tasks and asking follow-up questions, while the other researcher was mainly an observer. At times, something the participant said may have been uncertain and the second researcher asked further questions.

We posed this task, consisting of four parts, to 22 participants: 14 students in Grade 5, four students in Grade 8, and four students in Grade 11. For task 1a, the participant was shown a yellow paper square as the interviewer asked, "If I asked you what the area of this square was, what would I mean by area?" If the participant's response focused on multiplying sides of the square, then the interviewer asked, "What if you were talking to a second grader that does not know about multiplication? How would you describe area?" For task 1b, the interviewer showed the original, yellow paper square along with a red paper square of the same size. The interviewer folded the red paper square in half, vertically, making a rectangle. The participant was asked, "Compare the area of these two figures. How do you know?" For task 1c, the participant was shown the original, yellow paper square and a blue paper square of the same size. The blue paper square was folded in half, from corner to opposite corner, making a right triangle. Again, the interviewer asked, "Compare the area of these two figures. How do you know?" For the last part of Task 1, the participant was shown the rectangle from 1b and the right triangle from 1c and asked, "Compare the area of these two figures. How do you know?"





**Figure 1: Images of Folded Shapes for Task 1**

**Data Analysis**

After all interviews were completed, the team of three researchers worked through three stages of data analysis. During the first stage, the interviews were divided among the three researchers. For each interview, the assigned researcher watched it and created a written summary. Each summary consisted of a loose transcription of the interview along with words or phrases that described participants’ ways of reasoning. These words and phrases became codes that the researchers discussed, sorted into groups, and categorized into official codes. The updated codes were utilized when each interview and its summary was revisited again by the two other researchers who had not created it originally.

During the second stage of data analysis, all three researchers independently coded each task and then met to discuss those codes. For the final stage, pairs of researchers independently coded and then, again, all three researchers met to discuss. Interrater reliability was tracked for each interview. Scores ranged from 39 to 91%, with an average of 68%.

For the final stage of data analysis, pairs of researchers independently coded tasks and then together, compared codes. Interrater reliability scores ranged from 71 to 100%, with an average of 86%. If any discrepancies arose throughout this process, all researchers met to discuss and resolve them by revising or clarifying codes.

**Findings**

Overall, we identified two main categories to distinguish students’ reasoning about the areas of the rectangle and right triangle. Sixteen participants stated that the rectangle and right triangle had equal areas, and six participants stated that the rectangle and right triangle had unequal areas. However, within those two categories, we identified participants who wavered in their decision. This caused us to create subcategories. We describe below the nuances we found in students’ reasoning within each subcategory.

**Table 1: Participants’ Reasoning Categorization**

Participants who stated that rectangle and right triangle had <b>equal</b> areas	501*, 504, 505, 506, 508, 509*, 511*, 512*, 513, 514 802, 803* 1101, 1103, 1104, 1105
Participants who stated that the rectangle and right triangle had <b>unequal</b> areas	502, 503, 507, 510* 801*, 804

\*Participants who stated that the shapes' areas were both equal and unequal during the interview. The number of hundreds indicates the grade level. For example, participant 1101 was in Grade 11 at the time of the study.

### **Students' Reasoning Related to Both Shapes Having Equal Areas**

While 16 participants decided that the rectangle and right triangle had equal areas, 11 of them (504, 505, 506, 508, 513, 514, 802, 1101, 1103, 1104, 1105) promptly stated their reasoning. One participant (508) reasoned that the rectangle and right triangle are "equal because they were both half of the square." This response, with an emphasis on the relationship between each shape and the original square, was recorded by seven other participants (508, 513, 514, 1101, 1103, 1104, 1105). Four participants referred to the action of folding the paper square while comparing the areas of the halves. For example, one participant (504) stated "they're both still folded in half and they're both still the same size as the square if they're unfolded." Whereas three participants (505, 506, 802) used reasoning that both halves were the same because they had been folded from the square in different directions. For example, 505 stated that the rectangle and right triangle were "just folded in half two different ways."

Seven participants (501, 509, 510, 511, 512, 801, 803) expressed some hesitation when asked to compare the areas. In other words, they provided at least one reason each for the rectangle and right triangle to have both equal and unequal areas. Five participants (501, 509, 511, 512, 803) switched their reasoning from the rectangle and right triangle having equal areas to unequal areas during the interview, but in the end, stated that the halves had equal areas. Except for 803, participants switched back and forth between claims that shapes have equal areas or unequal areas, at least twice. Interestingly, within a matter of less than 4 minutes, 510 switched reasoning five times.

Of these five participants, two (511, 803) utilized numerical measurements to make a final decision about how the areas of the rectangle and right triangle compared. 511 began by stating that the areas of the shapes were the same because "they're both half of the square." Then, while 511 described the computational process of finding the area of the right triangle, "you multiply length and width to get the triangle, but divide it in half, because it's a triangle", 511 stopped and said "But the rectangle... I guess it wouldn't be then." 511 proceeded to then switch back to state the areas were different because "the triangle's sides are bigger." When the interviewer asked, "What are you grappling with?", 511 said, "I guess I just don't know the numbers... the length of the square... the width of it." The interviewer then asked 511 to describe the measurements she wanted. From the square, 511 stated that she wanted "the length of one side of it and the width of one side of it." 511 also asked for "the width" or "the bottom of the rectangle." After the interviewer supplied these measurements, 511 computed the area of the square, the rectangle, and the triangle. When 511 determined that the rectangle and triangle areas were 6 and 6, 511 stated that the areas would be equal. 803's interview began similarly with 803 stating that the areas were the same because "you folded it in half and since they're a square, they have equal portions." However, continuing that same statement, 803 said "but I don't think they are the same because you didn't fold a line, you folded a diagonal so the triangle would be bigger." The interviewer responded by asking about these conflicting conclusions, "What made you think they might have been the same at first?" 803 restated both ideas, the first being the areas are equal because the shapes are both half of the same square, and the second being that the triangle's area is greater because it was made by folding on a diagonal. Then, the interviewer asked, "Which one are you most convinced by?" to which 803 explained that she saw "both points" but was

more convinced by the areas being equal because she assigned values to the square and computed the areas of the rectangle and right triangle to realize they were the same.

The other three participants (501, 509, 512) who went back and forth but ultimately decided the areas were indeed equal used reasoning related to both shapes being half of the original square. Both 501 and 509 were only swayed once when deciding if the two areas were equal. At first, 501 was conflicted when it came to finding the area of the triangle, stating “it’d be hard because the triangle has points that aren’t the same as the square because it’s cut off.” Initially, 509 reasoned that the triangle’s area could be larger because “the triangle looks a little bigger than the rectangle.” 512 began by stating that since both shapes were half of the original square that their areas “might be equal.” The interviewer asked the participant to clarify his hesitance and 512 explained that the shape of the triangle made him doubt if its area was larger, “it looked like it could have had more area because it is more spread out.”

### **Students' Reasoning Related to Both Shapes Having Unequal Areas**

Six participants decided that the areas of the triangle had a larger area than the rectangle. However, four participants (502, 503, 507, 804) promptly stated that the rectangle and right triangle had unequal areas. At no point in the interview did these participants explicitly conclude that the areas of the rectangle and right triangle were equal, differentiating them from the reasoning exhibited in all other interviews. These participants exhibited diverse ways of claiming that the area of the right triangle was larger.

One participant (502) came to a swift conclusion stating that the area of triangle was bigger than the rectangle “from the way it was folded.” Two participants (503 and 507) argued that the right triangle had a larger area because it has a longer side than the rectangle. For the previous three parts of this task, both 503 and 507 focused their responses and comparisons on the computation involved in finding the areas. 503 continued this type of reasoning into the final part. For instance, 503 stated “I know the triangle is only half of the square, but then the triangle has a longer side which when you multiply it, it would get a bigger number than multiplying two of the same sides of the square that would not get a bigger number.” This computational way of reasoning about products of lengths also led 503 to make the claim that the triangle might have a larger area than the original square even after stating he knew the triangle “is only half the size of” the original square. 507 also attempted to reason about products. 507 first stated that the rectangle and right triangle were “both half the size of the square, so it just might be different of how you’re gonna get the area and the numbers might be a little bit different.” To explain how he knew that the areas of the two shapes were different, 508 asserted that the triangle “has one right angle and it’s got one longer side than the rectangle does.”

Another participant, 804, used reasoning related to an action to decide that the areas were different. When asked to compare the two shapes’ areas, 804 explained that “The triangle seems bigger. I think if I put the rectangle inside the triangle and cut it to like make it fit, the triangle would have more area.”

In contrast to 502, 503, 507, 804, when two participants (510, 801) were asked to compare the areas of the rectangle and right triangle, they went back and forth with their responses. However, they ultimately decided that the halves had different or unequal areas. In one interview, 510 presented several back-and-forth decisions as the interviewer asked for clarification. First, 510 stated that the right triangle was bigger because “it looks like it has more space than the rectangle.” When asked by the interviewer, “How do you know?”, 510 stated that the right triangle and rectangle were equal in area “because they both cover half the square evenly.” The interviewer repeated the participant’s two conflicting thoughts and asked, “Which

one are you most convinced by?” 510 responded that the two shapes had different areas and that the “triangle has a little more space than the rectangle.” The interviewer then asked, “So, you think the triangle looks like it has more space?”, to which 510 responded, “I kind of feel like it’s the rectangle, but I’m not sure.” Responding to another switch in reasoning, the interviewer asked, “What’s making you not confident in one or the other?”, 510 then described how “if you put the rectangle up to the triangle, there’s a little more space, a little triangle... but when you flip it over the other way, the rectangle takes over the whole thing.” Finally, the interviewer restated the prompt, “So if we compare the area of the triangle and the rectangle, do they have the same area or do they have different areas?”, 510 stated that “they have different areas” and that the right triangle has a larger area.

While the interview with 801 had fewer back and forth moments, 801’s final decision was also that the halves had unequal areas. However, 801 seemed to use information about the relationship between each shape and the original square, mainly the formulas to find the area of each shape. For instance, 801 stated that “the triangle has an area that is exactly half of the square” and since the rectangle “is already half the width, but the same length, so it’s already half the area.” When the interviewer asked, “so, are the areas different or the same?”, 801 said that the areas were different “because there are different ways to find the areas.” 801 went on to state that when you “visually compare them, the triangle seems to cover more area of the square.”

### Discussion

The data collected and analyzed from this task provided further evidence of what Simon claimed is a KDU: fraction as a quantity. As Simon (2006) expressed, students who responded that the rectangle and the right triangle were different, even after stating each were half of the square when presented separately, viewed fraction (half) as an arrangement. The students who stated that the rectangle and the right triangle were the same viewed fraction (half) as a quantity. However, the reasons for which a student believed each position to be true is what sets this study apart from what has been reported thus far.

Because Simon (2006) stated that an accumulation of experiences and activities was essential to a KDU, we were not surprised that the all four eleventh-grade students asserted that the rectangle and the right triangle were equal in area because they were both half of the same shape. This reasoning was exhibited by most students who believed that the rectangle and the right triangle had equal areas. Considering this study incorporated the action of folding of the squares into the rectangle and the right triangle, students also used reasoning about both folding and unfolding of the shapes. For instance, 504 stated “they’re both still folded in half and they’re both still the same size as the square if they’re unfolded,” and 505 stated that the halves are “just folded in half two different ways.” These two participants used folding and unfolding as a reason for the shapes to have equal areas. However, 502 used it as a reason for the right triangle to have a larger area “from the way it was folded.” The responses discussed so far were supported by generalizable knowledge. In other words, the students did not need to refer to the specific shapes but spoke in rules (e.g., “they are both half the same shape”). However, two fifth-grade students were more convinced to conclude that the areas of the two shapes were equal when they knew specific measurements and could calculate the areas of each.

Regarding the reasoning observed by the students who stated that the rectangle and the right triangle having unequal areas, there was also variety. Reasons revolved around visualization (i.e., superimposing shapes, folding, or cutting), computation, or using characteristics of the specific shapes. For example, 801 stated that when you “visually compare them, the triangle seems to

cover more area of the square,” 507 reasoned that the triangle “has one right angle and it’s got one longer side than the rectangle does.” As mentioned above, only one student, 502, referred to the action of folding, but in this case, to argue the opposite conclusion from the students above. Another student, 801, referred to the area formulas for each shape when she stated the triangle is bigger “because there are different ways to find the areas.” All these reasons either rely on seeing the specific shapes involved, analyzing characteristics of the shape, separate from the other, or anticipating actions being done to the specific shape.

### **Limitations and Directions for Future Research**

Due to the COVID-19 pandemic, all interviews were conducted over zoom. While this medium offered a way to continue to interview students when other formats were not available, it was difficult at times to clearly show the action of folding shapes and interpret student responses. For instance, when students offered reasoning related to turning or superimposing the shapes, the interviewer had to interpret the verbal directions the students offered. Future studies should conduct interviews in-person in which the student can hold and manipulate the shapes themselves to express their thoughts. Another limitation of this study was that it included a small sample of students, all recruited from the same school. Future studies may include a larger, more diverse population of students to interview.

### **Conclusion and Implications**

Simon (2006) claimed that it is essential to not only identify KDUs, such as fraction as a quantity, but also to describe the differences in the sophistication of students’ responses in elaboration of a KDU. In this study, we extended Simon’s work on KDUs to examine students’ reasoning about areas of squares, rectangles, and triangles. We noticed that there was variety in students’ comparisons of half as it related to areas of shapes. Specifically, even though students demonstrated that they viewed fraction as a quantity by stating that two halves, that appeared different, were the same, their reasons were different. Some reasoned about the action of folding and being a part of the same original shape and some reasoned about each shape’s numerical measurements. According to Simon (2006), identification of KDUs “is a way of specifying developmental steps” (p. 367) and further elaboration makes the steps in between more apparent. Additionally, participants’ reasoning, particularly the explanations in which participants went back and forth as they compared shapes, was significant and more thoroughly depicted than research that has been conducted thus far using similar tasks.

Both teachers and mathematics curriculum writers would benefit from knowing more about how students think about a fraction, either as an arrangement or as a quantity. According to Simon (2006), a KDU is not usually something that a student can learn solely from a teacher’s or peer’s explanation or demonstration or viewed from the perspective of the adult. These characteristics can make it challenging to develop instructional tasks that advance students’ understanding. These findings could influence a sequence of instruction involving measurement concepts. Future research may include teaching experiments to identify details related to specific tasks or interventions used to progress students’ reasoning from using characteristics of specific shapes toward generalization.

### **References**

Barrett, J. E., Cullen, C. J., Miller, A. L., Eames, C. L., Kara, M., & Klanderma, D. (2017). Area in the middle and later elementary grades. In J. E. Barrett, D. H. Clements, & J. Sarama (Eds.), *Children’s measurement: A longitudinal study of children’s knowledge and learning of length, area, and volume*. Journal for Research in Mathematics Education monograph series (Vol. 16, pp. 105–127). Reston, VA: National Council of Teachers of Mathematics.

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

- Hiebert, J., & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: An introductory analysis. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: the case of mathematics* (pp. 1–27). Hillsdale, NJ: Lawrence Erlbaum Associates, Inc.
- Simon, M. A., Tzur, R., Heinz, K., & Kinzel, M. (2004). Explicating a mechanism for conceptual learning: Elaborating the construct of reflective abstraction. *Journal for Research in Mathematics Education*, 35(5), 305–329. <https://doi.org/10.2307/30034818>
- Simon, M. A. (2006). Key developmental understandings in mathematics: A direction for investigating and establishing learning goals. *Mathematical Thinking and Learning*, 8(4), 359–371. [https://doi.org/10.1207/s15327833mtl0804\\_1](https://doi.org/10.1207/s15327833mtl0804_1)
- Simon, M. A., Placa, N., Avitzur, A., & Kara, M. (2018). Promoting a concept of fraction-as-measure: A study of the learning through activity research program. *Journal of Mathematical Behavior*, 52, 122–133. <https://doi.org/10.1016/j.jmathb.2018.03.004>
- Skemp, R. (1976). Instrumental understanding and relational understanding. *Mathematics Teaching*, 77, 20–26.
- von Glaserfeld, E. (1995). *Radical constructivism: A way of knowing and learning*. Washington, DC: Falmer.



# GEOMETRY STUDENTS' WAYS OF THINKING ABOUT ADINKRA SYMBOLS

Steven Boyce  
Portland State University  
sboyce@pdx.edu

Michael Ion  
University of Michigan  
mikeion@umich.edu

Keywords: Ethnomathematics, Geometry and Spatial Reasoning

## Background and Purpose

This research regards students' mathematical reasoning when engaging in explorations of the meanings and mathematical properties of Adinkra, which are cultural symbols of the Akan people of modern-day Ghana (Babbit et al, 2015). In the context of an online college geometry course, secondary preservice and inservice teachers formed conjectures of relationships between the mathematical aesthetics and cultural interpretations of five Adinkra symbols. These were chosen by the researchers to highlight and contrast connections to a focus Adinkra symbol, *Boa me na me moa wo*, which exhibits a near-symmetry property of *mutuality*. (Boyce et al, 2021). As a follow-up activity, students engaged in a series of asynchronous tasks: 1. creating their own symbol and conjecturing others' symbol's meanings (with justification) and 2. revealing their symbol's meanings and identifying transformational reasoning with others' symbols. We analyzed students' responses with the goal of identifying the qualities of geometric reasoning exhibited across these four follow-up tasks. Our research question is: what types of reasoning did Geometry students exhibit when describing symbols?

## Methods, Results, and Implications

Students' de-identified work on the tasks above was analyzed in the following manner. First, one researcher examined responses to task 1 and used open coding to categorize students' geometric justifications. This led to the following categories: transformational reasoning (regarding rotations, reflections, translations, or dilations), static figurative reasoning (regarding identification of particular figures, such as polygons or circles), and pattern reasoning (regarding sequences of (non-transformational) actions). For example, referring to a line of reflection would be transformational; identifying the number of triangles would be static figurative reasoning, and describing a figure's vertices increasing clockwise would be pattern reasoning. We also identified a fourth category: descriptions of figures exhibiting *mutuality*. A second researcher used *Label Studio* (Tkachenko et al, 2020) to analyze the frequencies categories across the two tasks, allowing comparisons prior to the instructions to identify transformational reasoning across the tasks. The results suggest that although transformational reasoning was available to participants, their inclination was to engage in other forms of reasoning prior to explicit instruction. We look forward to discussions with participants at the poster session about the potential for introducing Adinkra to teachers to support their students' engagement in varieties of geometric reasoning.

## Acknowledgments

The reported work is supported by NSF DUE-1725837. All opinions are those of the writers and do not necessarily represent the views of the Foundation.

## References

Boyce, S., Ion, M., Lai, Y., McLeod, K., Pyzdrowski, L., Sears, R., & St. Goar, J. (2021, May 6). Best-Laid co-plans for a lesson on creating a mathematical definition. AMS Blogs: On Teaching and Learning Mathematics. <https://blogs.ams.org/matheducation/2021/05/06/best-laid-co-plans-for-a-lesson-on-creating-a-mathematical-definition/>

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

- Babbitt, W., Lachney, M., Bulley, E., and Eglash, R. (2015), Adinkra mathematics: A study of ethnocomputing in Ghana. *Multidisciplinary Journal of Educational Research* 5(2), 10-35.  
<https://doi.org/10.17583/remie.2015.1399>.
- Tkachenko, M., Malyuk, M., Holmanyuk, A., & Liubimov, N. (2020). Label Studio: Data labeling software. Retrieved from <https://github.com/heartexlabs/label-studio>

# INVESTIGATING STUDENT'S UNDERSTANDING OF THE AREA AND PERIMETER OF RECTANGLES THROUGH PROBLEM-SOLVING

Kehinde Jegede  
Illinois State University  
kojeced@ilstu.edu

*Numerous studies (Anwar et al., 2016; Carpenter et al., 1980; Huang & Witz, 2013; Outhred & Mitchelmore, 1996; Machaba, 2016; Van de Walle, Karp & Bay-Williams, 2014; Winarti et al., 2012) have shown that students need help understanding the concept of area and perimeter. This study aimed to investigate what students know about finding the area and perimeter of rectangles through problem-solving. The participants for the study were five 4<sup>th</sup> Graders from a public school located in a mid-size suburb setting outside the Chicago metropolitan area. Participants were asked to solve five problem-solving tasks, after which an interview was conducted for three out of the five participants. Results showed that students did not understand the concept of the area and perimeter of rectangles. Implications for the study, as well as future research, were also discussed.*

Keywords: geometry, relational and instrumental understanding, area, perimeter, problem-solving

## Introduction/Statement of the Problem

Recommendations regarding geometry and measurement as essential content standards in kindergarten through 12<sup>th</sup>-grade mathematics are clearly stated in the National Council of Teachers of Mathematics documents (NCTM, 1989, 2000, 2010). It is believed that having geometric knowledge, relationships, insight, and the ability to measure helps children see that mathematics is practical in everyday life and that it also helps them develop many mathematical concepts and skills (NCTM, 1989, 2000, 2010). Research also shows that children learn about the properties of shapes and sharpen their intuitions and awareness of spatial concepts as they explore patterns and relationships with models, blocks, geoboards, and graph paper (King, Rojo, Bryant, 2022; Winarti et al., 2012). Area and perimeter skills have been highlighted as critical foundations of algebra and as a prerequisite for more advanced mathematics courses (National Mathematics Advisory Panel [NMAP], 2008). The learning of geometry and measurement encourages and develops students' problem-solving and mathematical thinking skills, as well as the ability to apply mathematical knowledge (Putra, Panjaitan, Putri, Wulandari, & Hermita, 2021).

Research in the field of mathematics education, locally and internationally, often reveals a poor understanding of the concepts of area and perimeter (Anwar, Yuwono, As'ari, & Dwi, 2016; Carpenter, Corbitt, Kepner Jr, Lindquist, & Reys; Huang & Witz, 2013; Machaba, 2016; Outhred & Mitchelmore, 2000; Winarti, Amin, Lukito, & Gallen, 2012). Results showed that most learners do not realize that it is possible to have many rectangles with the same area but different perimeters and vice versa (Outhred & Mitchelmore, 1996; Machaba, 2016; Van de Walle, Karp & Bay-Williams, 2014). Similarly, some learners thought that rectangles with the same area should possess the exact dimensions and have the same perimeter, and vice versa (Machaba, 2016). Another way that students demonstrated a lack of proper understanding is by giving procedural definitions (formula definitions) for the area and perimeter of rectangles (Huang & Witz, 2013; Machaba, 2016).

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

Unfortunately, a gap in the research is that the concept of area and perimeter is an underdeveloped skill area for many school-age children (King et al., 2022). Therefore, this study aimed to investigate 4<sup>th</sup> graders conceptual understanding of and their ability to solve problems related to the area and perimeter of rectangles by applying mathematical knowledge gained on area and perimeter to new experience. It is believed that this study will further emphasize the need to develop children's conceptual understanding of the area and perimeter of shapes (regular and irregular) and promote problem-solving to measure the current state of students in terms of their understanding of the area and perimeter of rectangles. To achieve the stated purpose of this study, I proposed the following research questions:

- What are students' conceptual understanding of the area and perimeter of rectangles?
- How did students' prior knowledge and /or conceptual understanding assisted them in solving tasks about the relationship between area and perimeter of rectangles?
- What knowledge if any were students' able to construct about the area and perimeter of rectangles?

### **Background of the Study**

Despite their importance in mathematics teaching and learning, studies have revealed a poor understanding of the concepts of area and perimeter (Anwar et al., 2016; Carpenter et al., 1980; Huang & Witz, 2013; Outhred & Mitchelmore, 1996; Machaba, 2016; Van de Walle, Karp & Bay-Williams, 2014; Winarti et al., 2012). According to (Winarti et al., 2012), some of the reasons for this continual source of confusion might be because (1) both area and perimeter involve measurement and (2) students are taught formulae for both concepts at about the same time, therefore getting formulae confused. Suggestions on how to address the issue were provided by some researchers (Carpenter et al., 1980; King et al., 2022; Putra et al., 2021; Winarti et al., 2012). For example, strategies that can be used to demystify area and /or perimeter include using manipulatives (Outhred & Mitchelmore, 2000; Winarti et al., 2012), focusing on the variable, using visual-chunking representations, and contextualizing instruction (King et al., 2022). Designing and implementing GeoGebra learning activities on the area and perimeter of rectangles have also been found to be helpful for primary school students (Putra et al., 2021). By using GeoGebra, students can increase their involvement in understanding the concepts of the area and perimeter of rectangles and visualize rectangular shapes of different sizes (Putra et al., 2021).

Problem-solving has also been found to aid students' understanding of the concept of as well as the development of relational understanding (Skemp, 1976) between the area and perimeter of rectangles (Carpenter et al., 1980; King et al., 2022). An effective way to achieve this is to contextualize instruction by incorporating real-world scenarios and experiences that would allow connection to student's prior knowledge (King et al., 2022; Machaba, 2016). Through problem-solving, students can apply knowledge and skills acquired during mathematics instruction to solve new tasks (Carpenter et al., 1980). Also, since students learn mathematics to be able to reason mathematically and not just to memorize mathematical formulas or facts, problem-solving has been found to help develop understanding and explain the processes used to arrive at solutions, rather than remembering and applying a set of procedures (Klerlein & Hervey, 2019; Skemp, 1976). Through problem-solving, students develop a relational understanding of mathematical concepts (Skemp, 1976), become more engaged (Bada, 2015), and appreciate the relevance and usefulness of mathematics (Klerlein & Hervey, 2019).

## Theoretical Framework

The theoretical framework for this study is based on Piaget's constructivist theory which focuses on children's ability to assimilate new learning into their existing schema.

Constructivism is a theory about how people learn (Machaba, 2016). It states that people construct their own understanding of the world through their experiences with the world (Bada, 2015; Hein, 1991; Suhendi & Purwano, 2018). When we encounter a new experience, we must reconcile it (assimilate) with our previous ideas and experiences (Bada, 2015; Machaba, 2016; Andang, 2018). The ability to reconcile a new experience (e.g., novice mathematical task) with existing schema is a way that students construct knowledge (Skemp, 1976). This view of learning sharply contrasts with one in which learning is the passive transmission of information from one individual to another, a view in which reception, not construction is key. For students to be able to construct knowledge, they must actively participate in the teaching and learning process (Bada, 2015; Hein, 1991). Some of the advantages of constructivist theory is that it produces students who are independent, able to think, able to express ideas and to solve problems (Bada, 2015; Machaba, 2016).

The process of assimilating or reconciling the unfamiliar with what one already knows in mathematics education is a way of identifying and establishing relationship(s) between mathematical topics or concepts (Machaba, 2016; Skemp, 1976). Identifying and establishing relationship(s) between mathematical topics or concepts creates a type of understanding known as relational understanding (Skemp, 1976). Skemp (1976), simply defined relational understanding as "knowing both what to do and why" (p. 2). Skemp expected that students in the mathematics class should not only know how to solve a problem (by using formulas or rules), but they should also be able to state or justify why the method(s) works. For example, Skemp mentioned that it is not sufficient for students to memorize the formula for calculating the area of a triangle =  $\frac{1}{2}$  base \* height but should also be able to relate this with finding the area of a rectangle (p. 9). In contrast to relational understanding, Skemp (1976), defined instrumental understanding as "the application of mathematical rules to find answers to mathematical tasks without reasons" (p. 2). For example, many learners know that the formula to calculate the area of a rectangle is length multiplied by breadth, but they do not know why this is so. He argues that learners should develop a relational understanding of area and perimeter.

For relational understanding to take place, students also need to have conceptual knowledge of mathematical topics or concepts (Skemp, 1976). This is because lack of conceptual understanding results into having misconceptions (Machaba, 2016). Studies have shown that the inability of learners to have a relational understanding of area and perimeter of figures is due to a misunderstanding of the concepts of area and perimeter (Outhred & Mitchelmore, 1996; Van de Walle et al. 2014). To have a conceptual understanding of area of a rectangle means to be able to find the amount of space inside the edges of the figure. Whereas having a conceptual understanding of the perimeter of a rectangle is to be able to find the measurement of the length of the edges of the figure (Machaba, 2016). To be able to have these conceptual understandings, students also need to be taught how to use tiles or unit cubes to cover up surfaces of rectangles as a way of finding their perimeters (Winarti et al., 2012) and /or areas (Outhred & Mitchelmore, 2000; Winarti et al., 2012). When students have good conceptual understanding of perimeter and area of rectangles, it becomes easy for them to see and understand why rectangles with same perimeter does not need to have same area measurement and vice versa. They also come to

understand that rectangles with the same area have dimensions that are factors of the fixed area (Machaba, 2016).

### Methodology

The research methodology used in this study was qualitative. It was organized around a problem-solving task administered to five 4th-grade students (two boys and three girls) between the age of 9-10 years as well as a clinical interview carried out with only three of the students (two boys and one girl). The study took place at a public school located in a mid-size suburb setting outside the Chicago metropolitan area. The student population consists of 48% female students and 52% male students. The school is racially diversified, with 75% White and 24.4% Minority Enrollment: 9.7% Hispanic/Latino, 8.5% two or more races, 3.8% Asian or Asian/Pacific Islander, and 2.3% Black or African American.

There are two parts to this study. In the first part, students were asked to answer four problem-solving tasks (Table 1) to test their understanding of and ability to solve problems related to the area and perimeter of rectangles. The students spent approximately 45 minutes on the tasks. The second part of the study was an interview session. Three students each participated in an interview that lasted for 15 minutes. The interview session (Table 2) enabled me to find out if the students had a conceptual understanding of the area and perimeter of rectangles, how each student's prior knowledge and /or conceptual understanding assisted to problem solve, and whether they were able to construct new knowledge. The interview session was audio recorded. The instruments used for data collection were problem-solving tasks and interview questions (see Table 1 for excerpts of interview questions).

**Table 1: Problem-Solving Tasks and Excerpt of Interview Questions**

The Perimeter and Area Puzzle	Excerpt of Interview Questions
Draw as many rectangles as you can that have a perimeter of 24 units. Find the area of each rectangle you drew. What do you notice about the rectangles?	Can you please read this question and explain to me in your own words what the problem is asking you to do?
Draw as many rectangles as you can that have an area of 24 sq. units. Find the perimeter of each rectangle you drew. What do you notice about the rectangles?	What does it mean to find the perimeter of a rectangle?
Find all the rectangles that have an area of 36 sq. units. Fill in the table with the length, width, and perimeter of each rectangle. What patterns did you notice?	What does it mean to find the area of a rectangle?
From your solutions to questions 1, 2, and 3 above, can you tell what happens to the rectangles when the area stays the same? Can you tell what happens to the rectangle when the perimeter stays the same?	Can you explain what you did and why you did it that way?
	Does this rectangle have a perimeter of 24

units?

What can you notice about the rectangles?

---

### **Data Analysis and Coding Scheme**

Two sets of data were analyzed. The first set of data was focused on the children's written solutions to the four problems. A student is said to have a conceptual understanding if s/he can come up with different rectangles (that has same perimeter but different area and vice versa) with the help of manipulative or by drawing rectangles with surfaces covered with squares. For the second research question, I identified students' prior knowledge: compose two-dimensional shapes, partition a rectangle into rows and column of same-size squares and count to find the total number of them, understand concepts of area and relate area to multiplication and to addition, and recognize perimeter as an attribute of plane figures (Common Core State Standards for Mathematics, 2010) used during problem-solving. By using their prior knowledge, students should be able to realize that: (1) we can have many rectangles with same perimeter but different areas and vice versa, (2) rectangles with the same area have dimensions that are factors of the fixed area (3) when the difference between the dimensions of a rectangle with a fixed area is the smallest, you will have the smallest perimeter (same is applicable when the difference between the dimensions of a rectangle with a fixed area is the largest), (4) given a fixed perimeter, the rectangle with the largest area will be the one with the dimensions that are closet together (a square), (5) given a fixed perimeter, the rectangle with the smallest area will be the one with the dimensions farthest apart. For the third research question, students can construct knowledge if they are able to establish the above relationships about the area and perimeter rectangles.

During the interview session, a student was said to have conceptual understanding if s/he didn't provide formula definitions of area and perimeter of rectangles (Machaba, 2016). The definitions or description could also be given in relation to when tiles or unit squares are used as coverings for the surface of rectangles. Two out of the questions asked during the interview are: (1) Can you explain what you did and why you did it that way? (2) What do you notice about the rectangles drawn? These two questions were asked to know if the students were able to form relationships between area and perimeter of rectangles (as stated above) using their prior knowledge (CCSSM, 2010). The interviews with the three students were transcribed. The reason for selecting these students was due to their responses to the tasks.

### **Results**

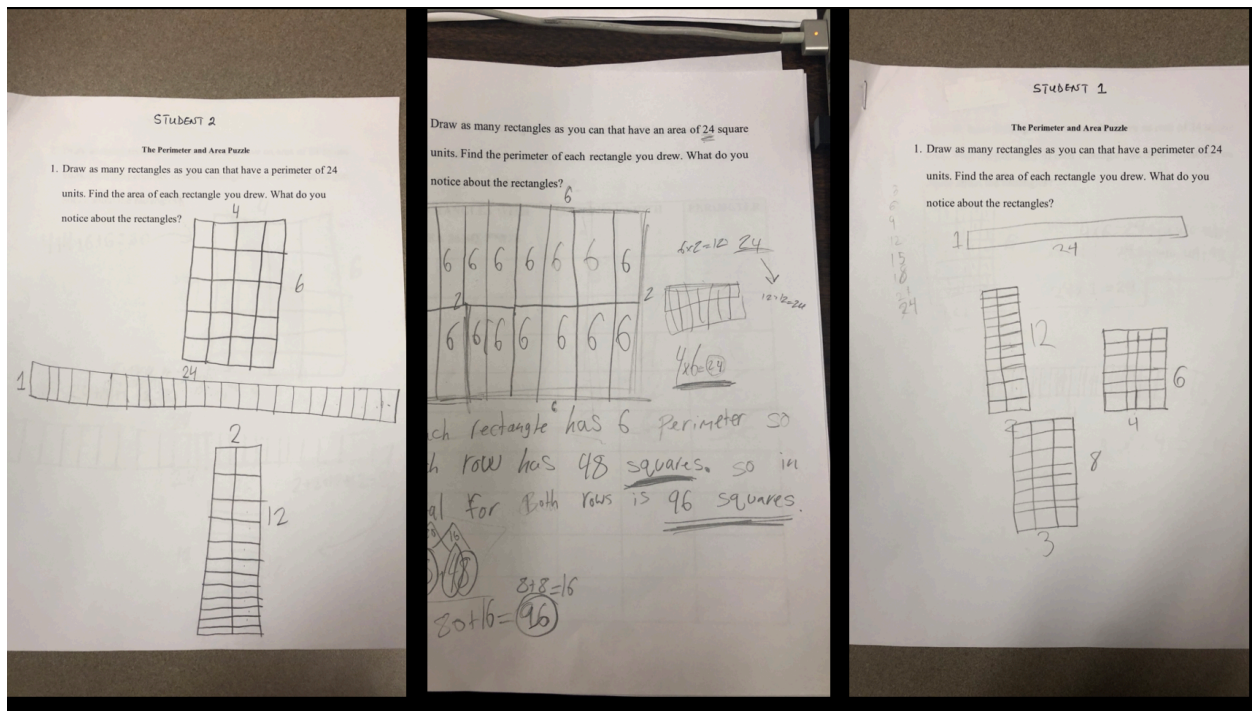
In this section, I will briefly discuss students' conceptual understanding of one of the problem-solving tasks (see Figure 1 & Table 1), type of prior knowledge used, whether students were able to establish relationship between area and perimeter using their prior knowledge, and their responses to part of the interview questions (see Table 2).

#### **Coded Student's Work**

It is important to mention that the students were already working on finding the area of rectangles using tiles/unit squares, completing multiplication facts, and solving multiplication equations before carrying out this study. As they began to solve the tasks, I provided them with manipulative. While student 1 and student 2 demonstrated a conceptual understanding of area of rectangles by drawing unit squares to cover the surfaces (Outhred & Mitchelmore, 2000), the question instead asked them to first draw as many rectangles as possible with perimeter 24 units, and then find the area of each rectangle. Based on this, it is unclear if the students had a

conceptual understanding of perimeter of rectangles. The solution provided by student 3 showed that she doesn't have a conceptual understanding of area and perimeter of rectangles (Outhred & Mitchelmore, 1996). Overall, I think all three students attended to finding the area of the rectangles because that was the topic they were learning at that time.

All three students work demonstrated the use of some prior knowledge of area and perimeter of rectangles. For instance, the students were able to compose two-dimensional shapes, partition a rectangle into rows and column of same-size squares (except for student 3) and count to find the total number of them, understand concepts of area and relate area to multiplication. But none of the students were able to recognize perimeter as an attribute of plane figures (Machaba, 2016). It was expected that the students would be able to relate area and perimeter of rectangles using their prior knowledge. Unfortunately, they were unable to notice any of such relationships. During the interview session on question 2 (not shown), only student 2 was able to notice that when the area is fixed, the rectangles have different perimeters. One of the reasons for this may be because the students lacked conceptual understanding of area and perimeter of rectangles.



**Figure 1: Sample of Student's Work**

**Student's Responses to Segment One of the Interview Questions.** The following session explains students' understanding of the concepts of the area and perimeter of rectangles (see Table 2 below). I purposely asked each of the student the first interview question to know if they have a good understanding of the questions. The ability to interpret and make sense of mathematical tasks is one of the skills required during problem-solving. (Carpenter et al., 1980; King et al., 2022). Contrary to some of the student's work (Figure 1 above), the interview session showed that only students' 1 and 2 had a conceptual understanding of what a perimeter of rectangle is. Student 3 however explained perimeter in terms of finding area. She said that "...No, the product, like times this and this". When asked about what it means to find the area a



rectangle, students' 1 and 2 answered by giving a procedural/formula definition (Machaba, 2016). Student 1 however gave a conceptual definition of area of a rectangle. He stated that "stuff on the inside...and you count on if it was a big 12 on that side and 12 on that side. That would be 24-foot area...cos it's the inside, not outside". It seems that the interview session helped one or two of the students to realize and correct some of the mistakes made in their work. Also, when student 1 was asked if the rectangle he drew in question 2 (not shown) had a perimeter of 24 units, he was able to provide a correct explanation that showed he had a conceptual understanding. He mentioned that "you would count 1234 to 24 for that then add then 4646". What this tells us is that student 1 counted the units squares and added up all the sides of the rectangles.

**Table 2: Student's Responses to Segment One of Interview Questions**

Interview Questions	Student's Responses	Conceptions
I: Explain to me in your own words what the problem is asking	S1: it's asking you to draw a rectangle with an outside of 24. S2: ...I think it's like, like, like the whole rectangle equals 34. No, 24, and just like perimeter of 24. S3: I should have done 12 times 12, but I did 4 times 2	
I: What does it mean to find the perimeter of a rectangle?	S1: it's asking you to draw a rectangle with an outside of 24. S2: $4 + 4 + 6 + 6$ . Like, add the top and bottom twice and then add them to each other. S3: perimeter is like the side kind of. No, the product, like times this and this	Conceptual understanding  Conceptual understanding  Lacked conceptual understanding.
I: What does it mean to find the area of a rectangle?	S1: stuff on the inside...and you count on if it was a big 12 on that side and 12 on that side. That would be 24-foot area...cos it's the inside, not outside. I: so, you mean for area you have to add? S1: no, you can use multiplication. S2: you have to times the top and the width. So, like the length and the width S3: the area I know for sure is	Conceptual understanding       Procedural understanding  Procedural understanding

	length times width, but I don't know about the perimeter	
I: Does this rectangle have a perimeter of 24 units?	S1: you would count 1234 to 24 for that then add then 4646	Conceptual understanding

---

### Discussions & Summary

Like previous research, the findings of this study showed that students lacked a conceptual understanding of the concept of the area and perimeter of rectangles which also prevented them from establishing relationships between area and perimeter of rectangles (Anwar et al., 2016; Carpenter et al., 2013; Machaba, 2016; Outhred & Mitchelmore, 1996; Winarti et al., 2012). The students' work also showed that they didn't attempt the part of the task which required that they write down what they noticed among the rectangles they created. For example, student's 3 work was totally wrong and was unrelated to the task. While she was being interviewed, she struggled to interpret the questions and demonstrated a poor understanding of area and perimeter of rectangles (Machaba, 2016; Outhred & Mitchelmore, 1996). When asked to interpret the questions, she stated that "I should have done 12 times 12, but I did 4 times 2". Student 3 thought 12 times 12 will give her a perimeter of 24 units.

However, during the interview session, after reading and interpreting the questions, only student 1 demonstrated conceptual understanding of area and perimeter. Student 2 was able to conceptualize perimeter but not the area as a measure of surface or a region of the rectangle. He described the area using the formula definition (Huang & Witz, 2013; Machaba, 2016). Generally, I would say that having a one-to-one discussion was helpful for the students (Cobb & Yackel 1996)

### Implications for Research

During the interview session, students had the opportunity to review their work and correct some of their mistakes. This shows the importance of one-to-one interviews, creating a psychological construct between the student and researcher (Cobb & Yackel, 1996). Since the study of geometry requires thinking and doing (NCTM, 2000), it is essential that both researchers and teachers engage students in problem-solving to develop their mathematical reasoning ability and for them to be able to apply mathematical knowledge.

Although the students' graded work indicated that they switched the definition for the area of a rectangle to mean the perimeter of a rectangle, this does not necessarily mean that they do not understand the concepts. During the interview, it was discovered that either the students did not understand the questions, or they interpreted the questions based on the current topic (finding the area of rectangles using arrays and equations for tens) they were learning. It is suggested that teachers or researchers should revise their tasks and activities so that they are interpreted by students as intended.

One limitation of the study is that some of the questions were not correctly worded, so students were unsure what to do or how to begin solving the problems. Future research should include more population of students (preferably lower grades), a pre-test, and a teaching session (depending on students' performance on the pre-test).

### References

Andang, S. P. (2018). Constructivist learning theory: The contribution to foreign language learning and teaching [Conference session]. The 1st Annual International Conference on Language and Literature. <https://doi.10.18502/kss.v3i4.1921>

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

- Anwar, R. B., Yuwono, I., As' ari, A. R., & Dwi, R. (2016). Mathematical Representation by Students in Building Relational Understanding on Concepts of Area and Perimeter of Rectangle. *Educational Research and Reviews*, 11(21), 2002-2008. <http://www.academicjournals.org/ERR>
- Bada, S. O. (2015). Constructivism learning theory: A paradigm for teaching and learning. *IOSR Journal of Research & method in Education*, 5(6), 66-70. <https://doi.10.9790/7388-05616670>.
- Carpenter, T. P., Corbitt, M. K., Kepner, H. S., Lindquist, M. M., & Reys, R. E. (1980). NAEP note: Problem-solving. *The Mathematics Teacher*, 73(6), 427-433. <https://www.jstor.org/stable/27962070>
- Cobb, P., & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational Psychologist*, 31(3-4), 175-190.
- George, E. H. (1991, October 15-22). The Museum and the needs of people [Conference session]. International Committee of Museum Educators Conference, Jerusalem, Israel. <http://www.exploratorium.edu/IFI/resources/constructivistlearning.html>
- Huang, H. M. E., & Witz, K. G. (2013). Children's Conceptions of Area Measurement and Their Strategies for Solving Area Measurement Problems. *Journal of Curriculum and Teaching*, 2(1), 10-26. <http://dx.doi.org/10.5430/jct.v2n1p10>
- King, S., Rojo, M., & Bryant, D. P. (2022). Demystifying Area and Perimeter: A Practitioner's Guide to Strategies That Work. *Intervention in School and Clinic*, 1-8. <https://doi.10.1177/10534512221093780>
- Klerlein, J., & Hervey, S. (2019). Mathematics as a complex problem-solving activity: Promoting students' thinking through problem-solving. *Generation Ready White Paper*.
- Machaba, F. M. (2016). The concepts of area and perimeter: Insights and misconceptions of Grade 10 learners. *Pythagoras*, 37(1), 1-11. <https://dx.doi.org/10.4102/pythagoras.v37i1.304>
- National Council of Teachers of Mathematics. (2010). *Common core state standards for mathematics*. Reston, VA: Author.
- National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author.
- National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics* Reston, VA: Author.
- National Mathematics Advisory Panel. (2008). *The final report of the national mathematics advisory panel*.
- Outhred, L., & Mitchelmore, M. (1996). Children's intuitive understanding of area measurement. In L. Puig, & A. Gutierrez (Eds.), *Proceedings of the 20th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 91-98). Valencia: PME
- Outhred, L. N., & Mitchelmore, M. C. (2000). Young children's intuitive understanding of rectangular area measurement. *Journal for Research in Mathematics Education*, 31(2), 144-167.
- Putra, Z. H., Panjaitan, I. O. D., Putri, N. A., Wulandari, T. R., & Hermita, N. (2021, October). Design and Implementation of GeoGebra Learning Activities of Area and Perimeter of Rectangles for Primary School Students. In *Journal of Physics: Conference Series*, 2049(1). <https://doi.10.1088/1742-6596/2049/1/012032>
- Skemp, R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching*, 77, 20-26.
- Van de Walle, J. A., Karp, K. S., & Bay-Williams, J. M. (Eds). (2014). *Elementary and middle school mathematics: Teaching developmentally*. Boston, MA: Pearson Education.
- Winarti, D. W., Amin, S. M., Lukito, A., & Van Gallen, F. (2012). Learning the concept of area and perimeter by exploring their relation. *Journal on Mathematics Education*, 3(1), 41-54.

## MENTAL OPERATIONS FOR ALTERING LENGTH AND PRESERVING ANGULARITY

Hamilton L. Hardison  
Texas State University  
HHardison@txstate.edu

*Angularity is a persistent quantity throughout K–12+ school mathematics, and many studies have shown that individuals often conflate angularity with linear attributes (e.g., the length of an angle model’s sides). However, few studies have examined the productive ways in which students might reason about angularity while attending to linear attributes like side lengths. Leveraging data from a yearlong teaching experiment with ninth-grade students, I present four mental operations that students indicated for altering lengths while preserving angularity. Additionally, I consider implications of these mental operations for teaching and research.*

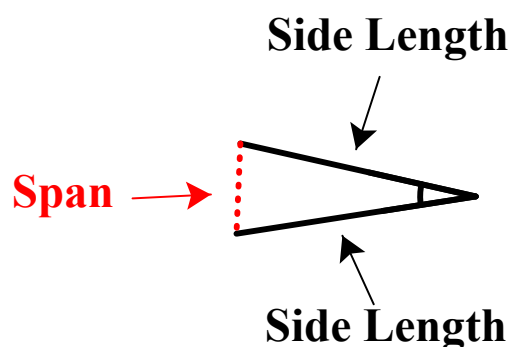
Keywords: Geometry and Spatial Reasoning, Cognition

Angularity is a persistent quantity in K–12+ school mathematics (Barabash, 2017), and considerably less has been written about individuals’ thinking regarding angular measure compared to other quantities (Smith & Barrett, 2017). Furthermore, scholars have explicitly called for additional research into how students quantify angularity (e.g., Moore, 2013; Smith & Barrett, 2017). Perhaps in response to these calls or the curricular salience of angularity, in recent years several researchers have conducted studies foregrounding angularity (e.g., Alyami, 2022; Germia, 2022; Hardison, 2018; Mullins, 2020). Despite this recent increase in studies and researchers focused on angularity, there remains a need for basic research elaborating the mental operations that play a role in individuals’ quantifications of angularity. For example, several studies discussed in the subsequent section have shown that individuals often problematically conflate angularity with linear attributes; however, little has been reported regarding whether and how individuals might productively attend to the length of an angle model’s sides, for example, when reasoning about angularity. This report offers progress toward this end by examining some ways in which students can productively attend to both linear and angular attributes. In particular, this study addresses the following question: What mental operations do students enact to preserve angularity while altering length?

### Some Relevant Extant Literature

Students’ tendencies to conflate angularity with numerous attributes (e.g., orientation, area, length, etc.) have been documented (Smith & Barrett, 2017). Here, I present a brief summary of research reporting on confluations involving length and angularity. In studies with students in elementary and middle grades, researchers have found students often conflate angularity with other attributes including two primary variations of linear attributes. First, students often judge which of two angles is larger by comparing the lengths of the angle model’s sides (Baya’a et al., 2017; Clements et al., 1996; Crompton, 2017; Devichi & Munier, 2013; Keiser, 2004; Lehrer et al., 1998). Second, students also tend to consider angularity to refer to the linear distance between an angle model’s sides (Baya’a et al., 2017; Keiser, 2000; 2004; Lehrer et al., 1998; Thompson, 2013); I refer to this second linear attribute as the *span* of an angle model (see Figure 0 wherein solid and dotted lines indicate perceptually available and absent segments, respectively). For example, when 60 elementary children enrolled in a three-year longitudinal study were asked to measure the amount of opening in various contexts including hinged wooden

jaws, an open door, or a bent straw, the children measured the span of these angle models 95% of the time (Lehrer et al., 1998). Lehrer and colleagues posited that children believed measuring length was “an adequate measure of angle” (p. 151), and they also found that conflating length and angularity was a persistent issue throughout the longitudinal study remarking, “the effects of length on children’s judgments about angles did not diminish during the 3 years of the study” (p. 149).



**Figure 0: Two Linear Attributes of an Angle Model: Side Length and Span**  
**Brief Theoretical Considerations**

The study at hand was heavily informed by a radical constructivist epistemology (von Glasersfeld, 1995) and principles of quantitative reasoning (Thompson, 1994; 2011). According to Thompson, an individual has constructed a quantity when they have conceived an attribute of an object along with an imagined process for measuring the attribute; such an imagined measurement process necessarily involves a mental comparison of at least two instantiations of the attribute at hand. Angularity is one example of an attribute that an individual might quantify; other such attributes include length, area, time, and speed. Because quantities exist in individuals’ minds, investigating how individuals conceive quantities necessarily involves attending to the mental operations particular attributes permit (from the perspective of that individual); mental operations include, for example, imagined physical actions like partitioning and iteration (Steffe & Olive, 2010). The collection of mental operations an attribute permits for an individual constitutes the individuals’ quantification for that attribute. Thus, quantities and quantifications necessarily vary, both across individuals and within an individual over time.

**Two Opposing Hypotheses Regarding Length and Angularity**

Given individuals’ propensities for conflating length and angularity, a reasonable hypothesis (at least at first blush) is that quantifications of length serve as obstacles for productively quantifying angularity. I refer to this hypothesis as *the angular interference hypothesis* (Hardison, in press). Educators working under the angular interference hypothesis might, for example, approach early angular-measure instruction in such a way as to divorce (as best as possible) students’ ways of thinking length and angularity. A second hypothesis, *the angular reorganization hypothesis*, stands in opposition to the angular interference hypothesis. The angular reorganization hypothesis posits that students’ quantifications of length can be used in service of quantifying angularity. Educators working under the angular reorganization hypothesis might, therefore, establish instructional goals different from those who prescribe to the interference hypothesis.

For the present study, the angular reorganization hypothesis was adopted. One reason for this choice involves considering students early experiences with angular measure; (see Hardison, 2018 for other reasons). Early experiences with angles often involve perceptually available angle models with finite side lengths (e.g., two drawn line segments sharing a common vertex or a pair of hinged wood chopsticks). As intimated in a previous section, conceiving of angular measure involves at least an implicit consideration of two instantiations of angularity, and two perceptually available angle models need not have sides of the same length. Because comparing extents of angularity across two such perceptually available angle models would require a cognitive means of rendering differences in length inconsequential, I conjectured that angularity-preserving, length-altering mental operations might play a critical role in individuals' initial quantifications of angularity. This conjecture is consistent with the angular reorganization hypothesis, and operations of this nature are the focus of this report.

### **Methods**

The data presented in the following sections was curated from a teaching experiment (Steffe & Thompson, 2000; Steffe & Ulrich, 2013) conducted in the southeastern U.S. with two ninth-grade students, Camille and Kacie, and transpiring over a single scholastic year. During the study, both students (a) were enrolled in a first-year algebra course and (b) had yet to experience a dedicated geometry course. The primary aim of the teaching experiment was to investigate how the students quantified angularity as well as the progressions transpiring in these quantifications during the study (see Hardison, 2018). The author served as teacher-researcher throughout the teaching experiment. During the study, students discussed their thinking as they interpreted and addressed mathematical tasks involving angle models, which included physical manipulatives (e.g., hinged wooden chopsticks), drawings, and virtual representations in dynamic geometry environments.

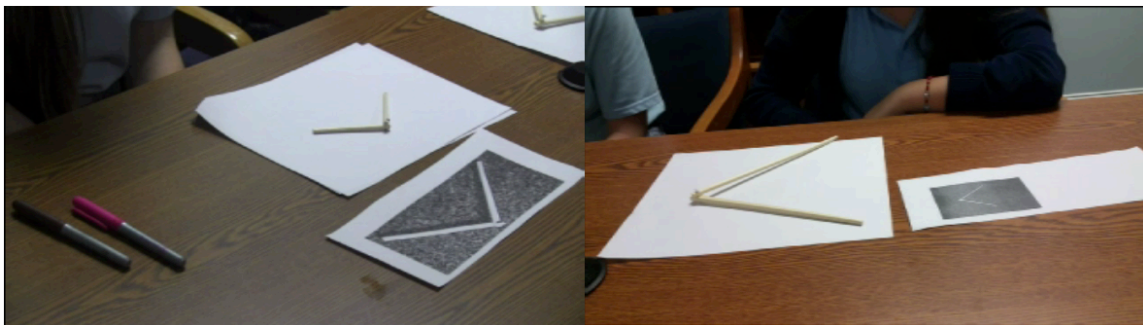
Each student participated in 2 initial interview sessions and 1 final interview session. Between these initial and final sessions, Camille and Kacie participated in 11 and 10 teaching sessions, respectively, which were conducted approximately once per week outside of their regular classroom instruction. Each session was video-recorded and approximately 30 minutes in length. Interview sessions were conducted with each student individually to understand (and not intentionally occasion change in) their ways of reasoning at the beginning and end of the teaching experiment. In contrast, during teaching sessions the teacher-researcher worked to both understand students' existing ways of reasoning and to engender productive changes in students' ways of reasoning; teaching sessions were conducted individually or in pairs. Data sources included recordings, digitized student work, and field notes. Conceptual analysis (Thompson, 2008; von Glasersfeld, 1995) was used to scrutinize records of students' observable activities (e.g., talk, gestures, written responses, etc.) both during the teaching experiment (on-going analysis) and afterwards (retrospective analysis). This report focuses on length-altering, angularity-preserving mental operations.

### **Findings**

From analyzing students' observable activities in the teaching experiment, I abstracted four mental operations the students enacted to alter linear attributes while preserving angularity. In the subsequent sections, I illustrate each of these mental operations using selected data from Kacie's first three teaching sessions, the first two of which were paired teaching sessions with Camille. These three teaching sessions were selected for this report because they were the first sessions in which students indicated length-altering, angularity-preserving mental operations.

### The First Mental Operation: Truncation (Teaching Session #1)

During Kacie and Camille's first paired teaching session, two pairs of hinged wooden chopsticks had been set to different extents of angularity from my perspective (see Figure 1; the photographs in the figure are from a previous task outside the scope of the present report). To investigate students' operations for comparing these extents of angularity, I asked the students which pair of wooden chopsticks were more open, Kacie's short chopsticks (left in Figure 1) or Camille's long chopsticks (right in Figure 1).



**Figure 1: Kacie's (left) and Camille's (right) chopsticks as set from a previous task.**

In this angular comparison task, Camille responded first and asserted Kacie's short chopsticks were more open. After a three second pause, Kacie hesitantly agreed the short chopsticks were more open. To understand how Camille was comparing extents of angularity, I asked her to explain her reasoning. Portions of Camille's response are described in Excerpt 1.

#### Excerpt 1: The truncation operation

C: I mean, I just kind of make this [long] one like if it was like that [short] one. So – kind of imagining it like that [*places marker across the top of her long chopsticks as if to render the long pair the same length as the short pair* (see Figure 2)]. So like this right here [*indicating the long chopsticks from vertex to marker*] is just like that [short] one, except that [short] one's more wider.



**Figure 2: Camille truncates her chopsticks (right) to the length of Kacie's chopsticks (left)**

From the interaction surrounding Excerpt 1, I inferred Camille mentally transformed the long chopsticks into the short chopsticks via the *truncation operation*, mentally shortening the long

chopsticks to a particular length, which was (in this case) the length as Kacie’s short chopsticks. That Camille instantiated the operation is supported both by her placement of the marker (Figure 2 right) and her explanation, “I just kind of make this one like it was like that one.” By truncating, Camille rendered the different lengths of the angle models inconsequential as she considered which angle model was more open.

Following further explanation from Camille, Kacie explained her reasoning and indicated a mental operation related to (but distinct from) truncation. Kacie’s explanation and this second mental operation are described in the section below.

### **The Second Mental Operation: Hyper-truncation (Teaching Session #1)**

As described above, Camille responded to the angular comparison task first, and Kacie paused before hesitantly agreeing with Camille. I infer Kacie’s hesitation resulted from considering Camille’s explanation, which likely differed from Kacie’s own initial reasoning. Support for this inference appears in Excerpt 2, wherein Kacie explains her reasoning.

#### **Excerpt 2: The hyper-truncation operation.**

K: I was like looking at how open these were [*repeatedly tracing out the span of the long chopsticks*]. But then I realized that that probably wouldn’t help me because it [the long pair] looks like it’s more open than mine because it’s bigger, like the chopsticks are longer than mine so it makes it look like it was more open. So then, I just kind of looked at like where they [*points to vertex of long pair*] cross or come together and saw that these [short pair] were like [*opens index fingers over the vertex*] not so like – I don’t know how to explain it. [6 sec pause] That mine weren’t so close together and these [long pair] were more – like there was littler space right here [*pointing to the interior near the vertex of the long pair as shown in Figure 3*] than there was in mine [*points similarly near the vertex of the shorter pair*].



**Figure 3: Kacie points to the interior near the vertex of the long chopsticks**



In Excerpt 2, Kacie indicated initially considering the span of the chopsticks and explained discarding this consideration due to differences in the lengths across the pairs of chopsticks. I suspect Kacie's attention to the differences in lengths was occasioned, at least in part, by Camille's assertion that the shorter chopsticks were more open than the longer chopsticks. Like Camille, Kacie indicated she had established a goal of rendering the lengths of the chopsticks inconsequential for comparing openness. Rather than truncate the longer chopsticks to match the length of the shorter chopsticks, Kacie mentally shortened both pairs of chopsticks as if trying to free them of length entirely. Although it is possible Kacie might have been imagining some previous experience involving angle markings (e.g., near-vertex arcs), this seems unlikely for two reasons: First, Kacie had not mentioned any such markings; Second, Kacie had not yet referred to the chopsticks as angles at this point in the teaching experiment. In discussing her comparison, Kacie was adamant that she limited her focus very near to the vertex: "I looked at like really close to where the chopsticks meet, but not like where they meet, like the space right after they meet." For this reason, I consider Kacie to have considered reducing the chopsticks to have sides of essentially infinitesimal length, which I refer to as *hyper-truncation*.

### **The Third Mental Operation: Elongation (Teaching Session #2)**

In the pair's second teaching session and at my request, Camille re-demonstrated her use of a marker to shorten the long chopsticks to match the lengths of the short pair. Because both students had, to this point, indicated shortening side lengths but not extending them, I requested the pair consider if it was possible to think of a way to "turn the shorter pair into the longer pair."

Following this request, Camille suggested cutting material from another chopstick and adding it to the shorter pair. Providing them with markers and paper, I asked the students to draw how they would add to material to the short chopsticks. Camille positioned the shorter chopsticks within the longer so that the vertices and sides were adjacent. Using a marker, Kacie linearly extended one side of the short chopsticks to match the length of the corresponding side of the long chopsticks.

Camille's verbal description and Kacie's physical activity indicated each student had constructed an *elongation operation*, which I consider to be the cognitive inverse of the truncation operation. An individual instantiates elongation when they mentally extend an angle model's sides to an arbitrary but particular length. Prior to this point in the teaching experiment, neither student had spontaneously indicated an elongation operation in any tasks involving these physical angle models.

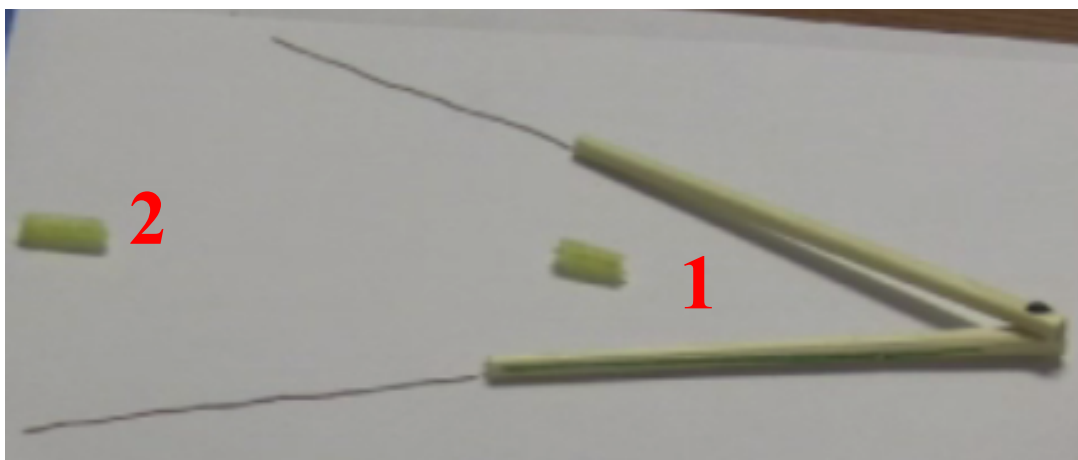
### **The Fourth Mental Operation: Hyper-elongation (Teaching Session #3)**

Because Camille was absent, I worked on-on-one with Kacie during her third teaching session and aimed to further examine how Kacie was thinking about various attributes of the chopsticks. In particular, I sought to investigate whether Kacie differentiated angularity from spanned area (i.e., the area of the triangle formed by an angle model's sides and span). To accomplish this, I presented Kacie with a pair of chopsticks and sequentially placed bits of pipe cleaner near the chopsticks. From my perspective, some bits were contained in the spanned area and some that were not. In our conversation, we referred to the bits of pipe cleaner as "dots," angularity as "openness," and spanned area as "area." I asked Kacie to consider whether each dot was in the openness and area of the chopsticks, and also to explain her reasoning.<sup>12</sup>

---

12 Footnote: I am grateful to Les Steffe for suggesting the use of this task for examining Kacie's operations for extending the sides of the chopsticks. After the teaching experiment, I discovered Silfverbeg & Joutsenlathti (2014) investigated prospective teachers' conceptions of angles as objects using a similar task.

When I placed the first dot within the spanned area of the chopsticks (from my perspective), Kacie explained that the dot was in both the openness and the area of the chopsticks (see “1” in Figure 4). When I placed a second dot beyond the endpoints of the chopsticks (see “2” in Figure 4), Kacie explained, “I wouldn’t consider it in the area, but I would consider it in the openness.” When I asked Kacie for additional elaboration, she continued, “The way I think of it it’s like these [sides] are still going on [*motions as if extending the sides of the chopsticks away from the vertex*].” At my request, Kacie used a marker to further illustrate her thinking and she drew in the segments shown in Figure 4. Although Kacie did not use, for example, arrows to visually indicate the sides of the angle model extended indefinitely, she clearly verbally indicated this was her intention. I refer to the mental indefinite extension of both sides of an angle model as *the hyper-elongation operation*. In this task, the hyper-elongation operation supported Kacie’s differentiation of at least two attributes: spanned area and angularity.

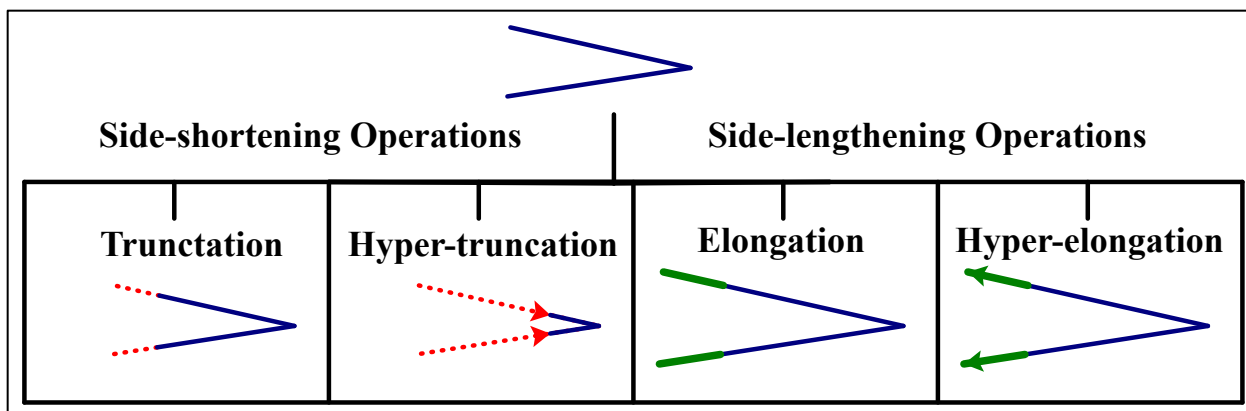


**Figure 4: Kacie hyper-elongates the chopsticks**

Kacie’s activities indicated she had made at least a temporary distinction between two different attributes: the openness and the spanned area. Previously in the teaching experiment, Kacie had used truncation to compare congruent spanned areas; by truncating the long chopsticks, Kacie had rendered the lengths of the angle models’ sides *inconsequential*.

#### **Discussion and Concluding Remarks**

In the preceding sections, I presented four mental operations the students enacted to alter lengths in angular contexts while preserving angularity: truncation, hyper-truncation, elongation, and hyper-elongation (Figure 5). In truncation and elongation, the sides of an angle model are mentally shortened or extended, respectively, to an arbitrary particular length (e.g., the length of another angle model). For hyper-truncation and hyper-elongation, the sides of angle model are mentally shortened or extended indefinitely; thus, hyper-truncation and hyper-elongation render an angle model’s sides either infinitesimal segments or infinite rays, respectively.



**Figure 5: Four length-altering, angularity-preserving mental operations**

Because comparing extents of angularity indicated by two perceptually available angle models necessarily involves considering finite, but not necessarily congruent lengths, I conjecture that these length-altering, angularity-preserving operations play a critical role in initial quantifications of angularity (i.e., those quantifications that involve gross or extensive quantitative operations; see author, year). As demonstrated in the findings above, these operations supported Kacie and Camille as they compared extents of angularity and worked to differentiate angularity from other attributes. Because I observed side-shortening operations to emerge spontaneously from students' goal-directed comparative activities in the teaching experiment, I conjecture side-shortening operations may spontaneously develop more naturally than side-lengthening operations, which required more targeted and intentional interventions for students to indicate during the teaching experiment. Furthermore, side-shortening operations require mentally discarding readily available perceptual material whereas the side-lengthening counterparts requires the mental insertion of material that is not perceptually available. In addition to systematically investigating both of these conjectures, future studies should examine the development of the four mental operations described in this report and others like them, particularly with students who have yet to receive classroom instruction on angle measure, as well as the affordances and limitations each operation occasion in students' quantifications of angularity.

### References

- Alyami, H. (2022). Preservice mathematics teachers' conceptions of radian angle measure. Unpublished doctoral dissertation, Purdue University.
- Barabash, M. (2017). Angle concept: A high school and tertiary longitudinal perspective to minimize obstacles. *Teaching Mathematics and Its Applications*, 36, 31–55.
- Baya'a, N., Daher, W., & Mahagna S. (2017). The effect of collaborative computerized learning using Geogebra on the development of concept images of the angle among seventh graders. In G. Aldon & J. Trgalová (Eds.), *Proceedings of the 13th International Conference on Technology in Mathematics Teaching* (208–215).
- Clements, D. H., Battista, M. T., Sarama, S., & Swaminathan, S. (1996). Development of turn and turn measurement concepts in a computer-based instructional unit. *Educational Studies in Mathematics*, 30(4), 313–337.
- Crompton, H. C. (2017). Using mobile learning to support students' understanding in geometry: A design-based research study. *Educational Technology & Society*, 20(3), 207–219.
- Devichi, C., & Munier, V. (2013). About the concept of angle in elementary school: Misconceptions and teaching sequences. *Journal of Mathematical Behavior*, 32, 1–19.
- Germia, E. (2022). Investigating elementary school students' reasoning about dynamic angles. Unpublished doctoral dissertation, Montclair State University.

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

- Hardison, H. L. (2018). Investigating high school students' understandings of angle measure [Doctoral dissertation, University of Georgia]. UGA Theses & Dissertations. [http://getd.libs.uga.edu/pdfs/hardison\\_hamilton\\_1\\_201805\\_phd.pdf](http://getd.libs.uga.edu/pdfs/hardison_hamilton_1_201805_phd.pdf)
- Hardison, H. L. (in press). Researching quantifications of angularity using genetic epistemology constructs. In P. C. Dawkins, A. J. Hackenberg, & A. Norton (Eds.), *Piaget's genetic epistemology in and for ongoing mathematics education research*.
- Keiser, J. M. (2000). The role of definition. *Mathematics Teaching in the Middle School*, 5(8), 506–511.
- Keiser, J. M. (2004). Struggles with developing the concept of angle: comparing sixth-grade students' discourse to the history of the angle concept. *Mathematical Thinking and Learning*, 6(3), 285–306.
- Lehrer, R., Jenkins, M., & Osana, H. (1998). Longitudinal study of children's reasoning about space and geometry. In R. Lehrer & D. Chazan (Eds.), *Designing learning environments for developing understanding of geometry and space* (pp. 137–167). Mahwah, NJ: Erlbaum.
- Moore, K. C. (2013). Making sense by measuring arcs: a teaching experiment in angle measure. *Educational Studies in Mathematics*, 83(2), 225–245.
- Mullins, S. B. (2020). Examining the relationship between students' measurement schemes for fractions and their quantifications of angularity. Unpublished doctoral dissertation, Virginia Tech.
- Silfverberg, H., & Joutsenlahti, J. (2014). Prospective teachers' conceptions about a plane angle and the context dependency of the conceptions. In C. Nichol, S. Oesterle, P. Liljedahl, & D. Allan (Eds.), *Proceedings of the Joint Meeting of PME-38 and PME-NA 36* (Vol. 5, pp. 185–192). Vancouver, Canada: PME.
- Smith, J. P., III., & Barrett, J. E. (2017). Learning and teaching measurement: Coordinating quantity and number. In J. Cai (Ed.), *Compendium for Research in Mathematics Education* (pp. 355–385). Reston, VA: NCTM.
- Steffe, L. P. & Olive, J. (2010). *Children's fractional knowledge*. New York: Springer.
- Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. Lesh & A. E. Kelly (Eds.), *Research design in mathematics and science education* (pp. 267–307), Hillsdale, NJ: Erlbaum.
- Steffe, L. P. & Ulrich, C. (2013). Constructivist teaching experiment. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 102–109). Berlin: Springer.
- Thompson, P. W. (1994). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 181–234). Albany, NY: SUNY Press.
- Thompson, P. W. (2008). Conceptual analysis of mathematical ideas: Some spadework at the foundations of mathematics education. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano & A. Sépulveda (Eds.), *Proceedings of the Annual Meeting of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 45–64). Morélia, Mexico: PME.
- Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In S. Chamberlin, L. L. Hatfield & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education: Papers from a planning conference for WISDOM<sup>2</sup>* (pp. 35–57). Laramie, WY: University of Wyoming.
- Thompson, P. W. (2013). In the absence of meaning. In K. Leatham (Ed.), *Vital directions for research in mathematics education* (pp. 57–93). New York, NY: Springer.
- von Glasersfeld, E. (1995). *Radical constructivism: A way of knowing and learning*. Washington, D. C.: Falmer.

## AGREEING ON OBJECTIVES OF GEOMETRY FOR TEACHERS' COURSES:

### FEEDBACK FROM INSTRUCTORS ON AN INITIAL LIST<sup>13</sup>

Michael Ion  
University of Michigan  
mikeion@umich.edu

Patricio Herbst  
University of Michigan  
pgherbst@umich.edu

Inah Ko  
University of Michigan  
inahko@umich.edu

Carolyn Hetrick  
University of Michigan  
hetrickc@umich.edu

*We report on an effort to vet a list of 10 student learning objectives (SLOs) for geometry courses taken by prospective geometry teachers. Members of a faculty online learning community, including mathematicians and mathematics educators who teach college geometry courses taken by prospective secondary teachers developed this list in an effort to reach a consensus that might satisfy various stakeholders. To provide feedback on the final list of 10 SLOs, we constructed and collected responses to a survey in which 121 college geometry instructors ranked a set of potential SLOs, including the 10 proposed SLOs as well as 11 distractors. The 10 SLOs were, for the most part, among the highest ranked by the sample.*

Keywords: Geometry and Spatial Reasoning, Research Methods

Secondary mathematics teachers' mathematical preparation is a critical issue influencing the quality of mathematics instruction. Mathematicians are essential stakeholders within higher education because these courses are often offered within mathematics departments; nevertheless, they are not the only stakeholders. Additional stakeholders include the teacher education programs that require students to take these courses and mathematics education scholars, who often operate at the nexus of disciplinary knowledge (i.e., mathematics) and teacher preparation. Numerous practical considerations guide the ongoing joint investment of mathematics departments and teacher education programs in such courses. Mathematics departments frequently face difficulties in being able to provide courses required for teaching accreditation. Related difficulties—particularly vis-à-vis college geometry courses (see Grover & Connor, 2000)—include finding faculty members who are willing to teach those courses and shaping the curriculum of those courses to attend to both the sensibility and knowledge generated by the discipline of mathematics and the need to develop capacity to handle the mathematical content of high schools. Still, mathematics departments are inherently invested in the articulation of secondary mathematics with undergraduate mathematics curricula, and so they continue to actively navigate dilemmas related to the varied constituencies of mathematics courses taken by prospective teachers. In prior work, we identified in our interviews with university geometry instructors describing their positions as having to manage five different tensions (Herbst et al., 2023). One of them, the *content* tension is subtended by two distinct perspectives that could guide the design of mathematics coursework taken by preservice mathematics teachers. The first emphasizes that those preparing to be high school mathematics teachers should study the same

---

<sup>13</sup> This research has been supported by NSF grant DUE- 1725837 to P. Herbst and A. Brown. All opinions are those of the authors and do not necessarily represent the views of the Foundation.

mathematics curriculum as mathematics majors, which can provide a comprehensive understanding of the field and its development then can be applied to the school curriculum (Matthews & Seaman, 2007). The other perspective emphasizes that high school mathematics teachers should focus on the mathematical knowledge needed for teaching, which includes understanding the high school mathematics curriculum (Ebby, 2000). The tension between determining course content purely based on disciplinary considerations and doing so attending to the diverse sources of the school curriculum is an ongoing challenge that mathematics departments must navigate in their efforts to provide courses necessary for teaching accreditation (Herbst et al., 2023; Brown et al., in press).

We consider the Geometry for Teachers (GeT) course to be a useful case for examining challenges to and opportunities for assessing and developing consensus across the multiple stakeholders involved with undergraduate mathematics curriculum. GeT courses are often required for teaching certification and are typically taught within mathematics departments at universities. Instructors are sometimes prepared as mathematicians and other times as mathematics educators. However, GeT courses typically have fewer students than other service courses in mathematics departments, such as calculus or linear algebra, making it difficult for mathematics departments to create local communities to support the course. This often means that decisions about the content of GeT courses are left up to the individual faculty members teaching them. While diversity in GeT curriculum is not in and of itself a problem, it is directly implicated in a dual set of related challenges: (a) secondary mathematics teachers have access to widely variable geometry coursework but are expected to teach a relatively cogent high school geometry curriculum and (b) assessing college geometry students' learning outcomes in some systematic way is complicated by the wide variability of curricula. So, in spite of some reasonable variations in curriculum and instruction, GeT courses present a compelling case for developing and sustaining some convergence around essential student learning outcomes.

To address the issue of increasing capacity for high school geometry instruction in the absence of resident GeT communities in mathematics departments, an inter-institutional online professional learning community (OPLC) of GeT instructors (GeT: A Pencil; see [getapencil.org](http://getapencil.org)) was formed and has been working together for the last past 5 years (An et al., 2023). The authors of this paper have been involved in efforts to convene and support the community in several ways, including facilitating interactions among participants and collecting data that informs their efforts. One issue that surfaced early in discussions within the OPLC was a lack of a clear shared understanding of what should be in the GeT course. A subgroup of 11 members of the community (SLO-WG, hereafter), including mathematicians and mathematics educators, have come to a consensus on a set of 10 Student Learning Objectives that they consider essential for any prospective secondary mathematics teacher to attain. For this list of objectives to have impact on the large systemic challenges named above, it was important to share it with the larger community of GeT instructors (including those outside of GeT: A Pencil) and obtain their feedback. As the group contemplated how they might share their work with and gauge reactions from this broader community, we considered how we might support their efforts to measure and systematically analyze notions of convergence. Thus, we designed and distributed a survey to a larger group of GeT instructors to provide feedback on the work of the SLO-WG: Would the community of GeT instructors at large endorse these 10 SLOs as those most important to aim for in a GeT course?

In this paper, we begin by providing a brief background of the SLO-WG and their approach for creating the list of SLOs for the GeT course. Next, we provide a perspective on the practical

challenges involved in the work of soliciting feedback on a consensus on a set of student learning objectives for a course, from the view of survey methodology design. We illustrate the use of Balanced Incomplete Block Design to elicit understanding at scale in a quick multiple-choice format. We conclude the paper with a discussion on the implications of this work with the teaching of GeT courses and how this framework could be used to examine sets of learning goals produced by other transdisciplinary communities interested in coming to a consensus on essential learning outcomes.

### Background on SLO-WG

Members of the SLO-WG decided to use a winnowing strategy to create a list of essential SLOs, with each group member contributing a set of student learning objectives they thought were essential to one master list (An et al., in press). Discussions and reflections on learning objectives drafted as part of earlier work guided their work. In developing learning objectives, instructors drew on or were informed by the following sources: (1) instructors' previous course syllabi and materials, (2) secondary geometry standards documents (e.g., CCSS-M; NGA; 2010; NCTM, 2000), (3) college geometry curricular guidelines and recommendations (e.g., CBMS, 2012; Venema et al., 2015), and (4) descriptive research on undergraduate geometry courses (e.g., Grover & Connor, 2000). Following the initial drafting, the SLO-WG culled the master list based on common themes. These themes became the focal point of subsequent meetings in which the group worked toward the development of common statements that all participants agreed were essential learning objectives. Ongoing discussions have been influenced by additional reflection and discussion on the interpretations of standards and guidance documentation.

**Table 1: Brief Statements of the 10 SLOs for the GeT Course Produced by SLO-WG**

SLO	Description	SLO	Description
1	Derive and explain geometric arguments and proofs.	2	Evaluate geometric arguments and approaches to solving problems.
3	Understand the ideas underlying current secondary geometry content standards.	4	Understand the relationships between axioms, theorems, and different geometric models in which they hold.
5	Understand the role of definitions in mathematical discourse.	6	Effectively use technologies to explore geometry and geometric relationships.
7	Demonstrate knowledge of Euclidean geometry, including its history.	8	Be able to carry out and justify basic Euclidean constructions.
9	Compare Euclidean geometry to other geometries such as hyperbolic or spherical.	10	Use transformations to explore definitions and theorems about congruence, similarity, and symmetry.

In Spring 2020, SLO-WG begun writing elaborations of the SLOs in a community newsletter and these have been collected in [getapencil.org](http://getapencil.org). These elaborations are seen as part of a living document and feedback is expected from all stakeholders that might result in continuous improvement of the list of SLOs as well as their elaborations. A companion edited book is also in development (Brown et al., in preparation) that will help extend the community of those interested in shaping an emerging consensus. To inform this emerging consensus, the SLO-WG posed the question: “What does the larger community of instructors who teach the geometry

course for secondary teachers think about the list of SLOs we have developed?” While the SLO-WG collected feedback on the SLOs in many modalities (e.g., dissemination in writing, delivering seminar talks, and hosting [getapencil.org](http://getapencil.org)), we took upon ourselves to construct and disseminate a survey to elicit feedback from a larger group of instructors of GeT courses.

When we began the challenge of gathering feedback from the greater community of educators on the proposed SLOs, we realized that developing a method for framing the task in a way that would elicit responses for a valid inference was critical. In the rest of this section, we describe some practical problems that constrain our ability to solicit and analyze open-ended feedback from individuals and undertake a more synthetic study of consensus and convergence.

### Practical Challenges

When we took on the task of asking the community of instructors at large for their views on the proposed SLOs, we realized that an important element was figuring out a strategy to pose the question that would arguably elicit answers for a valid inference. Some practical problems that we could anticipate were the following:

1. If we asked people whether they would endorse the 10 SLOs as the student learning objectives for all GeT courses, we could expect a wide range of negative answers—from those who would disown all 10 to those that would disown 1 of the 10. This strategy seemed biased against our goals.
2. If we asked people to list the 10 learning objectives they would support, then coded the responses and compared those with our list of 10, we would not be asking an endorsement question but seeking to reproduce the accomplishment of our working group but without the benefit of their discussion. This strategy seemed one that would not do what we wanted to do, regardless of possible outcomes (e.g., some objectives might find confirmation in their popularity).
3. A combination of the first and second strategy, whereby we'd give all 10 objectives and asked for participants to add what they saw missing seemed promising but also seemed that it could contain the same problems as 1 and 2 and perhaps others depending on what was asked of people.
  - a. If people were asked to choose 10 after listing what they wanted, we might be prompting bias toward their individual choices. While the SLOs that remained highly chosen (because kept in most lists) might be endorsed by the collective, the aggregate of written-in objectives might not be so easy to make sense of just yet.
  - b. If people were asked to cross from the 10 those they did not want and add however many they wanted, providing a list of possibly more than 10 choices, for us to later choose the 10 with the most choices, this might likely bias the task toward the initial 10.
4. We considered giving people the list of 10 and an equal number of distractors whose statements we drafted to read similar to statements of the 10 and considered having people rank order all SLOs and distractors to then tally the ranks and choose the 10 highest ranked. This strategy seemed unbiased but cognitively very demanding as the number of pairwise comparisons that would be entailed by any one accomplished ranking would be  $20 \times 19/2 = 190$ . That is, it would be hard to guarantee that after reading 20 statements and putting them in order of preference, the respondent would be able to defend the entailed preference order of any two potential objectives, considering how many pairs would need to be sorted. However, the idea of having a ranking task that did not make so high a cognitive demand on the participants seemed compelling.

These considerations took us to look for a survey design and statistical procedure that we could use to satisfy the following requirements:



1. To have a set of potential SLOs roughly twice as large as the 10 drafted by the WG, that is one in which it would be just as likely for a sampled set of 10 objectives to include an SLO or a distractor. Call the size of this set  $N$ .
2. To figure out how many objectives  $N$  should have, and how many sets of elements  $n < N$  (call this number of sets  $k$ ) we should have so that
  - a. Any one participant would be given one set of  $n$  elements to rank, sampled from the  $N$
  - b. The number of comparisons entailed by each individual ranking would be small enough not to cause high cognitive demands on raters
  - c. Any one pair of elements from the  $N$  would have equal chance of appear within all the  $k$  sets
3. If such a model existed, we would need to calculate how many people to ask to rank one set of  $n$  elements so that each of the sets had an equal chance to be used in the survey
4. We needed to figure out how to tally all the completed ranks so that an aggregate rank could be determined based on the aggregate of the ranks.
5. Further, if we wanted to figure out which 10 objectives out of the  $N$  ranked would be endorsed by the community, we would need to identify a statistical model and a sample size that could be used to evaluate the ranked list of size  $N$  to detect significant differences (e.g., between the objectives 10<sup>th</sup> and 11<sup>th</sup> ranked SLOs in the aggregate).

**Table 2: Distractor Statements of the 11 dSLOs Included in the Questionnaire**

<b>dSLO</b>	<b>Description</b>
11	Use geometric properties to find the measures of angles or sides
12	Distinguish between necessary and sufficient conditions in a mathematical claim
13	Understand the ideas underlying advanced Euclidean and absolute geometry
14	Make connections between geometry and other mathematical subjects such as linear algebra, mathematical modeling, and group theory.
15	Understand the importance and role of diagrams in geometric communication.
16	Effectively use digital proof tools to practice writing geometric proofs with feedback.
17	Understand the role played by practices like building structures, navigation, stargazing, and art in the historical development of geometry.
18	Be able to demonstrate that the three classical problems of geometry are not solvable with straightedge and compass
19	Prove theorems about circles and tangent lines.
20	Investigate advanced properties of projective geometry
21	Have experience with the mathematical modeling cycle in geometry problems.

### **Methods**

A Balanced Incomplete Block Design (BIBD) is a survey research strategy that ensures that all survey items are asked the same number of times while minimizing response biases (Alvo & Cabilio, 1991). Participants in a BIBD are randomly assigned to groups or blocks, and each group is shown a subset of the survey items. Each survey question is divided into a defined number of blocks, guaranteeing that all items are asked the same number of times while also allowing for fewer things to be included in each survey. This reduces respondent burden and survey fatigue while simultaneously guaranteeing that key survey items are not overlooked. To employ a BIBD in a questionnaire, we must determine the survey items that will be included and the number of blocks that will be used. A balanced incomplete block design is one in which all

pairs of treatments occur together in the same block at the same frequency; this number is indicated by  $\lambda$ . A total of  $nb$  judges rank  $t$  items  $k$  at a time based on  $n$  replications of a BIBD with  $b$  blocks. The BIBD must meet the following requirements: 1)  $bk = tr$  and 2)  $\lambda = r \times \frac{k-1}{t-1}$ . In our case, since we have 10 SLOs, in order to satisfy the conditions of this BIBD, we needed to come up with 11 ‘distractor’ SLOs (dSLOs, hereinafter) for a total of  $t = 21$  objects to rank. The remainder of the conditions are as follows: 1)  $b = 30, k = 7, t = 21, r = 10 \Rightarrow (30)(7) = (21)(10)$ ; 2)  $\lambda = 3; r = 10, k = 7, t = 21 \Rightarrow 3 = 10 \times \frac{(7-1)}{(21-1)} \Rightarrow 3 = 10 \times \frac{3}{10}$ .

**Table 3: Balanced Block Design for our Distribution of the Questionnaire**

block	1	2	3	4	5	6	7
1	2	5	10	11	17	19	20
2	3	6	11	12	18	20	21
3	4	7	12	13	19	21	15
4	5	1	13	14	20	15	16
5	6	2	14	8	21	16	17
6	7	3	8	9	15	17	18
7	1	4	9	10	16	18	19
8	3	4	8	13	17	19	20
9	4	5	9	14	18	20	21
10	5	6	10	8	19	21	15
11	6	7	11	9	20	15	16
12	7	1	12	10	21	16	17
13	1	2	13	11	15	17	18
14	2	3	14	12	16	18	19
15	1	6	9	12	17	19	20
16	2	7	10	13	18	20	21
17	3	1	11	14	19	21	15
18	4	2	12	8	20	15	16
19	5	3	13	9	21	16	17
20	6	4	14	10	15	17	18
21	7	5	8	11	16	18	19
22	1	2	4	8	9	11	21
23	2	3	5	9	10	12	15
24	3	4	6	10	11	13	16
25	4	5	7	11	12	14	17
26	5	6	1	12	13	8	18
27	6	7	2	13	14	9	19
28	7	1	3	14	8	10	20
29	1	2	3	4	5	6	7
30	8	9	10	11	12	13	14

In our questionnaire, we have  $t = 21$  treatments (i.e., number of all SLO and dSLO statements),  $k = 7$  statements per block (i.e., number of statements each participant gets to rank),  $r = 10$  times each SLO statement appears across the block design, and  $b=30$  blocks. Additionally,

$k = 3$ , how many times an item pair goes together in the block. The block design is in Table 3 below. Each block represents a set of statements that participants in the block had to rate against each other, answering the question: "Please rank the following statements of student learning outcomes of a geometry course in order of preference, where 1 is the highest (top) and 7 is the lowest (bottom) by dragging and dropping them into place." In the next section, we describe the findings from our study with 121 GeT instructors of the course nationwide, collected in 2022.

**Table 4: Descriptive Statistics of the Findings**

(d)SLO	Average	Rank	# ranked 1/7	# ranked 7/7	Std Dev.
1	2.1	1	19	0	1.29
2	2.5	3	21	3	1.93
3	3.1	6	7	1	1.66
4	2.4	2	17	1	1.69
5	2.7	4	10	0	1.40
6	3.5	9	5	2	1.67
7	2.8	5	12	1	1.72
8	4.3	13	1	2	1.64
9	3.5	8	4	1	1.52
10	3.6	10	6	1	1.69
11	4.6	15	0	7	1.65
12	4.3	14	1	3	1.63
13	3.3	7	7	6	2.03
14	5.0	17	1	5	1.41
15	3.7	11	3	2	1.80
16	6.3	21	0	23	1.05
17	5.3	18	1	12	1.58
18	6.2	20	0	22	0.92
19	4.2	12	3	2	1.58
20	6.1	19	0	18	1.07
21	4.7	16	3	9	1.95

### Findings

Overall, the group seemed to find consensus with the SLO-WG when comparing the brief statements of their 10 SLOs with the statements of the comparable distractors provided. In Table 4, some descriptive statistics about the average score given to each SLO (where lower score means higher priority to be in a course), where the SLO ranks with respect to the others, how often the SLO was rated the most or least important of the 7 provided to the participant, as well as a measure of variance for the item to show how varied the instructors were with respect to how they rated that SLO in comparison to others and amongst themselves.

Of the ten SLOs proposed, the 121 participants who took the survey on average ranked nine of the ten amongst the highest ranked statements. Additionally, all the SLOs that were ranked the most important of the seven 10 or more times came from the original list, and all the SLOs that

were ranked the least important of the seven 10 or more times came from the list of distractors. This helps demonstrate some key aspects of the construct and content validity of these SLOs.

### **Discussion and Limitations**

In this report, we provided a methodological framework for studying student learning outcomes at scale created by online learning communities. In designs where information needs to be gleaned from large amounts of researchers and practitioners without a tremendous number of resources or time, questionnaires such as these can provide a lever to gain a pulse on the field, and send a signal that this work of developing student learning objectives is being worked on. We continue to bolster the tremendous efforts of the members of this OPLC over the past few years in developing a set of student learning outcomes that can impact the teaching and learning of high school geometry. We hope this can be one of several ways to triangulate on a set of established student learning outcomes for a geometry course for teachers.

### **References**

- Alvo, M., & Cabilio, P. (1991). On the balanced incomplete block design for rankings. *The Annals of Statistics*, 19(3), 1597-1613.
- An, T., Berzina-Pitcher, I., Bigelow, V., Buchbinder, O., Herbst, P., Hetrick, C., Milewski, A., Miller, N., Prasad, P., Pyzdrowski, L., St Goar, J., Sears, R., Szydluk, S., & Vestal, S. (2023). A cross-institutional faculty online learning community: Community-guided faculty development in teaching college geometry for teachers. In S. Linder, C. Lee, S. Stefl, and K. High (Eds.), *Handbook of STEM Faculty Development* (pp. 325-336). IAP.
- An, T., Boyce, S., Brown, A., Buchbinder, O., Cohen, S., Dumitrascu, D., Escudero, H., Herbst, P., Ion, M., Krupa, E., Miller, N., Pyzdrowski, L., Sears, R., St. Goar, J., Szydluk, S., & Vestal, S. (in press). (Toward) Essential student learning objectives for teaching geometry to pre-service secondary teachers. *AMTE*.
- Brown, A., Herbst, P., Miller, N., & Pyzdrowski, L. (Eds., in preparation). *The GeT course: Resources and objectives for the geometry courses for teachers*.
- Brown, A., Herbst, P., and Ion, M. (in press). How instructors of undergraduate mathematics courses manage tensions related to teaching courses for teachers? In T. Lamberg et al. (Eds), *Proceedings of the 2023 PME-NA conference*. Reno NV.
- Conference Board of the Mathematical Sciences [CBMS] (Ed.). (2012). *The mathematical education of teachers II* (Vol. 17, Issues in Mathematics Education). American Mathematical Society.
- Ebby, C. B. (2000). Learning to teach mathematics differently: The interaction between coursework and fieldwork for preservice teachers. *Journal of Mathematics Teacher Education*, 3(1),
- Grover, B. W., & Connor, J. (2000). Characteristics of the college geometry course for preservice secondary teachers. *Journal of Mathematics Teacher Education*, 3, 47-67.
- Herbst, P., Brown, A. M., Ion, M., & Margolis, C. (in press, online first). Teaching geometry for secondary teachers: What are the tensions instructors need to manage? *International Journal of Research in Undergraduate Mathematics Education*, <https://doi.org/10.1007/s40753-023-00216-0>.
- Matthews, M., & Seaman, W. (2007). The effects of different undergraduate mathematics courses on the content knowledge and attitude towards mathematics of preservice elementary teachers. *Issues in the Undergraduate Mathematics Preparation of School Teachers*, 1.
- NCTM (2000). *Principles and standards for school mathematics*. Author.
- National Governors Association Center for Best Practices [NGA], & Council of Chief State School Officers [CCSSO]. (2010). *Common Core State Standards for Mathematics*. <http://www.corestandards.org/Math/>
- Venema, G., Barker, W., Farris, F., & Greenwald, S. (2015). *Geometry*. In MAA's Committee on the Undergraduate Program in Mathematics. Available at <https://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/cupm/2015-cupm-curriculum-guide>

# TEACHERS' REACTIONS TO ROUTINE AND ALTERNATIVE PRACTICES FOR PRESENTING PROOFS: A SURVEY EXPERIMENT

Bukola Ake  
University of Maine,Orono  
bukola.ake@maine.edu

Justin Dimmel  
University of Maine,Orono  
justin.dimmel@maine.edu

Patricio Herbst  
University of Michigan  
pgherbst@umich.edu

*We conducted a multimedia survey experiment with a nationally representative sample ( $n = 405$ ) of secondary geometry teachers. Participants were shown storyboard depictions of instructional episodes and asked to rate the appropriateness of the (hypothetical) teacher's actions using a Likert-like response format. We analyzed participants responses using ANOVA. The purpose of the experiment was to investigate how secondary geometry teachers expect students to communicate when presenting proofs during class. Our results (1) replicated findings from a prior investigation of what teachers expect when students present proofs and (2) investigated how geometry teachers reacted to instructional practices that attempted to steer student presentations of proofs toward disciplinary communication practices.*

Keywords: Mathematical Communication, Proof Transcriptions, Proof Presentation

## Introduction

According to the National Council of Teachers of Mathematics: "Communication is a fundamental aspect of mathematics and mathematics education" (NCTM, 2000, p.60). The communication standard of NCTM emphasizes how crucial it is for students to express their mathematical reasoning to their peers and teachers using mathematical language. Mathematical communication enhances students' cognitive functioning by helping them articulate their ideas and strategy (Kosko & Wilkins, 2010; Lee, 2006).

One context where communication is central in mathematics classrooms is when students share their proofs with an entire class, an activity we refer to as *presenting proofs* (Dimmel & Herbst, 2020). Proof plays an important role in creating, developing, and communicating mathematical knowledge and is considered an important aspect of students' mathematical learning experiences (Ball & Bass, 2003; Herbst & Brach, 2006; Stylianides, 2019). How do secondary geometry teachers expect students to communicate when presenting proofs? This study investigated this question through a multimedia survey experiment. It is both a replication and an iteration of a prior study (Dimmel & Herbst, 2020) that investigated what teachers expect from students when they present proofs in geometry.

## Background & Research Questions

Various modes of representation can be used when presenting proofs, including speaking, writing, diagramming, and gesturing (Stylianides, 2019). Several studies (e.g., Lai & Weber, 2014; Weber, 2004; Artemeva & Fox, 2011) have demonstrated how mathematicians employ various combinations of these modalities when presenting proofs in front of the class. For example, Weber (2004) examined the case of a professor presenting proofs in an introductory real analysis course. He found that the professor verbalized the proof's steps as they were written, offered commentary on the proof's development, and gave an overview of the proof before generating it.

Compared to the literature on students' written proofs, relatively little is known about how students present proofs (Kokushkin et al., 2022; Stylianides, 2019). Stylianides (2019) compared

the oral arguments that secondary students made in front of the class with the written arguments they created during small group work for the same assertions. While this study explored how students communicate their proofs when required by an expert to do so, it neither provided any insight into the broader communication practices typical in mathematics classrooms, nor did it investigate teachers' expectations during such presentations.

To describe these broader communication practices, Dimmel & Herbst (2020) conducted a two-part, mixed methods study. First, they analyzed video records of geometry classrooms when students were asked to present proofs. They found that, in contrast to the multimodal practices of mathematicians, students engaged in mark-for-mark reproductions of previously completed proofs—an action described by the authors as *proof transcription*. A key aspect of this practice is that proofs were not expected to be evaluated as mathematical arguments until the transcription was completed. Second, they conducted a multimedia survey experiment in which hypothetical episodes of geometry instruction, represented by cartoon storyboards, were shown to a sample ( $n = 60$ ) of midwestern secondary mathematics teachers. The storyboards showed students transcribing proofs and depicted different actions a teacher might take in response. Their study showed that the preferred action was no action – i.e., secondary mathematics teachers recognized proof transcription as an acceptable means for students to present proofs (Dimmel & Herbst, 2020). That teachers recognized proof transcription as routine is significant, because it suggests that there are limited opportunities for students to develop fluency with the multimodal communication practices for presenting proofs.

Our study builds on this prior work in two ways. One, we replicated the statistical tests described by Dimmel & Herbst (2018, 2020) with a larger, nationally representative sample ( $n = 405$ ) of secondary geometry teachers. The replication study investigated whether this representative sample of geometry teachers recognized proof transcription as a routine instructional practice during proof presentation. Two, this study used two additional storyboards created to depict instructional actions that attempted to steer students toward developing multimodal communication skills during their proof presentations in geometry classrooms. While the findings from the study conducted by Dimmel & Herbst (2020) offer some insight into secondary mathematics teachers' expectations regarding how their students' present proofs, the design of the storyboards leaves open the possibility that teachers were merely responding or attending to teaching actions that represented a departure from other classroom instructional norms—e.g., an expectation of how students should be treated when they are in socially vulnerable situations such as when sharing their written work at the board, rather than attending to the communication practice depicted in the instructional scenario. Therefore, this present study provided a different way of scaffolding expert communication practices by designing storyboards targeting alternative communication practices designed to steer students away from simply transcribing their written proofs toward a multimodal presentation of their proofs. These two storyboards were designed to investigate how teachers would react to instructional episodes that require that students' modes of communication approximate the disciplinary practices of mathematical communication rather than an unreflective transcription of proofs (e.g., giving a verbal overview of the plan for the proof before writing it on the board). We thus operationalized our overarching question about how students learn discipline-specific mathematical communication practices into two research questions:

How do secondary geometry teachers react to instructional scenarios that either foster or condone the hypothesized norm of proof transcription when students present proofs at the board?

How do secondary geometry teachers react to instructional scenarios that create opportunities for students to engage in approximations to disciplinary communication practices?

### **Theoretical Framework: Instructional Situations and Normative Actions**

We conceptualize teaching as a set of socially situated practices that occur in *instructional situations* (Herbst, 2006)—i.e., “identifiable segments of instruction that are organized around specific kinds of mathematical work, such as doing proofs in geometry or solving equations in algebra” (Dimmel & Herbst, 2020, p.7). Instructional situations are identifiable by their normative actions. Norms are familiar ways of observing, believing, evaluating, and behaving in an environment (Goodnough, 1971). They are defined in terms of the features of a social situation that not only regularly occur but that participants expect to occur (Garfinkel, 1963; Herbst & Chazan, 2003). The sociological view that people have unspoken expectations of how things should unfold has its roots in everyday experiences. For example, there are unspoken norms regarding how people stand in line when boarding a plane even though there are no explicitly written rules about the order in which people are to stand.

Instructional situations provide a way of framing the routine instructional activities of teachers and students in those situations. Thus, normative instructional activities, though invisible, can be investigated by observing how teachers react to scenarios that represent departures from those norms (Herbst & Chazan, 2015). The present study used the breaching experiment approach (Garfinkel, 1963) to investigate the normative ways secondary geometry teachers expect students to present proofs at the board. Specifically, we analyzed teachers’ reactions to scenarios that represented departures from the norm of proof transcription.

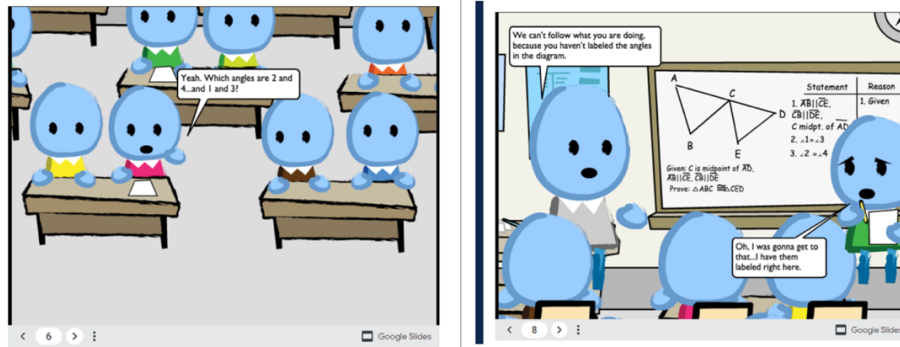
### **Methods: Multimedia Survey Experiment with Planned Comparisons**

We used a multimedia survey experiment with planned comparison between participants randomly assigned to different storyboard conditions. These different conditions were designed to probe teachers’ responses to instructional actions that either interfered (treatment) or did not interfere (control) with student presentations of proofs. We used storyboard representations, rather than video segments of real-world classrooms, so that we could create hypothetical classroom vignettes that realized the specific instructional actions we wanted to investigate (Dimmel & Herbst, 2020; Dimmel & Herbst, 2018; Herbst et al, 2011). The study’s goal was to determine whether participants’ reactions to the episodes differed depending on the instructional actions that were represented in the different scenarios.

### **Storyboard Design**

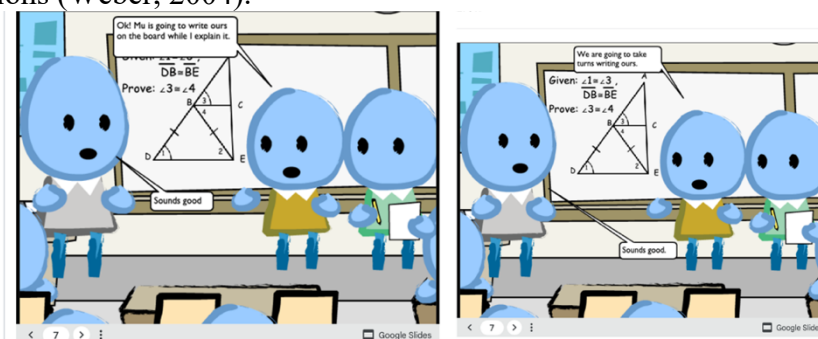
Storyboards consisted of 12 – 20 frames of classroom activity that were represented using simple cartoon graphics. They were designed in pairs: Up until a 3-5 frame segment of storyboard (i.e., the *segment of interest*), the storyboards in a pair were identical. The segments of interest for the storyboards were the segments of the storyboard that depicted specific teaching actions that were investigated during the survey experiment. In addition to the segment of interest, each storyboard pair featured a distracter segment where a teacher performed another distinct teaching activity. These activities were considered routine instructional activities, e.g., a teacher asking students to check their work, and they were the same for both the storyboards in a pair. The first and second storyboards targeted how students employed labels and reasons, respectively, when transcribing proofs. The treatment versions of these two storyboards show a teacher departing from the hypothesized norm by interfering with the student’s transcription in places where it could be described as lacking mathematical coherence from a disciplinary standpoint—e.g., referring to labeled angles in the written statements of a proof before adding

those labels in the diagram (Figure 1). The control versions show a teacher allowing the students to transcribe their proof without interference.



**Fig.1 Frames from the labels storyboard**

In addition to the replication of the original results, two additional storyboard pairs were designed to investigate how secondary geometry teachers would react to proof presentations that approximated disciplinary communication practices. Storyboard A (Figure 2) depicts an instructional scenario where two students presented a proof at the board. In the treatment version, one student writes the proof and the other verbalizes the proof as it is being written, while in the control version the students took turns transcribing the proof (Figure 2). Storyboard B compared two possible interventions that a teacher could make to steer a student's presentation of a proof toward the multimodal practices that are typical in the discipline. In treatment 1 (Fig. 3, right frame), the teacher asks the student to include appropriate congruence markings and to point at the diagram. In treatment 2, (Figure 3, left frame), the teacher asked the student to provide an overview of the proof strategy and use a conceptual register (Herbst, Kosko, & Dimmel, 2013) to describe the steps in the proof, rather than simply read the statements as they were written on the board. Comments on proof strategy and register switching are touchstones of mathematicians' proof presentations (Weber, 2004).



**Fig.2 Frames from Alternative Communication Storyboard A**



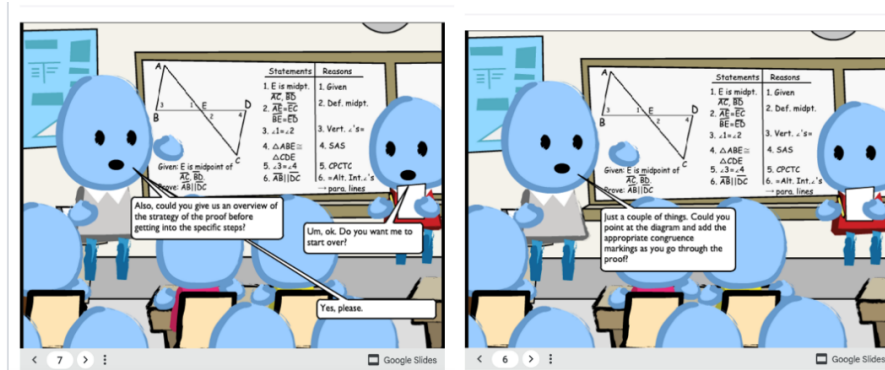


Fig.3 Frames from the Alternative Communication Storyboard B

### Participants and Data Collection

The participants for this study were 405 secondary mathematics teachers from 46 states within the United States. All participants were experienced geometry teachers with an average of 7 years of experience teaching geometry. 84.6% of them identified as White, 6.31% as Black, 2.78% as Asian, 2.02% as Hispanic, and 0.76% as other. 60.1% of them were female and 39.9% were male.

Using the storyboards described above as probes, data was collected using a multimedia survey assigned to designated experiment groups. The survey was administered remotely from 2015 – 2016 and was one of a suite of research instruments that were deployed in a large-scale study of the instructional practices of secondary mathematics teachers (Herbst et al., 2015). The experiment groups were designed such that each participant in the group viewed one and only one storyboard in each storyboard pair. Participants were asked the same set of open- and closed-ended questions after viewing each storyboard. For this study, we focused on participants' responses to three closed-ended questions that asked participants to rate, using a 6-valued Likert-like response format, the appropriateness of the teacher's actions. Participants were asked to rate the appropriateness of the teacher's actions across the entire episode and specifically for the actions depicted in the segment of interest and the distracter segment. The rating choices for the closed-ended rating questions were: 1 (very inappropriate), 2 (inappropriate), 3 (somewhat inappropriate), 4 (somewhat appropriate), 5 (appropriate), and 6 (very appropriate).

### Analysis

We analyzed participant responses to the closed-ended questions by creating planned comparisons within and between experimental groups. For both the replication storyboards, we generated two sets of hypotheses for the planned comparison: Across condition (three hypotheses, different participants) and within condition (two hypotheses, same participants). For the replication storyboards (Labels and Reasons), across conditions we hypothesized (1) that participants would rate the teacher's work more negatively in episodes that showed a departure from proof transcription than in episodes that showed students transcribing their proofs. We also hypothesized (2) that the mean ratings of *episode appropriateness* (EA) and *segment-of-interest appropriateness* (SIA) would be lower in episodes that depicted a departure from proof transcription compared to the episodes that showed transcription of proofs. Similarly, we hypothesized (3) that there would be no difference in the mean ratings of the *distracter segment appropriateness* (DSA) across conditions (since this action was the same across each storyboard pair). Within conditions, we compared the segment-of-interest and distracter ratings for both the

breach and routine conditions. We hypothesized (4) that the mean ratings of SIA would be lower than that of DSA in the treatment condition and (5) that there would be no difference between the ratings in the control condition (since these actions, though different, were both hypothesized to be routine teaching actions with the control conditions). For the alternative communication practice storyboards, we hypothesized that teachers would prefer proof presentations that hewed more closely to the proof transcription norm (the control version of Storyboard A, treatment 2 for Storyboard B). We tested this hypothesis by comparing mean appropriateness ratings across conditions for each of the three appropriateness measures. We conducted a repeated measures ANOVA with planned comparison to test our hypotheses.

## Results

### Replication Results

For the across conditions comparison, participant ratings of the treatment and control storyboards were as predicted for the episode and segment of interest for the two storyboards that replicated the original study. In comparison to the control storyboards, the treatment storyboards had significantly lower means on the episode appropriateness rating questions (Tables 1 and 2, row 1) and the segment of the interest rating questions (Tables 1 and 2, row 2). The distracter segment of the reason storyboard showed a significant difference in mean ratings (Tables 2, row 3). The effect-size statistical analysis showed that while the distracter segments of this storyboard had statistically significant differences in mean ratings, the size of the effect was small, with  $d = .278$ . These results thus support the existing literature that when participants view instructional episodes in which teachers allowed students to transcribe their proofs, the work of the teacher were rated higher than in those episodes in which the teacher interfered with the transcriptions.

**Table 1: Across condition pairwise comparison for Label Storyboard**

Appropriateness Measure	R (n=189)	B (n=212)	$\mu_1 - \mu_2$	Std. Error	Sig.	95% confidence Interval for Difference	
						Lower Bound	Upper Bound
Episode	4.233	3.684	0.549	.125	<.001	.303	.794
Segment of Interest	4.312	2.920	1.392	.143	<.001	1.111	1.674
Distracter Segment	4.741	4.675	0.066	.114	.561	-.158	.290

Based on estimated marginal means  
 \* The mean difference is significant at the .05 level

**Table 2: Across condition pairwise comparison for Reason Storyboard**

Appropriateness Measure	R (n=189)	B (n=212)	$\mu_1 - \mu_2$	Std. Error	Sig.	95% confidence Interval for Difference	
						Lower Bound	Upper Bound
Episode	3.884	3.052	0.832	.125	<.001	.585	1.078
Segment of Interest	3.984	2.717	1.267	.125	<.001	1.022	1.512
Distracter Segment	4.418	4.090	0.328	.115	.005	.102	.555

Based on estimated marginal means  
 \* The mean difference is significant at the .05 level

For the within condition comparison, the findings shown in Table 4 and 5 provide evidence that the mean ratings of the segment of interest (SIA) were lower than the mean ratings of the

distracter segments, for both the Labels and Reasons storyboards. Although this result supports our hypothesis for the treatment groups, it contradicted it for the control groups. The control storyboards depicted routine teaching actions for both the segment of interest and the distracter segments, hence we anticipated that there would be no significant difference between the mean ratings. An effect size analysis for both the Label and Reason storyboards ( $d = -0.340$  and  $-0.380$ , respectively) showed that the differences in the means between SIA and DSA were negligible, even if they were statistically significant.

**Table 3: Within condition pairwise comparison for Label Storyboard**

Condition	SIA	DSA	$\mu_1 - \mu_2$	Std. Error	Sig.	95% confidence Interval for Difference		Cohen's D
						Lower Bound	Upper Bound	
Routine	4.312	4.741	-.429	.105	<.001	-.635	-.223	-.340
Breach	2.920	4.675	-1.755	.099	<.001	-1.949	-1.560	-1.328

Based on estimated marginal means  
 \* The mean difference is significant at the .05 level

**Table 4: Within condition pairwise comparison for Reason Storyboard**

Condition	SIA	DSA	$\mu_1 - \mu_2$	Std. Error	Sig.	95% confidence Interval for Difference		Cohen's D
						Lower Bound	Upper Bound	
Routine	3.984	4.418	-.434	.085	<.001	-.601	-.267	-.380
Breach	2.717	4.090	-1.373	.080	<.001	-1.530	-1.215	-1.106

Based on estimated marginal means  
 \* The mean difference is significant at the .05 level

### Results: Alternative Communication Practices

In storyboard A, in comparison to the control storyboard, the treatment storyboard had significantly higher means on the episode appropriateness ratings (Table 5, row 1) and the segment of interest ratings (Table 5, row 2). These results were different from what we hypothesized. They suggest that secondary geometry teachers reacted positively to the instructional actions where the teacher enlisted the students to work together to present a multimodal (speaking and writing) version of the proof – we will return to this point in the discussion. The distracter segment showed a significant difference in mean ratings (Tables 5, row 3), however the effect-size analysis revealed that while the distracter segments of this storyboard had statistically significant differences in mean ratings, the size of the effect was small, with  $d = -0.25$ .

**Table 5: Across condition pairwise comparison for Communication Storyboard A**

Appropriateness Measure	R (n=189)	B (n=212)	$\mu_1 - \mu_2$	Std. Error	Sig.	95% confidence Interval for Difference	
						Lower Bound	Upper Bound
Episode	4.921	5.321	-.400	.086	<.001	-.570	-.230
Segment of Interest	4.111	4.962	-.851	.116	<.001	-1.079	-.623
Distracter Segment	4.979	5.212	-.233	.091	.011	.054	.413

Based on estimated marginal means  
 \* The mean difference is significant at the .05 level

In storyboard B, treatment 1 had significantly higher means on the episode appropriateness ratings (Table 6, row 1) and the segment of interest ratings (Table 6, row 2) in comparison to treatment 2. The distracter segment showed a significant difference in mean ratings (Tables 6, row 3) however the effect-size analysis revealed that the size of the effect was small, with  $d = .234$ . Both treatments depicted the teacher asking the students to engage in disciplinary communication practices. For treatment 1, these included gesturing while speaking and marking the diagram. For treatment 2, these included switching from a generic to a conceptual register and offering commentary on the overall plan for the proof. These results suggest that gesturing and marking were seen as more appropriate expectations for teachers to have of student presentations.

**Table 6: Across condition pairwise comparison for Communication Storyboard B**

Appropriateness Measure	T1 (n=189)	T2 (n=212)	$\mu_1 - \mu_2$	Std. Error	Sig.	95% confidence Interval for Difference	
						Lower Bound	Upper Bound
Episode	4.587	3.901	0.686	.119	<.001	.453	.920
Segment of Interest	4.275	3.472	0.803	.144	<.001	.521	1.086
Distracter Segment	5.132	4.910	0.222	.092	.016	.041	.403

Based on estimated marginal means  
 \* The mean difference is significant at the .05 level

### Discussion and Conclusion

The replication study offered a robust empirical test of the hypothesis that secondary geometry teachers expect student presentations of proofs to default to proof transcriptions (Dimmel & Herbst, 2020). This finding is significant because it provided additional evidence to support an observed social phenomenon. Given the ongoing replication crises in human subject research, such a result is non-trivial (Shrout & Rodgers, 2018). The iteration of the study that investigated teachers' reactions to alternative communication practices is significant because it provided evidence that teachers reacted positively to instructional actions that challenged students to develop their multimodal communication skills. In fact, across all four storyboard pairs, the highest mean ratings for the episode and segment of interest were linked to the storyboard that depicted two students working together to achieve a multimodal presentation of a proof. Also of note, each of the alternatives that were tested in the comparison for Storyboard B had higher mean ratings than the breaches that were depicted in the original study. The implication for teaching is that rather than concentrating solely on teaching students how to write proofs, teachers could provide opportunities for students to learn how to present proofs using approximations to the multimodal practices that are used by experts.

## Acknowledgement

This work is supported in part by grants from the Spencer Foundation (second author) and the National Science Foundation (Award #2145517, second author). This research includes secondary data analysis of a data corpus collected with the resources of NSF grant DRL- 0918425 to P. Herbst. All opinions are those of the authors and do not necessarily represent the views of the Spencer Foundation or the National Science Foundation.

## References

- Artemeva, N., & Fox, J. (2011). The writing's on the board: The global and the local in teaching undergraduate mathematics through chalk talk. *Written Communication*, 28(4), 345-379.
- Ball, D. L., & Bass, H. (2003). Making mathematics reasonable in school. A research companion to principles and standards for school mathematics, 27-44.
- Dimmel, J. K., & Herbst, P. G. (2020). Proof transcription in high school geometry: a study of what teachers recognize as normative when students present proofs at the board. *Educational Studies in Mathematics*, 105(1). <https://doi.org/10.1007/s10649-020-09975-y>
- Dimmel, J. K., & Herbst, P. G. (2018). What details do teachers expect from student proofs? A study of proof checking in geometry. *Journal for Research in Mathematics Education*, 49(3), 261–291. <https://doi.org/10.5951/jresmetheduc.49.3.0261>
- Garfinkel, H. (1963). A conception of and experiments with "trust" as a condition of concerted table actions. *The production of reality: Essays and readings on social interaction*, 381-392.
- Goodenough, W. (1971). *Culture, language, and society reading*. Massachusetts: Addison-Wesley.
- Herbst, P., & Brach, C. (2006). *Proving and Doing Proofs in High School Geometry Classes: What Is It That Is Going on for Students?* (Vol. 24, Issue 1).
- Herbst, P., & Chazan, D. (2003). Exploring the Practical Rationality of Mathematics Teaching through Conversations about Videotaped Episodes: The Case of Engaging Students in Proving. In *Source: For the Learning of Mathematics* (Vol. 23, Issue 1).
- Herbst, P., Kosko, K. W., & Dimmel, J. K. (2013). How Are Geometric Proof Problems Presented? Conceptualizing and Measuring Teachers' Recognition of the Diagrammatic Register. *North American Chapter of the International Group for the Psychology of Mathematics Education*.
- Herbst, P., & Chazan, D. (2015). Studying professional knowledge use in practice using multimedia scenarios delivered online. *International Journal of Research and Method in Education*, 38(3), 272–287. <https://doi.org/10.1080/1743727X.2015.1025742>
- Herbst, P., Chazan, D., Chen, C. L., Chieu, V. M., & Weiss, M. (2011). Using comics-based representations of teaching, and technology, to bring practice to teacher education courses. *ZDM*, 43, 91-103.
- Herbst, P. G. (2006). Teaching Geometry with Problems: Negotiating Instructional Situations and Mathematical Tasks. In *Source: Journal for Research in Mathematics Education* (Vol. 37, Issue 4).
- Kokushkin, V., Norton, A., Johnson, E., Alibali, M., Doolittle, P., Elgart, A., & Arnold, R. (2022). The Role of Students' Gestures in Offloading Cognitive Demands on Working Memory in Proving Activities.
- Kosko, K. W., & Wilkins, J. L. (2010). Mathematical communication and its relation to the frequency of manipulative use. *International Electronic Journal of Mathematics Education*, 5(2), 79-90.
- Lai, Y., & Weber, K. (2014). Factors mathematicians profess to consider when presenting pedagogical proofs. *Educational Studies in Mathematics*, 85, 93-108.
- Lee, C. (2006). *Language for learning mathematics: assessment for learning in practice: Assessment for learning in practice*. McGraw-Hill Education (UK).
- National Council of Teachers of Mathematics. (2002). *Principles and standards for school mathematics*. Reston, VA: Author.
- Shrout, P. E., & Rodgers, J. L. (2018). Psychology, Science, and Knowledge Construction: Broadening Perspectives from the Replication Crisis. *Annual review of psychology*, 69, 487–510. <https://doi.org/10.1146/annurev-psych-122216-011845>
- Stylianides, A. J. (2019). Secondary students' proof constructions in mathematics: The role of written versus oral mode of argument representation. *Review of Education*, 7(1), 156–182. <https://doi.org/10.1002/rev3.3157>
- Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

Weber, K. (2004). Traditional instruction in advanced mathematics courses: A case study of one professor's lectures and proofs in an introductory real analysis course. *Journal of Mathematical Behavior*, 23(2), 115–133.  
<https://doi.org/10.1016/j.jmathb.2004.03.001>

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

# “ $\frac{\pi}{5}$ ... WASN'T ON THE UNIT CIRCLE ”: A PRESERVICE MATHEMATICS TEACHER'S MEANINGS FOR RADIAN

Hanan Alyami  
Purdue University in Fort Wayne  
alyamih@pfw.edu

*Researchers have reported that preservice mathematics teachers' (PMTs') conceptions for radian angle measure are dominated by conceptions of degrees, memorization, and calculational strategies, and are always expressed in terms on  $\pi$ . In this report, I unpack the mathematical meanings of a PMT, Henry (pseudonym), as he engaged in three task-based interviews involving radian angle measure. In the first interview, Henry described radians procedurally, emphasizing memorized special angles written in terms of  $\pi$ . By the end of the final interview, Henry described radians conceptually, including generalizing special angles. I conclude by highlighting the importance of tasks designed to build and support conceptual understanding.*

Keywords: measurement, preservice mathematics teacher knowledge, proportional reasoning.

While a coherent understanding of radian angle measure is essential in higher mathematics such as trigonometry, precalculus, and calculus (Thompson et al., 2007), researchers have reported that learners struggle to conceptualize radian angle measure and how it relates to trigonometry (Akkoc, 2008; Çekmez, 2020; Fi, 2003; Moore, 2013, 2014; Moore et al., 2016; Tallman & Frank, 2020; Thompson, 2013; Topcu et al., 2006). Radian angle measure can be described by building on the conventional approach to measuring angles. Specifically, angles are measured relative to a benchmark associated with a circle centered at the angle's vertex. The angle's subtended arc is measured relative to a specific, yet arbitrary, fractional amount of the circle's circumference. In degrees, the subtended arc is measured relative to  $1/360^{\text{th}}$  of the circumference, while in radians, the subtended arc is measured relative to  $1/(2\pi)^{\text{th}}$  of the circumference. This convention makes angle measure a proportional relationship since the measure involves a fractional amount of a circumference. Moore's (2013) teaching experiment provided precalculus students the opportunity to use ideas of proportionality to conceptualize angle measure with this arc approach. However, in-service teachers questioned the practicality of using ideas of proportionality and the arc approach to measure angles for precalculus and calculus understanding (Thompson et al., 2007). Additionally, while preservice mathematics teachers (PMTs) procedurally converted between degrees and radians, they struggled to describe the results of their conversions as involving a proportional relationship (Akkoc, 2008; Çekmez, 2020; Fi, 2003; Topcu et al., 2006). However, when asked to use different diagrams to describe radian angle measure, some PMTs attended to proportionality (Alyami, 2022b). This finding suggests that PMTs' procedural conversion between units of angle measure might connect to conceptual understanding (Baroody et al., 2007; Maciejewski & Star, 2019; Nilsson, 2020; Nordlander, 2021; Star, 2005, 2007), a connection that is under-explored in the context of angle measure. In this report, I unpack one PMT's *mathematical meanings* for radian angle measure, a construct I will elaborate on in the next section. The research aim is to describe one PMT's meanings for radian angle measure through a series of three task-based interviews to unpack the connections between procedural and conceptual meanings. The research question guiding this

report is “What mathematical meanings does a PMT demonstrate upon engaging with a series of tasks that involve radian angle measure?”

### Theoretical Framing

#### Mathematical Meanings

I take a constructivist perspective and build on the work of Thompson et al. (2014), where mathematical meaning refers to assimilating to schemes associated with an understanding, including “the space of implications that the current understanding mobilizes” (Thompson et al., 2014, p. 13). This description of mathematical meanings builds on the duality between thinking and understanding, which are rooted in mental actions that are demonstrated through common cognitive characteristics given a mathematical situation (Harel, 2008). From Harel’s (2008) and Thompson et al.’s (2014) descriptions, I interpret mathematical meaning as the understanding that enables a learner to reason about a particular concept, including implications brought to bear from active reasoning about the mathematical concept. I also consider ways of thinking as the patterns a learner develops to reason about a particular concept, given a particular situation that evokes such reasoning. In this report, the concept is radian angle measure, and the active reasoning is expected to be brought forth by various mathematical tasks. For example, PMTs described radian angle measure with an emphasis on special angles written in terms of  $\pi$  (Akkoc, 2008; Fi, 2003; Topcu et al., 2006). Additionally, Akkoc (2008), Fi (2003), and Moore et al. (2016) reported that PMTs incorporated procedural calculations with radian angle measure using the unit circle (Figure 1), a circle diagram typically labeled with special angles in radians written as integer multiples of  $\frac{\pi}{6}$  and  $\frac{\pi}{4}$  radian. Another version of the unit circle depicts special angles in radians along with the equivalent measure in degrees.

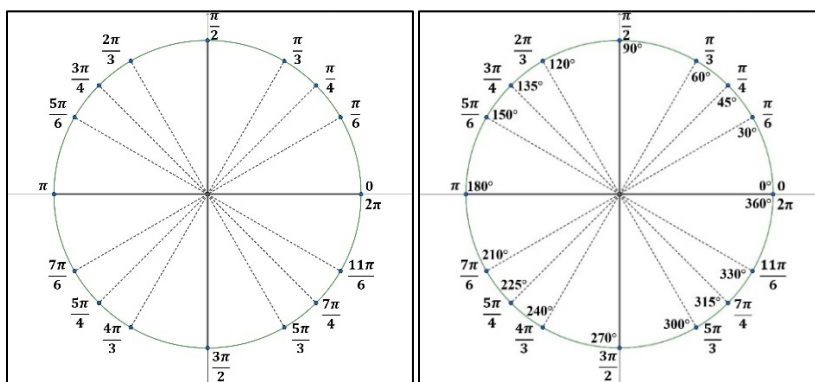


Figure 1. Typical Representations of the Unit Circle with Special Angles

The previous findings suggest that the PMTs’ previous experiences could have led to mathematical meanings and ways of thinking that incorporate procedures (e.g., conversion, calculations with the unit circle, memorizing special radian angles in terms of  $\pi$ , etc.) to reason about angle measure. Yet the National Council of Teachers of Mathematics (NCTM) calls for learners to move beyond procedural use of memorized facts, to flexibly applying mathematical problem-solving through a foundation of conceptual understanding (NCTM, 2014). Specifically, if applying procedures is expected to build on a foundation of conceptual understanding (NCTM, 2014), then attention to the amalgamation of procedural and conceptual meanings is needed.



## Proportional Reasoning and Angle Measure

Considering the convention for measuring angles described earlier, attending to ideas of proportionality is appropriate for this report. Thompson (2011) described proportional reasoning as involving multiplicative comparisons between two quantities. Measuring angles involves a multiplicative comparison between the angle's subtended arc and the circumference of the circle containing the arc, making angle measure a proportional relationship. The arc approach to angle measure demonstrates learners' use of fractions and ratios to reason about angle measure (Moore, 2013). Angle measure can also be described using other proportional reasoning concepts. For example, ninth-graders described angle measure using partitioning of familiar angles to create a  $1^\circ$  angle (Hardison, 2020). The ninth-graders attended to the measure of a  $1^\circ$  angle relative to familiar angles (i.e.,  $90^\circ$  as the right angle), demonstrating equipartitioning as a strategy for describing angle measure.

Based on the literature discussed above, I anticipate a PMT's thinking about radians would involve special angles written in terms of  $\pi$  (Akkoc, 2008), with attention to proportionality (Alyami, 2022b; Moore, 2013), and partitioning (Hardison, 2020).

## Methods

### Research Design

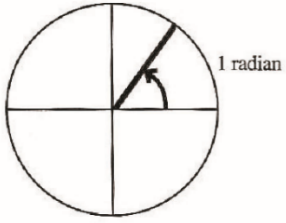
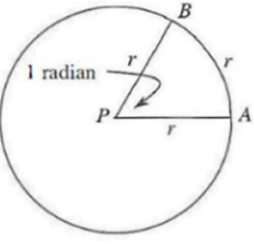
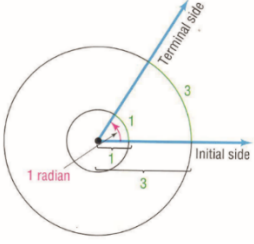
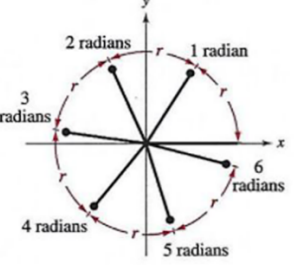
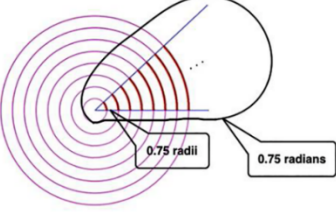
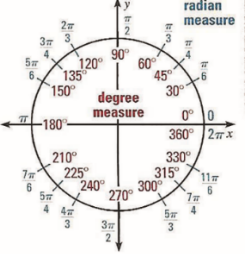
To examine the PMT's mathematical meanings about radian angle measure, I employed a qualitative case study design (Flyvbjerg, 2011) during participation in a series of task-based interviews involving radian angle measure (described below).

### Participant and Tasks

The participant in this report was Henry (pseudonym), a PMT enrolled in a mathematics teacher preparation program at a large Midwestern university. Henry volunteered to participate in three separate task-based interviews and was compensated for his time. I did not offer any learning sessions about radians prior to this research; however, Henry has likely learned about radian angle measure during his K-16 schooling. In the following sections, I describe each task and the timeline for Henry's participation.

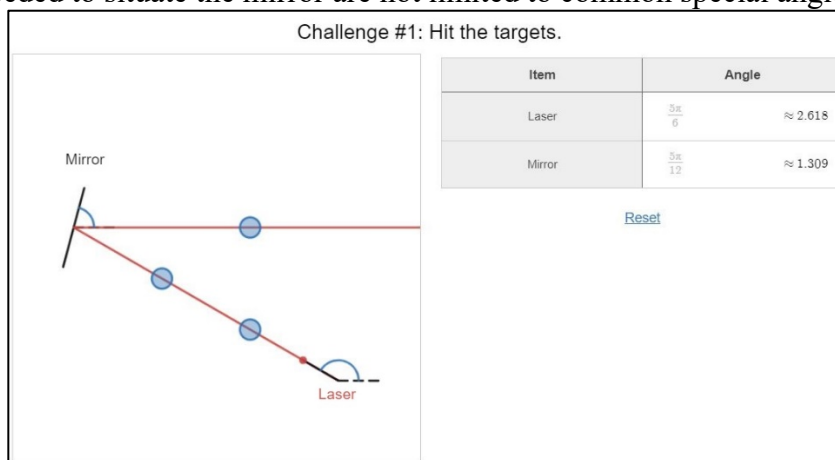
**Task 1.** The first task-based interview took place in spring of 2019. Henry was asked to describe radian angle measure through examining a series of radian diagrams (Alyami, 2022b). The diagrams were handed to Henry one at a time, in the order listed in Table 6.

**Table 6: Radian Angle Measure Representations Used for Task 1**

<p><b>Diagram 1</b></p>  <p>(Kysh et al., 2009, p. 47)</p>	<p><b>Diagram 2</b></p>  <p>(Kysh et al., 2009, p. 46)</p>	<p><b>Diagram 3</b></p>  <p>(Sullivan &amp; Sullivan, 2009, p. 355)</p>
<p><b>Diagram 4</b></p>  <p>(Larson et al., 2008, p. 259)</p>	<p><b>Diagram 5</b></p>  <p>(P. W. Thompson, personal communication, 23 February 2018)</p>	<p><b>Diagram 6</b></p>  <p>(Larson et al., 2010, p. 861)</p>

**Task 2.** The second task-based interview took place in spring of 2021. Henry was asked to describe what it would mean for an angle to have the measure of 1 radian. Henry also described how he would determine the measure of angles in radians, given specific measures in degrees (i.e.,  $360^\circ$ ,  $90^\circ$ ,  $72^\circ$ ,  $36^\circ$ ,  $112^\circ$ ,  $x^\circ$ ).

**Task 3.** The third and final task-based interview took place in fall of 2021. Henry engaged with a digital activity that involves radian angle measure and light reflection (Alyami, 2022a). Henry was asked to input angles measured in radians to situate a laser and one or two mirrors so the laser beam would pass through three stationary targets. A benefit of the *Radian Lasers* task is that the angles needed to situate the mirror are not limited to common special angles (Figure 2).



**Figure 2: A Radian Lasers Challenge, With the Mirror Angle not a Common Special Angle**

## Data and Analysis

Each interview followed the guidelines of structured, task-based interviews (Clement, 2000; Goldin, 2000), lasted approximately one hour, and was audio- and video-recorded. Each interview was transcribed, and the transcripts comprise the data for this report. To unpack Henry's meanings for radian angle measure, I attend to the strategies he used when addressing each task. By *strategies*, I refer to utterances and observable actions (including writing and sketching) made to address the various tasks. By characterizing Henry's strategies, I describe a collection of reasoning actions that were brought to bear through engagement with the tasks, representing the space of implication that resulted from assimilating to a scheme (i.e., Henry's meanings for angle measure). I used thematic analysis (Saldaña, 2013) to categorize Henry's mathematical meanings with attention to conceptual and procedural strategies. To answer the research question, the analysis focused on unpacking Henry's strategies throughout the three interviews "so that more can be gleaned from the data than would be available from merely reading, viewing, or listening carefully to the data multiple times" (Simon, 2019, p. 112).

### Findings

I organize this section in the chronological order of the interviews to illustrate the development in Henry's meanings for radian angle measure during the interviews. I then provide a summary of the findings in relation to the relevant literature in the discussion.

#### First Interview (Task 1)

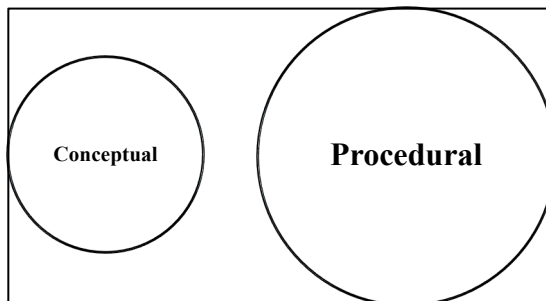
While Diagram 1 does not depict radian angle measure in terms of  $\pi$ , Henry described radian angle measure that involves  $\pi$ . He stated that "the unit of 1 radian doesn't really mean a whole lot...since [radian] is always measured in  $\pi$ ." Henry continued referring to  $\pi$  when describing radians from Diagrams 1 through 5 (none of which depicted radian angle measure in terms of  $\pi$ ). This suggests that Henry has developed thought patterns to reason about radian angle measure that involve angles measured in terms of  $\pi$ .

While Henry emphasized radian angle measure in terms of  $\pi$ , he identified the arc approach to radian angle measure when he was given Diagrams 2 through 5. For example, when looking at Diagram 2, Henry indicated that "there is the relationship between the lengths of the sides [points to the radius] and the arc length." When looking at Diagram 3, he elaborated that "the three to one ratio of the radii is the same as the three to one ratio [of] the arc lengths, and for me I'm just looking at this as a special case of that property in which the arc length just happens to be the same as the radius," which highlights his attention to the special case of equivalence. He gave the same response when describing radians in Diagram 4 with "each of these arc lengths is the same length as the radius," and in Diagram 5 as "each arc length is 0.75 of, the length of each arc is 0.75 times the radius." The previous statements suggest Henry's awareness of the multiplicative relationship between the arc length and the radius when measuring angles in radians.

Despite recognizing the proportionality involved when measuring angles in radians, Henry still preferred to think of radian angle measure in terms of  $\pi$  and questioned the practicality of such approach. This is evident from Henry's description of radians when looking at Diagram 6:

this is the most satisfying representation of radian, when you multiply it by  $\pi$  and you get easy fractions of a circle... you just have to immediately go to halfway around the circle is  $\pi$  ... you kind of just have to immediately identify  $\pi$  with the fraction of the circle and not spend time thinking about it's actually the physical length of the arc.

The previous excerpt demonstrates Henry’s preference for the unit circle representation, along with special angles written in terms of  $\pi$ . While Henry’s reference to fractions of a circle could be interpreted as partitioning, he emphasized the symbolic multiplication of  $\pi$  with symbolic association to the circle. By the end of this interview, Henry’s meanings for radian can be described as procedural, with awareness of conceptual meanings demonstrated by his attention to proportionality. Figure 3 represents Henry’s emphasis on procedural meanings for radian angle measure without connecting to the conceptual meanings he demonstrated.

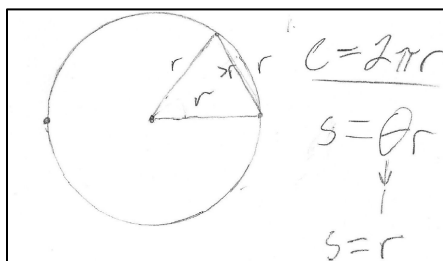


**Figure 3. A Model of Henry’s Meanings for Radian in the First Interview**

### Second Interview (Task 2)

When Henry was asked to describe what it would mean for an angle to have the measure of one radian, he stated “I remember this from the last study, but I’m going to take it a little bit to how I came to think of it after that study, how I came to think of it on my own.” This suggests that Henry’s description (below) resulted from reflecting after the interview for the first task:

we learn the formula for circumference before we learn anything about angles... the circumference is  $2\pi$  times the radius... this really didn’t hit me until... we were asked... what’s the length of this thing [points to arc in Figure 4]... it turns out that whatever that angle is, if you measure that angle in radians, then that’s just the length of the arc. I mean, still times  $r$ ... that was the part that kind of like, blew my mind... because... this circumference formula... [points to  $C = 2\pi r$ ] is just a special case of the more general formula  $S = \theta r$ ... this is what really made this significant for me, what happens if you just make this 1 [points to  $\theta$  in  $S = \theta r$ ]?... that arc length equals the length of the radius (Figure 4)... it’s just one radian.



**Figure 4. Henry Describing the Measure of 1 Radian Angle in Relation to the Arc Formula**

While the previous excerpt contains references to formulas and calculations, Henry’s description involves connections between the radian angle measure in relation to the arc length and circumference formulas. He recognized the circumference formula as a special case of the arc length formula, where the circumference is the arc length associated with a full rotation

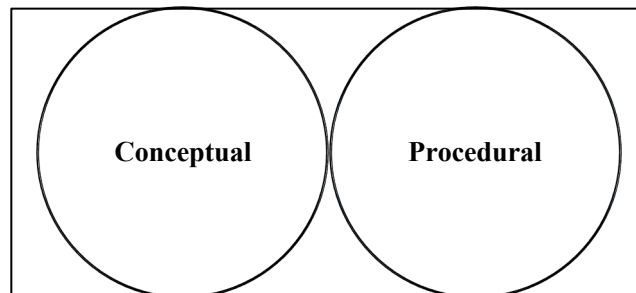
angle. Henry also uses the arc length formula to describe the measure of one radian as the situation when the arc length subtending a one radian angle equals the length of the radius.

While Henry was able to describe radian angle measure by describing the arc length in relation to the radius, he still referred to “easy fractions of  $\pi$ ” when given angle measure in degrees and was asked to describe the measure in radians. For example, when describing  $72^\circ$  and  $36^\circ$  in radians, Henry’s strategy involved using the measure relative to special angles that he memorized, such as  $\frac{\pi}{6}$ ,  $\frac{\pi}{3}$ , and  $2\pi$  (Figure 5). However, Henry was surprised by  $\frac{\pi}{5}$  radians, which satisfied the “easy fractions of  $\pi$ ” description, but he was not familiar with it. He wondered “why wasn't I able to look at 36 and immediately say, ‘Oh, that's  $\frac{\pi}{5}$ ?’” A justification he provided right away was “because it wasn't on the unit circle.”

The image shows two boxes of handwritten work. The left box contains the calculation for  $72^\circ$ :  $\frac{\pi}{3} + 12^\circ$  followed by  $= \frac{\pi}{3} + \left(\frac{2}{5} \cdot \frac{\pi}{6}\right)$ . The right box contains the calculation for  $36^\circ$ :  $30^\circ + \frac{1}{10}60^\circ$  followed by  $= \frac{\pi}{6} + \frac{1}{10}\left(\frac{\pi}{3}\right)$ .

**Figure 5. Henry’s Description of  $72^\circ$  and  $36^\circ$  as  $\frac{2}{5}\pi$  and  $\frac{\pi}{5}$  Radian Relative to Special Angles**

The previous examples demonstrate connections between Henry’s meanings for radian. Henry’s description of radian angle measure involved special angles written in terms of  $\pi$  along with awareness of proportionality. However, by the end of the second interview, Henry demonstrated various connections between ideas that involve radian angle measure. Figure 6 represents Henry’s meanings for radian angle measure after the second interview as involving both procedural and conceptual meanings with a connection between the two. Henry recognized the circumference formula as a special case of the arc length formula, and also recognized the limitation of describing radian angle measure as “easy fractions of  $\pi$ ” when he was only thinking about the few special angles typically depicted on the unit circle.



**Figure 6. A Model of Henry’s Meanings for Radian in the Second Interview**

### Third Interview (Task 3)

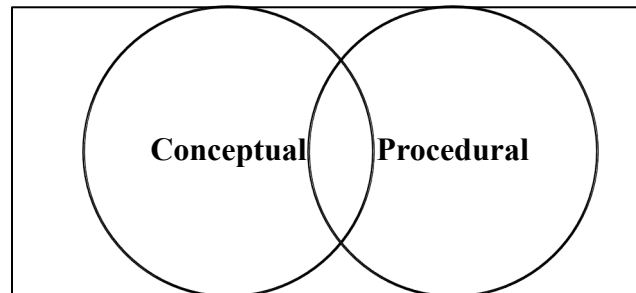
The *Radian Lasers* activity challenged Henry’s dependence on the special angles typically depicted on the unit circle diagram. For example, when working on Challenge 1 (Figure 2), Henry noticed that positioning the mirror at “ $\frac{\pi}{3}$  gives me too small of an angle,” while positioning the mirror at “ $\frac{\pi}{2}$  gives me too big of an angle, but each of those are off by the same amount, just in opposite directions. Okay, so I must just need the value in between those two.” After seeing that “the answer was  $\frac{5\pi}{12}$ ,” Henry explicitly stated,

once I saw that this is a problem about radians, I had already decided that the answer had to be on the unit circle, and so I looked for every one of those [special angles] before I even thought about, well, maybe I actually need to think about this problem, instead of just looking for the most applicable unit circle value.

Henry's statements illustrate both dependence on unit circle values and his acknowledgment of the limitations of such an approach. However, since the angles needed were not limited to common special angles, Henry reflected on the limitation of depending on unit circle values:

The knowledge that we're trying to give to students when we introduce the unit circle ... exercises like this will still get you to that same place ... but they won't lock you into like, the diagram. Like everyone knows the diagram, it's got all the multiples of  $\frac{\pi}{4}$  and all the multiples of  $\frac{\pi}{6}$ . That's the diagram ... but this develops a better intuition of actually knowing that ...  $\frac{\pi}{n}$  radians is one  $n$ th of the way around the semicircle.

Henry's statement suggests that even if students are introduced to the unit circle diagram, it can be done in a way that promotes conceptual meanings for radian, where students consider radian angle measure beyond the few special angles depicted on a typical diagram of the unit circle. While Henry initially used the special angles from the unit circle, he was able to flexibly think about radian angle measure beyond those special angles using conceptual meanings. Figure 7 represents Henry's meanings for radian angle measure after the third interview as involving overlapping procedural and conceptual meanings. Henry generalized radian angle measure as "easy fractions of  $\pi$ " by describing that any fraction of  $\pi$  radians represents an angle that is "one  $n$ th of the way around the semicircle."



**Figure 7. A Model of Henry's Meanings for Radian in the Third Interview**

### **Discussion**

This report aims to extend previous research exploring PMTs' conceptions of radian angle measure (Akkoc, 2008; Çekmez, 2020; Fi, 2003; Moore et al., 2016; Topcu et al., 2006) by tracking one PMT's mathematical meanings during engagement with three tasks involving radian angle measure. Henry's initial description of radians demonstrates the influence of common representations of radians in terms of  $\pi$  and in relation to the unit circle (Akkoc, 2008; Fi, 2003; Moore, 2013; Topcu et al., 2006). Additionally, during the first interview, Henry attended to the role of proportionality between the angle's subtended arc and radius (Alyami, 2022b). However, similar to in-service teachers in Thompson et al. (2007), Henry questioned the practicality of using ideas of proportionality and the arc approach to measure angles.

While not his preferred strategy, Henry elaborated on the relationship between the arc length and radius at the beginning of the second interview by relating the formulas for measuring arc

length and circumference. Using these relationships to describe one radian as the special case when the angle's subtended arc and the radius are equal reflects Henry's multiplicative comparison of the two lengths (Thompson, 2011). While Henry demonstrated this comparison procedurally using the formulas, his description incorporated conceptual strategies that focused on *how* and *why* these formulas are related (Nordlander, 2021).

During the second interview, Henry continued to describe radians as “easy fractions of  $\pi$ ,” with an emphasis on special angles depicted on a typical unit circle diagram. However, Henry acknowledged the limitation of this strategy upon encountering  $\frac{\pi}{5}$ , which he was not familiar with because “it was not on the unit circle.” This situation relates to how procedural strategies can be beneficial in limited situations (Thompson & Carlson, 2017). Eventually, Henry generalized his description of radians from “easy fractions of  $\pi$ ” to fractions that are “one  $n^{\text{th}}$  of the way around the semicircle,” suggesting a partitioning approach (Hardison, 2020). When reasoning about radians, procedural strategies might not be devoid of conceptual understanding (Baroody et al., 2007; Maciejewski & Star, 2019; Nilsson, 2020; Nordlander, 2021; Star, 2005, 2007). While Henry used conceptual strategies from the beginning, his procedural explanations outweighed his conceptual ones. Through exposure to situations with radians in contexts that challenged procedural strategies, such as non-familiar diagrams and measurement (e.g.,  $\frac{\pi}{5}$ ,  $\frac{5\pi}{12}$ ), Henry connected his procedural and conceptual meanings for radian angle measure.

The main implication from this report is the recommendation to capitalize on tasks that are designed to build and support conceptual understanding. Instead of tasks that emphasize procedures, using tasks that encourage learners to unpack the mathematical foundation behind the procedures potentially contributes to developing conceptual and productive meanings for radian angle measure.

## References

- Akkoc, H. (2008). Pre-service mathematics teachers' concept images of radian. *International Journal of Mathematical Education in Science and Technology*, 39(7), 857–878. <https://doi.org/10.1080/00207390802054458>
- Alyami, H. (2022a). A radian angle measure and light reflection activity. *Mathematics Teacher: Learning and Teaching Pre-K–12*, 115(6), 422–431. <https://doi.org/10.5951/MTLT.2021.0217>
- Alyami, H. (2022b). Defining radian: Provoked concept definitions of radian angle measure. *Research in Mathematics Education*. <https://doi.org/10.1080/14794802.2022.2041470>
- Baroody, A. J., Feil, Y., & Johnson, A. R. (2007). An alternative reconceptualization of procedural and conceptual knowledge. *Journal for Research in Mathematics Education*, 38(2), 115–131. <https://doi.org/10.2307/30034952>
- Çekmez, E. (2020). What generalizations do students achieve with respect to trigonometric functions in the transition from angles in degrees to real numbers? *The Journal of Mathematical Behavior*, 58. <https://doi.org/10.1016/j.jmathb.2020.100778>
- Clement, J. (2000). Analysis of clinical interviews: Foundations and model viability. In A. E. Kelly & R. A. Lesh (Eds.), *Research design in mathematics and science education* (pp. 547–589). Lawrence Erlbaum Associates.
- Fi, C. (2003). Preservice secondary school mathematics teachers' knowledge of trigonometry: Subject matter content knowledge, pedagogical content knowledge and envisioned pedagogy [Unpublished doctoral dissertation]. University of Iowa.
- Flyvbjerg, B. (2011). Case study. In N. Denzin & Y. Lincoln (Eds.), *The Sage handbook of qualitative research* (4th ed., pp. 301–316). Sage Publications.
- Goldin, G. A. (2000). A scientific perspective on structured, task-based interviews in mathematics education research. In A. E. Kelly & R. A. Lesh (Eds.), *Research design in mathematics and science education* (pp. 517–545). Lawrence Erlbaum Associates.
- Hardison, H. L. (2020). Acknowledging non-circular quantification of angularity. In A. I. Sacristán, J. C. Cortés-Zavala, & P. M. Ruiz-Arias (Eds.), *Mathematics education across cultures: Proceedings of the 42nd meeting of*

- the North American chapter of the International Group for the Psychology of Mathematics Education, Mexico (pp. 671–675). Cinvestav/AMIUTEM/PME-NA. <https://doi.org/10.51272/pmena.42.2020>
- Harel, G. (2008). DNR perspective on mathematics curriculum and instruction, part I: Focus on proving. *ZDM*, 40(3), 487–500. <https://doi.org/10.1007/s11858-008-0104-1>
- Kysh, J., Kasimatis, E., Hoey, B., & Sallee, T. (2009). *Pre-calculus with trigonometry: Volume 1. College Preparatory Mathematics Educational Program*.
- Larson, R., Boswell, L., Kanold, T., & Stiff, L. (2010). *Algebra 2*. Hold McDougal.
- Larson, R., Hostetler, R., & Edwards, B. (2008). *Precalculus with limits*. Houghton Mifflin Company.
- Maciejewski, W., & Star, J. R. (2019). Justifications for choices made in procedures. *Educational Studies in Mathematics*, 101(3), 325–340.
- Moore, K. C. (2013). Making sense by measuring arcs: A teaching experiment in angle measure. *Educational Studies in Mathematics*, 83(2), 225–245. <https://doi.org/10.1007/s10649-012-9450-6>
- Moore, K. C. (2014). Quantitative reasoning and the sine function: The case of Zac. *Journal for Research in Mathematics Education*, 45(1), 102–138. <https://doi.org/10.5951/jresematheduc.45.1.0102>
- Moore, K., LaForest, K., & Kim, H. (2016). Putting the unit in pre-service secondary teachers' unit circle. *Educational Studies in Mathematics*, 92(2), 221–243. <https://doi.org/10.1007/s10649-015-9671-6>
- National Council of Teachers of Mathematics (2014). *Procedural fluency in mathematics: A position of the National Council of Teachers of Mathematics*. <https://www.nctm.org/Standards-and-Positions/Position-Statements/Procedural-Fluency-in-Mathematics/>
- Nilsson, P. (2020). A framework for investigating qualities of procedural and conceptual knowledge in mathematics—An inferentialist perspective. *Journal for Research in Mathematics Education*, 51(5), 574–599.
- Nordlander, M. C. (2021). Lifting the understanding of trigonometric limits from procedural towards conceptual. *International Journal of Mathematical Education in Science & Technology*, 53(11). <https://doi.org/10.1080/0020739X.2021.1927226>.
- Saldaña, J. (2013). *The coding manual for qualitative researchers*. Sage.
- Simon, M. A. (2019). Analyzing qualitative data in mathematics education. In K. R. Leatham (Ed.), *Designing, conducting, and publishing quality research in mathematics education* (pp. 111–123). Springer.
- Star, J. R. (2005). Reconceptualizing procedural knowledge. *Journal for Research in Mathematics Education*, 36(5), 404–411. <https://doi.org/10.2307/30034943>
- Star, J. R. (2007). Foregrounding procedural knowledge. *Journal for Research in Mathematics Education*, 38(2), 132–135. <https://doi.org/10.2307/30034953>
- Sullivan, M., & Sullivan, M. (2009). *Precalculus enhanced with graphing utilities*. Prentice Hall
- Tallman, M., & Frank, K. (2020). Angle measure, quantitative reasoning, and instructional coherence: An examination of the role of mathematical ways of thinking as a component of teachers' knowledge base. *Journal of Mathematics Teacher Education*, 23(1), 69–95.
- Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In S. Chamberlin, L. L. Hatfield, & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education: Papers from a planning conference for wisdom* (pp. 35–57). University of Wyoming.
- Thompson, P. W. (2013). In the absence of meaning.... In K. R. Leatham (Ed.), *Vital directions for mathematics education research* (pp. 57–93). Springer New York.
- Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 421–456). National Council of Teachers of Mathematics.
- Thompson, P. W., Carlson, M. P., & Silverman, J. (2007). The design of tasks in support of teachers' development of coherent mathematical meanings. *Journal of Mathematics Teacher Education*, 10, 415–432.
- Thompson, P. W., Carlson, M. P., Byerley, C., & Hatfield, N. (2014). Schemes for thinking with magnitudes: A hypothesis about foundational reasoning abilities in algebra. In K. C. Moore, L. P. Steffe, & L. L. Hatfield (Eds.), *Epistemic algebra students: Emerging models of students' algebraic knowing* (Vol. 4, pp. 1–24). University of Wyoming.
- Topcu, T., Kertil, M., Akkoc, H., Yilmaz, K., & Onder, O. (2006). Pre-service and in-service mathematics teachers' concept images of radian. In J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková (Eds.), *Proceedings of the 30th conference of the International Group for the Psychology of Mathematics Education* (Vol. 5, pp. 281–288). *Psychology of Mathematics Education*. <http://files.eric.ed.gov/fulltext/ED496939.pdf>



---

# NUMBER CONCEPTS AND PROPORTIONAL REASONING

---

# A NEW CLASSIFICATION OF SEMANTIC STRUCTURES OF ONE-STEP WORD PROBLEM SITUATIONS

Oh Hoon Kwon  
Department of Mathematics  
University of Wisconsin – Madison  
kwon@math.wisc.edu

*A new classification for semantic structures of one-step word problems is proposed in this paper. The classification is based on illustrations of word problem situations in Common Core State Standards (CCSSM, 2010) and related historical studies (e.g. Weaver, 1973, 1979, 1982), as well as conceptual elaborations of embodied and grounded nature in Lakoff et al. (2000). The classification identifies two main classes: action on/change of an initial quantity and coordination/comparison of two quantities, providing a unifying characteristic of basic operations of quantities. This classification is more comprehensive and differentiated than the classification of CCSSM (2010) and Polotskaia et al. (2021), as it emphasizes conceptual demands of children's mathematics, coherence and continuity of progressions, and consistency with thinking modes and/or problem-solving strategies.*

Keywords: Number Concepts and Operations, Mathematical Representations, Cognition, Learning Trajectories and Progressions

## Introduction

Common Core State standards describe the situation types or the categories of word problems using two tables (CCSSM, 2010). The first table shows the distinct types of addition and subtraction situations: add to, take from, put together/take apart, and compare. The second table shows the distinct types of multiplication and division situations: equal groups of objects, arrays of objects, and comparison.

Having a thorough grasp of various problem types is of utmost importance for teachers as it enables them to discern the different approaches and methods employed by students when tackling problems. On occasion, the positioning of the unknown element within a problem significantly influences the strategies utilized and the level of complexity experienced. Moreover, there is a widely held belief that students should comprehend problems and devise solutions that align with their own understanding, rather than strictly adhering to a predetermined approach based on problem types. However, an alternative viewpoint can also suggest that if feasible, aligning problem types with thinking modes and strategies could enhance comprehension and empower students to effectively employ appropriate representations and modeling techniques.

There is an expected progression in comprehending different types of situations. Initially, students learn and solve problems related to situations involving whole numbers, which later advances to word problems incorporating fractions, integers, and eventually all rational and real numbers. Additionally, teachers typically introduce addition and subtraction situations in earlier grades, while multiplication and division situations are introduced in later grades. Despite research studies, such as Carpenter et al. (1999) and others, confirming that young children are capable of solving multiplication, division, and multistep problems by directly modeling the action or structure, some teachers and researchers still tend to believe that additive reasoning always precedes multiplicative reasoning. Our argument revolves around the notion that in order

to gain a comprehensive understanding of how the situation types in word problems contribute to students' comprehension, problem-solving methods, and thinking strategies associated with these situation types, it is imperative to illuminate the continuity in progressions and the shared characteristics of *operations involving quantities* grounded in human activity. These aspects have played a crucial role in enabling students to develop skills like direct modeling, while also guiding teachers in implementing suitable teaching approaches for their students.

This study, therefore, introduces a new classification of word problem situations based on the shared characteristics of operations involving quantities and the continuity of concepts across different levels of mathematics. To develop this classification, we analyze relevant studies on quantitative operations, word problem situation types, and embodied cognition. By examining various representations and the conversions utilized during operations involving quantities, we identify commonalities and ensure continuity in progression. The findings shed light on the current presentation of word problem situation types and propose a more consistent and continuous approach enriched by grounded and embodied cognition. This research opens avenues for future experiments, discussions, and reflections in this area.

### **The necessity of a new classification**

The types of word problems are closely connected to the semantic structures of quantity operations. In his work, Schwartz (1996) extensively examined the semantic aspects of quantities and their operations in mathematics. He proposed that the operations of quantities consist of two parts: numerical operations and operations on units of measurement, which extended the four basic operations of pure numbers. This idea highlighted the fact that multiplication and division operations of quantities could create new units of measurement, and were not merely repeated addition and subtraction. However, the addition and subtraction operations of quantities remained under the same umbrella category, similar to the traditional illustration of pure number operations or word problem types, whereas multiplication and division operations were classified under a different category. Schwartz did not specify any shared characteristics between these two categories in terms of quantity operations.

In the recent study of multiplicative structures, Polotskaia et al. (2021) have proposed a relational paradigm for understanding multiplicative structures, which includes three multiplicative relationships: multiplicative comparison, multiplicative composition, and cartesian product, and their corresponding visual models. Their view challenges traditional approaches to teaching word problem solving, which emphasize mastering elementary arithmetic operations before applying them to problem-solving. They highlighted an increasing number of studies that investigate the relationship between mathematical structures and word problem solving, such as the works of Cai et al. (2005), Ng & Lee (2009), and Verschaffel et al. (2010). Additionally, they contrasted two paradigms: the operational paradigm, which views arithmetic operations as the foundation for comprehending real-world scenarios that involve actions like adding, subtracting, comparing, and sharing, and the relational paradigm, which views relationships between three elements as the fundamental mathematical concepts, where two elements determine a unique third element as a function. The relational paradigm focuses on the idea of an operation as a function (Carragher et al., 2005) and enables various modes of thinking about arithmetic operations. Polotskaia et al. attempt to align multiplicative and additive structures using the relational paradigm and visual models. However, they still maintain a distinction between the two structures and propose three unique classes for multiplicative structure and its associated reasoning that do not apply to additive structure and its related reasoning.

Considering the fundamental but distinct roles of additive and multiplicative structures and their associated reasoning in teaching and learning mathematics, it is reasonable to investigate whether these structures share commonalities or similarities. Such an exploration could yield significant insights and may justify the development of a new classification scheme that highlights their shared features. This classification could pave the way for innovative discoveries and advancements in teaching and learning of mathematics.

### **Development of a new classification**

Expanding on the previous discussion, it is crucial to investigate the feasibility of uncovering a common theme that is applicable to both additive and multiplicative structures in the context of word problem situations or operations involving quantities. To identify such a theme, we are specifically focusing on the following two frameworks.

#### **Grounded and Embodied Nature of Operations of Quantities**

In their work, Lakoff and Núñez (2000) put forth a novel perspective regarding numerical operations, proposing that addition, subtraction, multiplication, and division are not separate entities but are instead rooted in our embodied experiences and perceptions of the physical world. They achieved this by utilizing semantic primitives and conceptual metaphors to map arithmetic operations onto source domains such as object collection, object construction, measuring stick, and motion along a path. These metaphors serve to illustrate the commonality across the four operations and challenge the traditional notion that addition and subtraction are fundamentally distinct from multiplication and division. Furthermore, the authors emphasize the strong connection between operations involving quantities and the same metaphoric source domains.


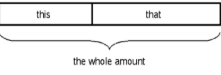
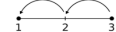

To exemplify the grounded and embodied nature of quantitative operations, let's examine the addition of two quantities. Imagine having two containers with different capacities, each holding varying amounts of water, and a third empty container large enough to hold the combined water. The action of pouring water between containers represents the source domain, while the addition operation signifies the target domain within this conceptual metaphor. Depending on the context and affordances present (as described by Gibson, 1979), we can pour smaller amounts into the larger container or merge the water from both containers to determine the total sum. Pouring smaller amounts into larger containers is typically easier for humans due to the enhanced visibility of changes and reduced risk of spillage. Through these everyday activities, we can deduce the fundamental principles that underlie quantitative operations.

#### **Unary(ish) vs. Binary Operations of Quantities**

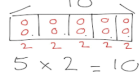

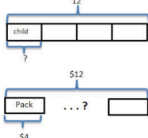
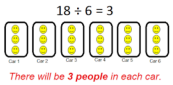
In his lesser-known but significant studies, J.F. Weaver (1973, 1979, 1982) emphasized the importance of students comprehending both binary-operation and unary(ish)-operator meanings of symbolic number sentences in the context of addition and subtraction. However, there is a lack of research on how young children interpret these types of sentences. It is uncertain whether binary and unary interpretations can develop simultaneously or if one needs to be learned before the other. Moreover, it is unclear when interference between the two interpretations may arise.

Although some questions remain unanswered, his studies have provided valuable insights into the operations of quantities, including not just addition and subtraction, but also multiplication and division. Based on this knowledge, we propose a new approach that is more logical and intuitive for understanding these operations. This approach is illustrated in Figure 3, and the symbolic representation shows that the operations of quantities can be viewed as

functions. This shared semantic structure leads to the following features of the four basic operations of quantities, as demonstrated in Figure 1 and Figure 2.

Addition ( $A+B=C$ )			Subtraction ( $A-B=C$ )		
	Action/Change-focused way	Comparison/Coordination-focused way	Action/Change-focused way	Comparison/Coordination-focused way	
Linguistic description	"Augend" + "Addend" = Result <b>Add</b> (Addend) <b>to</b> (Augend)	"Summand" + "Summand" = "Sum" <b>Add</b> (Summand) <b>and</b> (Summand)	"Minuend" - "Subtrahend" = Result <b>Subtract</b> (Subtrahend) <b>from</b> (Minuend)	Quantity - Quantity = "Difference" <b>Find the difference between</b> (Quantity 1) <b>and</b> (Quantity 2)	
Key Model					

**Figure 1. Addition and Subtraction of Two Quantities**

Multiplication ( $A \times B=C$ )			Division ( $A \div B=C$ )		
	Action/Change-focused way	Comparison/Coordination-focused way	Action/Change-focused way	Comparison/Coordination-focused way	
Linguistic description	"Multiplier" x "Multiplicand" = Result (or "Multiplicand" x "Multiplier" = Result in some countries) <b>Multiply</b> (Multiplicand) <b>by</b> (Multiplier), (Multiplier) <b>of</b> (Multiplicand), (Multiplier) <b>groups of</b> (Multiplicand)	"Factor" x "Factor" = "Product" <b>Multiply</b> (Factor 1) <b>and</b> (Factor 2)	"Dividend" $\div$ "Divisor" = Result <b>Divide</b> (Dividend) <b>by</b> (Divisor), <b>Divide/Partition</b> (Quantity) <b>into parts or portions</b> .	Quantity $\div$ Quantity = "Quotient" <b>Find the ratio/rate between</b> (Quantity 1) <b>and</b> (Quantity 2). <b>Distribute</b> (Quantity 1) <b>into</b> (Quantity 2). <b>How many times</b> (Quantity 1) <b>goes into</b> (Quantity 2)?	
Key Model					

**Figure 2. Multiplication and Division of Two Quantities**

Even though we juxtapose addition and subtraction together, and multiplication and division together, the main classes of action on/change of an initial quantity and comparison/coordination of two quantities are consistency throughout any operations of quantities.

	Operation # : $A \# B = C$	
Modes of Thinking	Action/Change-focused mode	Comparison/Coordination-focused mode
Structure	An operand with a numerical operator	Two operands with an operator sign
Symbolic description	$A \xrightarrow{\#B} C$	$\#(A,B) \longrightarrow C$
Problem Situation	Initial quantity with Change in the quantity results in Changed quantity	Initial quantity #1 together with Initial quantity #2 produce/result in New quantity
Number of involved quantities for input	Binary (Unary-ish)	Binary
Number of involved units of measurement/counting	<ul style="list-style-type: none"> <li>One unit (initial quantity and final quantity have the same unit. Change does not have any units, or has the same unit with the initial and final quantities. e.g. a location vector + translation vector = a new location vector, % + p.p. = % etc. )</li> </ul>	<ul style="list-style-type: none"> <li>One unit (addition, subtraction of two quantities of the same type)</li> <li>Two units (multiplication, division of two quantities of the same type) e.g. <math>2m \times 4m = 8 m^2</math></li> <li>Three units (all operations of two quantities of different types) e.g. 2 apples and 3 pears all together 5 fruits. Time x velocity = distance, etc.</li> </ul>
Main Idea	<ul style="list-style-type: none"> <li>Modification of an initial quantity</li> </ul>	<ul style="list-style-type: none"> <li>Amalgamation of two quantities to produce a single quantity, or</li> <li>Synthesis of two quantities to figure out the relationship between two.</li> </ul>
Related Advanced Math Idea	<ul style="list-style-type: none"> <li>"Reverse Engineering", Inverse Functional Thinking</li> </ul>	<ul style="list-style-type: none"> <li>Commutative Property, Associative Property</li> <li>Operations in MAGMAS</li> </ul>

**Figure 3. Two Operations of Quantities**

### Proposition of New Semantic Structure and Word Problem Situations

The proposed semantic structure and word problem situations in Figure 4 aim to provide a framework for classifying arithmetic word problems based on the types of actions or relationships involved. While some researchers, such as Carpenter et al. (1999), suggest that these classifications correspond to students' thinking about the problems, others, like Mulligan & Mitchelmore (1997), argue that they are somewhat arbitrary and can be extended, collapsed, or refined depending on the investigation's purpose. Multiple versions of these classifications exist, reflecting their ongoing development and refinement. Although the semantic structures of word problems may not always accurately reflect students' thinking or solution strategies, they can still be valuable tools for researchers seeking to understand and predict these thought processes, as well as for developing explicit models of knowledge structures and solution processes.

The common theme among the past and current classifications of semantic structure of the word problems is grouping of addition/subtraction, and multiplication/division based on the distinction between additive nature of thinking and multiplicative nature of thinking. However, those classifications hardly noticed a unifying nature of arithmetic operations regardless of the obvious types of operations or distinction between additive and multiplicative nature of thinking.

Arithmetic Operation	Addition 2+3		Subtraction 5-2		Multiplication 3×2		Division 6÷3	
Quantitative Operation (Mode of Thinking)	Semantic Structure / Word Problem Type / Problem Situation							
Action on / Change of initial quantity (operand and operator)	Add to	Initial quantity unknown	Take from	Initial quantity unknown	Multiply by / Scale up and down  3 groups of 2 are 6, Multiplication of 2 by 3 is 6.	Initial quantity unknown, <b>Classical PD</b>	Partition and select / Measure by cutting out	Initial quantity unknown
		Change unknown		Change unknown		Action unknown, <b>Classical MD</b>		Action unknown
		Result unknown		Result unknown		Result unknown <b>Classical Product 1</b>		Result unknown
Coordination of two quantities (operand and operand)	Put together	One quantity unknown	Take apart / Compare	One quantity unknown	Product / Times / Multiply and  3 times 2 is 6, Product of 3 and 2 is 6.	One quantity unknown	Distribute / Assign / Combine / Coordinate	One quantity unknown
		Result unknown		Result unknown		Result unknown <b>Classical Product 2</b>		Result unknown

**Figure 4. New Semantic Structure of Word Problem Situations**

### Discussion

In this section, we hope to show the usefulness of this new classification in several aspects of research studies by exploring some key signifying examples. These are parts of a larger on-going research project, and open to further studies and discussions.

#### Conceptual demands of children's mathematics

Our new classification provides a more explicit explanation of the conceptual demand in children's mathematics that was previously considered as their direct modeling or various strategies without understanding the reasons why children adopt those strategies or where they come from. Direct modeling involves modeling the action or relationships described in word problems, making the action or relationships depicted in word problems important clues to understanding why children take particular approaches. As a result, our new classification is particularly useful for analyzing these approaches. For instance, Carpenter et al. (1999) presented the following problem situation to observe children's use of the partitive strategy, a form of direct modeling of partitive division.

Mr. Franke baked 20 cookies. He gave all the cookies to 4 friends, being careful to give the same number of cookies to each friend. How many cookies did each friend get?

Three different types of strategies were introduced as variations of Partitive strategy.

Ellen counts out 20 counters. She placed the counters into 4 separate places one at a time. After she puts one counter in each spot, she starts over and adds another counter to each set, continuing this process until she has used up all the counters. Then she counts the counters in one pile and says, "5...each would get 5 cookies."

Based on our new classification, Ellen's problem can be categorized as a division problem that involves the coordination of two quantities. One of the quantities is clearly the number of counters, while the other quantity is the number of places or spots, even though they are not explicitly visible. The coordination of these two quantities is presented through actions such as

distributing, assigning, combining, coordinating two objects, and observing the relationship between them. This coordination is highlighted further in Rita's problem, as shown below.

First Rita counts out 20 counters. Then she selects 4 additional counters that are not part of the 20 to represent the 4 friends and puts them in separate places on the table. She deals the counters one by one to each of the 4 separate "friends" places on the table. When she has used up all 20 counters, she counts the number of counters in one of the groups, not counting the single counter that she first put out to identify the group and answers, "5."

Teacher: Good. I see how you got the 5, but can you tell me why you didn't count this [indicates the counter that represented the group]?

Rita: That's one of the friends.

Rita's case makes it clear that the division operation involves two quantities as inputs, represented by counters for cookies and friends. This is an example of coordinated division with two operands. The other cases are different in that they do not require two quantities to start with.

Kang counts out 20 counters. He places 4 counters in one group, 4 in another group, 4 in another, and 4 in another until he sees that there are 4 groups. At this point he sees that he had not used up all the counters, so he adds 1 counter to each group. Then he counts the counters in one of the groups and answers, "5."

Kang's strategy appears to be a combination of the measurement strategy, which involves repeated subtraction, and the partitive strategy, which involves partitioning, as described by Carpenter et al. (1999). It is possible that Kang initially selected four counters out of 20 to distribute among his friends later, but kept track of them mentally rather than physically. This approach does not involve coordination, assignment, or distribution of two quantities. However, Kang eventually realized that he had created four equal groups or partitions, which he identified as another type of object or quantity, and then attempted to coordinate the remaining counters with these four groups. This case demonstrates Kang's shift in thinking from an action or change of an initial quantity made with 20 counters to a coordination of two quantities made involving the leftover counters and the four groups he created.

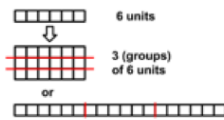
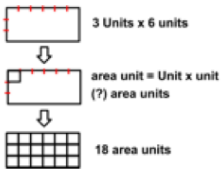
When dealing with questions that involve changing an initial quantity, division of a single quantity can be demonstrated through repeated subtraction or partitioning. For instance, starting with 20 counters, students can divide them into two equal groups of 10 counters each through partitioning. They can then further partition each of the 10-counter groups into two equal groups of 5 counters each. This is an example of division through repeated partitioning with only one quantity. No distribution to other objects/people is involved. To assume that the strategies employed by children to solve Partitive Division problems are variations of the Partitive strategy (Carpenter et al., 1999) would overlook important distinctions.

Based on this discussion, it can be inferred that the strategies devised by students are grounded in clear and intuitive principles, warranting their inclusion in a new classification. The conventional classification of partitive division and measurement division (quotative) primarily distinguishes these strategies based on the placement of unknowns within the multiplicative structure of mathematics, where the operator times the operand equals the product.



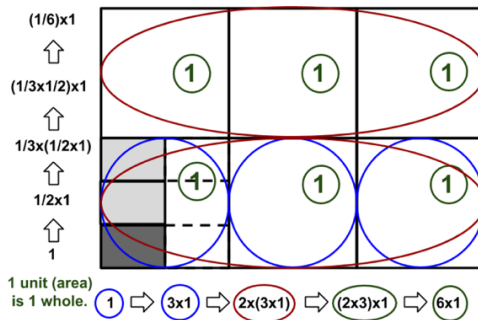
## Coherence and continuity of progressions

The new classification places significant importance on the coherence and continuity of progressions in semantic structures of operations. Throughout the progressions from additive reasoning to multiplicative reasoning in both numerical and quantitative senses, two main classes of actions are maintained: actions that involve changing an initial quantity, and actions that involve coordinating or comparing two quantities. These two classes have also been identified in visual representations of multiplication operation (Kwon et al., 2017, 2019). Specifically, the multiplication area model was reconstructed into two types: the length-to-area model and the area-to-area model, which are illustrated in Figure 5 below.

	Problem Types	Equal Groups	Arrays / Areas	Compare
	<b>Word Problems</b>	<p>A. There are 3 bags with 6 plums in each bag. How many plums are there in all?</p> <p>B. You need 3 lengths of string, each 6 inches long. How much string will you need altogether?</p>	<p>A. There are 3 rows of apples with 6 apples in each row. How many apples are there?</p> <p>B. What is the area of a 3 cm by 6 cm rectangle?</p>	<p>A. A blue hat costs \$6. A red hat costs 3 times as much as the blue hat. How much does the red hat cost?</p> <p>B. A rubber band is 6 cm long. How long will the rubber band be when it is stretched to be 3 times as long?</p>
<p><b>Area-to-Area Model (1 unit of measurement)</b></p> 	<b>Action on/Change of an initial quantity</b>	<p>A. 3 groups of 6 plums</p> <p>B. 3 (groups) of 6 inches</p>	<p>A. 3 groups of 6 apples</p>	<p>A. 3 groups of \$6</p> <p>B. 3 of 6cm</p>
<p><b>Lengths-to-Area Model (2 units of measurement)</b></p> 	<b>Operation/coordination of two quantities</b>	<p>A. 3 bags times 6 plums/bag</p> <p>B. 3 lengths times 6 inches/length</p>	<p>A. 3 rows times 6 apples/row</p> <p>B. 3 cm times 6 cm</p>	<p>A. 6 \$/blue hat times 3 blue hat/red hat</p>

**Figure 5. Multiplication Area Models with Multiplicative Word Problem Situations (Kwon et al., 2019)**

The semantic structures of multiplication rely on the differentiation between action on/change of an initial quantity and coordination of two quantities. This differentiation allows for the extension of the multiplication operation and its area model to fractions and beyond, without any confusion. As shown in Figure 4 below, the area-to-area model illustrates the connection between whole number multiplication and fraction multiplication, ensuring the coherence and continuity of progressions in semantic structures of multiplication.



**Figure 6. Whole Number Multiplication and Fraction Multiplication in Area-to-Area Model (Kwon et al., 2017, 2019)**

Coherence and continuity in operations extend to early algebra and algebra, encompassing both discrete and continuous models. Recent studies, such as those on the Davydov curriculum (e.g. Freiman, 2021; Mellone et al., 2021; Polotskaia et al., 2021; Coles, 2021, etc.), highlight the importance of maintaining coherence and continuity in the progressions of quantitative operations across various representations. A rapid transition from discrete to continuous objects and associated quantitative operations can aid early understanding of algebra, where variables are typically seen as continuously varying.

According to the development of the new classification, the way two quantities are processed is an essential component of quantity operations that extends beyond grade levels and into everyday human activities. This implies that the continuity between early and higher grades is not only related to objects, quantities, or raw materials, but also to the underlying concept that is deeply ingrained in the physical experiences of the body on how quantity operations are performed.

### Conclusion and Suggestion

The primary objective of this paper is to introduce a new and unifying classification of semantic structures in word problems, departing from the conventional focus on differentiating additive and multiplicative structures. This inclusive classification offers a more comprehensive and nuanced approach compared to previous classifications such as CCSSM (2010) and Polotskaia et al. (2021). It places emphasis on the conceptual foundation of children's mathematics, prioritizing coherence and continuity in the progressions of arithmetic operations and aligning with various thinking modes and problem-solving strategies. This innovative classification paves the way for further research, including exploring the use of technology incorporating hand and touch in problem-solving, investigating discrepancies between word problem structure/context and models/representations, examining different modes of thinking in mathematical modeling, and delving into inventive teaching and learning approaches for word problem solving.

### References

Cai, J., Lew, H. C., Morris, A., Moyer, J. C., Fong Ng, S., & Schmittau, J. (2005). The development of students' algebraic thinking in earlier grades: Lessons from China and Singapore. *Zentralblatt Für Didaktik Der Mathematik*, 37(1), 5–15.

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

- Carpenter, T. P., Fennema, E., Franke, M. L., Levi, L., & Empson, S. B. (1999). *Children's mathematics. Cognitively Guided Instruction*. Portsmouth, NH: Heinemann. Trad. de C. De Castro y M. Linares: Las Matemáticas que hacen los niños.
- Carragher, D., Schliemann, A., & Brizuela, B. (2005). Chapter 1: Treating the Operations of Arithmetic as Functions. *Journal for Research in Mathematics Education. Monograph, 13*.
- Coles, A. Commentary on a special issue: Davydov's approach in the XXI century. *Educ Stud Math 106*, 471–478 (2021). <https://doi.org/10.1007/s10649-020-10018-9>
- Common Core State Standards Initiative. (2010). *Common Core State Standards for Mathematics (CCSSM)*. Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers.
- Freiman, V., Fellus, O.O. Closing the gap on the map: Davydov's contribution to current early algebra discourse in light of the 1960s Soviet debates over word-problem solving. *Educ Stud Math 106*, 343–361 (2021). <https://doi.org/10.1007/s10649-020-09989-6>
- Gibson, J. J. (1979). *The Ecological Approach to Visual Perception*. Boston, MA: Houghton Mifflin.
- Kwon, O. H., Son, J. W., & I, J.Y. (2017). Revisiting Area Models for Fraction Multiplication. In E. Galindo & J. Newton, (Eds.), *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (p. 331). Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.
- Kwon, O. H., Son, J. W., & I, J.Y. (2019). Revisiting Multiplication Area Models for Whole Numbers, *The Mathematics Enthusiast: Vol. 16 : No. 1 , Article 17*.
- Lakoff, G., & Núñez, R. E. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. Basic Books.
- Mellone, M., Ramploud, A. & Carotenuto, G. An experience of cultural transposition of the El'konin-Davydov curriculum. *Educ Stud Math 106*, 379–396 (2021). <https://doi.org/10.1007/s10649-020-09942-7>
- Mulligan, J. T., & Mitchelmore, M. C. (1997). Young children's intuitive models of multiplication and division. *Journal for research in Mathematics Education*, 309-330.
- Ng, S. F., & Lee, K. (2009). The model method: Singapore children's tool for representing and solving algebraic word problems. *Journal for Research in Mathematics Education*, 40(3), 282–313.
- Polotskaia, E., & Savard, A. (2021). Some multiplicative structures in elementary education: a view from relational paradigm. *Educational Studies in Mathematics*, 106(3), 447-469.
- Schwartz, Judah. (1996). *Semantic Aspects of Quantity*.
- Verschaffel, L., Dooren, W., Greer, B., & Mukhopadhyay, S. (2010). Reconceptualising word problems as exercises in mathematical modelling. *Journal für Mathematik-Didaktik*, 31(1), 9–29.
- Weaver, J. F. (1973). The symmetric property of the equality relation and young children's ability to solve open addition and subtraction sentences. *Journal for Research in Mathematics Education*, 4(1), 45-56.
- Weaver, J.R. (1979), Addition, subtraction and mathematical operation. A paper presented at the Wingspread Conference, Racine, Wisconsin
- Weaver, J. F. (1982). Interpretations of number operations and symbolic presentations of addition and subtraction. In T. P. Carpenter, J. M. Moser, & T. A. Romberg (Eds.), *Addition and subtraction: A cognitive perspective* (pp. 60-66). Hillsdale, NJ: Erlbaum.

## ERROR PATTERNS OF AT-RISK STUDENTS IN SOLVING ADDITIVE WORD PROBLEMS BEFORE AND AFTER THE INTERVENTION

Yan Ping Xin  
Purdue University  
yxin@purdue.edu

Soo Jung Kim  
Purdue University  
kim2358@purdue.edu

Jingyuan Zhang  
Purdue University  
zhan3598@purdue.edu

Qingli Lei  
University of Illinois at  
Chicago  
qli@uic.edu

Büşra Yılmaz Yeniöglu  
Purdue University  
busrayenioglu@gmail.com

Samed Yenioglu  
Indiana University  
samedyenioglu@gmail.com

Signe Kastberg  
Purdue University  
skastber@purdue.edu

Bingyu Liu  
Jiaxing University  
beryl\_yu@qq.com

Xiaojun Ma  
The Altamont School  
busrayenioglu@gmail.com

*Students with learning disabilities/difficulties in mathematics often apply ineffective procedures to solve word problems. Given that current mathematics curriculum standards emphasize conceptual understanding in problem solving as well as higher-order thinking and reasoning, the purpose of this study was to evaluate the impact of a model-based problem-solving (MBPS) intervention program on elementary students' word problem-solving performance through analyzing the error patterns. Results indicate that after the MBPS intervention, participants significantly improved their problem-solving performance and made less errors on solving problems across a range of problem situations. Implications of the study will be discussed in the context of National Council of Teachers of Mathematics' calling for teaching big ideas to help students develop a deep understanding of mathematics knowledge.*

Keywords: Mathematical Representations, Number Concepts and Operations, Problem Solving, Special Education

### Introduction

According to the Nation's Report Card (National Assessment of Educational Progress [NAEP], 2022), mathematics scores of all students declined when compared to 2020. Particularly, lower performing students exhibited greater achievement decline than their average or high performing peers based on 2022 long-term trend mathematics assessments for age 9 students. Currently, majority (84%) of American 4<sup>th</sup> graders with disabilities performed below the proficiency level. By 8<sup>th</sup> grade, 93% of the students with disabilities performed below the Proficiency level (NAEP, 2022). In fact, students with learning disabilities or difficulties in mathematics (LDM) lag well behind their peers from very early on in their educational trajectory ; they often continue to fall further behind as they transition from elementary to secondary schools (Carcoba Falomir, 2019). These findings present a pressing issue for all teachers and educators, because legal mandates (e.g., Every Students Succeeds Act, 2015) and current standards (e.g., Common Core State Standards Initiative [CCSSI], 2012; National Council of Teacher of Mathematics [NCTM], 2000) require that **all** students, *including students with learning difficulties*, in the US be taught to high academic standards that will prepare them to succeed in college and careers. As mathematical problem solving is an important part of school

mathematics (Verschaffel et al., 2020), it is imperative that all students achieve proficiency in word problem solving.

Students with LDM often treat word problems mechanically and apply ineffective procedures such as searching for keywords to identify the operation. That is, they focus on whether to add, subtract, multiply or divide rather than whether or how the problem makes sense. When encountering a word problem, they often just find the numbers in the problem and apply an operation without comprehending the problem and understanding the mathematical relations in the word problem (Xin, 2008, 2007). On the other hand, NCTM (2000) and Common Core State Standards (CCSSI, 2012) both emphasize mathematical thinking and reasoning. Specifically, NCTM is calling for teaching “big ideas,” which is defined as “mathematical statements of overarching concepts that are central to a mathematical topic and link numerous smaller mathematical ideas into a coherent whole” (Caldwell, et al., 2011, p. 9). Mathematical big ideas draw students’ attention to fundamental concepts, link small fact/ideas together, and connect previously learned ideas to new concepts. As such, teaching big ideas can help students develop a deep understanding of mathematics knowledge (Caldwell et al., 2011).

### **Conceptual Framework: Perspectives in Math Word Problem Solving**

According to existing literature in mathematics education (de Corte & Verschaffel, 1987; Carpenter, & Moser, 1984), the semantic structure of the word problem significantly influenced young children problem solving strategies. Children apply a range of addition and subtraction strategies even before the formal education in the elementary school. Therefore, it is suggested that early math education pay attention to selection or construction of elementary arithmetic word problems. That is, it is preferred that the problems present in the textbook would reinforce the use of children’s pre-existing knowledge or make that knowledge useful and necessary. As such there are significant amount of research in psychology of math education and recently in the field of special education promoting the instruction or intervention that focuses on the semantic structure of the word problems (Verschaffel et al., 2020).

On the other hand, scholars in mathematics education support the emphasis on mathematical relations identified in word problem solving (Davydov, 1982; Savard & Polotskaia; 2017, Xin et al., 2008). To promote students conceptual understanding of mathematical problem solving, specifically designed tasks were presented to students for them to represent a range of variously constructed word problems in a cohesive mathematical model equation promoting students’ construction of the mathematical relationships (Savard & Polotskaia, 2017; Xin et al., 2008). These researchers argue that mathematical relations play a crucial role in mathematical learning; however, too often students were taught to rely on keywords or semantic feature of the word problem story in determining the operation for the calculation of the answer (Savard & Polotskaia, 2017).

### **Effective Instructional Features in Teaching Word Problem Solving**

To guide instructional practice for students with LDM, Institute of Educational Science (IES)’s latest Practice Guide suggests the use of *systematic instruction* through sequencing, using worked-out examples, providing visual and verbal supports, and teaching of precise mathematical language (Fuchs et al., 2021). As the outcome of a collaborative work from math education and special education we have developed, with the support from the National Science Foundation (NSF, Xin et al., 2015)<sup>i</sup>, a web-based computer tutor that emphasizes *conceptual model-based problem solving* (MBPS) (Xin, 2012). The MBPS program integrates research-based practices that are consistent with the latest IES practice guide (Fuchs et al., 2021),

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

including concrete (e.g., virtual manipulatives), representational (e.g., bar models), and abstract (mathematical model equations) instructional sequences, visual and linguistic support, and teaching of precise mathematical language.

Specifically, the MBPS program pay particular attention to making the reasoning behind mathematics explicit to students through nurturing fundamental mathematical ideas (e.g., the conception of number as a composite unit) that would lead to the additive reasoning (e.g., part and part makes up the whole or  $P+P=W$ ). As part of the MBPS approach, linguistic and visual support were integrated parts of the MBPS program. For instance, *Word Problem Story Grammar* prompting questions (Xin et al., 2008, 2012) were used as a series of linguistic scaffolds to facilitate students’ representation of word problems in mathematical model equations (e.g., part + part = whole) for accurate problem solving. In addition, visual were used to support student understanding of the part-part-whole (PPW) mathematical model. For instance, “name tags” generating from specific problem situations were used to denote each of the elements in the PPW diagram equation to facilitate the representation process.

Empirical studies have shown the effectiveness of MBPS in improving students’ word problem solving performance (Witzel et al., 2022; Xin et al., 2011, 2017, 2023). To understand the impact of the MBPS on students’ conceptual understanding of additive word problem solving, this study analyzed participating students’ error pattern when solving addition and subtraction word problems before and after the MBPS intervention. Analysis of error patterns assists identifying areas of instructional needs (Kingsdorf & Krawec, 2014). Specifically, we focused on the participating students’ success or error pattern as well as strategy use in solving additive word problems.

## Method

### Participants and Setting

This study was conducted within the larger context of the NSF-funded project<sup>1</sup> (Xin et al., 2015-2020). Participants included in this study were nine third graders with LDM from one elementary school in the mid-western United States. See Table 1 for demographic information of the participants.

**Table 1: Participant Demographics**

Variable/Name	S1	S2	S3	S4	S5	S6	S7	S8	S9
Gender	Female	Female	Male	Male	Male	Female	Male	Male	Female
Ethnicity	Hispanic	Black	Hispanic	Hispanic	White	White	White	Black	Black
Age(year-month)	8-11	9-5	8-8	8-8	8-6	8-3	8-9	9-1	8-1
Socioeconomic status	Low	Low	Low	Low	Low	Low	Low	low	Low
Years in special education	0	0	1 (LD)	0	3(LD)	0	0	3(LD)	0
RtI support	Tier 2	Tier 2	Tier 3	Tier 2	Tier 3	Tier 2	Tier 2	Tier 3	Tier 2
% in general Class	100%	100%	>80	100%	>80	100%	100%	>80	100%
OtisLennon/Full-scale	No test	76	77	61	90	79	72	82	No test
Verbal		74	79	73	92	81	73	77	
Performance		79	77	50	89	79	75	87	

Note. LD= learning disabilities; RtI = Response-to-Intervention; Tier = RtI Tiers

## Measures

To measure participating students’ performance before and after the intervention, we used a researcher-developed 14-item WPS criterion test (Xin et al., 2020). It involves eight part-part-whole problems (including *combine*, *change/join-in*, and *change/separate* story situations) with either the part or the whole as the unknown, and six additive compare problems (including “more than...” or “less than...” story situations) with either the *compared quantity*, *referent quantity*, or the *difference* as the unknown. The WPS test was designed in alignment with the NCTM and Common Core standards (CCSSI, 2012), which emphasize varying construction of word problems for assessing conceptual understanding of mathematics problem solving. Cronbach's Alpha of the criterion test was .86 and the test–retest reliability was .93 (Xin et al., 2020). Table 1 presents sample word problems included in the WPS Test.

**Table 1: Sample Word Problem Situations in the Test**

<i>Combine</i>	
“Whole” unknown CMB-W	Mr. Samir had 61 flashcards for his students. Mrs. Jones had 27 flashcards. How many flashcards do they have altogether?
“Part” unknown CMB-P	Together, Jamie and Daniella have 92 books. Jamie says that he has 57 books. How many books does Daniella have?
<i>Change-join in</i>	
“Whole” unknown ending amount CJ-WE	Leo has 76 math problems for homework. His Dad gives him 22 more problems to solve. How many math problems in total does Leo need to solve?
“Part” unknown change amount CJ-PC	Sam had 8 candy bars. Then Lucas gave him some more candy bars. Now he has 15 candy bars. How many candy bars did Lucas give Sam?
“Part” unknown beginning amount CJ-PB	Selina had several comic books. Then Andy gave her 40 more comic books. Now, Selina has 67 comic books. How many comic books did Selina have in the beginning?
<i>Change-separate</i>	
“Whole” unknown beginning amount CS-WB	Alex had many dolls. Then she gave away 12 of her dolls to her sister. Now Alex has 26 dolls. How many dolls did Alex have in the beginning?
“Part” unknown ending amount CS-PE	Davis had 62 toy army men. Then, one day he lost 29 of them. How many toy army men does Davis have now?
“Part” unknown change amount CS-PC	Ariel had 41 worms in a bucket for her fishing trip. She used many of them on the first day of her trip. The second day she had only 24 worms left. How many worms did Ariel use on the first day?
<i>Compare-more</i>	
Compared quantity unknown CM-C	Denzel has 28 toy cars. Gabrielle has 15 more toy cars than Denzel. How many toy cars does Gabrielle have?
Referent quantity unknown CM-R	Tiffany collects bouncy balls. As of today, she has 42 of them. Tiffany has 20 more balls than Elise. How many balls does Elise have?
Difference unknown CM-D	Logan has 52 rocks in his rock collection. Emanuel has 12 rocks in his collection. How many more rocks does Logan have than Emanuel?
<i>Compare-less</i>	
Referent quantity unknown CLS-R	Ellen ran 62 miles in one month. Ellen ran 29 fewer miles than her friend named Cooper. How many miles did Cooper run?
Compared quantity unknown CLS-C	Kelsie said she had 82 apples. If Lee had 32 fewer apples than Kelsie, how many apples did Lee have?
Difference unknown CLS-D	If Laura has 41 candy bars and another student named Paula has 70 candy bars, how many fewer candy bars does Laura have than Paula?

As for scoring, one point was given if a correct answer was given to a problem. In the case that the answer to the problem was incorrect, however, the algorithm or model equation was correctly set up, half point was awarded.

### MBPS Intervention

MBPS is web-based interactive tutoring program. Sessions were monitored by supervisors. The participants worked with the MBPS computer tutor “one-on-one” during the afterschool program, Monday through Thursday, for a total of 18 sessions (ranged from 15 to 23 across different individuals) with each session lasting for about 25 minutes. The session supervisor helped each of the participants log onto the MBPS computer tutor program in the beginning of each of the sessions. Then the student followed the direction of the computer tutor and engaged in the activities in Modules A through C. **Module A** engaged students in a series of activities involving the use of virtual manipulatives such as unifix cubes, to nurture fundamental mathematical ideas that are crucial for the development of additive reasoning and problem solving. It focuses on students’ conception of “number as the composite unit” (e.g., any number that is larger than 1 can be decomposed into a combination of two numbers, for instance, 4 is made of 3 and 1, or 2 and 2, or 1 and 3) and the development of multi-digit numbers as quantities of tens and ones. The aim of Module A is to challenge children’s counting acts to provoke changes in their mental operations, which will bring about the development of the composite unit. **Module B** engaged students in representing and solving various *combine* and *change* problem types (see Table 2 for problem types) using one cohesive mathematical model equation (part and part makes up the whole, or  $P + P = W$ ). **Module C** engaged students in representing and solving a range of *additive compare* problems using the same model equation, however, the denotations of each of the elements in the PPW diagram equation were adapted to the problem situations accordingly. Name tags were used to help students anchor “who has more?” which would be the “bigger” quantity or the “whole,” and “who has less?” which would be the “smaller” quantity or the “part.” After solving the comparison problems, students were given opportunities to represent and solve mixed additive word problems to further strengthen students’ construction of the mathematical model,  $P + P = W$ , for generalized problem solving. See Figure 1 for Sample screenshots of Module A (left column) and Module C (right column).

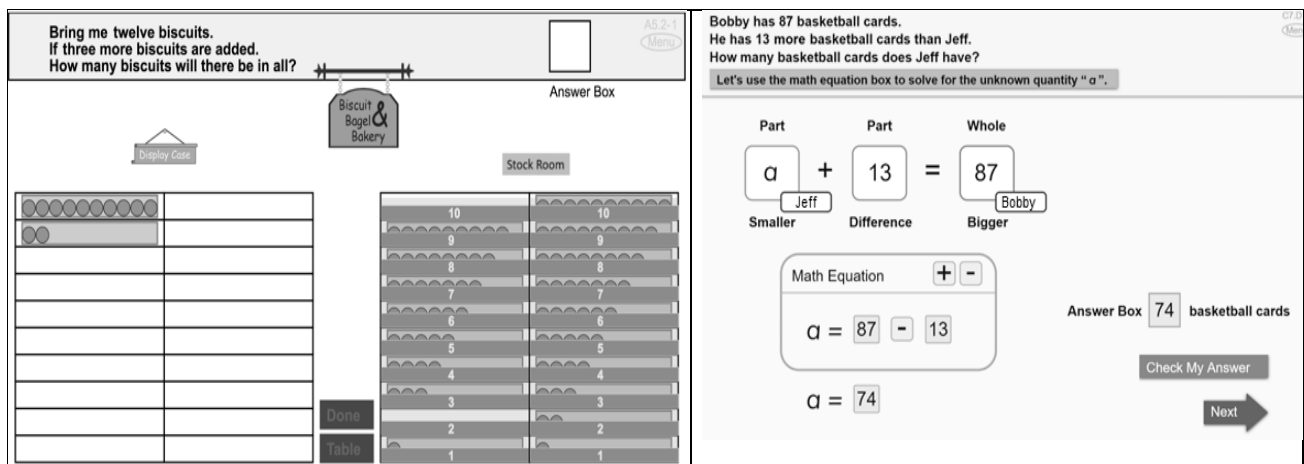
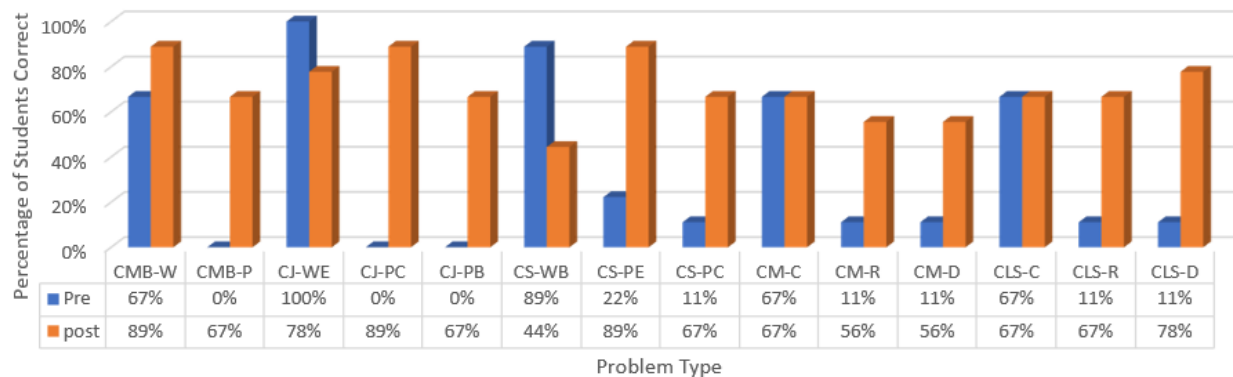


Figure 1: Screenshots of Modules A & C of the MBPS tutor (©Xin, Kastberg, & Chen,



### Results and Discussion

The purpose of this study was to explore the impact of web-based MBPS computer tutor on students' error patterns on solving a range of additive word problems. Overall, students significantly improved their performance from an average of 35.7% correct on pretest to an average of 71.4% correct on posttest. As for error patterns, results indicate that during the pretest, most of the participating students were just grabbing the numbers in the problem and adding them all together regardless of how the word problem was constructed and/or the mathematical relations described in the problem. After the MBPS intervention, students made their attempt to represent information, based on their comprehension of the problem, in the PPW diagram equation and then solve for the problem. Figure 2 presents percentage of students who correctly solved each of the 14 problems (the problem type is noted at the bottom of the bars, please refer to Table 1 for coding of problem types) before and after the intervention.



**Figure 2: Percentage of Students solved each of the problem types before (blue/dark color) and after (orange/light color) the MBPS Intervention**

As shown in Figure 2, after working with MBPS tutor, participants made less errors on solving problems across a range of problem situations (over 71% of all problem situations) except for the following two problem situations (a) *change-join*, ending total (whole) as the unknown (CJ-WE, e.g., Leo has 76 math problems for homework. His dad gives him 22 more problems to solve. How many math problems in total does Leo need to solve?), (b) Change-separate, “whole”/beginning amount unknown (CS-WB, e.g., Alex had many dolls. Then she gave away 12 of her dolls to her sister. Now Alex has 26 dolls. How many dolls did Alex have in the beginning?). Both problem types require adding the two given numbers to get the total (or “whole”). Upon careful examining students’ work in pretests, it was discovered that, during the pretest, most of the students simply took the two numbers given in the problem and added them together to get the answer. This senseless strategy would win them the luck in solving problems with above two problem situations. In addition, problems such as *change-join* with ending total (or “whole”) as the unknown (CJ-WE) are the easiest problem situations as students could either relying on the *keyword* (“more” or “total” signifies an operation of addition) to solve the problem, or blindly using addition to solve all problems would win them the luck. For the rest of the problem types, particularly those with missing part or missing addend problems (e.g., CMB-P, CJ-PC), problems with the beginning amount as the unknown (CJ-PB), comparison problems

with the referent quantity as the unknown (e.g., CM-R, CLS-R, the so-called “inconsistent language” problems, Lewis & Mayer, 1987; e.g., Ellen ran 62 miles in one month. Ellen ran 29 fewer miles than her friend named Cooper. How many miles did Cooper run?), it seems that the MBPS strategy benefited the students in solving these problems as shown in Fig.2 (significantly improved percentage of students who solved the problem correctly). However, for comparison problems with “consistent language” (e.g., Denzel has 28 toy cars. Gabrielle has 15 more toy cars than Denzel. How many toy cars does Gabrielle have?), students’ performance stayed the same after the intervention. It should be noted that to solve *compare* problems with “consistent language,” the “keyword” strategy would get them the correct answer, although it might involve no mathematical reasoning or understanding of the mathematical relations depicted in the problem. Figure 3 presents sample student work before (left column) and after the MBPS intervention (right column). As shown in Figure 3 (left column), during the pretest, the students simply added two numbers together for the answer regardless of the problem situations. After the intervention (right column), students used “name tags” to represent the information in the PPW diagram equation and then solve the problem.

Sample Student Work during Pretest	Sample Student Work after MBPS Intervention
<p>Together, Jamie and Daniella have 92 books. Jamie says that he has 57 books. How many books does Daniella have?</p> $\boxed{92} + \boxed{57} = \boxed{149}$ $\begin{array}{r} 92 \\ + 57 \\ \hline 149 \end{array}$ <p>ANSWER: <u>149 books</u></p>	<p>4. Davis had 62 toy army men. Then, one day he lost 29 of them. How many toy army men does Davis have now?</p> $\boxed{29} + \boxed{33} = \boxed{62}$ <p>Part      Part      Whole</p> <p>ANSWER: <u><del>33 toy army men</del></u></p>
<p>10. Tiffany collects bouncy balls. As of today, she has 42 of them. Tiffany has 20 more balls than Elise. How many balls does Elise have?</p> $\boxed{42} + \boxed{20} = \boxed{62}$ $\begin{array}{r} 42 \\ + 20 \\ \hline 62 \end{array}$ <p>ANSWER: <u>62 bouncy balls</u></p>	<p>10. Tiffany collects bouncy balls. As of today, she has 42 of them. Tiffany has 20 more balls than Elise. How many balls does Elise have?</p> $\boxed{a} + \boxed{20} = \boxed{42}$ <p>S      a      T</p> <p>ANSWER: <u><math>a = 42 - 20 = 22</math></u></p>
<p>9. Denzel has 28 toy cars. Gabrielle has 15 more toy cars than Denzel. How many toy cars does Gabrielle have?</p> $\boxed{28} + \boxed{15} = \boxed{43}$ $\begin{array}{r} 28 \\ + 15 \\ \hline 43 \end{array}$ <p>ANSWER: <u>43 toy cars</u></p>	<p>12. Ellen ran 62 miles in one month. Ellen ran 29 fewer miles than her friend named Cooper. How many miles did Cooper run?</p> $\boxed{62} + \boxed{29} = \boxed{91}$ <p>S      D      B</p> <p>ANSWER: <u><math>a = 62 + 29 = 91</math></u></p>

Figure 2: Sample Student Work Before and After the MBPS Intervention

## Implications for Practice

The MBPS intended to teach “big ideas” (e.g., “part and part makes up the whole”) in additive word problem solving to promote generalized problem-solving skills. The analysis of success or error pattern shows that participating students improved on solving most of the problems after the MBPS intervention. On the other hand, it should be noted that when teaching students a new strategy, perhaps it won’t be like switching a light bulb—“turning on” the new strategy and “turning off” the old strategy. Often there might be a delay of the use of the newly learned strategy or a mix-up in the use of newly learned strategy and the old strategy (Zhang et al, 2013). As teachers /educators make their effort to promote students’ conceptual understanding of word problem solving and make connections between mathematical ideas, it is important to connect the new concepts to students’ existing knowledge. Students should be provided with abundant learning opportunities, through teachers’ strategically designed learning tasks, for them to experience the advantages and/or the power of the new strategy, for instance, the MBPS which is applicable to solve a range of additive word problems, and therefore “undo” the existing “robust” however “ill” conceived strategies such as the keyword strategy.

## Acknowledgments

i) This research was partially supported by the National Science Foundation, under grant #1503451. The opinions expressed do not necessarily reflect the views of the Foundation.

©The screenshots presented in Figure 1 are copyrighted by the COMPS-RtI project (Xin et al., 2015). All rights reserved. Therefore, reproduction, modification, storage, in any form or by any means, is strictly prohibited without prior written permission from the Project Director Dr. Xin @yxin@purdue.edu.

## References

- Caldwell, J. H., Karp, K., & Bay-Williams, J. M. (2011). Developing essential understanding of addition and subtraction for teaching mathematics in prekindergarten–grade 1 (Essential understanding series). Reston, VA: National Council of Teachers of Mathematics.
- Carpenter, T. P., & Moser, J. M. (1984). The acquisition of addition and subtraction concepts in grades one through three. *Journal for Research in Mathematics Education*, 15(3), 179–202 <https://doi.org/10.2307/748348>
- Carcoba Falomir, G. A. (2019). Diagramming and algebraic word problem solving for secondary students with learning disabilities. *Intervention in School and Clinic*, 54(4), 212–218. <https://doi.org/10.1177/1053451218782422>
- Common Core State Standards Initiative (2012). Introduction: Standards for mathematical practice. Retrieved from <http://www.corestandards.org/the-standards/mathematics>
- Davydov, V. V. (1982). Psychological characteristics of the formation of mathematical operations in children. In T. P. Carpenter, J. M. Moser & T. A. Romberg (Eds.), *Addition and subtraction: cognitive perspective* (pp. 225–238). Hillsdale: Lawrence Erlbaum Associates.
- de Corte, E., & Verschaffel, L. (1987). The effect of semantic structure on first graders' strategies for solving addition and subtraction word problems. *Journal for Research in Mathematics Education*, 18(5), 363–381. <https://doi.org/10.2307/749085>
- Fuchs, L. S., Newman-Gonchar, R., Schumacher, R., Dougherty, B., Bucka, N., Karp, K. S., ... Morgan, S. (2021). *Assisting students struggling with math: Intervention in the elementary grades (WWC 2021006)*. Washington, DC: National Center for Education Evaluation and Regional Assistance (NCEE), Institute of Education Sciences, U.S. Department of Education. Retrieved from <http://whatworks.ed.gov/>.
- Kingsdorf, S., & Krawec, J. (2014). Error Analysis of Mathematical Word Problem Solving Across Students with and without Learning Disabilities. *Learning Disabilities Research & Practice (Wiley-Blackwell)*, 29(2), 66–74. <http://10.04.87/ldrp.12029>
- Lewis, A. B., & Mayer, R. E. (1987). Students’ miscomprehension of relational statements in arithmetic word problems. *Journal of Educational Psychology*, 79, 361-371.

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (Vol. 2)*. University of Nevada, Reno.

- National Council of Teachers of Mathematics (2000). Principles and standards for school mathematics. Author.
- National Assessment of Educational Progress result (NAEP, 2019). Extracted from <https://www.nationsreportcard.gov/mathematics/nation/scores/?grade=4>
- Savard, A., Polotskaia, E. (2017). Who's wrong? Tasks fostering understanding of mathematical relationships in word problems in elementary students. *ZDM Mathematics Education* 49, 823–833 (2017). <https://doi.org/10.1007/s11858-017-0865-5>
- Verschaffel, L., Schukajlow, S., Star, J., & Van Dooren, W. (2020). Word problems in mathematics education: A survey. *ZDM*, 52(1), 1–16. <https://doi.org/10.1007/s11858-020-01130-4>
- Witzel, B. S., Myers, J. A., & Xin, Y. P. (2021). Conceptually intensifying word problem solving for students with math difficulties. *Intervention in School and Clinic*. <https://doi.org/10.1177/105345122111047580>
- Xin, Y. P. (2012). Conceptual model-based problem solving: Teach students with learning difficulties to solve math problems. Rotterdam, Netherlands: Sense. doi: 10.1007/978-94-6209-104-7
- Xin, Y. P. (2008). The effect of schema-based instruction in solving word problems: An emphasis on pre-algebraic conceptualization of multiplicative relations. *Journal for Research in Mathematics Education*. 39, 526-551. <http://www.jstor.org/stable/40539313>
- Xin, Y. P. (2007). Word-Problem-Solving Tasks Presented in Textbooks and Their Relation to Student Performance: A Cross-Curriculum Comparison Case Study. *The Journal of Educational Research*, 100, 347-359.
- Xin, Y. P., Kastberg, S., & Chen, V. (2015). **Conceptual Model-based Problem Solving: A Response to Intervention Program for Students with Learning Difficulties in Mathematics (COMPS-RtI). National Science Foundation (NSF) funded project,**
- Xin, Y. P., Kim, S. J., Lei, Q., Liu, B., Wei, S., Kastberg, S. E., Chen, Y-J (2023). The Effect of Model-based Problem Solving on the Performance of Students Who are Struggling in Mathematics. *The Journal of Special Education*. <https://doi.org/10.1177/00224669231157032>
- Xin, Y. P., Kim, S. J., Lei, Q., Wei, S., Liu, B., Wang, W., Kastberg, S., Chen, Y., Yang, X., Ma, X., Richardson, S. E. (2020). The Impact of a Conceptual Model-based Intervention Program on math problem-solving performance of at-risk English learners. *Reading and Writing Quarterly: Overcoming Learning Difficulties*, 36(2), 104-123, <https://www.tandfonline.com/doi/full/10.1080/10573569.2019.1702909>
- Xin, Y. P., Tzur, Si, L. Hord, C., Liu, J., Park, J. Y. (2017). An intelligent tutor-assisted math problem-solving intervention program for students with learning difficulties. *Learning Disability Quarterly*, 40(1), 4-16. doi:10.1177/0731948716648740
- Xin, Y. P., Wiles, B., & Lin, Y. (2008). Teaching conceptual model-based word-problem story grammar to enhance mathematics problem solving. *The Journal of Special Education*, 42, 163-178. DOI: 10.1177/0022466907312895
- Xin, Y. P., Zhang, D., Park, J. Y., Tom, K., Whipple, A., & Si, L. (2011). A Comparison of Two Mathematics Problem-Solving Strategies: Facilitate Algebra-Readiness. *The Journal of Educational Research*, 104, 381-395. doi.org/10.1080/00220671.2010.487080
- Zhang, D, Xin, Y. P., & Si, L. (2013). Transition from Intuitive to Advanced Strategies in Multiplicative Reasoning for Students with Math Difficulties. *The Journal of Special Education*, 47(1), 50-64.

# EXPLORING THE ASSOCIATION BETWEEN UPPER ELEMENTARY SCHOOL STUDENTS' MATURE NUMBER SENSE AND GRADE-LEVEL MATHEMATICS ACHIEVEMENT

Patrick K. Kirkland  
University of Notre Dame  
pkirklan@nd.edu

Claire Guang  
University of Notre Dame  
cguang@nd.edu

Nicole M. McNeil  
University of Notre Dame  
nmcneil@nd.edu

*Students with mature number sense make sense of numbers and operations, use reasoning to notice patterns, and flexibly select the most effective and efficient problem-solving strategies (McIntosh et al., 1997; Yang, 2005). Despite being highlighted in national standards and policy documents (CCSS, 2010; NCTM, 2000), the association between students' mature number sense and other important outcomes is not well specified. For example, how does students' mature number sense relate to their grade-level mathematics achievement? We analyzed 153 upper elementary school students' scores on measures of mature number sense, fraction and decimal knowledge, multiplication fluency, and grade-level mathematics achievement. We found mature number sense to be measurably distinct from their fraction and decimal knowledge and uniquely associated with students' grade-level mathematics achievement.*

Keywords: Number Concepts and Operations; Cognition; Assessment; Rational Numbers

In the late 1980's, students' "number sense" emerged as a principal goal of mathematics education. While many high-quality mathematics teachers were certainly already pushing students to make sense of numbers and operations before then, "number sense" was not yet a commonly used expression. Specifically in the United States at this time, national curriculum frameworks (e.g., National Council of Teachers of Mathematics (NCTM), 1989) and government commissioned reports (e.g., National Research Council, 1989) emphasized "number sense" as a core objective of K-12 mathematics education. Students with mature number sense exhibit the disposition to *make sense of numerical situations* and use a rich conceptual understanding of number and operations to flexibly solve problems (McIntosh et al., 1997).

Despite this emphasis in the field, there is little evidence in the literature on how students' mature number sense develops over time or relates to central goals in mathematics education. For example, how do students' levels of mature number sense relate to their grade-level mathematics achievement? In prior work, we began to address this gap by examining how middle school students' mature number sense related to other theoretically related constructs (Kirkland et al., 2022). We aim to extend that work here to upper elementary school students, testing the hypothesis that upper elementary school students' mature number sense is both distinct from their fraction and decimal knowledge and mathematics achievement and uniquely associated with grade-level mathematics achievement.

## Background and Theoretical Framework

How number sense has been defined, operationalized, and measured looks very different across the disciplines of cognitive and developmental psychology, mathematics education, neuroscience, and special education (summarized in Whitacre et al., 2020). In this project, we want to clarify that our construct of interest is what Whitacre et al. (2020) have termed *mature number sense*, which we define in line with McIntosh et al. (1992) as "a person's general understanding of number and operations along with the ability and inclination to use this

understanding in flexible ways.” This focus is distinct from research on approximate number sense (e.g., Dehaene, 2001) or early number sense or numeracy (e.g., Jordan et al., 2009).

As a construct, mature number sense has often been subdivided into components (McIntosh et al., 1997; NCTM, 1989; Reys et al., 1999; Yang & Lin, 2015). Building from this work and in consultation with expert mathematics teachers, teacher educators, and mathematics education researchers, we specified a four-component framework of mature number sense for middle school as well as upper elementary students (Kirkland et al., *in press*) as follows:

3. Understanding basic number concepts and number magnitude: Strong mature number sense is characterized by a rich conceptual understanding of fractions, decimals, and whole numbers. Students use their understanding of place value and rational number magnitude to efficiently estimate results using concepts such as unit fractions.
4. Using multiple representations of a number: Strong mature number sense is characterized by proficiency in translating among multiple representations of rational numbers efficiently and flexibly to solve problems. Students use this understanding to translate between representations such as Arabic numerals and the number line.
5. Understanding the effect of arithmetic operations on numbers: Strong mature number sense is characterized by recognizing how the four core arithmetic operations affect whole numbers as well as fractions and decimals. Students demonstrate understanding that patterns observed with operations with whole numbers may not hold true for numbers between 0 and 1. Students use this understanding to efficiently estimate computational results and ensure the results make sense given the relationship between operations and rational numbers.
6. Understanding mathematical equivalence: Strong mature number sense is characterized by understanding the equal sign as a relational symbol, reflecting that the two sides of an equation are equal. Rather than approaching equations with an “operational” approach, students recognize patterns across the equal sign and use this relational thinking to flexibly solve problems (c.f. Jacobs et al., 2007).

We hypothesized, similar to Yang (2019), that mature number sense would be an overarching hierarchical latent construct, with the four components each theoretically related to it. That is, a student’s understanding of mathematical equivalence reflects their mature number sense as well as a more specific relational understanding of the equal sign. In our initial development of a brief assessment of mature number sense for middle school students (Kirkland et al., *accepted*) the structure of the response data reflected this hypothesized framework. A bifactor model with each component as a specific factor best fit the response data over theoretically related models. We then created an upper elementary form of the brief assessment, and the response data in our elementary validation study reflected the same structure. Here we used this upper elementary form of the assessment to further study mature number sense in 3<sup>rd</sup>-5<sup>th</sup> grade students.

### **Rationale for Current Study**

Mature number sense has been measured and studied internationally in the last three decades (summarized in Whitacre et al., 2020), most frequently in a single time point and with upper elementary and middle school students. Very few of these studies have analyzed any potential overlap with potentially related psychological constructs. For reference, Yang et al. (2008) is the only prior study we are aware of that examined mature number sense’s association with grade-

level mathematics achievement, and this was with 5<sup>th</sup> grade Taiwanese students. As a first step toward addressing this gap, we analyzed middle school students' ( $N = 129$ ) scores on measures of mature number sense, fraction and decimal knowledge, grade-level mathematics achievement, and addition fluency (Kirkland et al., 2022). We hypothesized that students with strong mature number sense are flexible problem solvers with a deep understanding of number and operation and, thus, that they would do well on more “traditional” school assessments. In contrast, throughout the literature, mature number sense is often juxtaposed with standardized school mathematics achievement and the algorithmic, overly procedural mathematics instruction common in schools (Whitacre et al., 2020). We found mature number sense to be measurably distinct from their fraction and decimal knowledge and uniquely associated with students' grade-level mathematics achievement. However, one could argue that this observed pattern may be drastically different in upper elementary (e.g., 3<sup>rd</sup>-5<sup>th</sup> grade) than in middle school (e.g., 6<sup>th</sup>-8<sup>th</sup> grade). In upper elementary, students are first introduced to multiplication and division as well arithmetic operations as fractions and decimals (CCSS, 2010). In addition, by the end of grade 3, students are expected to recall fluently all single-digit multiplication facts. One therefore might expect 3<sup>rd</sup>-5<sup>th</sup> grade students' multiplication fluency or basic fraction and decimal computation to be more aligned with grade-level mathematics achievement than with mature number sense. However, we are not aware of a study with upper elementary school students that examined the association between their mature number sense and either their fraction computation or their fluency with single-digit multiplication facts or how these related constructs may be associated with their overall grade-level mathematics achievement.

In this study, we examined how US upper elementary school students' mature number sense related to their grade-level mathematics achievement, controlling for a variety of other potentially related factors, including fraction knowledge and multiplication fluency. Our research questions were as follows: 1) Is the mature number sense of upper elementary school students measurably distinct from their fraction knowledge as well as their grade-level mathematics achievement? and 2) Is mature number sense in upper elementary school uniquely associated with students' standardized grade-level mathematics achievement, even after controlling for their fraction knowledge and multiplication fluency?

## Methods


### Participants

One hundred fifty-three upper elementary school students ( $N = 47$  in 3<sup>rd</sup> grade,  $N = 54$  in 4<sup>th</sup> grade,  $N = 52$  in 5<sup>th</sup> grade) from schools surrounding a university in the midwestern United States participated (48% identified as female; 81% identified as White, 12% Multiracial, 8% Hispanic, and 2% Black). Sessions took place after school hours in university building. Students were participants in timepoint 1 of a larger (still ongoing) longitudinal study. They completed eight measures over the course of two 45-minute sessions, scheduled about a week apart (median of 7 days between sessions). For this analysis, we focused on the four measures described below, but note that the results reported hold up to robustness checks that include all measures as well.

### Measures and Procedure

**Brief Assessment of Mature Number Sense: Elementary Form (Kirkland et al., *accepted*).** This is an electronic, 24 item multiple-choice test of students' mature number sense, aligned with our theoretical framework detailed above. Items differ from a traditional curriculum and are designed to specially assess students' number sense. Each item has a time limit of 60 seconds and students are not allowed to use paper and pencil to discourage the use of traditional

algorithms. Student’s total sum score is used in the analyses. In this study, students completed the upper elementary form of the brief assessment. This form includes 8 common linking items with the middle school form used in our prior work (Kirkland et al., 2022) This allows us to track students’ number sense development over time. Figure 1 includes an example item on the elementary form for each of the four components in our mature number sense framework. In the validation analysis, student scores on the elementary form brief assessment were reliable over time ( $r = 0.84$ ) and had evidence from student think-alouds, expert reviews, factor analyses, and item response theory analyses to support our validity argument.

Understanding the Effect of Operations on Numbers	Number Concepts and Magnitude
<p>What is true about the answer for <math>23 \div 4</math> (23 divided by 4)?</p> <p><input type="radio"/> The answer is in between 2 and 3</p> <p><input type="radio"/> The answer is in between 3 and 4</p> <p><input type="radio"/> The answer is in between 4 and 5</p> <p><input type="radio"/> The answer is in between 5 and 6</p>	<p>Which fraction is equivalent to (has the same value as) <math>\frac{9}{3}</math>?</p> <p><input type="radio"/> <math>\frac{1}{3}</math></p> <p><input type="radio"/> <math>\frac{10}{4}</math></p> <p><input type="radio"/> <math>\frac{6}{2}</math></p> <p><input type="radio"/> <math>\frac{4}{3}</math></p>
Multiple Representations of a Number	Mathematical Equivalence
<p>Which of the following numbers is the red arrow most likely pointing to on the number line below?</p>  <p><input type="radio"/> 100</p> <p><input type="radio"/> 250</p> <p><input type="radio"/> 500</p> <p><input type="radio"/> 900</p>	<p><math>34 + 19 = \underline{\quad} + 18</math></p> <p><input type="radio"/> 33</p> <p><input type="radio"/> 35</p> <p><input type="radio"/> 53</p> <p><input type="radio"/> 71</p>

**Figure 1: Sample Items Assessing Mature Number Sense Organized by Component**

**Massachusetts Comprehensive Assessment System (MCAS) Grade-Level Mathematics Test (2019).** Students completed the released 2019 MCAS paper test appropriate for their grade level. This is a freely available standardized test designed to assess student proficiency with grade-level mathematics standards. Student scores are converted to percent correct because the maximum possible correct differs by grade level. Students had no time limit to complete each section of the test and could use scratch paper but not calculators.

**Fraction and Decimals Assessment (adapted from the Rational Numbers Measure [Powell, 2014]).** This is a 28-item paper and pencil test of students’ knowledge of fractions, decimals, and percent. We adapted the Rational Numbers Measure (Powell, 2014) used in our prior work with middle school students to make it grade-level appropriate for 3<sup>rd</sup>-5<sup>th</sup> grade. To do so, we updated denominators to be those recommended by the Common Core State Standards (CCSS, 2010) for 4th grade, removed any percent or percentages question given that this content is not mentioned until 6th grade in CCSS, and reworded some word problems based on literacy levels for 3<sup>rd</sup>-5<sup>th</sup> grade. We chose to keep the bulk of the measure the same to allow for a more consistent comparison with our study of middle school students’ number sense. Students are asked to compare fractions, perform the four operations with both fractions and decimals, find



common denominators of fractions, and generate equivalent fractions. For example, one item was “ $1\frac{2}{8} + 3\frac{3}{8} = \underline{\hspace{2cm}}$ ” and another was “Write  $\frac{14}{100}$  as a decimal.” Students worked for 20 minutes or until they finished. They could use scratch paper but not calculators. Students received 1 point for each correct response.

**Multiplication Fluency Task (Burns et al., 2015; Nelson et al., 2013).** This measure includes all combinations of randomly presented single digit multiplication facts with the numbers 1-9. Students were tasked with solving as many correctly as they could in 1 minute. The order of the facts was predetermined randomly and then kept standard for all participants. Students received 1 point for each correct response.

### **Data Analyses**

To address the research question on whether mature number sense is measurably distinct from students’ general mathematics achievement, we first calculated the zero-order correlations between the constructs. We predicted mature number sense to be most closely related to students’ grade-level math achievement on the MCAS and scores on the Fraction and Decimal Assessment. We expected mature number sense to be least closely related to multiplication fluency. We then ran a series of partial correlation analyses. We analyzed the correlation between mature number sense and both other constructs after we control for students’ multiplication fluency scores. We then analyzed the associations between students’ grade-level math achievement, fraction and decimal computation, and mature number sense, controlling for each in turn. In each case, we predict the relationship between mature number sense and the other construct to remain significant, even after controlling for a strongly related third variable.

For additional evidence that mature number sense and fraction and decimal computation are distinct, we then conducted a series of factor analyses (Shaffer et al., 2016). We used a common method where we tested a constrained model with 2 latent factors set with a covariance equal to 1 and an unconstrained model where the 2 latent factors are allowed to freely covary with each other. If the unconstrained model provides a better fit through a RMSEA test that is significant, then the two measures of interest can be said to be distinct. We tested these two models using a chi-squared,  $\chi^2$ , difference test. Due to each grade’s different test for the MCAS, we are unable to perform this same analysis on grade-level mathematics achievement.

To address research question 2, we ran a partial correlation test on the association between grade-level mathematics achievement and fraction and decimal computation, controlling for mature number sense. We then ran a linear regression model, regressing mature number sense and other predictors measured in the study on students’ grade-level mathematics achievement.

### **Findings**

We first begin with the overall student performance on the measures in the study. Students solved on average 11.21 (47%) items correctly on our brief assessment of mature number sense ( $SD = 5.09$ ). Across the three grade levels, students solved on average 39% of the items correctly on the MCAS ( $SD = 0.22$ ). This differed dramatically by grade level, with 3<sup>rd</sup> graders outperforming ( $M = 50\%$  of items correct) both 4<sup>th</sup> graders ( $M = 35\%$ ) and 5<sup>th</sup> graders ( $M = 32\%$ ) on their respective grade level standards. Out of a maximum possible of 28 correct, students answered on average 4.43 items correct on the Fractions and Decimals measure ( $SD = 4.38$ ). While this number may appear very low as a percentage correct (16%), this measure was adapted from a measure designed to include many problems to help differentiate students’ fraction and

decimal knowledge up to the college level. Finally, on average, students answered correctly 9.08 ( $SD = 6.36$ ) single-digit multiplication problems on the multiplication fluency task.

To begin to address research question 1, the zero-order correlations between the measured constructs are summarized in Table 1 (\*  $p < 0.01$ , \*\* $p < 0.001$ ). As predicted, students' mature number sense scores correlated very highly with their fraction and decimal computation skills ( $r = 0.71$ ). However, the correlation with grade-level mathematics achievement ( $r = 0.46$ ) was much lower than expected given the observed correlation in middle school students ( $r = 0.76$ ). It also was lower than the correlation with multiplication fluency ( $r = 0.67$ ) and this difference was significant ( $t(152) = 2.81, p < .01$ ). Given the disparity observed between grade levels on the MCAS, we considered the correlation between the constructs for 4<sup>th</sup> and 5<sup>th</sup> grade students only. Here, the patterns were more in line with our predictions. Mature number sense was still strongly correlated with fraction and decimal computation ( $r = 0.70$ ) and multiplication fluency ( $r = 0.58$ ), but the correlation with grade-level achievement was stronger ( $r = 0.65$ ).

**Table 1: Correlation Between Measures in Study**

Construct	<i>M</i>	<i>SD</i>	1	2	3	4
1. Mature Number Sense	11.21	5.09	-			
2. MCAS – Grade-Level Achievement	39%	22%	0.46**	-		
3. Fraction and Decimal Computation	4.43	4.38	0.71**	0.30**	-	
4. Multiplication Fluency	9.08	6.36	0.67**	0.20*	0.63**	-

With all three grades included and controlling for students' multiplication fluency scores, students' mature number sense still correlated highly with their grade-level mathematics achievement ( $pr = 0.45, t(150) = 6.15, p < .001$ ) and their fraction and decimal computation ( $pr = 0.50, t(150) = 7.14, p < .001$ ). Relatedly, students' mature number still correlated highly with their multiplication fluency when controlling for either their grade-level mathematics achievement ( $pr = 0.66, t(150) = 10.83, p < .001$ ) or their fraction and decimal computation ( $pr = 0.40, t(150) = 5.33, p < .001$ ).

We then further examined the association between mature number and grade-level achievement. Controlling for students' fraction and decimal computation, their mature number sense scores still were significantly positively related with their grade-level achievement ( $r = 0.37, t(150) = 4.83, p < .001$ ). The same was true when controlling for grade-level achievement and examining mature number sense's relationship with fraction and decimal computation ( $r = 0.68, t(150) = 11.32, p < .001$ ). We would predict that if mature number sense and grade-level mathematics achievement are redundant constructs, this relationship would no longer be significant. Thus, mature number sense appears to be distinct from grade level achievement.

As discussed above, we then ran a series of confirmatory factor analyses for evidence on if mature number sense and fraction and decimal knowledge are distinct (Shaffer et al., 2016). These analyses provided evidence that the Fractions and Decimals Assessment and the Brief Assessment of Mature Number Sense measure distinct constructs,  $\chi^2$  difference = 67.03,  $p < 0.001$ ; *CIF* difference = 0.02 compared to benchmark of 0.002. This suggests the constructs, as measured in this study, are highly related, but not redundant.

To further examine the unique importance of mature number sense, we first tested the partial correlation between fraction and decimal computation and grade-level achievement.

Interestingly, when controlling for students' mature number sense, there is no longer a significant correlation between the two other constructs ( $r = -0.04$ ,  $t(150) = -0.50$ ,  $p = 0.62$ ). This is the same pattern we observed with middle school students as well, even with a weaker observed correlation between grade-level mathematics achievement and the other constructs with upper elementary school students.

We then ran a linear regression on students' grade-level mathematics achievement (Model  $R^2 = 0.23$ ). Results from the model are summarized in Table 2. Students' mature number sense was significantly positively related ( $B_{NS} = 2.51$ ,  $p < .001$ ) to students' grade-level mathematics achievement. However, students' fraction and decimal computation ( $B_{Fract} = 0.05$ ,  $p = 0.92$ ) and multiplication fluency ( $B_{Multi} = -0.66$ ,  $p = 0.06$ ) were not significantly related to achievement. To compare the relative importance of each regressor, we examined the semipartial correlation of each predictor in the model (Darlington & Hayes, 2016; Hayes & Rockwood, 2017). The semipartial correlation coefficient for mature number sense was significant ( $sr = 0.38$ ,  $t(149) = 4.95$ ,  $p < .001$ ). However, the semipartial correlation for neither fraction and decimal computation ( $sr = 0.01$ ,  $t(149) = 0.09$ ,  $p = 0.93$ ) nor multiplication fluency ( $sr = -0.14$ ,  $t(149) = -1.70$ ,  $p = 0.09$ ) were significant. We can interpret the ratio of the two semipartial correlations ( $0.38/0.01 = 38$ ;  $0.38/-0.14 = 2.7$ ) as a measure of the relative importance of each in explaining students' grade-level achievement. As another way to quantify mature number sense's importance, we ran an additional linear regression model with mature number sense removed from the model. The model  $R^2$  dropped from 0.23 to 0.09 (the same as  $sr_{ns}^2 = 0.14$ ) and this was a significant decrease in model fit ( $F(1, 149) = 27.46$ ,  $p < .001$ ).

**Table 2: Summary of Regression Model Predicting Grade-Level Mathematics Achievement**

Variable	<i>B</i>	<i>SE</i>	<i>t</i>	<i>p</i>
(Intercept)	0.16	0.040	4.02	<0.001
Mature Number Sense	0.03	0.005	5.24	<0.001
Fraction & Decimal Computation	0.001	0.005	0.10	0.92
Multiplication Fluency	-0.007	0.003	-1.93	0.06

### Discussion and Implications

From the evidence presented here, we see the same overall pattern of results as observed with middle school students and their mature number sense. Upper elementary students' mature number sense is distinct from their general grade-level mathematics achievement and fraction and decimal computation. Being identified as "strong with fractions" is not the same as displaying strong number sense. We also found students' mature number sense to be predictive within a single timepoint of their mathematics achievement, above and beyond their fraction and decimal computation and their single-digit multiplication fluency. When mature number sense was removed the model, there was a substantial and significant decrease in the explained amount of variance in students' grade-level achievement. Together, this suggests, that just as observed in middle school, students' mature number sense in upper elementary is uniquely predictive of their grade-level achievement, despite the significantly lower observed correlation between the two constructs.

The observed correlations between students' grade-level achievement and the other constructs in the study were much lower than expected or than observed with middle school

students. As we noted above, students in 3<sup>rd</sup> grade were much more successful on their grade-level mathematics achievement test ( $M = 50\%$  correct) than the 4<sup>th</sup> ( $M = 35\%$ ) and 5<sup>th</sup> graders ( $M = 32\%$ ). This was especially surprising to us as much of the 3<sup>rd</sup> grade content on the MCAS includes content that, based on the Common Core State Standards, would be brand new to students in 3<sup>rd</sup> grade: fraction magnitude, formal multiplication, division, and area. However, there were several items on multidigit addition and subtraction as well as interpreting bar graphs that students did very well on overall ( $M = 67\%$  correct). In addition, students did quite well on fraction items related to verbal situations (e.g., “There are 6 children on a bus. Each child is wearing a hat. What fraction of the children on the bus are wearing a hat?”) or basic area models ( $M = 75\%$  correct). Therefore, even though “fraction” does not formally appear in the Common Core State Standards until 3<sup>rd</sup> grade, the equal “partitioning” or “shares” of shapes in Grade 2 may prepare students well enough to solve initial fraction concept items. Further research into the strategies students used to solve these items would be necessary to better understand the mechanisms behind their strong performance.

Because students did better by grade on the brief assessment of mature number sense ( $M = 8.62$  in 3<sup>rd</sup>,  $11.93$  in 4<sup>th</sup>,  $12.98$  in 5<sup>th</sup>) and the 3<sup>rd</sup> grade MCAS was the easiest of the grade levels, the observed correlation between mature number sense and math achievement is much lower across 3<sup>rd</sup>-5<sup>th</sup> than within each grade level. Within 3<sup>rd</sup> grade for example ( $r = 0.75$ ), the correlation is in line with our initial expectations based on middle school students ( $r = 0.76$ ) and 4<sup>th</sup> and 5<sup>th</sup> graders ( $r = 0.65$ ). A deeper analysis of the 3<sup>rd</sup> grade MCAS performance in comparison to 4<sup>th</sup> and 5<sup>th</sup> grade MCAS performance is needed to shed light on the grade-level differences observed here.

Given the importance of fraction skill (e.g., Barbieri et al., 2021; Booth et al., 2014; Booth & Newton, 2012; Siegler et al., 2013) and multiplication fluency (e.g., Nelson et al., 2016) for students’ mathematics achievement, it is perhaps unanticipated that mature number sense was the strongest predictor of grade level mathematics achievement. Perhaps students’ ability to make sense of numbers and operations and use that knowledge flexibly to solve problems is a better indicator of future mathematics achievement than fraction or decimal computation or simple fact retrieval. Further research is needed to understand these relationships more fully, especially in a longitudinal setting where any potential mediating roles can be accurately tested.

Overall, we have initial evidence extending our understanding of mature number sense and its nomological network with other mathematical constructs across upper elementary and middle school. This is an important step forward to future research more fully examining how mature number sense might lay a foundation for future mathematics achievement or whether improving mature number sense might be a more important target for instruction than, say, procedural problem-solving skills with fractions. This is, however, only one initial study with a sample of students that may not be wholly representative of upper elementary school students more broadly. By continuing to characterize mature number sense’s relationship with other constructs, researchers can more fully explain how students develop a key proficiency in mathematics education and provide rigorous evidence to educators on how to help ensure all students have the disposition to *make sense of numerical situations*.

### Acknowledgements

This research was supported by National Science Foundation Grant DRL EHR 2100214.

## References

- Barbieri, C. A., Young, L. K., Newton, K. J., & Booth, J. L. (2021). Predicting Middle School Profiles of Algebra Performance Using Fraction Knowledge. *Child Development*, n/a(n/a), 1–22. <https://doi.org/10.1111/cdev.13568>
- Booth, J. L., & Newton, K. J. (2012). Fractions: Could they really be the gatekeeper's doorman? *Contemporary Educational Psychology*, 37(4), 247–253. <https://doi.org/10.1016/j.cedpsych.2012.07.001>
- Booth, J. L., Newton, K. J., & Twiss-Garrity, L. K. (2014). The impact of fraction magnitude knowledge on algebra performance and learning. *Journal of Experimental Child Psychology*, 118, 110–118. <https://doi.org/10.1016/j.jecp.2013.09.001>
- Burns, M. K., Ysseldyke, J., Nelson, P. M., & Kanive, R. (2015). Number of repetitions required to retain single-digit multiplication math facts for elementary students. *School Psychology Quarterly*, 30(3), 398–405. <https://doi.org/10.1037/spq0000097>
- CCSS. (2010). *Common Core State Standards for Mathematics*. Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers.
- Darlington, R. B., & Hayes, A. F. (2016). *Regression analysis and linear models: Concepts, applications, and implementation*. Guilford Publications.
- Dehaene, S. (2001). Précis of the number sense. *Mind & Language*, 16(1), 16–36.
- Hayes, A. F., & Rockwood, N. J. (2017). Regression-based statistical mediation and moderation analysis in clinical research: Observations, recommendations, and implementation. *Behaviour Research and Therapy*, 98, 39–57.
- Jacobs, V. R., Franke, M. L., Carpenter, T. P., Levi, L., & Battey, D. (2007). Professional Development Focused on Children's Algebraic Reasoning in Elementary School. *Journal for Research in Mathematics Education*, 38(3), 258–288. JSTOR. <https://doi.org/10.2307/30034868>
- Jordan, N. C., Kaplan, D., Ramineni, C., & Locuniak, M. N. (2009). Early math matters: Kindergarten number competence and later mathematics outcomes. *Developmental Psychology*, 45(3), 850–867. <https://doi.org/10.1037/a0014939>
- Kirkland, P.K., Cheng, Y., & McNeil, N.M. (in press). A Validity Argument for a Brief Assessment of Mature Number Sense. *Journal for Research in Mathematics Education*.
- Kirkland, P. K., Guang, C., Cheng, Y., Trinter, C., Kumar, S., Nakfoor, S., Sullivan, T., & McNeil, N. M. (2022). Middle School Students' Mature Number Sense is Uniquely Associated with Grade-Level Mathematics Achievement. *Proceedings of the Forty-Fourth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. <https://par.nsf.gov/biblio/10402434-middle-school-students-mature-number-sense-uniquely-associated-grade-level-mathematics-achievement>
- McIntosh, A., Reys, B., & Reys, R. (1992). A Proposed Framework for Examining Basic Number Sense. *For the Learning of Mathematics*, 12(3), 2–44.
- McIntosh, A., Reys, B., Reys, R., Bana, J., & Farrell, B. (1997). Number sense in school mathematics: Student performance in four countries. *ECU Publications Pre*. 2011. <https://ro.ecu.edu.au/ecuworks/6819>
- National Council of Teachers of Mathematics (NCTM). (1989). *Curriculum and evaluation standards for school mathematics*. National Council of Teachers of Mathematics.
- National Council of Teachers of Mathematics (NCTM). (2000). *Principles and Standards for School Mathematics*. NCTM.
- National Council of Teachers of Mathematics (NCTM). (2014). *Principles to Actions: Ensuring Mathematical Success for All*. National Council of Teachers of Mathematics. NCTM.
- National Research Council. (1989). *Everybody counts. A report to the nation on the future of mathematics education*. National Academy Press.
- Nelson, P. M., Burns, M. K., Kanive, R., & Ysseldyke, J. E. (2013). Comparison of a math fact rehearsal and a mnemonic strategy approach for improving math fact fluency. *Journal of School Psychology*, 51(6), 659–667. <https://doi.org/10.1016/j.jsp.2013.08.003>
- Nelson, P. M., Parker, D. C., & Zaslofsky, A. F. (2016). The Relative Value of Growth in Math Fact Skills Across Late Elementary and Middle School: Assessment for Effective Intervention. <https://doi.org/10.1177/1534508416634613>
- Powell, S. R. (2014). Rational numbers measure. Available from SR Powell.
- Reys, R., Reys, B., Emanuelsson, G., Johansson, B., McIntosh, A., & Yang, D. C. (1999). Assessing Number Sense of Students in Australia, Sweden, Taiwan, and the United States. *School Science and Mathematics*, 99(2), 61–70. <https://doi.org/10.1111/j.1949-8594.1999.tb17449.x>

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

- Shaffer, J. A., DeGeest, D., & Li, A. (2016). Tackling the Problem of Construct Proliferation: A Guide to Assessing the Discriminant Validity of Conceptually Related Constructs. *Organizational Research Methods*, 19(1), 80–110. <https://doi.org/10.1177/1094428115598239>
- Siegler, R. S., Fazio, L. K., Bailey, D. H., & Zhou, X. (2013). Fractions: The new frontier for theories of numerical development. *Trends in Cognitive Sciences*, 17(1), 13–19. <https://doi.org/10.1016/j.tics.2012.11.004>
- Whitacre, I., Henning, B., & Atabas, S. (2020). Disentangling the Research Literature on Number Sense: Three Constructs, One Name. *Review of Educational Research*, 0034654319899706. <https://doi.org/10.3102/0034654319899706>
- Yang, D.-C. (2005). Number sense strategies used by 6th-grade students in Taiwan. *Educational Studies*, 31(3), 317–333. <https://doi.org/10.1080/03055690500236845>
- Yang, D.-C., Li, M., & Lin, C.-I. (2008). A Study of the Performance of 5th Graders in Number Sense and its Relationship to Achievement in Mathematics. *International Journal of Science and Mathematics Education*, 6(4), 789–807. <https://doi.org/10.1007/s10763-007-9100-0>
- Yang, D.-C., & Lin, Y.-C. (2015). Assessing 10- to 11-year-old children's performance and misconceptions in number sense using a four-tier diagnostic test. *Educational Research*, 57(4), 368–388. <https://doi.org/10.1080/00131881.2015.1085235>

## FIRST GRADERS COORDINATION OF COUNTING AND MOVEMENTS ON A GRID WHEN PROGRAMMING WITH TANGIBLE BLOCKS

Abigail Erskine  
Purdue University  
aerskin@purdue.edu

Laura Bofferding  
Purdue University  
lbofferd@purdue.edu

Sezai Kocabos  
Purdue University  
skocabos@purdue.edu

Haoran Tang  
Purdue University  
tang474@purdue.edu

*As elementary students begin to program using tangible blocks, they must coordinate their use of counting with the movements, directions, and numbers they use to move a character. In our study, we analyzed 13 first graders' first attempts at coordinating these elements when playing a programming game on the iPad that used tangible programming blocks. We further analyzed how their programs changed over the six sessions. Our results highlight the challenges students faced when counting on a grid, representing movements with numbers, and distinguishing between movement blocks. We also present factors that influenced their improvements. The results indicate that game hints supported some students' use of numbers, while the highlighted path helped some and challenged others. Partner talk and having the opportunity to make iterative changes in their code also supported some groups.*

Keywords: Problem Solving, Computational Thinking, Computing and Coding, Number Concepts and Operations

Research in K-2 settings has shown a correlation between programming and mathematics scores (e.g., Grover, et al., 2016; Lewis & Shah, 2012; Kocabas et al., 2021). Students as young as three years old can learn to program in visual or tangible alternative programming environments such as Creative Hybrid environment for computer Programming (CHERP, Bers, 2010; Bers et al., 2014; Elkin et al., 2016; Sullivan & Bers, 2016), which pairs programming with a set of physical blocks. These and similar alternative environments reduce the chances of a program not being able to execute (Berland et al., 2013) and support young students by offering visual feedback and tangible manipulatives (Brusilovsky et al., 1997; Mladenovi et al., 2016). They also typically incorporate a physical or digital character (e.g., a turtle or a robot) and programming blocks to give the characters instructions. Blocks may include movements (e.g., walk, turn, jump), directional arrows, and numbers to indicate how many movements to execute (McNerney, 2004, see also Bofferding et al., 2022). One study found that following a one-week CHERP robotics and programming intervention, preschoolers and kindergarteners were more likely to have higher sequencing scores (Kazakoff et al., 2013). Likewise, first and third graders' performance in fixing mathematics bugs of double counting in the pretest were highly correlated with fixing programming bugs of double counting (Kocabas et al., 2021), indicating that learning programming can enrich mathematics content knowledge and problem-solving skills (Fessakis et al., 2013; Friend et al., 2018; Lewis & Shah, 2012). In one study, kindergarteners solved a series of programming puzzles to lead a ladybug to a leaf to hide herself, which also supported their counting and number comparison (Fessakis et al., 2013).

However, prior research has shown that young students struggle with counting in programming (e.g., Bofferding et al., 2020; Kocabas et al., 2019, 2021) and mathematics (e.g.,

Battista, 1999, 2010; Battista et al., 1998; Fuson, 2012). One counting issue younger students experience is double counting the same space or object twice. For example, three to five year olds made more double counting errors when objects were shown in a disorganized manner than when they were organized (Fuson, 2012; see also Kocabas et al., 2021).

Double counting often arises in structures with horizontal and vertical dimensions, such as arrays. Students have to monitor their count while coordinating their horizontal and vertical position (Risley et al., 2016). Children are good at keeping track of one thing at a time but may have more difficulty coordinating two aspects. For example, second graders who do not demonstrate columns and row structures may double count when columns and rows overlap (Battista, 1999, 2010; Battista et al., 1998). Likewise, when programming, first and third graders double counted the spaces when a column and a row overlapped on a programming path (Kocabas et al., 2019, 2021). Other counting issues may arise due to different movement pieces involved in programming (e.g., movements that move one space versus multiple spaces). The goal of this work was to explore early elementary students' counting related to space and movements in a tangible programming environment to illustrate how they navigate these issues and what helps them in this process. Therefore, we explored the following research questions:

1. When and how do first graders count the spaces they need Awbie to move in one or more directions?
2. How do they coordinate their counting of spaces and use of number blocks?
3. How do they interpret and coordinate differences in walking versus jumping with their counting and use of numbers?
4. What challenges do they face with counting and their use of programming blocks and how do they overcome them (if at all)?

## Methods

### Participants and Setting

We conducted this study in a midwestern elementary school in the United States where 11% of students were designated as English language learners, and 45% of students were eligible for free and reduced meals. The data in this study comes from a larger study focusing on 29 first and 28 third graders' commenting and debugging practices in programming and mathematics. Students participated in a pre-test, six 20-minute playing sessions, and a post-test in the larger study. For this analysis, we focus on 13 students from two of the first-grade classes (two of them left at different points in the study, so part of the analysis focuses on the remaining 11) as they played the game. During the sessions, each grade-level pair (and one student working alone) met with a researcher at a table in the hallway and worked to advance through different levels of the Coding Awbie game. Each session was video recorded, and researchers also took notes so we could analyze their choices in using the coding blocks to move the character (Awbie).

### Materials

The programming game students used, Coding Awbie, uses a series of movement blocks with direction arrows and number blocks that attach to the movement blocks to control how many times Awbie does that movement for each block. The movement blocks include a walk, jump, or grab, and the directions include up, down, left and right. The number blocks range from 1 to 5. Students can combine one movement block and a number or can stack several movement blocks with numbers to make longer programs (a movement block without a number defaults to a movement of 1). Students play the game by arranging their blocks in front of the iPad. A mirror



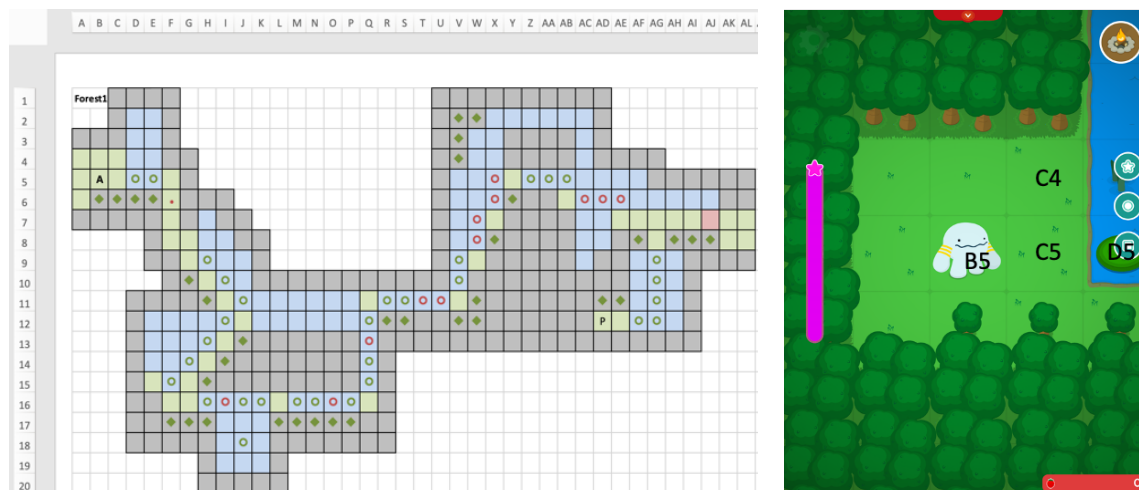
attached to the iPad captures the code and shows the potential path represented by the code highlighted on the iPad screen. When the student presses a tangible play button, the character moves on the screen. If a student blocks the mirror while playing, the iPad may misinterpret the code. Figure 1 shows an example program with three lines of code.



**Figure 1: Example of Programming Code**

The first line of code in Figure 1 shows *Walk Right 4*, which would move Awbie four spaces to the right from his current spot. The second line of code tells Awbie to *Jump Right 2* times. The jump command enables Awbie to skip a square, so Awbie would jump to the second square and then land on the fourth square from where he started. Awbie can jump over small items like bushes. Students have the ability to turn the arrow to change the direction, which we see in the third line of code where the arrow points down. The grab block keeps Awbie in his current space, but he grabs items (e.g., strawberries) in the square next to him in the given direction.

All groups started on the first level of the game, which we called Forest 1 (see Figure 2 for a map of the first level and screenshot of the starting point of each level). If they finished a level before their time was up, they continued on to the next level. They restarted any level they did not finish during their next session. The researchers encouraged pairs to work together and take turns using the coding pieces or running the program. Each of the six sessions lasted 20 minutes. For three of the six sessions, students spent part of the time studying and explaining worked examples of programs (5 to 8 min). We gave pseudonyms to students, so the pairs included Duck1 & Duck2, Duck3 & Duck4, Duck5 & Duck6, Bat1 & Bat2, Bat6 & Bat7, and Bat8. Bat2 left after the third session, at which point Bat8 played with Bat1. Bat7 left after session four, at which point Bat6 played alone.



**Figure 2: Forest 1 Map and Screenshot of Beginning Position**

*Description:* A = location where Awbie starts. Green space = Grass. Blue space = Water. Red = Strawberry. Circle = Lilypad. Dark green = Tree. P = Pie. Red space = End of level.

### Data Analysis

We were interested in how the student pairs played and navigated their counting and coordination of the movements. We coded for instances where students double counted one of the spaces when counting: the starting square (i.e., they included the square where Awbie was standing in their count of spaces to move) or a corner when changing directions (i.e., they counted the square as an ending point for one direction and a beginning point for the new direction). We also tried to make sense of other uses of numbers, such as whether they used numbers as labels for spaces to move to, using numbers as labels for the line of code, or using numbers in patterns or randomly. Finally, we also analyzed how they made sense of the distance they wanted Awbie to move and whether they used numbers that were too high or low, whether they broke up distances with two or more numbers, and whether and how they adjusted their counts.

Students' use of numbers could also vary depending on their interpretation of how the different movement pieces worked. For example, if they thought the jump moved one space (like a walk), their use of numbers would be different than if they knew the jump skipped over a space. Therefore, we also identified times when students were using the jump or grab blocks as if they moved just one space. For both counting and movement blocks, we also made notes for when their use changed and what factors might have influenced their change (e.g., the researcher said something, the game gave them a hint, etc.).

### Findings

#### Student Pairs' Attempts to Use Numbers and Movements on Session 1

Bat6 & Bat7 were the only ones who started off playing (did not do worked examples) and were also the only ones who scrolled ahead to see where the path went before running their first line of code. However, they initially organized their coding pieces from the bottom to the top and used a mixture of jump and walk commands with random numbers. Bat 7 said, "Awbie cannot jump over trees" when their first line of code (Jump Up 5) for their initial program caused Awbie

to bounce off the trees. They may have misinterpreted the jump as a walk and kept using it even when the hint showed up as Walk Right 1. They switched to only using the number 1 on their movements when getting the hint, but they often paid more attention to how many movements to make than on where they were going or stopping.

The other six groups initially only saw up to space D5 initially on the screen because they did not scroll to see where the path went; however, they had all explored some worked examples involving the coding pieces. Duck5 and Duck6 successfully moved Awbie to D5 (a lily pad on the water), but because he stopped there, Awbie fell in the water and went back to C5. Once they saw more of the path, they correctly used numbers and navigated the columns and rows to have Awbie Walk Right 3 to F5 and Walk Down 3 to F8. They were going to put a Jump Right 1 next but saw that the highlighted path showed his potential movement would land him in the water, so they changed it to Walk Right 1, ending at G8. At this point, one of the girls noticed that the path looked like stairs, and they were able to successfully program Awbie to Walk Down 1, Walk Right 1, multiple times in a row. They fell in the water a couple times because they did not move down enough times and ended on a lily pad, and they did not try the jump anymore that session.

The second group, Duck3 and Duck4 first made programs separately. Duck3 wanted to Walk Right 1 two times to get to the lily pad on D5 and then Jump Right 1 so the screen would move. Instead, they played Duck4's code, which involved a series of walks and jumps and got them to F5 (hitting a few trees that they had not seen). From F5, they played Walk Down 3, correctly counting to get to F8. Third, Bat8 kept the Jump Right 3 he had been using on a worked example before playing and used it again without knowing where it would go. He kept running previous lines of code multiple times until the researcher would remind him to change the code. For example, when he got to the part that looked like stairs, he correctly played Walk Right 1, Walk Down 1 and ran this code repeatedly, even after falling in the water several times at K12. Although he correctly counted spaces, he then added new movements onto part of his old code, so he often moved too far.

Fourth, and similar to Bat8, Bat1 and Bat2 also reused parts of their previous code. After initially jumping right and then left, Bat1 and Bat2 ended up one space away from their starting square. However, they quickly adjusted their code to successfully Walk Right 3 the correct number of spaces to get to F5. They kept the Jump Right 3 from their first line of code, which made them bounce off the trees, but then they correctly programmed Awbie to Walk Down 3. They often added or changed one correct movement but continued to keep some commands from previous lines of code and used numbers randomly, which hampered their progress. Fifth, Bat3 and Bat5's initial difficulty corresponded to them using the jump and walk blocks as if they both moved one space (potentially because they could not see where they were going). When that did not work for them, they tried just using jumps and added on numbers in sequence with random directions (i.e., Jump Right 1, Jump Right 2, Jump Up 3). They continued using sequences of numbers and a mix of walk and jump blocks.

Finally, Duck1 & Duck2 started off using numbers as locations for where they wanted Awbie to move to. For example, their first three lines of code were Walk Right 1, Walk Right 2, Walk Right 4, but they only needed the third line. Instead, because they had already moved right three, their third line of code made Awbie hit a tree and bounce back. Their continued number use appeared random, and Awbie often fell in the water before completing a step. Some of their difficulty may have been due to forgetting to turn the directional arrows.

### **Student Pairs' Progress Across Sessions 2-6**

**Bat6 and Bat7.** During sessions 2 and 3, Bat6 and Bat7 continued to build their programs from the bottom up (instead of top down) and sometimes scrolled to a future part of the path when building their code. Although they saw the structure in rows and columns, Bat6 thought the Jump block moved one space, plus often moved one space short. On the other hand, when Awbie was at G9, Bat7 thought Walk Right 3 would move Awbie diagonally along the lily pads. Bat6 tried to show her that it would make Awbie walk horizontally into the water, although she did not listen. They continued with these difficulties, reversing their code, even after following the program's hint showing them the correct way to combine pieces. After they examined worked examples at the beginning of session 4, they started placing their coding pieces in the correct order. Bat6 still thought the Jump would move one space until she played alone during session 5 (Bat7 moved) and noticed feedback from the game on where the jump would move Awbie. By this time, she also correctly counted the spaces to span longer distances.

**Bat1 and Bat2, Bat8.** Bat1 & Bat2 did not make much progress in sessions 2 and 3. From B5, they started off using Walk Right 1, Walk Right 1, Walk right 1, which did not get them far enough to F5. They kept trying to use a combination of jumps and walks with high numbers, and even ignored or did not follow the hint when it came up in both sessions. During these same sessions, Bat8 once again reused one set of codes until he got a hint from the game (Walk Right 1, Walk Down 1) at which point he repeated that code. At G16, when that code did not work, he changed the second line to Walk Right 1. Finally, he decided to change a number and did Walk Right 5 but also kept Walk Right 1, moving Awbie too far right and landing him in the water. He continued using this code (and going too far), sometimes changing the directions incorrectly or changing the code when he got a new hint. He repeated this same process in session 3, only he started using more jump blocks instead of walk blocks.

Starting at session 4, Bat1 and Bat8 played together because Bat2 moved. Bat1 primarily took charge of the pieces with input from Bat8. In session 4, after they got a hint to Walk Right 1, Walk Down 1, they often kept these pieces and just changed their numbers or directions. Therefore, even when they got to the point where they needed to Walk Up 5, they programmed Walk Up 5, Walk Up 5, once again using both pieces. They continued to change their code in reaction to Awbie not doing what they wanted after trying to reuse the code they already had, and on one occasion they double counted the starting square. By session 5, they were using the movements and numbers more intentionally, and had fewer instances of landing in the water. However, they still had an instance of double counting the initial square even in session 6. In both cases, the double counting happened on the first line of code for a new level.

**Bat3 and Bat5.** During sessions 2 and 3, Bat3 and Bat5 continued to use random movements, directions, and numbers without counting the spaces, although, Bat3 demonstrated some insight into how they could use a walk movement to get a strawberry they had jumped over. They stopped using numbers other than one after they got a hint to Walk Right 1, Walk Down 1 in session 3. In session 4, they used the movements more intentionally and started counting the number of spaces they needed to move. However, when they were at G16 and wanted to span the large distance to Q16 and then up to Q11, they ran out of Walk blocks. The researcher suggested they use numbers. They did this, plus also used Grab blocks, which they thought would move one space. They saw the structure in the columns and rows without any double counting, although they did think they only needed to move Awbie up four spaces instead of five from Q16 to Q11.

By session 5, they were more intentional and accurate in their use of numbers and directions. They started using the highlighted path, which helped them fix a Walk Down 3 (that was going

to make Awbie run into a tree) to Walk Down 2. They also used addition facts for larger distances in sessions 5 and 6. For instance, at G15, Bat3 mentioned that they needed to Walk Down 1 (pointing to the screen) and sideways 4.” Bat5 then changed the Walk Right 4 to Walk Right 5, considering the extra space they needed to move. While looking further on the path, they saw the additional spaces and added on another Walk Right 4 and Walk Right 1.

Sometimes, because they would alter previous lines of code without removing extra ones, their use of the highlighted path misled them into using numbers incorrectly. For example, at Q16, they had just finished running Walk Down 1, Walk Right 5, Walk Right 4, Walk Right 1. Since they need to go up, they turned the arrow on the first command up and counted the four lilypads. After moving the four from the third line of code up, they noticed the path did not go high enough and swapped it out for the five from the second command, giving them Walk Up 5, Walk Right 1, Walk Right 1, Walk Right 1. They then counted the four lilypads from R11 to U11 and put a four on their second line, then quickly realized it would need to be a five, giving the Walk Up 5, Walk Right 5, Walk Right 1, Walk Right 1. Because they still had the two extra Walk Right 1’s on the end, the path showed that Awbie would walk seven spaces right and hit a bush. They used trial and error to swap out numbers until they found that Walk Right 3 would work. Because they did not realize the last two Walk Right 1’s were showing up in the path, they ended up changing those so Awbie did not move as they expected.

**Duck1 & Duck2.** Although Duck1 and Duck2 appeared to be using numbers randomly in session 1, they started session 2 by considering how many spaces they wanted to move and used one block per space. Sometimes, they did not accurately account for how much space they needed to move. For example, when Awbie was at C5, they programmed him to Walk Right 1, Walk Right 1, Walk Down 1. This led Awbie to stop on a lilypad and fall into the water. Correcting their count the second time, they added in another Walk Right 1 block so that he would walk right three times before walking down. Later, when one partner was going to add an extra Walk Down 1, his partner corrected him, “I already did the down.” Another problem they encountered was thinking the jump would move one space. During session 3, when Awbie made it to I13, they once again confused the jump with a walk and also double counted the initial square, thinking Jump Left 2 would take them to H13 after which they planned to walk down. However, Awbie jumped into the water. Later, they iteratively increased the number on their Jump Right piece to figure out how it worked and correctly used three jump blocks in a row in their next session. Halfway through their time in session four, the game forced a hint that asked them to place Walk Up 5. From this point onward, they started combining some of their movements, and they used the numbers more strategically (e.g., Walk Right 2 instead of Walk Right 1 and Walk Right 1), although they sometimes added on an extra command or reused previous lines of code, that resulted in them falling in the water or moving too far.

**Duck3 & Duck4.** Primarily, Duck3 and Duck4 experienced difficulty counting the spaces, as they frequently double counted both Awbie’s initial square and corners during the remaining sessions. For example, they wanted to move Awbie from B5 to F5 and on to F8 but programmed him to Walk Right 5 by counting B5 and then Walk Down 4 by counting F5. Because of the double counting, they often moved Awbie too far. After trying the jump in session 1, they did not use the jump again until session 5 at which point they noticed that the jump skipped a space saying, “...ooo he jumped over.” On the second use of the jump in session 5, they used it correctly. On session 6, they used the jump correctly (reinforced by a hint from the game), although they made counting errors (e.g., using Jump Right 4 instead of Jump Right 3).

**Duck5 & Duck6.** In session 2, Duck5 and Duck6 started using addition facts for larger distances but also made a double counting error. When Awbie was at F16, they said, “We need to go six, but we do not have a number six” to get Awbie to L16. They used an addition fact to solve their problem, stating, “We can have four plus two,” but then they noticed they could move another five to Q16. Still looking ahead, Duck6 said they needed to go up five to Q11, but Duck6 double counted Q16, and said it was six. Throughout the other sessions they kept using addition facts to move larger distances and even used the jump block correctly in sessions 2 to 6 to skip a space. By their final session they were seeing distances rather than counting them, immediately making comments such as, “You need a three” when seeing the path.

### **Discussion and Future Directions**

Our analysis presents a fairly positive account of young students learning to coordinate counting, movements, and direction on a grid. Based on our analysis, we noticed that coordination with counting and movement is a multi-dimensional and essential skill which is needed by younger children who partake in programming. Although the students had some struggles with counting (Bofferding et al., 2020; Kocabas et al., 2019, 2021) and moving Awbie in the correct direction, they progressed in counting along the two dimensions and seeing sets of squares in terms of compositions of numbers. These results provide further evidence that programming activities can support mathematical concepts (Fessakis et al., 2013; Friend et al., 2018; Lewis & Shah, 2012).

There were cases where students double counted the starting space and corners and there were times when they under-counted the number of spaces they needed Awbie to move to land on green space or moved Awbie too far. Double counting errors were most common at the beginning of a level, potentially because they had a conflict in terms of how Awbie started or because they got used to Awbie’s movements as they moved within a level, resulting in fewer counting conflicts later. Students’ under- and over-counts often resulted from misreading the highlighted path showing the potential result of their program, a misinterpretation of a movement block (i.e., thinking the jump or grab blocks would move one space), or their desire to reuse previous code instead of counting again. Although reusing code is efficient, helping students check the feasibility of previous code is also important. The hints provided by the game often led them to change their approach, at least temporarily. In particular, the hints often helped students combine movements so that instead of using several walk blocks, they would use one with a number indicating the number of times they wanted to walk. Parts of the game where the character could move larger distances in one direction also encouraged students to use combinations of numbers, suggesting intentionally designed game features play an important role in both their understanding of the programming commands but also their use of numbers. Some students corrected each other, which helped them progress in their programming. These results suggest that providing students with ways of handling disagreements or talking about their reasoning when programming might be fruitful for encouraging productive peer talk and collaboration.

### **Acknowledgments**

This research was supported by NSF ITEST grant #1759254.

### **References**

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

- Battista, M. T., Clements, D. H., Arnoff, J., Battista, K., & Borrow, C. V. A. (1998). Students' spatial structuring of 2D arrays of squares. *Journal for Research in Mathematics Education*, 29(5), 503-532.
- Battista, M. T. (1999). Fifth graders' enumeration of cubes in 3D arrays: Conceptual progress in an inquiry-based classroom. *Journal for Research in Mathematics Education*, 30(4), 417-448.
- Battista, M. T. (2010). Thoughts on elementary students' reasoning about 3-D arrays of cubes and 306 vanderbilt. In Z. Usiskin, K. Andersen, & N. Zotto (Eds.), *Future curricular trends in school algebra and geometry: Proceedings of a conference* (pp. 183-199). IAP.
- Bers, M. U. (2010). The TangibleK robotics program: Applied computational thinking for young children. *Early Childhood Research & Practice*, 12(2). Retrieved from <http://ecrp.uiuc.edu/v12n2/bers.html>.
- Bers, M. U., Flannery, L., Kazakoff, E. R., & Sullivan, A. (2014). Computational thinking and tinkering: Exploration of an early childhood robotics curriculum. *Computers & Education*, 72, 145-157. <https://doi.org/10.1016/j.compedu.2013.10.020>
- Bofferding, L., Kocabas, S., Aqazade, M., Chen, L., & Haiduc, A. (2020, April 17–21). Exploring practices to support commenting and debugging in early years of tangible programming [structured poster session]. American Educational Research Association Annual Meeting. San Francisco, CA, United States. <http://tinyurl.com/yvd7ayh4> (Conference canceled)
- Bofferding, L., Kocabas, S., Aqazade, M., Haiduc, A., & Chen, L. (2022). The effect of play and worked examples on first and third graders' creating and debugging of programming algorithms. In A. Ottenbreit-Leftwich & A. Yadav (Eds.), *Computational thinking in PreK-5: Empirical evidence for integration and future directions. A Special Research Publication* (pp. 19-29). Association for Computing Machinery, Inc. and the Robin Hood Learning + Technology Fund.
- Elkin, M., Sullivan, A., & Bers, M. U. (2016). Programming with the KIBO robotics kit in preschool classrooms. *Computers in the Schools*, 33(3), 169-186. <https://doi.org/10.1080/07380569.2016.1216251>
- Fessakis, G., Gouli, E., & Mavroudi, E. (2013). Problem solving by 5–6 years old kindergarten children in a computer programming environment: A case study. *Computers & Education*, 63, 87-97.
- Friend, M., Matthews, M., Winter, V., Love, B., Moisset, D., & Goodwin, I. (2018, February). Bricklayer: Elementary Students Learn Math through Programming and Art. In *Proceedings of the 49th ACM Technical Symposium on Computer Science Education* (pp. 628-633).
- Fuson, K. C. (2012). *Children's counting and concepts of number*. Springer Science & Business Media.
- Kazakoff, E. R., Sullivan, A., & Bers, M. U. (2013). The effect of a classroom-based intensive robotics and programming workshop on sequencing ability in early childhood. *Early Childhood Education Journal*, 41(4), 245-255. <https://doi.org/10.1007/s10643-012-0554-5>
- Kocabas, S., Bofferding, L., Aqazade, M., Haiduc, A., & Chen, L. (2019). Students' directional language and counting on a grid. In Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (Eds.), *Proceedings of the 41st annual conference of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 431–432). St. Louis, MO
- Kocabas, S., Chen, L., Bofferding, L., Aqazade, M., & Haiduc, A. (2021). Identifying and fixing double counting errors in mathematics and programming. In Olanoff, D., Johnson, K., & Spitzer, S. (Eds), *Proceedings of the 43rd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 637–640). Philadelphia, PA
- Lewis, C. M., & Shah, N. (2012). Building upon and enriching grade four mathematics standards with programming curriculum. In *Proceedings of the 43rd ACM Technical Symposium on Computer Science Education* (pp. 57-62).
- Risley, R., Hodkowsky, N. M., & Tzur, R. (2016). Devin's Construction of a Multiplicative Double Counting Scheme: Dual Anticipation of Start and Stop. North American Chapter of the International Group for the Psychology of Mathematics Education.
- Sullivan, A., & Bers, M. U. (2016). Robotics in the early childhood classroom: Learning outcomes from an 8-week robotics curriculum in pre-kindergarten through second grade. *International Journal of Technology and Design Education*, 26(1), 3-20. <https://doi.org/10.1007/s10798-015-9304-5>

## MATHEMATIZING FAIRNESS: HOW ELEMENTARY STUDENTS DRAW ON FUNDS OF KNOWLEDGE WHILE SOLVING A MODELING TASK

Amy Been Bennett  
University of Nebraska-Lincoln  
abennett14@unl.edu

Julia Aguirre  
University of Washington  
jaguirre@uw.edu

Erin E. Turner  
University of Arizona  
eturner@arizona.edu

Elzena McVicar  
University of Washington  
elzenam@uw.edu

Erin Carll  
University of Washington  
ecarll@uw.edu

*Mathematical modeling is a process in which students investigate authentic problems and everyday situations using mathematics. In doing so, they bring their multiple mathematical knowledge bases and cultural funds of knowledge into their solution strategies. During a task called “Abuelo’s Birthday”, 297 students in grades 3-5 decided how to split the costs of a gift “fairly” and justified their work using early notions of ratio and proportional reasoning. We argue that these young students were successful with a complex task due to the way the realistic context connected to students’ lived experiences and funds of knowledge related to home and family life. We share rich examples of how students included elements outside of the task to justify and enhance their mathematical models and conclude with implications for the importance of modeling and non-routine tasks in elementary classrooms.*

Keywords: Elementary School Education, Modeling, Rational Numbers & Proportional Reasoning, Ethnomathematics

### Conceptual Framework

Mathematical modeling is internationally regarded as a beneficial mathematical practice for students across all grade levels (Schukajlow et al., 2018; Sriraman & English, 2010; Verschaffel & De Corte, 1997). In the U.S., the Common Core State Standards (2010) describe the mathematical practice of modeling as “the process of choosing and using appropriate mathematics and statistics to analyze empirical situations, to understand them better, and to improve decisions” (p.72), which is intended for grades K-12. At the elementary level, open-ended mathematics tasks that align with the modeling process can connect to children’s mathematical thinking in a variety of content areas, such as ratio and proportional reasoning. Moreover, the real-world contexts of modeling tasks can connect to disciplines and situations outside of the mathematics classroom. In this paper, we explore how the mathematical modeling process, students’ funds of knowledge, and proportional reasoning strategies intersect in one rich task.

### Mathematical Modeling

Mathematical modeling is defined in multiple ways in research literature and education standards. Modeling is a process for connecting the real world to the world of mathematics (Blum & Borromeo Ferri, 2009), or “a process in which students consider and make sense of an everyday situation that will be analyzed using mathematics for the purpose of understanding, explaining, or predicting something” (Anhalt et al., 2018, p. 202). During the modeling process, authentic situations are made sense of, simplified, modeled, analyzed, interpreted, and generalized. Mathematical modeling benefits students in multiple ways, including fostering creativity, problem solving, and communication (e.g., Chamberlin et al., 2022; Tidwell et al.,



2021). Modeling tasks are open-ended tasks that foreground diverse solution strategies and connections to multiple mathematical knowledge bases (Aguirre et al., 2013; Turner et al., 2012).

### **Cultural Funds of Knowledge**

Mathematics education scholars have found that students learn mathematics best when they work collaboratively on challenging tasks, draw on background knowledge and experiences, and connect mathematics activities to their everyday lives (e.g., Turner et al., 2012; Civil, 2002). Civil (2002) explored how children's mathematics learning and engagement can be better understood through the lens of *funds of knowledge* (Moll et al., 1995), as well as how mathematics activities can draw on students' cultural knowledge and experiences to help them connect school mathematics to authentic situations. This work emphasized that there are benefits to culturally responsive, community-based approaches to teaching mathematics, particularly for students from underrepresented groups in STEM fields.

Mathematical modeling tasks bring these features together in interdisciplinary tasks that connect mathematics content to real-world phenomena, such as using proportional reasoning to share the cost of a ride fairly (Sawatzki et al., 2019), using unit rates to upcycle plastic bags into jump ropes for a school (Turner et al., 2021), or addressing the clean water crisis in Flint, Michigan (Aguirre et al., 2019). At the elementary level, research has shown that mathematical modeling is accessible to young children and to students from a diverse range of mathematical and cultural backgrounds (e.g., English & Watters, 2005; Sawatzki et al., 2019; Verschaffel & De Corte, 1997). In this way, mathematical modeling tasks present an opportunity to draw on students' cultural funds of knowledge in the mathematics classroom.

### **Ratio and Proportional Reasoning**

Proportional reasoning is typically taught in middle (6-8) grades, after elementary students have a deeper understanding of multiplicative reasoning. Research literature documents elementary and middle grades students' struggles with ratio and proportional reasoning problems, illustrating how students sometimes rely on additive rather than multiplicative reasoning (e.g., Lo & Watanabe, 1997; Steinhorsdottir & Sriraman, 2009). Recent studies have demonstrated that younger students (grades 3-6) can successfully solve tasks related to real-world situations using multiplicative reasoning (e.g., Sawatzki et al., 2019).

Researchers have explored how children as young as first grade (about age 6) can understand some aspects of proportional reasoning, such as the Fittingness Principle and Covariation (Resnick & Singer, 1993). Fittingness refers to the notion that two (or more) things go together because their sizes or amounts are appropriate for one other. Covariation means when one quantity exists in an ordered series, it will covary, directly or inversely, with the other ordered quantity. This early reasoning is sometimes called *protoquantitative*, because it describes qualities or sets of objects, rather than precisely determining equivalent ratios with numeric values. Thus, Resnick and Singer (1993) suggested considering different approaches to help children develop flexible command of multiplicative structures, such as discussing multiple solutions to story problems. For instance, in fair sharing contexts, students may consider nonmathematical aspects of the situation to make sense of the quantities and their relations.

To this end, we posed the following two research questions:

- How do elementary students use proportional reasoning strategies to justify their notion of fairness in the context of solving a modeling task?
- How do elementary students' funds of knowledge and lived experiences shape their mathematical strategies and solutions?

## Methods

### Setting and Participants

The research presented here is part of a larger, multi-institutional project Mathematical Modeling with Cultural and Community Contexts (M2C3) in two diverse regions of the U.S. During the three years of this project, we worked with teachers and students at elementary schools that served multi-racial, multilingual, multicultural, and working-class communities. This project was motivated by prior research claiming that mathematics interest and learning improves when students can draw on their experiences and connect classroom mathematics to their everyday lives (e.g., Achmetli et al., 2019; Turner et al., 2012). We drew on elementary students' cultural funds of knowledge to create modeling tasks that fostered their mathematical thinking and their community-based ways of learning (e.g., Aguirre et al., 2013; Civil, 2002).

Teachers participated in a week-long summer workshop and ongoing monthly professional development sessions over two one-year cycles. During workshops, teachers engaged in solving and discussing modeling tasks; the Abuelo's Birthday task was one task out of several that teachers were introduced to and encouraged to use in their classrooms. In this task, students must create a fair plan for four grandchildren to share the costs of a birthday gift for their grandfather. The children are of different ages and have different weekly earnings, which challenges students to formulate models that differ from dividing the total cost into four equal parts (see Figure 1).

#### **Abuelo's Birthday Task** (adapted from Aguirre & del Rosario Zavala, 2013)

It is Señor Aguirre's 70<sup>th</sup> birthday. Four of his grandchildren want to buy him a photo printer so he can print photos of his family members. They found the printer on sale for \$120.

- Alex, a 9<sup>th</sup> grader, earns \$15 each week from babysitting.
- Sam, a 6<sup>th</sup> grader, earns \$10 each week taking care of his neighbor's pets.
- Elena, a 4<sup>th</sup> grader, earns \$5 each week doing jobs for an aunt.
- Jaden, a 1<sup>st</sup> grader, has no weekly job but has saved \$8 in her piggy bank.

One of the grandchildren says they should split the cost of the printer between them and each pay the same amount. Another grandchild says that is not fair and they should each pay different amounts. What do you think? What is *fair* in this situation?

**Figure 1. The task prompt for the Abuelo's Birthday Task**

The Abuelo's Birthday task is a complex task for elementary students because it relies on proportional reasoning, which they likely have not experienced in formal classroom settings. The Abuelo's Birthday task is not a traditional missing value task, where students need to set up and solve a single proportion; rather, it involves multiple relationships between quantities that covary. This structure requires students to create composite units, or "units of units" (Lamon, 1993), and decide how to iterate them for an unspecified amount of time. The openness of this task adds to its complexity, as students must make assumptions and decisions about how much each child should contribute and how long they will earn money toward the gift. The flexibility of strategies and solutions encourages students to engage in the modeling process, constantly making sense of, formulating, analyzing, and interpreting their model for "fairness".

### Data Collection and Analysis

For this paper, we focus on the dataset of classroom artifacts from the Abuelo's Birthday task, which consists of individual and group posters from student work during class. We collected artifacts from 297 students in grades 3-5 across 14 classrooms in two diverse regions of

the U.S. We analyzed 119 student work samples, often in the form of group posters, and excluded instances where teachers altered the original task for their classrooms, or the student work was illegible. We drew on Hatch’s (2002) notion of artifact analysis to engage in multiple rounds of coding student work for their mathematical strategies and other salient features of their solutions, such as explanations of “fairness”. We drew on the work from Aguirre et al. (2013) to justify our coding of students’ references to their cultural funds of knowledge. We initially used Lamon’s (1993) framework as a guide for classifying proportional strategies, but after iterative coding and discussions, we adapted our categories to more accurately describe the strategies we observed. For instance, we added the category Roughly Proportional to document students’ consistent use of protoquantitative reasoning. Table 1 first presents the five proportional strategies and then the five categories that describe other solution characteristics.

**Table 1: Codebook for classifying proportional strategies and other solution characteristics**

<b>Code</b>	<b>Description</b>
<b>All Equal</b>	The three older kids or all four kids contribute the same amount, with no connection to earnings; this can include Jaden or not.
<b>All Equal with Adjustments</b>	Initially used strategy “All equal” where everyone paid the same amount. Then, they realized that this will not work and made some adjustments to the existing strategy; they do not just start over with a new strategy (e.g., they see that Jaden cannot pay her \$30 and have Alex “cover” it).
<b>Roughly Proportional</b>	Must abide by protoquantitative principles, where children’s contributions are in the order of their earnings. May or may not include the number of weeks it takes to earn the money; includes solutions that start out using some proportional thinking, but final solution is not proportional due to major adjustments.
<b>Proportional with Adjustments</b>	Initially used a Proportional strategy, then adjusted plan (in a minor way) to make numbers friendly or sum to \$120 (e.g., all 3 older kids contribute for 3 weeks, and then Elena doesn’t contribute for week 4). Or, used a Proportional strategy but then re-distributed the excess earnings back to the grandchildren disproportionately.
<b>Proportional</b>	The three older kids’ contributions must be proportional to their earnings. Can use addition or multiplication and any representation; this can include Jaden or not. Solution should not include errors in the setup or selection of amounts.
<b>Over \$120?</b>	Does total earned exceed \$120?
<b>Explained Excess</b>	Did students show or describe what to do with money earned over \$120? If so, how did they use the excess money?
<b>Role of Jaden</b>	Does Jaden contribute money (in the final solution)? This includes paying for things besides the printer. If yes, how much did Jaden contribute?
<b>Cultural Funds of Knowledge</b>	Do students directly use reasoning or bring up aspects outside of the task prompt that may connect to home, family, or community?
<b>Statements of Fairness</b>	Do students explicitly say, “it is fair that...”, or, “it is not fair because...”? They may not use the word “fair” but imply it through statements like “older kids pay more, younger kids pay less”.

### Findings

We found that 90 of the 119 student work samples (76%) showed evidence of protoquantitative reasoning, attending to both the Fittingness Principle and Covariation (Resnick & Singer, 1993). This includes solutions classified as Proportional, Proportional with

Adjustments, and Roughly Proportional. As described above, the Abuelo’s Birthday task is a complex proportional reasoning task; thus, it is significant that the majority of elementary students in our study were successful with protoquantitative strategies. Table 2 presents the distribution of strategies across grades 3-5. It is noteworthy that students in all grade levels used proportional strategies.

**Table 2: Frequency of Math Strategies across all Student Work**

	<i>Grade 3</i>	<i>Grade 4</i>	<i>Grade 5</i>	<i>Total</i>
<i>Proportional</i>	10	15	12	<b>37</b>
<i>Proportional w/ Adjustments</i>	9	15	1	<b>25</b>
<i>Roughly Proportional</i>	18	10	0	<b>28</b>
<i>All Equal w/ Adjustments</i>	0	2	2	<b>4</b>
<i>All Equal</i>	1	10	1	<b>12</b>
<i>Other</i>	4	7	2	<b>13</b>
<b>Total</b>	<b>42</b>	<b>59</b>	<b>18</b>	<b>119</b>

Ultimately, we were interested in the evidence for how students drew on their cultural funds of knowledge, which we defined as explicit connections to home, family, and community that went beyond general notions of fairness. However, there were other characteristics of mathematical solutions that indicated students were drawing on outside knowledge and experiences (see Table 3). Many student solutions (53 of 119) across all mathematics strategies exceeded \$120 in their plan for buying Abuelo’s gift, and most of these solutions (30 of 53) explained what would be done with the extra funds, such as pay for a birthday card or cake, cover the sales tax on the gift, or save the money for another relative’s gift in the future.

Of the 119 solutions, 84 of them (71%) decided that Jaden should contribute to the gift. However, about half (44 of 84) decided that Jaden should only contribute a partial amount of her \$8. This is noteworthy since almost all of the 119 solutions showed the three older children contributing all of their earnings each week. This suggests that students differentiated between a constant rate of earnings for the three older children and the non-increasing amount in Jaden’s piggybank and demonstrated this distinction through their models.

**Table 3: Characteristics of Solutions and Fairness by Math Strategy**

	<i>Exceeded \$120</i>	<i>Explained Excess</i>	<i>Included Jaden</i>	<i>Direct Statement about Fairness</i>
<i>Proportional</i>	30	20	29	18
<i>Proportional w Adjustments</i>	9	3	23	18
<i>Roughly Proportional</i>	9	2	20	22
<i>All Equal &amp; All Equal w/ Adjustments</i>	5	5	12	11
<b>Total</b>	<b>53</b>	<b>30</b>	<b>84</b>	<b>69</b>

In the following subsections, we describe select student work samples, organized by their math strategy. These samples were chosen as representatives of the ways students drew on their funds of knowledge and mathematized fairness. An important assumption of this work is that students are always drawing on their experiences as they learn, but in some cases, such as these posters of student work, we have explicit evidence.

## Proportional Strategies

One key variation among student solutions was how they decided to include Jaden or not. In an example from a group of third graders (see the left image in Figure 2), Jaden's money was not included in the plan ("Jaden will not pay."), and so the older three children contributed proportional to their earnings over four weeks to earn exactly \$120.

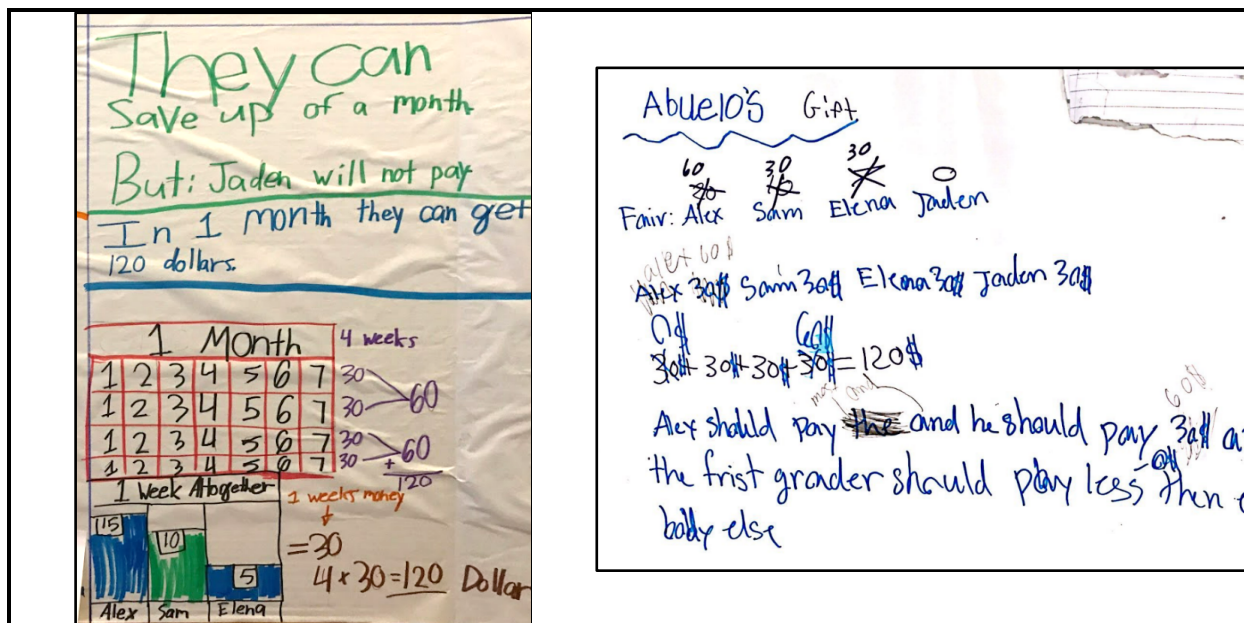


Figure 2. A Proportional strategy and an All Equal with Adjustments strategy

In contrast to the solution in Figure 2, a group of grade 4 students were purposeful to include Jaden in their proportional strategy, even in a small way. The older three children contributed all of their earnings over five weeks, but Jaden only gave \$2 (25%) of her savings. This suggests that these students drew on their experiences with siblings and family dynamics to decide what a young child with no weekly income should contribute. The total earnings for five weeks summed to \$192, which surpassed \$120 by \$72. The students decided that the grandchildren "can use [the extra] for tax or buy a cake". These students meaningfully made sense of the remainder of money in the context of buying a gift. Their realistic considerations connected to their funds of knowledge, which led to incorporating additional elements into the task (i.e., sales tax and a birthday cake).

In another proportional solution, a group of third graders wrote about how "Jaden can get money cleaning her room" if she needs to contribute more. They included an equation with parentheses to demonstrate how the three older children contributed at a constant rate over four weeks and deemed this "fair". Then, this group suggested how their plan could be applied to similar sharing situations, like sharing the cost of a room or buying "pizza, drinks, and Hershey's". They wrote, "Our plan can work in other situations because they can work the same amount and probably have enough money for room service."

Multiple groups expressed how their plan could inform other cost sharing situations between children, particularly for paying for some kind of lodging or "renting a beach house". Although we only have their final written work, it appears that these students shared about family

vacations, paying for rent, or other experiences where family members had to share the cost of a place to live. There were no references to traveling with family or paying for lodging in the task sheet, which indicates that students made (mathematical) connections between a grandparent's birthday and other family events.

### **Proportional with Adjustments Strategies**

In the Proportional with Adjustments category, students started with a proportional approach and then adjusted in a minor way, often referring to the realistic elements of the task context. One common strategy in this category involved decreasing the amount that Elena (the second youngest child) contributed so that Jaden could be included. One group of grade 4 students clearly wrote, "Fair: Little kids pay less. Big kids pay more," which emphasizes their understanding of protoquantitative fittingness and how it relates to fairness. In their model, Alex and Sam contributed earnings for four weeks (\$60 and \$40, respectively), but Elena only contributed for three weeks (\$15) to allow Jaden to contribute \$5 of her savings. This group made the assumption that it is important for all grandchildren to contribute, even if this makes the plan no longer proportional.

Two other groups of grade 4 students planned for one of the three older siblings to pay slightly less (than what is proportional) so that Jaden could contribute to the gift. Their solutions included several assumptions about the situation, including: that Abuelo will like the gift and will keep it (possibly thinking about gifts their own grandparents might like); that Abuelo lives in a different city and the children will need to ship the gift (connecting to experiences with mailing gifts and shipping costs); that they have ample time to earn the money; that the older siblings will be able to keep working and earn the money back that they contributed (possibly drawing on experiences with jobs); and that any extra earnings will cover sales tax. These assumptions influenced their mathematical models and demonstrated how students drew on their funds of knowledge about the gift-giving context.

In another solution, students made an initial assumption that Abuelo's birthday was in three weeks. They wrote equations to show each of the older children's weekly earnings multiplied by three, which gave a total of only \$90. The students realized that they still needed to account for \$30; they decided that Jaden would give her \$8, which left \$22 for the older children to cover. They wrote the assumption "that the older kids already had saved some money" so that they did not need to earn for additional weeks. From their savings, Alex would give \$9, Sam \$7, and Elena \$6. Rather than focusing on the lack of proportionality, we noted the attention to the Fittingness Principle and to their notion of fairness (i.e., that all children contribute, and the older children take on more financial responsibility), likely influenced by family experiences with siblings and cousins.

### **All Equal with Adjustments Strategies**

This strategy assumes a different notion of "fairness" than the previous two, since in an All Equal Strategy, all children contribute the same amount of money toward the gift. There were only a few examples of the All Equal with Adjustments strategy, but all of them made connections to students' funds of knowledge. In general, students initially used an All Equal strategy; then, they realized that this was not feasible or "not fair" and adjusted, usually reducing the amount that Jaden needed to contribute. In the image on the right of Figure 2, the students decided "Alex should pay most and he should pay \$60 and the first grader should pay less than everybody else." In their work, it is clear that they revised their initial plan by crossing out the \$30 for Alex and Jaden and writing the new amounts of \$60 and 0, respectively. In other words, they revised their model so that Jaden did not have to pay any of her savings and instead her

older sibling (or cousin) Alex paid for her part. In this example, when Jaden did not have enough money to pay her share, it became the responsibility of the oldest grandchild to cover it, rather than splitting her share among all three older children.

In another All Equal with Adjustments solution, each of the four children initially paid \$30; the group wrote a thorough description of how Jaden can earn the additional \$22:

"Each pay thirty dollars and little Jaden can get some money from her parents or can start a bake sale of box cookies or homemade cookies or maybe pie or lemonade to get thirty dollars. Or if Jaden can't get a job she can get money from her college account because she still has several years until college."

This excerpt is a rich example of how students related to Jaden and connected their own family experiences to hers. However, the students later made adjustments so that the older two children each paid \$50 and the younger two each paid \$10. Ultimately, everyone contributed, but more responsibility was given to the two older siblings.

### **Discussion and Conclusion**

We reiterate our finding that 90 of the 119 student work artifacts (76%) showed evidence of protoquantitative reasoning, attending to both the Fittingness Principle and Covariation (Resnick & Singer, 1993). This demonstrates that many young children enter the mathematics classroom with an understanding of quantity and covariation of units, which lays the foundation for multiplicative reasoning and proportional strategies. The Abuelo's Birthday task structure of listing grandchildren of different ages may have strengthened the notion of (direct) covariation, since the order of their ages aligned with the order of their earnings. Due to the open-ended nature of the Abuelo's Birthday task, students were able to draw on multiple strategies, representations, and prior mathematics concepts while attending to a realistic situation. This highlights the ability of tasks that are aligned with the modeling process to assess and revisit previous concepts but also introduce and build intuitive understanding of new math concepts. Our task, rooted in the authentic situation of sharing costs fairly, elicited informal notions of proportional reasoning that students had not experienced in formal classroom instruction (see Sawatzki et al., 2019). This is consistent with prior research claiming that mathematics problems involving realistic contexts help students develop deeper and stronger mathematical understandings (Lo & Watanabe, 1997; Verschaffel, & De Corte, 1997).

Students clearly communicated their model and justifications in the form of assumptions or sense-making about remainders (see Verschaffel, & De Corte, 1997). In our analysis of different strategies, we highlighted the strengths of students' work, rather than concluding that students were not demonstrating (formal) proportional reasoning. While students' mathematical solutions were justified by multiplicative and proportional strategies, their rationales were not exclusively mathematics-based, suggesting they were drawing on other mathematical knowledge bases (Turner et al., 2012) and their cultural funds of knowledge (Aguirre et al., 2013). Most of the student artifacts that exhibited evidence of students' funds of knowledge also included Jaden in their plans. This suggests that the non-routine structure of the Abuelo's Birthday task (i.e., including Jaden as a non-proportional component) added complexity to the task in a way that students had to draw on their experiences to make sense of. Our study provides evidence that young children across diverse cultural and linguistic backgrounds are capable of sophisticated mathematical reasoning while solving complex modeling tasks, and we encourage the use of similar tasks in elementary education settings.

## Acknowledgments

This work was part of the M2C3 Project supported by NSF #1561305. All conclusions and opinions are those of the authors and do not reflect the opinions of the NSF.

## References

- Achmetli, K., Schukajlow, S., & Rakoczy, K. (2019). Multiple solutions for real-world problems, experience of competence and students' procedural and conceptual knowledge. *International Journal of Science and Mathematics Education*, 17, 1605-1625. <https://doi.org/10.1007/s10763-018-9936-5>.
- Aguirre, J. M., & del Rosario Zavala, M. (2013). When equal isn't fair: Using ratios to scale up mathematical arguments. *Rethinking mathematics: Teaching social justice by the numbers*, 115-121.
- Aguirre, J. M., Turner, E. E., Bartell, T. G., Kalinec-Craig, C., Foote, M. Q., Roth McDuffie, A., & Drake, C. (2013). Making connections in practice: How prospective elementary teachers connect to children's mathematical thinking and community funds of knowledge in mathematics instruction. *Journal of Teacher Education*, 64(2), 178-192.
- Aguirre, J. M., Anhalt, C. O., Cortez, R., Turner, E. E., & Simic-Muller, K. (2019). Engaging teachers in the powerful combination of mathematical modeling and social justice: The Flint water task. *Mathematics Teacher Educator*, 7(2), 7-26.
- Anhalt, C., Cortez, R., & Bennett, A. B. (2018). The emergence of mathematical modeling competencies: An investigation of prospective secondary mathematics teachers. *Mathematical Thinking and Learning*, 20(3): 1–20. <https://doi.org/10.1080/10986065.2018.1474532>.
- Blum, W., & Ferri, R. B. (2009). Mathematical modelling: Can it be taught and learnt?. *Journal of mathematical modelling and application*, 1(1), 45-58.
- Chamberlin, S., Payne, A. M., & Kettler, T. (2022). Mathematical modeling: a positive learning approach to facilitate student sense making in mathematics. *International Journal of Mathematical Education in Science and Technology*, 53(4), 858-871. <https://doi.org/10.1080/0020739X.2020.1788185>.
- Civil, M. (2002). Culture and mathematics: A community approach. *Journal of Intercultural Studies*, 23(2), 133-148.
- Common Core State Standards Initiative. (2010). National Governors Association Center for Best Practices and Council Of Chief State School Officers. [http://www.corestandards.org/assets/CCSSI\\_Math%20Standards.pdf](http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf)
- English, L. D., & Watters, J. J. (2005). Mathematical modelling in the early school years. *Mathematics Education Research Journal*, 16(3), 58-79.
- Hatch, J. A. (2002). *Doing qualitative research in education settings*. Suny Press.
- Lamon, S. J. (1993). Ratio and proportion: Connecting content and children's thinking. *Journal for Research in Mathematics Education*, 24(1), 41-61.
- Lo, J. J., & Watanabe, T. (1997). Developing ratio and proportion schemes: A story of a fifth grader. *Journal for Research in Mathematics Education*, 28(2), 216-236.
- Moll, L., Amanti, C., Neff, D., & Gonzalez, N. (1992). Funds of knowledge for teaching: Using a qualitative approach to connect homes and classrooms. *Theory Into Practice*, 31, 132-141.
- Resnick, L. B., & Singer, J. A. (1993). Protoquantitative origins of ratio reasoning. In T. P. Carpenter, E. Fennema, & T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 107–130). Lawrence Erlbaum Associates, Inc.
- Sawatzki, C., Downton, A., & Cheeseman, J. (2019). Stimulating proportional reasoning through questions of finance and fairness. *Mathematics Education Research Journal*, 31(4), 465-484. <https://doi.org/10.1007/s13394-019-00262-5>.
- Schukajlow, S., Kaiser, G., & Stillman, G. (2018). Empirical research on teaching and learning of mathematical modelling: A survey on the current state-of-the-art. *ZDM*, 50, 5-18. <https://doi.org/10.1007/s11858-018-0933-5>.
- Sriraman, B., & English, L. (2010). Problem solving for the 21st century. In B. Sriraman & L. English (Eds.), *Theories of mathematics education: Seeking new frontiers* (pp. 263-286). Heidelberg, Germany: Springer.
- Steinhorsdottir, O. B., & Sriraman, B. (2009). Icelandic 5th-grade girls' developmental trajectories in proportional reasoning. *Mathematics Education Research Journal*, 21(1), 6-30.
- Tidwell, W., Anhalt, C. O., Cortez, R., & Kohler, B. R. (2021): Development of prospective elementary teachers' mathematical modelling competencies and conceptions, *International Journal of Mathematical Education in Science and Technology*, DOI: 10.1080/0020739X.2021.2005170.



- Turner, E. E., Drake, C., McDuffie, A. R., Aguirre, J., Bartell, T. G., & Foote, M. Q. (2012). Promoting equity in mathematics teacher preparation: A framework for advancing teacher learning of children's multiple mathematics knowledge bases. *Journal of Mathematics Teacher Education*, 15(1), 67-82.
- Turner, E. E., Roth McDuffie, A., Aguirre, J., Foote, M. Q., Chappelle, C., Bennett, A. B., Granillo, M., & Ponnuru, N. (2021). Upcycling plastic bags to make jump ropes: Elementary students leverage experiences and funds of knowledge as they engage in a relevant, community-oriented mathematical modeling task. In J. Suh, M. H. Wickstrom, & L. English (Eds.), *Exploring mathematical modeling with young learners*. New York, NY: Springer.
- Verschaffel, L., & De Corte, E. (1997). Teaching realistic mathematical modelling in the elementary school: A teaching experiment with fifth graders. *Journal for Research in mathematics education*, 28(5), 577-601.

## RELATIVE REASONING AND THE TRANSITION FROM ADDITIVE TO MULTIPLICATIVE THINKING IN PROPORTIONALITY

Jérôme Proulx

Laboratoire Épistémologie et Activité Mathématique, UQAM, Canada  
proulx.jerome@uqam.ca

*Abstract.* Research studies are abundant in pointing at how the transition from additive to multiplicative thinking acts as a core challenge for students' understanding of proportionality. This said, we have yet to understand how this transition can be supported, and there remains significant questions to address about how students experience it. Recent work on proportional reasoning has pointed to a type of strategy, called "relative", that appears to be lodged right between additive and multiplicative ways of thinking. This sort of "in-between" strategy raises significant interest and motivates further analysis. In this paper, I explore several of these relative strategies engaged in by a 13-year-old student, Marie, during a series of individual interviews. The analysis outlines several dimensions that can inform as much the transition from additive to multiplicative thinking than proportional reasoning itself.

Keywords: Proportionality; Relative reasoning; Additive thinking; Multiplicative thinking

Proportionality is as much a fundamental topic in school mathematics as it is an enduring challenge for students of various grades. This situation continues to raise the need to investigate ways in which proportionality can be made sense of and explored in the classroom. As such, research studies have for a long-time documented difficulties experienced by students when attempting to solve proportionality problems (e.g. Behr et al., 1992; Lamon, 2007). One main issue is that proportional problems require a shift from additive to multiplicative thinking (Van Dooren et al., 2010). Whereas most problems students face in their initial school years focus on additive situations, proportionality confronts them with multiplicative ones, and there remains significant questions as to how this transition is experienced and how it can be supported.

Work by Copur-Gencturk et al.'s (2022) has recently drawn attention to a type of strategy, called "relative", that appears to sit right in between additive and multiplicative thinking. For example, they gave to teachers the following rectangle problem:

The Science Club has four separate rectangular plots for experiments with plans.  
Which rectangle(s) looks more like a square? Explain your answer.

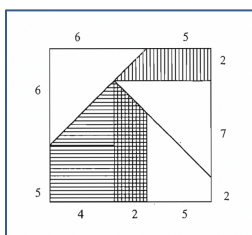
a) 1 foot by 4 feet	b) 17 feet by 20 feet
c) 7 feet by 10 feet	d) 27 feet by 30 feet

Usual type of strategies, ranging from incorrect (additive or else) to multiplicative, were used by the teachers. However, 17% of them engaged in another type of strategy, one that used the additive attributes of the problem, the difference of 3 feet, but went beyond it: the difference of 3 feet between both sides of the rectangle was assessed precisely in relation to the sides of each rectangle, when evaluating which one would look more like a square. Examples of answers were: "Because, since the side lengths are longer than all of the other rectangles, the 3 feet difference is less noticeable." (p. 8); "I think [option] d is most square because although the difference in feet of the dimensions is the same, the larger number will make it harder to tell that there is 3 feet difference." (p. 11). Copur-Gencturk et al. position these strategies as lying between additive and multiplicative ones, without necessary detailing more on them (their scientific objectives being

different). Hence, their work raises interest in better understanding these relative strategies and what they entail. Taking the form of a case study, this paper reports on the analysis of these kind of relative strategies, here engaged by a 13-year-old student, Marie, during a series of individual interviews focused on solving proportional problems. The consideration and investigation of her relative strategies offers enriched understandings of the mathematical strengths lying at the core of these sorts of “in-between” strategies, informing in turn matters concerning the additive and multiplicative transition in the unfolding of proportional reasoning in students.

### Transition from Additive to Multiplicative Thinking in Proportionality

The transition from additive to multiplicative thinking in proportionality is mostly known through the difficulties students experience: e.g., students use additive procedures in problems where multiplicative ones would be needed (Van Dooren et al, 2010). Solving problems in a proportional way requires the consideration of the multiplicative structure of the problem (Vergnaud, 1988), something that additive procedures do not necessarily succeed in doing. One main difficulty can be defined as the “additive invariant” conception, where students focus on the constant differences between quantities. Brousseau’s (1998) puzzle situation is a good example of such (Figure 1), where students are required to enlarge a square-shape puzzle, where a segment measuring 4 units in the initial puzzle has to measure 7 units in the new one. To do so, many students engage in additive thinking, adding 3 units to all other sides of the puzzle. This ends up modifying the initial square format, and the pieces no longer fit together.



**Figure 1: Illustration of Brousseau’s (1998) puzzle situation**

To inform the transition from additive to multiplicative thinking in proportions, numerous researchers have brought forth development models to help appreciate the many “levels” students are found to be working in concerning proportionality. Based on a variety of studies, Steinhorsdottir and Sriraman (2009) have evoked “three levels of strategies that students use as they grow in understanding proportional relationships” (p. 8). The first level is of qualitative nature, the second is additive, and the third multiplicative. To elaborate more precisely on each, consider the following “cat food problem” taken from their work:

It is lunchtime at the Humane Society. The staff has found that 8 cats eat 5 large cans of cat food. How many large cans of cat food would the staff members need to feed 48 cats? (p. 8)

The kind of qualitative strategy that some students could display would be centered on an appreciation of the relation existing between numbers without using direct calculations. For example, one student could say that “they would need a lot more cans because 48 cats is a lot more than 8 cats” (p. 8). Without being false, it might be said that it lacks precision as to how many cans of food will be needed. The second type of strategy is based on additive thinking, which has often been termed *building up* strategies (Hart, 1981; Pulos & Tourniaire, 1985). In

these kinds of strategies, students could, for example, add up values by jumps of 8 cats and 5 cans to arrive at the aimed-for number of cans for 48 cats: 8 for 5, 16 for 10, 24 for 15, 32 for 20, 40 for 25, and 48 for 30 (they could also combine these, like 8 for 5, 16 for 10, and directly 32 for 20, and another for 48 for 30). Although additive, these building up strategies are most often successful, especially when the number of jumps is described by a natural number (here 5). Finally, there is the level of multiplicative strategies, often represented by the following algebraic equation:  $8/5=48/x$ . In using this equation, students can focus on the ratio existing between 8 and 48 (6 times more) and adjust for 5 and 30. They can also focus on the ratio existing between 8 and 5 ( $5/8$  of) and adjust for 48 and 30. This said, those are only examples, as other strategies like the unit-rate or the cross-product could be engaged with.

Carpenter et al.'s (1999) model raise other distinctions for the additive dimensions. Whereas the first level concerns students' random calculations or focus on additive differences, the second level also concerns building up strategies; where students see the ratio of 8 for 5 as a single unit and can repeat it until the desired value is found (or can use the multiplication by a whole number, where 8 times 6 gives 48 and so 5 times 6 gives 30). These strategies are successful and intertwine with some multiplicative dimensions. However, in this second level, students are not able to work with fractional multiples, that is, when a problem does not imply the multiplication by a whole number. This ability would represent the third level, where a ratio is seen as a single unit but can also be repeated a noninteger number of times (or when fractioning the ratio is needed). E.g., if the problem called for 44 cats instead of 48 cats, students at level three could add 8 for 5 up until 40 for 25, and then split in two the ratio to get a 4 for 2.5, leading to 44 cats for 27.5 cans of food. Finally, at the fourth level, the focus goes beyond the ratio as a single unit and into a consideration of the multiplicative relations between the numbers in the problem. Hence, the whole-number multiplicative relation from 8 to 48 or the fractional one from 8 to 5 can both be considered in solving the problem for finding the desired value.

Another developmental model frequently referred to is the Piaget sequences on proportional reasoning (Inhelder & Piaget, 1958; Piaget et al., 1967). Slightly different, Piaget's model also has four steps ending in a mastery of the multiplicative relations in the problem. Piaget mentions as well that the first steps taken in solving a proportional problem are often additive, focusing on respecting the differences between the numbers in the problem. Then students migrate to what he calls pre-proportionality, which still uses additive, and incorrect, thinking, but adapts the numbers in relation to their size: the additive difference need not be constant and varies with the size of the numbers. To understand the difference between these two steps, imagine that in the catfood problem there is a third house with 24 cats, and one wonders how many cans of foods would be needed. Students in the first step would focus on the additive difference of 3 between 5 and 8, and would reproduce it between 24 and 21 cans, as well as between 48 and 45. Students in the pre-proportionality mode could, however, focus on the fact that the third house seems somewhat "in the middle" of the first one of 8 cats and of the second one of 48; hence the number of cans would have to represent that middleness. Whereas the first house of 8 cats needs 5 cans of food, the second one could use 15 and the third 25: differences change and follow the size of the numbers associated with it. This pre-proportionality step is followed with the logical proportions step. This time, students consider the set of (multiplicative) relations existing between the four numbers in a proportion. Thus, they understand that increases or decreases in any number cause corresponding increases or decreases in the other. Students in the logical proportions step often succeed with simple problems (ratios of 1 for 2 or matching numbers like 2 and 8), and mostly develop local understandings of the problem at hand: the functioning of

these relations are not extended to other problems, nor conceived as a general law of proportions. This abstract level (e.g.  $8/5=48/x$ ) would constitute the final and fourth step. As Pulos and Tourniaire (1985, p. 187) explain: “According to Piaget, adolescents’ proportional reasoning develops from a global compensatory strategy, often additive in nature, to an organized proportional strategy without generalization to all cases, to finally the formulation of the law”.

Other models and theories have been proposed over the years, refining or offering other distinctions between their levels (e.g., Resnick & Singer’s, 1988, protoquantitative reasoning). These help flag the significance of mastering the multiplicative relations between the numbers in proportionality problems and raise the fact that there exists important steps to get from an additive take on the problem to a consideration of its multiplicative structure, and then to be able to reason proportionally on it. As this transition continues to need to be better understood, efforts being made toward investigating it have led to important advances (e.g., Singh, 2000, study using Steffe’s coordination of units; Van Dooren et al., 2010, study of students’ strategies in proportional and non-proportional problems). These studies raise recognitions of how additive and multiplicative strategies differ, how students’ strategies can oscillate between the two, as well as how specific characteristics of the problems can create difficulties for students (e.g., numbers, context). These studies also show how hard it is to pinpoint at specific elements that can play a role in (understanding) this transition, and how there continues to be much to be understood about it. Can the investigation of relative strategies contribute to understandings about this transition? This is the declared intention of this paper, through exploring examples of strategies that one student, Marie, engaged in when solving proportional problems.

### **Methodological Considerations**

The data analyzed in this paper comes from a larger study on proportionality, consisting of two phases: a first one about conducting individual interviews and a second one consisting of classroom-wide experiments. Because the objective of this paper is to scrutinize the use of relative strategies for solving proportional problems, it only reports on selected strategies that Marie (13-year-old student, Grade-8), engaged in during the first phase of individual interviews. Over a period of three months, interviews were conducted with Marie, who had recently been taught proportions during her school year. The interviews were conducted by the PI, using an online environment for recording purposes (except for one face-to-face interview). The interviews consisted of presenting a problem to the student, who had to solve it and then explain her answer (orally, using paper, or drawing on the computer). Each interview lasted between 20 to 30 minutes, depending on the students’ availability. In total, Marie solved 16 problems. These were chosen or inspired from a review of the current literature, which abounds in problems on proportionality, and mostly ranged from missing-value problems, comparison problems, and transformation problems (as classified in Lesh et al., 1988). This paper reports on some of Marie’s strategies with comparison problems, in which she engaged in relative strategies.

The data reported on are circumscribed and only concern strategies engaged in by Marie that align with Copur-Gençturk et al.’s synthetic description: engaging with additive attributes while considering the relative value of these same additive attributes in relation to other quantities in the problem. As such, the data is analyzed through elements outlined about the transition from additive to multiplicative thinking. Note that the goal here is not to report on all of Marie’s strategies, nor to discuss the evolution that might have happened or the long-term outcomes of this work with her. Instead, the scientific intention is to gain a better understanding and make sense of relative strategies when solving proportionality problems, as some sort of case study. As

Lamon (1994) and Singh (2000) argue, the detailed analysis and precise identification of students' ideas and thinking processes can enhance understandings of how to work with specific mathematical content. This is particularly so, Lamon flags, if these processes and ideas can act as connectors for relating different mathematical dimensions (here, for additive and multiplicative thinking). In other words, the intention of the data analysis is to directly address and explore relative strategies, and investigate how these can inform understandings of proportionality, ways of working with it, and the transition from additive to multiplicative thinking.

### Investigation of Marie's Relative Strategies

In the following, three examples of relative strategies engaged by Marie are presented. These are discussed and analyzed through aspects of the additive and multiplicative transition.

#### The Rectangle Problem

The rectangle problem was given to Marie during her interviews (using meters instead of feet). Marie's response is that the 27x30 rectangle is the one closer to a square. She explains that even though all rectangles have a difference of 3 meters between their sides, these 3 meters would appear particularly small for the 27x30 rectangle, and big for the 1x4 rectangle. Using her fingers (Figure 2a), she shows what the 27x30 rectangle would look like and how, because each side is almost the same, the 3 meters would appear small. She then contrasts this for the 1x4 rectangle. Keeping one finger up from her 27x30 representation, she elongated from the tip of that finger to simulate how far apart the 4-meter side is from the 1-meter one (Figure 2b).



Figure 2: Marie's illustration of (a) the 27x30 rectangle, (b) the 1x4 rectangle

This was Marie's way to show how "not-square" the 1x4 rectangle is, and how the difference of 3 would be greater in this case, making it obvious that it is a rectangle, because it is "4 times more" as a length. She continues by saying that it would be the same for the 17x20, meaning that the 3 meters would also make it more apparent *than the 27x30* that it is a rectangle, and again for the 7x10 where the 3 meters would make "even more" apparent (than this latter 17x20) that it is a rectangle. Her explanations offer a sort of gradient of rectangles, from being more rectangle-like with 1x4, toward a lesser one with 27x30. She concluded by insisting that the sides of the 27x30 rectangle are almost the same, but with a difference of "merely 3 small meters", adding that with the side of 27 meters, the 3 meters is "almost nothing, it is a mini-fraction of it".

Marie's strategy appears to go beyond a quantification of the difference between the ratios forming the sides of the four rectangles (for then comparing with a 1 for 1 ratio for a square). Her strategy is focused on a relative quality of the constant difference of 3 between the sides of the rectangle. This translates by a movement toward getting closer and closer to a square, the more the dimensions of the rectangle enlarge: the "same" difference of 3 is at times small, at times big, bigger, "merely 3 small meters", or "almost nothing", in relation to the dimensions of the rectangle. The consideration of this additive invariance of 3 meters is something that can lead

students into difficulties, where they could answer that all rectangles are as “square-ish”. But Marie’s take on this additive difference of 3, although invariant, is not conceived in absolute terms: it is considered in relation to other quantities that are themselves varying (i.e. each rectangle’s side dimensions), which in turn make this 3 vary *qualitatively*. It thus makes these 3 meters a kind of variable entity in relation to each rectangle to which it gets associated. But this 3 meters does not vary in terms of quantity since it is the same numerical 3 everywhere. It varies in terms of its relative nature, its quality. It somehow stops being a plain constant 3 meters and merges into a qualitatively variable 3 meters, relative to its referent, that is, the dimensions of the rectangle from which it comes. Marie’s relative strategy raises a qualitative-quantitative interplay, leading her beyond an additive strategy in considering values as relative in the problem. She also engaged in similar relative strategies when faced with a purely numerical problem and even one without numbers. The following two are examples of such.

### The Group Increment Problem

The following problem was given to Marie (inspired by MEO, 2012):

Which one of the two groups of persons experienced the most important increase?  
The Frimousses were 3 and are now 9  
The Grippettes were 100 and are now 150.

A focus on the numerical differences, before and after the transformation, would lead one to assert that the increase of 50 for the Grippettes is a more significant one than the one of 6 for the Frimousses. Marie did not opt for this additive route and engaged as much relatively as multiplicatively in it. She expressed that it is the Frimousses who have a more important increase, since 9 is 3 times 3 and 150 is only 1.5 times 100. She was then questioned on the fact that from 100 to 150 there is a difference of 50, and that from 3 to 9 there is only a difference of 6. She responded by saying that it is the increase one needs to focus on, and not the 50 nor the 6.

Marie: it is not the number that counts, it is the biggest increase that we look for. In the 1<sup>st</sup> group there is 3 times more people than before, and in the 2<sup>nd</sup> group there is to the half more.

Her explanations reveal two types of multiplicative considerations: there is the “3 times more”, and there is a blend of an additive and a multiplicative entry with her “to the half more”. In the first case, the difference of 6 is not explicitly considered, where in the second case the difference of 50 is. This shows how she here oscillates between an additive and a multiplicative mode. Her “to the half more” concerns the 50 and is as much a multiplicative relation (50 is half of 100), as it is an additive relation (50, this half, is added to 100). Questioned about what she meant by “looking for the increase”, she added how considering the increase leads one to go beyond the numbers and consider them in relation to another number (i.e. the initial value):

Marie: well, it is not linked to the number itself [the numerical value of the increase], it is related to how much times more it is [than the initial value].

Marie’s strategy highlights again the difference between the numbers themselves, a quantity in the absolute, and that same number “in relation” to another one, which gives this number its true value in the problem. This points anew to the varying quality that a number can take in these situations, and how its value is relative. Counterintuitively, an increase of 6 is now somehow becoming bigger than one of 50, when seen in relation to where it came from; in the same way that getting to 9 is more impressive than getting to 150, since what counts is relative to where it started. Here, this 50 becomes a relative number, a “relative 50”, in relation to 100.

These explanations illustrate at least a part of what underlies Marie’s relative consideration of the numbers at stake, and how it goes beyond an additive consideration of them: she considers not only the number itself, but the number in its relation to other numbers. There is more: considering the relation between quantities is not enough, since an additive relation is still a relation! Marie’s take on the relativity focuses on the *multiplicative relation* that exists between these numbers: the 3 times 3, the half of it, etc. Hence, the relativity of the numbers in this proportional problem is of a multiplicative nature. More than seeing 6 as big because of 3, and 50 as small because of 100, the relativity for each number emerges from a multiplicative connection: where 9 is three times as 3, and 50 is only half of 100. This leads her to bridge additive and multiplicative issues as well as to establish differences between them. She can see how the additive differences can be considered, and also sees them in a multiplicative way in relation to, relative to, another number. This became salient when, at the end of the interview, she was challenged once more on these same matters. The following discussion took place:

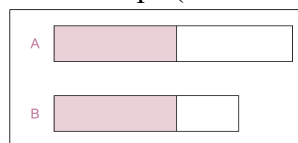
PI: So, if you had 3 pairs of earrings, you would prefer that your dad buys you 6 more pairs than if you have 100 pair of earrings and your dad buys you 50 more?

Marie: Oh no! The thing is not that he buys me 6 more, but that he buys me 3 times more!

Marie is here making quite plain how her relative connotation of the additive difference in the problem is grounded in multiplicative matters. This shows how her consideration of quantities as relative is a move toward multiplicative thinking itself, about considering the multiplicative relations that exist within the quantities in the problem; and how these relations transform the qualitative value, thence the quantitative value, of the numbers.

### **The Color Comparison Problem**

Marie engaged in similar relative proportional reasoning in the following problem without numbers, where she had to compare colored strips (from MEO, 2012; inspired by Small, 2008).



**Which figure contains more color?**

Marie asserted that the B strip had more color, relative to its length: “they both have the same amount, but B is smaller”. She explained that both colored parts were the same, but that she considered them in relation to their respective strip, which made them different. The amount of color was seen as constant, but became relative to its entire strip, hence part of a different whole (and not considered in the absolute). She explained this by referring to surfaces and volumes:

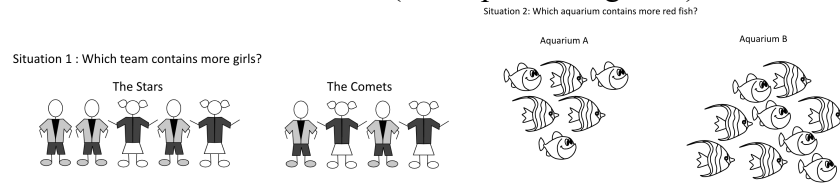
Marie: they have the same surface of color. But the cylinder A is bigger so the fraction of color will be smaller whereas in B it will be bigger.

Her reference to the fractioning of the strip illustrates again how the notion of relativity is connected to a multiplicative consideration of the situation. The length of color is not viewed as absolute, but is in relation to another length, of which it is a part, and that modulates its value. Her reference to “a fraction of” points to how the same length of color becomes smaller when taken in relation to another length, which makes it a fraction of it.



### Relative Proportional Reasoning and Multiplicative Thinking

Not only are the quantities relative in the examples of Marie's relative strategies, but these quantities become grounded in multiplicative considerations. Numbers, amounts, differences, etc., are not considered in absolute terms, but are constantly placed in relation to other ones that modulate their value in the problem: it is a part of, it is in relation to, etc. This relativity appears to act as a core element for going beyond a unique additive consideration of numbers for embarking in multiplicative thinking. These quantities are not, again, taken as absolutes, but become relative through their multiplicative link with another quantity. This multiplicative link transforms their worth in the problem. To better appreciate this difference, consider the following problems from Van de Walle et al. (2008, p. 168; Figure 3).



**Figure 3: Reproduction of comparison problems from Van de Walle et al. (2008)**

To engage additively in these problems, one needs to consider the number of girls or of fish, and then compare them. The answer to “which has more?” becomes one of comparing the numbers obtained and asserting which is the biggest. However, engaging proportionally in these problems means considering the number of girls or of fishes in relation to another amount, which in turn modulates its value. For the fish problem, this other amount can be the number of white fish or the total number of fish in each aquarium. An additive answer leads one to say that there are 4 fish in B and 3 in A; hence aquarium B has more red fish, because 4 and 3 are considered in absolute terms. A proportional answer, however, would lead one to say that e.g. aquarium A has 3 red fish for 3 white fish, whereas aquarium B has 4 red fish for 5 white fish; hence B has more white fish than red, and thus aquarium A has more fish when the other number of fish is considered. The answer to these problems becomes relative to these other amounts in it.

It is this relativity that not only leads beyond an additive take on the problem, but also to the consideration of the relation between the quantities in the problem: here a 1 to 1 or a 3 to 3, for example, in the fish situation, or what Marie has called “a fraction of” in the color problem. Being “relative to” leads to consider, literally, the relation that exist between the quantities. Marie's relative strategies thus taps on some important reflections on the transition from additive to multiplicative thinking in proportionality. Proportional reasoning is not only about quantities, it is about quantities in relation to other ones, hence relative, in their multiplicative relation. This is what happens in Marie's relative strategies: problems can be solved additively, but it is when they are considered under their multiplicative nature that they become proportional.

### Concluding Remarks

That students experience difficulties in proportional reasoning is not new. One main obstacle has often been seen as a matter of transitioning from additive to multiplicative thinking, and how there seem to be a wide gap lying between these. The notion of relative strategies raised by Copur-Gencturk et al. seems to have some potential for contributing to our understanding of this difficult transition, being situated right in between both. As shown, Marie's relative strategies call upon the additive and multiplicative structure of the problem, while controlling the quantities

or magnitudes of the problem proportionally. The “answers” given were not obtained uniquely through calculations but understood in relation to other quantities and magnitudes in the problems to solve. This might offer conceptual keys for informing the difficult transition from additive to multiplicative thinking. Marie’s relative strategies point to how the passage from additive to multiplicative called for a qualitative shift, and how it connected the two. This qualitative shift on numbers might be an element to reflect on concerning the multiplicative structure that proportional problem requires: 3 is not merely a 3, it is a 3 in relation to a 27, to a 9, etc.; hence it is multiplicatively connected to another quantity in the problem, related to it. Obviously representing an avenue requiring more study, the consideration of aspects of relative strategies has potential to push further our current understandings of the transition from additive to multiplicative thinking in proportionality.

### References

- Behr, M., Harel, G., Post, T., & Lesh, R. (1992). Rational number, ratio, and proportion. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 296-333). New York: Macmillan.
- Brousseau, G. (1998). *Théorie des situations didactiques*. Éditions La Pensée Sauvage: Grenoble.
- Carpenter, T., Fennema, E., Franke, M. L., Levi, L., & Empson, S. (1999). *Children’s mathematics: Cognitively guided instruction*. Portsmouth: Heinemann.
- Copur-Gençturk, Y., et al. (2022). A closer look at teachers’ proportional reasoning. *International Journal of Science and Mathematics Education*.
- Hart, K.M. (1981). Ratio and proportion. In K.M Hart (Ed.), *Children’s understanding of mathematics 11–16* (pp. 88-101). Oxford: Murray.
- Inhelder, B., & Piaget, J. (1958). *The growth of logical thinking from childhood to adolescence*. New York: Basic Books.
- Lamon, S. (1993). Ratio and proportion: Connecting content and children’s thinking. *Journal for Research in Mathematics Education*, 24(1), 41–61.
- Lamon, S. (1994). Ratio and proportion: cognitive foundations in unitizing and norming. In J. Confrey (Ed.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 89-120). Albany: SUNY Press.
- Lamon, S. J. (2007). Rational numbers and proportional reasoning: toward a theoretical framework for research. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 629–667). Information Age Publishing.
- Lesh, R., Post, T., & Behr, M. (1988). Proportional reasoning. In J.Hiebert & M.Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 93-118). Reston, VA: NCTM.
- Ministère de l’éducation de l’Ontario [MEO]. (2012). *Qu’est-ce que le raisonnement proportionnel?* Ministère de l’éducation de l’Ontario.
- Piaget, J., Grize, J., Szeminska, A., & Bangh, V. (1977). *Epistemology and psychology of functions* (F. Castelanos & V. Anderson, Trans.). Dordrecht, The Netherlands: Reidel.
- Resnick, L.B., & Singer, J.A. (1994). Protoquantitative origins of ratio reasoning. In T.P. Carpenter et al. (Eds.), *Rational numbers: an integration of research* (pp. 107–130). LEA.
- Singh, P. (2000). Understanding the concept of proportion and ratio constructed by two grade six students. *Educational Studies in Mathematics*, 43(3), 271-293
- Steinthorsdottir, O.B. & Sriraman, B. (2009). Icelandic 5th-grade girls’ developmental trajectories in proportional reasoning. *Mathematics Education Research Journal*, 21(1), 6-30.
- Tourniaire, F., & Pulos, S. (1985). Proportional reasoning: a review of the literature. *Educational Studies in Mathematics*, 16(2), 181-204.
- Van de Walle, J., & Lovin, L.A. (2008). *L’enseignement des mathématiques: l’élève au centre de son apprentissage (6<sup>e</sup> à 8<sup>e</sup> année)*. ERPI: Montreal, Quebec.
- Van Dooren, W., De Bock, D., & Verschaffel, L. (2010). From addition to multiplication ... and back. *Cognition and Instruction*, 28(3), 360-381.
- Vergnaud, G. (1988). Multiplicative structures. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in middle grades* (pp. 141–161). Reston, VA: NCTM.

### ROLE OF AUTHENTIC CONTEXTS IN PROPORTIONAL REASONING

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

Debasmita Basu  
Eugene Lang College of Liberal Arts, The  
New School  
basud1@newschool.edu

Hong Nguyen  
Eugene Lang College of Liberal Arts, The  
New School  
nguyh376@newschool.edu

Keywords: Rational Numbers & Proportional Reasoning, Integrated STEM/STEAM, Curriculum, Middle School Education.

Proportional reasoning entails a multiplicative relationship between two quantities. Two quantities are proportional if the ratio between the quantities remains the same irrespective of their actual measures (Lamon, 1993). Despite being a fundamental concept that underpins higher-level mathematical and scientific understanding, students often struggle to distinguish between proportional and non-proportional situations and use additive thinking rather than multiplicative reasoning in situations involving proportional quantities (Sowder et al., 1998). Ben-Chaim et al. (1998) have discussed the effectiveness of different curricula in developing students' proportional reasoning skills. They said, familiar and authentic contexts can engage students in proportional reasoning and help them develop their own problem-solving tools. In line with this, the current study designed a proportional reasoning lesson on the context of added sugar to explore how *the relatable context of added sugar provides students with a meaningful platform to reason proportionally?*

According to Hyman (2008), food nutrient labels are excellent tools for teaching mathematics and promoting healthy eating habits in children. Consequently, in this study, two middle school students, Lenny and Kaya, were shown pictures of four food products and instructed to record (i) *the number of servings in each food product* and (ii) *grams of added sugar in each serving*. They were informed that one teaspoon is equivalent to four grams of added sugar and was asked to use this information to calculate the total teaspoons of added sugar in the whole package of each food product.

Lenny first calculated the number of teaspoons of added sugar present in one serving of Oreo cookies. He formulated an equation using grams of added sugar in each serving of Oreo and grams of added sugar in one teaspoon and used cross-product algorithm (Cramer, Post, Graeber, 1993) to find that one serving of Oreo contains  $\frac{17}{4}$  teaspoons of added sugar. When asked to explain his reasoning, Lenny stated that he used comparison to keep the known quantities in the numerator (top) and unknown quantities in the denominator (bottom) and performed cross-multiplication to reach the solution. Lenny used the same strategy to examine the total teaspoons of added sugar present in the entire packet of Oreo. Kaya followed a different approach to compute the total quantity of added sugar in one packet of Oreos. She used the "stretcher and shrinker" concept, proposed by Lamon (1993), to scale up the 4.25 teaspoons of added sugar in one serving of Oreo by the ten servings in the whole package.

In this study, we did not provide any instructions or draw students' attention to the concept of proportional reasoning. Our goal was to allow students to develop their mathematical strategies and insights to address the issue of added sugar. Although students did not articulate the term proportionality explicitly during their engagement with the tasks, results of the study suggest that students used the concept of proportionality to calculate the quantities of added sugar present in the food products. This supports the idea presented by Ben-Chaim et al. (1998) that using familiar and genuine contexts such as added sugar provides students with a natural and meaningful way to reason proportionally.

## References

- Ben-Chaim, D., Fey, J. T., Fitzgerald, W. M., Benedetto, C., & Miller, J. (1998). Proportional reasoning among 7th grade students with different curricular experiences. *Educational Studies in Mathematics*, 36, 247-273.
- Cramer, K., Post, T., & Graeber, A. O. (1993). Connecting research to teaching: Proportional reasoning. *The Mathematics Teacher*, 86(5), 404-407.
- Hyman, B. (2008). Integrating math and nutrition education: Teaching with the FDA food label. *American Journal of Health Education*, 39(2), 113-117.
- Lamon, S. J. (1993). Ratio and proportion: Connecting content and children's thinking. *Journal for research in mathematics education*, 24(1), 41-61.
- Sowder, J., Armstrong, B., Lamon, S., Simon, M., Sowder, L., & Thompson, A. (1998). Educating teachers to teach multiplicative structures in the middle grades. *Journal of Mathematics Teacher Education*, 1(2), 127-155.

## STUDENTS' PRE-INSTRUCTIONAL IDEAS ABOUT INTEGER MULTIPLICATION

Lisa Lamb  
San Diego State University  
Lisa.Lamb@sdsu.edu

Carlos Alejandro de Alba  
San Diego State University  
cdealba3923@sdsu.edu

Jessica Bishop  
Texas State University  
jbishop01@txstate.edu

Keywords: Cognition, Learning Trajectories and Progressions, Number Concepts and Operations

Students often learn a rule for multiplying integers. For example, one student solved  $-4 \times -3 = \underline{\quad}$  by sharing that “two negatives make a positive... I just learned a rule. I don’t remember why.” We became curious about how students, before school-based instruction, solved these types of problems, so we conducted semi-structured pilot interviews with eight students in grades 3–6 to explore their reasoning about integer multiplication.

The research about students’ ways of solving integer multiplication problems is rare (see Sfard, 2007; Carpenter & Wessman-Enzinger, 2018). Researchers have, however, documented Ways of Reasoning (WoRs) that students use to solve additive integer open number sentences (Lamb et al., 2018). These WoRs include order-based, analogy-based, formal, computational, and emergent. We hypothesized that some of these WoRs could be extended to integer multiplication. The strategies shared are from one fourth grader, N, and two fifth graders, K and P. They solved a variety of open number sentences, including  $4 \times -3 = \underline{\quad}$ ,  $-4 \times 3 = \underline{\quad}$ , and  $-2 \times \underline{\quad} = 6$ .

### Three Productive Strategies

N, K, and P used two productive WoRs: analogy-based reasoning and formal reasoning. For  $4 \times -3 = \underline{\quad}$ , N, K, and P all drew upon their understanding of multiplication as “groups of” to inform their responses. That is, the students used analogy-based reasoning about negative numbers, reasoning about them as entities that could be manipulated discreetly to come up with an answer. For example, N worked from an equal-groups meaning of multiplication saying that  $4 \times -3$  means adding negative 3 four times. The three students also used strategies that reflected formal reasoning. One was the use of the commutative property to conjecture about how integer multiplication worked with a negative multiplicand. When solving  $-4 \times 3$ , P said, “Not sure. Maybe -12 because negative 4 groups of 3. Since negative 4 groups of 3 [voice trails off]. Wait, no! The answer should be -12 because of the associative [*sic*] property. I can flip the problem around and get the same answer so 3 groups of negative 4. The answer is -12.” The other use of formal reasoning was logical necessity, wherein the students compared their reasoning on one problem to a related problem. All three students approached  $-2 \times \underline{\quad} = 6$  similarly. N reasoned, “It could be -3 but I don’t know really. If it was 3 in the blank then the answer [product] would be negative 6, not 6.” She eventually decided the only feasible answer was -3. N compared the given problem,  $-2 \times \underline{\quad} = 6$ , to a related problem,  $-2 \times 3$ , and reasoned that the blank could not be a ‘normal’ 3 because the product of  $-2 \times 3$  was negative. For her, the only other reasonable possibility was -3. She generalized her answer and conjectured that the product of two negatives is a positive (or regular number). She then productively used her conjecture to reason through all other problems involving the product of two negatives.

### Implications

We suggest that the selection and order in which students solve problems matters: start with a positive multiplicand, unknown product; then positive multiplier, unknown product; then

positive product, unknown multiplier. Know that many students can reason robustly about these problems. Finally, provide opportunities for students to make their thinking visible in class.

### References

- Carpenter, C. H., & Wessman-Enzinger, N. M. (2018). Grade 5 Students' Negative Integer Multiplication Strategies. In T.E. Hodges, G.J. Roy, & A.M. Tyminski, (Eds.) Proceedings of the 40th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (pp. 139-146). University of South Carolina & Clemson University.
- Lamb, L. L., Bishop, J.P., Philipp, R.A., Whitacre, I., & Schappelle, B.P. (2018). A Cross-Sectional Investigation of Students' Reasoning About Integer Addition and Subtraction: Ways of Reasoning, Problem Types, and Flexibility. *Journal for Research in Mathematics Education*, 49(5), 575-613.
- Sfard, A. (2007). When the rules of discourse change, but nobody tells you: Making sense of mathematics learning from a commognitive standpoint. *The journal of the learning sciences*, 16(4), 565-613.

## THE IMPACT OF OPERATION STRATEGIES ON THE CONVICTION OF PRESERVICE TEACHERS' BELIEFS ABOUT "LEARNING STYLES"

Bernard Smith  
University of Central  
Arkansas  
Bsmith25@cub.uca.edu

Jordan Allred  
University of Central  
Arkansas  
Jallred5@cub.uca.edu

Lisa Skultety  
University of Central  
Arkansas  
lskultety@uca.edu

Keywords: Number Concepts and Operations, Preservice Teacher Education, Teacher Beliefs

The myth that everyone has a "learning style" or primary way of learning, such as visual, kinesthetic, or auditory is prevalent in education, with up to 80-95% of educators believing that learning styles play a role in students' learning and is a meaningful way to differentiate instruction for all learners (Nanceckivell et al., 2020). However, after decades of research, there is little to no evidence that providing instruction in someone's "learning style" impacts their ability to learn new information (Pashler et al., 2008). In teaching for specific learning styles, teachers may make instructional choices to support one student's learning style while inadvertently withholding methods and strategies from others despite research showing that it is beneficial for all students to interact with information using multiple "styles" (Mayer, 2002). While the research has been clear for a while, it is common for teacher preparation programs to incorporate learning styles in coursework (Lethaby & Harries, 2015; Tardif et al., 2015). However, explicit instruction on learning styles may not be necessary for preservice teachers (PSTs) to solidify their belief that some students are visual, auditory, or kinesthetic learners.

In this study, we ask, *How and why do PSTs' views about the importance of students' learning style change during an elementary mathematics content course on early numeracy and integer operations?* At the start and end of the course, sixty-seven PSTs shared what they knew about learning styles, if they identified as a particular "type" of learner, how important a student's learning style is for their success in mathematics (on a scale from 0-5, with 5 as critically important) and an explanation for their rating. The PSTs' rating of the importance of learning styles in mathematics were essentially equivalent at the start and end of the course with a mean rating of 4.2 and 4.1 respectively. However, there was a shift in PSTs' explanation of their score at the end of the course. At the start of the semester PSTs' justifications were broad and unspecific, such as, "Your learning style helps you process information." At the end of the semester, PSTs' reasoning for their numerical score frequently (64%) referenced their experiences with reasoning strategies to solve integer operation problems. During the course, PSTs solved problems with various strategies, including drawn models, diagrams, arrays, and manipulatives, which PSTs considered to be strategies for "visual learners." As they understood the mathematical concepts at depth they had not experienced before, PSTs frequently identified as a "visual" learner themselves, which, in turn, confirmed the importance of learning styles. One PST shared that this course demonstrated the "power of visual models for visual learners." While exposure to such strategies was clearly impactful, reinforcing the "learning style" myth was an unintended consequence. The exposure to a vast array of more visual and physical strategies for integer operations is a cornerstone for elementary PST preparation, but such unintended consequences should be considered and explicitly addressed throughout such courses. Readings

and discussion around the research on learning styles and the importance of conceptual understanding and multimodality in mathematics learning may be beneficial.

### References

- Lethaby, C., & Harries, P. (2015). Learning styles and teacher training: Are we perpetuating neuromyths? *ELT Journal*, 70, 16–27. <http://dx.doi.org/10.1093/elt/ccv051>
- Mayer, R. E. (2002). Multimedia learning. In B. H. Ross (Ed.), *Psychology of Learning and Motivation*, 41, 85–139. [http://dx.doi.org/10.1016/S0079-7421\(02\)80005-6](http://dx.doi.org/10.1016/S0079-7421(02)80005-6)
- Nancekivell, S. E., Shah, P., & Gelman, S. A. (2020). Maybe they're born with it, or maybe it's experience: Toward a deeper understanding of the learning style myth. *Journal of Educational Psychology*, 112(2), 221-235. <http://dx.doi.org/10.1037/edu0000366>
- Pashler, H., McDaniel, M., Rohrer, D., & Bjork, R., (2008). Learning styles: Concepts and evidence. *Psychological Science in the Public Interest*, 9(3), 105-119. <http://dx.doi.org/10.1111/j.1539-6053.2009.01038.x>
- Tardif, E., Doudin, P. A., & Meylan, N. (2015). Neuromyths among teachers and student teachers. *Mind, Brain and Education*, 9, 50–59. <http://dx.doi.org/10.1111/mbe.12070>



## USING NUMBER TALKS TO REASON ABOUT EARLY NUMBER CONCEPTS IN AUTHENTIC WAYS

Jane M. Keiser  
Miami University

JaneKeiser@MiamiOH.edu

Suzanne R. Harper  
Miami University

Suzanne.Harper@MiamiOH.edu

Dana C. Cox  
Miami University

Dana.Cox@MiamiOH.edu

Keywords: Instructional Activities and Practices, Preservice Teacher Education, Number Concepts and Operations, Elementary School Education.

Number talks are an effective pedagogical strategy for exposing emergent numeracy for consideration within a community of learners. Number Talks (Humphries & Parker, 2015) have been described as “short, mathematical, whole-class discussions during which students solve problems and share their ways of seeing and reasoning about mathematics” (Gerstenschlager & Strayer, 2019, p. 363). Summarized by Matney et. al (2020), there are five essential components of number talks:

- The classroom should be a safe learning environment;
- Students indicate their readiness to engage through the use of hand signals;
- The teacher acts as a facilitator that reinforces the socio-mathematical norms and poses purposeful, strategic questions;
- Students are encouraged to use mental mathematics to arrive at a solution; and
- Talks should include purposeful computation problems.

While all are important to the goal of building number sense, it is the fifth component that we take up here for consideration in the context of working mathematically with preservice elementary teachers. Our goal is to provide a setting in which adult learners can experience the authentic “invention” of computational strategies similar to that experienced by young learners (Carpenter et al., 1999) without being hindered by algorithmic familiarity and rote application of procedures.

Seeking ways to make the familiar unfamiliar and providing more authentic experience of learning new mathematics, we translated existing curricular tasks using Schifter and Fosnot’s (1993) XManian numeration system. This system immerses Elementary Mathematics Preservice Teachers in a base five number system (using A, B, C, D, and O as digits). We build up numeracy starting with basic counting before exploring patterns on a number chart, skip counting, and eventually incorporating operations. We have seen the productive struggle that occurs as they make sense of how to operate within this foreign system, but it remained difficult for adult learners to express emergent thinking verbally for others to consider or critique.

Number Talks provide the forum for these explanations. In this presentation we will share our experiences with number talks in a content course on number and operation for preservice elementary mathematics teachers. In this context, we observed PSTs utilizing more rudimentary strategies that were neither rote nor algorithmic such as skip counting, compensation, equal additions for subtraction, and doubling for multiplication. They more readily deconstructed numbers into place values and flexibly utilized mathematical properties when appropriate. Translating the tasks typically used in number talks for elementary students into unfamiliar number systems has had the added benefit of establishing patterns of thinking in adult learners that more authentically match those of their elementary counterparts.

## References

- Carpenter, T.P., Fennema, E., Franke, M.L., Levi, L., & Empson, S.B. (1999). *Children's mathematics: Cognitively guided instruction*. Heinemann.
- Gerstenschlager, N. E., & Strayer, J. F. (2019). Number talks for statistics and probability. *Mathematics Teaching in the Middle School*, 24(6), 362-368. <https://doi.org/10.5951/mathteacmidscho.24.6.0362>
- Humphreys, C., & Parker, R. (2015). *Making number talks matter*. United States: Stenhouse Publishers.
- Matney, G.T., Lustgarten, A., & Nicholson, T. (2020). Black holes of research on instructional practice: The case of number talks. *Investigations in Mathematics Learning*, 12(3),1-15. <https://doi.org/10.1080/19477503.2020.1804273>
- Schifter, D. & Fosnot, C.T. (1993). A sample mathematics lesson for teachers: Xmania. In *Reconstructing Mathematics Education: Stories of Teachers Meeting the Challenge of Reform* (pp. 41-62). New York: Teachers College Press.

---

# PRE-CALCULUS AND HIGHER- LEVEL MATH

---

## A LONGITUDINAL STUDY OF WOMEN'S SENSE OF BELONGING IN AN ACTIVE LEARNING UNDERGRADUATE CALCULUS COURSE

Casey R. Griffin  
University of Delaware  
crgrif@udel.edu

*Low sense of belonging has been identified as a key reason for women's decisions to leave STEM majors. Prior studies indicate that incorporating active learning opportunities may support students' sense of belonging. Calculus is an especially ripe setting for investigating STEM students' sense of belonging, as it often acts as a gatekeeper course for STEM majors. This mixed methods study investigates changes in women's sense of belonging during the first semester of a year-long active learning Calculus course. Women who reported large changes were interviewed to investigate the influence of active learning opportunities on their sense of belonging. Results indicate an initial significant increase followed by a plateau in women's sense of belonging. Six women who experienced changes in their sense of belonging agreed to be interviewed, and this analysis is ongoing.*

Keywords: affect, emotion, beliefs and attitudes, calculus, gender, undergraduate education

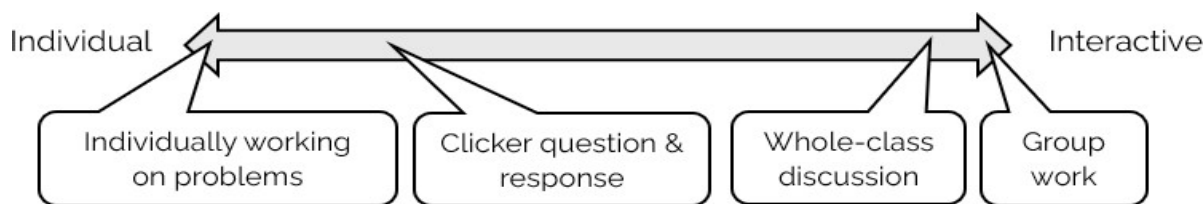
The underrepresentation of women in STEM careers has been attributed to a lack of women pursuing undergraduate STEM degrees (Carmichael, 2017; Chamberlain, 2017). Not only do fewer women than men enter into undergraduate STEM majors, but women leave STEM majors at a higher rate than men – especially after taking Calculus (Chen et al., 2013; Eagan et al., 2016; Seymour & Hunter, 2019). Calculus is a critical junction because it is a required course for STEM majors and is typically taken as a pre- or co-requisite for more advanced STEM coursework, often gatekeeping students who perform poorly from continuing on their STEM trajectory. Consequently, a significant leak occurs in the STEM pipeline at Calculus.

Prior research identifies low sense of belonging as a key reason why women decide to leave STEM majors (Seymour & Hunter, 2019; Shapiro & Sax, 2011). One feels a *sense of belonging* when they feel like “an accepted member of an academic community whose presence and contributions are valued” (Good et al., 2012, p. 701). Strayhorn (2012) argues that a sense of belonging can be so essential that one cannot engage in a space without feeling a sense of belonging to that space. For students, this might mean they have trouble listening to a lecture or studying for an exam without first feeling a sense of belonging. Further, research indicates that students with a stronger sense of belonging are more likely to persist in STEM (Rainey et al., 2018; Seymour & Hunter, 2019). While feeling a low sense of belonging can be an issue for all students, women tend to report a lower sense of belonging than men in STEM (Rainey et al., 2018; Shapiro & Sax, 2011).

Prior studies have found that sense of belonging can change over time, though these changes have typically been negative (Anderman, 2003; Hausmann et al., 2007). However, researchers found that decreases in sense of belonging were buffered by a respectful learning environment (Anderman, 2003) and perceptions that the instructor cared about the students' learning (Hausmann et al., 2007). These studies not only provide evidence of sense of belonging's malleability but also suggest that sense of belonging can be influenced by external factors like learning environment and instructor care. Researchers have identified other contributors to students' sense of belonging, including academic and social integration (Lewis & Hodges, 2015;

Tinto, 1975). Tinto (1975) describes academic integration as students' intellectual and academic development and faculty concern for students' development, and social integration as students' social connectedness with their peer group and faculty. Rainey et al. (2018) confirmed the influence of both academic and social factors in their interview study with 201 college students. They found that students most frequently cited interpersonal relationships and perceived competence as contributors to their sense of belonging in STEM. Further, students who left STEM reported feeling a low sense of belonging and attributed it primarily to a lack of interpersonal relationships and perceived competence.

One way that Calculus instructors may be able to support their students' sense of belonging is to provide opportunities for students to engage in active learning. Active learning opportunities engage students in "the process of learning through activities and/or discussion as opposed to passively listening to an expert" (Bonwell & Eison, 1991, p. iii). Active learning has been shown to benefit students' achievement, persistence, and sense of mastery (Freeman et al., 2014; Lahdenpera et al., 2019; Rasmussen et al., 2019). However, active learning is quite a broad category, and there lacks consensus on the particular types of active learning opportunities that best support students. Active learning opportunities can be categorized along a continuum from solo activities (e.g., individually working on problems, responding to clicker polls) to interactive activities (e.g., engaging in whole-class discussion or group work) (See Figure 1.; Griffin, 2021). I hypothesize that interactive activities that engage all learners and allow students to academically and socially integrate have the most potential to support students' sense of belonging.



**Figure 1: Active Learning Continuum**

This study investigated the following research questions: (1) For undergraduate women enrolled in a year-long active learning Calculus course, in what ways, if any, does their sense of belonging change during the first semester of the course? and (2) For women who reported significant changes in their sense of belonging, to what do they attribute those changes?

## Methods

### Setting and Participants

This study was conducted at a mid-Atlantic research university during the 2022-2023 academic year. The university offers a two-semester Integrated Calculus course designed for students who place into Pre-calculus but need Calculus for their intended major. This year-long course is typically taken by first-year students intending to major in STEM. The first semester of the course develops differential Calculus and the second semester develops integral Calculus, with each course weaving in necessary pre-Calculus topics in a "just in time" approach. The course was designed to provide frequent opportunities for students to

engage in active learning, primarily through small group work, whole-class discussions, and short, interactive lectures.

Note that each of these activities would fall toward the Interactive end of the Active Learning Continuum (Figure 1) described above. The course is highly coordinated – in addition to common textbooks and exams, instructors teach from common lesson plans specifying which problems to work on each class, and whether they will be discussed as a whole class or in small groups. Thus, the teaching methods employed by all instructors teaching the course are consistent.

In the Fall 2022 semester, two sections of the course were offered, taught by two permanent faculty, with 63 and 64 students enrolled in each section. Participants were students enrolled in both sections of the course. Early in the fall semester, students received an email inviting them to participate in the study by completing a survey. The survey asked students to identify their gender by selecting all that apply from the following options: Man, Woman, Transgender, Non- binary, Not Listed (please specify), and Prefer Not to Disclose. For the purpose of this study, only students who selected at least Woman and completed all three surveys were considered for analysis. For the fall semester, N=41.

### **Data Collection and Analysis**

The survey was distributed and completed electronically using Qualtrics, a web-based survey tool, to all students enrolled in the course three times over the first semester of the course – beginning, middle, and end of the Fall 2022. To measure students' sense of belonging, the survey included Good et al.'s (2012) Mathematical Sense of Belonging (MSoB) scale. The MSoB portion consisted of 28 Likert items asking students to indicate the extent to which they agree with statements about their sense of belonging on a scale of 1=*Strongly Disagree* to 6=*Strongly Agree*. To analyze these data, for each student I calculated the mean of the student's responses to the 28 items to obtain a sense of belonging score. I then used these scores to conduct a repeated- measures ANOVA to determine if significant changes were occurring between each pair of time- points – beginning and middle, and middle and end.

Women whose sense of belonging changed between any pair of time points were asked to participate in an interview to further examine what contributed to those changes. To select these women, I first found the absolute value of the difference in sense of belonging between each pair of consecutive time-points. I sorted the differences after each time-point from largest to smallest to identify women who experienced large changes (whether positive/increase or negative/decrease) and asked them to participate in a 45-minute interview over Zoom. Interviews took place after the second and third surveys. Interview participants were asked to (1) describe what "sense of belonging" means to them and how they feel when they have a sense of belonging, (2) tell a story about something that happened in class during which they felt their sense of belonging being impacted, and (3) agree or disagree with the change in their sense of belonging as reported in the survey and explain why. These responses will be analyzed using theory-guided analysis, with *academic*, *social*, and *learning environment* as a priori codes, while welcoming emergent themes.

### **Results**

Students' responses to the MSoB portion of the survey were used to measure their sense of belonging at three time-points over one semester in a year-long active learning Calculus

course. Preliminary results indicate an initial significant increase with a large effect size ( $d=.79$ ; Cohen, 1988) followed by a plateau in women’s sense of belonging during the Fall 2022 semester (See Table 1.).

Of the women who showed particularly large changes in their sense of belonging, six agreed to be interviewed – only one was interviewed after the second survey, and all six were interviewed after the third survey. Preliminary results indicate that in all six cases, these women described themselves as feeling more comfortable speaking up (e.g., seeking and/or providing help) when they have a sense of belonging. According to one student, she felt comfortable speaking up even if she was wrong: “I’m not afraid to, like, speak, I think, also not afraid to like, make a mistake...even if I say something wrong, it’s still like I still feel comfortable, or I just laugh about it, and so I feel like I belong because I feel free to like say something that is wrong...It’s a hard class, it’s okay to make mistakes.” In their stories describing a time in which they felt their sense of belonging being impacted, three students talked about a time they were able to provide help to their groupmates, and that being able to show that they understood the material made them feel like they belonged. In reference to helping her classmates, one student said, “When people know my abilities and don’t underestimate me, I feel a lot more valued.” Four of these students experienced initial growth followed by a decline in their sense of belonging, and each of these students tied this pattern to the difficulty of the material they were learning in class and the extent to which they understood the material. When asked about the changes in her sense of belonging, one student responded, “I think it really all had to do with our course material. As far as the social aspect of it, I don’t think anything has changed.” Further analysis of interview data is ongoing, and I anticipate drawing connections between these women’s sense of belonging and their opportunities to interact with their classmates during class.

**Table 1: Changes in Women’s Sense of Belonging during Fall 2022**

	Week 1	Week 7	Week 14
Sense of Belonging	3.9312	4.5488***	4.6159

*Note. Asterisks are used to denote the p-values (\* for  $p < .05$ , \*\* for  $p < .01$ , and \*\*\* for  $p < .001$  significance levels) corresponding to the significance of the difference between that time-point and the previous time-point.*

### Discussion

As Strayhorn (2012) argues, students’ sense of belonging is essential to their engagement and motivation in class. Research has shown that students, particularly women, feel a low sense of belonging in STEM, and this is a key reason for women’s decisions to leave STEM (Rainey et al., 2018; Seymour & Hunter, 2019; Shapiro & Sax, 2011). However, adjusting instructional approaches to incorporate more opportunities for students to interact offers promise in supporting students’ sense of belonging. In this study, women enrolled in a year-long active learning Calculus class reported a significant increase in sense of belonging from the beginning to the middle of the fall semester, followed by a plateaued sense of belonging from the middle to the end of the semester. These results are promising and show that women’s sense of belonging can in fact show positive change and that the increase can be maintained. Preliminary results of the interview analysis suggest that for women who reported large changes, opportunities to interact by sharing ideas and providing help supported their

sense of belonging. Further, perhaps incorporating opportunities to interact could be a way to support students' perceived competence when the course material gets more difficult. While this study's design does not allow for causal claims, these findings suggest a link between opportunities for women to interact with each other, their level of perceived competence, and their sense of belonging in Calculus. Further analysis will shed light on specific aspects of the learning opportunities they experienced that were particularly impactful to changes in their sense of belonging.

## References

- Anderman, L. H. (2003). Academic and social perceptions as predictors of change in middle school students' sense of school belonging. *The Journal of Experimental Education*, 72(1), 5–22.  
<https://psycnet.apa.org/doi/10.1080/00220970309600877>
- Bonwell, C. C., & Eison, J. A. (1991). *Active learning: Creating excitement in the classroom*. School of Education and Human Development, George Washington University.
- Carmichael, S. G. (2017). Women dominate college majors that lead to lower-paying work. Retrieved from Harvard Business Review website: <https://hbr.org/2017/04/women-dominatecollege-majors-that-lead-to-lower-paying-work>
- Chamberlain, A. (2017). The pipeline problem: How college majors contribute to the gender pay gap. Retrieved from Glassdoor website: <https://www.glassdoor.com/research/app/uploads/sites/2/2017/04/FULL-STUDY-PDF-Gender-Pay-Gap2FCollege-Major-1.pdf>
- Chen, X. (2013). STEM attrition: College students' paths into and out of STEM fields (NCES 2014-001). National Center for Education Statistics, Institute of Education Sciences, U.S. Department of Education.
- Cohen, J. (1988). *Statistical power analysis for the behavioral sciences* (2nd ed.). Hillsdale, NJ: Lawrence Erlbaum Associates, Inc. <https://doi.org/10.4324/9780203771587>
- Eagan, Kevin; Stolzenberg, Ellen Bara; Bates, Abigail K.; Aragon, Melissa C., Suchard, Maria Ramirez; Rios-Aguilar, C. (2016). *American Freshman: National Norms fall 2015*. Los Angeles: Higher Education Research Institute, UCLA.
- Freeman, S., Eddy, S. L., McDonough, M., Smith, M. K., Okoroafor, N., Jordt, H., & Wenderoth, M. P. (2014). Active learning increases student performance in science, engineering, and mathematics. *Proceedings of the National Academy of Sciences of the United States of America*, 111(23), 8410–8415.  
<https://doi.org/10.1073/pnas.1319030111>
- Good, C., Rattan, A., & Dweck, C. S. (2012). Why do women opt out? Sense of belonging and women's representation in mathematics. *Journal of Personality and Social Psychology*, 102(4), 700–717.  
<https://doi.org/10.1037/a0026659>
- Griffin, C. (2021). Calculus instruction and female sense of belonging. In Olanoff, D., Johnson, K., & Spritzer, S. M. (Eds.) *Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (p. 1331-1335). Philadelphia, PA.  
[http://www.pmena.org/pmenaproceedings/PMENA\\_43\\_2021\\_Proceedings.pdf](http://www.pmena.org/pmenaproceedings/PMENA_43_2021_Proceedings.pdf)
- Hausmann, L. R. M., Schofield, J. W., & Woods, R. L. (2007). Sense of belonging as a predictor of intentions to persist among African American and white first-year college students. *Research in Higher Education*, 48(7), 803–839. <https://doi.org/10.1007/s11162-007-9052-9>
- Lahdenperä, J., Postareff, L., & Rämö, J. (2019). Supporting quality of learning in university mathematics: A comparison of two instructional designs. *International Journal of Research in Undergraduate Mathematics Education*, 5(1), 75–96. <https://doi.org/10.1007/s40753-018-0080-y>
- Lewis, K. L., & Hodges, S. D. (2015). Expanding the concept of belonging in academic domains: Development and validation of the Ability Uncertainty Scale. *Learning and Individual Differences*, 37, 197-202.  
<https://doi.org/10.1016/j.lindif.2014.12.002>
- Rainey, K., Dancy, M., Mickelson, R., Stearns, E., & Moller, S. (2018). Race and gender differences in how sense of belonging influences decisions to major in STEM. *International Journal of STEM Education*, 5(10).  
<https://doi.org/10.1186/s40594-018-0115-6>
- Rasmussen, C., Apkarian, N., Hagman, J. E., Johnson, E., Larsen, S., & Bressoud, D. (2019). Characteristics of precalculus through calculus 2 programs: Insights from a national census survey. *Journal for Research in Mathematics Education*, 50(1), 98–111. <https://doi.org/10.5951/jresmetheduc.50.1.0098>
- Seymour, E., & Hunter, A.-B. (Eds.). (2019). *Talking about leaving revisited: Persistence, relocation, and loss in*
- Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.



undergraduate STEM education. Boulder, CO: Center for STEM Learning. <https://doi.org/10.1007/978-3-030-25304-2>

Shapiro, C. A., & Sax, L. J. (2011). Major selection and persistence for women in STEM. *New Directions for Institutional Research*, 14(7), 5–18. <https://doi.org/10.1002/ir>

Strayhorn, T. L. (2012). *College students' sense of belonging: A key to educational success for all students*. New York, NY: Routledge. <https://doi.org/10.4324/9780203118924>

Tinto, V. (1975). Dropout from higher education: A theoretical synthesis of recent research. *Review of Educational Research*, 45(1), 89–125. <https://doi.org/10.3102/00346543045001089>

## UTILITY VALUE INTERVENTION TO SUPPORT UNDERGRADUATE STUDENT INTEREST, ENGAGEMENT, AND ACHIEVEMENT IN CALCULUS AND CALCULUS-BASED PHYSICS

Viviane Seyranian, Ph.D.  
Cal Poly Pomona  
vseyranian@cpp.edu

Ian Thacker, Ph.D.  
UT San Antonio  
ian.thacker@utsa.edu

Nina Abramzon  
Cal Poly Pomona  
nabramzon@cpp.edu

Alex Madva, Ph.D.  
Cal Poly Pomona  
ammadva@cpp.edu

Paul Beardsley, Ph.D.  
Cal Poly Pomona  
pmbeardsley@cpp.edu

*The purpose of this study was to help undergraduate STEM students at a Hispanic-serving institution make connections between calculus and physics content and their lives using a utility-value intervention. As part of either a Calculus II or a calculus-based Newtonian Physics course, 471 undergraduate students were randomly assigned to either read essays written by peers emphasizing the usefulness of their coursework in their daily life or to a control group. We found that students in the intervention condition reported significantly higher levels of utility value, midterm grades, and earned higher course grades. A path analysis revealed that utility value indirectly improved achievement through interest and engagement factors. Findings support predictions stemming from Expectancy-Value theory and offer practitioners with principles for leveraging perceived utility value, interest, engagement, and achievement.*

Keywords: expectancy-value theory, utility value, calculus, physics

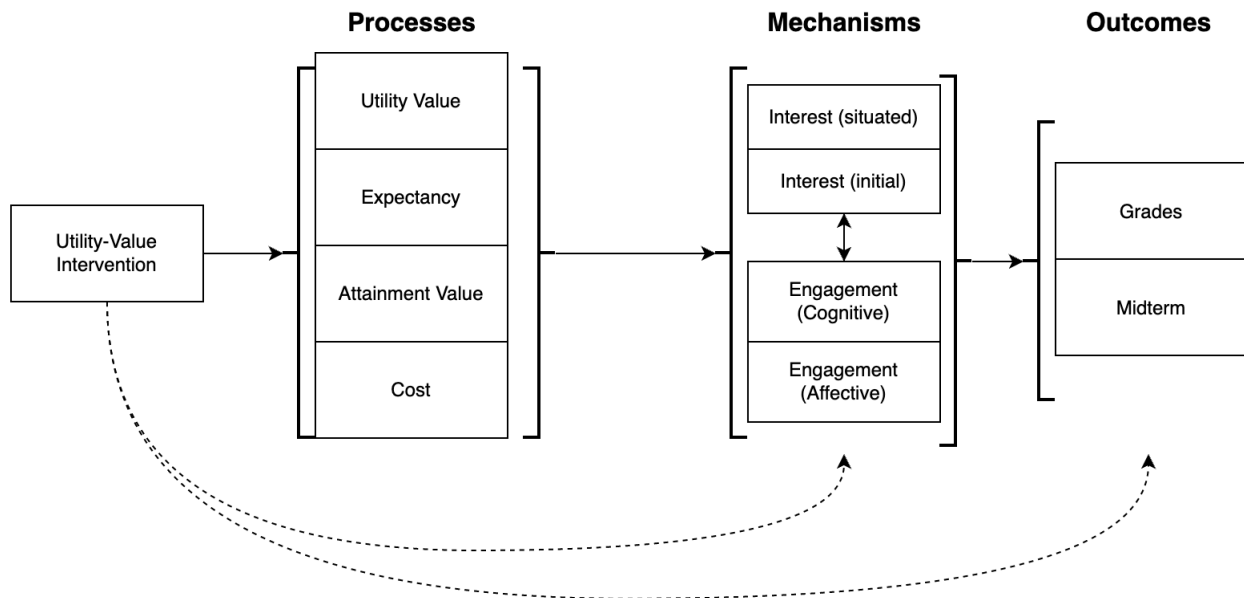
A wealth of research in education shows that student motivation is a key variable in fueling learning and successful academic performance (e.g., Gottfried et al., 2013; Vu et al., 2022). Student motivation is augmented when students place *value* on the content that they are learning (Wigfield et al., 2017). The expectancy-value theory (Eccles et al., 1983) underlines three ways that value is related to student motivation. First, students may be motivated by experiencing the *intrinsic value* of what they are learning or finding an activity enjoyable for its own sake. Second, they could be motivated by *attainment value*, which is viewing the learning as important to their identity. Finally, learning may be motivated by *utility value* or finding the learning experience useful now or in the future.

Drawing on utility value component of expectancy-value theory (Eccles et al., 1983), Utility Value (UV) interventions have been designed by social psychologists as interactive classroom-based assignments to help students make connection between their lives and the content they are learning (Hulleman & Harackiewicz, 2022). The assignments ask students to write an essay (e.g., Harackiewicz et al., 2016), a letter (e.g., Aron et al., 1991) or read quotations (e.g., Kosovich et al. 2019) about the utility of the course material. The interventions should be personal, specific, and context relevant to be effective. Overall, a recent meta-analysis of randomized field experiments shows that UV interventions significantly increase student learning and interest (Hulleman et al., 2018) and may be particularly effective for underrepresented students in higher education (Harackiewicz et al., 2016).

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

Hulleman and Harackiewicz (2022) propose a logic model that outlines the processes and mechanisms through which UV interventions influence grades, persistence, and career intentions. Drawing on the variables from this logic model, we put together a UV intervention model (Figure 1) that we tested using path model analyses with specific process and mechanism variables that could explain the relationship between UV intervention and outcomes. Specifically, drawing from expectancy-value theory (Eccles et al., 1983), the model predicts that the UV intervention would elicit higher levels of perceived utility value and expectancy, and lower cost (process variables). These processes variables, in turn, should predict student interest and engagement (mechanisms) which are predicted to increase learning outcomes such as grades.

**Figure 1.** Hypothesized Utility-Value Intervention Logic Model



*Note.* Model was inferred from the Hulleman & Harackiewicz (2022). All dependent variables used in the model were posttest scores. Not shown in this figure are pretest scores which were included in the model as covariates predicting each posttest value (namely, this pertains to utility value, expectancy, value, cost, situated and initial interest, cognitive and affective engagement, and midterm grades).

In this model, interest and engagement are viewed as important mechanisms that are signaled by values and predict learning outcomes. *Interest* is defined as stable tendencies to engage with a subject over time (individual interest), or as a short-term phenomenon that is aroused by context (situational interest; Durik & Harackiewicz, 2007; Rotgans & Schmidt, 2017). Similarly, *Engagement* is multidimensional and is regarded in the literature as having behavioral, emotional, and cognitive dimensions (Fredricks et al., 2004). Behavioral engagement includes actions such as attendance and participation. Positive emotional (or affective) engagement is characterized by experiencing joy or excitement. Cognitive engagement is the willingness to engage in effortful tasks, purposiveness, strategy use, and self-regulation.

Despite theory that predicts relationships between processes, mechanisms, and academic outcomes, few, if any, empirical studies have synthesized these constructs and tested this model.

### **Current Research**

The current study involves a randomized controlled trial examining the efficacy of a UV intervention in math and physics courses at a Hispanic-Serving Institution. It contributes

important insights to the literature on UV interventions in three important ways. First, the current study seeks to test the effectiveness of UV interventions in critical transition *math* (Calculus II) and *physics* (Newtonian Mechanics) courses at a Hispanic-Serving Institution in the Western United States. This allows us to examine whether the intervention is effective with diverse populations (particularly Latino/a/x) which is a gap in the UV literature. Second, the current study examines the efficacy of quotations as a variation of the UV intervention. Rather than write essays themselves, students read 4-6 quotations from students who have previously completed the course. Previous research supports the efficacy of quotation utility value in math courses (Gaspard et al., 2015; Kosovich et al. 2019), but no previous research has examined quotations in a four-year university context. Third, this study empirically includes both process and mechanism variables outlined in the Utility-Value Intervention Logic Model (see Figure 1), which, to our knowledge, have not been simultaneously tested in a single model.

### **Research Questions**

1. Will the UV intervention improve students' achievement outcomes (reported midterm grades, course grades) compared with a control?
2. Will the hypothesized relationships between task value processes and achievement outcomes be mediated by mechanisms of interest and engagement? (see Figure 1)

Based on predictions posed by the UV Intervention Logic Model (Hulleman et al., 2021), we hypothesized that the UV intervention would significantly improve students' achievement outcomes and this effect would be mediated by situated and initial interest as well as cognitive and affective engagement (see Figure 1).

## **Methods**

### **Study Context**

To answer our research questions, we initially recruited 518 undergraduate students from a general, calculus-based physics course (Newtonian Mechanics) and a Calculus II course. Five different instructors taught five sections of the physics course. One calculus instructor taught two sections of Calculus II and one instructor taught one section. We excluded 48 students from the study, 40 who did not meet our inclusion criteria because they were enrolled in two participating courses (both physics and mathematics), and 8 who opted not to participate. After these restrictions, our analytic sample was  $N = 471$ . Of this group, 5 were excluded from analyses when course grade was the main outcome because their grades were either unavailable, students did not consent to our using their grade, or they received a non-traditional course grade. Furthermore, 67 students did not complete a posttest, so were excluded from analyses where the main outcome was captured at posttest (i.e., midterm grades, career interest, and utility value).

Participating students were about 20.5 years old, first-generation college students (40% female (30%), male (70%), Hispanic (39%), Asian (33%), Black (4%), Native American (5%), Pacific Islander (11%), White (41%), Other race (6%). Most completed the intervention as part of a calculus-based Newtonian Physics course (86%), the remaining from a Calculus II course (14%). Over the course of the Fall 2020 semester, students were emailed four survey links 1–4 weeks apart. Students were emailed a pretest survey at the beginning of the semester, two waves of intervention links, and a posttest survey that was nearly identical to the pretest near the end of the semester. Below we provide details about the intervention, pretest, and posttest surveys.

### **The Utility Value Intervention**

As part of their regular coursework, on two separate weeks, students were given a link to a survey where they completed an assignment that was required for a small portion of course credit. Students were assigned to either the UV intervention group or a control group for both

assignments. Students in the control group were also asked to process and evaluate textbook excerpts from either a Newtonian physics textbook or from a Calculus II textbook depending on the course that the student was enrolled in. Students in the treatment group were similarly presented with texts presenting student quotations, but in this case, they emphasized the utility value of topics specific to either Newtonian Physics or Calculus II, depending on the course that the student was enrolled in, and asked a series of questions encouraging students to process and evaluate the student quotations (see Table 1 for examples).

Students in the intervention group engaged with a total of six student quotations (3 quotations X 2 waves) written from the perspective of peers at their university explaining how a particular topic they learned in their Calculus II or Physics course “is relevant in [their] own life.” The quotations were developed in a separate study over the course of several iterations of focus group interviews and were intended to pertain to topics they had recently covered in class. Some of the quotations were written by actual students (and edited by the research team) and some were created by the researchers. In all cases, students in the utility value intervention were asked to reflect on and engage with the student quotations by rating whether they “liked” them, explained why or why not, and then they ranked the quotations from their most to least favorite.

**Table 1. Sample Student Quotations from Utility Value Intervention and Control Group**

Course	Example Utility Value Narrative
Calculus-Based Physics	I loved playing soccer in high school. We had a great team in my senior year and were fighting really hard to win our final tournament game. I got the ball and saw an opening so I was taking the ball down the field as fast as I could and when I kicked the ball to my teammate, my ankle just gave out. I had to be carried off the field without being able to finish my last soccer game. Turns out I had torn the ligament in my ankle. I took me weeks to recover and a lot of physical therapy. At CPP, when I was studying for Physics 1510 one night, it hit me that the rotational forces could explain my ankle injury. The movement of the ankle is connected to work and energy, momentum and rotational motion. The ligament in my ankle was bent in a direction approximately perpendicular to the long bones of the leg. This tore my ligament because of the rotation of the bones beyond what was physiologically allowed. My doctor had me wear ankle wraps for a few weeks after the injury to provide some extra pressure to support my ankle, so it endures less force during recovery. It was not fun being injured and I hated being in crutches but now I know that I going to pursue a career in biomechanics so I can work on technologies to help athletes with injuries. (Jenna, Age 20, major: Mechanical Engineering)
Calculus II	A lot of people will say that you never need high level math in real life. This is true for most people, but if you are taking calc 2 then you’re probably going to take harder math classes in the future and you WILL need what you learn. I am a mechanical engineering major and I use calc 2 all the time. I use integration by parts in my physics classes. Calc 2 is also the foundation for three variable calculus, which is basically 3D math. Any real-life object is going to be 3D so calc 2 and 3 is really important for modeling real things. I also know that other high level math courses like linear algebra, differential equations, even statistics all involve integration and sequences and series. It’s like calc 2 is another step on the mathematical ladder. Without it, you can’t get to the next step. (Esmeralda Lopez, 20, major: Mechanical Engineering).

*Note.* All student names shown were pseudonyms.

## Materials

As noted, the pretest was identical to the posttest, both of which consisted of 9 questionnaires (Utility value, expectancy, attainment value, cost, interest, cognitive engagement, affective engagement, STEM career interest, and a demographics questionnaire). The posttest also included a demographics questionnaire, which included self-reported grades for course exams and a question requesting permission to use students’ course grades in our analyses.

### *Outcome Measures*

**Grade.** With students' consent, their course grades were accessed for use as a main outcome in our study. Grades were on a four-point scale (e.g., 4.0 = A+, 2.19 = C-).

**Self-Reported Midterm Grades.** In the posttest given near the end of the semester, we asked students to report a percentage score for their first and second midterm. The second midterm score was used as a main outcome and the first midterm as a covariate, when applicable.

**Utility Value.** UV was measured using four items adapted from Hulleman et al. (2010). This scale asked students to report their feelings that what they were learning was relevant to their life, and consisted of four items that differed depending on which course the student was taking (e.g., "[Calculus II/Physics] can be useful in everyday life." Responses were on a five-point agreement scale and were reliable at conventional levels ( $\alpha = .84$  pre,  $.90$  post). Given that the interventions target UV, we used this variable as an outcome in a manipulation check, but also used the variable as a covariate in central analyses predicting academic performance.

### **Predictors**

**Expectancy.** Expectancy was measured using three items adapted from Kosovich et al., (2015) at pre- and posttest. Items asked participants to rate their agreement on a five-point scale to statements about their expectancy for success (e.g., "I believe that I can be successful in my [Calculus II/Physics] class"). The scale was reliable at conventional levels ( $\alpha = .92$  pre,  $.93$  post).

**Attainment Value.** Attainment value was measured at pre- and posttest using three items from Kosovich et al., (2015) measuring perceived importance of the class on a five-point agreement scale (e.g., "I think [Calculus II/Physics] class is important.";  $\alpha = .92$  pre,  $.92$  post).

**Cost.** Perceived cost was measured at pre- and posttest using four items from Kosovich et al., (2015) measuring on a five-point agreement scale, the perceived cost of time and effort required for success (e.g., "I have to give up too much to do well in my [Calculus II/Physics] class."  $\alpha = .86$  pre,  $.85$  post).

**Interest (situated and initial).** STEM interest was measured at pre- and posttest with a scale adapted from Hulleman et al., (2010) that assesses both situational and initial interest. The initial interest scale consisted of eight items assessing interest in the subject (e.g., "I think [Calculus II/Physics] is an interesting subject.") on a seven-point agreement scale ( $\alpha = .93$  pre,  $.94$  post). The situated interest subscale consisted of five items measuring interest as situated in the course (e.g., "I think that what we're learning in this class is fascinating."  $\alpha = .89$  pre,  $.88$  post) on the same response scale.

**Cognitive Engagement.** Cognitive engagement was measured at pre- and posttest using 16 adapted items from two cognitive engagement scales (Greene & Miller, 1996; Pintrich et al., 1991). Items assessed participants' reported levels of cognitive effort when learning (e.g., "I make sure I understand material that I learn in [Calculus II/Physics]",  $\alpha = .89$  pre,  $.90$  post).

**Affective Engagement.** Affective engagement was measured at pre- and posttest using four items adapted from Thompson (2007). The items asked participants to report on a five-point scale (1 = *Never* to 5 = *Always*) how often they felt "inspired," "determined," "attentive," and "active" ( $\alpha = .79$  pre,  $.81$  post).

**Demographic Factors.** Lastly, participants completed demographics items. In the current study, we included factors of race/ethnicity, gender, and first-generation status as covariates. For analyses, we recoded gender as a dichotomous variable (1 = *Female*, otherwise 0), as well as race/ethnicity (1 = *African/African American/Black or Hispanic/Latino*, 0 = otherwise), and First-Generation College student status (coded as 1 for students who reported that neither of their parents received a degree from a four-year college or graduate degree, and 0 otherwise).

## Results

Independent samples t-tests revealed that there were no significant differences in pre-test scores between control or treatment groups with the analytic sample for all outcomes (all  $p$ s > .31) and many predictors ( $p$ s > .060). The exceptions were that pre-test situated interest ( $M_{\text{control}} = 5.12$ ;  $M_{\text{treatment}} = 4.84$ ;  $p = .03$ ), pretest expectancy ( $M_{\text{control}} = 4.92$ ,  $M_{\text{treatment}} = 4.68$ ;  $p = .014$ ), and cognitive engagement ( $M_{\text{control}} = 4.71$ ,  $M_{\text{treatment}} = 4.51$ ,  $p = .028$ ) were higher in the control group than in the treatment group. To account for these differences and to improve power, we included pre-test scores as covariates in all models.

### Effects of our Intervention on Utility Value and Academic outcomes (RQ1)

To test the effects of the intervention, we ran linear regressions with academic achievement outcomes and utility value as main outcomes. We also included instructor fixed effects to account for classroom-level differences (e.g., to account for differences in teachers' grading patterns). In separate models, the outcomes were course grade, reported second midterm grades, and utility value. To improve power, we included pre-intervention scores as covariates in all models, with the exception of course grades because course grades were not assigned prior to the intervention. To assess the effectiveness of the math versus physics intervention, we also tested models after including course type (physics or math) as a moderator of the intervention effects. See Table 2 for a summary of all regression models with all relevant coefficients.

**Table 2.**  
OLS Regression Analyses After Adjusting for Teacher Fixed Effects (N = 471)

	Course Grade		Reported Midterm 2 Grade			Utility Value		
Treatment	<b>0.188~</b> <b>(0.105)</b> <b>p = .077</b>	<b>0.255*</b> <b>(0.113)</b> <b>p = .025</b>	<b>3.523*</b> <b>(1.702)</b> <b>p = .040</b>	<b>3.445*</b> <b>(1.670)</b> <b>p = .040</b>	<b>4.875**</b> <b>(1.817)</b> <b>p = .008</b>	0.087 (0.136) p = .525	<b>0.271**</b> <b>(0.101)</b> <b>p = .008</b>	<b>0.279*</b> <b>(0.110)</b> <b>p = .012</b>
Pretest Score				0.081*** (0.022) p < .001	0.073** (0.023) p = .002		0.731*** (0.043) p < .001	0.731*** (0.043) p < .001
Treatment * Math Course		-0.5 (0.308) p = .105			-9.088~ (4.653) p = .052			-0.054 (0.278) p = .847
Observations	465	465	321	321	321	361	330	330
R-Squared	0.007	0.013	0.014	0.053	0.065	0.001	0.479	0.479

*Note:* Reference group for “Treatment” variable is “Control,” and for “Math Course” is “Physics Course” ~ $p < .1$ ; \* $p < .05$ ; \*\* $p < .01$ ; \*\*\* $p < .001$

We found that post-test **utility value** was significantly higher in the treatment condition when compared with the control condition ( $d = .08$ ) after adjusting for pre-test utility value. This finding remained significant after including course type as a moderator. These findings serve as a manipulation check to ensure that the intervention was working as expected.

We found a marginally significant effect of treatment on **course grades** ( $d = .17$ ). This effect became significant after adjusting for the treatment's interaction with course type. This suggests that the treatment was slightly less effective in the math class, but not significantly so.

We found that reported **midterm grades** were significantly higher among individuals in the treatment condition ( $d = .26$ ) before and after adjusting for prior midterm scores. That is, individuals in the intervention condition reported scoring about 3.4 percentage points higher than those in the control condition. The effect remained significant after including course type as a moderator, revealing marginally stronger effects ( $p = .052$ ) of the treatment in the physics courses when compared with the math courses.

We also ran exploratory analyses in which we reran the previous models but with race, gender, first generation college student status, and number of treatments as moderators of the effects of the intervention. We found no significant moderation effects (all  $p > .087$ ).

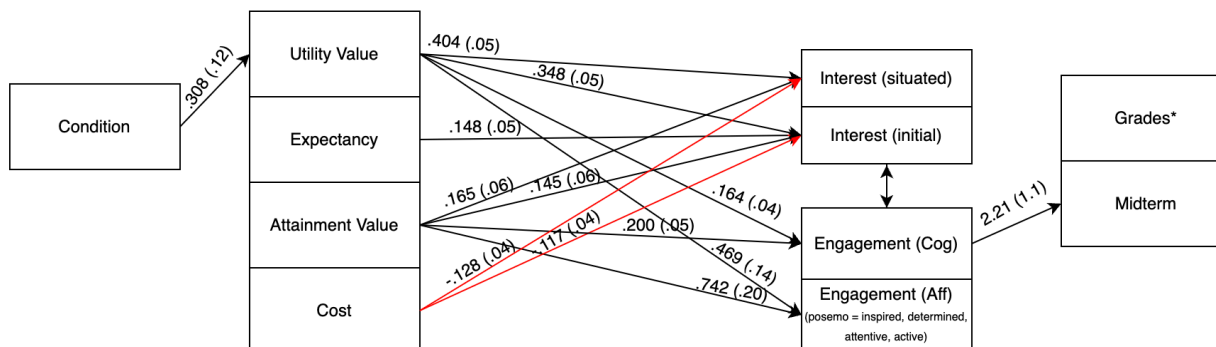
### **Path Analysis (RQ2)**

We ran a path model to investigate which relationships hypothesized by the UV Intervention Logic Model were significant for our intervention. Specifically, we tested the hypothesized model shown in Figure 1, which depicts the intervention condition predicting process variables (UV, expectancy, attainment value, and cost), followed by mechanism variables (STEM interest [initial and situated]), and engagement (cognitive and affective), followed by academic outcomes (course grades and midterm grades). In all cases, scores at posttest were of primary interest, although pretest scores were included as covariates predicting each posttest value whenever available (namely, pertaining to UV, expectancy, attainment value, cost, situated and initial interest, cognitive and affective engagement, and midterm grades). We allowed for all variables at each of the process, mechanism, outcome stages to correlate and initially included pre-intervention variables to covary with their post-intervention counterpart whenever available in all models. See Table 3 for intercorrelations between variables. All analyses were done with the "lavaan" package in R version 4.0.2 (Rosseel, 2012).

As is typical of path modeling, we underwent model modifications to improve the model fit. The initial model had fit statistics that were not satisfactory at conventional levels (Chi-sq = 802,  $df = 129$ , CFI = .816, TLI = .73, AIC = 16534, RMSEA = .141; Hu & Bentler, 1999). As such, we computed the modification indices and included additional paths between pretest scores with all modification indices larger than 45. Fit improved (Chi-sq = 359,  $df = 113$ , CFI = .93, TLI = .89, AIC = 16023, RMSEA = .091). Figure 2 shows the full path model with all coefficients.



**Figure 2. Path Model with Significant Paths Shown**



*Note.* Only paths that are significant at the .05 level are shown, red paths are used when coefficients are negative. All variables shown represent values at posttest. Not shown in this figure are pretest scores which were included in the model as covariates predicting each posttest value

**Interventions influenced processes.** As depicted in the figure, there was a significant effect of UV condition on reported UV ( $p = .008$ ), with no significant effects on expectancy, attainment value, or cost ( $ps = .573, .473, .692$  respectively).

**Processes predicted mechanisms.** These four process variables were significantly associated with mechanism variables. Namely, UV was significantly and positively associated with situated and initial interest, and cognitive and affective engagement (all  $ps < .001$ ). Expectancy positively predicted initial interest ( $p = .004$ ), but not situated interest, or cognitive or affective engagement ( $ps = .139, .062, .957$  respectively). Attainment value positively predicted situated and initial interest and cognitive and affective engagement (all  $ps = .009$  and  $.015$ , and less than  $.000$  and  $.000$  respectively). Cost negatively predicted situated and initial interest (both  $ps = .001$ ), but did not significantly predict cognitive or affective engagement ( $ps = .156$  and  $.724$  respectively).

**Mechanisms predicted outcomes.** Of the process variables, reported midterm grades were significantly and positively predicted by cognitive engagement ( $p = .047$ ), but not by situated or initial interest, or affective engagement ( $ps = .386, .160, \text{ and } .750$  respectively). Grades were not significantly predicted by any of the mechanism variables of situated / initial interest or cognitive / affective engagement ( $ps = .737, .290, .747, \text{ and } .976$  respectively).

### Significance

We sought to investigate whether a UV intervention would improve a diverse group of students' achievement outcomes in advanced calculus and physics, and whether such relationships were mediated by student interest and engagement in course content. We found that students who were asked to engage with student quotations emphasizing useful applications of content they were learning in their course (either relating to calculus 2 or Newtonian physics) indeed significantly improved perceptions of utility value of course content compared with a control group. Students in the UV intervention condition also had higher end-of-course grades compared with the control group ( $d = 0.17$ ) and reported scoring about a quarter of a standard deviation higher on their second midterm exams ( $d = 0.26$ ). These findings are consistent with prior research showing that utility value interventions can support achievement outcomes for undergraduate students in STEM (c.f., Hulleman et al., 2021) and support the idea that, to improve motivation and achievement, it is important for practitioners to remind students how concepts they are learning will be useful for their lives (c.f. Thacker et al., 2022).

Furthermore, we tested predictions that interest and engagement may be important mechanisms underlying relations between utility value and achievement outcomes. We found that the relationships predicted by the Utility Value Intervention Logic Model had satisfactory fit. Indeed, psychological processes of value perceptions were significant predictors of psychological mechanisms (such as individual/situated interest and affective/cognitive engagement) which significantly predicted achievement outcomes (grades and self-reported midterm scores). Although we did not find significant *indirect* effects of the intervention on achievement through process and mechanism variables (likely due to only modest effects of the intervention), our findings provide emerging evidence that interest and engagement may be important underlying mechanisms by which utility value operate.

Although our findings span a semester of growth, future research might investigate more long-term effects of UV interventions using longitudinal methods. Future research might also use qualitative methods to further illuminate why UV interventions lead to improved achievement outcomes, and in particular, explore to what extent students perceive such interventions to support their interest and engagement.

### References

- Aron, A., Aron, E. N., Tudor, M., & Nelson, G. (1991). Close relationships as including other in the self. *Journal of Personality and Social Psychology*, 60(2), 241-253. <https://doi.org/10.1037/0022-3514.60.2.241>
- Durik, A. M. & Harackiewicz, J. M. (2007). Different strokes for different folks: How individual interest moderates the effects of situational factors on task interest. *Journal of Educational Psychology*, 99(3), 597-610.
- Eccles, J., Adler, T. F., Futterman, R., Goff, S. B., Kaczala, C. M., Meece, J. L., & Midgley, C. (1983). Expectancies, values, and academic behaviors. In J. T. Spence (Ed.), *Achievement and achievement motives: Psychological and sociological approaches* (pp. 75–146). San Francisco: W. H. Freeman.
- Fredricks, J. A., Filsecker, M., & Lawson, M. A. (2016). Student engagement, context, and adjustment: Addressing definitional, measurement, and methodological issues. *Learning and Instruction*, 43, 1-4.
- Greene, B. A., & Miller, R. B. (1996). Influences on achievement: Goals, perceived ability, and cognitive engagement. *Contemporary Educational Psychology*, 21(2), 181-192.
- Harackiewicz, J.M., Canning, E.A., Tibbetts, Y., Priniski, S.J., & Hyde, J.S. (2016). Closing achievement gaps with a utility-value intervention: Disentangling race and social class. *Journal of Personality and Social Psychology*, 111, 745-765. <https://doi.org/10.1037/pspa0000049>
- Hu, L. T., & Bentler, P. M. (1999). Cutoff criteria for fit indexes in covariance structure analysis: Conventional criteria versus new alternatives. *Structural Equation Modeling: A Multidisciplinary Journal*, 6(1), 1-55.
- Hulleman, C. S., Godes, O., Hendricks, B. L., & Harackiewicz, J. M. (2010). Enhancing interest and performance with a utility value intervention. *Journal of Educational Psychology*, 102(4), 880.
- Hulleman, C. S., & Harackiewicz, J. M. (2021). The utility-value intervention. In G. M. Walton & A. J. Crum (Eds.), *Handbook of wise interventions: How social psychology can help people change* (pp. 100–125). The Guilford Press.
- Hulleman, C.S., Wormington, S. V., Tibbetts, C. Y., & Philipoom, M. (2018, August). A meta-analytic synthesis of utility-value interventions in education. Paper presented at the bi-annual meeting of the International Conference on Motivation. Aarhus, Denmark.
- Gaspard, H., Dicke, A.-L., Flunger, B., Brisson, B. M., Häfner, I., Nagengast, B., & Trautwein, U. (2015). Fostering adolescents' value beliefs for mathematics with a relevance intervention in the classroom. *Developmental Psychology*, 51(9), 1226-1240. <https://doi.org/10.1037/dev0000028>
- Gottfried, A. E., Marcoulides, G. A., Gottfried, A. W., & Oliver, P. H. (2013). Longitudinal pathways from math intrinsic motivation and achievement to math course accomplishments and educational attainment. *Journal of Research on Educational Effectiveness*, 6(1), 68–92. <https://doi.org/10.1080/19345747.2012.698376>.
- Kosovich, J.J., Hulleman, C.S., Phelps, J., & Lee, M. (2019). Improving algebra success with a utility-value intervention. *Journal of Developmental Education*, 42 (2), 1-10.
- Pintrich, P.R., Smith, D.A.F., Garcia, T., McKeachie, W.J. (1991). *A manual for the use of the Motivated Strategies for Learning Questionnaire (MSLQ)* (Tech. Report No. 91-B-004). Board of Regents, University of Michigan, Ann Arbor, MI.
- Rosseel, Y. (2012). lavaan: An R package for structural equation modeling. *Journal of Statistical Software*, 48, 1-36.
- Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

- Rotgans, J. I., & Schmidt, H. G. (2017), Interest development: Arousing situational interest affects the growth trajectory of individual interest. *Contemporary Educational Psychology*, 49, 175-184.
- Thacker, I., Seyranian, V., Madva, A., Duong, N., & Beardsley, P. (2022). Social connectedness in physical isolation: Online teaching practices that support underrepresented undergraduate students' feelings of belonging and engagement in STEM. *Education Sciences*, 12(2), 61. <https://doi.org/10.3390/educsci12020061>
- Thompson, E. R. (2007). Development and validation of an internationally reliable short-form of the positive and negative affect schedule (PANAS). *Journal of Cross-Cultural Psychology*, 38(2), 227-242.
- Vu, T., et al. (2022). Motivation-Achievement Cycles in Learning: a Literature Review and Research Agenda. *Educational Psychology Review* 34 (1), 39–71. <https://doi.org/10.1007/s10648-021-09616-7>
- Wigfield, A., Rosenzweig, E. Q., & Eccles, J. S. (2017). Achievement values. In A. Elliot, C. Dweck, & D. Yeager (Eds.), *Handbook of competence and motivation: Theory and application* (pp. 116-134). Guilford Press New York, NY.

# CHARACTERIZING CALCULUS I STUDENTS' PARTICIPATION IN THE PARALLEL SPACES OF COURSEWORK AND INQUIRY-ORIENTED SUPPLEMENTAL INSTRUCTION

Karmen Yu  
Montclair State University  
Yuk4@montclair.edu

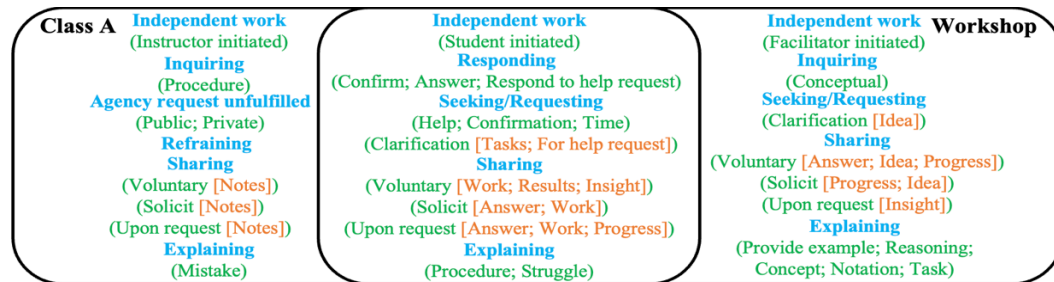
Keywords: Calculus, Problem Solving, Undergraduate Education, Classroom Discourse.

Calculus has long been a “gateway course” to STEM fields in postsecondary education (Hagman et al., 2017). Informed by the MAA’s recommendations (Bressoud et al., 2015), researchers at Mountain State University (pseudonym) designed a model of supplemental instruction to address this pressing issue. This model features peer-facilitated workshops where Calculus I students work in groups on inquiry-oriented, groupworthy tasks (Cohen & Lotan, 2014).

The purpose of this poster is to present preliminary findings of a study that is seeking to answer the question, *How do undergraduate students experience and navigate their learning of calculus in the parallel spaces of coursework and inquiry-oriented supplemental instruction?* These findings are characterizations of students’ participation that emerged (Corbin & Strauss, 2014) from my analysis and that will be used to answer this question.

This study uses a multiple-case study methodology (Merriam, 1998), is framed from a situated perspective (Lave & Wenger, 1991), and uses Holland et al.’s (1998) concept of figured worlds to analyze changes in agentic participation and in relation to identity formation (Vågan, 2011). The participants are Calculus I students who attended the workshops as part of their course requirements. Video recordings were taken of 24 classes, six workshops, and three focus group interviews (Creswell, 2012).

Preliminary findings of the more common forms of participation are depicted in the Venn diagram in Figure 1. Due to space constraints, we only highlight the variety of agentic opportunities that invite students’ participation in each of the two spaces, the contrasts among them, and their complementary nature relative to learning calculus with understanding.



**Figure 1: Common forms of agentic participation across the two spaces.**

As this study progresses, these findings will be used to explore how students navigate the contrasting norms of participation in the two spaces, if there is any “crossover” effect (e.g., due to a desire for more agentic forms of participation in a setting that often renders them passive), and whether there might be new opportunities for students’ agentic participation embedded in

conventional coursework. These findings should be of interest to undergraduate mathematics educators who wish to correct the historic trend of low performance in calculus.

### References

- Bressoud, D., Mesa, V., & Rasmussen, C. (Eds.). (2015). *Insights and recommendations from the MAA national study of college calculus*. The Mathematics Association of America.
- Cohen, E. G., & Lotan, R. A. (2014). *Designing groupwork: Strategies for the heterogeneous classroom* (3rd ed.). Teachers College Press.
- Corbin, J., & Strauss, A. (2014). *Basics of qualitative research: Techniques and procedures for developing grounded theory* (4th ed.). Sage.
- Creswell, J. W. (2012). *Educational research: Planning conducting and evaluating quantitative and qualitative research* (4th ed.). Pearson.
- Hagman, J. E., Estrella, J., & Fosdick, B. K. (2017). Factors contributing to students and instructors experiencing a lack of time in college calculus. *International Journal of STEM Education*, 4(1), 15.
- Holland, D., Lachicotte Jr., W., Skinner, D., & Cain, C. (1998). *Identity and agency in cultural worlds*. Harvard University Press.
- Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. Cambridge University Press.
- Merriam, S. B. (1998). *Qualitative research and case study applications in education: Revised and expanded from case study research in education*. Jossey-Bass Publishers.
- Vågan, A. (2011). Towards a sociocultural perspective on identity formation in education. *Mind, Culture, and Activity*, 18(1), 43-57.

# EXAMINING OPPORTUNITIES TO LEARN DEFINITE INTEGRALS IN WIDELY USED CALCULUS TEXTBOOKS

Dae S. Hong  
University of Iowa  
dae-hong@uiowa.edu

*This study explores opportunities to learn definite integrals in three widely used textbooks in the U.S. Definitions, worked examples, and exercise problems were coded using research – based cognitive resources in definite integrals. The results show that limited opportunities for students to explore multiplicative relationship between two quantities and adding small pieces to understand definite integrals.*

Keywords: cognitive resources, integral, learning opportunities, textbooks

## Introduction

In the United States, Calculus 1 is a core and beginning course for students heading into the disciplines of science, technology, engineering, and mathematics (STEM). Many experts in teaching calculus view an understanding of the integral as a necessary central concept or skill for genuine understanding of first-year calculus (Jones, 2013; Sofronas et al., 2011). While understanding of the integral is necessary, several existing studies demonstrate students' challenges as well as types of knowledge needed to understand the integral concept (Jones, 2013; Mahir, 2009; Sealey, 2014), illustrating that, in many cases, students are able to correctly integrate functions but do not know how to interpret it in context and tend not to use types of knowledge needed to understand integral conceptually. In this study, we examined how widely used calculus textbooks provide learning opportunities to students. These are our research questions that we attempted to answer.

- What units of knowledge about definite integral do widely used calculus textbooks provide to students?
- How do widely used calculus textbooks provide opportunities for students to be familiar with adding up pieces symbolic form to introduce the concepts of integral?

## Related literature

### How students learn integral concepts

According to experts in teaching calculus, there are three significant interpretations of definite integrals: 1) the integral as net change or accumulated total change, 2) the integral as area, and 3) techniques of integration (Ely, 2017; Sofronas et al., 2011). Previous studies show that students are able to procedurally calculate integrals but are not able to explain what they are doing and are not able to define the definite integral (Rasslan & Tall, 2002). What we can learn from these studies is students are very familiar with certain interpretations of the integral (area under a curve, anti – derivative, or techniques of integration), but it is challenging for them to understand why the definite integral represents area under a curve or in other contexts such as physics and engineering (Ely, 2017; Jones, 2015; Sealey, 2014). More recently, in addition to pointing out students' familiarity to certain interpretations of integrals, researchers have explored questions about types of knowledge students need and coordinate to expand their understanding

of definite integral (Larsen, Marrongelle, Bressoud, & Graham, 2016). This idea led to examination of small units of ideas that students need to activate to expand their understanding of definite integral (Jones, 2013) because it might be too simplistic to say that understanding mathematical topics requires a single idea (Harlow & Bianchini, 2020). With examination of small units of ideas, in addition to think about area under a curve, and anti – derivative, we can look at more fine-grained element of knowledge students need to coordinate to expand meaning of definite integrals. Sealey (2014) described four different layers – limit, summation, function, and product – in understanding definite integrals. Among these, students struggled most to understand the product layer that describes the product of two quantities,  $f(x_i)$  and  $\Delta x$  (Sealey, 2014). Jones (2013) found students' various cognitive resources (“fine-grained” elements of knowledge in a person’s cognition) in understanding integrals, claiming that students may demonstrate challenges in understanding integrals because productive cognitive resources were not activated. He observed a variety of cognitive resources activated by students in terms of symbolic forms, the perimeter and area symbolic form, the function matching symbolic form, and the *adding up pieces* symbolic form (Jones, 2013). Among these, Jones (2013) found that adding up pieces was most productive in understanding definite integral. Since recognizing the multiplicative relationship in adding up pieces is useful in applied contexts in physics and engineering (Ely, 2021; Jones, 2013; Sealey, 2014), with adding up pieces, students may possess units of knowledge that enable them to expand their understanding of definite integrals. Thus, being able to use adding up pieces along with the product layer may be a key units of knowledge students need to possess (Jones, 2015).

### **Exploring calculus students’ learning opportunities**

As Reeves, Carnoy, and Addy (2013) described, we can examine content coverage (list of topics and subtopics covered), content exposure (amount of time spent to instruction), and (c) content emphasis (which topics are selected for emphasis) to examine students’ learning opportunities. When particular procedures, algorithms, and problems are presented in textbooks, teachers and students can have potential learning opportunities to become familiar with those procedures and problems. When teachers prepare and plan their lessons, they select (and modify) tasks and activities from textbooks in planning their lessons (Remillard & Heck, 2014). When selected tasks and activities are included in the lessons and enacted, what is included from textbooks and lessons transform into learning opportunities for students. Additionally, when some tasks are assigned as homework or when students read textbooks to study on their own, those can be transformed into learning opportunities for students. We think of calculus students’ opportunities to become familiar with the concepts of definite integrals. If widely used textbooks give more attention to the perimeter and area symbolic form or the function matching symbolic form than adding up pieces (less coverage, exposure and emphasis) and calculus instructors use those ideas, students’ understanding may be limited opportunities to those ideas that they learn from textbooks. Limited coverage in widely used textbooks may lead them to have units of knowledge allow them to interpret integrals as area under a curve or anti – derivative but be unable to interpret definite integrals in other contexts or not feel the need to learn and interpret integral in other contexts.

## **Methods**

### **Data source**

Three widely used Calculus 1 books, Calculus 7E 7th Edition (Stewart, 2010, abbreviated to SC), Calculus Single & Multivariable 7th Edition (Hughes-Hallett et al., 2017, abbreviated to HC), and Thomas’ Calculus 13th Edition (Thomas et al., 2014, abbreviated to TC) were selected

for analysis (Bressoud, 2011), we analyzed Riemann sum and the first integral sections from each textbook. In total, 128 integral tasks (both worked examples and exercise problems) from Stewart, 156 tasks from Thomas, and 113 tasks from Hughes-Hallett et al. were analyzed.

### Textbook analysis

We referred to several textbook content studies to shape our analytic framework (Charalambous et al., 2010; Hong, 2023; Hong, Choi, Runnalls, & Hwang, 2018; Son & Hu, 2016). First, how the three textbooks introduce definite integral concepts was examined. Second, we examined each worked example and exercise problem and coded each item (one item can be one worked example or one exercise problem). We were interested in finding problems and worked examples that explore adding up pieces and the product layer (multiplicative relationship) in addition to the familiar interpretations of area under a curve and anti-derivatives (or techniques of integrations).

### Examples of coding

Table 1 are codes that we used and terms/mathematical expressions that we looked for. Examples of these codes are provided in this section.

**Table 1: Analytic Framework**

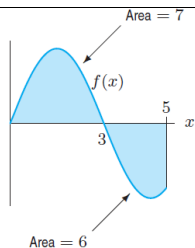
Code	Terms/mathematical expression
Adding up pieces	Adding rectangles under a curve as the number of rectangles increases
	Multiplication between quantities relevant to definite integrals
	Meaning of each symbol
Area under a curve	Finding area under a curve without connecting it to adding rectangles
Anti-derivative (function matching)	Finding anti-derivative to evaluate integral
Matching mathematical symbols to integral	Matching each mathematical symbol in Riemann sum to integral without describing what each means
Evaluating Riemann sum or integral with finite number of rectangles	Evaluating Riemann sum or integral with several rectangles
Using integral properties	Verifying various properties of integrals

Table 2 shows examples of each code that we used for integral tasks from the three textbooks. These codes show types of knowledge that students will be exposed to when they use these textbooks.

**Table 2: Coding examples**

Item	Code
If your CAS can draw rectangles associated with Riemann sums, use it to draw rectangles associated with Riemann sums that converge to the integrals. Use $n = 4, 10, 20,$ and $50$ subintervals of equal length in each case. $\int_0^1 (x^2 + 1) dx = \frac{4}{3}$ (TC, p. 274)	Adding up pieces
What is the area between the graph of $f(x)$ in Figure 5.35 and the $x$ -axis, between $x = 0$ and $x = 5$ ? (HC, p. 287)	Area under a curve





Prove that $\int_a^b x \, dx = \frac{b^2 - a^2}{2}$ (SC, p. 377)	Anti-derivative (function matching)
Express the limits in Exercises 1-8 as definite integrals.	Matching mathematical symbols to integral
$\lim_{\ p\  \rightarrow 0} \sum_{k=1}^n c_k^2 \Delta x_k$ , where p is a partition of [0, 2] (TC, p. 274)	
Estimate $\int_0^1 e^{x-2} \, dx$ using n = 5 rectangles to form a left-hand sum (HC, p. 288).	Evaluating Riemann sum or integral using several rectangles
Use the properties of integrals to verify the inequality without evaluating the integrals.	Using integral properties
$\int_0^1 \sqrt{1+x^2} \, dx \leq \int_0^1 \sqrt{1+x} \, dx$ (SC, p. 378).	

The first example shows that evaluating definite integral can be done with several rectangles. Students are asked to think about what will happen if more rectangles are added, which gives them ideas of adding up small pieces to get area. The second example is about area under a curve. This example does not ask students to relate the area to rectangles. It simply asks them to see the shaded regions represent the area under a curve. The third example can be answered if students know how to find an anti-derivative of  $x$ . Knowing how to find anti-derivative of  $x$  will be enough to answer this correctly. The fourth example asks students to match Riemann sum to definite integral, matching each symbol to the corresponding symbol of definite integral. Although they may see the product layer (how  $f(c_k)$  are replaced  $f(x)$  and how the subinterval widths  $\Delta x_k$  become the differential  $dx$ ), it is possible that students just simply match each symbol in Riemann sum to each symbol in integral. The fifth example asks students to use five rectangles, a finite number, instead of conceptualizing many rectangles to estimate  $\int_0^1 e^{x-2} \, dx$ . The last example asks students to verify an inequality using one of the properties of integral. This can be answered correctly once the graphs of two functions are drawn. This requires understanding of area, but this example may not lead students to think about infinitely many rectangles.

### Reliability

We reviewed previous studies carefully to develop our codes (Jones, 2013; Sealey, 2014). We were able to develop a few codes as we examined textbooks. Once developed, two independent readers reviewed the textbooks several times and compared findings to ensure reliability of results. The initial agreement rate for task discussions was 95%. Discussion was done until there was 100% agreement.

## Results

### Introduction and definition of definite integral

Figure 1 shows the definition of definite integral from the three textbooks.

Stewart	Thomas	Hughes-Hallett
---------	--------	----------------

<p>For now, <math>dx</math> the symbol has no meaning by itself; <math>\int_a^b f(x)dx</math> is all one symbol. The <math>dx</math> simply indicates that the independent variable is <math>x</math>. So Definition 2 says that the definite integral of an integrable function can be approximated to within any desired degree of accuracy by a Riemann sum.</p> <p>If the subinterval widths are <math>\Delta x_1, \Delta x_2 \dots \Delta x_n</math>, we have to ensure that all these widths approach 0 in the limiting process (p. 366- 368).</p>	<p>The sum symbol <math>\Sigma</math> is replaced in the limit by the integral symbol <math>\int</math>, whose origin is in the letter "S" The function values <math>f(c_k)</math> are replaced by a continuous selection of function values <math>f(x)</math>. The subinterval widths <math>\Delta x_k</math> become the differential <math>dx</math>. It is as if we are summing all products of the form <math>f(x) \cdot dx</math> as <math>x</math> goes from <math>a</math> to <math>b</math>. (p. 263)</p>	<p>Now we take the limit of these sums as <math>n</math> goes to infinity. If <math>f</math> is continuous for <math>a \leq t \leq b</math>, the limits of the left- and right-hand sums exist and are equal. The <i>definite integral</i> is the limit of these sums.</p> <p>The " <math>\int</math> " notation comes from an old-fashioned "S," which stands for "sum" in the same way that <math>\Sigma</math> <b>does</b>. The "dt" in the integral comes from the factor <math>\Delta t</math> (p. 282).</p>
--	---	---

**Figure 1: Introduction of definite integral**

Prior to introducing definite integrals, all three textbooks had a section introducing Riemann sum and showed how  $f(c_k) \cdot \Delta x_k$  would provide the area of each rectangle. Moreover, all three textbooks explored the distance traveled problems using the product layer,  $f(c_k) \cdot \Delta x_k$ . Additionally, SC and TC included several problems (similar to example 1 in Table 2) that asked students to use computer or graphing calculators to describe adding up pieces idea. We were very interested in exploring how three textbooks connect Riemann sum to integrand and differential in definite integrals. After introducing Riemann sum, all three textbooks describe the infinite process of dividing an interval to smaller pieces showing the hint of describing "adding up pieces." For example, SC describes  $\Delta x_n$  as widths approaching zero while HC describes  $dt$  in the integral comes from the factor  $\Delta t$ . However, only TC includes explanations of what  $f(x) \cdot dx$  means and where  $dx$  came from (connecting  $f(c_k) \cdot \Delta x_k$  and  $f(x) \cdot dx$ ) while the other two textbooks' explanations are unclear about the product layer.

### Worked example

Table 3 shows the distribution of worked examples by each code. In these worked examples, the adding up pieces symbolic form is not found. Figure 2 shows worked example from SC.

**Table 3: Distribution of worked examples**

Code	Stewart	Thomas	Hughes-Hallett
Adding up pieces	2	1	0
Area under a curve	2	3	3
Anti-derivative	0	0	0
Matching mathematical symbols to integral	2	0	0
Evaluating Riemann sum or integral with several rectangles	6	3	5

Using integral properties	3	2	0
Total	15	9	8

The worked example in Figure 2 shows how to set up an expression from an integral. It describes dividing an interval and adding those, but the product layer is not clearly described. It appears that students are asked to match each symbol of the integral to summation symbols. To complete this task, students refer to Theorem 4, which is about converting the definite integral to Riemann sum.

**Set up an expression for  $\int_1^3 e^x dx$  as a limit of sums.**

**Solution:** Here we have  $f(x) = e^x$ ,  $a = 1$ ,  $b = 2$  and  $\Delta x = \frac{b-a}{n} = \frac{2}{n}$

So  $x_0 = 1$ ,  $x_1 = 1 + \frac{2}{n}$ ,  $x_2 = 1 + \frac{4}{n}$ ,  $x_3 = 1 + \frac{6}{n}$ , and  $x_i = 1 + \frac{2i}{n}$

From Theorem 4 we get  $\int_1^3 e^x dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_k) \Delta x = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(1 + \frac{2i}{n}) \frac{2}{n} =$   
 $\lim_{n \rightarrow \infty} \frac{2}{n} \sum_{i=1}^n e^{1+2i/n}$  (SC, p. 371)

**Figure 2: A worked example from SC.**

#### Exercise problems

**Table 4: Percent distribution of exercise problems**

Code	Stewart	Thomas	Hughes-Hallett
Adding up pieces	12 (11%)	18 (12.2%)	0
Area under a curve	14 (12.3%)	41 (27.9%)	35 (33.3%)
Anti-derivative	3(32.6%)	0	0
Matching mathematical symbols to integral	23 (20.3%)	9 (6.1%)	0
Evaluating Riemann sum or integral with several rectangles	46 (40.6%)	22 (14.9%)	70 (66.7%)
Using integral properties	15 (13.2%)	57 (38.8%)	0
Total	113	147	105

Table 4 shows the distribution of exercise problems from the three textbooks. Both SC and TC includes several problems similar to Figure 3. In the process of solving the problem in Figure 3, students need to think about the number of rectangles (adding up pieces). Thus, they can see that adding infinitely many rectangles will give the area under a curve. Although students will see the idea of adding up pieces, it is not clear if they are able to see the product layer. They may see it when they create rectangles, but it may be useful to ask students to describe the area of each rectangle in Figure 3, how each area can be expressed as a product of two quantities, and how the product is related to integrand and differential.

Exercises 95-98, use a CAS to perform the following steps:

Plot the functions over the given interval

Partition the interval into  $n = 100, 200,$  and  $1000$  subintervals of equal length, and evaluate the function at the midpoint of each subinterval.

Compute the average value of the function values generated in part (b).

Solve the equation  $f(x) = (\text{average value})$  for  $x$  using the average value calculated in part (c) for the  $n = 1000$  partitioning.

$$f(x) = \sin x \text{ on } [0, \pi] \text{ (TC, p. 274)}$$

**Figure 3: Adding up pieces problem from TC**

For SC, the most frequent task was evaluating Riemann sum with several rectangles.

Use the Midpoint Rule with the given value of to approximate the integral. Round the answer to four decimal places.  $\int_2^{10} \sqrt{x^3 + 1} dx, \quad n = 4$  (SC, p. 377)

**Figure 4: Exercise problem from SC**

For TC, the most frequent task was using integral properties. Those properties are order of integration, zero width interval, constant multiple, sum, and difference and additivity. Although these properties are important and valuable, students do not have opportunities to explore adding up pieces and the multiplicative relationship.

Use the rules in Table 5.4 and Equations (1)-(3) to evaluate the integrals in Exercises 41-50.

$$\int_0^2 5x dx \text{ (TC, p. 271)}$$

**Figure 5: Exercise problem from TC**

For HC, the most frequent task was evaluating Riemann sum with several rectangles (similar to Figure 4).

**Summary and Discussion**

With our results, we can think about the types of knowledge that students will be exposed to when they either use these textbooks on their own or their instructors' use these textbooks. With Riemann sum and problems similar to Figure 3, students will be exposed to adding up pieces but it is not clear how students would make a connection between Riemann sum and integral notations. With what we found from these textbooks, they will be familiar with converting Riemann sum to integral, area interpretations, using a few rectangles to find the area under a curve. Although these are important units of knowledge that students need to understand and

they may be able to use and activate those units of knowledge, it is challenging for students to expand their understanding of definite integral as these textbooks provide limited opportunities for them to become familiar with the product layer and adding up pieces. TC is the only textbook that provides learning opportunities to explore both the product layer (definition) and adding up pieces (exercise problems). Although TC's definition shows how  $f(c_k)$  are replaced by  $f(x)$ , how the subinterval widths  $\Delta x_k$  become the differential  $dx$ , and what  $f(x) \cdot dx$  means, the exercise problems do not clearly provide students with opportunities to understand the meaning of these symbols. To provide more precise learning opportunities for students, it may be useful to ask them to describe the area of each rectangle, how each area can be expressed as a product of two quantities, and how the product is related to integrand and differential. When these opportunities are added to what three textbooks present to students, they may be able to have units of knowledge needed to understand more clearly what it means to add infinitely many rectangles and how the area of a rectangle can be formed. As described earlier, students struggled most with the product layer but those students who used adding up pieces often correctly identified the product and how it can be interpreted in definite integrals (Jones, 2013; Sealey, 2014). Thus, it will be useful to modify tasks in these textbooks to more clearly make the connection among those rectangles, the product, integrand and differential as researchers recommended (Jones, 2013; Sealey, 2014). Although it is critical to think about how we teach calculus, we can realistically think about what really happens in calculus classrooms. As Wagner (2018) mentioned, when students are able to correctly evaluate definite integral with anti-derivative, they may not feel the need to use and understand the product layer or adding up pieces ideas. Our findings showed one possible area that may contribute to learning challenges in previous studies. Our findings from these widely used textbooks is a first step to think about how to revise and modify curriculum materials to improve teaching and learning of mathematics (Thompson & Harel, 2021) and how our results contribute to further discussion. Thompson and Harel (2021) pointed out that successful curricula require interactions among a triad of elements: a) production of national mathematics curriculum documents, b) mathematics education research, and c) teachers' understandings of content, cognition, and pedagogy. Adding what we know about these textbooks to the results of existing studies is an initial step to think about possible ways to revise integral lessons in calculus textbooks.

## References

- Bressoud, D. (2011). The calculus I instructor. Retrieved from [https://www.maa.org/external\\_archive/columns/launchings/launchings\\_06\\_11.html](https://www.maa.org/external_archive/columns/launchings/launchings_06_11.html)
- Charalambous, C. Y., Delaney, S., Hsu, H.-Y., & Mesa, V. (2010). A Comparative Analysis of the Addition and Subtraction of Fractions in Textbooks from Three Countries. *Mathematical Thinking and Learning*, 12(2), 117-151. doi:10.1080/10986060903460070
- Ely, R. (2017). Definite integral registers using infinitesimals. *The Journal of Mathematical Behavior*, 48, 152-167. doi:<https://doi.org/10.1016/j.jmathb.2017.10.002>
- Ely, R. (2021). Teaching calculus with infinitesimals and differentials. *ZDM – Mathematics Education*, 53(3), 591-604. doi:10.1007/s11858-020-01194-2
- Harlow, D. B., & Bianchini, J. A. (2020). Knowledge-in-Pieces—Andrea A. diSessa, David Hammer. In B. Akpan & T. J. Kennedy (Eds.), *Science Education in Theory and Practice: An Introductory Guide to Learning Theory* (pp. 389-401). Cham: Springer International Publishing.
- Hong, D. S. (2023). Examining Opportunities to Learn Limit in Widely Used Calculus Textbooks. *International Journal of Science and Mathematics Education*, 21(3), 881-898. doi:10.1007/s10763-022-10273-7
- Hong, D. S., Choi, K. M., Runnalls, C., & Hwang, J. (2018). Do textbooks address known learning challenges in area measurement? A comparative analysis. *Mathematics Education Research Journal*, 30(3), 325-354. doi:10.1007/s13394-018-0238-6
- Jones, S. R. (2013). Understanding the integral: Students' symbolic forms. *The Journal of Mathematical Behavior*,
- Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

- 32(2), 122-141. doi:<https://doi.org/10.1016/j.jmathb.2012.12.004>
- Jones, S. R. (2015). Areas, anti-derivatives, and adding up pieces: Definite integrals in pure mathematics and applied science contexts. *The Journal of Mathematical Behavior*, 38, 9-28. doi:<https://doi.org/10.1016/j.jmathb.2015.01.001>
- Larsen, S., Marrongelle, K., Bressoud, D., & Graham, K. (2016). Understanding the Concepts of Calculus: Frameworks and Roadmaps Emerging From Educational Research. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 526-550). Reston, VA: National Council of Teachers of Mathematics.
- Mahir, N. (2009). Conceptual and procedural performance of undergraduate students in integration. *International Journal of Mathematical Education in Science and Technology*, 40(2), 201-211. doi:10.1080/00207390802213591
- Rasslan, S., & Tall, D. (2002). Definitions and images for the definite integral concept. Paper presented at the Proceedings of the 26th conference of the International Group for the Psychology of Mathematics Education, Norwich, United Kingdom.
- Reeves, C., Carnoy, M., & Addy, N. (2013). Comparing opportunity to learn and student achievement gains in southern African primary schools: A new approach. *International Journal of Educational Development*, 33(5), 426-435. doi:<https://doi.org/10.1016/j.ijedudev.2012.12.006>
- Remillard, J. T., & Heck, D. J. (2014). Conceptualizing the curriculum enactment process in mathematics education. *ZDM*, 46(5), 705-718. doi:10.1007/s11858-014-0600-4
- Sealey, V. (2014). A framework for characterizing student understanding of Riemann sums and definite integrals. *The Journal of Mathematical Behavior*, 33, 230-245. doi:<https://doi.org/10.1016/j.jmathb.2013.12.002>
- Sofronas, K. S., DeFranco, T. C., Vinsonhaler, C., Gorgievski, N., Schroeder, L., & Hamelin, C. (2011). What does it mean for a student to understand the first-year calculus? Perspectives of 24 experts. *The Journal of Mathematical Behavior*, 30(2), 131-148. doi:<https://doi.org/10.1016/j.jmathb.2011.02.001>
- Son, J.-W., & Hu, Q. (2016). The initial treatment of the concept of function in the selected secondary school mathematics textbooks in the US and China. *International Journal of Mathematical Education in Science and Technology*, 47(4), 505-530. doi:10.1080/0020739X.2015.1088084
- Thompson, P. W., & Harel, G. (2021). Ideas foundational to calculus learning and their links to students' difficulties. *ZDM – Mathematics Education*, 53(3), 507-519. doi:10.1007/s11858-021-01270-1
- Wagner, J. F. (2018). Students' Obstacles to Using Riemann Sum Interpretations of the Definite Integral. *International Journal of Research in Undergraduate Mathematics Education*, 4(3), 327-356. doi:10.1007/s40753-017-0060-7

# EXPLICATING STUDENTS' UNIT STRUCTURES IN COMBINATORIAL CONTEXTS

Joseph Antonides  
Virginia Tech  
jantonides@vt.edu

Erik S. Tillema  
Indiana University Bloomington  
etillema@indiana.edu

Keywords: Advanced Mathematical Thinking, Number Concepts and Operations, Cognition

Units coordination is a theory that explains how people construct and organize units to solve mathematical problems (Hackenberg & Sevinc, in press). Grounded in Piagetian constructivism (Piaget, 1970; von Glasersfeld, 1995), and building out of Steffe's (1992, 1994) work on children's development of number concepts and operations, units coordination has proliferated throughout the mathematics education research literature as a powerful way to explain and predict students' available ways of additive and multiplicative reasoning (e.g., Hackenberg & Tillema, 2009; Norton et al., 2015; Ulrich, 2015, 2016). While substantial research has investigated the role of units coordination in student reasoning across domains, much is still unknown about the unit structures involved in combinatorial contexts and the mental operations used to create them. We view this as an important gap in the research literature, since combinatorics is a domain in which there is great potential for students to construct and coordinate highly complex, multi-dimensional unit structures.

This presentation focuses on two students' combinatorial unit structures and their operations for constructing them. Specifically, the students (both preservice elementary teachers) were engaged in a 1-on-1 teaching experiment (Steffe & Thompson, 2000) focusing on student reasoning about permutations (cf. Antonides, 2022). Drawing on mental operations identified by Tillema (2013) and Antonides and Battista (2022a, 2022b), we compare the students' reasoning about the following tasks. On the Towers Task, specific values of  $n$  were given.

*Ten Squares Task:* Suppose you have 10 squares, each of a different color. You want to arrange the squares next to each other in a line. How many different arrangements can you make?

*Towers Task:* Suppose you have  $n$  different colors of squares. How many different towers  $n$ -squares-high could you make if each tower contains 1 square of each color?

Claire, who was assessed as operating at Stage 3 of units coordination, and who had some prior experience with combinatorics (through formula-based instruction), assimilated each task using her concept of factorial. More importantly, she was able to not only write a symbolic multiplicative expression to solve each task, but also explain the multiplicative relationships embedded within successive stages of the counting process (cf. Lockwood, 2013). We interpreted Claire's reasoning as evidence of her construction of a *generalized slot structure*: a cognitive structure that enables students to assimilate combinatorial situations, involving any number of units, using multiplicative concepts and reasoning. By contrast, Ashley (who was assessed as operating at advanced Stage 2, possibly emergent Stage 3 of units coordination) reasoned there would be  $10 \times 10$  different possibilities for the Ten Squares Task—what Antonides (2022) called an objects-times-positions scheme. She used similar reasoning to solve a variety of follow-up tasks, including Towers Tasks. For  $n = 3$ , for instance, she reasoned there would be  $3 \times 3$  different towers 3-squares-high, since there are 3

colors available and 3 positions into which each color can be placed. We illustrate and discuss Ashley's generative activity as she constructed towers 1-by-1 using her available concepts and operations.

### References

- Antonides, J. (2022). A cognition-based analysis of undergraduate students' reasoning about the enumeration of permutations (Unpublished doctoral dissertation). The Ohio State University.
- Antonides, J., & Battista, M. T. (2022a). A learning trajectory for enumerating permutations: Applying and elaborating a theory of levels of abstraction. *Journal of Mathematical Behavior*, 68, 101010. <https://doi.org/10.1016/j.jmathb.2022.101010>
- Antonides, J., & Battista, M. T. (2022b). Spatial-temporal-enactive structuring in combinatorial enumeration. *ZDM Mathematics Education*, 54(4), 795–807. <https://doi.org/10.1007/s11858-022-01403-0>
- Hackenberg, A. J., & Sevinc, S. (in press). Students' units coordinations. To appear in P. Dawkins, A. Norton, & A. Hackenberg (Eds.), *Piaget's genetic epistemology for mathematics education research*. Springer.
- Hackenberg, A. J., & Tillema, E. S. (2009). Students' whole number multiplicative concepts: A critical constructive resource for fraction composition schemes. *Journal of Mathematical Behavior*, 28(1), 1-18.
- Lockwood, E. (2013). A model of students' combinatorial thinking. *Journal of Mathematical Behavior*, 32(2), 251–265. <https://doi.org/10.1016/j.jmathb.2013.02.008>
- Norton, A., Boyce, S., Ulrich, C., & Phillips, N. (2015). Students' units coordination activity: A cross-sectional analysis. *Journal of Mathematical Behavior*, 39, 51-66. <https://doi.org/10.1016/j.jmathb.2015.05.001>
- Piaget, J. (1970). *Genetic epistemology* (E. Duckworth, Trans.). Norton.
- Steffe, L. P. (1992). Schemes of action and operation involving composite units. *Learning and Individual Differences*, 4(3), 259–309. [https://doi.org/10.1016/1041-6080\(92\)90005-Y](https://doi.org/10.1016/1041-6080(92)90005-Y)
- Steffe, L. P. (1994). Children's multiplying schemes. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 1-39). State University of New York Press.
- Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 267–306). Lawrence Erlbaum Associates.
- Tillema, E. S. (2013). A power meaning of multiplication: Three eighth graders' solutions of Cartesian product problems. *Journal of Mathematical Behavior*, 32(3), 331–352. <https://doi.org/10.1016/j.jmathb.2013.03.006>
- Ulrich, C. (2015). Stages in constructing and coordinating units additively and multiplicatively (part 1). *For the Learning of Mathematics*, 35(3), 2–7.
- Ulrich, C. (2016). Stages in constructing and coordinating units additively and multiplicatively (part 2). *For the Learning of Mathematics*, 36(1), 34–39.
- von Glasersfeld, E. (1995). *Radical constructivism: A way of knowing and learning*. Falmer Press.



## FRAMING INSTRUCTIONAL TASKS FOR INTERACTION WITH CONTENT: INTRODUCING DERIVATIVES GRAPHICALLY WITH INQUIRY

Saba Gerami  
University of Michigan  
[sgerami@umich.edu](mailto:sgerami@umich.edu)

*In this study, I present how eight U.S. college calculus instructors with different patterns of inquiry practices used instructional situations to frame instructional tasks for introducing derivatives graphically to students. During four interviews, the instructors proposed up to eight tasks for introducing derivatives physically, graphically, verbally, and symbolically (Zandieh, 2000). The findings focus on the two tasks proposed by each instructor that centered the graphical representation of the derivative: derivative at a point as the slope of a tangent line, and derivative as a function presented by its graph. While no two instructors proposed the same tasks with the same instructional situations for teaching these concepts, they relied on graphing, conjecturing, and calculating situations to frame their tasks.*

Keywords: Undergraduate Education, Calculus, Instructional Activities and Practices, Mathematical Representations

This study contributes to our understanding of teaching of calculus and implementing inquiry in undergraduate mathematics classrooms by investigating the ways calculus instructors frame instructional tasks in relation to the content to teach a fundamental calculus concept—derivatives—with inquiry. There is still very little basic research on the instruction of calculus and even less on those that promote ‘active learning’ (Larsen et al., 2017). The existing research has mainly attended to the learning of calculus ideas and the content and cognitive orientation of non-instructional tasks (e.g., exams and textbooks; Tallman et al., 2016; White & Mesa, 2014).

This scarcity of knowledge about what happens in calculus classrooms is discouraging, given that the course has been notorious for being a gateway for many students to follow a STEM pipeline or graduate from college (Blair et al., 2018; Bressoud, 2012). While lecture still dominates calculus lessons (Laursen, 2019), implementing more inquiry has been promoted to make calculus more accessible. However, research on inquiry teaching in undergraduate mathematics, which has mainly focused on pedagogical variables rather than the content, does not always suggest positive student outcomes (e.g., Johnson et al., 2020). Alternatively, I attend to the content instructors and students interact with in these classrooms to operationalize inquiry teaching in relation to the content. Using inquiry as the context to study calculus tasks is valuable, as it allows uncovering a wide range of calculus-specific instructional tasks, given that inquiry in undergraduate mathematics education is often characterized by students’ discovery of mathematical ideas through engagement with sequences of mathematical tasks (Laursen & Rasmussen, 2019). Here, I specifically focus on the tasks that instructors proposed for teaching the graphical representations of derivatives: slope of tangent line, as derivative at a point; and the derivative function, as a graph. More precisely, I address the following research question: How do college Calculus I instructors, who teach with various inquiry approaches, frame instructional tasks to introduce the graphical representations of the derivative?

## Literature Review

The ample research on the learning of derivatives graphically suggests its importance for students' understanding of derivatives (e.g., García-García & Dolores-Flores, 2021; Ubuz, 2007).

Borji et al. (2018) and Hähkiöniemi (2006) confirmed the importance of the graphical representations of derivatives by carrying out teaching interventions designed with the APOS (Action, Process, Object, and Schema) theory that emphasized visual representations. In Borji et al.'s study, students who used graphing software performed better on derivatives tests later. Hähkiöniemi found that students could recognize derivatives visually early on, concluding that visual representations are a good fit for introducing and developing understanding of derivatives. However, the scarce research on teachers' actual teaching of derivatives graphically shows that teachers do not approach them similarly. Kendal and Stacy (2001) compared two high school calculus teachers' use of graphical, symbolic, and numerical representations with a computing and graphing tool. The teachers used the graphing aspect of the tool differently based on their beliefs about the importance of various representations and whether their students were capable of learning different representations simultaneously. Park (2015) studied how three college calculus instructors defined derivative at a point and as a function, showing that: 1) although they used secant and tangent lines to graphically illustrate derivative at point, they did not make the connections to symbolic representations explicit; and 2) they often showed derivative as a function with zero, positive, or negative values of the derivative "rather than numerical values that change over an interval" (p. 248). Delos and Thomas (2003) studied four teachers' teaching of derivatives, finding that while there was not a common way of teaching derivatives among the teachers, all four used the approach of moving secants toward a tangent. The study presented in this paper dives deeper into instructors' use of graphical representations for teaching derivatives by revealing the different ways they frame their instructional tasks to teach slope of tangent line as derivative at a point and the graph of derivative function.

## Theoretical Framework

Goffman defines framing as the explicit or implicit answer to the question "What is it that's going on here?" that individuals must answer in a social situation to proceed "with the affairs at hand" (1974/1986, p. 8). In education research and the learning sciences, framing has been used in various ways to: theorize students' understanding of themselves and the intellectual activities they engage with in the classroom (Greeno, 2009), investigating opportunities to learn within learning environments (Hand et al., 2012), characterizing teachers' shaping of mathematics and science lessons and activities (Engle, 2006; Herbst et al., 2020), and bridging learning in mathematics and sciences between contexts (Chapman, 2022; Hammer et al., 2005). For the purposes of this study, I use Herbst and colleagues' (2020) notion of framing a mathematical problem, originally introduced by Bateson (1954/2003) and later developed by Goffman, to identify how interactions within a specific moment in time during calculus instruction are interpreted and governed. To unpack college calculus instructors' framing of instructional tasks for teaching derivatives with inquiry, I answer Goffman's question of 'what is going on here' with two types of framing: *Framing for Interaction with Content* and *Framing for Social Interaction* (also known as activity structure; Doyle, 1984). Due to space, I only focus on the former here. To frame a task for interaction

with the content, I use instructional situations (or situations as I use them interchangeably) within the notion of didactical contract, a contract with an often-implicit set of expectations that regulates the relationship between the teacher and students regarding the work of teaching and learning mathematics (Brousseau, 1984).

Instructional situations as “*distinct types of problems* [emphasis added] used in a course of studies” are the customary ways by which the teacher’s and students’ actions and interactions are *framed* into appropriate units of work regarding the knowledge at stake (Herbst et al., 2020, p.

5); given an instructional situation, students know what kind of problem they are presented with and what kind of mathematical work and interactions they should prototype (Herbst & Chazan, 2012). By focusing on these subject-specific framings in tasks, we can see what learning opportunities are offered to students during instruction (Herbst et al., 2018).

### Methods

The data for this study comes from a larger project that seeks to understand how eight U.S. calculus instructors frame their instructional tasks for teaching derivatives with inquiry. The eight instructors were purposefully selected using cluster analysis from a pool of 48 calculus instructors to represent different patterns of inquiry-oriented practices (Shultz, 2020). The participants were tenured and had more than four semesters experience teaching Calculus I with inquiry. Five participants used he/him pronouns (Justin, Adrian, Barry, Matthew, Gopher; pseudonyms); one used she/her (Monica); one used they/them (Max); and one used all pronouns (Alex). I conducted four semi-structured 1–2-hour long interviews with each instructor, as they proposed up to eight tasks, organized by Zandieh’s (2000) framework (each interview was dedicated to one representation of the derivative; Gerami, 2023). Zandieh’s framework organizes students’ conceptions of derivatives by representation (graphical, verbal, physical, symbolic) and process-object layers (ratio, limit, function). The process-object layers are hierarchical, as each layer is found by taking the process of that layer over the previous layer as an object. For example, the limit layer is found by the *process of finding the limit of the ratio as an object*. The limit layer corresponds to when the denominator approaches zero; the function layer is presented as an array of numbers or set of ordered pairs of differences. The instructors were asked to propose a task (from their teaching materials or to create new ones they could use in the future) that would help transition students’ conceptions from one layer to the next (the target layer) within a representation. Although not prompted, they could also reach out to other representations as needed. Here, I focus on the two tasks that the instructors proposed during Interview 2 for the graphical representation: from the ratio to the limit layer via Prompt 3 (“Assume that you have already taught about slope of secants and want students to learn about slope of tangents. Propose a task involving the graph of a function, where students have to figure out the slopes of some tangent lines”), and from the limit to the function layer via Prompt 4 (“Assume that you have already taught about slope of tangents and want students to learn about the graph of the derivative function. Propose a task involving the graph of a function, where students have to visualize the graph of the derivative function.”).

To analyze framing for interaction with content, I used inductive/deductive hybrid thematic analysis, which entails using pre-ordinate themes “through the application of an explicit theoretical framework developed through engagement with the literature” (the deductive element) to generate themes from the data (the inductive element; Proudfoot, 2022).

Because instructional situations have not yet been identified in the context of calculus instruction, I used instructional situations identified in other content areas (mainly geometry and algebra) to find eight generic types of problems that students could do in any mathematics classroom: *graphing*; *calculation*; *exploration & conjecturing*; *doing proofs*; *generating a new definition or installing a new concept*; *installing a new theorem, property, or formula*; *solving equations with known methods*; and *solving word problems*. I used these to identify the emerging instructional situations (i.e., the mathematical work that students are expected to do) in each task using the written descriptions of the tasks and triangulated the findings by listening to the interviews and reading the transcripts to find information about the situations that instructors mentioned but did not include in the task description.

### **Findings**

The eight instructors proposed a variety of tasks for the two prompts (Prompt 3: ratio  $\rightarrow$  limit and Prompt 4: limit  $\rightarrow$  function), all of which consisting of three generic types of problems— graphing, calculating, and conjecturing. Although some instructors used similar calculus-specific instructional situations within the three generic problem types, nearly no two instructors proposed the same tasks consisting of all the same instructional situations. Although I cannot show all 16 tasks due to space, I overview the similarities and differences among the instructional situations found within each. I also differentiate between explicit and embedded situations: a situation is explicit if students know what type of problem they are working on after reading a task, whereas a situation is embedded within an explicit (larger) situation if while working on the explicit problem, students find out that they must solve another smaller problem in order to solve the explicit situation.

### **Tasks Proposed for Prompt 3: Ratio $\rightarrow$ Limit**

To frame the tasks proposed for prompt 3, the instructors relied on a variety of graphing, calculating, and conjecturing situations involving the limit layer (the target layer), with all but two instructors (Alex and Justin) also explicitly using graphing and calculating situations at the ratio layer as precursors to the situations with the target layer (Figure 1). The tasks proposed consisted of 2-4 explicit situations, with only Justin using two to frame his task.

**Situations involving the target layer (limit)**

❖ **Graphing**

- Graphing tangent line at a point-Alex, Barry, Matthew

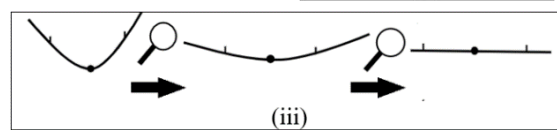
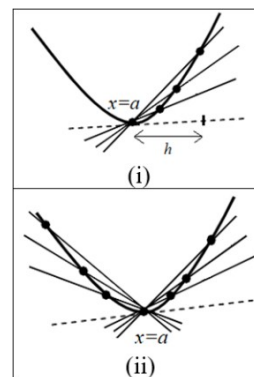
❖ **Conjecturing**

- Conjecturing how a smooth function’s graph looks like if we zoom in enough (iii)-Gopher

- Conjecturing slope of a tangent line at a point from a pattern of slopes of secants
  - $h$ -units distanced to the right, with  $h$  decreasing-Adrian, Justin
  - to the left and right side of a fixed point (ii)-Monica, Max

❖ **Calculating:**

- Calculating slope of tangent line at a point
  - by calculating the limit of slope of secants  $h$ -units distanced to the right side of a fixed point with  $h$  approaching zero (i)-Gopher, Justin
- Estimating slope of tangent line at a point
  - by estimating slope of the line on plain background-Alex
  - {by estimating slope of the line on plain background} (finding equation of a line)-Alex
  - by calculating slope of secants  $h$ -units distanced to the right side of a fixed point for very small  $h$ -Alex
  - from pattern of slope of secant lines to the left and right side of a fixed point-Matthew
  - by zooming in (iii)-Barry



**Situations not involving the target layer (ratio)**

❖ **Graphing**

- Graphing secant lines

- to the left and right side of a fixed point (ii)-Matthew, Monica, Max
- $h$ -units distanced to the right side of a fixed point with  $h$  decreasing (i)-Barry, Adrian

❖ **Calculating:**

- Calculating slope of secant lines between two points-Adrian, Max
  - to the left and right side of a fixed point (ii)-Monica, Matthew, Gopher
  - $h$ -units distanced to the right, with  $h$  decreasing- Gopher
  - {to the left and right side of a fixed point (ii)} (within finding equation of a line)-Monica

Note1. Brackets, [xxx], denote embedded calculus-specific instructional situations. Braces, {xxx}, denote an embedded calculus-specific instructional situations within non-calculus-specific situations, which is noted in parenthesis outside the braces.

Note2. The images are adapted from Haghjoo et al. (2023).

**Figure 1. Calculus-Specific Instructional Situations within Tasks Proposed for Prompt 3**

Matthew, Monica, Max, Barry and Adrian started their tasks at the ratio layer (not the target layer) by asking students to graph secant lines given two points, one of which being the point they wanted students to eventually find the slope of the tangent line at (*Graphing secant lines or segments*). Matthew, Monica, and Max asked students to use points on the function on

both sides of the fixed point, while Barry and Adrian defined the second points  $h$ -horizontal-units to the right, with  $h$  decreasing (image (ii) and (i) in Figure 1 respectively). Monica, Max, Matthew, and Adrian also asked their students to find the slopes of the secant lines either explicitly (*Calculating slope of secant lines between two points*) or as an embedded situation (finding equation of a secant line). From these instructors, Barry and Matthew went on to ask students to draw a tangent line at the point (*Graphing tangent line at a point*), thus involving the target layer (limit) in a graphing situation. Alex asked students to draw tangent lines to the graph of a continuous function, but they did so without mentioning secants first. Given that tangent lines as derivative at a point are the target layer of the task, it is interesting that only Alex, Barry, and Matthew explicitly asked students to draw tangent lines. Although Gopher and Justin used graphs as a representation, they did not frame the tasks with any graphing situations, meaning that they did not expect students to do the work of graphing to complete their tasks.

Although only three instructors used a graphing situation at the limit layer (target), all the instructors relied on conjecturing and/or calculating situations to engage students with slope of tangent. A conjecturing situation involving the limit layer, *how a smooth function's graph looks like if we zoom in enough*, was used by Gopher at the beginning of his ask after providing students with a quadratic graph in a graphing application: “What do you observe about how the graph appears as you view it more and more closely [at  $xx = 0.8$ ]?”. However, Gopher’s situations later in the task did not require students to use their conjectures to find the slope of tangent at a point, as he asked them to *calculate slope of secant lines*. The other conjecturing situation—*conjecturing slope of a tangent line at a point from a pattern of slopes of secants*—was used by Adrian, Monica, Justin, and Max. While Adrian and Monica had their students graph and calculate slopes of secant lines before this situation, Justin and Max had their students use technology to collect slopes of secants (after finding the slope of one secant line, Max’s students would use Excel to collect the slope of the remaining secants; Justin’s students would use an interactive graph with one fixed point [Figure 3a] to collect the slopes of secant lines by moving the second point closer to the first). Adrian, Monica, and Max ended their tasks with this situation, while Justin and Gopher continued their tasks with one more situation: *Calculating slope of tangent line at a point by calculating the limit of slope of secants  $h$ -units distanced to the right side of the point with  $h$  approaching zero*. Given that Justin and Gopher expect students to find an exact value using limits, I named this situation calculating; I use “estimating” to define the other calculating situations in which instructors did not expect exact values (estimating is still under the umbrella of calculating because students use known formulas to estimate).

Alex, Matthew, and Barry used the *estimating slope of tangent line at a point* situation to have their students find slope of tangents. After having students draw tangent lines at various points to a continuous graph on plain background, Alex asked them to find the slope of the tangent lines. Given that the graph is not on grid paper with labeled axes, students must estimate the slopes. Alex finished the task by asking students: “If the graph of a function weren’t readily available, how would we go about finding the slope of a line tangent to its graph?”. Because Alex expected students to write the equation of average rate of change and come to the idea that

$\Delta xx$  or  $h$  must be “very very small,” I captured this situation as *estimating slope of secants  $h$ -units distanced to the right side of a fixed point for very small  $h$* . Matthew had a very similar approach as those who used *conjecturing slope of a tangent line at a point from a pattern of*

*slopes of secants*, because students started the task by drawing secants and finding their slopes. However, the task directly told students that they are approximating slope of a tangent line and that they should find an overestimate and an underestimate; thus, given the wording of the task, students do not need to make a conjecture about the slope of the tangent line (e.g., whether it exists, whether it is smaller/larger than slope of nearby secants). Lastly, Barry also asked his students to *estimate the slope of tangent line at a point by zooming in* the graph and calculating the slope once the function looks linear.

#### **Tasks Proposed for Prompt 4: Limit → Function**

To frame the tasks proposed for prompt 4, all instructors except Justin, used a variety of graphing situations involving the function layer (the target layer), with Adrian, Gopher and Max also explicitly using calculating situations at the limit layer as precursors to the graphing situations (Figure 2). Justin was the only instructor who used conjecturing, and only conjecturing, to frame his task. The tasks consisted of 1-2 explicit situations, with only Adrian using three to frame his task.

Within the graphing situations, the most common situation was *Graphing  $ff'$  given the graph of  $f$* . Although the tasks looked similar on the surface, there is more nuance to this instructional situation when looking at the work that the instructors expect of their students. Adrian, Barry, Max, and Monica provided graphs of  $f$  that were either on a grid or plain background, meaning that students should either *estimate* or *calculate* slopes of tangent lines at various points before plotting the slopes as points on the graph of  $ff'$  and drawing a curve that fits the points. This means that these situations include embedded calculating situations (*Estimating slopes of tangents* and *calculating exact slopes of tangents*) within them because the instructors did not explicitly allude to these situations in the tasks' descriptions. On the other hand, Gopher used a similar situation (*Graphing  $ff'$  by plotting slope of tangents at various points and drawing a curve that fits best through the points*) but pre-framed the situation differently. He first explicitly asked students to calculate slope of tangent lines at multiple given points using the limit definition of the derivative (*Calculating the slope of tangent at an arbitrary point  $a$  using the limit definition*) as a precursor to the graphing situation without involving the target layer.

Gopher then instructed students how to draw  $ff'$  given their findings from the previous step: "Plot the values of the slope at each value of  $t$ . Find an equation that fits your data."

<p><b>Situations involving the target layer (function)</b></p> <p>❖ <b>Graphing</b></p> <ul style="list-style-type: none"> <li>• Graphing <math>ff'</math> given the graph of <math>ff</math> <ul style="list-style-type: none"> <li>◦ [by estimating slopes of tangents] and plotting slope of tangents at various points and drawing a curve that fits best through the points-<i>Barry, Max &amp; Monica</i></li> <li>◦ [by calculating exact slopes of tangents] and plotting slope of tangents at various points and drawing a curve that fits best through the points-<i>Adrian &amp; Monica</i></li> </ul> </li> <li>• Graphing <math>ff'</math> by plotting slope of tangents at various points and drawing a curve that fits best through the points-<i>Gopher</i></li> <li>• Graphing <math>ff</math> given the graph of <math>ff'</math> [by estimating <math>yy</math>-values of <math>ff'</math> as slopes of tangents of <math>ff</math>] and drawing a curve with estimated slopes for their tangents-<i>Barry</i></li> <li>• Graphing <math>ff</math> given information about the function's critical points and plotting the points and drawing a curve that fits best through the points -<i>Alex</i></li> <li>• Graphing <math>ff</math> given the formula of <math>ff'</math> by [finding the function's critical points] and plotting the points and drawing a curve that fits best through the points -<i>Alex</i></li> </ul> <p>❖ <b>Conjecturing</b></p> <ul style="list-style-type: none"> <li>• Conjecturing the relationship between two graphs (of <math>ff</math> and <math>ff'</math>)-<i>Justin</i></li> </ul>
<p><b>Situations not involving the target layer (limit)</b></p> <p>❖ <b>Calculating:</b></p> <ul style="list-style-type: none"> <li>• Calculating slope of tangent line at a point given graph of <math>ff</math>-<i>Adrian</i></li> <li>• Calculating <math>ff'(aa +/ -)</math> for given number <math>aa</math> using the limit definition-<i>Adrian</i></li> <li>• Calculating the slope of tangent at an arbitrary point using the limit definition-<i>Gopher</i></li> </ul> <p><i>Note.</i> Brackets, [xxx], denote embedded instructional situations.</p>

**Figure 2. Calculus-Specific Instructional Situations within Tasks Proposed for Prompt 4**

Adrian, like Gopher, also used calculating situations at the limit layer as precursors to the graphing situation at the function (target) layer. He started the task with the formula and graph of  $ff(xx) = |xx - 2|$ , followed by two calculating situations: *Calculating slope of tangent line at a point given graph of  $ff$  (not 2)* and *Calculating  $ff'(2+)$  and  $ff'(2-)$  for given number  $aa$  using the limit definition*. He finished the task by asking students to define  $ff'$  symbolically via graphing it. After realizing that the derivative does not exist at  $xx = 2$  (because  $ff'(2+) \neq ff'(2-)$  by the definition of differentiability at a point), students would use the previous calculations to draw  $ff'$  before and after  $xx = 2$ . Therefore, it seems that given his choice of function (absolute value with two straight lines meeting at a non-differentiable corner), he used the opportunity of graphing  $ff'$  to have students practice using the limit definitions of derivative and encounter a situation where  $ff$  is not a smooth curve. Monica was the only other instructor that used a non-smooth function in his task of graphing  $ff'$  given  $ff$ , but did not provide any scaffolds like Adrian.

After asking students to graph  $ff'$  given the graph of  $ff$ , Barry asked his students to do the graphing in the reverse direction in the second part of his task (*Graphing  $ff$  given the graph of  $ff'$* ) using a similar approach of estimating slopes in his first part of the task. Alex, however, started his task by asking students to draw the graph of  $ff'$  given information about the original function's critical points, without first asking them to draw  $ff'$  given  $ff$ 's graph. They then continued the task by drawing  $ff$  using the formula of  $ff'$ , wanting students to find the



critical points themselves. When asked about why they have students go from  $ff'$  to  $ff$ , rather than  $ff$  to  $ff'$ , they said: “I don't think if you know the graph of a function, graphing the derivative [is] as a meaningful, like, I don't see that as a question that many students are asking ... we know something about the derivative and you don't know something about the function, that's a meaningful epistemological position that they're often in.” Alex then explained that they would only do an example later from  $ff$  to  $ff'$  if a student asked about it.

Justin used a conjecturing situation in his task by utilizing a graphing application (Figure 3b). Students were told to open the application, type in various functions of their choice (the blue box in Figure 3b) and play with the application to find out what the application does. As students moved the blue point on the blue graph of  $ff(xx)$  from left to right, another function without any label appeared in green ( $ff'$ ). Moreover, a small red line segment tangential to the function's graph moved with the blue point with its approximated slope next to it. Although the task clearly involves graphical representations of  $ff$  and  $ff'$ , I captured the situation as conjecturing, instead of graphing, because the work students do in this task is not graphing, but observing the two graphs and making a statement about how, they think, the graphs are related.

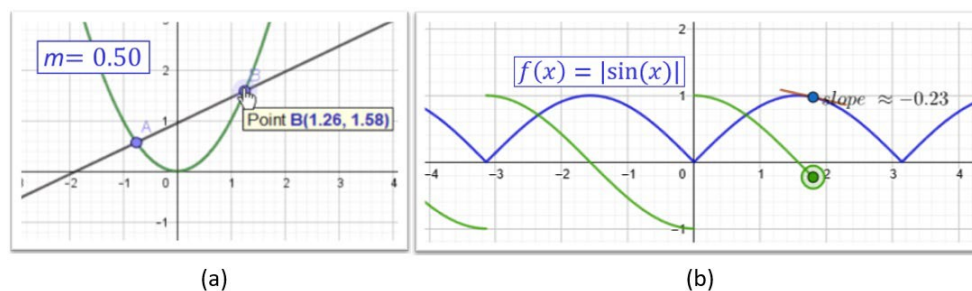


Figure 3. The graphing applications designed by Justin for Prompt 3 (a) and 4 (b)

### Discussion and Conclusion

In this paper, I describe the ways eight college Calculus I instructors across the U.S., who teach with inquiry, frame their instructional tasks to introduce the graphical representations of the derivative at a point (Prompt 3) and as a function (Prompt 4). The study goes beyond the prior research on identifying the representations of derivatives that students engage with and investigates the *types* of calculus-specific mathematical problems (i.e., instructional situations) they work on. The instructors mainly relied on three generic types of problems: graphing, calculating, and conjecturing. While the instructors used a variety of calculus-specific situations in their tasks for both prompts, they used more unique situations to introduce the slope of tangent line as derivative at a point, than derivative as a function. The findings add to those of Delos and Thomas's (2003); even within the graphical representation, teachers have other ways of introducing derivative in addition to secants approaching a tangent. Although the tasks were designed for teaching with inquiry, most situations asked students to apply what they know (calculating, graphing). No instructor used *installing* situations (defining a new idea); they instead seemed to rely on conjecturing in the tasks and installed the new ideas via lecture themselves. Moreover, while all tasks were composed of multiple situations, the teachers made most situations explicit. Thus, inquiry can be operationalized as the number of situations instructors summon in their tasks and how explicit they make them; a task would be more inquiry-oriented if it has more situations and/or those situations are more implicit than

explicit.

### Acknowledgments

Funding for this work was provided by the University of Michigan's Rackham Graduate School to the author. I sincerely thank Dr. Vilma Mesa, Dr. Patricio Herbst, and the RTMUS lab for their continuous guidance and feedback.

### References

- Bateson, G. (1974/2003). A theory of play and fantasy. In Caro, I., & Read, C. S. (Eds.), *General semantics in psychotherapy Charge: Selected writings on methods aiding therapy* (pp. 267–283). Institute of General Semantics.
- Blair, R. M., Kirkman, E. E., Maxwell, J. W., & American Mathematical Society. (2018). Statistical abstract undergraduate programs in the mathematical sciences in the United States: Fall 2015 CBMS survey. American Mathematical Society.
- Borji, V., Font, V., Alamolhodaei, H., & Sánchez, A. (2018). Application of the complementarities of two theories, APOS and OSA, for the Analysis of the university students' understanding on the graph of the function and its derivative. *EURASIA Journal of Mathematics, Science and Technology Education*, 14(6), 2301-2315. <https://doi.org/10.29333/ejmste/89514>
- Bressoud, D. (2012). Characteristics of successful programs in college calculus. [http://www.macalester.edu/~bressoud/talks/2012/CSPCC\\_NCTM.pdf](http://www.macalester.edu/~bressoud/talks/2012/CSPCC_NCTM.pdf)
- Brousseau, G. (1984). The crucial role of the didactical contract in the analysis and construction of situations in teaching and learning mathematics. In H.-G. Steiner (Ed.), *Theory of mathematics education* (pp. 110–119). Occasional paper 54. Bielefeld, Germany: IDM.
- Chapman, K. (2022). “Wait—it’s a math problem, right?”: Negotiating school frames in out-of-school places. *Educational Studies in Mathematics*, 109(3), 661-676. <https://doi.org/10.1007/s10649-021-10099-0>
- Delos, S. A., & Thomas, M. O. (2003). Perspectives on the teaching of derivative with graphics calculators. *Australian Senior Mathematics Journal*, 17(1), 40-58. <https://search.informit.org/doi/10.3316/aeipt.129370>
- Doyle, W. (1984). How order is achieved in classrooms: An interim report. *Journal of Curriculum Studies*, 16(3), 259–277.
- Engle, R. A. (2006). Framing interactions to foster generative learning: A situative explanation of transfer in a community of learners' classroom. *The Journal of the Learning Sciences*, 15(4), 451-498. [https://doi.org/10.1207/s15327809jls1504\\_2](https://doi.org/10.1207/s15327809jls1504_2)
- García-García, J., & Dolores-Flores, C. (2021). Pre-university students' mathematical connections when sketching the graph of derivative and antiderivative functions. *Mathematics Education Research Journal*, 33, 1-22. <https://doi.org/10.1007/s13394-019-00286-x>
- Gerami, S. (2023, February). Calculus I Instructors' Use of Representations in Instructional Tasks: Introducing Derivatives with Inquiry. *Research in Undergraduate Mathematics Education (SIGMAA-RUME)*. Omaha, NE.
- Goffman, E. (1986). *Frame analysis: An essay on the organization of experience*. Northeastern University Press. (Original work published 1974)
- Greeno, J. G. (2009). A theory bite on contextualizing, framing, and positioning: A companion to Son and Goldstone. *Cognition and Instruction*, 27(3), 269-275. <https://doi.org/10.1080/07370000903014386>
- Haghjoo, S., Radmehr, F., & Reyhani, E. (2023). Analyzing the written discourse in calculus textbooks over 42 years: The case of primary objects, concrete discursive objects, and a realization tree of the derivative at a point. *Educational Studies in Mathematics*, 112(1), 73-102. <https://doi.org/10.1007/s10649-022-10168-y>
- Hähkiöniemi, M. (2006). The role of representations in learning the derivative. In M. Johnsen-Høines & A. B. Fuglestad (Eds.), *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 73–80). PME.
- Hammer, D., Elby, A., Scherr, R. E., & Redish, E. F. (2005). Resources, framing, and transfer. *Transfer of Learning from a Modern Multidisciplinary Perspective*, 89.
- Hand, V., Penuel, W. R., & Gutiérrez, K. D. (2012). (Re) framing educational possibility: Attending to power and equity in shaping access to and within learning opportunities. *Human Development*, 55(5-6), 250-268. <https://doi.org/10.1159/000345313>
- Herbst, P., & Chazan, D. (2012). On the instructional triangle and sources of justification for actions in mathematics teaching. *ZDM*, 44(5), 601-612. <https://doi.org/10.1007/s11858-012-0438-6>

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

- Herbst, P., Boileau, N., & Gürsel, U. (2018). Examining the work of teaching geometry as a subject-specific phenomenon. In *International perspectives on the teaching and learning of geometry in secondary schools* (pp. 87-110). Springer, Cham.
- Herbst, P., Chazan, D., & Boileau, N. (2020). Framing, normativity, and serviceability in teachers' decision-making during lessons. In *Proceedings of the 14th International Congress on Mathematical Education (ICME-14)*. Shanghai, China.
- Johnson, E., Andrews-Larson, C., Keene, K., Melhuish, K., Keller, R., & Fortune, N. (2020). Inquiry and gender inequity in the undergraduate mathematics classroom. *Journal for Research in Mathematics Education*, 51(4), 504-516. <https://doi.org/10.5951/jresmetheduc-2020-0043>
- Kendal, M., & Stacey, K. (2001). The impact of teacher privileging on learning differentiation with technology. *International Journal of Computers for Mathematical Learning*, 6, 143-165. <https://doi.org/10.1023/A:1017986520658>
- Larsen, S., Marrongelle, K., Bressoud, D., & Graham, K. (2017). Understanding the concepts of calculus: Frameworks and roadmaps emerging from educational research. In J. Cai (Ed.) *Compendium for research in mathematics education* (pp. 526-550). National Council of Teachers of Mathematics.
- Laursen, S. (2019). Levers for change: An assessment of progress on changing STEM instruction. American Association for the Advancement of Science.
- Laursen, S. & Rasmussen, C. (2019). I on the prize: Inquiry approaches in undergraduate mathematics. *International Journal of Research in Undergraduate Mathematics*, 5(1), 1-18. <https://doi.org/10.1007/s40753-019-00085-6>
- Park, J. (2015). Is the derivative a function? If so, how do we teach it?. *Educational Studies in Mathematics*, 89, 233-250. <https://doi.org/10.1007/s10649-015-9601-7>
- Proudfoot, K. (2022). Inductive/Deductive hybrid thematic analysis in mixed methods research. *Journal of Mixed Methods Research*. <https://doi.org/10.1177/15586898221126816>
- Shultz, M. (2020). The rationality of college mathematics instructors: The choice to use inquiry-oriented instruction [Unpublished doctoral dissertation]. University of Michigan.
- Tallman, M. A., Carlson, M. P., Bressoud, D. M., & Pearson, M. (2016). A characterization of calculus I final exams in US colleges and universities. *International Journal of Research in Undergraduate Mathematics Education*, 2(1), 105-133. <https://doi.org/10.1007/s40753-015-0023-9>
- Ubuz, B. (2007). Interpreting a graph and constructing its derivative graph: stability and change in students' conceptions. *International Journal of Mathematical Education in Science and Technology*, 38(5), 609-637. <https://doi.org/10.1080/00207390701359313>
- White, N., & Mesa, V. (2014). Describing cognitive orientation of Calculus I tasks across different types of coursework. *ZDM – Mathematics Education*, 46(4), 675-690. <https://doi.org/10.1007/s11858-014-0588-9>
- Zandieh, M. (2000). A theoretical framework for analyzing student understanding of the concept of derivative. *CBMS Issues in Mathematics Education*, 8, 103-127.

## GUIDED REINVENTION AS TEACHING PRACTICES THAT AIM TO PROMOTE STUDENT LEARNING TO DEFINE

Jungeun Park  
University of Delaware  
jungeun@udel.edu

Michael Oehrtman  
Oklahoma State University  
michael.oehrtman@okstate.edu

Jason Martin  
University of Central Arkansas  
jasonm@uca.edu

Douglas Rizzolo  
University of Delaware  
drizzolo@udel.edu

*Our study explores teaching practices that aim to promote students' learning to define an analytical object. A Calculus II instructor conducted a teaching experiment (TE) in which 11 students reinvented a formal definition of a limit over five class periods with the instructor's guidance. During the TE, the instructor's teaching practices were based on what principles of definitions from the literature inform about defining discourse. Our analysis of the instructor's teaching practices and students' follow-up work revealed several teaching practices that seem to promote development of students' narratives towards a formal definition of an analytical object: Providing testing methods to check if students' narratives can be considered as a completion of the defining task, asking students to place components of their definitions on graphs with related quantities, and asking them to reflect their illustration of the definition in their written definition.*

Keywords: Calculus, Classroom Discourse, Undergraduate Education

Recently, there has been growing interest in students' learning at times when the rules of mathematical discourse change and the teaching practices promoting such learning. The Commognition approach (Sfard, 2008) defines such learning as meta-level learning because it involves students learning about new rules of discourses as well as expanding and changing properties of mathematical objects that the students are already familiar with (Valenta & Enge, 2022). The instructor's role in meta-level learning is particularly important because those new discursive rules are historically established and thus difficult for learners to discover on their own or appreciate their value in contrast to the ones that are familiar to them (Nachlieli & Elbaum-Cohen, 2021; Valenta & Enge, 2022). There have been recent studies about the teaching practices that aim to promote meta-level learning and calls for more studies (Cooper & Lavie, 2021; Nachlieli & Elbaum-Cohen, 2021; Schüler-Meyer, 2020; Valenta & Enge, 2022).

In this study, we consider students' learning to define as meta-level learning because this learning involves students learning about the rules of defining discourse, which impacts both their defining procedure and the end product, i.e., their definitions as well as expanding their narratives about mathematical objects. Studies about students' engagement with definitions have examined their difficulties with formal definitions and how they come to agreement on how to use mathematical words according to their formal definitions (e.g., Tabach & Nachlieli, 2015) and how students develop their own definition based on iterative refinements of their intuitive thinking about a mathematical object (Oehrtman et al., 2014; Swinyard, 2011; Schuler-Meyer, 2018 & 2020; Swinyard & Larsen, 2012). This study aims to expand our understanding of students learning to define by examining teaching practices that aim to promote students'

development of a definition of an analytic object in a guided reinvention setting with the research question:

What teaching practices can promote students' reinvention of a formal definition of an analytic object?

### **Theoretical Background**

We adopted commognition as our theoretical framework by conceptualizing learning to define as meta-level learning and building on existing studies about teaching practices that promote meta-level learning. We adopted guided reinvention as an instructional approach that could promote meta-level learning. Moreover, we built on existing guided reinvention studies that reported how students developed their narratives about mathematical objects and expand those results by investigating the teaching practices that could promote such development.

### **Learning and Teaching in Commognition**

Commognition is a theoretical framework that views mathematics as a type of discourse characterized by its distinctive use of words and visuals, narratives about mathematics objects, and routines that are task-procedure pairs. A task is a setting where a learner considers "herself bound to act," and a procedure is the prescription of the actions "that fits both the present performance and those on which it was modeled" from her past experience (Lavie et.al., 2019, pp. 160-161; Sfard, 2008). Commognition defines learning as change in one's discourse: at the object-level which involves new words, routines, narratives about familiar objects and at the meta-level which involves changes in meta-rules, which impact routines. The role of experts is important in meta-level learning because such learning is often not self-motivated by students due to the fact that those rules are historically developed and the value of adopting them is not clear to non-experts (Nachlieli & Tabach, 2012; Valenta & Enge, 2022). Commognition defines teaching as "communicational activity the motive of which is to bring the learners' discourse closer to a canonical discourse" (Tabach & Nachlieli, 2016, p. 299) and teaching practice also as task-procedure pairs, i.e., "the task as seen by the performing teacher together with the procedure she executed to perform that task" (Nachlieli & Elbaum-Cohen, 2021, p. 3).

Teaching practices that promote meta-level learning became the subject for a growing field of research (e.g., Cooper & Lavie, 2021; Nachlieli & Elbaum-Cohen, 2021; Schüler-Meyer, 2020; Valenta & Enge, 2022). These studies have revealed teaching practices that help students transition from their old discourse that is no longer aligned with meta-rules of the target discourse towards the new discourse by making "boundaries between previous and new discourses" (Nachlieli & Elbaum-Cohen, 2021, p. 11) clear, by providing students tasks with which they could use features of their old discourse (e.g., routines) in emerging ways that are "appropriate in the eyes of the teacher" (Cooper & Lavie, 2021, p. 3; Schluer-Meyer, 2020), and helping them interpret the task towards a more formal discourse (Valenta and Enge, 2022).

### **Defining as Meta-level Learning & Guided Reinvention as Teaching Practice Promoting It**

The meta-discursive nature of defining and its importance as a mathematical practice have been established (Martín-Molina, 2018; Martín-Molina, 2020; Ouvrier-Buffet, 2011; Zandieh & Rasmussen, 2010). Studies have examined the characteristics of experts' or students' defining processes (Martín-Molina, 2018; Martín-Molina, 2020; Ouvrier-Buffet, 2011; Zandieh & Rasmussen, 2010). There have been recent studies considering defining as meta-level learning, which involves learning about rules about definitions and constructing a definition according to them (Martín-Molina, 2020; Schüler-Meyer, 2018). Those meta-rules are informed by properties of definitions such as that they should be "unambiguous and precise", "operable, so that it can be decided whether an object belongs to a category or not", "complete and sufficient" (Schüler-

Meyer, 2020, p. 238), and “invariant under a change of representation” (Zaslavsky & Shir, 2005, p. 320). Activities aiming to produce a definition that can be endorsed in formal mathematical discourse involves adoption of metarules that those properties inform (Schüler-Meyer, 2020).

We adopted a guided reinvention instructional approach in which students reinvent a formal definition of an analytical object: the limit of a sequence. In this approach, students learn about mathematical objects through “a process by which students formalize their informal understandings and intuitions” through their own experiences and activities with carefully designed instructor’ guidance (Gravemeijer et al. 2000, p. 237). Guided reinvention studies have investigated the development of students’ narratives towards a formal definition from the constructivist view by analyzing the data with pairs of students (Oehrtman et al., 2014; Swinyard, 2011; Swinyard & Larsen, 2012). Recent guided reinvention studies addressed the meta-discursive nature of defining through commognition (Schuler-Meyer, 2018 & 2020). Schuler-Meyer (2020) guided secondary students towards formal mathematical discourse by using a task where students could use their familiar secondary mathematics routines of “categorizing, describing, symbolizing, and calculating” in a specific task situation (i.e., epsilon-strip activity) and attempt to transform them into routines of formally defining limits.

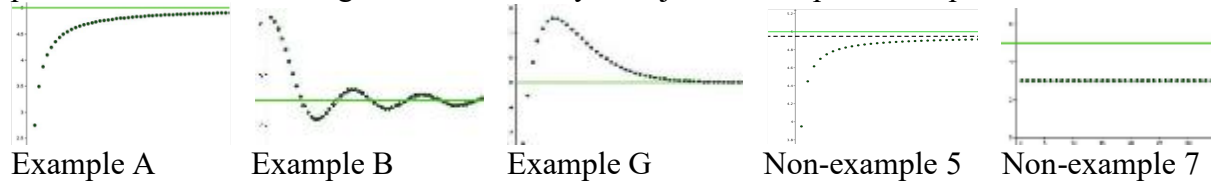
Our study also views learning to define as meta-level learning and aims at examining teaching practices that can promote it. Like Valenta and Enge (2022) who documented teaching practices aiming at helping students in a proving context transition from providing an empirical argument using examples to providing a deductive argument “to show the validity of the concluding narrative” (p. 5), we examined the teaching practices that promote students learning to define (a limit in our study) by transitioning from their intuitive and descriptive narrative toward a formal narrative about quantities and relations between quantities. We are particularly interested in how the metarules of defining, which are informed by the principles of definitions from the literature, could guide such teaching practices and their potential impact on students’ work. In comparison to Schuler-Meyer (2020) where students’ defining activities built on their completion of a specific task, which involves the components and deductive order of a formal definition of limit, we explored more general aspects of students’ defining process and the teaching practices that seemed to help students progress toward a formal definition. The results of this study will also contribute to the field by providing a mechanism behind students’ learning trajectory that was revealed in existing guided reinvention studies by providing teaching practices that might have helped such learning.

### **Research Design**

We collected data in a Calculus II class at a public U.S. university where an instructor (the 2<sup>nd</sup> author, the TR- teacher researcher) conducted the guided reinvention TE in which 11 students generated definitions of limit for five class periods (50-75 minutes). Students confirmed that they had not seen a formal definition of limit before. In the beginning of the TE, students were asked to generate a broad range of example and non-example graphs of sequences converging to 5 (Figure 1) and discuss their convergent or non-convergent behaviors. Then, students worked in small groups (TE group – GTE, G1 and G2) to define a limit of a sequence. We videorecorded the activity of one group of 4 students (GTE with participants – P1, P2, P3, and P4) and whole class interactions with the TR, transcribed the recordings using Transana, and collected and digitized copies of students’ written work.

For analysis, we catalogued the GTE’s individual and group definitions, their interactions with the instructor and other groups, problems the GTE identified with their definitions, solutions that they suggested, and problematic issues with their definition (issues that experts

may recognize but the GTE had not noticed). We analyzed teaching practices that aimed to promote students' learning to define an analytic object as task-procedure pairs. We first



**Figure 1. Selected example and non-example graphs used in TE (solid horizontal line: 5)**

examined the sequences in data where the TR intervened in students' work by looking at the TR's actions, writing the general procedure based on the actions and trying to identify the task that the TR wanted to accomplish. We, then identified meta-level or object-level rules that seemed to inform each teaching practice. We also documented what seemed to be an impact of the teaching practice on students' subsequent work. It should be noted that in the beginning of TE, the TR laid the groundwork for students to adopt some of the metarules, such as that a definition should "include all examples and all nonexamples," "not be synonymous," and "be concise and precise" by discussing them with students. These can be considered as a teaching practice prompting students' learning to define. However, we decided to focus on the teaching practices in the results that are responsive to what students created as definition to document a potential impact on their work. During TE, the TR used the metarules in his teaching practice to help students shift their narratives about the limit towards a formal definition.

### Results

This section will report the teaching practices that aim to promote students' reinvention of a formal definition. For each teaching practice, we present the status of students' work at the time it was observed, describe the teaching practice, and explain what seems to be its impact on students' work. Due to limited space, we only present teaching practices (TP) that seemed to lead to substantial changes in students' subsequent definitions.

#### **TP1: Communicating to students that their definition should not exclude examples or include non-examples.**

Consistent with the results of other studies, the GTE's beginning definitions were descriptive of salient examples including "terms ...ultimately approach 5 as  $n$  increases," to which the TR asked if their definition captures an example. Table 1 summarizes this teaching practice.

TR: I could see a student coming here who hadn't seen your definition before, and saying something to the extent of, well it appears to me that these, since you described it [Graph B] as terms go away from 5, and then they come back and then they go away and then they come back, uh that the terms in B, the values of them, the a sub  $n$  values... don't always approach 5. [6 second pause] (Day 1, 45 minutes)

**Table 1: TP1: Communicating to students that definitions should not exclude any examples.**

Task	Procedure	Empirical Observation
Leading students to realize that their definition does not include all	TR uses a hypothetical student outside the GTE, pointing out that their	"A student...something to the extent of ... terms go away from 5, and then they come back...don't

examples, thus violating Metarule 1.	definition excludes an example.	always approach 5.”
---	------------------------------------	---------------------

This teaching practice provided a method for students to check if their definition can be considered as a completion of the defining task in terms of meeting the metarule that a definition should include all examples and exclude all non-examples. This teaching practice seemed to have an impact on the GTE’s work, as in follow-up conversations, they used this method to test their revised definition that included “the distance between the terms and 5 decreasing,” ultimately rejecting it because, “That’s not true either cause the distance is increasing at the last point” (P1).

**TP 2: Communicating to students that words in definition should be quantified through placing them on graphs**

In the GTE’s initial definitions, they used “ultimately” with “terms...approach” and justified it with inclusion of graph G. The TR then asked what and where “ultimately” is on the graph:

- TR: In G, what's this ultimately?
- P2: Because it starts out not approaching 5, like you don't think it is.
- TR: Yeah. Can you get specific, what count one, two, three, four, five, you know, where- what's this ultimately? In G, what's this ultimately?
- P3: It approaches from a one and a two, but then from a three through four, five, six, it's going away from 5.
- P1: It'd be six on.
- TR: Six on. Is that what you think ultimately was getting at?...I'm looking for you kinda illustrating this definition [points to a GTE definition] on here [waves hand over graphs B, G & 5 on their board] (Day 2, 35 minutes).

In this excerpt, the TR explicitly chose one of the GTE student’s responses to “where/what ultimately is” (P1) and asked them to illustrate the word on the graph. Table 2 summarizes this teaching practice.

**Table 2: TP2: Communicating to students that words in definition should be quantified through placing them on graphs**

Task	Procedure	Empirical Observation
Making student aware of quantities related to words that are used in their definition	The TR asked students to place the words on graphs with specific numbers that are associated with it.	“What's this ultimately?...Can you get specific, what count one, two, three, four, five, you know, where- what's this ultimately?” ([3-5 & 6])

In contrast to TP1 which provided students a method to test if they completed the task of defining, TP2 provides a procedure through which they can quantify not-yet quantified words in the definition, thus progress towards a formal definition. Note that TP2 is based on the following metarule of defining discourse: components in written definition should have places on graphs. This metarule is our operationalization of a principle that a definition should be “invariant under a change of representation” (Zaslavsky & Shir, 2005, p. 320). This invariance principle was applied when a form of students’ narratives changed from written to graphical. In subsequent



conversations, the GTE placed ultimately on graph G (Figure 2), and added “a specific value for  $n$ ” in their next definition:

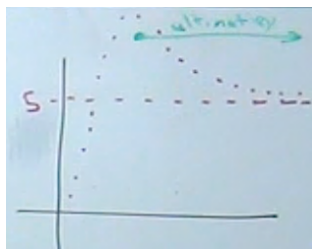


Figure 2. GTE’s illustration of “ultimately” (Day 2, 41 minutes)

**GTE Definition 6:** There is a specific value for  $n$ , such that  $a_n$  for all  $n$ 's after that is so close to 5 that it basically equals 5. (like  $0.999=1$ ) (Day 2, 43 minutes).

Although there is no direct evidence that the phrase “a specific value for  $n$ ...for all  $n$ 's after that” is transformed from “ultimately”, the phrase seems closely related to them identifying “ultimately” with “six on” and drawing it as an arrow starting at a specific value of  $n$  and on, which seemed to be an impact of TP2.

**TP 3: Asking students to illustrate not-yet-defined elements on graphs and then to capture the illustration to explain the elements**

On Day 4 of the TE, the GTE was provided the opportunity to read other groups’ definitions, and decided to adopt some of their notations in their own definition:

**GTE Definition 16:** If  $|a_n - 5| \leq \varepsilon$  for any value of  $n$ ,  $n_c < n$  with error  $\rightarrow 0$ , then  $a_n$  converges to 5 ( $n_c = \text{some } n$ ) (Day 4, 8 minutes).

In this definition,  $n_c$  and  $\varepsilon$  are not explained but the GTE did not seem to notice this problem even after illustrating the definition on graphs multiple times. To address this issue, the TR asked them to illustrate the definition on graphs explicitly including those components, then to check if their definition captured their illustrations, and then to revise it if not.

TR: So where exactly is  $n$  sub  $c$ ? ...

P1: I think it would be this one [points to the 14th point]. Well, that's the last  $n$  that all the  $n$ 's after that [same point], the terms for these  $n$ 's would be within the error bound.

TR: So then how would you label  $n$  sub  $c$ ?

P3: Well this would be 14 [labels  $n_c$  on the  $n$  axis] I think. That'd be the  $n$  sub  $c$ .

...

TR: Do you feel like your definition captures everything you’ve illustrated because you keep saying there is this  $n$  sub  $c$  [points to  $n_c$  in Graph B], and then you said all the terms [looking at P1]

...

P1: All the terms after it aren’t within-

TR: All the terms after it- so do you think this definition [waving hand over their definition] captures that idea that-

P1: We should probably add that though, all the terms after it?

In this excerpt, the TR asked the GTE to illustrate an undefined term and describe the illustration, and then ask them to reflect the descriptions in their definition, which Table 3 summarizes.

**Table 3. TP3: Asking student to illustrate a not-yet-defined component on graphs and then asking them if their definition captures how they illustrated the component.**

Task	Procedure	Empirical Observation
Leading students to realize an undefined component in their definition and have them define it.	Asking students to illustrate the definition including the not-yet-defined component on graphs and asking them if their definition captures how they illustrated the component.	“Where exactly is $n$ sub $c$ ?” [After GTE answered] “Do you think this definition captures that idea?”

We note that TP3 is also based on a metarule of defining discourse: a written definition should capture how it is illustrated on example and non-example graphs. This metarule is also our operationalization of the principle that a definition should be “invariant under a change of representation” (Zaslavsky & Shir, 2005, p. 320), previously mentioned in TP2. This time, it was applied when the form of students’ narratives changed from graphical to written. This teaching practice seemed to have an impact on the GTE’s definition, as after this conversation, they revised their definition consistent with how they illustrated  $n_c$ :

**GTE Definition 17: If  $|a_n - 5| \leq \varepsilon$  for an  $n_c$  being an  $n$  that all terms after  $n_c$  are within the given error bound then  $\lim_{n \rightarrow \infty} a_n = 5$  (Day 4, 29 minutes).**

Using the same practice, by directing the GTE’s attention to the component that they constantly placed first while illustrating their definition on graphs, which was “ $\varepsilon$ ”, the TR asked GTE to reflect the illustration of  $\varepsilon$  in their definition. This teaching practice also seemed to have an impact; students eventually revised their definition by starting with “within a given error bound”.

### Discussion and Conclusion

The goal of this study was to identify teaching practices that aim to promote students’ learning of defining. We analyzed five days of a guided reinvention TE in which students reinvented a formal definition of limit, identified such teaching practices (TP), and here we reported on three of the practices that seemed to lead to substantial changes in students’ subsequent definitions:

- TP1: Communicating to students that definitions should not exclude any examples.
- TP2: Communicating to students that words in definition should be quantified through placing them on graphs.
- TP 3: Asking students to illustrate not-yet-defined elements on graphs and then to capture the illustration to explain the elements.

These teaching practices show how metarules of defining could be used to help students transition from their informal descriptive narrative about a mathematical object towards formal ways of talking about it, i.e., by defining the object using quantities and their relationships. First, TP1, which provided students with a method to test if their definition could be considered as completion of the task of defining, is based on the metarule that a definition should include all examples and exclude all nonexamples. The instructor used TP1 to point out that students' initial definition violated this metarule, so it could not be considered as completion of the defining task. Students adopted this testing method to test their subsequent definitions, and rejected them when they violated this rule. In comparison, TP2 and TP3 provided students with guidance on how to proceed to create a definition that meets the metarule. TP2 and TP3 were based on our operationalization of another principle of definition, namely that a definition should be "invariant under a change of representation" (Zaslavsky & Shir, 2005, p. 320), which enables and reflects flexibly moving between a written definition of an analytic object (e.g., limits) and its graphical mediation on a plane. Specifically, TP2 seemed to impact students' subsequent work where they started to quantify not-yet quantified words in their definition, thus moving towards a formal definition of limit, which consists of quantities and their relations (e.g., Changes in the GTE's definitions: "ultimately" → "six on" → an arrow on the graph in Figure 1 → "a specific value for  $n$ , such that  $a_n$  for all  $n$ 's after that" → " $n_c \dots$  all terms after  $n_c$ "). TP3 also seemed to impact their subsequent work, such as when they explained previously undefined terms (i.e.,  $n_c$  and  $\varepsilon$ ) in their definition by reflecting how they placed those terms on graphs while illustrating their written definition. We also view TP2 and TP3 as using components of students' existing discourse (not-yet quantified or explained terms in their current definition) according to routines governed by the metarules of the targeting discourse (i.e., formal discourse of defining), by having students treat those terms as if they were already part of a formal discourse (e.g., asking them to place not-yet quantified terms like "ultimately" on a graph with numbers and place not-yet explained terms " $n_c$  and  $\varepsilon$ " based on how they think about them as if they were explained in the definition). Such uses of components of existing discourse in the context of emerging discourse "in a manner that is appropriate in the eyes of the teacher" (Cooper & Lavie, 2021, p. 3) could be considered another way of helping students transition to more formal discourse governed by new metarules. Activities that are interdiscursive in nature should be further investigated in terms of their role and impact on development of students' defining narratives in general and their definitions of the limit in particular. We suggested that those teaching practices explain a mechanism behind students' reinvention of a formal definition of limit. In particular, we suggested that changes of students' narratives about the limit seemed to be impacted by those practices based on metarules of defining.

Our study contributes to the ongoing discussion about teaching practices that promote students' metalevel learning (Martín-Molina, 2020 & Schüler-Meyer, 2020) by providing empirical evidence of such practices in the context of defining, which is evidence that existing studies have called for (Nachlieli and Elbaum-Cohen, 2021). It also contributes to the existing guided reinvention research, which has documented learning trajectories through which students develop a formal definition of limit, by providing a potential mechanism through documenting teaching practices that seem to promote such development. We also note that such practices, which are based on metarules of defining, provided an alternative way to use and communicate those rules with students instead of by explicitly telling students about the rules and asking them

to adjust their definition accordingly, which Schüler-Meyer (2020) found “very demanding for the teacher and students” (p. 245).

## References

- Cooper, J., & Lavie, I. (2021). Bridging incommensurable discourses – A commognitive look at instructional design in the zone of proximal development. *The Journal of Mathematical Behavior*, 61, 1–9. <https://doi.org/10.1016/j.jmathb.2020.100822>
- Gravemeijer, K., Cobb, P., Bowers, J., & Whitenack, J. (2000). Symbolizing, modeling, and instructional design. In P. Cobb, E. Yackel, & K. McClain (Eds.), *Symbolizing and communicating in mathematics classrooms: Perspectives on discourse, tools and instructional design* (pp. 225–273). Mahwah, NJ: Erlbaum and Associates.
- Lavie, I., Steiner, A., & Sfard, A. (2019). Routines we live by: From ritual to exploration. *Educational Studies in Mathematics*, 101(2), 153–176. <https://doi.org/10.1007/s10649-018-9817-4>
- Martín-Molina, V., González-Regaña, A. J., & Gavilán-Izquierdo, J. M. (2018). Researching how professional mathematicians construct new mathematical definitions: A case study. *International Journal of Mathematical Education in Science and Technology*, 49(7), 1069-1082.
- Martín-Molina, V., González-Regaña, A. J., Toscano, R., & Gavilán-Izquierdo, J. M. (2020). Differences between How Undergraduate Students Define Geometric Solids and What Their Lecturers Expect from Them through the Lens of the Theory of Commognition. *EURASIA Journal of Mathematics, Science and Technology Education*, 16(12), 1-10.
- Nachlieli, T., & Elbaum-Cohen, A. (2021). Teaching practices aimed at promoting meta-level learning: The case of complex numbers. *The Journal of Mathematical Behaviour*, 62, Article 100872. <https://doi.org/10.1016/j.jmathb.2021.100872>
- Nachlieli, T., & Tabach, M. (2012). Growing mathematical objects in the classroom—The case of function. *International Journal of Educational Research*, 51, 10–27.
- Oehrtman, M., Swinyard, C., & Martin, J. (2014). Problems and solutions in students' reinvention of a definition for sequence convergence. *The Journal of Mathematical Behavior*, 33, 131-148. <https://doi.org/10.1016/j.jmathb.2013.11.006>
- Ouvrier-Buffet, C. (2011). A mathematical experience involving defining processes: In-action definitions and zero-definitions. *Educational Studies in Mathematics*, 76(2), 165-182. <https://doi.org/10.1007/s10649-010-9272-3>
- Schüler-Meyer, A. (2018) Defining as discursive practice in transition—upper secondary students reinvent the formal definition of convergent sequences. In V. Durand-Guerrier, R. Hochmuth, S. Goodchild & N.M. Hogstad (Eds.) *Proceedings of INDRUM 2018. Second conference of the International Network for Didactic Research in University Mathematics*. Kristiansand, Norway: University of Agder and INDRUM, pp. 537–546.
- Schüler-Meyer, A. (2020). Mathematical routines in transition: facilitating students' defining and proving of sequence convergence. *Teaching Mathematics and its Applications: An International Journal of the IMA*, 39(4), 237-247. <https://doi.org/10.1093/teamat/hrz019>
- Sfard A. (2008). *Thinking as communication*. Cambridge, UK: Cambridge University Press.
- Swinyard, C. (2011). Reinventing the formal definition of limit: The case of Amy & Mike. *Journal of Mathematical Behavior*, 30, 93–114.
- Swinyard, C., & Larsen, S. (2012). What does it mean to understand the formal definition of limit? Insights gained from engaging students in reinvention. *Journal for Research in Mathematics Education*, 43(4), 465–493.
- Tabach, M., & Nachlieli, T. (2015). Classroom engagement towards using definitions for developing mathematical objects: The case of function. *Educational Studies in Mathematics*, 90, 163-187. DOI 10.1007/s10649-015-9624-0
- Tabach, M., & Nachlieli, T. (2016). Communicational perspectives on learning and teaching mathematics: Prologue. *Educational Studies in Mathematics*, 91(3), 299–306. DOI 10.1007/s10649-015-9638-7
- Valenta, A., & Enge, O. (2022). Teaching practices promoting meta-level learning in work on exploration-requiring proving tasks. *The Journal of Mathematical Behavior*, 67, 1-15 <https://doi.org/10.1016/j.jmathb.2022.100997>
- Zandieh, M., & Rasmussen, C. (2010). Defining as a mathematical activity: A framework for characterizing progress from informal to more formal ways of reasoning. *Journal of Mathematical Behavior*, 29(2), 57-75. doi:10.1016/j.jmathb.2010.01.001
- Zaslavsky, O., & Shir, K. (2005). Students' conceptions of a mathematical definition. *Journal for Research in Mathematics Education*, 36(4), 317-346.

# IMPLICATIONS OF FASTER/SLOWER LANGUAGE ON UNDERGRADUATE PRECALCULUS STUDENTS' GRAPHING

Irma E. Stevens  
University of Rhode Island  
irma.stevens@uri.edu

*Researchers have recommended using tasks that support students in reasoning covariationally to build productive meanings for graphs, rates of change, exponential growth, and more. However, not many recent studies have been done to identify how students reason when engaging in covariational reasoning tasks in undergraduate precalculus courses. In this study, I analyze submitted classwork, including video submissions of that work, in an applied precalculus undergraduate course. In comparing current literature on students' covariational reasoning with these students' responses, there is some overlap that this study provides additional insights to, and there are also unique ways of reasoning these students exhibited tied to understanding the steepness of slope as being associated with ideas of speed. This study contributes to knowledge about how students develop covariational reasoning.*

Keywords: Cognition, Undergraduate Education, Precalculus, Advanced Mathematical Thinking

## Introduction

Attempts to study students' covariational reasoning—students' reasoning about how quantities change together—has been going on for decades. Back in 2002, Carlson et al. (2002) proposed a framework for covariational reasoning. This framework was built after interviewing precalculus students on what has been aptly referred to as the Bottle Problem. In this problem, students graph volume vs. height of a given image of a slice of a bottle. Variations of the Bottle Problem have been used with a variety of populations (with K-12 students, undergraduate mathematics students, preservice teachers, etc.) and contexts (in the US and internationally). The framework developed from Carlson et al.'s analysis of students' responses to the Bottle Problem has served several researchers for the past few decades in understanding how students reason covariationally. Moreover, in 2015, Thompson & Carlson (2015) proposed a new framework for (co)variational reasoning that attends to more nuanced mental actions—such as the construction of multiplicative objects and distinguishing between variation and covariation.

The study described in this report builds on the work of these and other researchers' understanding of covariational reasoning by using these newer ideas on covariational reasoning. In particular, the newer ideas are used to explain a common way of reasoning seen in the Bottle Problem with students in an undergraduate precalculus classroom—one of which was a way of reasoning that was offered by a student in the initial Carlson et al. (2002) paper to introduce the framework. Below is the excerpt from that paper (Carlson et al, 2002, p. 366) in which the student describes their reasoning for constructing a graph with a concave down curve.

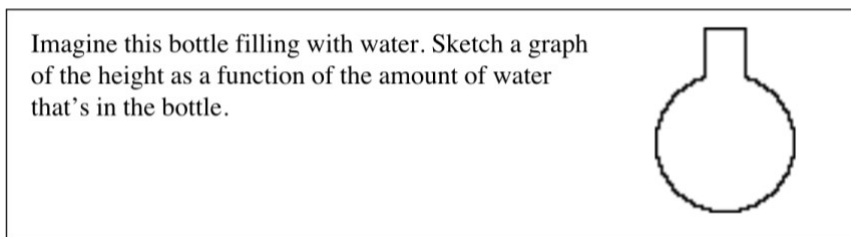
B: OK, the more water, the higher the height would be [MA2]. In terms of height of the water, that is what we are talking about. If you are talking about the height left over, that is basically decreasing. Right here the height will be zero and the volume is zero.

As you go up, a little more height increases and the volume increases quite a bit [MA3], so the amount by which the height goes up is not as fast [MA3]. Once you get there [pointing to halfway up the spherical part of the bottle], the height increases even slower [MA3]. I guess from here to there height increases the same as the volume increases, and once you

get here it increases slower [MA3]. No, I am wrong. So, every time you have to put more and more volume in to get a greater height towards the middle of the bottle and once you get here, it would be linear, probably [pointing to the top of the spherical portion]. So, it's always going up [tracing his finger along the concave-down graph], then it would be a line.

Int.: So, what does the graph look like?

B: Like this [pointing to the concave-down graph he has constructed], but it has a straight line at the end.



**Figure 1: The Bottle Problem from Carlson et al. (2002, p. 360).**

The goal of this study was to collect students' responses to the Bottle Problem from precalculus students and identify the covariational reasoning the students exhibited. In doing so, the following research question emerged and will be addressed in this paper: How do students who draw graphs for a situation involving two different rates reason covariationally? To answer this question, I will report on what literature on students' quantitative and covariational reasoning has said about this situation in the past, including ideas of iconic translations, shape thinking, and thematic reasoning. I will then describe the methods of how I collected data on this problem and coded it to pull out examples similar to the ones described in the research question and identify a unique but related way of reasoning the students exhibited. I will then report on the students who exhibited this reasoning and conclude by describing how these students justified their graphs and relate it to the existing research on covariational reasoning. I conclude by introducing a new way of understanding students' reasoning about concavity and linearity in graphs that has implications for future research on students' meanings for graphing.

### **Theoretical Framework**

In 2002, Carlson et al. (2002) proposed a framework for covariational reasoning that included five mental actions: "coordinating the value of one variable with changes in another" (MA1), "coordinating the direction of change of one variable with changes in another variable" (MA2), "coordinating the amount of change of one variable with changes in the other variable" (MA3), "coordinating the average rate-of-change of the function with uniform increments of change in the input variable" (MA4), and "coordinating the instantaneous rate of change of the function with continuous changes in the independent variable for the entire domain of the

function” (MA5) (p. 357). In classifying students based on this framework, they recognized that students can exhibit behaviors indicating particular mental actions without providing adequate evidence they “possessed an understanding that supported the behavior” (p. 358), citing Vinner’s (1997) notion of pseudo-analytical behavior. They thought that the student in the excerpt above may fit this category of reasoning.

In moving forward with research on student’s covariational reasoning, Moore & Thompson began to distinguish between evidence of understanding vs. pseudo-analytical behavior in students’ graphing activities. In 2015, they introduced the notion of static and emergent shape thinking. Static shape thinking entails thinking about a graph “as an object in and of itself”, possessing properties associated with learned facts. An example in the paper was that of a student referencing “the sine graph” as a “graph everyone knows about” and that has a particular shape and orientation. Emergent thinking emphasizes the creation of a trace constructed via representing a covariational relationship between quantities. Students’ reasoning in the sine example might include highlighting magnitudes and constructing a point on the graph simultaneously representing both quantities’ magnitudes (i.e., a multiplicative object (Piaget, 1970)), and then varying the quantities’ magnitudes to construct a trace of a line with that point.

Their distinction was one contribution of many to research interested in making sense of graphical reasoning. For example, Stevens et al. (2016) looked at students’ justifications of curvature, Frank (2017) looked at the process of constructing a multiplicative object, Paoletti (2020) looked at the impact of inverse functions on students’ representational activity, Johnson (2022) looked at the impact of varying representations, and Ellis et al. (2013) focused on students’ understanding particular kinds of relationships being represented (e.g., exponential). Moore and colleagues also continued to learn about students’ construction of relationships between quantities via figurative and operative thought (Liang & Moore, 2020; Moore et al., 2019). This research included attending to iconic translations and thematic associations as evidence of figurative thought, thought focused on the “figure to ground” (Thompson, 1985).

Beyond understanding students’ representational activities, colleagues also worked to develop the construct of covariational reasoning. For example, Castillo-Garsow, Johnson, & Moore (2013) introduced the notion of chunky and smooth shape thinking, attending to the discrete vs. continuous ways of thinking involved in reasoning about varying objects. Ely & Ellis (2018) also considered this in Calculus contexts in terms of a “zooming in” on functions’ graphs. Meanwhile, work continued to develop the initial framework posed by Carlson et al. (2002), and in 2017, Thompson & Carlson (2017) proposed a new framework for covariational reasoning,

revised by attending to “students’ variational reasoning separately from covariational reasoning” and to “how students coordinate their images of quantities’ values varying” by considering the variational reasoning and their construction of a multiplicative object (Thompson & Carlson, 2017; p. 440). The resulting levels of covariational reasoning include: no coordination, precoordination of values, gross coordination of values, coordination of values, chunky continuous covariation, and smooth continuous covariation (Thompson & Carlson, 2017, p. 441).

The aforementioned body of literature has influenced this study. First, the Carlson et al. (2002) covariational reasoning provided an initial framework to analyze students’ responses. The work on students’ static and emergent shape thinking and figurative and operative thought provided a lens to tease apart students who focused on building quantitative relationships between quantities versus representing learned properties. The work on



understanding the impact and importance of gaining evidence for students' reasoning via changing the representation systems helped make sense of the severity of students' association to particular shapes. The work on understanding various relationships helped to unpack the role in which relationship was being described was impacting the resulting graphical representation. Lastly, the updated Thompson & Carlson (2017) framework provided a way to make sense of students' actions in ways that did not rely solely on directional reasoning and amounts of change reasoning, but rather their coordination of quantities and their imagery of variation.

### **Methods**

The study was conducted with 39 students from an undergraduate applied precalculus course at a medium-sized public university in the northeastern U.S. The course is coordinated and the instructor of the course is the researcher. The students in this course are not intended Mathematics or Engineering students, but rather the students with a variety of other majors (e.g., Biology, Conservation, Pharmaceutical Sciences, Marine Affairs, Journalism, Communication Studies, Psychology). The course consisted of students self-reported as White (27), Black (2), Hispanic (1), 2 or more races (8), and not specified (1). Throughout the semester, coursework was collected from the students, and the focus of this study is on the third class assignment of four total class assignments (see task description in the following section). All assignments had two parts: one done in class through groupwork and one outside of class time. Both parts were submitted individually online. The last two assignments included a video submission in which the students talked through their solutions on Part II of the assignment.

In analyzing students' covariational reasoning for this study, the researcher analyzed the 33 submitted written work with accompanying video submissions of Assignment 3. The work was analyzed using Carlson et al.'s (2002) framework. Additionally, using thematic analysis (Corbin & Strauss, 2009), notes were taken on similar work within and across those categories (e.g., mention of speed) and compared to literature on covariational reasoning. The researcher then identified that several of the students reasoned similar to the student mentioned in Carlson et al. (2002). The researcher then used the ideas of shape thinking, co(variational) reasoning, and slope/rate of change to characterize students' reasonings in ways that are viable based on their responses.

### **Task Descriptions**

In this section, I will summarize the goals of the first three assignments to provide a better understanding of the assignments collectively, and then I will focus on the Bottle Problem assignment, indicating adaptations that were made to target the precalculus concepts of polynomial and exponential growth.

### **Overview of Class Assignments**

There were four class assignments throughout the course of the semester. Part I questions primarily focused on exploring a dynamic context, identifying quantities, and representing relationships. Part II primarily asked for more pointed questions about the relationships between quantities. In Assignment 3, the students received various cross-sections of bottles, descriptions of bottles, or graphs relating the height and volume of the bottle and asked to

either draw the corresponding volume-height graph or the corresponding bottle. In Part II of the assignment, the students were asked to submit a 3-5 minute video in which they talked through their various graph and bottle constructions for the Part II bottles.

### **Class Assignment 3: The Bottle Problem**

In the Bottle Problem, the students were given cross-sections of bottles, descriptions of bottles, or graphs of the volume-height relationship for bottles. Part I adopted materials used by Moore and colleagues from the NSF funded *Advancing Reasoning Project*. In Part II, the prompts included asking for specific reference to Direction and Amounts of Change talk. The following narrative was provided to introduce this talk to the students:

**We** create graphs to represent quantities' measures and how these measures change together. So, when we talk about graphs, we should talk about quantities' measures and relationships. We call this **covariation**. There are numerous ways of talking about covariation including: **Correlation** As quantity A changes, quantity B also changes. **Directional** As quantity A increases (or decreases), quantity B increases (or decreases). **Amounts of Change** As quantity A *increases by equal increments*, the *amount of increase (or decrease)* in quantity B *decreases (or increases)*. **Rates of Change** As quantity A increases, quantity B increases (or decreases) at an increasing (or decreasing) rate.

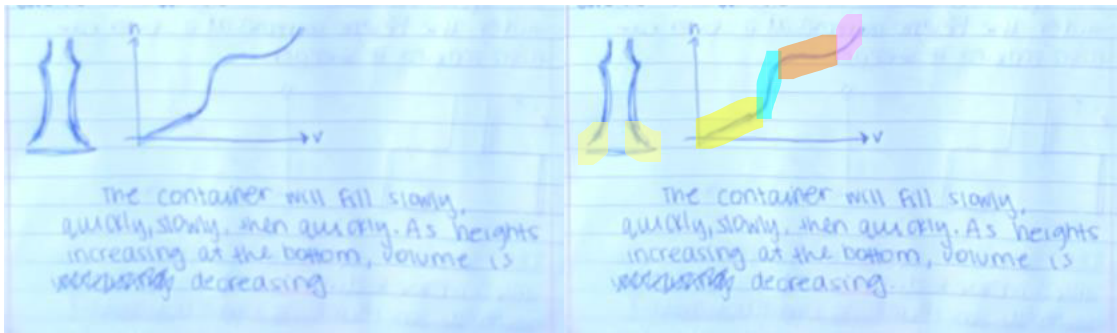
Part II of the assignment had three problems. In the first problem, the students were asked to sketch the volume-height graph and draw the cross-section of a bottle (a) from a volume-height graph of their own choosing, (b) a bottle in which the bottle doubles in volume for every additional inch of height, (c) a bottle in which the for each inch of height the bottle increases, the volume of the bottle increases two more inches cubed in volume than the previous increase. In the second problem, the students drew their own cross-section of a bottle and created the corresponding height-volume graph. In all these problems, the students were also asked to include Direction and Amounts of Change talk using color-coding. In the third problem, the students were asked to create a 3–5-minute video talking through their work in Part II using Direction and Amounts of Change talk. Only responses to Part II 1a and 2 are reported here.

### **Results**

Although all the student work was analyzed using the Carlson et al. (2002) framework for covariational reasoning, the results reported here are two common themes of work in which students used some sort of covariational reasoning in their argument but also offered different justifications for the curvature of their graphs for at least some regions of their graphs. In particular, these categories are justifications based on faster/slower language (thematic associations) and justifications based on the shape of the bottle (iconic) (related to the aforementioned description of the student from Carlson et al. (2002).

#### **Thematic Justification: Faster/Slower**

Ten of the 33 students mentioned some of descriptions that associated the words *faster* and *slower* with the steepness of slope. For four of these students, these descriptions were associated with graphs that were sectioned into linear components (with, from the researcher's perspective, some moderate curvature between the sections that was unaddressed in any students' descriptions). Julia, for instance, gave the following verbal description associated with the work she produced in Figure 3.



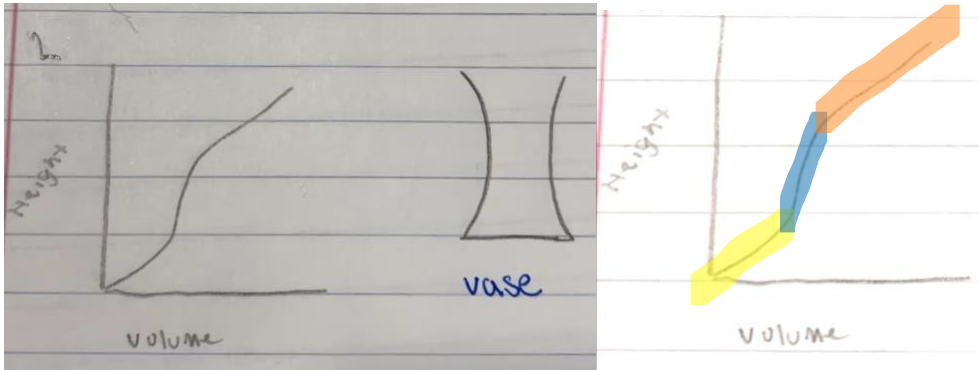
**Figure 3: (Left) Julia’s Bottle and Graph and (Right) Additional Highlights Added by Author to Represent the Sections of the Graph Referenced by Julia.**

[Julia’s Written Response in Figure 3: “The container will fill slowly, quickly, slowly, then quickly. As the height’s increasing at the bottom, volume is decreasing.”]

Julia: And as for this one, the container will fill slowly, then quickly, then slowly, and then quickly again. And you can see this because at the base [points to the bottom of the cross section], even though it’s wider [makes angled in motion with hands], it doesn’t allow as much water in ’cause it narrows and it becomes slimmer [motioning along the middle of the bottle]. So this would fill in very slowly [starts moving fingers alongside and up the cross section of the bottle] but not-but at a more constant rate [pointing to the first linear region on the graph], and then when it changes directions again [motions up the remainder of the bottle], that’s when this changes directions [motions up the remainder of the graph]. Yep. Then as height’s increasing at the bottom, volume is decreasing.

Lorella and Leonhard produced similar bottles and graphs and gave similar justifications. When describing the middle section of her cross section and associated slanted line on her graph, Lorella stated that the volume will “start to fill up quicker and then as shown in the graph right here, is the portion that kinda closes in”, pointing to the middle region of her graph. Leonhard offered the following description of the vase and his graph seen in Figure 4.

Leonhard: And as for this one, the container will fill slowly, then quickly, then slowly, and then quickly again. The vase fills slowly, then quickly, then slowly. With the curve starting out less steep as we can see here [pointing to the first section of the curve], then steeper [pointing to the middle section of the curve]-steeper, very steep, then less steep [pointing to the third section of the curve]. Which is the same as it is here [pointing to the first section of the curve]. Which is why it will be the vase right here [pointing to diagram of cross section of vase].

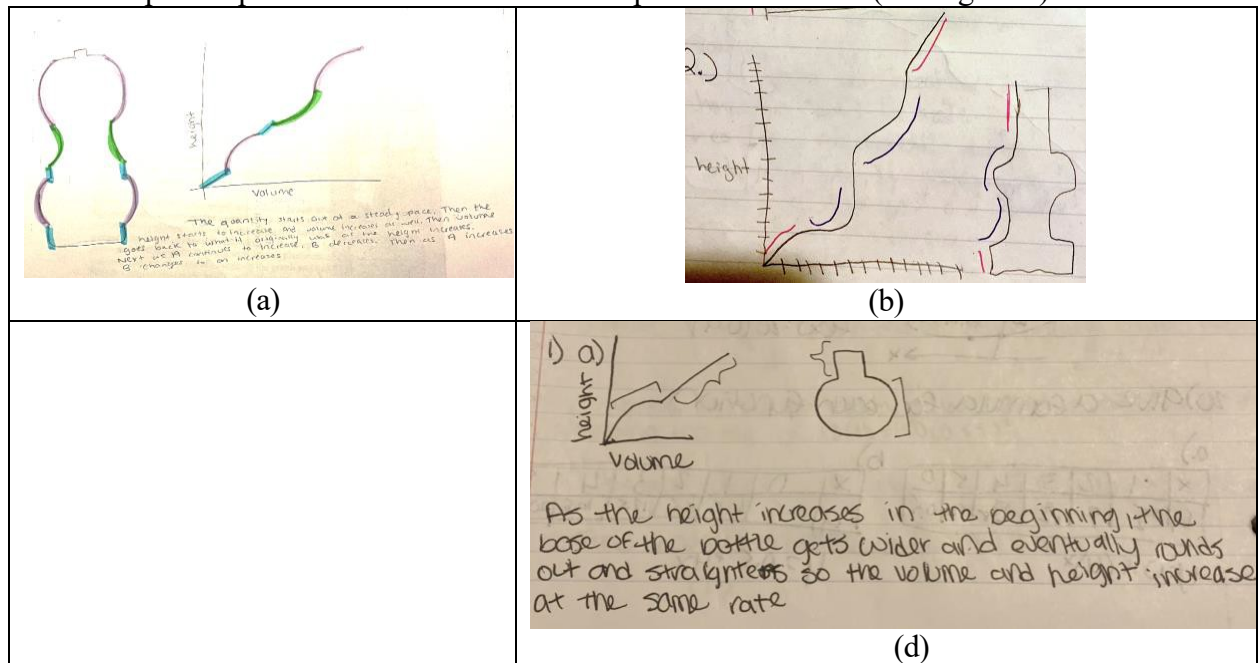


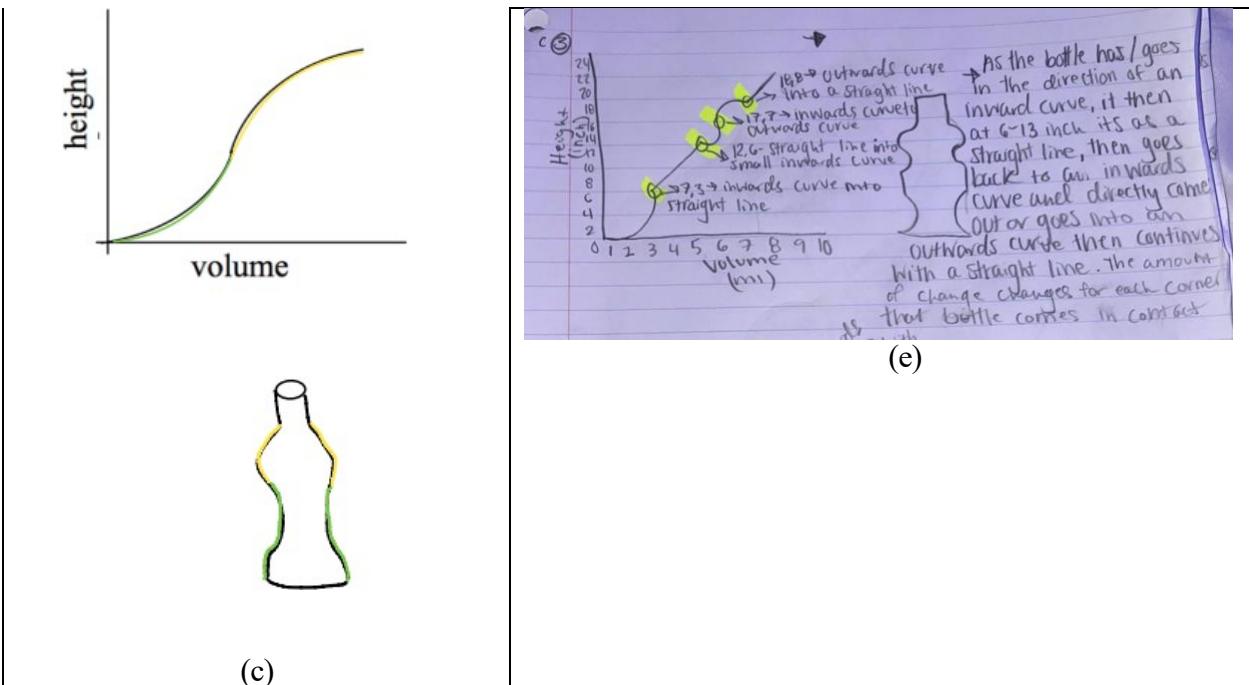
**Figure 4: (Left) Leonhard’s Bottle and Graph with (Right) Additional Highlights Added by Author to Represent the Three Sections of the Graph Referenced by Leonhard**

These students all exhibited reasoning which involved an imagery of pouring in water that filled the glasses “slowly” or “quickly”. Also, although volume was the quantity they were referencing as filling “quickly”, a *higher* volume to height rate with the provided axes would imply a *less* steep slope based on their description. Thus, all these students associated speed with the steepness of a slope (without attention to the axes labels)-faster speed results in a steeper slope. Moreover, the “steeper” or “less steep” regions were referencing *regions* of the bottle (not *points*), and thus occurring over intervals in the corresponding graph.

**Iconic Translations: Decreasing then Increasing**

The results in this section correspond to the result that Carlson et al. (2002) reported for Student B discussed in the introduction. Although the bottles are different, like Student B, these five students all used a single concave up or down curve to represent the height and volume relationship of a spherical and an inward curved portion of a bottle (see Figure 5).





**Figure 5: Bottles and Graphs (a) Louise, (b) Elizabeth, (c) Sandra, (d) Hertha, and (e) Alan.**

In all the bottles in Figure 5, students associate concave down and up curves with regions that normatively have changes from increasing to decreasing (or vice versa) rates of height with respect to volume. Below are some excerpts from the various students describing their work:

Louise (5a): [Referencing the green section on her bottle] The volume now starts to decrease, so rate of change quantity, everything, the volume is now decreasing as the height is increasing. [Referencing the purple section above the green section on her bottle] But then as the height increases, the volume then starts to increase and then increases.

Elizabeth (5b): And then for question 2, I did this demonstrated with the pink being the straight line [pointing at leftmost pink region on graph] because it goes straight out, and then when there's a rate of change in the curve [pointing at the left purple region on the graph], it is demonstrated by the purple [pointing to the bottom purple curve on the cross section]. And this side it changed again [pointing to the right purple region on the graph], so it's purple here [pointing to the top purple curve on the cross section]. And then straight up [pointing at the right pink region on the graph] so it's pink [pointing at the top pink region on the cross section]. And that is what that bottle looks like.

Sandra (5c): The graph, it doesn't have any straight lines in this, so I knew it was gonna be a curvier bottle. So the green is going inwards so you see that inwards bottle there, and then it goes outwards, making it up to the cap of the bottle.

Hertha (5d): So the bottle does round out and at the bottom and it's straight at the top and you can see that in the graph. So at the top, it's increasing at more of the same rate, and at the bottom, the volume is increasing significantly faster than the height.

From all these various pieces of work, we see evidence of iconic translations, static shape thinking associating a curve in the bottle with a similarly represented curve (albeit diagonally placed) on a graph. However, when analyzing their descriptions, as well as the description B gave from the Carlson et al., (2002) article, these students are referencing quantities varying. Specifically, they talk about height and/or volume increasing. Instead of the focus being on speed, the focus here is on the shape of the bottle as height (or volume) increases. There are references to the bottle getting “wider” or have an “inwards curve” vs. an “outwards curve” as that quantity varies. Thus, the iconic translations the students are making are occurring in sections that they consider as one quantity is varying. For covariational reasoning to occur, two quantities need to be related, but here instead the students associate features (not quantities) as one quantity increases. So referencing features such as having “more volume than height” or “volume increasing significantly faster than the height” are not quantitative comparisons of quantities, but rather descriptions of characteristics associated with particular shapes in a graph. What makes it more sophisticated than some other iconic translations described in the literature review (with purely figurative reasoning) is twofold: (i) all the graphs are monotonically increasing indicating some awareness of the varying (particularly the increase) in one quantity, and (ii) like the previous examples, a bulging out seems to always be represented by a concave down shape to be associated with a decreasing, in a way similar to the speed decreasing was getting represented by less steep slopes.

### **Discussion**

Altogether, as seen in the results, the discussed corpus of literature enabled the identification of nuances in students’ construction of a static object with apparent covariational reasoning (based on the Carlson et al. (2002) framework). Namely, rather than only characterizing the work of iconic translations or thematic associations, or simply as pseudo-analytical, the additional insights offered by the Thompson & Carlson (2015) frameworks enabled me to distinguish variational reasoning from covariational reasoning in ways that still attend to the students’ descriptions of variation in their justifications, distinguishing figurative reasoning from operative reasoning. These nuances brought to light connections students were making about how to represent differing speeds graphically (e.g., faster region implies steeper line segment, changing from faster to slower implies a concave down curve). The contribution of this study is noting that the students’ reasoning, although seemingly thematic, involves imagery of a changing speed occurring in a chunky matter—that is, with intervals of linear segments (thematic examples) or regions of steepness (iconic examples) whose steepness can be compared to one another to make claims about how changing quantities are related. The faster and slower language supported them to an extent, even the students making seemingly iconic translations based on shape of an object. I encourage continued research on how the development of chunky and smooth thinking might support student learning of various function types, especially during the transition from precalculus to calculus, when they will be introduced to tangent lines and instantaneous rates of change as resources to thinking about how quantities change together.

### **References**

Castillo-Garsow, C., Johnson, H. L., & Moore, K. C. (2013). Chunky and smooth images of change. *For the Learning of Mathematics*, 33(3), 31–37.

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

- Corbin, J. M., & Strauss, A. (2008). *Basics of qualitative research: Techniques and procedures for developing grounded theory* (3rd ed.). Thousand Oaks, CA: Sage.
- Ellis, A.B., Ozgur, Z., Kulow, T., Williams, C.C., & Amidon, J. (2013). An exponential growth learning trajectory. In Lindmeier, A. M. & Heinze, A. (Eds.), *Proceedings of the 37th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, 273–280. Kiel, Germany: PME.
- Ely, R., & Ellis, A.B. (2018). Scaling-continuous variation: A productive foundation for calculus reasoning. In A. Weinberg, C. Rasmussen, J. Rabin, M. Wawro, & S. Brown (Eds.), *Proceedings of the twenty-first Annual Conference on Research in Undergraduate Mathematics Education*, 1180–1188. San Diego, CA: San Diego State University.
- Frank, Kristin M. (2017). *Examining the development of students' covariational reasoning in the context of graphing*. Unpublished Ph.D. dissertation, School of Mathematical and Statistical Sciences, Arizona State University. (Marilyn Carlson, co-advisor).
- Johnson, H. L. (2022). Task design for graphs: Rethink multiple representations with variation theory. *Mathematical Thinking and Learning*, 24(2), 91–98. <https://doi.org/10.1080/10986065.2020.1824056>
- Liang, B., & Moore, K. C. (2020). Figurative and operative partitioning activity: Students' meanings for amounts of change in covarying quantities. *Mathematical Thinking and Learning*. <https://doi.org/10.1080/10986065.2020.1789930>
- Moore, K. C., Stevens, I. E., Paoletti, T., Hobson, N. L. F., & Liang, B. (2019). Pre-service teachers' figurative and operative graphing actions. *Journal of Mathematical Behavior*, 56, 100844. <https://doi.org/10.1016/j.jmathb.2019.01.008>
- Moore, K. C., & Thompson, P. W. (2015). Shape thinking and students' graphing activity. In T. Fukawa-Connelly, N. E. Infante, K. Keene & M. Zandieh (Eds.), *Proceedings of the 18th Meeting of the MAA Special Interest Group on Research in Undergraduate Mathematics Education*, pp. 782–789. Pittsburgh, PA: RUME.
- Paoletti, T. (2020) Reasoning about relationships between quantities to reorganize inverse function meanings: The case of Arya. *The Journal of Mathematical Behavior*, 57, 1–24.
- Piaget, J. (1970). *Genetic Epistemology*. New York: W. W. Norton & Company, Inc.
- Stevens, I. E., & Moore, K. C. (2016). The Ferris wheel and justifications of curvature. In M. B. Wood, E. E. Turner, M. Civil, & J. A. Eli (Eds.), *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 644–651). Tucson, AZ: The University of Arizona.
- Thompson, P. W. (1985). Experience, problem solving, and learning mathematics: Considerations in developing mathematics curricula. In E. A. Silver (Ed.), *Teaching and learning mathematical problem solving: Multiple research perspectives* (pp. 189–243). Hillsdale, NJ: Erlbaum.
- Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 421–456). Reston, VA: National Council of Teachers of Mathematics.
- Vinner, S. (1997). The pseudo-conceptual and the pseudo-analytical thought processes in mathematics learning. *Educational Studies in Mathematics*, 34, 97–129.

## LEARNING ASSISTANT- STUDENT INTERACTION IN CALCULUS: A CRITICAL DISCOURSE ANALYSIS

Rebecca Butler

University of New Hampshire  
rebecca.butler@unh.edu

Orly Buchbinder

University of New Hampshire  
orly.buchbinder@unh.edu

*Learning Assistants (LAs) are undergraduate peer-tutors who, having successfully passed a particular course, return to assist with teaching that course. Through their work across many STEM courses, LAs have been shown to have positive effects on several student outcomes, but little is known about why LAs' presence in classrooms is positively associated with these outcomes. This study provides a novel perspective on this issue by critically analyzing a portion of classroom dialog between an LA and a student in a Calculus I course. The language used by these interlocutors was analyzed with attention to the social and informational aspects of the dialog, examining both the relationship between the student and the LA, and the ways they frame mathematical content. These findings have implications for the future study of LAs' practice and bear relevance to the improvement of LA educational programs.*

Keywords: Calculus, Classroom Discourse, Undergraduate Education

Introductory courses in science, technology, engineering, and mathematics (STEM) at the university level in the United States have repeatedly been showed to pose barriers to students' continued participation in these fields (Wu et al., 2018). Subsequently, there have been many calls to reform these courses to increase the number and diversity of technically oriented college graduates who are prepared for careers in STEM fields (Reinholz et al., 2021; Talanquer, 2014). Some of the research-backed approaches to reform involve student-centered instruction emphasizing active learning, group problem-solving, and studio course designs (Freeman et al., 2014; Kim et al., 2013). One way to support facilitation of active-learning approaches has been to incorporate near-peer tutors into course design. Near-peer tutors are undergraduate students who have previously passed the course and who aid in course instruction. They have been implemented across disciplines in various ways, such as peer-supported learning outside or during class instruction (Adreanoff, 2016). While implementation models can vary greatly, they have been consistently shown to support active learning pedagogies and foster student success at the undergraduate level (Williams & Fowler, 2014; Wilson & Varma-Nelson, 2016).

One form of near-peer tutoring, which appears to be particularly effective in fostering aspects of active learning in undergraduate STEM courses is the Learning Assistant (LA) model (Knight et al., 2015). In this model, LAs synchronously aid in course instruction by facilitating small group work. In addition, outside classroom time, LAs are supported through enrollment in a pedagogy course and regular meetings with the instructional staff of the course. In the pedagogy course, LAs are introduced to concepts related to teaching and learning, often with attention to theories of learning and findings from cognitive science, and critically reflect upon their teaching in light of these theoretical concepts. In meetings with their instructional team, LAs review the mathematical content of the course in order to be able to support teaching of that content. This combination of content preparation, synchronous teaching practice, and pedagogical education distinguishes the LA model from other forms of near-peer tutoring (Otero et al., 2010).

Since its inception in 2003 at the University of Colorado - Boulder physics department, the LA model has spread across disciplines, institutions, and countries. Interest in the LA model



partly stems from a body of literature documenting positive outcomes for students in LA-supported courses, including a range of social, conceptual, and academic benefits (Barrasso & Spiliotis, 2021). Notably, Herrera et al. (2018) found that, across institutions, students in LA-supported physics courses scored higher on post assessment of conceptual understanding of introductory physics, compared to students in collaborative courses without LAs. This finding is particularly interesting as it suggests that the outcomes seen in LA-supported courses are not the mere consequence of task alterations or group collaboration but can be attributed to LAs.

While positive outcomes of the LA model are well documented, the mechanisms responsible for them are not well understood, and are likely to be multifaceted involving various social, contextual, and contentual factors (Talbot et al., 2016). Still, LA classroom practice is hypothesized to be central to this phenomenon, contributing to these positive outcomes through their deep and frequent interactions with students during class (Hernandez et al., 2021). Studies examining LA-student interactions are mainly focused on LAs' classroom actions (Thompson et al., 2020; Knight et al., 2015), and found that in addition to facilitating tasks, providing feedback, and increasing time spent in group discussions, LAs acted as mentors, offering students advice about the course. Yet, the finer details of LA-student interactions remain unexplored.

This study provides a new perspective on LA-student engagement by beginning to analyze the language LAs use in the classroom, through the lens of Critical Discourse Analysis (CDA). CDA pays particular attention to the ways in which language relates identity, information, and action within specific contexts (Gee, 2014a); thus, it captures both the social and contentual features of an LA's classroom language. Attending to these two features is critical for understanding LAs' classroom role, since LAs are both temporally closer to the learning process and relationally closer to students. This study offers a fine-grained analysis of the language used by one LA with a student in a university Calculus I course. Specifically, we ask:

5. What social identities and relationships are enacted and sustained by the LA and student through their language?
6. How do the LA and student use language to foreground and background particular mathematical content and concepts?

### **Theoretical Perspectives & Literature Review**

This focus on classroom language when exploring LA practice is motivated by Vygotsky's sociocultural perspective which conceptualizes learning as a social process mediated by individual's interaction with people and tools in social contexts. Learning occurs when an individual interacts with a "more knowledgeable other" (MKO), such as a teacher or a peer, around tasks and concepts that an individual may not be able to do on their own but is able to achieve with MKO's help. The language used by MKO to facilitate the learning process plays an essential role in an individual's learning, since "just as a mold gives shape to a substance, words can shape an activity into a structure" (Vygotsky, 1978, p.28).

In classrooms, LAs act as MKOs supporting student learning, and have a unique relationship with the students and the course's content. LAs are closer to students than course instructors and/or graduate teaching assistants (GTA). LAs share a peer status with the students in the course by virtue of being undergraduate students at the same institution and commonly participating in the student culture and subcultures of the institution. With respect to content, LAs have some mathematical authority since they have passed the course but are still relatively close to the experiences of first-time learners. As such LAs are less likely to suffer from the

expert blind spot and may attend to issues missed by other instructors. These relationships between LAs, students, and content create a unique power dynamic within the classroom; LAs are not so powerful as to be intimidating, but not so powerless as to be unreliable sources of aid (Robertson et al., 2014; Hernandez et al., 2021).

To capture the nuances of this power dynamic through scrutiny of the linguistic choices made by LAs and students participating in classroom dialogue, we utilized Gee's (2014a) Critical Discourse Analysis (CDA) framework. Gee views language as a tirade of "ways of saying (informing), doing (action), and being (identity)" (Gee, 2014a, p.8) integrated with each other through grammar. CDA is a form of discourse analysis, which ties deep functional linguistics analysis with critical social concepts such as power and identity (Lin, 2014).

The use of CDA is gaining traction in mathematics education. Recent studies used CDA to explore teachers' discourse around race and gender (McNeill et al., 2022) and teachers' linguistic moves which invite or discourage students to participate in classroom discourse (Wagner & Herbel-Eisenmann, 2008). With respect to studying near-peer tutoring, DiMaio (2020) used CDA to understand how near-peer tutors in university writing centers utilize politeness to navigate between their collaborative roles as peers and authoritative roles as teachers.

Considering the social functions of language and LA's unique social positioning in the classroom as near-peer tutors, CDA provides a useful lens for scrutinizing the language of LA-students interactions. Thus, CDA provides our study with a means to understand the relational dynamics between LAs and students in undergraduate mathematics classrooms.

## Methods

### Setting

This study is a part of a larger project aiming to improve teaching and learning in introductory STEM courses at a large public university in the northeast of the United States. One aspect of this initiative was introducing an LA program to the mathematics department to support Calculus I instruction. The LA program followed research-based best practices, with novice LAs enrolled in a pedagogy course (along with near-peer tutors of other STEM courses) in which they learned about general principles of teaching and learning. The LAs also attended a weekly meeting with a course instructor to discuss upcoming content.

The Calculus I course followed a lecture-recitation model, with students attending a large lecture (~ 160 students) led by faculty members three days a week and a smaller recitation (~ 20 students) led by GTAs and LAs twice a week. In these recitations, students worked in small groups on activities or problem sets (common for all recitations) related to the content of the lecture. Another aspect of the larger project was introducing six specially designed conceptually oriented activities for students to engage with during recitation sessions. The GTAs launched the activities, facilitated student group work, directed whole class discussions, and proctored quizzes. The LAs' primary role was to support students while they worked in small groups. Thus, the LAs had a less authoritative role than the GTAs, as they were not responsible for grading students, coordinating course logistics, or delivering content. Still, LAs had ample opportunities to form relationships with students and the GTA, since each LA was assigned to particular recitation sections for whole semester.

### Participants

This inquiry was conducted in Fall 2022; it focuses on a single LA's interaction with a small group of students in a Calculus I recitation section. The LA, Nia, identifies as a woman of color from southeast Asia; she is a biology major sophomore in her first semester as an LA. She took Calculus I in the previous semester and was recruited based on strong recommendations from her

calculus GTA and professor. The GTA that Nia worked with was a last-year mathematics Ph.D. with the six years of experience in a variety of courses in the mathematics department.

For data collection, we randomly selected one group from the recitation, who remained consistent throughout the semester. The group consisted of four freshmen STEM majors: three female and one male, all identified as white. The students seemed to communicate and collaborate well on activities. There was a tendency for the three female students to work together, while the male student worked independently. He appeared to prefer working at his own pace, often raising his hand for help from the GTA or LA, while the female students tended to consult one another. Despite the split along gendered lines, this dynamic did not appear tense, with the none of the group members dominating discussion.

### Data Collection

The group was recorded six times during the semester when the conceptually oriented activities were implemented. The recordings were done with a tabletop 360 video camera, which can simultaneously capture interactions among multiple interlocutors even in noisy classroom environments (Buchbinder et al., 2021). This paper is focused on the fine-grained analysis of one recitation session, that occurred around the middle of the semester. The class setting was typical, with students working on a worksheet in their small groups for most of the class, followed by a short class discussion of the answers. The activity worksheet on the topic of logarithmic differentiation (Figure 1) was designed by the project's research team.

**Directions:** Working with the peers in your group, solve the following problems. Make sure to show and justify all your work. Next, make sure everyone in the group understands the solution and participates. Finally, be prepared to report your answers to the whole class.

A logistic growth model of population growth reflects the fact that some populations cannot grow without bound, such as populations with limited resources like space or food.

The equation gives an example of a logistic growth model:

$$P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{-Mt}}, \quad M, P_0 > 0 \text{ with } M > P_0,$$

where  $t$  is measured in years and has domain  $[0, \infty)$ .

- To understand what the constant  $P_0$  represents, Evaluate  $P(t)$  at  $t = 0$ , in another words  $P(0)$ .
- To understand what the constant  $M$  represents, find  $\lim_{t \rightarrow \infty} P(t)$ . Why do you think we often refer to the constant  $M$  as the "carrying capacity" of the model?
- Compute  $P'(t)$
- What would  $P'(t)$  equal if  $P(0) = M$ ? Does this make sense given what the model represents?
- Write a story about a population this model might describe.

**Figure 1: Logarithmic Differentiation Activity**

This paper presents the fine-grained analysis of a three-minute video of the fifty-minute class session. The clip was selected because it was Nia's only sustained interaction with this group in the lesson. The interaction occurred toward the end of the class session. While Nia initially approached the whole group, she decided to work one-on-one with the male student - Ken. Nia and Ken's discussion focused on part (c) of the activity which calls for using logarithmic differentiation to find the derivative of the given logistic growth model. Nia and Ken's conversation about this problem was analyzed using CDA.

## Analytic Framework

According to Gee (2014a) CDA “is a reflexive, reciprocal process in which we shuttle back and forth between structure (form, design) of a piece of language and the situated meanings it is attempting to build about the world, identities, and relationships in a specific context” (p.148). It involves continued engagement with the data and the surrounding context (Mullet, 2018), with the research questions operationalized through examination of personal pronouns and specialist and vernacular term use. The process began with the first author creating an extended transcript of the selected video clip which included clarifications of referents (“this”, “here”, “what we did before”), gestures (pointing, writing instructions) and emotions (joking, frowning) extracted from the context of the 360-degree video. Constructing such transcript is a theoretically driven process informed by research goals (Ochs, 1979). In our case, the decision to shape the transcript around text accuracy, referents of speech, and clarity of social meaning reflect this study's focus on mathematical content, social relationships, and personal identities.

The transcript was then segmented into units of analysis: idea units and stanzas. Idea units (Gee, 2014a; Gee, 2014b) are complete or partial clauses identified by short breaks in speech. These breaks typically go unrecognized by the listener, but they are key indicators of when new information is introduced by the speaker. Analyzing the discourse at this fine-grained level is important for detecting small changes in what the interlocutors talk about and how. The idea units were then grouped into stanzas. A stanza is a sequence of idea units related to one specific topic, event or character (Gee, 2014b). This larger unit of analysis allows tracking the overall movement of the dialogue, identifying the major events and shifts in conversation. The 77 idea units identified in the transcript, were organized into eight stanzas averaging 14.25 idea units each (ranging in length from four to 18 idea units).

The segmented transcript was analyzed in two ways. To address the first question about the identities and relationships enacted and sustained throughout the dialog, the personal pronouns used by the speakers were examined. Personal pronouns are an important articulation of individuals’ representation, giving insight into how the speakers socially position themselves and each other in the classroom (Wagner, 2007). The language around these pronouns, especially the actions of these subjects, provides additional detail regarding the nature of their social positioning. To address the second research question about mathematical content and concepts, the analysis focused on mathematical terms used by the speakers, operationalized through Gee’s (2014a) notion of specialist and vernacular language. Specialist terms are more formal words often used by an expert in some area, while vernacular terms are ones used by a novice or layman. The words around these terms were also analyzed to illuminate their function in the dialog. It is important to note that the two lines of analysis - the personal pronouns and specialist and vernacular terms – are complementary, since the types of terms one uses reflect their social positioning and, reflexively, one’s social positioning affects their authority surrounding terminology use. Thus, both analyses were used to answer the two research questions.

## Results

### Identities and Relationship

Throughout the interaction, Ken’s identity was as an active author of mathematics, as evidenced by Ken and Nia’s personal pronoun use (Table 1). Nia consistently used the personal pronoun “you”, positioning Ken as the central agent in the mathematical process. This positioning was simultaneously enforced by Ken, who primarily uses the personal pronoun “I”, indicating that the mathematics belongs to him, and he is the one responsible for doing mathematics. Nia’s language further reinforces Ken’s identity as an active author in several

ways. She repeatedly created space for Ken to carry out mathematics and communicate his ideas via explicit direction (e.g., “Do it. Tell me what it is.”) and asking questions (e.g., "How would you..."), thus placing the responsibility for doing the mathematical work upon Ken. Nia also identified the products of Ken’s work as his own (e.g., “You have...”, “You did...”) rather than independently existing or communally owned entities (e.g., “It is...”, "We have...”).

**Table 1: LA and Student Personal Pronoun Use**

Personal Pronoun	Nia <i>Number of Occurrences</i>	Ken <i>Number of Occurrences</i>
I	1	7
Me	1	0
Us	4	0
We	3	1
You	14	0

Nia’s primarily identity in the interaction is that of mathematically authoritative MKO. This is evident in Nia’s responses to Ken’s work, which were often short, direct evaluations such as “correct” or “no”. Nia’s role as a MKO is illustrated in Table 2, as she demonstrates that she understands what Ken knows and what he is confused about. Nia provided Ken with a similar but more familiar to him example to work through. This bridging between what Ken knows and what he can do with assistance is precisely the role of an MKO (Vygotsky, 1978). Nia takes up her role as an MKO by taking an authoritative stance with respect to the mathematics involved in the conversation. This is seen in her evaluation of Ken’s work (Line 8) and in directing him to carry out mathematical processes (Line 6). This type of interaction, when Ken brings up his confusion and Nia responds with a similar example for Ken to work through, happened three times through this short videoclip, solidifying Nia’s role as an authoritative MKO.

**Table 2: Excerpt of LA-Student Dialog**

Line	Speaker	Text
1	Ken	But still it's like it's two parts here. [ <i>Ken referring to <math>\ln(P_0 + (M - P_0)e^{-Mt})</math> when taking the derivative in part (c) of the worksheet</i> ]
2	Nia	Yeah...OK so let's do...how would you differentiate the natural log of $x^2 + 3$ ?
3	Ken	Where?
4	Nia	This one, $x^2 + 3$ . [ <i>Nia has written an example <math>\ln(x^2 + 3)</math> on a blank sheet of paper</i> ]
5	Ken	This on bottom and derivative up top.
6	Nia	Do it. Tell me what it is.
7	Ken	$2x$ and one over $x^2 + 3$
8	Nia	Correct. Now you want to do the same thing with that. It’s literally the exact same thing. You have a constant, you have a coefficient multiplied by your function.

These identities of Ken as an active author of mathematics and of Nia as an authoritative MKO, someone who evaluates and guides what Ken is doing, illuminate the hierarchical element

in their relationship. Nia presents herself as the mathematical authority, which is acknowledged by Ken. This is evident in the excerpt in Table 2 when, in response to Ken’s concern, Nia introduced a related example for Ken to work through and was responsible for judging the validity of his work. Ken willingly follows Nia’s example and directions, whether or not he understands their relevance, trusting Nia to provide information that is productive to his learning. Nia’s authority with respect to mathematics is also evident in the number and the variety of specialist terms she used, especially compared to those used by Ken (Table 3). Moreover, when Ken used specialist terms, these terms were typically introduced by Nia first (e.g., Nia: “so you have to derive both sides of the equation” Ken: “Derive them, right... Yeah, yeah”).

Despite a notable hierarchy in Nia and Ken’s interactions, there are aspects of their relationship that are notably “peer”. This is evident in Nia’s initial approach to the group asking, “How are you guys doing?”. This informal and broad question mirrors the everyday language of college students addressing one another. Ken picks up this informality, responding with a joke about being stuck on a problem: “I just think I’m such a genius. But...” (pause). The peer nature of this relationship is also seen through Nia’s ability to understand the vernacular terms Ken uses (Table 3). Returning to the excerpt in Table 2, note that Nia has no trouble understanding that when Ken says, “two parts,” he refers to the two expressions which are added together within the natural log. This is not a trivial translation, as there were several other ways in which Ken could have been separating this expression into “two parts.” Nia’s ability to understand Ken’s language reflects her keen understanding of student difficulties when taking the derivative of natural logs. There is a strong sense of Nia’s mathematical authority within this interaction, but we also see her and Ken engage as peers in their mathematical and non-mathematical talk.

**Table 3: LA and Student Vocabulary through the Dialog**

	Specialist Terms	Vernacular Terms
Nia	Coefficient, constant, derive, differentiate, equation, function, minus, multiplied, natural log, numerator, product rule	Something over something else
Ken	Coefficient, constant, derive, derivative, function, product rule	On bottom, up top, two parts, over

### Mathematical Content

Nia and Ken primarily frame mathematical content through a binary classification of right or wrong. At the beginning of the interaction, Ken talked about his work as “completely like wrong.” In response, Nia began to go through the answers on his worksheet, pointing out “that is correct, that is correct, that is correct,” before arriving at the question Ken thought was wrong. Then, Nia began reviewing what Ken has done on the question so far, focusing on the “right steps” that he has completed. Both Nia and Ken used language to classify Ken’s productions as correct or incorrect. This theme is continually enacted in the interaction, with Nia frequently phrasing her feedback to Ken as “correct” or “no”, and Ken focused on obtaining “the right answer.” “Right and wrong” were foregrounded by both Nia and Ken as the primary features of concern within the process of doing mathematics rather than specific content or affective qualities of the work, such as effort.

In addition, throughout this interaction, Nia centered mathematics as a rule-based process. This is evident through Nia’s phrasing of mathematics in terms of verbs rather than nouns,

saying things like “you have to derive” instead of “you have to take the derivative”. Using verbs instead of nouns to frame specialist mathematical terms suggests that these are not static concepts but active processes that must be carried out by an agent.

Nia further framed these processes as sets of rules which can be applied across various contexts. At one point in the interaction, Nia referred to Ken’s previous work saying, “but you remember how we’ve had issues with this before.” By this, she identified a previously seen analogous structure to invoke processes related to solving the problem at hand. This is also seen in the excerpt in Table 2, Line 8, when Nia explained that the same rules maybe be used to solve the worksheet problem and the example she provided. The stance that Nia conveys toward mathematics as a rule-based process, is consistent with her positioning herself as an authoritative MKO in the interaction, meaning that she has the status to say what mathematics is and how it is done. However, it is unclear whether Ken embraces Nia’s stance toward mathematics, since during their interaction, he primarily asked questions rather than explaining his work to Nia.

### **Discussion**

The micro-level analysis of LA-student interactions offers important insights into the nature of LAs’ classroom practices. Through her language use, Nia enacts and sustains her identity as an authoritative MKO while simultaneously identifying Ken as an active author of mathematics. These identities are also sustained through Ken’s language. These identities imply a clear hierarchical relationship, with Nia positioned as the primary mathematical authority, but this relationship also has peer elements. Nia and Ken also use language to foreground mathematics as essentially right or wrong, with Nia further framing mathematics as a rule-based process.

While this analysis is focused on a single LA and student across just a three-minute episode, our results point to several educational implications. First, Nia is a clear example of how LAs can act as MKOs in the classroom. Nia was able to capitalize on her knowledge and recent experience as a learner in Calculus I course to quickly assess Ken’s knowledge, identify the origin of his difficulty, and provide appropriate examples to help him identify the analogy and make progress on his work. Though brief, this interaction exemplifies how LAs can effectively facilitate student engagement in class. Further, the results of this study can be useful for educators of LAs, who can use these results to facilitate discussions in LAs pedagogy courses around the issues of identity, power dynamics, and perspectives on mathematics as a discipline.

This study provides a fresh perspective on the classroom practices of LAs. Previous inquiries into LA practice have provided evidence that LAs can facilitate student engagement within the classroom through actions such as asking questions and providing feedback to students (Thompson et al., 2020; Knight et al., 2015). This study looks beyond categorization of LAs’ actions, explicating the nature of the relationships that these actions generate. Detailing the types of LAs’ relationships with students and how they frame course content, in the way utilized in the current study, is a vital step toward understanding the mechanisms behind the positive effects of LAs on student outcomes reported in the literature (Barrasso & Spilios, 2021). In addition, our study illustrates the utility of CDA for analyzing LA classroom practice, and the power of this analytic approach to elicit a wealth of information about the social and contentual dynamics between the student and the LA. Through our ongoing inquiry into small group discourse using CDA, we hope to provide further insights into LAs’ classroom roles and the ways in which they impact student learning process in Calculus classrooms.

## Acknowledgments

This research was supported by the National Science Foundation, Award No. 2013427. The opinions expressed herein are those of the authors and do not necessarily reflect the views of the National Science Foundation.

## References

- Andreanoff, J. (2016). The impact of a peer coaching programme on the academic performance of undergraduate students: a mixed methods study. *Journal of Learning Development in Higher Education. Special Edition: Academic Peer Learning*, 2(1), 1-27.
- Barrasso, A. P., & Spilios, K. E. (2021). A scoping review of literature assessing the impact of the learning assistant model. *International Journal of STEM Education*, 8(12), 1-18. <https://doi.org/10.1186/s40594-020-00267-8>
- Buchbinder, O., Brisard, S., Butler, R., McCrone, S. (2021). Preservice secondary mathematics teachers' reflective noticing from 360-degree video recordings of their own teaching. *Journal of Technology and Teacher Education*, 29(3), 279-308.
- DiMaio, L. (2020). *Negotiating difference: A critical discourse analysis of writing center interactions between peer tutors and multilingual tutees*. [Doctoral Dissertation, West Chester University]. Digital Commons at West Chester University.
- Freeman, S., Eddy, S. L., McDonough, M., Smith, M. K., Okoroafor, N., Jordt, H., & Wenderoth, M. P. (2014). Active learning increases student performance in science, engineering, and mathematics. *Proceedings of the national academy of sciences*, 111(23), 8410-8415. <https://doi.org/10.1073/pnas.1319030111>
- Gee, J. P. (2014a). *An introduction to discourse analysis: Theory and method*. Routledge.
- Gee, J. P. (2014b). *How to do discourse analysis: A toolkit*. Routledge.
- Hernandez, D., Jacomino, G., Swamy, U., Donis, K., & Eddy, S. L. (2021). Measuring supports from learning assistants that promote engagement in active learning: Evaluating a novel social support instrument. *International Journal of STEM Education*, 8(22), 1-17. <https://doi.org/10.1186/s40594-021-00286-z>
- Herrera, X., Nissen, J., & Van Dusen, B. (2018, August 1-2). *Student outcomes across collaborative-learning environments* [Paper presentation]. Physics Education Research Conference Proceedings, Washington D. C., United States. <https://doi.org/10.48550/arXiv.1808.07076>
- Kim, K., Sharma, P., Land, S. M., & Furlong, K. P. (2013). Effects of active learning on enhancing student critical thinking in an undergraduate general science course. *Innovative Higher Education*, 38(3), 223-235. <https://doi.org/10.1007/s10755-012-9236-x>
- Knight, J. K., Wise, S. B., Rentsch, J., & Furtak, E. M. (2015). Cues matter: Learning assistants influence introductory biology student interactions during clicker-question discussions. *CBE Life Sciences Education*, 14(4), 1-14. <https://doi.org/10.1187/cbe.15-04-0093>
- Le Roux, K. 2008. A critical discourse analysis of a real-world problem in mathematics: looking for signs of change. *Language and Education*, 22(5), 307-326. <https://doi.org/10.1080/09500780802152663>.
- Lin, A. (2014). Critical discourse analysis in applied linguistics: A methodological review. *Annual Review of Applied Linguistics*, 34, 213-232.
- McNeill, R. T., Leyva, L. A., & Marshall, B. (2022). "They're just students. There's no clear distinction": a critical discourse analysis of color-evasive, gender-neutral faculty discourses in undergraduate calculus instruction. *Journal of the Learning Sciences*, 31(4), 630-672. <https://doi.org/10.1080/10508406.2022.2073233>
- Mullet, D. R. (2018). A general critical discourse analysis framework for educational research. *Journal of Advanced Academics*, 29(2), 116-142. <https://doi.org/10.1177/1932202X18758260>
- Ochs, E. (1979). Transcription as theory. *Developmental pragmatics*, 10(1), 43-72.
- Otero, V., Pollock, S., & Finkelstein, N. (2010). A physics department's role in preparing physics teachers: The Colorado learning assistant model. *American Journal of Physics*, 78(11), 1218-1224. <https://doi.org/10.1119/1.3471291>
- Oughton, H. (2007). Constructing the 'ideal learner': A critical discourse analysis of the adult numeracy core curriculum. *Research in Post-Compulsory Education*, 12(2), 259-275. <https://doi.org/10.1080/13596740701387536>
- Pimm, D. (1991). Discourse analysis and mathematics education: An anniversary of sorts. In N. Ellerton, & K. Clements (Eds.), *Mathematics and language: A review of language factors in mathematics learning*. Geelong, Australia: Deakin University Press.
- Planas, N., Morgan, C., & Schütte, M. (2018). Mathematics education and language: Lessons and directions from two decades of research. *Developing research in mathematics education*, 196-210.
- Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.



- Reinholz, D. L., White, I., & Andrews, T. (2021). Change theory in STEM higher education: A systematic review. *International Journal of STEM Education*, 8(37), 1-22. <https://doi.org/10.1186/s40594-021-00291-2>
- Robertson, A. D., Eppard, E. P., Goodhew, L. M., Maaske, E. L., Sabo, H. C., Stewart, F. C., Wenzinger, S. T. (2014). Being a Seattle Pacific University Learning Assistant: A transformative experience of listening and being heard. In *American Physical Society Forum on Education Newsletter (Summer 2014)*.
- Rogers, R., Schaenen, I., Schott, C., O'Brien, K., Trigos-Carrillo, L., Starkey, K., & Chasteen, C. C. (2016). Critical discourse analysis in education: A review of the literature, 2004 to 2012. *Review of Educational Research*, 86(4), 1192-1226.
- Talbot, R. M., Doughty, L., Nasim, A., Hartley, L., Le, P., Kramer, L., Kornreich-Leshem, H., & Boyer, J. (2016). Theoretically framing a complex phenomenon: Student success in large enrollment active learning courses. *2016 Physics Education Research Conference Proceedings*. <https://doi.org/10.1119/perc.2016.pr.081>
- Talanquer, V. (2014). DBER and STEM education reform: Are we up to the challenge?. *Journal of Research in Science Teaching*, 51(6), 809-819. <https://doi.org/10.1002/tea.21162>
- Thompson, A. N., Talbot, R. M., Doughty, L., Huvad, H., Le, P., Hartley, L., & Boyer, J. (2020). Development and application of the action taxonomy for learning assistants (ATLAs). *International Journal of STEM Education*, 7(1), 1-14. <https://doi.org/10.1186/s40594-019-0200-5>
- Vygotsky, L. (1978). *Mind in society: The development of higher psychological processes*. (M. Cole, V. John-Steiner, S. Scribner, E. Souberman, Eds.). Harvard University Press.
- Wagner, D. (2007). Students' critical awareness of voice and agency in mathematics classroom discourse. *Mathematical thinking and learning*, 9(1), 31-50. <https://doi.org/10.1080/10986060709336604>
- Wagner, D., & Herbel-Eisenmann, B. (2008). "Just don't": The suppression and invitation of dialogue in the mathematics classroom. *Educational Studies in Mathematics*, 67(1), 143-157. <https://doi.org/10.1007/s10649-007-9097-x>
- Williams, B., & Fowler, J. (2014). Can near-peer teaching improve academic performance?. *International Journal of Higher Education*, 3(4), 142-149.
- Wilson, S. B., & Varma-Nelson, P. (2016). Small groups, significant impact: A review of peer-led team learning research with implications for STEM education researchers and faculty. *Journal of Chemical Education*, 93(10), 1686-1702. <https://doi.org/10.1021/acs.jchemed.5b00862>
- Wu, X., Deshler, J., & Fuller, E. (2018). The effects of different versions of a gateway STEM course on student attitudes and beliefs. *International Journal of STEM Education*, 5(44), 1-12. <https://doi.org/10.1186/s40594-018-0141-4>

# PERCEPTIONS OF LEARNING IN A CALCULUS COURSE INFUSED WITH MULTIMODAL WRITING

Aaron Trocki  
Elon University  
atrocki@elon.edu

*Student multimodal writing has gained attention in the teaching and learning of mathematics with the anticipation that these assignments will benefit student engagement and learning in a variety of ways. This study investigates students' perceptions of learning in a post-secondary calculus course, which contained four multimodal writing assignments. A course designed with multimodal writing is described along with student-generated products and feedback. Study results suggest that multimodal writing may assist students with understanding the complexities of calculus content and increase collaboration and engagement.*

Keywords: Calculus, Undergraduate Education, Instructional Activities and Practices

Numerous researchers have recommended engaging post-secondary students in writing-to-learn activities in various forms such as reflective writing (e.g. Ryan, 2011). In recent years, multimodal writing has garnered attention with student access to readily available digital and technological tools (e.g. Bezemer, & Kress, 2008). Shepherd (2018) explains that students often have extensive extracurricular experience with digital and multimodal writing but struggle to see its connection to academic writing. Multimodal writing represents an opportunity to build on students' extracurricular understandings and interests to promote learning within a particular subject. Multimodal writing is writing that integrates text, images, visuals, and sounds (e.g. short video reflections, or blog posts). The potential benefits of multimodal writing for student learning have been explored and often focus on 21<sup>st</sup> century literacy skills (e.g. Ball, Sheppard, & Arola, 2018). Some research has been documented on effects of multimodal writing in mathematics learning at precollegiate levels (e.g. Freeman, Higgins, Horney, 2016; Pytash, Kist, & Testa, 2017), but limited research has been documented in post-secondary mathematics.

In mathematics education, a growing body of research has prioritized the challenge of engaging all learners (e.g. Schoenfeld, 2016) at the post-secondary level. Braun (2014) explored student engagement and learning through writing and identified four types of mathematics writing: personal, expository, critical, and creative. Multimodal writing can take on any of these types but differs in its considerations of varying audiences, expectations, and formats (e.g. digital video). The goal of this study is to investigate how the inclusion of multimodal writing may affect students' perceptions of learning in a post-secondary calculus course. Study components addressed the following research questions:

- 1) How may post-secondary students' perceptions of learning mathematics change in a calculus course infused with multimodal writing assignments?
- 2) What are post-secondary students' perceptions of the usefulness of multimodal writing assignments in a calculus course?

## Theoretical Framing

Two constructs grounded study design to address the research questions. Cobb's complementary perspective (1994) and Anderson and Krathwohl's (2001) revised version of Bloom's Taxonomy. Analyzing students' perceptions of learning was viewed through the

complementary perspective of constructivist and sociocultural theories of learning as conceptualized by Cobb (1994). Cobb accounts for both what students learn and how they interact and communicate (often through writing) to do so. A revised version of Bloom's Taxonomy also provided a lens for understanding student learning as increases in knowledge (e.g. calculus concepts) and engagements in cognitive processes (e.g. analyze). Anderson and Krathwohl's (2001) taxonomy includes four major knowledge dimensions: factual, conceptual, procedural, and metacognitive. It also contains six major categories of cognitive processes: remember; understand; apply; analyze; evaluate; and create (Anderson & Krathwohl, 2001). The processes are hierarchal in complexity with create representing the highest complexity. For the purposes of this study, their six cognitive processes with definitions were included directly into data collection instruments for student consideration. Definitions of these processes are appropriately provided in the results section of this paper.

## Methods

### Context and Participants

This study took place in one section of an applied calculus course at a midsize private university. The instructor had taught this course previously three times before revising to include multimodal writing assignments. The course was designed with foci of disciplinary content and student multimodal writing. The instructor aimed to engage all students while attending to the intense learning demands of a post-secondary calculus course. To help with the challenge of meeting these learning demands, students read the first seven chapters of McGuire's (2018) book, *Teach Yourself How to Learn: Strategies You Can Use to Ace Any Course at Any Level*. The book includes strategies focused on metacognition, mindset, and emotions. Over the first ten weeks of the semester, students wrote a short reading reflection on each assigned chapter. The reflection for the chapter on emotions was adapted into a multimodal writing assignment.

A particular demand in this applied calculus course was students responding to many contextual calculus problems. Such problems regularly require solution strategies that are not prescribed in the problem. For instance, one exam problem in this course is as follows.

A restaurant chain is considering adding a vegetarian dinner to the menu. The marketing manager collects data to determine the following equations.

$$\text{Revenue: } R(x) = 0.075x^2 + 41.74x \text{ dollars when } x \text{ dinners are sold}$$

$$\text{Cost: } C(x) = 3993.37 + 13585 \ln(x) \text{ dollars for } x \text{ dinners}$$

How many dinners does the restaurant need to sell to maximize profit?

Solution strategies for this question can vary and may include writing a profit equation. Questions of this nature were typical on tests and exams in this applied calculus course and were challenging for many students. Furthermore, thinking required to solve such problems would likely be characterized in higher cognitive processes (e.g. evaluating) as found on the revised Bloom's Taxonomy (Anderson & Krathwohl, 2001).

Twenty-one students completed this course and represented a variety of majors along with varying levels of preparedness to engage with calculus concepts. Of the twenty-one students, seventeen agreed to participate in the study at the beginning of the semester. Twelve students were in the first year and five were in their second year of college.

Four multimodal writing assignments were created by the calculus instructor in consultation with a writing professor. These assignments were completed by all students in the course and are summarized in Table 1.

**Table 1: Multimodal Writing Assignments**

<b>Assignment Name</b>	<b>Description</b>
1) Write Letter Home (individual)	Students explained scope and purpose of course, their approaches to learning, and how they planned for success.
2) Produce a Podcast Episode (groups of two or three)	Students wrote a script to explain two major concepts to audience of future applied calculus students. Podcast had interview format and encouraged students to have fun and incorporate humor. Podcast recordings were submitted.
3) Create Educational Materials (groups of two or three)	Students wrote full explanations of how to approach and carry out all steps in answering a multipart difficult calculus word problem about absolute maximums and minimums. Audience was future applied calculus students in need of tutoring.
4) Write an Email (individual)	Students read the chapter, “How Your Emotions Affect Your Motivation and Learning” from the book, <i>Teach Yourself How to Learn: Strategies You Can Use to Ace Any Course at Any Level</i> (McGuire, 2018). Students wrote a reading reflection in the form of an email to a classmate who forgot to read the assigned chapter. Informal writing style typical of an email was encouraged.

Students wrote for a real audience in the sense that they were aware that the instructor planned to use their multimodal writing products with future applied calculus students. Students were also encouraged to share their products with family members in the first assignment and with classmates in the other three assignments for studying purposes. For example, groups of students exchanged products for assignment three to help prepare for an exam.

#### **Data Collection and Analysis**

Data was collected in three ways: pre-course questionnaire, post-course questionnaire, and student multimodal writing submissions. Seventeen students completed the pre-course questionnaire. Ten of these students completed the post-course questionnaire. Questionnaire data was collected anonymously. The lower number of post-course questionnaire responses was due to student attrition and a group of student athletes traveling during the last week of class. Questionnaires were parallel in form and were written based on the study’s two research questions. For example, consider the parallel structure of a pre-course questionnaire prompt and a post-course questionnaire prompt in Table 2.

**Table 2: Sample Parallel Questionnaire Prompts**

Pre-Course Question Sample	Think about what was required of you to demonstrate on tests/exams <b>in previous math classes</b> . Rank the following requirement from 1-never to 5-very often. Creating: Putting elements together to form a coherent or functional whole;
----------------------------	---

	reorganizing elements into a new pattern or structure through generating, planning, or producing.
Post-Course Question Sample	Think about what was required of you to demonstrate on tests/exams <b>in this class</b> . Rank the following requirement from 1-never to 5-very often. Creating: Putting elements together to form a coherent or functional whole; reorganizing elements into a new pattern or structure through generating, planning, or producing.

Like the example in Table 2, the other questions on both questionnaires were parallel in structure and contained ranked responses. One exception was the following additional question prompt on the post-course questionnaire, “*How was this class different than your conception of 'typical' math classes? Explain.*” To analyze student responses to this prompt, each sentence for each student response was categorized under unfavorable conception or favorable conception. Selected sentences are provided in the results section.

Each questionnaire contained a section on students’ perceptions on the six major categories of cognitive processes (Anderson & Krathwohl, 2001) and a section on writing in mathematics classes. Quantitative analysis included tallying response frequencies for each questionnaire prompt that contained ranked responses. For example, response frequency results for a particular question could take the form, 1: 5.9%, 2: 11.8%, 3: 41.2%, 4: 35.3%, and 5: 5.9%. The rankings of all ranked-response questions contained the language, “*Rank the following requirement from 1-never to 5-very often*” with the exception of one ranked-response question that contained more descriptive language for the ranking. This language is shared in the results section.

The multimodal writing products students submitted were also treated as data and analyzed. These products were analyzed for themes related to student learning of calculus concepts and their perceptions of learning. This thematic analysis followed Creswell’s (2013) recommendations to manage data, add memos on the artifacts, and classify the memos under larger themes. Themes were documented for each multimodal writing assignment submission and then noted for their frequency of occurrence across student submissions for each assignment.

### Results

Results from the thematic analysis of student multimodal writing submissions are documented in Table 3. Themes related to both learning calculus content and perceptions of learning. For each assignment, the two or three themes with the highest occurrence frequencies are provided.

**Table 3: Learning Themes for Multimodal Assignment Submissions**

Assignment Name	Learning Theme
1) Write Letter Home (individual)	<ul style="list-style-type: none"> <li>• Focus on calculus as a study of change</li> <li>• Appreciation/surprise regarding use of technology to replace tedious math work by hand</li> <li>• Recognition that this course is different in activities and expectations from other math courses taken</li> </ul>
2) Produce a Podcast Episode (groups of two)	<ul style="list-style-type: none"> <li>• Focus on concept of derivative and its applications</li> <li>• Incorporate podcast theme of how to be successful in applied</li> </ul>

or three)	calculus
3) Create Educational Materials (groups of two or three)	<ul style="list-style-type: none"> <li>• Explanation of difference between absolute and relative extrema</li> <li>• Mention metacognitive strategies from book, <i>Teach Yourself How to Learn: Strategies You Can Use to Ace Any Course at Any Level</i> (McGuire, 2018)</li> <li>• Focus on checking reasonableness of answers</li> </ul>
4) Write an Email (individual)	<ul style="list-style-type: none"> <li>• Connect reading contents to learning current course topics</li> <li>• Encouraged application of reading contents to test preparation (e.g. deeply understanding underlying concepts)</li> </ul>

Themes related to both what and how students learn. Themes provide evidence that multimodal writing assignments promoted knowledge gains of calculus concepts and productive approaches to learning.

Quantitative analysis of results from the pre-course and post-course questionnaires demonstrated effects to students' perceptions of learning. Results for cognitive process questions are summarized in Table 4, which also includes the questions used in the questionnaires.

**Table 4: Comparisons of Cognitive Processes Ranked-Response Questions**

Question	Percentage Frequency for Rankings
<b>Pre-Course 1:</b> Think about what was required of you to demonstrate on tests/exams <b>in previous math classes</b> . Rank the following requirement from 1-never to 5-very often. Remembering: retrieving, recognizing, and recalling relevant knowledge from long-term memory.	1: 0% 2: 5.9% 3: 11.8% 4: 52.9% 5: 29.4%
<b>Post-Course 1:</b> Think about what was required of you to demonstrate on tests/exams <b>in this class</b> . Rank the following requirement from 1-never to 5-very often. Remembering: retrieving, recognizing, and recalling relevant knowledge from long-term memory.	1: 0% 2: 0% 3: 20% 4: 70% 5: 10%
<b>Pre-Course 2:</b> Think about what was required of you to demonstrate on tests/exams <b>in previous math classes</b> . Rank the following requirement from 1-never to 5-very often. Understanding: constructing meaning from oral, written, and graphic messages through interpreting, exemplifying, classifying, summarizing, inferring, comparing, and explaining.	1: 0% 2: 5.9% 3: 41.2% 4: 35.3% 5: 17.6%
<b>Post-Course 2:</b> Think about what was required of you to demonstrate on tests/exams <b>in this class</b> . Rank the following requirement from 1-never to 5-very often. Understanding: constructing meaning from oral, written, and graphic messages through interpreting, exemplifying, classifying, summarizing, inferring, comparing, and explaining.	1: 0% 2: 0% 3: 10% 4: 50% 5: 40%
<b>Pre-Course 3:</b> Think about what was required of you to demonstrate on tests/exams <b>in previous math classes</b> . Rank the following requirement from 1-never to 5-very often. Applying: Carrying out or using a procedure	1: 0% 2: 0% 3: 11.8%

through executing, or implementing.	4: 47.1% 5: 41.2%
<b>Post-Course 3:</b> Think about what was required of you to demonstrate on tests/exams <b>in this class</b> . Rank the following requirement from 1-never to 5-very often. Applying: Carrying out or using a procedure through executing, or implementing.	1: 0% 2: 0% 3: 0% 4: 30% 5: 70%
<b>Pre-Course 4:</b> Think about what was required of you to demonstrate on tests/exams <b>in previous math classes</b> . Rank the following requirement from 1-never to 5-very often. Analyzing: Breaking material into constituent parts; determining how the parts relate to one another and to an overall structure.	1: 5.9% 2: 11.8% 3: 41.2% 4: 35.3% 5: 5.9%
<b>Post-Course 4:</b> Think about what was required of you to demonstrate on tests/exams <b>in this class</b> . Rank the following requirement from 1-never to 5-very often. Analyzing: Breaking material into constituent parts; determining how the parts relate to one another and to an overall structure.	1: 0% 2: 0% 3: 10% 4: 60% 5: 30%
<b>Pre-Course 5:</b> Think about what was required of you to demonstrate on tests/exams <b>in previous math classes</b> . Rank the following requirement from 1-never to 5-very often. Evaluating: Making judgments based on criteria and standards through checking and critiquing.	1: 5.9% 2: 17.6% 3: 29.4% 4: 41.2% 5: 5.9%
<b>Post-Course 5:</b> Think about what was required of you to demonstrate on tests/exams <b>in this class</b> . Rank the following requirement from 1-never to 5-very often. Evaluating: Making judgments based on criteria and standards through checking and critiquing.	1: 0% 2: 11.1% 3: 11.1% 4: 44.4% 5: 33.3%
<b>Pre-Course 6:</b> Think about what was required of you to demonstrate on tests/exams <b>in previous math classes</b> . Rank the following requirement from 1-never to 5-very often. Creating: Putting elements together to form a coherent or functional whole; reorganizing elements into a new pattern or structure through generating, planning, or producing.	1: 0% 2: 18.8% 3: 62.5% 4: 18.8% 5: 0%
<b>Post-Course 6:</b> Think about what was required of you to demonstrate on tests/exams <b>in this class</b> . Rank the following requirement from 1-never to 5-very often. Creating: Putting elements together to form a coherent or functional whole; reorganizing elements into a new pattern or structure through generating, planning, or producing.	1: 0% 2: 0% 3: 30% 4: 40% 5: 30%

Regarding cognitive processes, comparisons of pre-course frequencies to post-course frequencies show a general trend of higher post-course frequencies in categories closer to “very often” or in the “very often” category. This finding also stood out for the top three cognitive processes of analyzing, evaluating, and creating. This trend gives evidence of students’ perception of greater demands for higher cognitive processes as described by Anderson & Krathwohl (2001) on assessments in this calculus course compared to previously completed mathematics courses.

Students also responded to ranked-response questions regarding writing in mathematics with quantitative findings summarized in Table 5.

**Table 5: Comparison of Writing Usefulness Ranked-Response Questions**

Question	Ranking Descriptions and Percentage Frequencies
<p><b>Pre-Course Question:</b> Think about the writing you have completed <b>in your prior experiences learning math</b>. Use the following scale to rank how helpful this writing was for your learning and understanding of math.</p>	<ul style="list-style-type: none"> <li>• Not helpful for my learning and understanding of math: 25%</li> <li>• Somewhat helpful for my learning and understanding of math: 62.5%</li> <li>• Helpful for my learning and understanding of math: 25%</li> <li>• Significantly helpful for my learning and understanding of math: 0%</li> </ul>
<p><b>Post-Course Question:</b> Think about the writing you have completed <b>in this class</b>. Use the following scale to rank how helpful this writing was for your learning and understanding of math.</p>	<ul style="list-style-type: none"> <li>• Not helpful for my learning and understanding of math: 10%</li> <li>• Somewhat helpful for my learning and understanding of math: 30%</li> <li>• Helpful for my learning and understanding of math: 30%</li> <li>• Significantly helpful for my learning and understanding of math: 30%</li> </ul>

These parallel pre-course and post-course questions captured students' perceptions of the helpfulness of writing for their learning and understanding of mathematics. Ranked-response frequencies demonstrate a strong trend in students' perceptions towards higher levels of helpfulness in writing completed in the applied calculus course in this study compared to previous mathematics courses.

Finally, students responded to a post-course question about the calculus class in this study. Responses fell into one of two general categories of conception: unfavorable or favorable. Selected excerpts under each category have been revised for grammar and are summarized in Table 6.

**Table 6: Students' Perceptions of the Applied Calculus Course**

<p><b>Post-Course Final Question: How was this class different than your conception of 'typical' math classes? Explain.</b></p>
<p><b>Category 1: Unfavorable</b></p> <ul style="list-style-type: none"> <li>• More graphs than I expected, and I thought there would be more equations.</li> <li>• It wasn't very different than any other math class I've taken other than it was a weird teaching style where it wasn't a lot of doing examples but more just definitions. I didn't love it with this teaching style.</li> <li>• You really need to work as a group to understand the material. I could not do it by myself like I could with Geometry and Precalculus.</li> </ul>



### **Category 2: Favorable**

- This class challenged me but also helped me gain new study skills.
- We learned about real life situations, such as maximizing profit when talking about a company.
- In this math class we have more writing assignments that forces students to actually understand what they are talking about.
- There were less practice problems and more class discussions.
- The way that this class is different from my prior "typical" math classes is that when learning the material, it is more explanation based rather than practicing examples.

In addition to addressing what calculus content they were to learn (e.g. maximizing profit), student responses often addressed how writing and discussing influenced how they learned. This outcome aligns with Cobb's (1994) complementary view of learning. Favorable responses outweighed unfavorable responses by an approximate ratio of three to one. This ratio along with details included in response excerpts give some evidence that students perceived this applied calculus course with its multimodal writing as beneficial to their learning.

### **Discussion**

Study results suggest that the applied calculus course, which contained four multimodal writing assignments, affected students' perceptions of learning mathematics in a number of ways. Students recognized a greater emphasis on providing explanations compared to previous learning experiences in mathematics. In general, students also responded favorably to the multimodal writing they experienced as evidenced by increases in their perceptions of the usefulness of writing for learning mathematics. Finally, perceptions of what is required for students to demonstrate on assessments trended towards higher levels of cognitive processes as described by Anderson and Krathwohl (2001) at the end of the course.

These promising results are limited as they are based on a small sample within one applied calculus class. Further, other variables such as style of instruction and readings from McGuire's (2018) book influenced students' perceptions of learning as well. However, this study provides some evidence of the potential for multimodal writing assignments to positively affect student perceptions of learning post-secondary mathematics.

Instructors of applied calculus and other post-secondary mathematics are encouraged to consider experimenting with multimodal writing assignments with their students. Doing so may help students transfer extracurricular knowledge as recommended by Shepherd (2018) and communicate about their learning of mathematics. Study results have an additional implication of prompting further research that investigates the potential benefits of multimodal writing for engaging students in learning post-secondary mathematics.

### **References**

- Anderson, L., & Krathwohl, D. (2001). *A Taxonomy for Learning, Teaching and Assessing: A Revision of Bloom's Taxonomy of Educational Objectives*. New York: Longman.
- Ball, C., Sheppard, J., & Arola, K. (2018). *Writer/designer: A guide to making multimodal projects* (2nd ed.). Boston, MA: Bedford/St. Martin's Press.
- Bezemer, J., & Kress, G. (2008). Writing in multimodal texts: A social semiotic account of designs for learning. *Written Communication*, 25(2), 166-195. Retrieved from <https://doi.org/10.1177/0741088307313177>
- Braun, B. (2014). Personal, Expository, Critical, and Creative: Using Writing in Mathematics Courses. *PRIMUS*, 24, 447-464.
- Cobb, P. (1994). Where is the mind? Constructivist and sociocultural perspectives on mathematical development. *Educational Researcher*, 23(7), 13-20.

- Creswell, J. (2013). *Qualitative Inquiry and Research Design: Choosing Among Five Approaches*. (3rd ed.) Thousand Oaks, CA: SAGE Publications, Inc.
- Freeman, B., Higgins, K., & Horney, M. (2016). How students communicate mathematical ideas: An examination of multimodal writing using digital technologies. *Contemporary Educational Technology*, 7(4), 281-313. Retrieved from <https://dergipark.org.tr/en/pub/cet/issue/29516/316668>
- McGuire, S. (2018). *Teach Yourself How to Learn: Strategies You Can Use to Ace Any Course at Any Level*. Stylus Publishing, LLC.
- Ryan, M. (2011). Improving reflective writing in higher education: a social semiotic perspective. *Teaching in Higher Education*, 16(1), 99-111. DOI: [10.1080/13562517.2010.507311](https://doi.org/10.1080/13562517.2010.507311)
- Pytash, K., Kist, W., & Testa, E. (2017). Remixing my life: The multimodal literacy memoir assignment and STEM. *Journal of Adolescent & Adult Literacy*, 61(2), 163-172. <https://doi.org/10.1002/jaal.633>
- Schoenfeld, A., (2016). What makes for powerful classrooms - and what can we do, now that we know?. *MAA Focus*, 36(2), 11-14.
- Shepherd, R. (2018). Digital writing, multimodality, and learning transfer: Crafting connections between composition and online composing. *Computers and Composition* 48, 103-114.

## PROOF WITHOUT CLAIM: A NOVEL TOOL FOR EXPLORING STUDENTS' CONCEPTIONS OF PROOFS

Alison Mirin  
The University of Arizona  
amirin@math.arizona.edu

Dov Zazkis  
Arizona State University  
dzazkis@asu.edu

Andre Rouhani  
Arizona State University  
arouhani@asu.edu

*In order to learn more about student understanding of the structure of proofs, we generated a novel genre of tasks called “Proof Without Claim” (PWC). Our work can be viewed as an extension of Selden and Selden’s (1995) construct of “proof framework”; while Selden and Selden discuss how the structure of a proof can be discerned by the claim it proves, we leverage the consideration that one can discern a claim by the structure of its proof. In these PWC tasks, the student is presented with a proof with any explicit mention of its claim being proven removed and is instructed to discern the proof’s claim. We introduce the construct of Personal Proof Framework (PPF) to describe the varied ways in which students relate the claim being proven to the content of the proof.*

Keywords: Reasoning and proof, Undergraduate education.

Understanding how to comprehend and write proofs is an essential skill for transitioning to advanced mathematics (Stylianides et al., 2017). One tool aimed at helping students transition to being effective proof writers is based on Selden and Selden’s (1995) construct of *proof frameworks*. This construct accounts for the fact that the logical structure of a theorem (a claim) can influence the structure of its proof. For example, if we are proving an implication of the form “for all  $x$ ,  $P(x)$  implies  $Q(x)$ ”, we might start our proof with the assumption that  $P(x)$  is true for an arbitrary  $x$  in the universe of discourse, and end with the conclusion that  $Q(x)$  is true. Similarly, we might begin writing a proof by contradiction of a statement  $X$  by starting with something akin to “Toward a contradiction, assume not- $X$ ”. As discussed by Selden et al. (2018), this structure provides a valuable way for students to start their proofs via outlining. Some authors have indeed observed some improvements in student proving when these proof frameworks are explicitly taught (Gaber et al., in press).

Selden and Selden (2015) describe distinctions between “proof validation”, “proof comprehension”, and “proof construction”. Proof validation is the process by which one reads a proof and reaffirms its soundness, proof comprehension is the process by which one “understands” (p.10) a proof, and proof construction is the process of writing a proof. Selden and Selden (ibid.) call for more research on the interrelationships between these aspects of proof. Our study looks at the relationships between students’ proof comprehension, proof construction, and proof validation. Specifically, we address the questions: how do students interact with and conceptualize common proof frameworks? How do students relate the claim being proven to the content of a valid proof?

We do this by leveraging Selden and Selden’s (1995) construct of *proof frameworks* to study how students conceptualize the connections between a claim being proven and its proof. While Selden and Selden (2018) explore how students can start with a claim and use its logical structure to outline a proof that coheres with that structure, our study does the reverse. Specifically, we examine how students discern the claim being proven when given only the content of a proof. For example, if a proof begins with “toward a contradiction, suppose not  $X$ ” and ends with “therefore, we have found a contradiction”, we can discern that the claim being

proven is “X”. In other words, the logical structure of the claim appears in *both* the claim and in the body of the proof.

In order to facilitate our investigation, we generated a novel genre of task we call Proof Without Claim (PWC). These tasks begin with a correct proof of some claim. Guiding sentences which sometimes bookend a proof and allude to what is being proven by stating “we will prove that X” and “thus we have shown that X” are removed. Students are given the remainder of the proof and tasked with figuring out and then writing down what claim the original proof proved. Observing what students pay attention to when figuring out these PWC tasks thus provides a lens into their conceptions of how proofs prove, and particularly how a proof is related to the claim it is proving. Instead of prescribing what connections students should ideally see between a claim and its proof, we study what connections students do see. This necessitates a shift from thinking about proof frameworks in the normative sense to what we call Personal Proof Frameworks (PPFs). By investigating what students write down as the proof’s claim, we are investigating an aspect of *proof construction*. By investigating how students read the provided valid proof and relate it to the claim, we are investigating an aspect of students’ *proof comprehension* and *proof validation*. Hence, our work can be viewed as looking at the relationships between proof comprehension, validation, and construction (Selden & Selden, 2015).

### Literature

One important aspect of student conceptions of proving is the role equations and computations play within proofs. Several studies have identified differences between expert and novice provers’ reading behaviors that suggest that experts and novices might understand the role of equations and computations differently. This research includes eye-tracking studies that revealed differences in where readers’ attention is focused; experts tended to spend less time looking at equations and calculations than novice readers (Ingles & Alcock, 2012). While experts tended to read nonlinearly and attend to the structure of a proof, students tended to read linearly (*ibid.*). Read aloud studies have revealed that novices tend to sound out equations by vocalizing the individual symbols, while experts tend to verbalize the purpose or meaning of the equations/calculation (Shepherd & van de Sande, 2014).

Recent work has posited that some students’ prior experiences with calculation-centered mathematics have significant influence on their interpretations of the purpose of equations within proofs. For example, in a number theory context, an equation might serve no calculational purpose and instead be used to infer divisibility properties of variables in that equation. Dawkins and Zazkis (2021), conducted a study of students’ in-the-moment understanding of proofs. They displayed proofs line-by-line with the previously read lines visible but subsequent lines hidden. This allowed the researchers to observe that some students assumed equations in a proof would be used for future calculation, despite the non-calculational purpose regarding divisibility being explicitly stated and accomplished within the proof. This result suggests that these students conceptualize equations within proofs as being limited to a purely calculational role.

Other research gives insight into what students might focus on when reading proofs and when identifying a proof’s central argument. For example, in Zazkis and Villanueva (2016), students often did not connect an informal argument of a result to a more formalized version of that same argument (the proof). Specifically, students were given an informal argument for a theorem and two correct formal proofs of that theorem. Only one of these formal proofs mirrored the informal argument. However, students generally did not correctly identify which of the two proofs was a formalization of the argument. Students tended to focus on (what we consider to be) a superficial detail of the proof to make their assessments instead of its core argument. In our

study, we seek to better understand how students interpret another aspect of proof: the connection between the content of a proof and the claim that it proves.

### Framework and Methods

In this study we are in a sense doing the reverse of what Selden et al. (2018) did with their *proof frameworks* construct (described in more detail above). While they suggest that students start writing the structure of a proof by using the structure of its claim, we have students discern the claim from the content of the proof. So, we use their construct as a starting point for our work. The proof framework construct described in Selden and Selden (1995) can be conceptualized as a normative expert view on the connection between the claim and set up of a proof based on the proof technique being implemented. Since we are instead interested in student conceptions of the connections between a proof and the claim it proves, we propose an anti-deficit construct that we call *personal proof framework (PPF)*. This is a student's construal of the connection between the proof and the claim being proven, which is reflected in their approaches to PWC tasks.

A recruitment email was sent to all instructors of Anonymous State University's (ASU) introduction to proof course (David & Zazkis, 2020). The email requested that the instructors inform their students of the opportunity to take part in the study. All interested students who responded were interviewed – consequently, six students participated. All participants were compensated for their time in accordance with our Institutional Review Board (IRB) approval. Each interview consisted of an orientation task in which the student was shown a complete proof and asked to identify what the proof proves: i.e., what claim or theorem the proof validates. Further, the student was asked to identify the location(s) in the proof where the claim being proven was stated. The student was then shown a version of that same proof with the first and last line removed. These removed sentences stated what would be proven and what had just been proven, respectively. The student was then told that all remaining proofs in the interview would be of the second type with the removed lines, and that their task was to figure out what the proof was proving in order to fill in these missing sentences (PWC tasks). It was explained to the participant that all the proofs presented are error-free and do indeed successfully prove a claim. Interviews were video, screen, and audio recorded. The first two authors collaboratively analyzed the interviews in a way that is consistent with Simon's (2019) three layers of analysis. This involved iteratively documenting and interpreting interviewees' processes for determining what a proof in a PWC tasks was proving and comparing these individual processes to each other to identify similarities and differences. The collection of PWC tasks implemented a variety of proof techniques, including induction, contradiction, direct proof, and disproof by counterexample. However, due to space constraints, we will primarily focus on the task in which students were presented with a standard proof by contradiction that the square root of two is irrational.

### Results

We propose three categories of Personal Proof Frameworks (PPFs) as explanations for how students interact with PWC tasks. Although we describe these frameworks within the context of our PWC tasks, we hypothesize that these conceptions extend to how students view proofs more generally. Our hypothesis is somewhat confirmed by the fact that students responded similarly to the standard proof (without its claim removed) in the orientation task (discussed more below).

- 1) **Calculational Personal Proof Framework (Calculational PPF):** This category is reminiscent of results from the aforementioned reading studies wherein students primarily focus on the calculations within proofs, treating them as the primary aspect of a proof. In our study, this

framework specifically refers to a conception that proofs are calculations, and that what is being proven by a proof thus must be what is calculated by the equations and computations it contains. Two participants demonstrated Computational PPFs (Maria and Guillermo).

- 2) Cohesive Personal Proof Framework (Cohesive PPF): With this conception of proof, the proof technique used and the details of how the proof is proven are both integral to understanding what is being proven. Two students demonstrated Cohesive PPFs (Javon and June).
- 3) Claim-Setup Personal Proof Framework (Claim-Setup PPF): With this conception, the setup and conclusion of a proof are all that is needed to discern what is being proven. The details in the middle of a proof may be needed for the proof to be complete and correct, but they are not needed for determining the claim being proven. Two participants demonstrated Claim-Setup PPFs (Thomas and Manuel).

Although at face value it might appear that these categories are mutually exclusive, it is important to note that novice provers often have varying familiarity with different proof techniques and mathematical objects. So, it might be perfectly reasonable for a student to demonstrate one conception when working on some proof and then another when they are working within a different context or technique. Nonetheless, within the confines of the tasks in this study, participants each fit into one of the three categories.

### Computational Personal Proof Framework

Undergraduate mathematics students beginning to take proof-centered courses have typically had years of experience interacting with equations in calculational contexts. Thus, it would make sense for some students to focus on equations while overlooking what may be inferred from the sentences around them. Consider the following excerpt from Guillermo's interaction with Task A, which presented the student with the proof that  $\sqrt{2}$  is irrational.

Guillermo: So we are pretending that root 2 is rational.

Interviewer: What do you mean by pretending?

Guillermo: "Suppose to the contrary," for the sake of the proof we are pretending. Because it isn't normally rational. Since they are saying "suppose to the contrary" that means that in this world. Rad 2 is normally irrational

Interviewer: Square root of 2 is irrational.

Guillermo: Okay, I think this is saying that this is proving algebraically that if you have the quotient of two squared numbers that you can take the square root of both of those numbers. And if you have it set equal, you can take the square root of that particular number. I believe that's what it's trying to prove. We are pretending that root 2 is rational (highlights the centered equations  $\sqrt{2} = \frac{p}{q}$  and  $2 = \frac{p^2}{q^2}$ ).

Interviewer: So, what would you write in the first box?

Guillermo: We will prove that r equals p over q implies that r squared equals p squared over q squared (see Figure 1 below for Guillermo's inscriptions).

In the excerpt above, the interviewee notices the contradiction setup at the beginning of the proof when he refers to "pretending that root 2 is rational." However, the square root of two is not part of what he concludes the proof is proving. He instead shows evidence of a Computational PPF when he focuses on the two equations present in the proof and infers what the proof proves is instantiated in those equations. Focusing on this computation, Guillermo concludes that the fact that squaring both sides of an equation yields a new valid equation is what is being proven,

instead of the validity of this operation being utilized as part of a larger proving process.

**We will prove that**

$$\sqrt{r} = \frac{p}{q} \equiv r = \frac{p^2}{q^2}$$

Suppose to the contrary that  $\sqrt{2}$  is rational. This means  $\sqrt{2}$  is the quotient of two relatively prime integers—let these be  $p$  and  $q$ . We have

$$\sqrt{2} = \frac{p}{q}$$

Squaring both sides, we see that

$$2 = \frac{p^2}{q^2}$$

**Figure 1: Guillermo’s work on Task A.**

Here, we can see that Guillermo focused on the equation and ignored much of the rest of the proof when deciding what the proof is proving. In so doing, he demonstrated a Computational PPF. As a consequence of this PPF, he did not see the argument in the proof as being about the infeasibility of those quantities being relatively prime. For him, the supposition that  $\sqrt{2}$  is rational seemed not to be relevant for determining the claim being proven – instead, the claim being proven was simply being done in the counterfactual “world” in which  $\sqrt{2}$  is rational.

**Cohesive Personal Proof Framework**

A Cohesive PPF involves conceptualizing a proof as a whole, in which every part of the proof is relevant to the claim being proven. During the orientation task, Javon demonstrated this outlook when he stated that, “You kind of have to look through the whole proof, because we are trying to get to just one center goal.” In fact, Javon highlighted both the beginning, the end, and parts of the middle in the orientation task when asked where he saw what parts of the proof indicate what was being proven.

**Orientation Task**

*What does this proof prove?*

We will prove that the sum of two even integers is even. Let  $n$  and  $m$  be two even integers. This means there exist integers  $l$  and  $k$  such that  $n = 2l$  and  $m = 2k$ . Thus, their sum is  $n + m = 2l + 2k = 2(l + k)$ . Since  $l$  and  $k$  are integers, so is  $l + k$ . We have that  $n + m = 2(l + k)$ , twice an integer. So  $n + m$  is even. Therefore, the sum of two even integers is even.

### Figure 2: Javon’s work on the orientation task.

Excerpts from later in the interview help illustrate his cohesive proof framework in other tasks. When presented with Task A, Javon initially used the first line of the proof to hypothesize that the claim being proven is that the square root of 2 is irrational. However, he explained that he was “more comfortable” with his answer if he reads through the entire proof:

Javon : I’m going to look through the question just a little bit more. Um... I can also see that, um, we, uh, can’t have a ratio in a, uh, base form, so we can see p and q are relatively prime so we can’t simplify the function any more, which of course means that square root of two is not rational.

Consistent with his prior statement regarding the importance of the whole proof, he continued to read the proof to glean how it achieves the contradiction. It is important to note that, when writing down his answer, he included aspects of the middle of his proof in the second claim box. Specifically, he wrote “ We have proven that  $\sqrt{2}$  cannot be represented by a simplified ratio of two integers, thus  $\sqrt{2}$  is not rational.” In doing so, he documented not just what was proven, but also *how* it was proven. He went beyond simply formulating the claim to be the last sentence in the proof by including this extra information about how the claim was justified. We hypothesize that he did this because of past experiences with proof validation and proof comprehension. While the PWC are a type of task that he has not experienced prior to this interview, he had significant experience with proof comprehension and validation tasks from his coursework. Proof comprehension and validation involves parsing the details of a proof, so it seems reasonable that he applied strategies from past experiences with proof comprehension to PWC tasks.

#### Claim-Setup Personal Proof Framework:

The Claim-Setup PPF closely mirrors what Selden and Selden (1995) call a “proof framework”. That is, there is a structural relationship between the claim being proven, the proof technique, and the beginning and ending portions of the proof. In other words, students with this framework readily see the beginning and ending of a proof as mirroring the structure of the claim being proven. Thus, if the task is simply to glean what is being proven in a PWC task, there is little to be gained from looking at the middle of the proof.

In Task A, Manuel immediately underlined the first (“suppose to the contrary that  $\sqrt{2}$  is irrational”) and last (“contradicting our supposition”) line of the proof and did not annotate anywhere in the middle. He then explained:

Manuel: We have what I would call a contradiction. Proof by contradiction. And so we assume that somewhat contrary, something is not true. So, we assume that radical two *is* rational, but then we proved something, and it ended up being a contradiction. And therefore, that’s not gonna work. Therefore, proof by contradiction proves that the hypothesis is true. Meaning, we have proven that radical 2 is irrational.

Having a Claim-Setup PPF does not guarantee that a student can easily, confidently, and accurately determine what a proof is proving. Although students with this framework understand the structure of the proof as mirroring the claim, there are a multitude of ways that students might interpret the structure of the proof. For example, while Thomas also demonstrated a Claim-Setup PPF, his response to Task A diverged greatly from Manuel’s, even though Thomas



also focused on the beginning and ending portion of the proof. While thinking about the task, Thomas explained how he was using the beginning and ending portion:

Interviewer: What's standing out?

Thomas: So a lot of what I'm looking for is things that I can use in the beginning in the end.

Although he focused on the beginning and ending sentence, he felt conflicted about how to relate them. This appeared to arise from his own interpretation of the word "supposition" at the end of the proof ("contradicting our supposition that  $p$  and  $q$  are relatively prime"). He interpreted this use of the word "supposition" to indicate that " $p$  and  $q$  are relatively prime" was the hypothesis of a conditional being proven. He identified the claim as a conditional that was being proven by contradiction: "From the concluding sentence, we kind of contradicted one of our givens".

However, he had trouble reconciling the generality of  $p$  and  $q$  with the specificity of  $\sqrt{2}$ :

Thomas: I'm having trouble figuring out what the *then* portion of the proof is, in the sense that it seems like I would assume from the first sentence, where it says "suppose to the contrary that the square root of 2 is rational", that I would believe is kind of the, I don't know, I don't remember the exact terminology for it, it's the *then* portion of the conditional statement. But, it says that the square root of 2 is rational, and that seems more like, specific than if we start with a proof and we have  $p$  and  $q$  where we prove that the square root of *some* number is rational, rather than square root of 2 is rational."

We can see that Thomas's PPF was a Claim-Setup Proof Framework. He focused on the word "suppose" and used the beginning and end of the proof to suggest that the claim being proven was "if  $p$  and  $q$  are relatively prime, then the square root of 2 is irrational (or perhaps rational)."

### **Interpretation of Claims**

Additionally, it bears mentioning that determining the immediate claim being proven does not necessitate a particular interpretation of this claim. It seems reasonable to believe that a student might identify a conditional (for example) being proven but still interpret its meaning or applicability differently than a mathematician would. Indeed, this phenomenon occurred with Task D, a PWC that proved that a particular linear function is injective. The word "injective" was not used – it was proven that the implication "if  $x_1=x_2$ , then  $f(x_1)=f(x_2)$ " holds. For example, Manuel, who demonstrated a Claim-Setup PPF, used the beginning and ending and identified the proof as proving the implication, but then explained "there is a function  $f(x)$  where any unique input  $x$  gives a unique output  $f(x)$ ". In other words, despite readily identifying the implication being proven, Manuel interpreted it to be an existence statement that described meeting the criteria for being a function. Here, we see an analogue to the Shepherd and van de Sande (2014) work by noting that interpreting the literal statement being proven does not necessarily determine an interpretation of that statement.

### **Future Directions and Discussion**

From our data, it can be seen that students vary in their conception of how a proof relates to the claim being proven. Students with a Claim-Setup PPF focused on the beginning and ending portions of a proof to discern the logical structure and content of the claim. This tended to involve paying attention to the language being used in the proof (e.g., words like "suppose"). We can contrast this approach markedly with students who used the Computational Proof Framework. These students often ignored the verbiage of the proof and focused on the symbolism and equations. Like the students in the eye-tracking studies (e.g., Ingles & Alcock, 2012), they honed

in on the equations and computations and appeared to view them as constituting the proof. Such students seemed not to connect the calculations within a proof to the overall structure of the proof and the verbiage surrounding the equations. Finally, students with a Cohesive PPF often needed to make sure they understood every or several of the sentences in the proof before feeling comfortable enough to confidently state what the proof was proving. It appears that their sensemaking of the middle portion of the proof (including the computations and equations) aided them in constructing the claim. Unlike students with a Claim-Setup PPF, these students tended not to stop and take note of structure-indicating words such as “suppose”. Future work can investigate how such students interpret such words.

We saw evidence that students were being influenced by their prior experience with mathematics. For students with the Computational PPF, prior experiences in calculation-based courses appeared to be playing a role. This observation aligns with Dawkins and Zazkis’ (2021) claim that students’ prior calculational experiences were influencing the fact that students tended to view equations within proofs as existing solely for computational purposes. Similarly, we noticed that students with the Claim-Setup PPF appeared to be influenced by their experiences with textbooks or instruction that prompted them to identify “givens” and “goals” when producing a proof. We note that, although the students with the Claim-Setup PPF appeared to exclusively use the beginning and end of the proof to write their claim, they still sometimes read portions of the middle of the proof. We hypothesize that such students are influenced by the fact that much of their past experiences with reading proofs involved proof validation and proof comprehension.

Importantly, even though for mathematicians the structure of the claim mirrors the structure of a proof that proves it, students do not readily make the same connections. While Selden and Selden (2018) posit that it can be useful to teach students to start their proofs using proof frameworks, we believe our PWC tasks can be similarly useful. PWC tasks have the utility of being includable on assessments as short-answer or multiple-choice questions. Future research can study the how PWC tasks can be implemented for instructional or assessment purposes.

### References

- David, E. J., & Zazkis, D. (2020). Characterizing introduction to proof courses: a survey of US R1 and R2 course syllabi. *International Journal of Mathematical Education in Science and Technology*, 51(3), 388-404.
- Dawkins, P.C., & Zazkis, D., (2021). Using moment-by-moment reading protocol to understand students’ processes of reading mathematical proof. *Journal for Research in Mathematics Education*, 52(5), 510-538.
- Gaber, Kirsh, & Zigdon (in press). Applying Proof Frameworks Is Harder for Students than We Think. *Proceedings of 25<sup>th</sup> Annual Conference on Research in Undergraduate Mathematics Education (RUME)*.
- Inglis, M., & Alcock, L. (2012). Expert and Novice Approaches to Reading Mathematical Proofs. *Journal for Research in Mathematics Education*, 43(4), 358–390. <https://doi.org/10.5951/jresematheduc.43.4.0358>
- Selden, J., & Selden, A. (1995). Unpacking the logic of mathematical statements. *Educational Studies in Mathematics*, 29, 123-151.
- Selden, A., Selden, J., & Benkhalti, A. (2018). Proof frameworks: A way to get started. *PRIMUS*, 28(1), 31–45. <https://doi.org/10.1080/10511970.2017.1355858>
- Selden, A., & Selden, J. (2015). Validation of Proofs as a Type of Reading and Sense-Making: Illustrated By An Empirical Study (No. 2015-4). Tennessee Technological University. <https://www.tntech.edu/cas/pdf/math/techreports/TR-2015-4.pdf>
- Shepherd, M. D., & van de Sande, C. C. (2014). Reading mathematics for understanding—From novice to expert. *The Journal of Mathematical Behavior*, 35, 74–86. <https://doi.org/10.1016/j.jmathb.2014.06.003>
- Simon, M. A. (2019). Analyzing Qualitative Data in Mathematics Education. In K. R. Leatham (Ed.), *Designing, Conducting, and Publishing Quality Research in Mathematics Education* (pp. 111–122). Springer International Publishing. [https://doi.org/10.1007/978-3-030-23505-5\\_8](https://doi.org/10.1007/978-3-030-23505-5_8)

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

- Stylianides, G. J., Stylianides, A. J., & Weber, K. (2017). Research on the teaching and learning of proof: Taking stock and moving forward. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 237–266). National Council of Teachers of Mathematics.
- Zazkis, D., & Villanueva, M. (2016). Student conceptions of what it means to base a proof on an informal argument. *International Journal of Research in Undergraduate Mathematics Education*, 2, 318-337.

# UNDERSTANDING ONE CALCULUS INSTRUCTOR'S CLASS PRACTICES USING A POSSIBLE STUDENTS' COGNITIVE MODEL

Richard Velasco  
University of Oklahoma  
richard.velasco@ou.edu

Dae S. Hong  
University of Iowa  
dae-hong@uiowa.edu

*In this study, we examined one experienced mathematician's class practices, with particular attention to cognitive model described in genetic decomposition. Our findings indicate that students only had limited opportunities to be familiar with the first three steps in genetic decomposition, which may potentially lead students to answer limit tasks correctly, but not necessarily having a deep conceptual understanding behind those tasks. With limited opportunities, it will be challenging for students to overcome well-known learning challenges in limit.*

Keywords: cognitive model, genetic decomposition, limit concepts, mathematician's class practices

## Introduction

Among several topics in calculus, the concept of limit is considered fundamental to the study of advanced mathematics, including derivatives, integrals, and series (Bezuidenhout, 2001; Cornu, 1991). In recent years, attempts have been made to understand how students come to understand the concepts of limit by exploring possible cognitive model (Larsen et al., 2016). Using genetic decomposition, a construct of APOS theory (Action, Process, Object and Scheme), some students were able to overcome challenges of understanding infinite process of finding limit (Swinyard, 2011; Swinyard & Larsen, 2012). This suggests that genetic decomposition can be a useful tool for calculus instructors in designing limit tasks and lessons to support students (Arnon et al., 2014a; Trigueros, 2022). In this study, we examined one mathematician's teaching practices in presenting the concept of limit as a springboard to further discussions about what and how we teach foundational topics in calculus as what students experience in their classes is an important influencing factor that allow students to complete the calculus sequence successfully (Hagman, Johnson, & Fosdick, 2017). The following research questions guided our present study:

- How does an experienced mathematician introduce the concept of limit?
- How are students' learning challenges and cognitive model treated in limit lessons?

## Relevant literature

### Students' understanding of the limit concept

Students often think the limit of a function cannot be equal to a number, the limit is unreachable (Güçler, 2013; Szydlik, 2000), only continuous functions have limits (Bezuidenhout, 2001), limit is an approximation so it will not be equal to a number (Nagle, Tracy, Adams, & Scutella, 2017; Roh, 2008), and the limit is the same as the value of the function at the point (Nagle et al., 2017). To overcome these difficulties, students need to be able to coordinate how two varying quantities,  $x$  and  $y$ , behave (Jones, 2015; Thompson & Carlson, 2017). However, the mathematical notation, limit  $\lim_{x \rightarrow c} f(x) = L$ , itself seems to emphasize how  $x$  approaches a number, domain process, without clearly describing how  $y$  behaves, range process.

Studies show that students are very familiar with plugging in  $x$  values to find the limit, but unable to or do not describe how  $y$  values change (Cottrill et al., 1996; Keene et al., 2014).

### **Developing a cognitive model for students' understanding**

Recent studies have moved towards to understanding and describing how students come to understand limit (Cottrill et al., 1996). One such attempt used APOS theory that describes models of what might be going on in the mind of a student when he or she is trying to learn a mathematical concept and uses these models to design instructional materials (Arnon et al., 2014b). In the process of understanding students' constructions of mathematical concepts, genetic decomposition plays an important part in APOS theory as genetic decomposition describes the mental structures and mechanisms that a student might need to construct in order to learn a specific mathematical concept (Arnon et al., 2014a). Although genetic decomposition is not unique, when it is empirically tested, it can serve as a guide for instruction of particular mathematical concept (Trigueros, 2022).

### **Genetic decomposition of limit**

Genetic decomposition for the limit describes a sequence of mental constructions students may make in the process of coming to understand the limit concept informally and formally (Figure 1).

1. The action of evaluating  $f$  at a single point  $x$  that is considered to be close to, or even equal to,  $a$ .
2. The action of evaluating the function  $f$  at a few points, each successive point closer to  $a$  than was the previous point.
3. Construction of a coordinated schema as follows.
  - a. Interiorization of the action of Step 2 to construct a domain process in which  $x$  approaches  $a$ .
  - b. Construction of a range process in which  $y$  approaches  $L$ .
- c. Coordination of (a), (b) via  $f$ . That is, the function  $f$  is applied to the process of  $x$  approaching  $a$  to obtain the process of  $f(x)$  approaching  $L$ .
4. Perform actions on the limit concept by talking about, for example, limits of combinations of functions. In this way, the schema of Step 3 is encapsulated to become an object.
5. Reconstruct the processes of Step 3(c) in terms of intervals and inequalities. This is done by introducing numerical estimates of the closeness of approach, in symbols,  $0 < |x - a| < \delta$  and  $|f(x) - L| < \epsilon$ .
6. Apply a quantification schema to connect the reconstructed process of the previous step to obtain the formal definition of limit.
7. A completed  $\epsilon - \delta$  conception applied to a specific situation

**Figure 1: Refined genetic decomposition of limit (Cottrill et al. 1996, p. 177–178).**

Figure 1 illustrates genetic decomposition used by Cottrill et al. (1996). Genetic decomposition can be revised when needed. For example, step 1 in Figure 2 was added to the previous preliminary decomposition because some students evaluated function at a single point (Arnon et al., 2014a; Cottrill et al., 1996). Also, the coordinated process (step 3) replaced the single process as well because students tended to construct a separate Process for  $x$  approaching a number apart from application of the function  $f$  (Arnon et al., 2014c; Cottrill et al., 1996). Since process of finding limit is not restricted to a finite computation, students who are only able to

evaluate functions with a finite number of values are at the Action stage. At this stage, they are only able to evaluate a function at a few finite points without considering how  $y$  values behave, so it is possible for them to think that the limit and the value of the function at a point are equal to each other (Cottrill et al., 1996; Nagle et al., 2017). In order to move to the Process stage (interiorizing the action is needed), students need to be able to know what will happen to the function when evaluating it at infinitely many points as they construct both domain and range processes (Cottrill et al., 1996; Dubinsky, 1991). However, interiorizing this process to think about the infinite process of domain and range is difficult for students (Swinyard, 2011; Swinyard & Larsen, 2012). Often, *close as we want*, *close enough*, or *sufficiently close to* are used describe the infinite process, but they are vague and it is often very challenging for students to conceptualize the meaning of “close enough” (Keene et al., 2014). When they are able to coordinate and interiorize both processes, students are able to think that the limit can be reached, but it may not necessarily be equal to the value of the function. With empirical evidence for the first four steps (Cottrill et al., 1996), those four steps can be considered as a useful model of students’ cognition that we may see in most students’ work (Arnon et al., 2014a). Thus, it may be beneficial for calculus instructors to use the first three steps in genetic decomposition to design limit lessons to provide learning opportunities for students to possibly overcome learning challenges, as others have suggested (Cottrill et al., 1996; Nagle et al., 2017).

### Methods

#### Data sources

A Midwestern research university in the United States was the setting for this study. The calculus instructor, Dr A (a pseudonym), has taught Calculus I several times and has a PhD in mathematics with a specialization in topology. Each of Dr A’s classes was 50 minutes long. The source of data came from class video and audio recordings. All calculus classes taught by Dr A were videotaped. For this study, we examined four video and audio recordings during the initial introduction of limit concept since Dr. A spent four lessons to cover limit topics. In all, we examined discussions of 32 implemented tasks. In addition to videotaped lessons, we interviewed Dr. A to see if she had any particular reasons to make her instructional decisions.

#### Analysis of Implemented Tasks

For our analysis of Dr. A’s lessons, we used the first three steps in Cottrill et al’s genetic decomposition.

**Table 1: Analytic Framework**

Genetic Decomposition	
Cognitive model	Step 1: evaluating function at just one point. Step 2: evaluating function at several points Step 3: coordinating and interiorizing both domain and range processes

Table 1 describes our analytic framework. For each of these areas, we examined Dr. A’s lessons and task implementations to see if her students had opportunities to experience and discuss first three steps of cognitive model. We carefully examined each task discussion to determine if students had opportunities to use a cognitive model described in genetic decomposition. If students are just familiar with step 1 of genetic decomposition, it is possible for students to think that the limit and value of a function are equal to each other (Cottrill et al., 1996). Even when students are able to be at step 2, without being able to coordinate both domain

and range processes, they could still think that the limit and value of a function are equal to each other (Cottrill et al., 1996). When students are at step 3, being able to interiorize both domain and range processes, they might be able to see the infinite process of finding the limit and conceptualize “close enough” because you can’t plug in numbers infinitely many times, calculations need to be contemplated (Cottrill et al., 1996).

### Examples of analysis

The unit of analysis is discussion of one task. For each unit of analysis, the analytic framework was employed.

Dr. A: Very good. So what you do is your look when  $x$  goes very very close to 1 what does it happen to  $y$  and I’ve attempted to indicate the  $y$  approaches 2. Now I am going to change the graph, what is the limit of  $f$  of  $x$  as  $x$  approaches 2?

S: 4?

Dr. A: I have four answer,

Dr. A: I have three answer,

Dr. A: I have does not exist answer ... Alright the key is to remember that you do not care what happens at 1 you just care what happen near 1. What happens near 1? what does  $f$  of, I mean, near 2. What does  $f$  of  $x$  approach  $x$  is getting close to do?

Dr. A: Three. The value of  $f$  of three of  $f$  of 2 is four that actually completely does not matter as far as limit is concerned. So what happen happens at the number is irrelevant. All you care about is what happen when you get close to the number. You can also get asked about limits at the infinity. For example, what is the limit of  $f$  of  $x$  as  $x$  goes to the infinity. So you should thinking about  $x$  gets very very large. What does  $x$ , what does  $f$  of  $x$  approaches when  $x$  gets very very large?

S: Zero

Dr. A: Excellent.

### Figure 2: A sample discussion of a task

Figure 2 shows implementation of a task. Dr. A drew a graph to demonstrate how to find the limit when a graph is given. We observed several different findings here. First, she described both domain and range processes, “ $f(x)$  approaches  $L$  as  $x$  approaches  $a$ ” to describe limit. She used the term “very, very close” (or “very, very large”) to describe closeness. She also mentioned that the limit and the value of the function are not necessarily equal to each other. Dr. A provided learning opportunities to address important issues in learning the concept of limit. However, we reviewed the discussion more in depth to see more precise learning opportunities that Dr. A’s students have. In the discussion, some students mentioned limit is 3, 4 and does not exist. Then, here is what she said to describe the limit of this function.

The value of  $f$  of three of  $f$  of 2 is four that actually completely does not matter as far as limit is concerned. So what happens at the number is irrelevant. All you care about is what happen when you get close to the number.

She mentioned that “what happens at the number is irrelevant,” and “all you care about is what happen when you get close to the number” to describe the limit is not always equal to the

value of the function at that point, but describing what it precisely means to be close enough and how  $x$  and  $y$  behave near 2 using table of values would be more helpful. Students can be asked to confirm that they can always do better as they coordinate domain and range processes to describe the process if limit exists. Such discussion gives them opportunities to think about the difference between what happens to the function at 2 and what happens when  $x$  is getting closer and closer to 2; ultimately, why limit exists while function value is not 3. In previous studies, using the zoom feature helped students interiorize and conceptualize the processes (Swinyard, 2011; Swinyard & Larsen, 2012). We are not suggesting that the formal  $\epsilon$  and  $\delta$  definitions need to be mentioned here, but with how this task was presented and discussed, they may not be able to conceptualize closeness (it is not clear how close you have to be to have limit), coordinate both domain and range processes, or understand why the function value does not matter when finding limit. Lastly, students were able to find the limit at 2 with a graph. However, this can leave students at Step 1 of genetic decomposition, evaluating functions at one point.

### Coding Reliability

We reviewed previous studies carefully to develop our codes (Hong, 2023; Nagle et al., 2017; Swinyard & Larsen, 2012). Once they were developed, two independent readers watched and read the lessons and transcripts several times and compared findings to ensure reliability of results. The initial agreement rate for task discussions was 93.5% (30 out of 32). Discussion was done until there was 100% agreement.

### Results

In this section, we describe our findings from our analysis of Dr. A’s four limit lesson tasks. We describe learning opportunities that students have to understand and use research – based students’ thinking. Table 2 shows the results of our analysis. One clear finding is many tasks were discussed at Step 1 of genetic decomposition. We will discuss these in more detail.

**Table 2: Percent distribution of class discussions.**

Cognitive model	
Genetic Decomposition: Step 1	Genetic Decomposition: Step 2
26 (81%)	3 (9.4%)

### Genetic decomposition

Dr. A: So here’s an example. Alright, [pauses] I’m going to go through this one example using the rules. So you have a bunch of functions added and subtracted from each other. So the first rule told you that the limit of two functions added is the sum of the limits. So this splits up into limit of  $2x^2$  minus limit of  $3x$  plus limit of 4, this is all  $x$  goes to 5. So this is by the first two rules that told you that you can subtract and add the limits. Alright, what rule do we apply to this kind then? You can think of it as multiplying or you can think of it as a constant times the function. And you know that constants can come out – this was like, rule 3 or so... And we can apply the same rule to the next kind?

Dr. A: So here’s an example. Alright, [pauses] I’m going to go through this one example using the rules. So you have a bunch of functions added and subtracted from each other. So the first



rule told you that the limit of two functions added is the sum of the limits. So this splits up into limit of  $2x^2$  minus limit of  $3x$  plus limit of 4, this is all  $x$  goes to 5. So this is by the first two rules that told you that you can subtract and add the limits. Alright, what rule do we apply to this kind then? You can think of it as multiplying or you can think of it as a constant times the function. And you know that constants can come out – this was like, rule 3 or so... And we can apply the same rule to the next kind?

Dr. A: Alright, and now I think we're done because we have the rule that says that the limit is  $x = a$  of  $x^2$  is  $a^2$ . We have it for general  $n$  so the limit of  $x$  is  $x^2 = 5$  is what?

S: 25

Dr. A: And what is the limit of  $x$  is  $x=5$ ?

S: 5

Dr. A: And what is the limit of the constant? The constant itself.

**Figure 3: Dr. A's discussion of a task**

Figure 3 describes discussion of the task  $\lim_{x \rightarrow 5} 2x^2 - 3x + 4$ . Dr. A said the following, "Alright, and now I think we're done because we have the rule that says that the limit is  $x = a$  of  $x^2$  is  $a^2$ . We have it for general  $n$  so the limit of  $x$  is  $x^2 = 5$  is what?" This task was discussed right after discussing several limit laws. This discussion is mostly about Step 1, evaluating a function at one point, of genetic decomposition. There was no consideration of how  $x$  and  $y$  behave, what it means to having  $x$  closer and closer to 5 or interiorizing the process. Among the 32 tasks that we examined, 26 task discussions were very similar to this. Students are mostly becoming familiar with plugging in numbers to evaluate functions, Step 1 of genetic decomposition. We do acknowledge that once students see some limit tasks and limit laws, it is realistic to think that calculus instructors mostly discuss plugging in a number, and other procedures such as factoring and dividing by a term; however, at this point in the lesson, students have a very limited number of learning opportunities to think about coordinating domain and range processes and conceptualizing closeness.

**Coordinating domain and range processes**

The following shows the typical way Dr. A introduces and describes how  $x$  and  $y$  behave.

Dr. A: Okay, so today, we'll start calculating limits. We'll actually do this algebraically without, um, without needing to graph things.

Dr. A: So if I do want the limit  $x = 1$  of  $x^2 + 1$ , the way we did this – like, two seconds ago – was we graphed the function which was a continuous function which means that you can draw it the graph without picking up your pencil – for now – and, um, we saw that the  $x=1$  the  $y$  value was 2 and because this is all nice and there's no holes or breaks in the graph, this told us that limit is actually 2 because when  $x$  gets close to 1,  $y$  got close to 2. Yes?

Dr. A: So when things are nice, you can actually just plug in a number into the equation and that will tell you what the limit is. And that's basically what we're going to do today.

**Figure 4: Dr. A's discussion of another task**

Figure 4 describes Dr. A's class discussion of  $\lim_{x \rightarrow 1} x^2 + 1$ . Dr. A mentioned how  $x$  and  $y$  behave, "x gets close to 1, y got close to 2" (two other tasks are also discussed this way). However, discussion of this task did not describe both processes in more detail. Instead, she

mentioned, “So when things are nice, you can actually just plug in a number into the equation and that will tell you what the limit is.” This may give students the idea that limit is simply plugging in numbers into  $x$  and the limit is the same as the value of the function at that point. Coordinating domain and range processes was mentioned briefly, and the main focus was more about plugging in numbers rather than describing how to coordinate  $x$  and  $y$  values – step 1 of genetic decomposition.

### **Conceptualizing closeness**

Defining “close” in an iteratively restrictive manner is one way to help students operationalize the notion of infinite closeness. This leads to the idea that every possible requirement for closeness must be met for the limit to exist.

Dr. A: Let’s see how far we get.  $F$  of  $x$ , as  $x$  goes to negative 1 from the negatives. Is this big enough can you guys see this or show they are omitted?  
Dr. A: alright.  $x$  is going to -1 from the negatives  
S: 2  
Dr. A: -1 from the negatives. I love it. -1 from the positives.  
S: 2  
Dr. A: Fantastic. Grey person. -1.  
Dr. A: Yes.  
Dr. A: -1 would be 2 I guess?  
Dr. A: Sounds great.  
Dr. A: the limit for -4 does not exist?  
Dr. A: Very nice. The limit to -4 does not exist because from the left it’s -2 from the right it’s -3.  
Dr. A: does not exist.  
T: What I said a second ago was incorrect, -4 from one side is 3 from one side is 4 and four also does not exist and that one from one side is -3 and from the other side

### **Figure 5: Dr. A’s discussion of several limit tasks**

Figure 5 shows the discussion of several tasks that students need to find the limit from a graph. Dr. A mentioned “going to -1 from negatives” or similar terms. There were 23 tasks that were discussed this way, where students needed to find the limit using graphs. Although they correctly found the limit using graphs, without more clearly describing infinite process, as the value of the function can be easily found from the graph, it is possible for students to think that limit is the same as the value of the function or limit is about finding the value at one point.

### **Interview with Dr. A**

With our findings, we were very interested in knowing what influenced her instructional decisions. We asked Dr. A about what shapes her instructional practices. Here is what she said about what her students usually want in calculus class.

They do not want concepts. They want to follow procedures. They often want to know what is going to be on the exam.

This is very similar to what other researchers found out about how students forced instructors to alter their instructional decisions (Hemmi, 2010; Kontorovich, 2016). We also asked her whether she is familiar with APOS theory and cognitive model proposed in genetic decomposition. She said although she is not familiar with APOS theory, she thinks that using students’ thinking might be beneficial in teaching calculus.

I definitely think that building understanding using their own thinking might be only valuable way to teach calculus. If I do not have pressure to how much material that I have to cover, I will definitely figure out what students understand and start from that point to entirely build everything on understanding.

Her comments show that she has similar issues as other studies pointed out and her class practices might be dictated by other factors that force her to alter her pedagogies.

### **Summary and Discussion**

Our study examined how one mathematician presents the concepts of limit. Our results are very similar to previous studies that describe what mathematicians do in calculus classes (Güçler, 2013; Park, 2015; Wagner, 2018). Our results show her students focused on finding the value of the function at one point (step 1 of genetic decomposition – action stage) or performing procedures. We do not claim that finding limit by plugging in numbers or procedures of factoring and dividing by a term are not necessary; however, they were discussed with limited discussions of a possible research-based findings of how students may come and understand limit concept. Students only accustomed to plugging in values to find the limit will find it difficult to coordinate both domain and range processes, conceptualize infinite process of closeness, and overcome the challenge that limit and the value of the function may not be equal to each other. Although calculus students may successfully answer many limit tasks on assessments, they most likely lack conceptual understanding. What we observed from Dr. A's four lessons was limited opportunities for students to be familiar with three steps in genetic decomposition. It would be useful for students to conceptualize and interiorize these concepts with tables of values to more precisely see what happens to each function if  $x$  values get closer and closer to a number. In previous studies, using the zoom feature was helpful for students to interiorize and conceptualize the processes (Swinyard, 2011; Swinyard & Larsen, 2012). Although using  $\epsilon$  and  $\delta$  may not be necessary, it would be useful to have tasks or activities that allow students to see those processes and have opportunities to use three steps in genetic decomposition to possibly avoid students' learning challenges in limit.

We also need to consider what shapes calculus instructors' class practices. The natural questions are "Is that what happens in most calculus classes?" and "Are there any pedagogical reasons that shape Dr. A's class practices?" Other studies often indicated that college mathematics instructors often altered their class practices because of their students' needs or limited content knowledge (Hong & Choi, 2019; Hong & Lee, 2023). Moreover, other researchers also indicated that it is natural to think that when important conceptual ideas are not being assessed, calculus instructors and students focus on what will be assessed on exams (Tallman et al., 2021). For limit, when simply plugging in numbers, factoring or dividing provides correct answers to assessment items, they will be less likely to feel the need to consider other methods, which can greatly influence what and how calculus instructors plan and implement their lessons. Dr. A's students lack prerequisite content knowledge and they pay more attention to what will be on exams than the concepts behind limit, which all contributed to her class practices. As we mentioned, Dr. A and other mathematicians have other issues to consider when plan their lessons. Then, is it fair or realistic to expect mathematicians like Dr. A to implement teaching practices that build on findings from limit research? However, one thing that is worthwhile to mention is Dr. A also acknowledges that using students' own thinking will be beneficial in teaching calculus. Although this study only examined one mathematician in one country, Dr. A's teaching practices can be quite widespread in other calculus classes, as Wagner (2018) suggested. Finally, we acknowledge that the results of this study are just limited to

practices of one mathematician and we examined one genetic decomposition. Although other mathematicians may have similar issues that shape their instructional practices, it will be interesting to examine other mathematicians' practices as well. As Dr. A mentioned, more mathematicians might feel the need of using students' thinking in developing their lessons.

### References

- Arnon, I., Cottrill, J., Dubinsky, E., Oktaç, A., Fuentes, S. R., Trigueros, M., & Weller, K. (2014a). Genetic Decomposition. In *APOS Theory: A Framework for Research and Curriculum Development in Mathematics Education* (pp. 27-55). New York, NY: Springer New York.
- Arnon, I., Cottrill, J., Dubinsky, E., Oktaç, A., Fuentes, S. R., Trigueros, M., & Weller, K. (2014b). Introduction. In *APOS Theory: A Framework for Research and Curriculum Development in Mathematics Education* (pp. 1-4). New York, NY: Springer New York.
- Arnon, I., Cottrill, J., Dubinsky, E., Oktaç, A., Fuentes, S. R., Trigueros, M., & Weller, K. (2014c). The Teaching of Mathematics Using APOS Theory. In *APOS Theory: A Framework for Research and Curriculum Development in Mathematics Education* (pp. 57-91). New York, NY: Springer New York.
- Bezuidenhout, J. (2001). Limits and continuity: some conceptions of first-year students. *International Journal of Mathematical Education in Science and Technology*, 32(4), 487-500. doi:10.1080/00207390010022590
- Cornu, B. (1991). Limits. In D. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 153-166). Dordrecht: Springer Netherlands.
- Cottrill, J., Dubinsky, E., Nichols, D., Schwingendorf, K., Thomas, K., & Vidakovic, D. (1996). Understanding the limit concept: Beginning with a coordinated process scheme. *The Journal of Mathematical Behavior*, 15(2), 167-192. doi:[https://doi.org/10.1016/S0732-3123\(96\)90015-2](https://doi.org/10.1016/S0732-3123(96)90015-2)
- Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. In D. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 95-123). Dordrecht: Springer Netherlands.
- Güçler, B. (2013). Examining the discourse on the limit concept in a beginning-level calculus classroom. *Educational Studies in Mathematics*, 82(3), 439-453. doi:10.1007/s10649-012-9438-2
- Hagman, J. E., Johnson, E., & Fosdick, B. K. (2017). Factors contributing to students and instructors experiencing a lack of time in college calculus. *International Journal of STEM Education*, 4(1), 12. doi:10.1186/s40594-017-0070-7
- Hemmi, K. (2010). Three styles characterising mathematicians' pedagogical perspectives on proof. *Educational Studies in Mathematics*, 75(3), 271-291. doi:10.1007/s10649-010-9256-3
- Hong, D. S. (2023). Examining Opportunities to Learn Limit in Widely Used Calculus Textbooks. *International Journal of Science and Mathematics Education*, 21(3), 881-898. doi:10.1007/s10763-022-10273-7
- Hong, D. S., & Choi, K. M. (2019). Challenges of maintaining cognitive demand during the limit lessons: understanding one mathematician's class practices. *International Journal of Mathematical Education in Science and Technology*, 50(6), 856-882. doi:10.1080/0020739X.2018.1543811
- Hong, D. S., & Lee, J. K. (2023). Contrasting Cases of College Calculus Instructors: Their Preferences and Potential Pedagogy in Teaching Derivative Graphs. *International Journal of Mathematical Education in Science and Technology*. doi:10.1080/0020739X.2022.2120838
- Jones, S. R. (2015). Calculus limits involving infinity: the role of students' informal dynamic reasoning. *International Journal of Mathematical Education in Science and Technology*, 46(1), 105-126. doi:10.1080/0020739X.2014.941427
- Keene, K. A., Hall, W., & Duca, A. (2014). Sequence limits in calculus: using design research and building on intuition to support instruction. *ZDM*, 46(4), 561-574. doi:10.1007/s11858-014-0597-8
- Kontorovich, I. (2016). The answer depends on your lecturer. *Research in Mathematics Education*, 18(3), 283-298. doi:10.1080/14794802.2016.1234405
- Larsen, S., Marrongelle, K., Bressoud, D., & Graham, K. (2016). Understanding the concepts of calculus: Frameworks and roadmaps emerging from educational research. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 526-550). Reston, VA: National Council of Teachers of Mathematics.
- Nagle, C., Tracy, T., Adams, G., & Scutella, D. (2017). The notion of motion: covariational reasoning and the limit concept. *International Journal of Mathematical Education in Science and Technology*, 48(4), 573-586. doi:10.1080/0020739X.2016.1262469
- Park, J. (2015). Is the derivative a function? If so, how do we teach it? *Educational Studies in Mathematics*, 89(2), 233-250. doi:10.1007/s10649-015-9601-7

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

- Roh, K. H. (2008). Students' images and their understanding of definitions of the limit of a sequence. *Educational Studies in Mathematics*, 69(3), 217-233. doi:10.1007/s10649-008-9128-2
- Swinyard, C. (2011). Reinventing the formal definition of limit: The case of Amy and Mike. *The Journal of Mathematical Behavior*, 30(2), 93-114. doi:<https://doi.org/10.1016/j.jmathb.2011.01.001>
- Swinyard, C., & Larsen, S. (2012). Coming to Understand the Formal Definition of Limit: Insights Gained From Engaging Students in Reinvention. *Journal for Research in Mathematics Education*, 43(4), 465-493. doi:10.5951/jresmetheduc.43.4.0465
- Szydlik, J. E. (2000). Mathematical Beliefs and Conceptual Understanding of the Limit of a Function. *Journal for Research in Mathematics Education*, 31(3), 258-276. doi:10.2307/749807
- Tallman, M. A., Reed, Z., Oehrtman, M., & Carlson, M. P. (2021). What meanings are assessed in collegiate calculus in the United States? *ZDM – Mathematics Education*, 53(3), 577-589. doi:10.1007/s11858-020-01212-3
- Thompson, P. W., & Carlson, M. (2017). Variation, Covariation, and Functions: Foundational Ways of Thinking Mathematically. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 421-456). Reston, VA: National Council of Teachers of Mathematics.
- Trigueros, M. (2022). APOS Theory and the Role of the Genetic Decomposition. In Y. Chevallard, B. Barquero, M. Bosch, I. Florensa, J. Gascón, P. Nicolás, & N. Ruiz-Munzón (Eds.), *Advances in the Anthropological Theory of the Didactic* (pp. 61-74). Cham: Springer International Publishing.
- Wagner, J. F. (2018). Students' Obstacles to Using Riemann Sum Interpretations of the Definite Integral. *International Journal of Research in Undergraduate Mathematics Education*, 4(3), 327-356. doi:10.1007/s40753-017-0060-7

## A LONGITUDINAL STUDY OF WOMEN'S SENSE OF BELONGING IN AN ACTIVE LEARNING UNDERGRADUATE CALCULUS COURSE

Casey R. Griffin  
University of Delaware  
crgrif@udel.edu

*Low sense of belonging has been identified as a key reason for women's decisions to leave STEM majors. Prior studies indicate that incorporating active learning opportunities may support students' sense of belonging. Calculus is an especially ripe setting for investigating STEM students' sense of belonging, as it often acts as a gatekeeper course for STEM majors. This mixed methods study investigates changes in women's sense of belonging during the first semester of a year-long active learning Calculus course. Women who reported large changes were interviewed to investigate the influence of active learning opportunities on their sense of belonging. Results indicate an initial significant increase followed by a plateau in women's sense of belonging. Six women who experienced changes in their sense of belonging agreed to be interviewed, and this analysis is ongoing.*

Keywords: affect, emotion, beliefs and attitudes, calculus, gender, undergraduate education

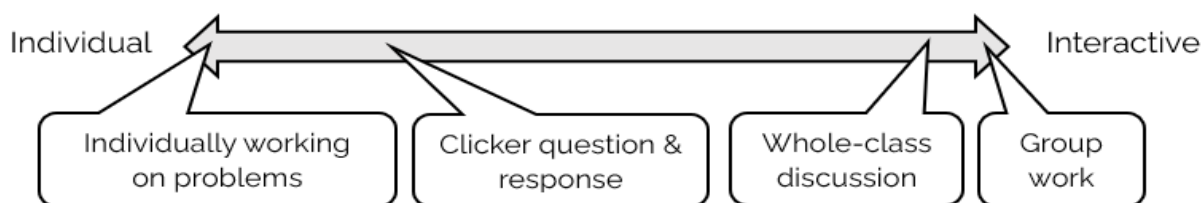
The underrepresentation of women in STEM careers has been attributed to a lack of women pursuing undergraduate STEM degrees (Carmichael, 2017; Chamberlain, 2017). Not only do fewer women than men enter into undergraduate STEM majors, but women leave STEM majors at a higher rate than men – especially after taking Calculus (Chen et al., 2013; Eagan et al., 2016; Seymour & Hunter, 2019). Calculus is a critical junction because it is a required course for STEM majors and is typically taken as a pre- or co-requisite for more advanced STEM coursework, often gatekeeping students who perform poorly from continuing on their STEM trajectory. Consequently, a significant leak occurs in the STEM pipeline at Calculus.

Prior research identifies low sense of belonging as a key reason why women decide to leave STEM majors (Seymour & Hunter, 2019; Shapiro & Sax, 2011). One feels a *sense of belonging* when they feel like “an accepted member of an academic community whose presence and contributions are valued” (Good et al., 2012, p. 701). Strayhorn (2012) argues that a sense of belonging can be so essential that one cannot engage in a space without feeling a sense of belonging to that space. For students, this might mean they have trouble listening to a lecture or studying for an exam without first feeling a sense of belonging. Further, research indicates that students with a stronger sense of belonging are more likely to persist in STEM (Rainey et al., 2018; Seymour & Hunter, 2019). While feeling a low sense of belonging can be an issue for all students, women tend to report a lower sense of belonging than men in STEM (Rainey et al., 2018; Shapiro & Sax, 2011).

Prior studies have found that sense of belonging can change over time, though these changes have typically been negative (Anderman, 2003; Hausmann et al., 2007). However, researchers found that decreases in sense of belonging were buffered by a respectful learning environment

(Anderman, 2003) and perceptions that the instructor cared about the students' learning (Hausmann et al., 2007). These studies not only provide evidence of sense of belonging's malleability but also suggest that sense of belonging can be influenced by external factors like learning environment and instructor care. Researchers have identified other contributors to students' sense of belonging, including academic and social integration (Lewis & Hodges, 2015; Tinto, 1975). Tinto (1975) describes academic integration as students' intellectual and academic development and faculty concern for students' development, and social integration as students' social connectedness with their peer group and faculty. Rainey et al. (2018) confirmed the influence of both academic and social factors in their interview study with 201 college students. They found that students most frequently cited interpersonal relationships and perceived competence as contributors to their sense of belonging in STEM. Further, students who left STEM reported feeling a low sense of belonging and attributed it primarily to a lack of interpersonal relationships and perceived competence.

One way that Calculus instructors may be able to support their students' sense of belonging is to provide opportunities for students to engage in active learning. Active learning opportunities engage students in "the process of learning through activities and/or discussion as opposed to passively listening to an expert" (Bonwell & Eison, 1991, p. iii). Active learning has been shown to benefit students' achievement, persistence, and sense of mastery (Freeman et al., 2014; Lahdenpera et al., 2019; Rasmussen et al., 2019). However, active learning is quite a broad category, and there lacks consensus on the particular types of active learning opportunities that best support students. Active learning opportunities can be categorized along a continuum from solo activities (e.g., individually working on problems, responding to clicker polls) to interactive activities (e.g., engaging in whole-class discussion or group work) (See Figure 1.; Griffin, 2021). I hypothesize that interactive activities that engage all learners and allow students to academically and socially integrate have the most potential to support students' sense of belonging.



**Figure 1: Active Learning Continuum**

This study investigated the following research questions: (1) For undergraduate women enrolled in a year-long active learning Calculus course, in what ways, if any, does their sense of belonging change during the first semester of the course? and (2) For women who reported significant changes in their sense of belonging, to what do they attribute those changes?

### **Methods**

#### **Setting and Participants**

This study was conducted at a mid-Atlantic research university during the 2022-2023 academic year. The university offers a two-semester Integrated Calculus course designed for students who place into Pre-calculus but need Calculus for their intended major. This year-long course is typically taken by first-year students intending to major in STEM. The first semester of the course develops differential Calculus and the second semester develops integral Calculus,

with each course weaving in necessary pre-Calculus topics in a “just in time” approach. The course was designed to provide frequent opportunities for students to engage in active learning, primarily through small group work, whole-class discussions, and short, interactive lectures. Note that each of these activities would fall toward the Interactive end of the Active Learning Continuum (Figure 1) described above. The course is highly coordinated – in addition to common textbooks and exams, instructors teach from common lesson plans specifying which problems to work on each class, and whether they will be discussed as a whole class or in small groups. Thus, the teaching methods employed by all instructors teaching the course are consistent.

In the Fall 2022 semester, two sections of the course were offered, taught by two permanent faculty, with 63 and 64 students enrolled in each section. Participants were students enrolled in both sections of the course. Early in the fall semester, students received an email inviting them to participate in the study by completing a survey. The survey asked students to identify their gender by selecting all that apply from the following options: Man, Woman, Transgender, Non-binary, Not Listed (please specify), and Prefer Not to Disclose. For the purpose of this study, only students who selected at least Woman and completed all three surveys were considered for analysis. For the fall semester,  $N=41$ .

### **Data Collection and Analysis**

The survey was distributed and completed electronically using Qualtrics, a web-based survey tool, to all students enrolled in the course three times over the first semester of the course – beginning, middle, and end of the Fall 2022. To measure students’ sense of belonging, the survey included Good et al.’s (2012) Mathematical Sense of Belonging (MSoB) scale. The MSoB portion consisted of 28 Likert items asking students to indicate the extent to which they agree with statements about their sense of belonging on a scale of 1=*Strongly Disagree* to 6=*Strongly Agree*. To analyze these data, for each student I calculated the mean of the student’s responses to the 28 items to obtain a sense of belonging score. I then used these scores to conduct a repeated-measures ANOVA to determine if significant changes were occurring between each pair of time-points – beginning and middle, and middle and end.

Women whose sense of belonging changed between any pair of time points were asked to participate in an interview to further examine what contributed to those changes. To select these women, I first found the absolute value of the difference in sense of belonging between each pair of consecutive time-points. I sorted the differences after each time-point from largest to smallest to identify women who experienced large changes (whether positive/increase or negative/decrease) and asked them to participate in a 45-minute interview over Zoom. Interviews took place after the second and third surveys. Interview participants were asked to (1) describe what “sense of belonging” means to them and how they feel when they have a sense of belonging, (2) tell a story about something that happened in class during which they felt their sense of belonging being impacted, and (3) agree or disagree with the change in their sense of belonging as reported in the survey and explain why. These responses will be analyzed using theory-guided analysis, with *academic*, *social*, and *learning environment* as a priori codes, while welcoming emergent themes.

### **Results**

Students’ responses to the MSoB portion of the survey were used to measure their sense of belonging at three time-points over one semester in a year-long active learning Calculus course. Preliminary results indicate an initial significant increase with a large effect size ( $d=.79$ ; Cohen,



1988) followed by a plateau in women’s sense of belonging during the Fall 2022 semester (See Table 1.).

Of the women who showed particularly large changes in their sense of belonging, six agreed to be interviewed – only one was interviewed after the second survey, and all six were interviewed after the third survey. Preliminary results indicate that in all six cases, these women described themselves as feeling more comfortable speaking up (e.g., seeking and/or providing help) when they have a sense of belonging. According to one student, she felt comfortable speaking up even if she was wrong: “I’m not afraid to, like, speak, I think, also not afraid to like, make a mistake...even if I say something wrong, it’s still like I still feel comfortable, or I just laugh about it, and so I feel like I belong because I feel free to like say something that is wrong...It’s a hard class, it’s okay to make mistakes.” In their stories describing a time in which they felt their sense of belonging being impacted, three students talked about a time they were able to provide help to their groupmates, and that being able to show that they understood the material made them feel like they belonged. In reference to helping her classmates, one student said, “When people know my abilities and don’t underestimate me, I feel a lot more valued.” Four of these students experienced initial growth followed by a decline in their sense of belonging, and each of these students tied this pattern to the difficulty of the material they were learning in class and the extent to which they understood the material. When asked about the changes in her sense of belonging, one student responded, “I think it really all had to do with our course material. As far as the social aspect of it, I don’t think anything has changed.” Further analysis of interview data is ongoing, and I anticipate drawing connections between these women’s sense of belonging and their opportunities to interact with their classmates during class.

**Table 1: Changes in Women’s Sense of Belonging during Fall 2022**

	Week 1	Week 7	Week 14
Sense of Belonging	3.9312	4.5488***	4.6159

*Note. Asterisks are used to denote the p-values (\* for  $p < .05$ , \*\* for  $p < .01$ , and \*\*\* for  $p < .001$  significance levels) corresponding to the significance of the difference between that time-point and the previous time-point.*

### Discussion

As Strayhorn (2012) argues, students’ sense of belonging is essential to their engagement and motivation in class. Research has shown that students, particularly women, feel a low sense of belonging in STEM, and this is a key reason for women’s decisions to leave STEM (Rainey et al., 2018; Seymour & Hunter, 2019; Shapiro & Sax, 2011). However, adjusting instructional approaches to incorporate more opportunities for students to interact offers promise in supporting students’ sense of belonging. In this study, women enrolled in a year-long active learning Calculus class reported a significant increase in sense of belonging from the beginning to the middle of the fall semester, followed by a plateaued sense of belonging from the middle to the end of the semester. These results are promising and show that women’s sense of belonging can in fact show positive change and that the increase can be maintained. Preliminary results of the interview analysis suggest that for women who reported large changes, opportunities to interact by sharing ideas and providing help supported their sense of belonging. Further, perhaps incorporating opportunities to interact could be a way to support students’ perceived competence when the course material gets more difficult. While this study’s design does not allow for causal claims, these findings suggest a link between opportunities for women to interact with each

other, their level of perceived competence, and their sense of belonging in Calculus. Further analysis will shed light on specific aspects of the learning opportunities they experienced that were particularly impactful to changes in their sense of belonging.

## References

- Anderman, L. H. (2003). Academic and social perceptions as predictors of change in middle school students' sense of school belonging. *The Journal of Experimental Education*, 72(1), 5–22.  
<https://psycnet.apa.org/doi/10.1080/00220970309600877>
- Bonwell, C. C., & Eison, J. A. (1991). *Active learning: Creating excitement in the classroom*. School of Education and Human Development, George Washington University.
- Carmichael, S. G. (2017). Women dominate college majors that lead to lower-paying work. Retrieved from Harvard Business Review website: <https://hbr.org/2017/04/women-dominatecollege-majors-that-lead-to-lower-paying-work>
- Chamberlain, A. (2017). The pipeline problem: How college majors contribute to the gender pay gap. Retrieved from Glassdoor website: <https://www.glassdoor.com/research/app/uploads/sites/2/2017/04/FULL-STUDY-PDF-Gender-Pay-Gap2FCollege-Major-1.pdf>
- Chen, X. (2013). STEM attrition: College students' paths into and out of STEM fields (NCES 2014-001). National Center for Education Statistics, Institute of Education Sciences, U.S. Department of Education.
- Cohen, J. (1988). *Statistical power analysis for the behavioral sciences* (2nd ed.). Hillsdale, NJ: Lawrence Erlbaum Associates, Inc. <https://doi.org/10.4324/9780203771587>
- Eagan, Kevin; Stolzenberg, Ellen Bara; Bates, Abigail K.; Aragon, Melissa C., Suchard, Maria Ramirez; Rios-Aguilar, C. (2016). *American Freshman: National Norms fall 2015*. Los Angeles: Higher Education Research Institute, UCLA.
- Freeman, S., Eddy, S. L., McDonough, M., Smith, M. K., Okoroafor, N., Jordt, H., & Wenderoth, M. P. (2014). Active learning increases student performance in science, engineering, and mathematics. *Proceedings of the National Academy of Sciences of the United States of America*, 111(23), 8410–8415.  
<https://doi.org/10.1073/pnas.1319030111>
- Good, C., Rattan, A., & Dweck, C. S. (2012). Why do women opt out? Sense of belonging and women's representation in mathematics. *Journal of Personality and Social Psychology*, 102(4), 700–717.  
<https://doi.org/10.1037/a0026659>
- Griffin, C. (2021). Calculus instruction and female sense of belonging. In Olanoff, D., Johnson, K., & Spritzer, S. M. (Eds.) *Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (p. 1331-1335). Philadelphia, PA.  
[http://www.pmena.org/pmenaproceedings/PMENA\\_43\\_2021\\_Proceedings.pdf](http://www.pmena.org/pmenaproceedings/PMENA_43_2021_Proceedings.pdf)
- Hausmann, L. R. M., Schofield, J. W., & Woods, R. L. (2007). Sense of belonging as a predictor of intentions to persist among African American and white first-year college students. *Research in Higher Education*, 48(7), 803–839. <https://doi.org/10.1007/s11162-007-9052-9>
- Lahdenperä, J., Postareff, L., & Rämö, J. (2019). Supporting quality of learning in university mathematics: A comparison of two instructional designs. *International Journal of Research in Undergraduate Mathematics Education*, 5(1), 75–96. <https://doi.org/10.1007/s40753-018-0080-y>
- Lewis, K. L., & Hodges, S. D. (2015). Expanding the concept of belonging in academic domains: Development and validation of the Ability Uncertainty Scale. *Learning and Individual Differences*, 37, 197-202.  
<https://doi.org/10.1016/j.lindif.2014.12.002>
- Rainey, K., Dancy, M., Mickelson, R., Stearns, E., & Moller, S. (2018). Race and gender differences in how sense of belonging influences decisions to major in STEM. *International Journal of STEM Education*, 5(10).  
<https://doi.org/10.1186/s40594-018-0115-6>
- Rasmussen, C., Apkarian, N., Hagman, J. E., Johnson, E., Larsen, S., & Bressoud, D. (2019). Characteristics of precalculus through calculus 2 programs: Insights from a national census survey. *Journal for Research in Mathematics Education*, 50(1), 98–111. <https://doi.org/10.5951/jresmetheduc.50.1.0098>
- Seymour, E., & Hunter, A.-B. (Eds.). (2019). *Talking about leaving revisited: Persistence, relocation, and loss in undergraduate STEM education*. Boulder, CO: Center for STEM Learning. <https://doi.org/10.1007/978-3-030-25304-2>
- Shapiro, C. A., & Sax, L. J. (2011). Major selection and persistence for women in STEM. *New Directions for Institutional Research*, 14(7), 5–18. <https://doi.org/10.1002/ir>
- Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

- Strayhorn, T. L. (2012). *College students' sense of belonging: A key to educational success for all students*. New York, NY: Routledge. <https://doi.org/10.4324/9780203118924>
- Tinto, V. (1975). Dropout from higher education: A theoretical synthesis of recent research. *Review of Educational Research*, 45(1), 89–125. <https://doi.org/10.3102/00346543045001089>

---

# STATISTICS, PROBABILITY AND DATA SCIENCE

---

**ENGAGEMENT IN THE INSTEP PROFESSIONAL LEARNING PLATFORM: DEVELOPING EXPERTISE  
TO TEACH DATA AND STATISTICS**

Gemma F. Mojica  
NC State University  
gmjojica@ncsu.edu

Emily Thrasher  
NC State University  
epthrash@ncsu.edu

Adrian Kuhlman  
NC State University  
akuhlma@ncsu.edu

Bruce Graham  
NC State University  
bmgraha2@ncsu.edu

Hollylynn S. Lee  
NC State University  
hstohl@ncsu.edu

Michelle Pace  
NC State University  
bmmurphy@ncsu.edu

*In this study, 82 middle and high school teachers engaged with the InSTEP online professional learning platform to develop their expertise in teaching data science and statistics. We investigated teachers' engagement within the platform, aspects of the platform that were most and least effective in building teachers' expertise, and the extent to which teachers' self-efficacy changed. Using mixed methods, we collected, analyzed and integrated multiple data sources.*

Keywords: Professional Development, Data Analysis and Statistics, Online and Distance Education

### **Introduction**

Many recognize the importance of preparing students for pathways that lead to careers in data science and statistics (DS&S) and improving data literacy for all, including individuals who do not pursue data intensive careers. However, graduates are generally underprepared in statistics and data skills (Finzer, 2013), making a career pathway in a data intensive career unattainable for many (Kwasny, 2015). This is in spite of decades long support for the inclusion of statistics and data in K-12 education (NCTM, 2000), efforts by the American Statistical Association to have guidelines and support for K-12 (e.g., Bargagliotti et al., 2020; Franklin et al., 2007; 2015), the adoption of Common Core State Standards (2010), and many state-level efforts to include DS&S in grades 6-12 (e.g., Jones, 2018). To prepare students who are data literate and ready to pursue careers requiring DS&S skills, *teachers* need to effectively integrate data experiences into an already packed mathematics curriculum, when they themselves may be underprepared to do so. In fact, teachers are often trained in programs that do not support becoming effective statistics teachers with a robust understanding of key data and statistics ideas (Groth & Meletiou-Mavrotheris, 2018; Justice et al., 2018; Lovett & Lee, 2018; Zieffler et al., 2018).

Recently, there have been some well documented large-scale professional learning experiences that have supported teachers' learning to teach statistics. The Stats4STEM online learning environment for AP statistics students (Simoneau, 2018) has a teacher training component and teacher message board to support teachers who are implementing Stats4STEM resources with students. Lee and Mojica have designed and implemented larger scale online professional learning focused on teaching statistics in grades 6-14, serving over 4000 teachers. Over multiple studies, they have found that teachers increased: their confidence to teach statistics, beliefs and practices in using real data, investigative approaches, and use of visualization tools, and developing statistical habits of mind such as attending to variability and embracing uncertainty (e.g., Lee et al., 2017; Lee et al., 2020; Mojica et al., 2018).

Our work is situated in an online professional learning experience, Invigorating Statistics and Data Science Teaching through Professional Learning (InSTEP), aimed at supporting teachers in

developing their expertise to teach DS&S. We investigated the following research questions: *RQ1: How do teachers engage in the InSTEP online professional learning platform focused on teaching DS&S? RQ2: What portals are most effective and least effective in supporting teachers' professional growth to teach DS&S? RQ3: After engaging in the InSTEP online professional learning platform, to what extent are there changes in teachers' self-efficacy to teach DS&S?*

### **Theoretical Framework**

Over four years, the NSF-funded InSTEP project designed and implemented an online professional learning platform to support 6-12+ grade teachers in developing expertise to teach DS&S. We see the InSTEP online professional learning platform as an affinity space (Gee, 2005), where teachers with diverse experiences and backgrounds engage with the space to build their expertise and learn about innovative approaches to teaching DS&S. An important component of an affinity space is its content, or *generator*, which is the InSTEP platform itself. Teachers engage with the content and possibly each other through *portals*. While the InSTEP platform can also be viewed as a portal since this is where teachers acquire resources and tools, portals can also be subspaces of the larger space. Examples of portals in the InSTEP platform include engagement with data investigations, discussion forums, microcredentials and individual resources or tools. Other important aspects of an affinity space are the *internal* grammar, the way the space is designed and organized, and the *external* grammar, how individuals interact with the content or others.

### **InSTEP Platform Design**

We believe that teachers can support students' learning about DS&S by engaging them in data investigations using a Data Investigation Process (Lee et al., 2022) and developing statistical and data habits of mind and dispositions. Thus, an emphasis in professional learning in the InSTEP platform is providing learning experiences that support developing the skills and knowledge to apply innovative approaches to teaching DS&S with data investigations. Within the platform teachers have three opportunities to learn about teaching DS&S by engaging with *data investigations*, *learning modules* and *microcredentials*. Teachers can learn at their own pace and choose any pathway that meets their goals. While teachers can choose linear pathways, the platform is designed to allow teachers to pick and choose in any order. *Data investigations* provide opportunities for teachers to engage in solving problems themselves with data by using a Data Investigation Process (Lee et al., 2022). This involves solving a problem with large, real, messy data using a technology tool. Often the tool is the Common Online Data Analysis Platform (CODAP). *Learning modules* are organized by 7 interrelated dimensions of learning environments that support the teaching and learning of DS&S, adapted from Ben Zvi et al. (2018): data and statistical practices, central statistical ideas, data, technology, tasks, argumentation and discourse, and assessment. Within each dimension, there are multiple learning modules. Modules contain *essential* and *extended* resources. *Microcredentials*, performance assessments, provide opportunities for teachers to learn and demonstrate their competency about ideas presented in data investigations and learning modules that relate to teaching DS&S.

While it is beyond the scope of this paper to discuss the design of the InSTEP platform in more detail, it is important to highlight an important aspect of the platform. A major goal is to personalize learning to support teachers' professional learning goals through customized recommendations. While the development of customization is ongoing, in our initial study, we used multiple surveys and assessments to measure teachers' expertise and confidence. We refer

to these throughout this report as personalization surveys. We created logic models to map results from these personalization surveys to provide recommendations to engage with specific content in the platform related to data investigations and resources in learning modules.

## Methods

In this section, we describe our context and participants, data collection and analysis.

### Context and Participants

During Fall 2022, 82 teachers participated in professional learning using the InSTEP online professional learning platform from August to December. Teachers were recruited nationally through DS&S education networks, including K-12 district and state educational professionals, educators and scholars in higher education and formal and informal networks (e.g., Messy Data Coalition). Almost 200 teachers indicated interest, and 99 teachers were selected to participate with an initial goal of 75. Eight teachers never logged in, and eight withdrew from the study. Throughout, unless otherwise noted,  $n = 82$ . Incentives were provided for participation in professional learning and the research activities described below.

Teachers provided demographic information when indicating their interest in participating in our study. Most were mathematics teachers (70%), 24% were science teachers and 6% were both mathematics and science teachers. About half (51%) of these teachers focused on high school, 43% focused on middle school and 6% indicated they focused on both middle and high school. Primary responsibility and years of teaching experience were only provided by 79 teachers. They were overwhelmingly classroom teachers (91%), where 5% identified as mathematics coaches/district supervisors, 3% were professional development leaders, and 1% was a college or graduate student. Teaching experience ranged from novice to very experienced, ranging from 1 to 38 years ( $M = 17$ ,  $SD = 8.25$ ). Over three-quarters of these teachers had not previously engaged in professional learning related to teaching DS&S.

### Data Collection and Analysis

We used mixed methods to collect, analyze, and integrate multiple sources of data: post-experience survey, self-efficacy survey, microcredential responses, and interviews. All 83 teachers were invited to participate in each of these research activities.

**Post-experience survey.** In the final weeks of the study, 45 teachers responded to a survey to share their experiences engaging in professional learning using the InSTEP platform. They responded to 13 Likert scale questions using a 6-point or 7-point scale (very ineffective to very effective), with multiple items for each question, and 4 open-ended questions. To analyze responses related to the effectiveness of features of the platform for Likert scale questions, frequencies and percentages for each rating were calculated per item. A grounded approach was used to analyze open-ended responses, including using open coding and constant comparative methods to identify emergent themes (Glaser & Strauss, 1967; Strauss & Corbin, 1990).

**Self-efficacy survey.** The *Self Efficacy for Teaching Statistics (SETS) Survey* (Harrell-Williams et al., 2019) was used to measure teachers' self-confidence to teach DS&S, a 44-item survey where teachers identify their confidence to teach specific topics/tasks using a 6-point Likert scale (ranging from 1-not at all confident to 6-completely confident) aligned with the GAISE framework (Franklin et. al., 2007, Bargagliotti et. al., 2020). The instrument provides an overall score of a teacher's confidence to teach statistics and sub-scale scores that correspond to Levels A-C in the GAISE framework. Level A is considered more concrete and introduces

students to the problem-solving process, and level C is considered the most abstract with full development of statistical literacy (Bargagliotti et. al., 2020). Teachers ( $n = 41$ ) completed SETS both before and after their professional learning experience using the InSTEP platform.

Initial analysis of the 44 6-point Likert scale items included calculating pre- and post-scores that corresponded to the 6-point Likert scale. These were calculated by summing each teacher's responses and dividing each sum by the number of items. Using the same procedures, sub-scale scores were also calculated for each teacher. A gain score was calculated for each teacher as the difference of post- and pre- scores for each item. Means were computed for pre-, post-, and gain scores, and a Matched Pairs t-test was conducted to test for the significance from pre- to post-.

**Microcredential response.** Twenty-six teachers submitted a microcredential, a performance assessment to demonstrate competency related to ideas presented in the module focused on key ideas, habits of mind and dispositions of DS&S. We analyzed one of four parts of the microcredential where teachers were asked: How do you plan to use your understanding of what statistics and data science are, especially how it is used in careers and data, as well as statistical habits and dispositions, to provide instructional support for students' learning of statistics and data science? As with the open-ended post-implementation survey questions, we utilized grounded theory, including using open coding and constant comparative methods to identify emergent themes (Glaser & Strauss, 1967; Strauss & Corbin, 1990). A data-driven codebook was also developed with definitions and examples (DeCuir-Gunby et al., 2010).

**Interview.** Four sets of interviews were conducted throughout the study. Three sets of interviews used semi-structured protocols: focus group interviews, individual interviews with wireframes (i.e., mock-ups of the platform), and individual interviews with the development platform, an early version of the platform. A fourth and final interview was conducted at the conclusion of the study, where 11 teachers participated in a 1-hour interview using a structured interview protocol to understand teachers' experiences using the InSTEP platform. Interviews were viewed and analyzed by summarizing the responses to each question from the interviews.

## Findings

We share findings related to teachers' engagement in the InSTEP platform, portals from the InSTEP platform that were most effective and least effective in supporting professional learning, and the extent to which teachers' self-efficacy changed after engaging in professional learning.

### Engagement in the InSTEP Online Professional Learning Platform

Over a four-month period, over half the teachers engaged in professional learning utilizing data investigations and learning modules. Sixty two percent of the teachers engaged with the platform for 10 hours, 54% engaged in professional learning for 20 hours, and 15% completed 30 hours of professional learning. Approximately 34 hours of professional learning were available in the learning modules at the end of the study.

Fewer teachers engaged with microcredentials. While only 32% of teachers submitted a microcredential, 46% of those who submitted ( $n = 26$ ) were successful and earned a microcredential certificate of completion. Only four teachers who were initially unsuccessful receiving the microcredential certificate of completion resubmitted a response. Of those who resubmitted, 75% eventually earned a microcredential certificate of completion. A majority (81%) of microcredential responses were submitted in the final weeks of the professional learning experience; thus, a low resubmission rate may in part have been impacted by time.



Thirteen percent of teachers completed 20 hours of professional learning and earned a microcredential certificate of completion. Interestingly, no one who engaged in professional learning for only 10 hours earned a microcredential certificate of completion. For those who engaged in less than 10 hours of professional learning, only one teacher also earned a microcredential certificate of completion. Four percent of teachers engaged in 30 hours of professional learning also earned a microcredential certificate of completion.

Analysis is ongoing to examine datalogs to investigate teachers' pathways and engagement with specific resources and tools in the learning modules, as well as other features of the InSTEP platform. This will be shared in future research reports.

### **Impact on Teachers' Expertise to Teach DS&S**

Across multiple sources of data, we examined the InSTEP platform features and opportunities to engage with data investigations, learning modules and microcredentials that were most effective and least effective in supporting teachers' development in their expertise to teach DS&S. Here we summarize themes that emerged.

**Data investigations and technology tools were critical in developing teachers' expertise.** Teachers identified that engaging in data investigations and using technology tools were critical in supporting them in applying what they learned about teaching DS&S to their practice. In an open-ended question in the post-experience survey, teachers were asked if there were any specific activities, frameworks, resources, or supports in the InSTEP professional learning platform that were critical in helping them apply what they learned to their practice. Data investigations were identified most frequently, followed by CODAP and/or other technology tools. Thirty-two percent of respondents indicated that data investigations, many specifically identifying the Roller Coaster Data Investigation, were essential to their learning. Twenty-nine percent indicated that CODAP and 9% identified other technology tools (e.g., Gapminder and Tuva) as being a key factor in their learning. Overall, when specifically asked about the effectiveness of data investigations on the post-implementation survey, 96% of teachers suggested that the data investigations were effective or very effective in supporting their professional learning. In interviews, teachers also indicated that engaging with data investigations and technology tools, CODAP in particular, supported their professional growth.

**Engaging with microcredentials was a positive experience and opportunity to reflect.** For teachers who attempted the microcredential, 65% described the experience as positive in the post-experience survey, while 26% suggested the experience was negative and 9% were neutral. In open-ended responses on this survey, 39% of teachers who attempted a microcredential indicated that doing so provided an opportunity to reflect on their practice, and 17% described the experience as a way to get feedback. In the wireframe and development platform interviews, teachers indicated that it was important for them to get feedback in their engagement with the platform. In the final interviews, over a third of teachers (36%) also indicated that the microcredentials were a tool for reflection on practice and professional growth.

An analysis of the microcredential responses where teachers were asked how they plan to use their understanding of DS&S, as well as statistical habits and dispositions, to support students' learning of DS&S provided insight into impact on their practice. In their reflections, about half the teachers who submitted a microcredential indicated that they planned or were already providing opportunities to make data relevant to their students (54%), use real (46%) and use large data (38%). While only 19% of teachers discussed engaging students in four or more of the

six phases of a Data Investigation Cycle (Lee et al., 2022), 69% reflected on engaging students in the *consider and gather data* phase. Their responses about the importance of making data relevant to students was highly connected to asking students to collect their own data. Over a third (36%), also indicated they planned to or were providing opportunities for students to *explore and visualize data*. Typically, teachers in mathematics classes emphasize the *analysis* of data (Zieffler et al., 2018). There is evidence that these teachers valued exploring and visualizing data and not just providing opportunities to select a model. Over a quarter of teachers (26%) also indicated that they would provide opportunities for students to engage with data analysis or other technology tools. Lastly, in relation to developing dispositions to support students' learning of DS&S, two dispositions emerged as being important to teachers. Thirty percent of teachers reflected that they planned to or were providing opportunities to encourage skepticism, and 30% also wanted to develop students' curiosity.

**Discussion forums and playlists were less effective in supporting professional learning.**

Other features of the InSTEP platform were less effective in supporting teachers' professional learning. In the post-experience survey, 47% of teachers indicated that they never used discussion forums, and only 29% of teachers found them to be effective or highly effective. While teachers valued being able to save resources and tools as described above, the ability to save resources and tools to a playlist was not as highly valued. Fifty-eight percent of teachers never used the ability to save resources and tools to a private playlist, and 64% of teachers never utilized the ability to create a public playlist. Less a third of teachers rated the ability to create private and public playlists as effective or highly effective, 31% and 24%, respectively. This was surprising since during the wireframe and platform development interviews a majority of teachers indicated that they thought saving resources as a playlist would be a feature they would use. However, only a couple of teachers shared this view in the final interview which aligns with the post-implementation survey results.

**Customized recommendations supported personalizing teachers' professional learning.**

Teachers highlighted that completing personalization surveys was an important part of supporting them in their professional growth. In the post-implementation survey, where teachers were asked about the effectiveness of the personalization surveys, 78% indicated they were effective or very effective. In the interviews, half of the teachers described using the personalization surveys as reflection tools to assess their confidence, knowledge and practice and then as a tool to reevaluate where they were at the end of their learning experience.

By our design, the personalization surveys were highly connected to recommendations provided to teachers in the platform. In the post-implementation survey, where teachers were asked about the effectiveness of the recommendations, 82% stated that recommendations were effective or highly effective. Some teachers also indicated that they used recommendations to direct their learning pathway in the platform during their interviews.

**Features related to tracking progress were effective in supporting professional growth.**

Teachers identified several features of the InSTEP platform as being effective. In the post-implementation survey, the following features were rated by teachers as being effective or highly effective: ability to track progress within a module (using a sidebar to track individual resources and tools) (93%), ability to track progress on a Dashboard (89%), ability to earn certificates of completion (for every 10 hours of professional learning) (89%), ability to save resources (76%), and ability to view results from personalization surveys (71%). In final interviews, these features

were not directly addressed by most teachers. However, 33% of those interviewed indicated that they began their professional learning experiences using the Dashboard since they could track what had been completed and what was incomplete. In previous interviews with wireframes of the platform and the development platform, teachers overwhelmingly indicated that the ability to track progress within a module for individual resources and on a Dashboard was helpful.

**Time is an important factor in teachers’ professional learning.** Teachers identified time as a factor that impacted their professional learning experience. In two of the four open-ended questions on the post-experience survey, time generated the greatest number of codes in both instances. In one instance, teachers reflected on what was missing and what supports/resources they wish had been provided. Eleven percent of teachers indicated they needed more time. In the other instance, teachers explained why they did not attempt a microcredential, and 17% of the teachers indicated that they needed more time, while 13% of teachers indicated they needed to resubmit microcredential responses. In focus group interviews, wireframe interviews and development platform interviews, time was often discussed. Teachers frequently pointed out that their time was extremely limited and that having an estimate of time needed to engage with specific resources would help them in selecting resources given any time restrictions.

**Impact on Self-efficacy to Teach DS&S**

We investigated the impact of the InSTEP professional learning experience by examining the mean pre-, post- and gains. Before the professional learning experience, teachers rated their overall confidence to teach statistics as somewhat confident (score of 3.38, see Table 1), and they concluded their experience between confident (score of 4) and very confident (score of 5) in their self-efficacy to teach statistics with an average gain of just over 1 Likert point (1.14 gain). All teachers, except three, improved in their confidence to teach statistics with two teachers improving their average by three Likert points. Using a matched-pairs t test, the gains in confidence between pre- and post- survey are statistically significant.

Similar trends can be seen in all three sub-scale scores. Level A had the highest pre- and post- scores with average confidence starting slightly below teachers being confident (score of 3.89) to teach Level A statistical content and ending the experience with teachers' average confidence growing to just a little below very confident (score of 4.87). Interestingly, level B and level C topics saw similar gains with an increase in 1.22 and 1.17 Likert points of confidence, respectively. Teachers began the professional learning experience slightly more than somewhat confident (3.48) and ending at an average a little below very confident (4.70) for level B content. Level C content, on average, began at a lower confidence level (just below somewhat confident, 2.92) and ended at confident to teach level C statistical ideas (4.09). However, the higher standard deviation for the gains scores at level C indicates more variability in gains by the teachers. All three sublevels had statistically significant gains when tested with a matched pairs t-test (Table 2).

**Table 1: Descriptive Statistics of Likert scale items**

		Pre	Post	Gains
Overall	Mean	3.38	4.51	1.14
	Standard Deviation	1.16	0.99	0.83
Level A	Mean	3.89	4.87	0.98
	Standard Deviation	1.02	0.90	0.90

Level B	Mean	3.48	4.70	1.22
	Standard Deviation	1.14	0.95	0.87
Level C	Mean	2.92	4.09	1.17
	Standard Deviation	1.39	1.26	0.98

**Table 2: Matched Pairs t -Test**

	Test Statistic	p-value
Overall	8.8072	6.589e-11
Level A	6.9611	2.125e-8
Level B	8.9656	4.082e-11
Level C	7.6373	2.468e-9

### Discussion

We found that some portals in the InSTEP platform were more effective in supporting teachers' professional learning related to teaching DS&S than others. Engaging in data investigations, using technology tools and using microcredentials as an opportunity to reflect on practice were the most effective portals in the InSTEP platform. Teachers who reflected on their practice through microcredentials indicated they would provide or were already providing opportunities to engage students with relevant, real, large data using a data investigation process or phases of a data investigation process. This is similar to findings by Lee et al. (2017) and Mojica et al. (2018) where teachers also increased their beliefs and practices related to using real data and investigative approaches, as well as their confidence to teach statistics. In our work, teachers also significantly increased their confidence to teach statistics. Yet, other portals in the InSTEP platform were less effective in supporting learning, such as discussion forums. One surprising finding is that teachers did not value or utilize discussion forums. This is in contrast to other work on teachers' learning, where there is evidence that interactions in discussion forums supported their professional learning (e.g., Park, 2015; Revere & Kovach, 2011).

Our results have several implications for DS&S professional learning and designing DS&S learning environments. Key features that supported online learning that we identified could be used to design other online learning environments. The internal grammar, or the way we structured elements of the InSTEP platform, such as personalized recommendations and tracking features, which teachers found effective, can be applied to other professional learning contexts, whether online or not.

While we were encouraged that a little over half of the teachers engaged with data investigations and resources in learning modules for 10 and 20 hours of professional learning, those teachers were provided with monetary incentives and certificates of completion where they were able to earn professional development hours or Continuing Education Credits (CEUs). Since InSTEP is freely available to the public, we will need to find new ways to encourage engagement in the platform. Time was identified as an important factor in teachers' engagement. With the public launch in March 2023, there are no plans to close the platform. Perhaps this will provide sufficient time for teachers to engage with the platform in ways that fulfill their needs. There is also a concern about ways to increase the number of teachers who engage with

microcredentials since this involved a small subset of teachers who were receiving incentives.

### Acknowledgments

This study was supported by the NSF under Grant No. 1908760 awarded to NC State University. Any opinions, findings, and conclusions or recommendations expressed herein are those of the principal investigators and do not necessarily reflect the views of the NSF.

### References

- Bargagliotti, A., Franklin, C., Arnold, P., Gould, R., Johnson, S., Perez, L., & Spangler, D. (2020). Pre-K-12 Guidelines for assessment and instruction in statistics education II (GAISE II). American Statistical Association and National Council of Teachers of Mathematics. [https://www.amstat.org/asa/files/pdfs/GAISE/GAISEIIPreK-12\\_Full.pdf](https://www.amstat.org/asa/files/pdfs/GAISE/GAISEIIPreK-12_Full.pdf)
- Ben-Zvi, D., Gravemeijer, K., & Ainley, J. (2018). Design of statistics learning environments. In D. Ben-Zvi, K. Makar, & J. Garfield (Eds.), *International handbook of research in statistics education* (pp. 473–502). Springer. [https://doi.org/10.1007/978-3-319-66195-7\\_16](https://doi.org/10.1007/978-3-319-66195-7_16)
- DeCuir-Gunby, J. T., Marshall, P. L., & McCulloch, A. W. (2010). Developing and using a codebook for the analysis of interview data: An example from a professional development research InSTEP. *Field Methods*, 23(2), 136-155. <https://doi.org/10.1177%2F1525822X10388468>
- Finzer, W. (2013). The data science education dilemma. *Technology Innovations in Statistics Education* 7(2), 1-9.
- Franklin, C., Kader, G., Mewborn, D., Moreno, J., Peck, R., Perry, M., & Scheaffer, R. (2007). Guidelines for assessment and instruction in statistics education (GAISE) Report: A Pre-K-12 curriculum framework. American Statistical Association. [https://www.amstat.org/asa/files/pdfs/GAISE/GAISEPreK-12\\_Full.pdf](https://www.amstat.org/asa/files/pdfs/GAISE/GAISEPreK-12_Full.pdf)
- Gee, J. P. (2005). Semiotic social spaces and affinity spaces. In D. Barton, & K. Tusting (Eds.), *Beyond communities of practice* (pp. 214–232). Cambridge University Press.
- Glaser, B. J. & Strauss, A. (1967). *The discovery of grounded theory: Strategies for qualitative research*. Aldine.
- Groth, R., & Meletiou-Mavrotheris, M. (2018). Research on statistics teachers' cognitive and affective characteristics. In D. Ben-Zvi, K. Makar & J. Garfield (Eds.), *International handbook of research in statistics education* (pp. 327-355). Springer. [https://doi.org/10.1007/978-3-319-66195-7\\_10](https://doi.org/10.1007/978-3-319-66195-7_10)
- Harrell-Williams, L. M., Lovett, J. N., Lee, H. S., Pierce, R. L., Lesser, L. M., & Sorto, M. A. (2019). Validation of scores from the high school version of the self-efficacy to teach statistics instrument using preservice mathematics teachers. *Journal of Psychoeducational Assessment*, 37(2), 194-208. <https://doi-org.prox.lib.ncsu.edu/10.1177/0734282917735151>
- Jones (February 19, 2018). Big Data classes a big hit in California high school. EdSource. <https://edsourse.org/2018/big-data-classes-a-big-hit-in-california-high-schools/593838>
- Justice, N., Zieffler, A., Huberty, M., & delMas, R. (2018). Every rose has it's thorn: Secondary teachers' reasoning about statistical models. *ZDM—The International Journal on Mathematics Education*, 50(7), 1253–1265. <https://doi.org/10.1007/s11858-018-0953-1>
- Kwasny, M. J. (2015). Statistics in K–12: Educators, students, and us: Challenges and resources for K–12 educators and students. *CHANCE*, 28(4), 26-29.
- Lee, H. S., Lovett, J. N., & Mojica, G. (2017). Characterizing impacts of online professional development on teachers' beliefs and perspectives about teaching statistics. In Galindo, E., & Newton, J., (Eds.), *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 407-414). Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators. <https://files.eric.ed.gov/fulltext/ED581323.pdf>
- Lee, H. S., Mojica, G. F., & Lovett, J. N. (2020). Examining how online professional development on teachers' beliefs and perspectives about teaching statistics. *Online Learning*, 24(1), 5-27. <https://doi.org/10.24059/olj.v24i1.1992>
- Lee, H. S., Mojica, G. F., Thrasher, E. P., & Baumgartner, P. (2022). Investigating data like a data scientist: Key practices and processes. *Statistics Education Research Journal*, Special Issue: Research on Data Science Education, 21(2), 3. <https://doi.org/10.52041/serj.v21i2.41>
- Lovett, J. N., & Lee, H. S. (2017). New standards require teaching more statistics: Are preservice secondary mathematics teachers ready?. *Journal of Teacher Education*, 68(3), 299-311. <https://doi.org/10.1177%2F0022487117697918>

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

- Mojica, G. F., Lee, H. S., & Lovett, J. N. (2018). Designing spaces to support teacher learning about teaching statistics. In T. E. Hodges, G. J. Roy, & A. M. Tyminski (Eds.), *Proceedings of the 40th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 410-413). Greenville, SC: University of South Carolina & Clemson University.  
<http://www.pmena.org/pmenaproceedings/PMENA%2040%202018%20Proceedings.pdf>
- National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Author
- National Governors Association Center for Best Practice & Council of Chief State School Officers. (2010). *Common core state standards for mathematics*. Author. <http://www.corestandards.org/Math/>
- Park, J. Y. (2015). Student interactivity and teacher participation: An application of legitimate peripheral participation in higher education online learning environments. *Technology, Pedagogy & Education*, 24(3), 389-406. <https://doi.org/10.1080/1475939X.2014.935743>
- Revere & Kovach, 2011 Revere, L., & Kovach, J. V. (2011). Online technologies for engaged learning: A meaningful synthesis for educators. *Quarterly Review of Distance Education*, 12(2), 113-124.
- Strauss, A. L. & Corbin, J. (1990). *The basics of qualitative analysis: Grounded theory procedures and techniques*. Sage.
- Zieffler, A., Garfield, J., & Fry, E. (2018). What is statistics education? In D. Ben-Zvi, K., Makar, & J. Garfield (Eds.), *International handbook of research in statistics education* (pp. 37-70). Springer.  
[https://doi.org/10.1007/978-3-319-66195-7\\_2](https://doi.org/10.1007/978-3-319-66195-7_2)

# INVESTIGATING PROBABILISTIC CONTEXTS USING COMPUTER SIMULATIONS

Karoline Smucker  
Eastern Oregon University  
ksmucker@eou.edu

Azita Manouchehri  
The Ohio State University  
manouchehri.1@osu.edu

*Simulations are considered essential instructional tools in teaching probability and statistics, with computer simulations in particular receiving emphasis in recent years. Despite this, little research has been reported on how learners use computer simulations to solve problems and investigate stochastic contexts. Clinical interviews with four preservice secondary mathematics teachers investigated this issue as they worked with a series of probabilistic tasks in CODAP. This report presents findings based on participants' performance on one of the tasks, which involved making predictions about a school board election based on simulated polling results.*

Keywords: Probability, Data Analysis and Statistics, Technology, Problem Solving

Simulations are important tools in probability and statistics, with research suggesting that both physical and virtual tools can support student learning in these areas (Konold & Kazak, 2008; Rossman & Chance, 2014; Watson & Chance, 2012). There seems to be agreement that computer simulations in particular can play a key role, allowing for large quantities of data to be collected and organized quickly (Chance et al., 2007; Garfield et al., 2015). Several studies have looked at how learners use specific simulation tools to solve problems (Konold et al., 2011; Lee et al., 2010; Stohl & Tarr, 2002), suggesting that decisions about data collection and organization impact learners' conclusions. It has also been reported that task context influences simulation use; for example Nilsson (2014) hypothesizes that probabilistic experimentation varies based on whether the context involves a known or unknown distribution. As these kinds of simulations become more accessible, it is important to understand the specific practices learners may use in order to support their development of statistical reasoning (Ben-Zvi & Garfield, 2004).

This study investigated preservice teachers' interactions with tasks that relied on computer simulations created in the Common Online Data Analysis Platform (CODAP), free online data analysis software. Using a clinical interview methodology (Goldin, 2000), four preservice secondary teachers engaged with seven tasks to develop a model of their problem-solving practices using the simulation environment. This paper will focus on their approaches to one of the tasks to respond to the following research question: how are computer simulations used as a tool for problem-solving in stochastic contexts involving unknown probability distributions?

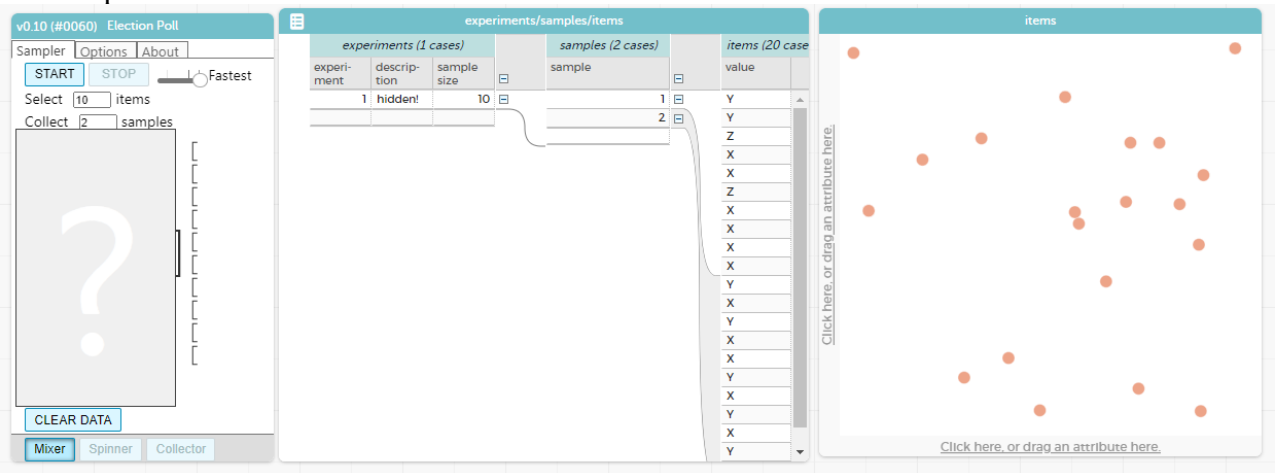
## Theoretical Perspectives

Research suggests that computer simulations can facilitate the problem solving process in probabilistic contexts (Chance et al., 2007; English & Watson, 2016; Saldanha, 2016; van Dijke-Droogers et al., 2020), serving as tools that mediate learning (Kazak et al., 2015; Vygotsky, 1987). It is also acknowledged that a variety of factors may impact simulation use, including the task context and affordances of the technology (Bumbacher et al., 2018; Konold et al., 2011; Lee et al., 2010). In conceptualizing how technology may assist in learners' stochastic problem solving process, the GAISE II Framework for statistical inquiry proposes a four-stage process for data investigations: formulating questions, collecting/considering data, analyzing data, and interpreting results (Bargagliotti et al., 2020). Based on empirically grounded models of statistical inquiry (i.e. Wild & Pfannkuch, 1999), movement through the stages of GAISE II will vary depending on what occurs during an investigation. Since technology may be utilized as a

tool in statistical inquiry, it is important for teachers and researchers to consider how learners use computer simulation environments to collect, organize, analyze, and interpret their data.

### Methods

Participants included four preservice mathematics teachers at a secondary teacher preparation program at a large university in a Midwestern state who had just completed their student teaching. A series of five semi-structured clinical interviews were conducted with each of the participants during summer 2021 (virtual via Zoom). The goal of the interviews was to capture how participants approached seven tasks involving simulations designed in CODAP. Clinical (or task-based) interviews are an effective method to investigate how humans think about problems in the context of specific tasks (Goldin, 2000). Data consisted of video recordings of the Zoom sessions, along with pictures of participants’ work and screenshots of their actions in CODAP. Analysis was completed consecutively for each participant, using the GAISE II framework as a starting point in terms of identifying the potential stages of their work (Bargagliotti et al., 2020). An analysis table was created documenting the participants’ activities, outlining their approaches and themes prominent in their interactions with the simulations.



**Figure 1: Election Prediction task environment**

This report will consider the work of the four participants (Alex, Bridget, Erik, and Diana, all pseudonyms) on one task. The context (see Figure 1) included a “?” sampler in CODAP (underlying distribution hidden), which represented voters for a school board election. The distribution represented 48%, 44%, and 8% support for Candidates X, Y, and Z, respectively. Participants were asked to determine what they could conclude about the election using the sampler, which allowed them to collect data on voter preferences for both individual voters (“items”) and groups of voters (“samples”). The graph could be organized by placing attributes onto the axes and had options for adding values like counts and percentages.

### Results

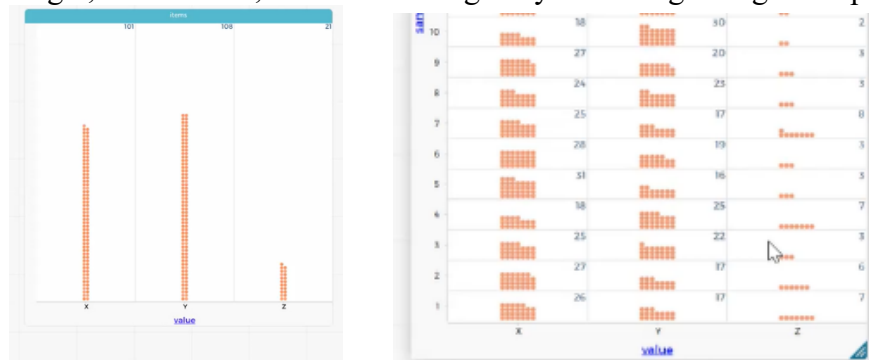
Participants began by collecting data to explore/organize the environment, then used an iterative approach to data collection and analysis. They organized and evaluated their data in different ways. These diverse approaches impacted not only the conclusions they made but also their confidence in them. These themes will be elaborated on in the following sections.

#### Exploration/Organization

A key stage of investigation of the Election Prediction Task involved exploring the context through initial testing with a small number of trials. This occurred for all participants and was typical of their work throughout the interviews. Erik began by collecting 10 trials, “just to get an



idea” of the sampler’s output. Bridget also began with 10 voters, saying, “I don't want to do too many right now, because I feel like that would be overwhelming”. Alex “play[ed]” with the sampler, collecting 1, then 4 trials, while Diana began by collecting a single sample of size 50.



**Figure 2: Graphs by Bridget (a, left) and Diana (b, right)**

Participants considered how to best organize the data after their initial trials. For example, Bridget decided to create a graph showing candidate frequencies (with counts, Figure 2a), saying, “I kind of want to organize this graph over here to count the different values”. Alex and Erik created graphs which were similar to Bridget’s, with Alex adding percentages to his. Diana decided that she would organize her data by both candidate frequencies and sample, adding these attributes to her graph axes before collecting additional samples of the same size (Figure 2b).

### **Iterative sampling and analysis**

Another feature of the participants’ work was the iterative approach they used towards sampling and analysis. They collected data in stages, adding new data to what they had previously gathered. This often started with a smaller number of trials, then increased as they added organizational tools like counts and percentages to their graphs. After collecting 3 samples of size 10, Bridget decided to switch to samples of size 50, saying, “So I think now, I might want to start upping the sample size”. Similarly, after collecting 10, then 40 trials, Erik decided to collect an additional 150 (200 total).

Participants added their new data to their previous samples as they considered the potential election results, but their interpretations of this varied. Bridget and Eric sought “consistency” in results as additional data were added. They used their graphs as a validation tool, for example, Eric noted that the shape of his graph had remained the same between 50 and 200 total trials and concluded that he might not need to collect any more data because of this. Bridget considered the “leading” candidate at various stages, saying that “I would want to see like the same thing happening multiple times” to be confident in any conclusions.

At times Bridget seemed unsure of how to interpret her overall data since it had been collected over several iterations – should it be viewed as a single large sample, or interpreted as a series of smaller samples? After 230 trials (3 sets of 10, 4 sets of 50, see Figure 2a) Bridget considered whether X or Y was “favored” to win the election. She noted that while Y was in the lead overall, she recalled several samples (of size 10) where X had been more frequent. This discrepancy was one reason it was difficult for her to decide if X or Y was actually “favored”.

This contrasts with Diana, whose graph displaying frequencies for samples of size 50 (Figure 2b) emphasized the results of each sample rather than showcasing the overall frequencies. She considered which candidate had been the “winner” in each sample and used this to quantify her beliefs about the likelihood that each candidate would win the election.

Alex, on the other hand, had added percentages to his display and focused on the overall results in his analysis. As he observed changes in the percentages with additional data (considering 225, then 425, then 1000 total trials), Alex believed that his poll results were “approaching” the true preferences of voters. He stated that the percentages appeared to be changing less with additional data, which meant his results were becoming more accurate.

### **Conclusions/Confidence**

The ways that the participants collected, and organized data impacted their conclusions as well as their confidence in their accuracy. As noted previously, Erik and Bridget relied on observing consistent results. Bridget felt quite confident that Candidate Z would lose based on Z being consistently the candidate with the lowest frequency. However, she felt she could not decide between Candidates X and Y since the results had been gone back and forth as she “watched” the polling data populate her graph. Erik observed the consistent shape of his graph as data were added; this prompted him to conclude Candidate Y would likely win after 200 trials.

Alex gained confidence with increased data. As he considered his final round of data collection (1000 total trials), he noted that even with additional data the percentages (47% X, 45% Y, 8% Z) were unlikely to “change much”, anticipating up to a 0.5% adjustment. Based on this, he concluded that it was likely that Candidate X would be the winner of the election.

Diana defined a calculation for her confidence in certain candidates winning the election. Because she had collected samples of the same size and organized her data to show the results of each, she used this information to gauge her confidence. According to her, since Candidate Z was never the winner (lost in 21/21 samples), she concluded that she was 100% confident that Candidate Z would lose. Candidate X had “won” in  $15/21 = 71.4\%$  of her trials, so she concluded that she was around 70% confident that Candidate X would win. Her choice to display her data broken down by sample served as a key feature of Diana’s justification of her conclusions.

### **Discussion**

This study provides insight on the ways that computer simulations may be used in problem solving in stochastic contexts involving an unknown underlying distribution. Individuals began by exploring the context through the collection of a few trials in order to consider how to best organize their data. After this, movement between data collection, analysis, and interpretation (Bargagliotti et al., 2020) was fluid as they iteratively added data to their displays. Participants differed in terms of how they interpreted their additional data. Their conclusions and confidence were impacted by both their choices in terms of organization/analysis (i.e. deciding to graph frequencies for each sample or a reliance on percentages) and their interpretation of additional data as it was added to their displays (i.e. focusing on graph consistency or reasoning that results were becoming more accurate with increased data).

In addition, findings suggest that learners may not harness the power of computer simulations to efficiently collect and organize large amounts of data without support and guidance, a key affordance of these environments (Chance et al., 2007). Even with CODAP’s ability to collect significantly more data, several hundred trials were considered to be sufficient by some participants, leading to inaccurate or ambiguous conclusions.

This paper considers the work of only four individuals on a single task, limiting its scope. However, analysis of the other tasks within the larger study has shown some similar tendencies, in particular the iterative approach to data collection and analysis shown here. Future work will compare participants’ work across tasks and consider the impact of the task context on their approaches. In addition, future testing of the tasks with different groups (for example secondary students) would provide insight on the extent to which these conclusions apply more broadly.

## References

- Bargagliotti, A., Franklin, C., Arnold, P., Johnson, S., Perez, L., & Spangler, D. A. (2020). Pre-K-12 guidelines for assessment and instruction in statistics education II: (GAISE II): A framework for statistics and data science education. American Statistical Association.
- Ben-Zvi, D., & Garfield, J. (Eds.). (2004). *The Challenge of Developing Statistical Literacy, Reasoning, and Thinking*. Springer Science and Business Media.
- Bumbacher, E., Salehi, S., Wieman, C., & Blikstein, P. (2018). Tools for science inquiry learning: Tool affordances, experimentation strategies, and conceptual understanding. *Journal of Science Education and Technology*, 27(3), 215–235. <https://doi.org/10.1007/s10956-017-9719-8>
- Chance, B., Ben-Zvi, D., Garfield, J., & Medina, E. (2007). The role of technology in improving student learning of statistics. *Technology Innovations in Statistics Education*, 1(1), 1–24. <http://dx.doi.org/10.5070/T511000026>
- English, L. D., & Watson, J. M. (2016). Development of probabilistic understanding in fourth grade. *Journal for Research in Mathematics Education*, 47(1), 28. <https://doi.org/10.5951/jresmetheduc.47.1.0028>
- Garfield, J., Le, L., Zieffler, A., & Ben-Zvi, D. (2015). Developing students' reasoning about samples and sampling variability as a path to expert statistical thinking. *Educational Studies in Mathematics*, 88(3), 327–342. <https://doi.org/10.1007/s10649-014-9541-7>
- Goldin, G. A. (2000). A scientific perspective on structured, task-based interviews in mathematics education research. In *Handbook of research design in mathematics and science education* (pp. 517–545).
- Kazak, S., Wegerif, R., & Fujita, T. (2015). Combining scaffolding for content and scaffolding for dialogue to support conceptual breakthroughs in understanding probability. *ZDM*, 47(7), 1269–1283. <https://doi.org/10.1007/s11858-015-0720-5>
- Konold, C., & Kazak, S. (2008). Reconnecting data and chance. *Technology Innovations in Statistics Education*, 2(1).
- Konold, C., Madden, S., Pollatsek, A., Pfannkuch, M., Wild, C. J., Ziedins, I., Finzer, W., Horton, N. J., & Kazak, S. (2011). Conceptual challenges in coordinating theoretical and data-centered estimates of probability. *Mathematical Thinking and Learning*, 13(1–2), 68–86. <https://doi.org/10.1080/10986065.2011.538299>
- Lee, H. S., Angotti, R. L., & Tarr, J. E. (2010). Making comparisons between observed data and expected outcomes: Students' informal hypothesis testing with probability simulation tools. *Statistics Education Research Journal*, 9(1), 68–96.
- Nilsson, P. (2014). Experimentation in probability teaching and learning. In E. J. Chernoff & B. Sriraman (Eds.), *Probabilistic Thinking* (pp. 509–532). Springer.
- Rossmann, A. J., & Chance, B. L. (2014). Using simulation-based inference for learning introductory statistics: Using simulation-based inference. *Wiley Interdisciplinary Reviews: Computational Statistics*, 6(4), 211–221. <https://doi.org/10.1002/wics.1302>
- Saldanha, L. A. (2016). Conceptual issues in quantifying unusualness and conceiving stochastic experiments: Insights from students' experiences in designing sampling simulations. *Statistics Education Research Journal*, 15(2), 81–105.
- Stohl, H., & Tarr, J. E. (2002). Developing notions of inference using probability simulation tools. *Journal of Mathematical Behavior*, 21, 319–337.
- van Dijke-Droogers, M., Drijvers, P., & Bakker, A. (2020). Repeated sampling with a black box to make informal statistical inference accessible. *Mathematical Thinking and Learning*, 22(2), 116–138. <https://doi.org/10.1080/10986065.2019.1617025>
- Vygotsky, L. S. (1987). *Thinking and Speech*. In R. W. Rieber & Carton, A. S. (Eds.), *The Collected Works of L. S. Vygotsky* (Vol. 1). Plenum Press.
- Watson, J. M., & Chance, B. (2012). Building intuitions about statistical inference based on resampling. *Australian Senior Mathematics Journal*, 26(1), 6–18.
- Wild, C. J., & Pfannkuch, M. (1999). Statistical thinking in empirical enquiry. *International Statistical Review*, 67(3), 223–248.

# LEVELS OF UNDERSTANDING WHEN SOLVING A DESCRIPTIVE STATISTICS PROBLEM

Abraham E. Guerra  
Universidad de Guadalajara  
abraham.guerra5480@alumnos.udg.mx

Claudia M. Orozco  
Universidad de Guadalajara  
claudia.orozcor@academicos.udg.mx

Alexander Yakhno  
Universidad de Guadalajara  
alexander.yakhno@cupei.udg.mx

Humberto Gutiérrez  
Universidad de Guadalajara  
humberto.gpulido@academicos.udg.mx

Keywords: Instructional Activities and Practices, Data Analysis and Statistics.

## Introduction

There are difficulties in college students about building, understanding, and solving problems in class. Consequently, interpretation of their results is deficient. In attention to the above, a didactic sequence was designed, and it proposes a guided re-invention of statistic concepts based on “realistic” contexts, according to the Realistic Mathematics Education (RME) theory. According to Freudenthal (1991), mathematics should be learned as an activity of mathematizing reality, through compliance with six principles: activity, reality, levels, intertwinement, interactivity, and guidance (Van den Heuvel-Panhuizen & Drijvers, 2020). The aim of this paper is to analyze levels of understanding, that emerge when solving a didactic sequence about descriptive statistics problem. The students must perform statistical analysis using statistical measures that can describe the data sample, such as median, range, standard deviation, etc.

## Methodology

The proposal was applied to a 10-student group of engineers, solved in 3 teams of 2-3 members. The didactic sequence consists of 4 activities, and two sessions of formalization of concepts, one after every two activities. In which the students must solve a problem situation, consists of estimating the time without failure of two motors for heavy machinery samples, and the students must propose the warranty time of each model. The data was collected through tape recorders, worksheets, and observations. The data analysis was performed under the MRE approach, describing compliance with the assumptions and, mainly, the levels of compression.

## Results

We found evidence that the didactic sequence complies with the principles of the RME for learning descriptive statistics. It should be noted that in the first activity, students gave interpretations of the environment and some initial and unstable ideas about the possible solution to the problem, placing them in the situational level; but as they progressed in the activities, statistical concepts about measures of central tendency and dispersion emerged in a natural and unsophisticated language, giving rise to the reference level; finally, after group discussion, students managed to solve the problem, providing the time of guarantee of the engines, using some descriptive statistics, leaving evidence of the general level. The formal level was not achieved with the activities, but in the phases of discussion and formalization of concepts of the didactic sequence.

## References

Freudenthal, H. (1991). *Revisiting Mathematics Education: China Lectures*, Kluwer, Dordrecht, Reidel Publishing.  
Van den Heuvel-Panhuizen, M., & Drijvers, P. (2020). Realistic mathematics education. *Encyclopedia of mathematics education*, 713-717. [https://doi.org/10.1007/978-3-030-15789-0\\_170](https://doi.org/10.1007/978-3-030-15789-0_170)

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.



# NIVELES DE COMPRENSIÓN QUE EMERGEN AL RESOLVER UNA SECUENCIA DIDÁCTICA PARA EL APRENDIZAJE DE ESTADÍSTICA

## UNDERSTANDING LEVELS THAT EMERGE WHEN SOLVING A DIDACTIC SEQUENCE FOR LEARNING STATISTICS

José Luis Pineda Pastor Universidad de Guadalajara jose.pineda4867@alumnos.u dg.mx	Claudia Orozco Rodríguez Universidad de Guadalajara claudia.orozcor@academicos.u dg.mx	Humberto Gutiérrez Pulido Universidad de Guadalajara humberto.gpulido@academicos. udg.mx
---	---	---

Palabras clave: Análisis de Datos y Estadísticas, Metacognición

### Introducción

Los estudiantes pueden tener dificultades en el aprendizaje de la estadística, en ocasiones derivadas de la enseñanza de conceptos y procedimientos estadísticos sin contextualizar, y por medio de la memorización (Altaylar y Kazak, 2021; Çakmak y Durmuş, 2015; Garfield y Ben-Zvi, 2008). La Educación Matemática Realista (EMR) define que la enseñanza de las matemáticas debe estar conectada con la realidad, permanecer cercana y ser relevante para los alumnos (Bressan et al., 2016). Para esto, se propuso la implementación de una secuencia didáctica, diseñada bajo los 6 principios de la EMR, para el aprendizaje de la estadística descriptiva. Estos principios son: actividad, realidad, interacción, interconexión, reinención y nivel (Treffers, 1978). Este último principio cuenta con 4 niveles de comprensión: situacional, referencial, general y formal (Pérez y Vázquez, 2016), utilizados para el análisis de los datos y para el cumplimiento de nuestro objetivo que es analizar la comprensión por el principio de niveles de un grupo de estudiantes de ingeniería ante la resolución de la secuencia didáctica.

### Metodología

La secuencia didáctica se implementó, por equipos, en un grupo de 10 estudiantes de Ingeniería en Logística y Transporte, de una universidad pública en México. Esta consistía en la solución de una situación problema referente a la “inoculación transversal de huitlacoche en Texcoco, Edo. De México y Celaya, Gto.” Los equipos compararon la “producción de huitlacoche en dos localidades” y “dos métodos de inoculación”; así, debían tomar una decisión considerando o no el riesgo. Los análisis fueron hechos utilizando diagrama de puntos, de caja y bigotes, medidas de tendencia central y rango. Los datos se analizaron con base en las características de los modelos de los estudiantes, categorizados en los niveles de comprensión.

### Resultados

Al inicio de la secuencia, las respuestas de los estudiantes solo interpretaron el problema, dando descripciones e interpretaciones utilizando lenguaje natural sin mencionar los conceptos estadísticos involucrados, por lo que su nivel de comprensión se categorizó en el situacional. Posteriormente, los estudiantes presentaron modelos informales (diagramas de caja y bigote, sumatoria, medidas de tendencia central, dispersión, etc.) para representar el problema y obtener la producción total de huitlacoche, mostrando evidencia del nivel referencial. Al avanzar en la resolución de las actividades, se observó que los modelos del nivel anterior, sobre los conceptos estadísticos, fueron utilizados en situaciones similares, dando lugar al nivel general. Aunque el nivel formal no emergió por sí solo en la resolución de las actividades, con la interacción entre pares y la reinención guiada por el docente se logró la comprensión formal de cada concepto. A lo largo de la secuencia se observó el cumplimiento de los principios de la EMR, por ejemplo,

para la reinención guiada, emergieron varios conceptos en un lenguaje natural, mostrando evidencia del principio, pues los estudiantes reutilizaron y adaptaron sus concepciones previas para construir a nuevos conocimientos.

### Referencias

- Altaylar, B., y Kazak, S. (2021). The Effect of Realistic Mathematics Education on Sixth Grade Students' Statistical Thinking. *Acta Didactica Napocensia*, 14(1), 76-90.
- Çakmak, Z., y Durmuş, S. (2015). Determining the concepts and subjects in the area of learning statistics and probability that 6-8th grade math students have difficulties. *Abant İzzet Baysal Üniversitesi Eğitim Fakültesi Dergisi*, 15(2), 27–58.
- Garfield, J., y Ben-Zvi, D. (2008). *Developing students' statistical reasoning*. New York: Springer Science and Business Media B.V.
- Bressan, A., Gallego, M., Pérez, S., y Zolkower, B. (2016). Educación matemática realista bases teóricas. *Educación*, 63, 1-11.
- Pérez, A., y Vásquez, N. (2016). Educación Matemática Realista: un enfoque para desarrollar habilidades de Matematización con alumnos de secundaria.

## SCHEMA REVISION ON MONTY HALL PROBLEM

Yuriko Hoshiya Brown  
Pennsylvania State University  
yhb100@psu.edu

*The importance of statistical and mathematical literacy has been argued for a long time from social justice and equity perspectives. In this study, the famous brain teaser in math literacy, the Monty Hall Problem is used to investigate if schema revision is possible with a minimal intervention. The participants are 55 undergraduate students. Three participants' schema revision was observed during the assessment with a questionnaire focused on reversible thinking, i.e., what would have caused the favorable future outcome. The result could lead to reform in teaching and learning on reversible thinking in counterintuitive statistical and mathematical problems.*

Keywords: cognition, probability, problem solving, mathematical literacy

The importance of statistical and mathematical literacy has been argued for a long time. Statistical literacy is crucial for individuals to be able to read, evaluate, and make informed decisions based on the statistical arguments they are bombarded with on a daily basis (Weiland, 2017).

Monty Hall hosted a game show on American television called “Let’s Make a Deal” from 1963 for nearly 30 years. Contestants on this show were often faced with a dilemma in which they had to decide whether to stick with an initial choice or switch to an alternative. What contestants should do in this situation sparked a heated debate in 1991 among academics as well as lay people. Many researchers (Miller & Sanjurjo, 2019; Hirano et al., 2017; De Neys & Verschueren, 2006; Krauss & Wang, 2003; Tubau & Alonso, 2003; Ichikawa & Shimojo, 1989; Verschueren, 2006; Lucas et al., 2009; Gilovich et al., 1995; Franco-Watkins et al., 2003; D’Ariano et al., 2002) in many disciplines, for example, mathematics, statistics, psychology and social science have studied the problem for the last 30 years. Ichikawa and Shimojo (1989) pointed out the importance of understanding the Monty Hall Problem from the public mathematics/statistics literacy perspective.

The question known today as the Monty Hall Problem (MHP), the Monty Hall Dilemma, or the Three Door Problem is the following:

Suppose you are on a game show, and you are given the choice of three doors. Behind one door is a car and behind the others are goats. Assume that the car is equally likely in any three doors. You pick a door, say, Number 1, and the host, who knows what’s behind the doors, opens another door, say, Number 3, which has a goat. He then says to you, “Do you want to switch to door Number 2?”. Is it to your advantage to switch?

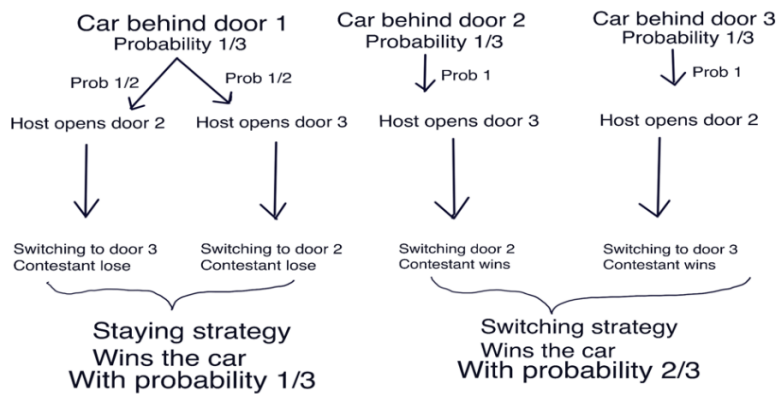
While the intuitively compelling answer is that either of the two remaining doors leads to the same chance of winning the car, actually, their chance doubles if the contestant switches from door Number 1 to Number 2. People, even mathematically inclined academics, typically get this problem wrong at first, and furthermore even many mathematicians have expressed disbelief when told the correct answer (Vazsonyi, 1999). It is not only difficult to find the correct solution to the problem, but it is more difficult to make people accept the solution.

Several assumptions are needed to solve this problem.



- A car is equally likely to be placed behind any door at the beginning.
- Monty, the game show host, knows where the car is. Therefore, he always opens the door which has a goat after the contestant picks one door initially.
- If the contestant initially picks the door which has a car, the host randomly chooses one of the two remaining doors.

There are two main approaches to understand the MHP. The first one is to list every possible outcome if the contestant initially picks door Number 1. There are exactly 4 scenarios. Out of which the probability of winning by switching is 2/3 (Figure 1).



Possible outcomes of contestant choosing door 1 initially

**Figure 1: All 4 possible outcomes of contestant choosing door Number 1 initially**

The other solution is calculating the conditional probability by Bayes' Theorem. Many studies from many perspectives have been done on MHP to examine the dilemma/counterintuitive-ness and look for ways to help people understand the correct solution. However, almost no study has been done in mathematics education community on MHP and no study has focused on *reversible thinking*, which means focusing on what would have caused the desirable outcome in the future. In this study, Cognitive Schema Theory (CST) is used to identify specific cognitive mechanisms that underlie schema construction and revision during working on MHP. What is very interesting, and fascinating is that most people, including many professional mathematicians, could not accept the correct solution of MHP even after thinking it over carefully for a while. This study intends to examine students' schema revision on MHP. The participants of this study are 55 undergraduate students. The research question is the following:

- Can we observe participants' schema revision (changing their opinions) when they are given a questionnaire focused on *reversible thinking*, without training or explaining the answer?

This question will contribute to pedagogical reform on how to help students solve counterintuitive problems in mathematics as well as in public statistical literacy. Many studies were conducted from psychological perspectives on why people resist switching to the alternative, and why we find correct solutions counterintuitive on the MHP. Also, much research

was done on people's performance after providing explanations or training on the MHP game, however, no study has been done on people's performance by aiming to trigger their appropriate cognitive field.

### Literature Review

Why does the correct solution of the MHP feel counterintuitive? The MHP became very famous and controversial for the column in *Parade* magazine of Marilyn Vos Savant. In three of her weekly columns, Vos Savant (1990a, 1990b, 1991) explained the correct solution of the MHP, and she received numerous passionate responses in disagreement. Granberg and Brown (1995) conducted experimental study on MHP. In the study, they asked participants if they would stick to door number 1 or switch to door number 2. The study found that 13% indicated to switch to door number 2. People's tendency to stick to their initial pick has been reported despite the fact they believe the chance is 50/50 to win the car either for stick or switch. Granberg and Brown (1995) also found that participants would feel worse if they switched from a door with the car behind it than if they stick to a door with a goat behind it. This may be due to the status quo bias which is an emotional bias of people's preference to stick with the choice of door they have initially made. Gilovich et al. (1995) asked if people reduce dissonance more for their errors of commission than their errors of omission, and argued that all other things being equal, people prefer to make errors through inaction (staying) as opposed to action (switching).

Some studies (Franco-Watkins et al., 2003; Hirano et al., 2016; Tubau & Alonso, 2013; Granberg & Brown, 1995) conducted experiments using computer programs or card games to let participants play the game or watch the simulation numerous times to experience the odd in the MHP. Result of such experiments increased the proportion of switching responses but did not change erroneous intuitions or did not increase understanding of the mathematical reasons why switching (to door number 2) increase their chance of winning the car. This suggests that there is dissociation between implicit knowledge gained from the task and explicit understanding as to why switching was the best strategy (Franco-Watkins et al., 2003).

De Neys and Verschueren (2006) presented an empirical test of the involvement of working memory resources in MHP reasoning. Working memory component is a system of human cognitive capacity, and they found that participants who gave the correct switching response had a significantly large working memory capacity than participants who reasoned erroneously.

Can people gain an understanding of the probabilistic structure of the MHP? Is it possible to develop appropriate ways to present and explain the MHP such as to eliminate the typical resistance to the switch decision? Krauss and Wang (2003) identified correct reasonings to solve the MHP and conducted a study to answer these questions. In the study, they provided the Bayes's formula sheet, pictures of all possible scenarios of locations of a prize in the MHP game, a chart of mental models based on the frequency simulation and wrote the problem from the perspective of the game show host, to train the participants to evaluate their learning. They reported that before training, none of the 110 participants solved the MHP with a correct justification for the switch choice. However, after giving them training, they found that the frequency simulation and perspective change training only improve participants' performance if presented in combination. The result was that 10~46% of the participants improved their performance on related or similar problems.

Tubau and Alonso (2003) also conducted a study on how to improve people's performance on the MHP. They investigated the effects of practicing the simulation game, making explicitly describing the structure, or enhancing the representation of the different possibilities of the MHP, on reaching and stating the correct answer. They found that representing the MHP as a game

between two adversaries (decision maker and informant perspectives) significantly improved the participants' reasoning, that is 25% of them stated a correct reasoning. Moreover, by showing the participants the explicit all possible outcomes of the game with the adversary perspective improved the participants' performance more significantly, to 60%. They noted that the direct game practicing of the MHP simulation did not improve their performance on reasoning a correct answer. Therefore, the literatures confirmed that it is possible, although it is still difficult, to learn and understand a correct reasoning to solve the MHP even with training and showing the explicit all possible outcomes.

### **Intuition**

The commonly accepted definition of intuition is the ability to understand something immediately, self-evidently without the need for conscious reasoning. Fischbein (1999) claimed that intuitions are very sensitive to the influences of a representation of the context, especially because no logical, formal support intervenes. Moreover, intuitions are very resistant to change because intuitions are related to well-structured systems of our cognitive-behavioral, adaptive activity. This implies that intuitions change together with the entire adaptive system to which they belong, namely cognitive field currently activated. Intuitions are profoundly related to cognitive field. Intuitions are the cognitive counterpart of some structural schemata, and they may be manipulated by non-adequate schemata (Fischbein, 1999). Since for most of us, the correct solutions of the MHP feel counterintuitive, with his assumptions, it may be that the original presentation of the MHP triggers non-adequate schema, a cognitive field, for most of us. Then, is it possible for students to activate a propriate cognitive field to understand the probability structure of the MHP so that they can solve it intuitively, without training or without showing all the explicit outcomes, even for those who think that there is 50/50 chance in winning the car by staying?

### **Theoretical Framework: Cognitive Schema Theory (CST)**

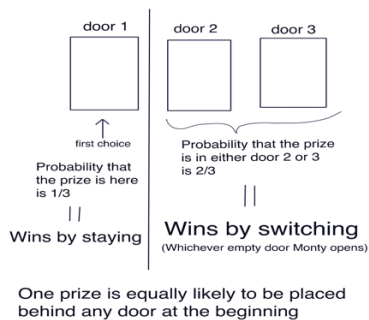
Derry (1996) explained CST as an important theoretical perspective that has significant potential for building conceptual bridges among cognitive constructivism viewpoints. The purpose of CST is to identify specific cognitive mechanism that underlie schema construction and revision. A schema is a program which enables the individual to a) record, process, control and mentally integrate information, and b) to react meaningfully and efficiently to the environmental stimuli (Fischbein, 1999). From this definition, one may see that a schema is similar to a computer software, for example, schema construction is considered as download a programming software. The theory assumes that long-term memories store previously learned schemas and working memories represent a person's span of immediate attention. Cognitive activities, like thinking and learning, take place in working memory, where prior knowledge schemas are activated in response to environmental input, providing context for interpreting experience and assimilating new knowledge (Derry, 1996). According to CST, learning involves constructing three types of schemas that interact during the learning process. They are *memory objects*, *mental models*, and *cognitive fields*. Memory-object represents the permanent results of learning stored in memory and thus constitute the collection of all knowledge and preconceptions that one might use to interpret any situation. For example, "the total probability of a set of related events sums up to one, i.e., 1/2: 1/2, or 1/3: 2/3." is a memory object. Mental modeling can be viewed as a process of constructing a mental representation of a situation. For example, a process of thinking that 'the probability of one box contains a prize is  $\frac{1}{2}$  if there are two boxes and one prize is randomly placed in one of them' is a mental model. The goal of mental modeling is to form an understanding of a phenomenon. Cognitive field is a mental program which activates

relevant preconceptions in response to a particular event, such as a problem posed, that makes certain memory objects more available for use than others. Cognitive field mediates experience and learning. Starting to think a posed problem triggers activation of a cognitive field, then it set out the memory objects which are readily available for modeling the problem. The cognitive field determines what interpretations and understandings of the problem are probable. The MHP asks us if we want to stay or switch. This wording may trigger a cognitive field of choosing one out of two, namely, the 50/50 schema. Therefore, we could say, an inappropriate cognitive field gets activated when the MHP is posed for the first time for most people.

Radical constructivists believe that students must actively reflect on the fit of new information or experience to currently activated logical schema in order to update or revise their cognitive structure. On the contrary, CST says that schema revision does not always require deep reflection, direct contradiction to prior beliefs, or feelings of disequilibrium. Schema revision requires the activation of relevant appropriate cognitive fields with introduction of useful, understandable information at critical times during the model construction process. Therefore, it may be possible for students to activate an appropriate cognitive field for MHP by providing them useful, understandable instruction. Moreover, it may be possible to promote schema revision of students who thought that the remaining doors have the same chances of winning the car by re-presenting the problem in a way to trigger appropriate cognitive field.

### Method

So far, the most successful mental model to increase people’s understanding of the mathematical structure of the MHP was the model used by Krauss and Wang (2003). They formulated the MHP using manipulations in natural frequencies and perspective change. They asked the participants guiding frequency questions to motivate them to construct the mental model. The 1/3: 2/3 model used in this study was modified from their model (Figure 2). The objective of this study is to see if the participants who first triggered inappropriate cognitive field could activate the 1/3: 2/3 mental model by a questionnaire focused on reversible thinking. The participants of this study are 55 undergraduate students in precalculus and in business calculus. They are first-year students. Three written assessments are provided and given to the students each consecutive week for 3 weeks as bonus quizzes. The first assessment includes the original MHP and some extra questions to evaluate student’s memory objects on probability. This assessment is to evaluate if they have basic probability schema. Also, the students are asked if they knew of a similar problem already. Students are asked to explain (in writing) their answers.



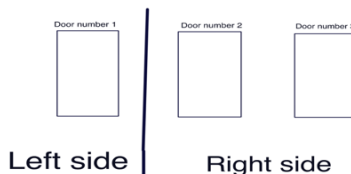
**Figure 2: The 1/3: 2/3 Mental Model**

## Second Assessments

The second assessment is the following: Instead of a car, a diamond is used.

1. What is the probability that door number 1 has the diamond?
2. What is the probability that the contestant didn't choose the door with the diamond?

Let's imagine a line between door number 1 and the other doors. Now there are two sections, the right side has two doors (2 & 3) and the left side has door 1. (Figure 3)



**Figure 3: Three doors with a line**

3. If YOU can bet on the two doors on the right side (2 & 3) for the diamond, or one door (number 1) on the left side, which would you choose? Circle one.  
Right side   Left side   It doesn't matter   I don't know
4. What is the probability that the diamond is on the right side?
5. What is the probability that the diamond is in one of the doors, number 2 or 3?  
At least, one of the door number 2 or 3 is empty, right? (Because there is only one diamond)

Then, the host reveals that door number 3 is empty.

6. Now the host gives the contestant another chance. Which door should the contestant choose?
7. Based on your answer in 6, suppose the contestant followed your advice, what is the probability that the contestant wins the prize?
8. Please explain your answer.

## Third Assessment

The third assessment is the following:

Suppose you won the diamond by switching to door number 2. Do you agree/disagree with the following statements: Circle one.

1. This means that you picked an empty door at first. Agree, Disagree, Don't know
2. Winning by switching = Picking an empty door at first. Agree, Disagree, Don't know
3. The probability of winning by switching = The probability of picking an empty door at first. Agree, Disagree, Don't know
4. What is the probability of picking an empty door at first?
5. What is the probability of winning the diamond by switching?

Finally, let me ask again. (The original MHP is asked again here) Would you switch to door number 2? Please explain your answer.

## Analysis

The first assessment determines if a participant knows the problem already and knows the correct answer (to switch). Also, it determines if a participant has basic probability memory object, which means that the probability of door number 1 containing the prize is  $1/3$  if the prize is randomly placed, and the probability of door number 1 not containing the prize is  $2/3$ . The second assessment is given to determine if providing Figure 3, the line between door number 1 and 2, helps them launch the correct, or inappropriate cognitive field. The third assessment is to

see if reversible thinking in the ‘winning by switching’ case helps them launch the right cognitive field.

### Results

On the first assessment, out of 55 participants, 4 participants indicated that they knew similar problems and knew that they should switch. 2 participants who didn’t know the MHP still answered both the second and the third assessments correctly. Most of the remaining 49 participants answered problem 1, 2 and 3 on the second assessment correctly. Out of these 49, 17 participants thought the odds were 50/50 incorrectly, and 28 participants correctly, on problem 4 in the second assessment. However, all these 28 participants indicated that the odds were 50/50 after the host revealed door number 3 was empty. One participant answered correctly on both assessments but didn’t explain the reasoning in detail to evaluate further. However, 3 participants indicated that they didn’t know the MHP or similar problems before, they stated that this is a 50/50 situation for the first and second assessments but answered correctly with the correct reasoning on the third assessment.

#### The Three Participants (Table 1)

1. Participant  $\alpha$  stated that they didn’t know the problem, and “it doesn’t matter if you switch or stay because it is a 50/50 situation” on the first assessment. On the second assessment, they answered problem 1 – 5 correctly to indicate that  $\alpha$  thinks that the probability that the prize is in one of the two doors, number 2 or 3 is  $2/3$ , but after the host revealed that door number 3 is empty,  $\alpha$  stated that it is still a 50/50 situation. Then, on the third assessment,  $\alpha$  answered correctly on all the questions.  $\alpha$  changed their mind to “switch” from “it doesn’t matter, it’s 50/50”.  $\alpha$  wrote that “because you first picked a door on a  $1/3$  chance, and then switch (to door number 2) after an empty door is shown, boost switching to the other increase the odds”.
2. Participant  $\beta$  indicated the situation is a 50/50 on the first assessment. On the second assessment,  $\beta$  answered problem 1- 4 correctly. However, on problem 5,  $\beta$  stated that the probability that the prize is in one of doors number 2 or 3, is  $1/2$  and explained that “it is  $1/2$ , or 50/50, because door 3 is empty which leaves 2 doors and a 50% chance to win the diamond”. However, on the third assessment,  $\beta$  answered all questions correctly and explained that the probability of winning the diamond by switching is  $2/3$  “because switching after he (the host) has opened one of the doors actually gives me a higher chance of winning the diamond”.
3. Participant  $\gamma$  first indicated that  $\gamma$  would stay with door number 1 because the host knows where the prize is and hoping to trick the contestant.  $\gamma$  indicated that the probability that the prize is on the right side of the line is  $1/2$  on the second assessment.  $\gamma$  wrote that there is  $1/2$  probability that the diamond is in either door 1 or door 2. However, on the third assessment,  $\gamma$  answered all problem correctly, and wrote that the reason for switching is “there is a  $2/3$  probability of picking an empty door first”.

On the first and second assessments, participants  $\alpha$  and  $\beta$  launched wrong cognitive field of ‘the situation is 50/50’ after the host opened door number 3. Participant  $\gamma$  on the other hand, launched the wrong cognitive field after the line was shown between door number 1 and 2. However, all three participants’ activating the appropriate cognitive field was observed on the third assessment. This may be evidence that the questionnaire in the third assessment helped them launch the correct cognitive field.

**Table 1: Analysis of the three participants  $\alpha$ ,  $\beta$  and  $\gamma$**

	Assessment 1	Assessment 2	Assessment 3

	Memory Object	Cognitive Field	Mental Model	Cognitive Field	Cognitive Field
Participant $\alpha$	50/50 1/3:2/3 confirmed	Activated 50/50 schema	Line between Door 1 and 2&3 1/3:2/3 model	Door 3 revealed empty activated 50/50 schema	Reversible Thinking activated 1/3:2/3 schema
Participant $\beta$	50/50 1/3:2/3 confirmed	Activated 50/50 schema	Line separates right and left 50/50 model	Activated 50/50 schema	Reversible Thinking activated 1/3:2/3 schema
Participant $\gamma$	50/50 1/3:2/3 confirmed	Activated Tricky Host schema	Line separates right and left 50/50 model	Activated 50/50 schema	Reversible Thinking activated 1/3:2/3 schema

### Discussion

The Monty Hall Problem is notorious for being mistaken for a 50/50 situation and people have difficulty accepting the solution even after being shown the correct answer. In terms of Cognitive Schema theory, most of the participants launched an inappropriate cognitive field after the host reveals one empty door, or for some, after the line is introduced to separate 3 doors into two sections, right side, and left side. Either way, the wordings of “switch or stay”, “right side or left side”, or “door 1 or 2” seem to trigger the 50/50 situation schema. In this study, three participants’ schema revision was observed during the third assessment. The third assessment focused on reversible thinking about the ‘winning by switching’ case. The questions were chosen to encourage participants to focus on what would have caused (in the past) the favorable outcome in the future. This perspective may have activated the appropriate cognitive field to these 3 participants to revise their inappropriate schema (50/50 situation) to the correct schema.

This study focused on schema revision with only a written questionnaire on reversible thinking, a minimum intervention. Students’ difficulty and importance in reversible thinking in mathematics has been documented (Tall & Razali, 1993; Hirashima & Kurayama, 2011; Warren, Cooper & Lamb, 2006). However, future research is needed to further evaluate the role of reversible thinking in people’s schema revision. The result of this study could lead to reform in teaching and learning counterintuitive statistical or mathematical problems, and on public statistical literacy.

### References

- D’Ariano, G. M., Gill, R. D., Werner, M. K., Werner, R. F., Kummerer, B., & Maassen, H. (2002). The quantum Monty Hall problem. *Quantum Physics*, <https://doi.org/10.48550/arXiv.quant-ph/0202120>.
- De Neys, W., & Verschueren, N. (2006). Working memory capacity and a notorious brain teaser: The case of the Monty Hall Dilemma. *Experimental Psychology*, 53(2):123-131.
- Derry, S. (1996). Cognitive schema theory in the constructivist debate. *Educational Psychologist*, 31(3/4), 163-174.
- Fischbein, E. (1999). Intuition and schema in mathematical reasoning. *Educational Studies in Mathematics*, 38:11-50.
- Franco-Watkins, A. M., Derks, P. L., & Dougherty, M. R. P. (2003). Reasoning in the Monty Hall problem: Examining choice behavior and probability judgements. *Thinking and Reasoning*, 9(1), 67-90.

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

- Gilovich, T., & Chen, S. (1995). Commission, omission, and dissonance reduction: Coping with regret in the “Monty Hall” problem. *Personality and Social Psychology Bulletin*, Vol. 21, 2:182-190.
- Granberg, D., & Brown, T.A. (1995). The Monty Hall dilemma. *Personality and Social Psychology Bulletin*, 21, 711-723.
- Hirashima, T., & Kurayama, M. (2011). Learning by Problem-Posing for Reverse-Thinking Problems. In: Biswas, G., Bull, S., Kay, J., Mitrovic, A. (eds) *Artificial Intelligence in Education. AIED 2011. Lecture Notes in Computer Science*, vol 6738. Springer, Berlin, Heidelberg. [https://doi.org/10.1007/978-3-642-21869-9\\_18](https://doi.org/10.1007/978-3-642-21869-9_18)
- Hirano, T., Murphy, T. L., & Masaki, H. (2017). Brain activities associated with learning of the Monty Hall Dilemma task. *Psychophysiology*, 2017:54:1359-1369. <https://doi.org/10.1111/psyp.12883>
- Ichikawa, S., & Shimojo, S. (1989). Intuitive reasoning about probability: Theoretical and experimental analyses of the “problem of three prisoners”. *Cognition*, 32, 1:1-24.
- Krauss, S., & Wang, X. T. (2003). The psychology of the Monty Hall Problem: Discovering psychological mechanisms for solving a tenacious brain teaser. *Journal of Experimental Psychology*, 132, 1:3-22.
- Lucas, S., Rosenhouse, J., & Schepler, A. (2009). The Monty Hall Problem, reconsidered. *Mathematics Magazine*, vol. 82, 5:332-342.
- Miller, J. B., & Sanjurjo, A. (2019). A bridge from Monty Hall to the Hot Hand. *The Journal of Economics Perspective*, vol. 33, 3:144-162.
- Tall, D. & Razali, M., R. (1993). Diagnosing Students’ difficulties in Learning Mathematics. *International Journal for Mathematical Education in Science and Technology*, 24(2), 209-222. <https://doi.org/10.1080/0020739930240206>
- Tubau, E., & Alonso, D. (2003). Overcoming illusory inferences in a probabilistic counterintuitive problem: The role of explicit representations. *Memory & Cognition*, 31(4), 596-607.
- Vazsonyi, A. (1999). Which door has the Cadillac? *Decision Line*, 30(1):17-19.
- Vos Savant, M. (1990a). Ask Marilyn. *Parade Magazine*, p.15.
- Vos Savant, M. (1990b). Ask Marilyn. *Parade Magazine*, p.25.
- Vos Savant, M. (1991). Ask Marilyn. *Parade Magazine*, p.12.
- Warren, E. A., Cooper, T. J., & Lamb, J. T. (2006). Investigating Functional Thinking in the Elementary Classroom: Foundations of Early Algebraic Reasoning. *Journal of Mathematical Behavior*, 25, 208-223. <https://doi.org/10.1016/j.jmathb.2006.09.006>
- Weiland, T. (2017). Problematizing Statistical Literacy: An Intersection of Critical and Statistical Literacies. *Educational Studies in Mathematics*, 96:33-47.



## SUPPORTING PRESERVICE SECONDARY MATHEMATICS TEACHERS IN HAVING CRITICAL DATA TALK DISCUSSIONS

Nina G. Bailey  
Montclair State University  
baileyn@montclair.edu

Allison W. McCulloch  
University of North Carolina Charlotte  
allison.mcculloch@charlotte.edu

With calls for critical statistical literacy (CSL) on the rise (e.g., Bargagliotti, 2020; National Governors Association Center for Best Practice & Council of Chief State School Officers, 2010), it is vital that preservice secondary mathematics teachers (PSTs) learn to support their students' development of CSL. Yet, many of these calls were criticized for promoting vague skepticism of statistics which can result in "[exposing] unfairness in a dataset but falls short of seeking accountability for justice" (Rubel et al., 2021, p. 219). Thus, the field needs to research ways to support PSTs develop more than superficial CSL. This is complicated by research that suggests PSTs are not comfortable discussing or teaching social justice topics (e.g., Simic-Muller et al., 2015). This study aims to describe how an introduction to the Critical Statistical Literacy Habits of Mind Framework (CSLHM; Bailey & McCulloch, 2022, 2023) supported PSTs' facilitation of a social justice centered data talk (Boaler et al., 2021).

CSLHM (Bailey & McCulloch, 2022, 2023) are the robust thinking behaviors used to make sense of statistical messages with a focus on how the statistical message is used to uphold or dismantle structures of inequity. The CSLHM framework is a set of six habits of mind with descriptions and guiding questions. Components of the CSLHM include questioning sample size and methods, recognizing appropriate statistics, desiring additional information, acknowledging alternate explanations, recognition of one's own sociopolitical consciousness, and employing active citizenry.

Our study took place in a senior secondary mathematics method course in Fall 2021 with 10 PSTs. PSTs were introduced to the CSLHM framework, participated in a model data talk (Boaler et al., 2021), and then planned and enacted their own data talk. They were asked to select a data representation for their data talk that was of interest from a social justice perspective and to use the CSLHM framework as they planned. Our research questions are as follows: (a) What types of data representations do PSTs select for a data talk? (b) Which CSLHM do PSTs enact as they think through (i.e., plan) their selected data representation? And how robustly? (c) When enacting their data talks, in what ways do PSTs support student development of CSLHM?

Findings show that PSTs selected a wide variety of topics and graph types all relevant to social justice, yet only one PST selected a data representation that included race. While making sense of their data representation, most PSTs enacted several if not all of the CSLHM. Similarly, the PSTs planned questions to support their students' development of at least some CSLHM, and several PSTs incorporated all CSLHM. The most commonly enacted CSLHM was recognizing appropriate statistics and representations. During their enactment of the data talk, PSTs did not enact all CSLHM; however, this was expected as the PSTs only had 15 minutes and used student responses to steer the conversation. Reflections on the data talk project demonstrated that the PSTs saw value in the CSLHM framework. One PST explicitly indicated that they are ready to tackle a more controversial topic after completing this project. Detailed findings and implications for both research and teacher education will be presented.

## References

- Bailey, N. G. (2022, November). Investigating critical statistical literacy habits of mind [Poster session]. 44th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Nashville, TN.
- Bailey, N. G., & McCulloch, A. W. (2023). Describing critical statistical literacy habits of mind. *The Journal of Mathematical Behavior*, 70. <https://doi.org/10.1016/j.jmathb.2023.101063>
- Bargagliotti, A., Franklin, C., Arnold, P., Gould, R., Johnson, S., Perez, L., & Spangler, D. A. (2020). Pre-K-12 guidelines for assessment and instruction in statistics education II: A framework for statistics and data science education. National Council of Teachers of Mathematics. [https://www.amstat.org/asa/files/pdfs/GAISE/GAISEIIPreK-12\\_Full.pdf](https://www.amstat.org/asa/files/pdfs/GAISE/GAISEIIPreK-12_Full.pdf)
- Boaler, J., LaMar, T., & Williams, C. (2021). Making sense of a data-filled world. *Mathematics Teacher: Learning and Teaching PK-12*, 114(7), 508–517. <https://doi.org/10.5951/MTLT.2021.0026>
- National Governors Association Center for Best Practice & Council of Chief State School Officers. (2010). *Common core state standards for mathematics*.
- Rubel, L. H., Peralta, L. M., Herbel-Eisenmann, B., Jiang, S., Kahn, J. B., Lim, V. Y. (2021). Theorizing data science education: An intersectional feminist perspective on data, power, and “playing the game”. In D. Olanoff, K. Johnson, & S. M. Spitzer (Eds.), *Proceedings of the 43rd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 217–221). Philadelphia, PA.
- Simic-Muller, K., Fernandes, A., & Felton-Koestler, M. D. (2015). “I just wouldn't want to get as deep into it”: Preservice teachers' beliefs about the role of controversial topics in mathematics education. *Journal of Urban Mathematics Education*, 8(2). <https://doi.org/10.21423/jume-v8i2a259>

---

# WORKING GROUP PAPERS

---

## AESTHETIC AND AFFECTIVE DIMENSIONS OF MATHEMATICS LEARNING

Leslie Dietiker  
Boston University  
dietiker@bu.edu

Meghan Riling  
Vanderbilt  
University  
meghan.riling@van  
derbilt.edu

V. Rani Satyam  
Virginia  
Commonwealth University  
vrsatyam@vcu.edu

*Students' aesthetic and affective responses are interrelated and both central to mathematics learning. This working group will continue the conversation begun in 2022 to explore the connection between the affective and aesthetic dimensions of mathematics education, and how connecting these dimensions can help to understand how students experience mathematics. The goals of this working group are to evaluate the state of the field, build shared terms, and identify research questions for further inquiry.*

Keywords: Affect, Emotion, Beliefs, and Attitudes; Curriculum; Instructional Activities and Practices

### Background

Here is a summary of the purpose of this working group, from Satyam et al. (2022):

The role of affect in mathematical experiences and learning has become abundantly clear over time (e.g., Cheeseman & Mornane, 2014; DeBellis & Goldin, 2006; Malmivuori, 2001). *Affect* has been defined as all aspects of experience that involve feeling (McLeod, 1988); this ranges from deeply held, long duration constructs such as beliefs, attitudes, math anxiety, and motivation, to shorter-term and in-the-moment feelings such as emotions and engagement (Grootenboer & Marshman, 2016; McLeod, 1992; Middleton et al., 2017). In math, where success versus failure is starkly visible, students' affective responses can be quite strong (Boaler, 2015) and impactful (Grootenboer & Marshman, 2016; Op 't Eynde et al., 2006).

Yet, similar to how we would also examine a piece of art for explanations for an individual's gasp in a museum, we argue that researchers also need to attend to the nature of the mathematical experience for explanations of how it potentially impacted students (e.g., inspiring a question). We refer to the way a lesson supports the felt impulses that compel (or impede) a student to continue to progress (or not) through an experience as its *aesthetic dimensions* (Dietiker, 2015). Some researchers have recently begun to study the aesthetic potential of mathematical learning environments (e.g., Dietiker, 2016; Sinclair, 2001), learning for example how the design of technological tools can offer surprise and appeal (Sinclair et al., 2009). While still emerging, the field is learning how to design and enact what Sinclair (2001) calls "aesthetically-rich" mathematical experiences, which she describes as those that "enable children to wonder, to notice, to imagine alternatives, to appreciate contingencies and to experience pleasure and pride" (p. 26).

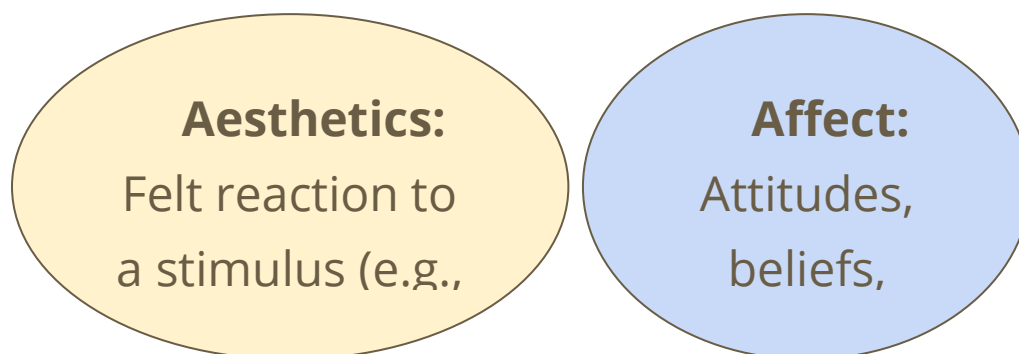
This working group will continue to explore the linkage between affect and aesthetic to tackle multiple dimensions of one of the most significant dilemmas facing mathematics classroom practice today: poor student experiences with mathematics that fuel negative student dispositions. Through our working group design, we plan to bring together graduate students, early career and senior faculty together, to solidify the current state of the field and help form informal and formal connections for future research endeavors.

## History of Working Group at PME-NA 2022

The goals of the first working group were to bring together researchers in order to evaluate the state of the field in regard to aesthetic and affective issues, build community through structured but also informal conversation and sharing each other's work, negotiate shared terms and definitions (given the various constructs for similar ideas), and brainstorm some research questions. We met over three days and had a mixture of university faculty and graduate students in attendance.

### Day 1: An Anchoring Experience

On the first day, we focused on both building community and anchoring our future discussions on aesthetic and affect in a common mathematical experience. The session started with introductions of participants with details about individual research and teaching that related to the working group topic. The group then turned its attention to the major questions: *What is Affect? What are Aesthetics?* Preliminary definitions were introduced: Aesthetics was defined initially as a “felt reaction to a stimulus (e.g., surprise, wonder, or ‘a sense of fit’)”, while affect was defined initially as “attitudes, beliefs, emotions, motivation, engagement, etc.” These definitions were represented in disjointed circles, as shown in Figure 1, to stimulate discussion about questions such as: What is the relationship between the two constructs? Does one belong within the other? Do they overlap? Are they separate?



**Figure 1: Diagram used to stimulate discussion on the relationship between aesthetics and affect.**

Next, with a beginning understanding emerging, the group split into four smaller groups to collaborate on a common mathematical exploration of shapes with a hinged mirror. Using only the prompt, “discuss with your group what you see and share any wonderings,” the four groups spent at least 30 minutes excitedly sharing ideas as they worked together to figure out how adjusting the angle and placement of the hinged mirrors were influencing the resulting geometric designs. Numerous exclamations erupted when sudden results surprised members. In a debrief of the activity, many people recognized combinations of *feelings* such as frustration and joy, along with *aesthetic pulls* such as curiosity about the possibility of certain shapes

Following analyzing the anchoring experience with the hinged mirrors, the whole group then worked to further refine our understanding of aesthetics and affect. Dr. Satyam shared several ways affect has been defined: “Aspects of human experience that involve feeling”, based on McLeod (1988), “The wide range of beliefs, feelings, and moods that are generally regarded going beyond the domain of cognition” (direct quote from McLeod, 1992, p. 576). Dr. Satyam

also introduced categories of the affective domain in mathematics from McLeod (1992): *beliefs* (about mathematics, self, mathematics teaching, and the social context), *attitudes* (e.g., dislike of geometric proof), and *emotions* (e.g., joy or frustration). Interestingly, the emotions category of McLeod’s (1992) framework included “aesthetic responses to mathematics” (p. 578).

Finally, Dr. Dietiker introduced several definitions of aesthetic in order to enable a discussion about similarities and differences in how aesthetic has been conceptualized:

- “Aesthetic understanding is a rich network of conceptual knowledge combined with a deep appreciation for the beauty and power of ideas that literally transform one’s experiences and perceptions of the world” (Girod et al., 2003, p. 578)
- “The way a lesson supports the felt impulses that compel (or impede) a student to continue to progress (or not) through an experience” (Dietiker, 2015)
- “Aesthetically-rich” mathematical experiences: Those that “enable children to wonder, to notice, to imagine alternatives, to appreciate contingencies and to experience pleasure and pride” (Sinclair, 2001, p. 26)

The session ended with an overview of the remaining schedule of the working group.

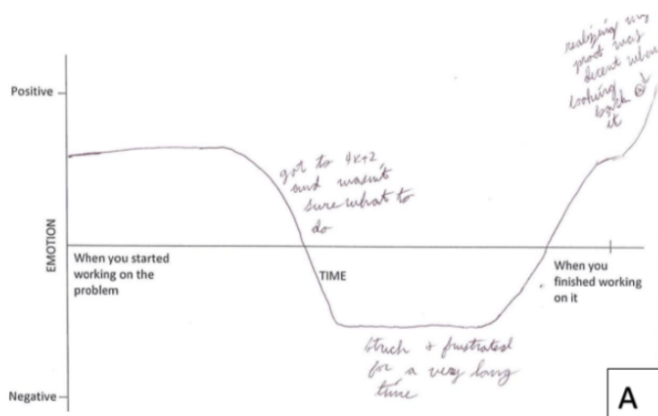
### Day 2: Artifact Stations

On day two, we facilitated extended introductions, in which participants shared their research and teaching interests and explained how they relate to aesthetics and affect. We intentionally wanted this to occur after our participants had an experience together first (the previous day), so that participants would have shared language and experiences when learning about each other’s work and interests.

The main activity of this session was to look at “artifacts”: Data where aesthetics and/or affect was apparent, as well as tools designed to measure these ideas. We had four artifacts in total: (a) a classroom video showing a student’s aha moment pertaining to algebra, (b) a survey for lesson experience, (c) a set of emotion graphs for a proof task (see Figure 2), and (d) a transcript of an interview with a multivariable calculus student. In our introductory survey, we informed participants they could bring an artifact of their own to share with the working group; this last artifact, the transcript, was provided by one of the working group participants.

TASK: A perfect square is any number that can be written as  $n^2$ , for some integer  $n$ . Prove the following statement:

If  $a$  and  $b$  are odd perfect squares, then their sum  $a + b$  is never equal to a perfect square.



**Figure 2: An undergraduate transition-to-proof task and one student's associated *emotion graph*, drawn to reflect their affect throughout engaging in the task.**

We arranged a station for each artifact. Participants rotated through the stations in small groups, with about 10 minutes to examine and discuss each artifact. Provided these guiding questions to spark discussion at the stations:

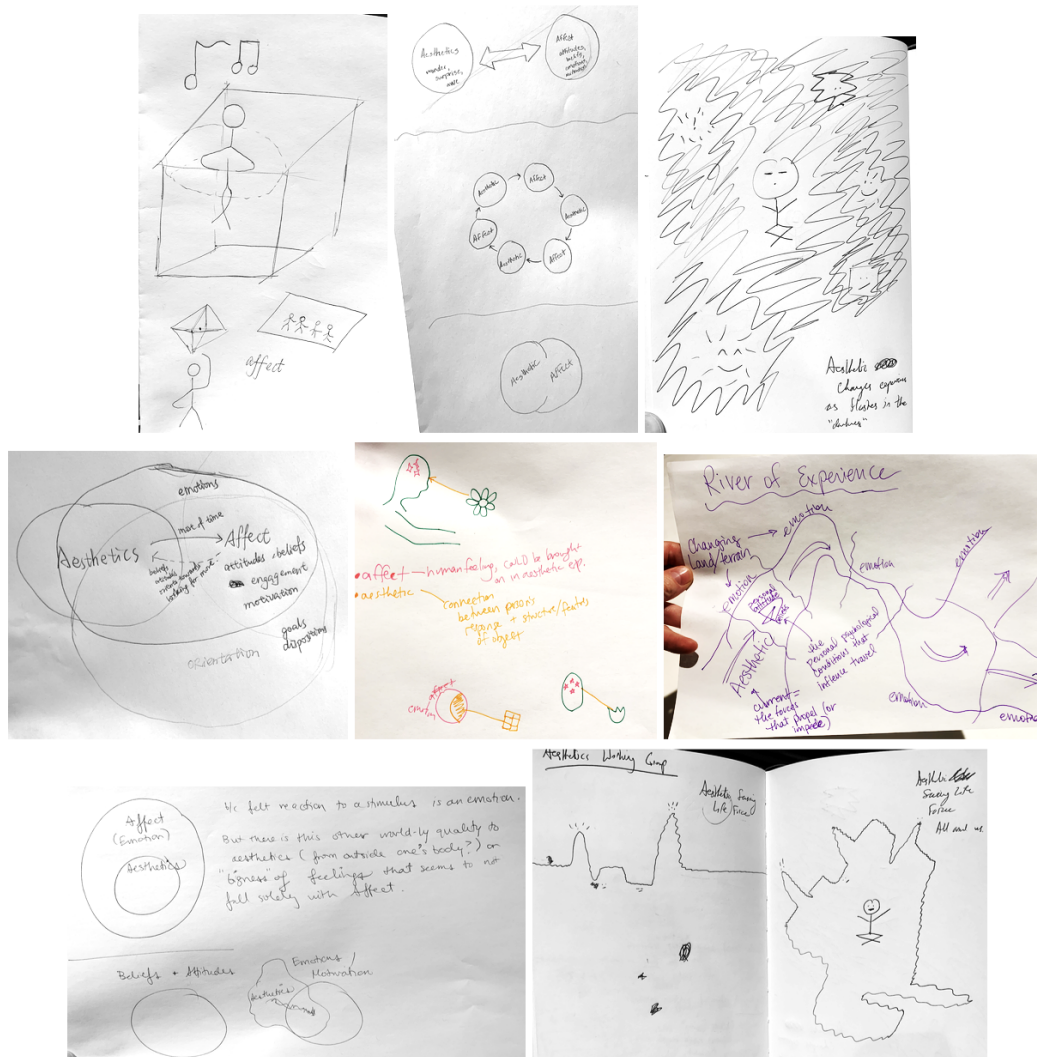
1. Where do you see affect present?
2. Where do you see aesthetics present?
3. How do these two perspectives inform what we know about this artifact?
4. What new questions does this raise?

In the following whole group discussion, participants shared their observations and judgements from the artifacts. Our whole group discussion about the final stations question served as a segue into day three, as we brainstormed questions of interest to the group and directions for future research.

**Day 3: Comparing and Contrasting Conceptualizations**

On day three, we worked to consolidate our earlier brainstormed ideas and set directions for future research and collaboration. We asked participants to create a visual that represented their current understanding of the relationship between affect and aesthetics within mathematics education (see Fig. 3). These visuals were diverse in form. Some were abstract diagrams (e.g., Venn diagram), whereas others were more representational (e.g., a river). All participants had the chance to review all created visuals during a gallery walk. During a subsequent discussion, participants noted several themes that emerged across the visuals, such as the centrality of the human figure, the strong connection between aesthetics and affect, and the temporal nature of both constructs. One proposed, though not universally endorsed, relationship between the two constructs was that aesthetics may be considered as a subset of affect. Other participants

considered the two constructs to both stand on their own, but to influence one another.



**Figure 3: A selection of participants’ visual representations of affect and aesthetics.**

Next, participants working in small groups wrote open questions related to aesthetics and affect in mathematics education. This work was conducted in shared online documents, in order to enable future potential collaboration. Participants’ questions included both those that were theoretical (e.g., “How are these (affect and aesthetic) impacted by humanistic vs cognitive perspectives? How is that divide explaining how these constructs emerged historically but does this divide simplify what we can see or do attend to?”) and empirical (e.g., “We wondered how teachers may also impact and experience aesthetic - and how they may learn to address aesthetic concerns across their career?”) in nature.

Finally, we shared two tools designed to support ongoing research and collaboration: a spreadsheet with participants’ contact information, and a shared folder for participants to share



their work related to affect and aesthetics.

### Next Steps

This working group addressed the continued need for focus on students' mathematical dispositions, and the potential enabled by identifying the shared interests in aesthetics and affect. Our aim is to enable new solutions to a persistent problem that can become possible by bringing together these two domains. We plan to continue this working group next year, expanding our focus to useful tools and methodologies.

### Session Organization

Prior to the conference, we will survey our participants to gather information on their research interests specific to aesthetics and affect, student age group (elementary, secondary, postsecondary), and whether they are already doing work in this area or interested in it as a new topic. We will welcome participants to bring an artifact they would like to work on with the group for feedback, e.g., a task they wish to refine, to improve the aesthetic or affective response.

Day 1: We will start by having participants introduce themselves and share their research interests (15 min). We will all engage in an anchoring mathematical activity aimed toward producing emotions and aesthetic reactions for 60 minutes. We'll end with an overview of the goals of this working group and will offer opportunities for participants to share a piece of their work for Day 2 for the next session (15 min).

Day 2: We will split the session into slots for participants to share an item or research they've been working on related to aesthetics and affect, leaving time for discussion across the shared works. An underlying goal for Day 2 is for participants to notice common ideas across researchers, even within differently named constructs.

Day 3: We will start a collective project for after the working group, for interested participants. Options include writing a conference proposal grant, a collective writing venture such as an article, and/or creating powerful activities for students based on participants' teaching interests. We will maintain a Google Drive folder, where Google Docs with work that we generate together will be kept as a record after the conference. There will also be a folder for participants to share their relevant published work with the group to facilitate spread of ideas, especially from new scholars. We will provide space for participants to list their names and contact information next to any research questions/directions they'd like to pursue.

### References

- Boaler, J. (2015). *What's math got to do with it?: How teachers and parents can transform mathematics learning and inspire success*. Penguin.
- Cheeseman, J., & Mornane, A. (2014). Primary students' perceptions of their mathematics learning. In J. Anderson, M. Cavanaugh, & A. Prescott (Eds.), *Curriculum in focus: Research guided practice (Proceedings of the 37<sup>th</sup> annual conference of the Mathematics Education Research Group of Australasia)* (pp. 135-142). Sydney: MERGA.
- DeBellis, V. A., & Goldin, G. A. (2006). Affect and meta-affect in mathematical problem solving: A representational perspective. *Educational Studies in Mathematics*, 63, 131–147.
- Dietiker, L. (2015). What mathematics education can learn from art: The assumptions, values, and vision of mathematics education. *Journal of Education*, 195(1), 1–10.
- Dietiker, L., Richman, A. S., Brakonieccki, A., & Miller, E. R. (2016). Woo! Aesthetic variations of the “same” lesson. In M. B. Wood, E. E. Turner, M. Civil, & J. A. Eli (Eds.), *Proceedings of the 38th annual meeting of the North American chapter of the International Group for the Psychology of Mathematics Education (PME-NA)* (pp. 66–73).
- Grootenboer, P., & Marshman, M. (2016). The affective domain, mathematics, and mathematics education. In *Mathematics, affect and learning* (pp. 13-33). Springer, Singapore.
- Malmivuori, M. L. (2001). The dynamics of affect, cognition, and social environment in the regulation of personal
- Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

- learning processes: The case of mathematics. Doctoral dissertation.
- McLeod, D. B. (1988). Affective issues in mathematical problem solving: Some theoretical considerations. *Journal for Research in Mathematics Education*, 19(2), 134-141.
- McLeod, D. B. (1992). Research on affect in mathematics education: A reconceptualization. *Handbook of research on mathematics teaching and learning*, 1, 575-596.
- Middleton, J. A., Jansen, A., & Goldin, G. A. (2017). The complexities of mathematical engagement: Motivation, affect, and social interactions. *Compendium for research in mathematics education*, 667-699.
- Op't Eynde, P., Corte, E. D., & Verschaffel, L. (2006). "Accepting emotional complexity": A socio-constructivist perspective on the role of emotions in the mathematics classroom. *Educational Studies in Mathematics*, 63(2), 193-207.
- Sinclair, N. (2001). The aesthetic "is" relevant. *For the learning of mathematics*, 21(1), 25-32.
- Sinclair, N., Healy, L., & Sales, C. O. R. (2009). Time for telling stories: Narrative thinking with dynamic geometry. *ZDM*, 41(4), 441-452.

## FINAL REPORT: CONCEPTUALIZING ETHICS, AUTHENTICITY, AND EFFICACY OF SIMULATIONS IN TEACHER EDUCATION

Carrie Wilkerson Lee  
East Carolina University  
leecarr16@ecu.edu

Liza Bondurant  
Mississippi State  
University  
lb2206@msstate.edu

Bima Sapkota  
The University of Texas  
Rio Grande Valley  
bima.sapkota@utrgv.edu

Heather Howell  
ETS  
hhowell@ets.org

Yvonne Lai  
University of Nebraska-Lincoln  
yvonnexlai@unl.edu

*This working group was a continuation of working groups in 2019 and 2021 that initially aimed to focus on equity in simulations of practice in mathematics teacher education. We began by discussing our conceptualizations of simulations and equity. Next, we reflected on the lack of work that currently exists at the intersection of simulations and equity as well as our limited collective expertise in this space. We proposed the following areas of potential research: Access, Design, Affective Domains, Teaching Practices, Assessment, Critical Conversations. Attendees self-selected into focus groups and met to discuss their current work and how future work could focus more on equity and access. At the conclusion of our time together we developed a plan for achieving our key goal of disseminating a book that documents the landscape of the field.*

Keywords: Equity, Inclusion, and Diversity, Instructional Activities and Practices, Preservice Teacher Education

### Introduction

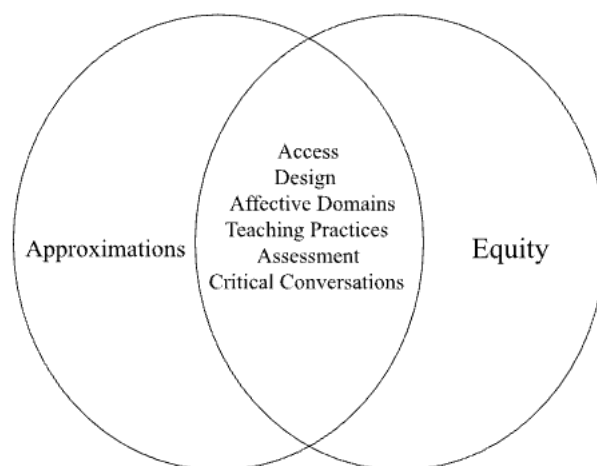
We draw our work from practice-based approaches to mathematics teacher education (Ball & Cohen, 1999). According to this approach, educator preparation programs (EPPs) should focus the work of preservice teachers (PSTs) on practicing the complexities of teaching (Kavanagh & Danielson, 2020). Grossman and colleagues' (2009) framework of pedagogies of practices conceptualizes how representations, decompositions, and approximations of practice can engage PSTs in aspects of teaching with varying levels of complexity and authenticity (Ball & Cohen, 1999; Ball & Forzani, 2009; Grossman et al., 2009; Zeichner, 2012). In our proposal, we focus on approximations of practice.

Approximations refer to the practices that are more or less proximal to those of the profession (Grossman et al., 2009). Mathematics teacher educators (MTEs) can vary the level of authenticity and complexity by focusing only on certain practices (e.g., decomposing) and providing scaffolding (Tyminski et al., 2014). MTEs can intentionally begin with less complex or authentic experiences initially and gradually increase the levels of complexity and authenticity over time (Bannister et al., 2018; Grossman et al., 2009).

Some MTEs express concerns about the limitations of approximations. Klein and Taylor (2017) argue that the simplified nature of approximations may not prepare PSTs for the social and cultural complexities of classrooms. Trent (2013) points out that approximations do not sufficiently prepare PSTs in transiting into their teaching roles. Particularly, MTEs have cautioned that PSTs may struggle to adapt approximated practices into enacted practices in school contexts (Zeichner, 2012). Few approximation studies have a strong equity framing

(Buttimer et al., 2022; Self & Stengel, 2020). In fact, the field has been criticized for foregrounding Whiteness (Daniels & Varghese, 2020), pushing aside issues of equity and justice (Philip et al., 2019), and stereotypically representing students (Baker-White, 2021; Bondurant & Reinholz, 2023).

Drawing upon the literature and our past work, we seek to explore the affordances and constraints of varying levels of complexity and authenticity in simulations (Bondurant et al., 2021; Howell et al., 2019; Howell & Mikeska, 2021). Furthermore, we aimed to expand the literature at the intersection of approximations and equity. Here, we report on an overview of our participants' critical reflections on how simulating the social and cultural components of classrooms might (a) cause biases and stereotypes about traditionally marginalized learners to surface, be exaggerated, or perpetuated as well as (b) strategies for mitigating this from occurring. Finally, we report on our dissemination plans.



**Figure 1: Research Space**

### Leadership Preparation for PME-NA

Leading up to the PME-NA 44 Conference the leadership team met for two main purposes. First, we met to develop a proposal for a special issue. Over the course of several weeks, we collaboratively developed a strong proposal and sixteen scholars in the field agreed to serve as reviewers for our proposed special issue. Although the special issue editors did not select our proposal, composing the proposal pushed us to conceptualize six topics at the intersection of equity and simulations (see Table 1). Secondly, we met to establish a detailed plan for our time together at PME-NA. The topics we developed for the special issue proposal laid the groundwork for our conversations at PME-NA 44. Despite our initial special issue proposal's rejection, the process of detailing how we would approach an edited collection of work in this space inspired us to keep this as a goal. We entered PME-NA eager to continue this important work.

**Table 1: Topics at the Intersection of Equity and Simulations**

Access	cost, program variability, differentiation of and access to experiences, disenfranchisement
Design	authenticity of representations, scenarios, students,

	cultural contextualization and relevance
Affective Domains	biases, beliefs, identity, positionality
Teaching Practices	demonstrating high expectations for all learners, positioning all learners as experts. planning culturally relevant curricula, implementing culturally relevant curricula, leveraging early conceptions and prior knowledge, soliciting equitable participation
Assessment	equitable assessment strategies, providing asset-based feedback, blinded grading practices, writing formative and summative assessments, justifying assessment policies
Critical Conversations	meeting with stakeholders, critical race theory, social emotional learning (SEL), book banning, bullying, othering, tracking, social justice

### Grouping Work Summary

Fourteen scholars shared openly and thoughtfully over the course of our three sessions. We attribute our increase in attendance to the saliency of approximation work and also to our efforts to publicize our group in advance of the conference. The conference committee disseminated a promotional graphic we created to invite attendees to join us. We also asked our colleagues who participate in the Association of Mathematics Teacher Educators (AMTE) Community Circle on simulations to participate in our PME-NA working group. We were excited to see returning members, colleagues from the AMTE Community Circle, and new participants join us. On day one, the leadership team began by sharing our research journey with both digital and non-digital simulations over the past four years. Next, we reached a consensus on our terminology and goals. We decided to use the broader terminology of *Approximations of Practice* to describe the space we are exploring. Moreover, we shared the dimensions of variability we developed in 2019 to situate our work in this diverse space (Howell et al., 2019). The majority of our time on day one was spent sharing our approximations research, with equity as our unifying lens. We had fruitful discussions surrounding how we can foreground equity and access in our approximations work by considering what mathematics is taught, whose mathematics is taught, and how mathematics is taught (Aguirre et al., 2013).

On days two and three we divided into three groups based on participants' interests in topics proposed by the leadership team (Table 1). Participants were most interested in Design, Affective Domains, and Teaching Practices. In each group, participants shared in depth descriptions of their work and how it fits within the space. We provided groups with prompts to begin outlining a 1–2-page prospectus (see Figure 2).

Please include the following:

- I. Proposed authorship with affiliations

- |       |   |
|-------|---|
| II.   | What type of approximation does this prospectus address/feature (e.g., digital with live actor(s), in person rehearsal, online practice space, etc.)? |
| III.  | How does the prospectus connect to equity?  |
| IV.   | What type of chapter does the prospectus describe (e.g., a specific research study, a call to action, a theoretical piece, a literature review, etc.) |
| V.    | Statement of issue or topic to be explored  |
| VI.   | Key literature that guides your thinking  |
| VII.  | Brief summary of methods  |
| VIII. | Description of impact   |

**Figure 2: Approximations and Equity Prospectus Outline**

Collaborating on the prospectus outlines provided opportunities for us to critically reflect on our work. Many participants began expressing concern about the lack of focus they currently place on issues of equity and access. As we came together at the end of the second day, we grappled with the concerns we were feeling. Our whole group discussion led us to consider a different direction for the proposed collection of work, which we continued to mold on day three. As a team we decided that an edited book, which we further articulate in the next section, would be the best outlet for our work. We reached a consensus that the edited book would include approximation work from scholars at different places along the continuum of foregrounding equity. We envisioned that this edited book would provide the field with an overview of the current landscape as well as our aspirational equity-focused vision. We discussed how authors and editors could serve as critical friends during the writing and editing process, pushing each other towards our collective goal of foregrounding equity in our approximations work.

**Results from the Working Group: A Shift in Focus**

We collectively decided to make a number of key shifts to our initial vision and mission. One collective decision we made was to shift to the broader term *approximations of practice* rather than the term *simulations*. As noted in our prior working group, the field lacks clarity on the distinction between the two (e.g., using simulations to refer to only digital simulations). We chose to use the broader term *approximations* to avoid the risk of potential authors, particularly new collaborators, not classifying their work in this space due to our use of the narrower term *simulations*. We also noted that while equity considerations might play out differently in different approximation formats, few of them are unique to format alone, and there are cases in which the contrast between different approaches or technologies might yield useful insights.

A second key shift was the decision to produce an edited book rather than a special issue. This decision was based on four key factors that emerged during our discussions. First, the enthusiastic responses from potential reviewers coupled with the eagerness of working group participants to contribute made it clear that an edited book would be needed to provide ample space for contributions. Secondly, we realized that working alone scholars may struggle to realize our goal of sharing work that places a dual emphasis on equity and approximations. We recognize that equity is a journey, not a destination. Therefore, scholars in this space are continuously striving for growth. We discussed how authors and editors could serve as critical friends during the writing and editing process, encouraging each other to continue and extend their work towards our collective goal of foregrounding equity in our approximations work. We

decided that an edited book would be needed to share approximation work from scholars at different places along the continuum of foregrounding equity in their work. Third, we decided that an edited book would allow us to organize both empirical and theoretical work in this space into thematic sections without the limitation of a single editorial introduction, which is often a constraint of special issues. Finally, we felt an edited book would create space to be provocative in what we ask the field to focus on moving forward.

Our resulting proposed outline is an edited book with three sections, each of which would consist of an introductory chapter followed by 3-5 short chapters. The first is a landscape section, that seeks to broadly demonstrate how approximations of practice are being used in the field. This section would include empirical pieces that showcase approaches that are different from one another. Some approaches that foreground equity would be discussed in this section. However, we would not limit this section to equity-focused uses because the current state of the field does not reflect this focus. Overall, in this section, we aim to help the reader understand the breadth of work that characterizes approximations of practice for mathematics PSTs.

The second section shifts to asking how the field is or could be foregrounding equity in approximations work. Our vision for this section, because of the limited work that focuses clearly at the intersection of approximations and equity in mathematics education, is to pair willing authors in the mathematics education approximations space with co-authors with equity-focused research experience. We believe collaborators will serve as critical friends, encourage each other to extend their research, and expand the work at the intersection of approximations and equity. We hope these unique chapters will provide examples of how to do the hard work of placing a dual emphasis on equity and approximations. Potential authors will be asked to self-identify their willingness to have their work featured in this section in their chapter proposal. The editors plan to support the co-authoring teams intensively throughout the process.

The third section will focus on what equity-focused approximations should or could be in the future. This section is less likely to feature specific projects or empirical work, although forward thinking and provocative projects might be featured here. We would also welcome short chapters exploring where scholars would recommend focusing, or innovative approaches that might address challenges in ways that have not yet been operationalized or used in the field.

### **Conclusion**

Our leadership team left Nashville eager to take action on our plans to edit a book. We created documentation of our new framework and shared it with working group members. We have also continued conversations via email, virtual biweekly meetings, and collaborative projects. Additionally, we have met quarterly through AMTE's Simulations Community Circle. We are currently working with IGI Global: International Academic Publisher to finalize a book proposal. With our evolved framework, we are motivated to bring voices together in the space of approximations of practice and equity to inform the work of MTEs. There is a plethora of important work to do in this space, and we aim to invite others to join us in our journeys towards foregrounding equity in approximations of practice.

### **References**

- Aguirre, J. M., Mayfield-Ingram, K., & Martin, D. B. (2013). Identities, agency, and mathematical proficiency: What teachers need to know to support student learning. In *The impact of identity in K-8 mathematics learning and teaching: Rethinking equity-based practices*. The National Council of Teachers of Mathematics, Inc.
- Baker-White, E. (2021, December 1). "This Is Blackface": White actors are playing Black characters in virtual reality diversity training. BuzzFeed News. <https://www.buzzfeednews.com/article/emilybakerwhite/diversity->

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

[training-mursion-vr-white-actors](#)

- Ball, D. L., & Cohen, D. K. (1999). Developing practice, developing practitioners: Toward a practice-based theory of professional education. *Teaching as the learning profession: Handbook of policy and practice*, 1, 3-22.
- Ball, D. L., Forzani, F. M. (2009). The work of teaching and the challenge for teacher education. *Journal of Teacher Education*, 60(5), 497-511. <https://doi.org/10.1177/0022487109348479>
- Bannister, N., Kalinec-Craig, C., Bowen, D. & Crespo, S. (2018). Learning to Notice and Name Students' Mathematical Strengths: A Digital Experience. *Journal of Technology and Teacher Education*, 26(1), 13-31. Waynesville, NC USA: Society for Information Technology & Teacher Education. Retrieved February 20, 2023 from <https://www.learntechlib.org/primary/p/181150/>.
- Bondurant, L., Howell, H., Kwon, M., Wilkerson, C. L., & Lai, Y. (2021). Preservice teacher learning of practice through simulated teaching experiences before, during, and after COVID. In Olanoff, D., Johnson, K., & Spitzer, S.M. (2021). *Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Philadelphia, PA. Retrieved February 20, 2023 from <https://www.pmena.org/pmenaproceedings/PMENA%2043%202021%20Proceedings.pdf>.
- Bondurant, L., & Reinholz, D. (2023). "Rahul is a Math Nerd" and "Mia Can Be a Drama Queen": How Mixed-Reality Simulations Can Perpetuate Racist and Sexist Stereotypes. *Mathematics Teacher Educator*, 11(3).
- Buttimer, C. J., Littenberg-Tobias, J., & Reich, J. (2022). Designing Online Professional Learning to Support Educators to Teach for Equity During COVID and Black Lives Matter. *AERA Open*, 8. <https://doi.org/10.1177/23328584211067789>
- Daniels, J. R., & Varghese, M. (2020). Troubling practice: Exploring the relationship between Whiteness and practice-based teacher education in considering a raciolinguicized teacher subjectivity. *Educational Researcher*, 49(1), 56-63. <https://doi.org/10.3102/0013189X19879450>
- Grossman, P., Hammerness, K., & McDonald, M. (2009). Redefining teaching, re-imagining teacher education. *Teachers and Teaching: Theory and Practice*, 15(2), 273-289. <https://doi.org/10.1080/13540600902875340>
- Howell, H., Lai, Y., & Lee, C. (2019). Simulations of practice for the education of mathematics teachers. In Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. St Louis, MO: University of Missouri. Retrieved February 20, 2023 from <https://www.pmena.org/pmenaproceedings/PMENA%2041%202019%20Proceedings.pdf>
- Howell, H., & Mikeska, J. N. (2021). Approximations of practice as a framework for understanding authenticity in simulations of teaching. *Journal of Research on Technology in Education*, 53(1), 8-20. <https://doi.org/10.1080/15391523.2020.1809033>
- Kavanagh, S. S., & Danielson, K. A. (2020). Practicing justice, justifying practice: Toward critical practice teacher education. *American Educational Research Journal*, 57(1), 69-105. <https://doi.org/10.3102/0002831219848691>
- Klein, E. J., & Taylor, M. (2017). Teacher educators struggling to make complex practice explicit: Distancing teaching through video. *Studying teacher education*, 13(3), 312-330. <https://doi.org/10.1080/17425964.2017.1366307>
- Philip, T. M., Souto-Manning, M., Anderson, L., Horn, I., J. Carter Andrews, D., Stillman, J., & Varghese, M. (2019). Making justice peripheral by constructing practice as "core": How the increasing prominence of core practices challenges teacher education. *Journal of Teacher Education*, 70(3), 251-264. <https://doi.org/10.1177/0022487118798324>
- Self, E. A., & Stengel, B. S. (2020). *Toward anti-oppressive teaching: Designing and using simulated encounters*. Harvard Education Press.
- Trent, J. (2013). From campus to classroom: A critical perspective on approximations of practice in teacher education. *Research Papers in Education*, 28(5), 571-594. <https://doi.org/10.1080/02671522.2012.710246>
- Tyminski, A. M., Zambak, V. S., Drake, C., & Land, T. J. (2014). Using representations, decomposition, and approximations of practices to support prospective elementary mathematics teachers' practice of organizing discussions. *Journal of Mathematics Teacher Education*, 17(5), 463-487. <https://doi.org/10.1007/s10857-013-9261-4>
- Zeichner, K. (2012). The turn once again toward practice-based teacher education. *Journal of teacher education*, 63(5), 376-382. <https://doi.org/10.1177/0022487112445789>

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.



## APPROXIMATIONS OF PRACTICE AND EQUITY: SURVEYING THE SPACES

Bima Sapkota  
The University of Texas Rio Grande Valley  
bima.sapkota@utrgv.edu

Carrie Wilkerson Lee  
East Carolina University  
leecarr16@ecu.edu

Liza Bondurant  
Mississippi State University  
lb2206@msstate.edu

Heather Howell  
ETS  
hhowell@ets.org

*After three years of meeting as a working group at PME-NA, we will be editing a book on how several models of approximations of practice support or hinder the creation of equitable mathematics learning and thinking spaces. Participants and working group leaders will serve as critical friends as we use a fishbowl format and small groups to provide spaces for refinement and challenge. Participants will leave with a network of thought partners and a draft chapter. At the end of the working group, we will identify overarching themes about affordances and limitations of several models of digital and non-digital forms of approximations of practices in order to develop preservice teachers (PSTs') equitable teaching practices.*

Keywords: Equity, Inclusion, and Diversity, Instructional Activities and Practices, Preservice Teacher Education

### Background

In the last two decades, mathematics teacher education programs have emphasized designing and implementing practice-based experiences (Ball & Cohen, 1999; Janssen et al., 2015; Zeichner, 2012). Approximations of practice, a type of practice-based experience, “opportunities for novices to engage in practices that are more or less proximal to the practices of a profession” (Grossman et al., 2009, p. 2058); however, there is a risk of oversimplifying the social and cultural aspects of classrooms (Grossman et al., 2009; Zeichner, 2012). Thus, mathematics teacher educators (MTEs) have begun foregrounding their approximation research on several aspects of equity, such as how students’ identities might influence their thinking and learning (Shah, 2017), what counts as knowledge, and who generates it (Gutiérrez, 2018).

Equity-focused approximations have the potential to prepare PSTs to use equitable practices in their planning, instruction, and assessment when working with diverse students, including those from marginalized populations. As we mentioned earlier, there is limited research on how different digital and non-digital forms of approximations provide PSTs with opportunities to develop equitable teaching practices. Self’s & Stengel’s (2020) SHIFT project and Buttner and colleagues’ (2022) MIT Teaching Systems Lab projects are two examples of equity-focused digital simulations projects, but more attention is needed to fully understand the potential work that could occur in this space.

We anticipate MTEs interested in foregrounding equity in their approximations work will be eager to participate in our working group. We plan to dedicate our time together at PME-NA to writing, providing feedback, and revising chapters for an edited book. Participants will serve as “critical friends” (Schuck et al., 2008) who “[act as] sounding board, offer opportunities for reflection, [will be] co-learners, and will ask challenging questions” (Russell & Schunk, 2004, p.

2013). We envision this working group as a space, where MTEs challenge each other to reflect on their current work and collaborate to develop approximations that foreground equity.

### **Focus of the Work**

This working group is an extension of a group that began in 2019. During the 2022 conference our group engaged sixteen participants. Collectively, the leadership team and participants decided to produce an edited book (Wilkerson et al., 2023). We are working with IGI publishers on an edited book. We will send a call for chapter proposals out in May 2023. Chapter proposals will be due December 2023 and authors will be notified in February 2024. Our meeting times during PME-NA will provide a collaborative space for authors to shape their work. We plan to solicit chapter proposals on professional listservs, Twitter, in the Association of Mathematics Teacher Educators (AMTE) *Connections* newsletter, and AMTE Simulations Community Circle. Our PME-NA sessions will focus on these key questions that stem from the organization of the book (outlined more in depth in our 2022 Final Report):

- 1) What are several models of approximations of practice currently used in the field?
- 2) What are unique affordances and limitations of those approximations to foreground equity in teacher preparation?
- 3) How design elements of approximations could be altered in order to optimize PSTs' opportunities to develop equitable teaching practices?

### **Organization and Plan for Active Engagement**

The working group sessions will be structured parallel to the edited book's three sections. The first session will highlight the breadth of work that characterizes PSTs approximations. The second and third sessions will push beyond the current approximations work to forefront equity. Specifically, we will provide space for authors to collaborate and critically analyze how equity can be foregrounded in their work. Time will be spent writing, providing feedback, and revising.

#### **Session 1: Approximations of practice as spaces to forefront equity**

During Session 1 the leadership team will provide a brief overview of current research in approximations and our group's plans to publish an edited book. We will engage participants in an activity to place their work within a conceptual map of the space of approximations. We will form Critical Friend Groups (CFGs) based on participants' interests. CFGs will meet to share their work and how they are currently attending to equity. At the conclusion of the first session, participants will be given a CFG document to focus their work during Session 2.

#### **Session 2: Critical conversations to expand our spaces**

Prior to PMENA, we will select one approximation article and during Session 2 the leadership team will use this article to model the process of engaging in critical conversations to foreground equity. We will use a modified CFG protocol (Braaten, Grandados & Bradford, 2022) to facilitate critical conversations on how the study could foreground equity. We will set norms and allow time for multiple points of view to be shared in a productive and transformative way. This activity models the process authors of chapters in the second section will use.

#### **Session 3: Future spaces within approximations of practice**

During Session 3 CFGs will engage in provocative and aspirational conversations regarding our visions for the future. Everyone from graduate students to veteran scholars will be provided space and time to share their perspectives and goals. Editors will also meet one on one with participants who would like feedback on their chapter proposals.

## References

- Ball, D. L., & Cohen, D. K. (1999). Developing practice, developing practitioners: Toward a practice-based theory of professional education. In L. Darling-Hammond & G. Sykes (Eds.), *Teaching as the learning profession* (pp. 3–31). Jossey-Bass.
- Braaten, M., Granados, E., & Bradford, C. (2022). Making sense through dissonance during preservice teacher preparation. *Teaching and Teacher Education, 109*, 103541. <https://doi.org/10.1016/j.tate.2021.103541>
- Grossman, P., Compton, C., Igra, D., Ronfeldt, M., Shahan, E., & Williamson, P. (2009). Teaching practice: A cross-professional perspective. *Teachers College Record, 111*(9), 2055-2100. <https://doi.org/10.1177%2F016146810911100905>
- Gutiérrez, R. (2018). When mathematics teacher educators come under attack. *Mathematics Teacher Educator, 6*(2), 68-74. <https://doi.org/10.5951/mathteaceduc.6.2.0068>
- Janssen, F., Grossman, P., & Westbroek, H. (2015). Facilitating decomposition and recomposition in practice-based teacher education: The power of modularity. *Teaching and Teacher Education, 51*, 137-146. <https://doi.org/10.1016/j.tate.2015.06.009>
- Russell, T., & Schuck, S. (2004). How critical are critical friends and how critical should they be? In D. L. Tidwell, L. M. Fitzgerald, & M. L. Heston (Eds.), *Journeys of hope: Risking the journey of self-study in a diverse world. Proceedings of the Fifth International Conference on Self-Study of Teacher Education Practices, Herstmonceux Castle, East Sussex, England* (pp. 213–216). University of Northern Iowa.
- Shah, N. (2017). Race, ideology, and academic ability: A relational analysis of racial narratives in mathematics. *Teachers College Record, 119*(7), 1–42. <https://doi.org/10.1177/016146811711900705>
- Schuck, S., Aubusson, P., & Buchanan, J. (2008). Enhancing teacher education practice through professional learning conversations. *European Journal of Teacher Education, 31*(2), 215–227. <https://doi.org/10.1080/02619760802000297>
- Wilkerson, C.L., Bondurant, L., Sapkota, B., Howell, H. (2023). Final report: Conceptualizing ethics, authenticity, and efficacy of simulations in teacher education. In T. Lamberg (Eds.), *Proceedings of the forty-five annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Reno, NV.
- Zeichner, K. (2012). The turn once again toward practice-based teacher education. *Journal of Teacher Education, 63*(5), 376-382. <https://doi.org/10.1177%2F0022487112445789>

## COMPLEX CONNECTIONS: REIMAGINING UNITS CONSTRUCTION AND COORDINATION FOR MKT AND COMBINATORIAL REASONING

Karen Zwanch  
Oklahoma State University  
karen.zwanch@okstate.edu

Sarah Kerrigan  
George Fox University  
skerrigan@georgefox.edu

Beth L. MacDonald  
Illinois State University  
blmacd1@ilstu.edu

Steven Boyce  
Portland State University  
sboyce@pdx.edu

Joseph Antonides  
Virginia Tech  
jantonides@vt.edu

Cody Harrington  
SUNY Cortland  
Cjjorgen@gmail.com

Cameron Byerley  
University of Georgia  
cameron.byerley@gmail.com

Erik Tillema  
Indiana University Bloomington  
etillema@iu.edu

*Units construction and coordination is a powerful tool for modeling mathematical thinking about whole numbers and fractions. This working group will support collaboration of mathematics educators to make connections between foundational contexts for units coordination, and emerging contexts such as combinatorial reasoning, mathematical knowledge for teaching, and communicating mathematical ideas in our daily lives.*

Keywords: Learning Theory, Number Concepts and Operations, Learning Trajectories and Progressions, Cognition

### Background

The purpose of the Complex Connections working group (WG) is to provide a space in which mathematics educators can collaborate on the ways that we perceive units construction and coordination to apply across contexts. This year, we broaden units construction and coordination to include combinatorial reasoning, the ways that teachers' units construction and coordination inform their mathematical knowledge for teaching, and how units coordination research might contribute to a more just society. This paper presents an overview of existing research related to these three areas, and a summary of our planned WG activities.

Research indicates that students' combinatorial reasoning is related to their units construction and coordination. Both Antonides (2022; Antonides & Battista, 2022a) and Tillema (2013, 2018, 2020) have analyzed how students establish unit structures in their solution of combinatorics problems, including how these unit structures are related to, and often eclipse those that are most frequently discussed in research literature on whole number and fractions (e.g., Tillema & Gatzka, 2016). Antonides and Battista's work focuses on how students form and structure units in the context of their spatial and temporal activity, which they call spatial-temporal-enactive structuring (Antonides & Battista, 2022b). Tillema's current work (e.g., Tillema and Burch, 2022) focuses on how the unit structures students create in combinatorial contexts can support their generalization of structures and systems related to polynomials.

Current research into teachers' mathematical knowledge for teaching (MKT) finds that teachers' mathematical knowledge directly affects their mathematical pedagogy (Hill & Ball, 2004), which can also be linked to teachers' units coordination (Harrington, 2021). As teachers construct more levels of units into their assimilatory schemes, they transition into higher levels of mathematical reasoning, affording them the ability to recognize and analyze levels of units

within their students' mathematical reasoning (Harrington, 2021). By identifying the linkages between MKT and teachers' levels of units coordination, researchers can engage teachers in mathematical tasks and interventions that might lead to higher levels of units coordination, and stronger MKT. To date, there has been little research into these linkages, but one study (Harrington, 2021) recently utilized interviews with teachers to analyze evidence into the teachers' levels of units coordination, their analysis of students' levels of units coordination and the teachers' growth in units coordination and MKT over time.

Finally, we acknowledge that mathematics education research related to student thinking, such as units coordination research, and research related to "equity have often been cast as separate or even competing" (Adiredja, 2019, p. 402). However, cognitive researchers such as Hackenberg et al. (2020) and Ellis (2022) attend to both cognition and equity. Our WG aims to explore this connection more deeply, by leveraging units coordination as a tool to support equitable research, instruction, and dissemination of mathematical information societally.

#### Day 1: Extending to MKT

On day 1, participants will discuss the theoretical frameworks of MKT and units coordination. Author 6 will present videos of elementary teachers discussing student reasoning, looking for evidence of the teachers' levels of units coordination and their recognition of levels of units coordination in their students' reasoning. Participants will break into small groups to discuss and analyze the videos. The session will culminate with a discussion of the theoretical connections and implications of teachers' units coordination on MKT.

#### Day 2: Extending to Combinatorial Reasoning

On day 2, we will discuss how units coordination may be involved in student reasoning about combinatorial enumeration. Author 5 and Author 8 will present on their research on student reasoning within the realm of enumerative combinatorics. Activities during this day will include small group discussions about combinatorial tasks. Participants in small groups will consider combinatorial counting tasks, and unit coordinating operations that may be involved in reasoning about these tasks. Then, participants will view excerpts from students' combinatorial reasoning, and to consider what units coordinations were involved. Finally, participants will hypothesize how students' combinatorial reasoning, and units coordination, might lead to construction of a "multiplicative slot structure." For instance, assimilating counting tasks as multiplicative structures modeled by a three-factor multiplicative structure, such as  $a \times b \times c$ .

#### Day 3: Extending to Disseminating Information to Adults

On day 3, we will discuss how units coordination schemes allow citizens to make sense of real-world information about risk and health. Byerley will share results of a controlled experimental trial involving the Relative Risk Tool (Joshua et al., 2022). The tool shows the utility of using models of mathematical thinking to make critical information accessible to citizens with a wide variety of mathematical thinking. We will discuss research goals from both cognitive and sociopolitical perspectives (i.e., Gutiérrez, 2013; Steffe, 2017) and consider how second order models can contribute to a more just society.

On day 3 we will also share new and continuing opportunities for scholarship stemming from our collaborations. We will begin by introducing our current efforts, including updates on the content of our units coordinating website and progress of our co-editing a special issue on units construction and coordination. We will then facilitate discussion about next steps, to include transitioning from a working group to a research colloquium, preparing for additional collaborations into research and teaching, and planning for additional publication opportunities.

## References

- Adiredja, A. P. (2019). Anti-deficit narratives: Engaging the politics of research on mathematical sense making. *Journal for Research in Mathematics Education*, 50(4), 401-435.
- Antonides, J. E. (2022). *A cognition-based analysis of undergraduate students' reasoning about the enumeration of permutations* (Unpublished doctoral dissertation, The Ohio State University).
- Antonides, J., & Battista, M. T. (2022a). A learning trajectory for enumerating permutations: Applying and elaborating a theory of levels of abstraction. *Journal of Mathematical Behavior*, 68, 101010. <https://doi.org/10.1016/j.jmathb.2022.101010>
- Antonides, J., & Battista, M. T. (2022b). Spatial-temporal-enactive structuring in combinatorial enumeration. *ZDM Mathematics Education*, 54(4), 795-807. <https://doi.org/10.1007/s11858-022-01403-0>
- Ellis, A. (2022). Decentering to build asset-based learning trajectories. In A. E. Lischka, E. B. Dyer, R. S. Jones, J. N. Lovett, J. Strayer, & S. Drown (Eds), *Proceedings of the 44<sup>th</sup> Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Nashville, TN: Middle Tennessee State University.
- Gutiérrez, R. (2013). The sociopolitical turn in mathematics education. *Journal for Research in Mathematics Education*, 44(1), 37-68.
- Hackenberg, A. J., Creager, M., & Eker, A. (2021). Teaching practices for differentiating mathematics instruction for middle school students. *Mathematical Thinking and Learning*, 23(2), 95-124. <https://doi.org/10.1080/10986065.2020.1731656>
- Harrington, C. (2021). *Progressions in mathematical reasoning: A case study of two teachers' levels of units coordination* (Publication No. 28416231) [Doctoral dissertation, University of Colorado Denver] ProQuest Dissertation & Theses Global.
- Hill, H. and Ball, D. (2004). Learning mathematics for teaching: Results from California's mathematics professional development institutes. *Journal for Research in Mathematics Education*, 35, 330-351.
- Joshua, S., Drimalla, J., Horne, D., Lavender, H., Yon, A., Byerley, C., ... & Moore, K. (2022). Exploring relative size and relative risk. *Mathematics Teacher: Teaching and Learning PK-12*, 115(5), 339-350.
- Steffe, L. P. (2017). Psychology in mathematics education: Past, present, and future. In E. Galindo & J. Newton (Eds), *Proceedings of the 39<sup>th</sup> Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.
- Tillema, E. S. (2013). A power meaning of multiplication: Three eighth graders' solutions of Cartesian product problems. *Journal of Mathematical Behavior*, 32(3), 331-352. <https://doi.org/10.1016/j.jmathb.2013.03.006>
- Tillema, E. S. (2018). An investigation of 6<sup>th</sup> graders' solutions of Cartesian product problems and representation of these problems using arrays. *Journal of Mathematical Behavior*, 52, 1-20. <https://doi.org/10.1016/j.jmathb.2018.03.009>
- Tillema, E. S. (2020). Students' solution of arrangement problems and their connection to Cartesian product problems. *Mathematical Thinking and Learning*, 22(1), 23-55. <https://doi.org/10.1080/10986065.2019.1608618>
- Tillema, E. S., & Burch, L. J. (2022). Using combinatorics problems to support secondary teachers' understanding of algebraic structure. *ZDM Mathematics Education*, 54(4), 777-793. <https://doi.org/10.1007/s11858-022-01359-1>
- Tillema, E. S., & Gatza, A. (2016). A quantitative and combinatorial approach to non-linear meanings of multiplication. *For the Learning of Mathematics*, 36(2), 26-33.

## COMPLEX CONNECTIONS WORKING GROUP: OUTCOMES AND PROGRESS

Sarah Kerrigan  
George Fox University  
skerrigan@georgefox.edu

Karen Zwanch  
Oklahoma State University  
karen.zwanch@okstate.edu

Beth L. MacDonald  
Illinois State University  
blmacd1@ilstu.edu

Steven Boyce  
Portland State University  
sboyce@pdx.edu

Diana Moss  
University of Nevada, Reno  
dmoss@unr.edu

Claudia M. Bertolone-  
Smith  
Chico State University  
cmbertolone-  
smith@csuchico.edu

Cameron Byerley  
University of Georgia  
cameron.byerley@gmail.com

Christopher Roman  
Portland State University  
roman7@pdx.edu

Jeffery Grabhorn  
Portland State  
University  
jeff4@pdx.edu

*The Complex Connections: Reimagining Units Construction and Coordination working group has been running since PMENA in 2018. Previous years have focused on developing a website to share research articles and share completed work and facilitating collaboration between researchers and educators regarding students' units construction and coordination. This goal aligned with Steffe's (2017) call for work focused collaboration between researchers and educators to advance units construction and coordination for all learners. The group has grown beyond the initial topic of individuals' units coordination in whole number and fraction settings to discuss units coordination in algebraic and covariational reasoning.*

Keywords: Learning Theory, Learning Trajectories and Progressions, Number Concepts and Operations

### **Progress from Previous Years**

The initial goals of this working group were to extend and widen units coordination research, teaching, and learning to broader audiences. The focus is to broaden the horizons of units coordination beyond the primary work done with whole numbers and fractions in elementary and middle school mathematics to explore connections with other populations and content domains. Here we summarize the progress made each year to give context to how the group has evolved over the years but maintained the original roots of extending and widening units coordination.

### **Progress Made in 2018 (Year 1)**

The first year of the group consisted of five leaders initiating an open invitation to researchers and educators to discuss units coordination in their teaching and research. The sessions were focused on defining future goals for the group and discussing challenges with defining and using units coordination in research and teaching. Another focus of the group was to discuss how we as a research and education community share our work to make it more accessible. Out of this goal of accessibility came one of the main outcomes from the first year, a units coordination website (<https://unitscoordination.wordpress.com/>). The website, put together by the original leaders, started as a place to collect works on units coordination in one location as a resource for new and current members of the education community. It also included the list of the original leaders to help with open communication and facilitate collaboration.

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

### **Progress Made in 2019 (Year 2)**

The second year saw an increase in membership to the WG, drawing both novices and experts alike in levels or familiarity with units coordination (Figure 1). This year the sessions during the conference included discussing the challenges with assessing students' units coordination and discussing ways to make assessments more accessible for teachers. Activities included engaging in video and written data of student work on fractions tasks to discuss affordances and constraints of various tasks for assessment. The group spent much of the time discussing Pre-service teachers' units coordination on the various tasks. This led to conversations between both novices and experts and teachers and researchers about the application of units coordination within and outside the classroom.

The leaders also shared updates made to the website, including the addition of 10+ articles shared by members and updating membership to include participants from the previous year. Another outcome from this year was various continued conversations throughout the year between researchers that met during the conference to continue the conversation about new topics to explore regarding units coordination.



**Figure 1: Attendees at the 2019 Complex Connections Working Group**

### **Progress Made in 2020 (June 2021, Year 3)**

The third year of the group saw a shift in leadership to include a new member and a shift in focus to open the discussion for topics outside of fractions. The group met in a hybrid format with most joining online. It also saw an increase in membership, including an increase in non-Native English speakers which required the group to reflect on the accessibility of its work and lead to innovations to increase access. Discussions during this working group focused on defining what a “unit” consisted of and how it evolves in different contexts. The group expanded from fractions and whole numbers to discuss new topics including integers and covariational reasoning. We examined tasks and student work on those tasks that were not explicitly designed with the units coordination framework in mind. This broadened the conversation about units coordination in different contexts to drive discussion of the theoretical underpinnings connecting these new contexts with what units are present and how students leverage those units in their work.

From the third year's discussions, the group launched new investigations into different topics connecting to units coordination with a more cohesive framework for “what constitutes a unit”

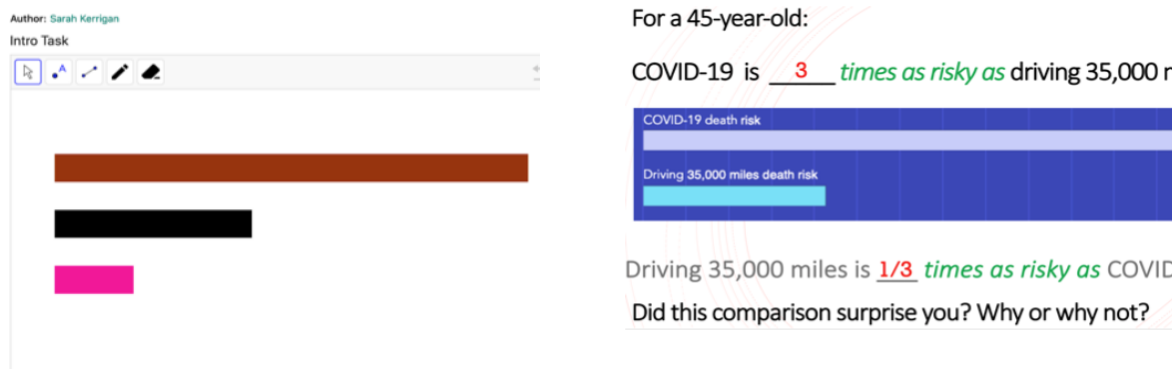


and “how units are constructed.” Out of this come five new publications, two new software apps designed to measure individual’s units coordination, and several new investigations that carried into the next several years of the working group.

### **Progress Made in 2021 (October 2021, Year 4)**

In the fourth year of the WG, we again met in a hybrid format with equal numbers in person and online. The first two days centered around pre-service teachers’ responses to fractions tasks. The group analyzed responses and discussed the differences in the pre-service teachers’ use of unit structures on the various tasks. This promoted discussion around assessment of units coordination in older populations of students and opened new topics such as using units coordination as a teaching tool in elementary classrooms.

The third day, the group discussed new developments made in applets designed for engaging in mathematical reasoning. One was designed specifically for assessing units coordination that was an adaption of a previously validated written assessment (Norton et. al., 2015). The other was for developing proportional reasoning related to interpreting data from COVID-19 and was designed to leverage units coordination theory. On the last day, the group also spent time discussing future collaborations, including a proposal for special journal issue.



**Figure 2: left) Screenshot of applet shared by Sarah Kerrigan; right) Screenshot from applet shared by Working Group Leader Cameron Byerley**

### **Progress Made Throughout the November 2022 Working Group (Year 5)**

In fall of 2022, the working group again met in a hybrid format at the conference. There were 27 attendees, including three leaders in the online format and several new researchers to the group. The main leadership for presenting involved new researchers that discussed their extensions of unit coordination to algebraic and covariational reasoning. The group also continued focusing on new collaborations with further discussion of the special issue. In the next section, we elaborate more on the outcomes from this most recent conference.

#### **Day One Progress and Outcomes**

In 2022, the WG added two members to the leadership team. The first day of the conference was led by Karen Zwanch, a new leadership team member. The purpose of day one was to extend ideas of units coordination to algebraic reasoning. The group began with introductions and a summary of old business, including updates to the website. The WG was attended by members in person as well as online. The WG was attended by 27 participants across the conference, and the participants ranged from WG leaders, to expert researchers in units coordination, to graduate students familiar with units coordination from their coursework, to

mathematics educators unfamiliar with how units coordination might apply to their teaching and research. Therefore, we devoted time on day one to introducing definitions and terminology related to units coordination to ensure that everyone's participation could be meaningful. This involved both small and large group sharing revolving around a published definition of units coordination.

Next, we moved to the topic of connecting units coordination to ideas of algebraic reasoning. As a group, we defined algebraic reasoning and identified the types of student behaviors that might be taken as evidence of algebraic reasoning. Following an agreed definition, we viewed two algebraic reasoning tasks that involved systems of linear equations in two and three variables. The group brainstormed ways that students who operate with different levels of units might engage with the tasks, and what types of behaviors the students might demonstrate in response to the tasks. This included ideas about the students' operations with unknown quantities, equation writing, algebraic substitution strategies, and guess and check strategies. Then, written work produced by middle-grades students on the two systems of linear equations tasks was shared with the working group. Small and large group discussion continued by examining how the students' written work related to the levels of units with which they operated.

### **Day Two Progress and Outcomes**

The second day of the conference was led by Sarah Kerrigan and extended units coordination to covariational reasoning. The group started with an introduction to covariational reasoning through engaging with a dynamic task to determine the relationship between various quantities in the animation. People first discussed in small groups their observations before sharing out their ideas of how the different quantities changed and related to one another. This led to a discussion about where people saw different units being constructed and coordinated within their covariational reasoning.

After this introduction activity, the group briefly reviewed Carlson and colleagues' (2002) covariational framework to be used in the next activity. Kerrigan shared video data of an eighth-grade student engaging in a different dynamic task where a shape increased in size jumping back and forth between a square and a rectangle, tripling in area every change. The student was asked how the area changed as time went on and how the amounts of change changed. Participants of the working group watched clips of the student's responses and were asked to identify their covariational reasoning and how the student used their units structures in the covariational reasoning. Again, there were small group discussions before the groups shared out their thoughts.



**Figure 2: Screen Shots of the First Three steps of Kerrigan's Covariation Task**

Out of these various discussions, a variety of questions concerning new units coordinating structures and application of units arose. This led to new insights into different types of units and transformations of units, opening up new avenues for further research

collaborations. It also returned to the main motivation of the original working group to investigate questions such as “what constitutes a unit?” and “how can these new units be coordinated in novel contexts?”

### **Day Three Progress and Outcomes**

On the third day of the WG, we returned to discussion and sharing that revolved around algebraic reasoning. On day one, participants expressed continual interest in watching how the students solved algebraic reasoning tasks, rather than only viewing students’ written work. Therefore, Karen Zwanch led a portion of day three as well in which she provided video of sixth-grade students working with different manipulatives, drawings, and equations to solve problems of algebraic generalizations. These videos led to a third day of vigorous and involved discussion among novices and experts in units coordination. Furthermore, along with the tasks and written work shared on day one, these videos contributed to the goals of the working group by examining what constitutes a unit in the context of algebraic reasoning, and how units coordination applies in contexts other than whole number and fractional reasoning.

Following the discussion of algebraic reasoning videos, Steve Boyce and Beth MacDonald led the remainder of the day by sharing information about the upcoming special journal issue on the topic of units coordination. This involved questions from participants in the working group, sharing of goals and information pertinent to the special issue, and time to formulate research collaborations among the group. Among these collaborative conversations, mathematics educators discussed potential extensions of units coordination to topics of covariational reasoning, algebraic reasoning, geometry and measurement, combinatorial reasoning, place value, and teacher education.

### **Conclusion**

Across the five years that the Complex Connections Working Group has met at PME-NA, we have produced research collaborations, teaching and assessment tools, electronic tools, and a website which acts as a repository and hub for mathematics educators interested in learning more about and contributing to the study and application of units coordination. Additionally, we have worked together to broaden the definition of units coordination to move beyond the foundational concepts of whole number and fractional reasoning in the elementary and middle grades to pre-service teachers, and to concepts of algebraic and covariational reasoning. Finally, we have met our previous year’s goal of developing a special journal issue, which will be published in 2024. The work of this group, as documented in this report, is immense and spans various research topics in different mathematical contexts and student populations, as well as a diverse audience of researchers (experts and novices) and educators. The future work of this group aims to continue to explore the power of units coordination in modeling student thinking with a focus on understanding how units coordination can be used to promote equitable instruction of mathematics that engages all learners.

### **References**

- Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying Covariational Reasoning While Modeling Dynamic Events: A Framework and a Study. *Journal for Research in Mathematics Education*, 33(5), 352–378. <https://doi.org/10.2307/4149958>
- Norton, A., Boyce, S., Phillips, N., Anwyll, T., Ulrich, C., & Wilkins, J. L. M. (2015). A written instrument for assessing students’ units coordination structures. *IEJME-Mathematics Education*, 10(2), 111-136.
- Steffe, L. P. (2017). Psychology in Mathematics Education: Past, Present, and Future. (Eds.), *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 27-56). Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

## CONCEPTUALIZING THE ROLE OF TECHNOLOGY IN EQUITABLE MATHEMATICS CLASSROOMS

Nicholas Witt  
Western Michigan University  
nicholas.e.witt@wmich.edu

Kayla Chandler  
East Carolina University  
chandlerk@ecu.edu

Jennifer Suh  
George Mason University  
jsuh4@gmu.edu

Allison McCulloch  
UNC Charlotte  
allison.mcculloch@uncc.edu

Karen Hollebrands  
NC State University  
kfholleb@ncsu.edu

Charity Cayton  
East Carolina University  
caytonc@ecu.edu

Jon Davis  
Western Michigan University  
jon.davis@wmich.edu

*This new working group will examine ways in which technology can be leveraged as a tool to support equitable opportunities to learn for every mathematics student. We plan to engage participants in discussion regarding current technology- and equity-centered frameworks that are used, develop technologically enhanced tasks to support equity, and determine a series of next steps toward furthering the goal of better understanding and bolstering existing technology-centered frameworks to be more aligned with equitable teaching. The group aims to provide the community with a welcoming space for diverse voices to generate a line of research specifically focused on bridging equity-centered frameworks with technology-centered frameworks.*

Keywords: Technology, Equity

Calls for mathematics educators to incorporate technology have become focal points in teacher preparation (AMTE, 2017; NCTM, 2020). Simultaneously recommendations for the incorporation of equitable teaching practices have largely been made independent of technology (e.g., NCTM, 2020). However, recent considerations of how technology can be a tool to support equity have been initiated (Author Z, 2022) and recommended (AMTE, 2022). This new working group plans to lay out varied perspectives on how we envision work with technological tools to be a means for providing more equitable learning opportunities for mathematics students. The working group is comprised of both early career and experienced scholars who've focused a significant amount of their scholarly activity on examining how technological tools are utilized in K-12 curriculum and classrooms, how teachers learn to teach with new technological tools, and how technological environments support student learning. Some of us have discussed together the ways we see technology as a tool for promoting equity in mathematics classrooms, but we are in the beginning stages of developing a line of research specifically focused on bridging equity-centered frameworks with technology-centered frameworks. Thus, we propose this working group as starting point for developing a research program dedicated to defining, describing, and exploring new connections between equity-minded teaching and learning and equitable uses of technology in K-12 classrooms. We invite scholars of all backgrounds to contribute to the conversation, help refine tasks and interventions, and plan research agendas centered on exploring how technology can promote equitable classrooms.

## Theoretical Background

Prior related work has focused on developing tools for analyzing technologically enhanced lessons for their alignment with equity-centered teaching practices (Author Z et al, 2022; Authors et al., 2021a). Author Y (2022) explored how technology can be a tool support the development of a sociopolitical disposition. Authors X (2017; 2015) developed a framework for analyzing tasks for their ability to promote high-cognitive demand tasks, which we see as a necessary component of equitable instruction, but the hope is to focus and perhaps better align this framing with other dimensions of equitable mathematics instruction. Authors (2021b) have provided an overview of frameworks on how the incorporation of technology is framed by mathematics teacher educators, but the role of equitable practices is unclear in these frameworks.

To frame our equity-related work, we consider several dimensions that have been shown to promote equitable mathematics teaching and learning. In particular, we see: the TRU framework constructs of Access, Agency, and Authority (Schoenfeld & The Teaching for Robust Understanding Project, 2016); the roles Power and Positioning (Esmonde & Langer-Osuna, 2013); Identity (Aguirre et al., 2013) and Teaching Mathematics for Social Justice (Gutstein, 2003) as starting points for this discussion and will connect these constructs to what we conjecture are technologically-enhanced approaches to promoting them. In our sessions we hope to foster an environment where others contribute their equity-centered framing to help refine our thinking about what it means to equitably teach mathematics with technology.

### Focus and Goals of the Working Group

The goal of this working group is to lay the groundwork for a research program dedicated to better understanding how technology can be used as a tool to support equity in mathematics education. While existing frameworks dedicated to equitable mathematics teaching and teaching with technology have developed independently of one another, we aim to bolster existing technology-centered frameworks with considerations of equity from various dimensions. In support of this goal, we will begin by presenting recent related research and frameworks that we feel align with equitable teaching. Then we will provide examples of technologically enhanced tasks and interventions that we believe could support equity. The interactive nature of the sessions will be focused on gathering ideas for developing tasks, generating research ideas and questions, and forming subgroups to explore the various ways in which technology and equity may intersect. We hope others will join us in this initial phase and in future PME-NA Conferences to develop this research program.

**Table 1: Plan for Engagement**

	Activities	Next Steps
Session One	<ol style="list-style-type: none"> <li>1. Introductions and goals</li> <li>2. Our work around technology and equity</li> <li>3. Questions and small group discussions about technology and equity frameworks</li> </ol>	<ol style="list-style-type: none"> <li>1. Come to Session Two prepared to share how/what other frameworks may inform your work.</li> </ol>
Session Two	<ol style="list-style-type: none"> <li>1. Small group discussions about which frameworks we should consider</li> <li>2. Group debrief to share current stances on how technology can support equity</li> </ol>	<ol style="list-style-type: none"> <li>1. Think about what subgroups would be useful to examine various perspectives on equity.</li> </ol>
Session Three	<ol style="list-style-type: none"> <li>1. Form subgroups and plan</li> <li>2. Subgroup share-out</li> <li>3. Reflections and future work plan</li> </ol>	<ol style="list-style-type: none"> <li>1. Implement subgroup plan</li> <li>2. Prepare to share subgroup work next year</li> </ol>

## References

- Aguirre, J. M., Mayfield-Ingram, K., & Martin, D. B. (2013). *The impact of identity in K-8 mathematics: Rethinking equity-based practices*. National Council of Teachers of Mathematics.
- Association of Mathematics Teacher Educators (2022). *Press Release: AMTE statement on technology*. <https://amte.net/news/2022/06/press-release-amte-statement-technology>
- Association of Mathematics Teacher Educators. (2017). *Standards for preparing teachers of mathematics*. <https://amte.net/standards>
- Author Z et al. (2022).
- Author Y (2022).
- Authors (2021a).
- Authors (2021b).
- Authors X (2017).
- Author (2015).
- Esmonde, I., & Langer-Osuna, J. M. (2013). Power in numbers: Student participation in mathematical discussions in heterogeneous spaces. *Journal for Research in Mathematics Education*, 44(1), 288-315.
- Gutstein, E. (2003). Teaching and learning mathematics for social justice in an urban, Latino school. *Journal for Research in Mathematics Education*, 34(1), 37-73.
- National Council of Teachers of Mathematics (NCTM). (2020). *Standards for the preparation of secondary mathematics teachers*. <https://www.nctm.org/Standards-and-Positions/CAEP-Standards/>
- Schoenfeld, A. H., & the Teaching for Robust Understanding Project. (2016). *An introduction to the teaching for robust understanding (TRU) framework*. Graduate School of Education. [https://www.map.mathshell.org/trumath/intro\\_to\\_tru\\_20161223.pdf](https://www.map.mathshell.org/trumath/intro_to_tru_20161223.pdf)

## WORKING GROUP REPORT: TAKING UP MATHEMATICS EDUCATION RESEARCH SPACES AS RESISTANCE: TOWARD DISABILITY JUSTICE

Kathryn R. Westby  
Michigan State University  
westbyka@msu.edu

Alison Mirin  
University of Arizona  
amirin@math.arizona.edu

Rachel Lambert  
UC Santa Barbara  
rlambert@ucsb.edu

Paulo Tan  
University of Missouri SL  
paulotan@umsl.edu

Cathery Yeh  
University of Texas Austin  
cathery.yeh@utexas.edu

Katherine E. Lewis  
University of Washington  
kelewis2@uw.edu

Annette Bagger  
Örebro University  
Anette.Bagger@oru.se

Amber Candela  
University of Missouri SL  
candela@umsl.edu

Jessica Hunt  
North Carolina State  
Jhunt5@ncsu.edu

Erica N. Mason  
University of Illinois  
enmason@illinois.edu

Juuso Nieminen  
Hong Kong University  
juuso@hku.hk

Alexis Padilla  
University of New Mexico  
apadilladiv@gmail.com

Kai Rands  
Independent Scholar  
randske@gmail.com

James Sheldon  
University of Arizona  
jsheldon@email.arizona.edu

*In continuing with past working groups on research at the intersections of disability studies and mathematics education, this working group met for PME-NA 44 centering on Disability Justice as a framework to share new developments from group members, to make new connections, and work toward future directions. Over the course of the three days, we built community, broke up into subgroups, and planned for next steps beyond the bounds of the working group meetings.*

Keywords: students with disabilities; special education; equity, inclusion, and diversity

### **History, Motivation, and Focus of the Group**

Our PME-NA working group formally met from 2016 to 2018. A core group consisting of faculty, graduate students, disability activists, and classroom teachers discussed theoretical perspectives of disability they employed within their work in mathematics education, current projects they were engaged in related to disability and mathematics education, and critical issues members often found themselves confronting in taking up this complex work.

We plan to continue and expand collaborations between members of this working group with the intent of looking ahead to how this group will structure research through intersectional disability justice approaches to address the enduring challenge of reframing disability in mathematics education. This group will create a space in which all involved can study with and from each other with the purpose of establishing collaborative and generative work throughout the year. The group provides opportunities to connect with others and engage in ongoing

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

projects. It also helps us nurture our growing community of scholars with interest in disability justice, mathematics education, and critical special education.

### **Theoretical Background**

We center on Disability Justice, a theoretical framework developed through the grassroots efforts of disabled queer activists and disabled people of color (Sins Invalid, 2019). It focuses on the relationship between ableism and other interlocking systems of oppression, such as racism, sexism, imperialism, settler colonialism, and capitalism. We specifically draw upon the 10 principles for disability justice created by Patty Berne and others from Sins Invalid (Sins Invalid, 2019): intersectionality, leadership of the most impacted, anti-capitalist politics, cross-movement solidarity, recognizing wholeness, sustainability, cross-disability solidarity, interdependence, collective access, and collective liberation. Given the legacy of harm done to disabled communities, Disability Justice's centralization of the needs and experiences of folk experiencing intersectional oppression allows researchers to recognize the intersecting legacies of white supremacy, colonial capitalism, gender oppression, and ableism to understand how people's bodies and minds are labeled "deviant", "unproductive" and/or "invalid." (Sins Invalid, 2019). Traditionally, researchers determine the questions asked, the methods of data collection, and the meaning made of the data, with no input from marginalized "subjects". We acknowledge that disabled students and communities of color have gifts (Annamma & Morrison, 2018) such as unique mathematical ideas and perspectives and solutions to challenge current structures of ableism (Author, Date). Bringing these ideas into the research process can shift power. For example, we call attention to an emancipatory research study (Author, date) where Author (Author's description) was positioned as the inquirer and Author (Author's description) as the expert. We aim to build out theories of intersectional disability justice in and through mathematics education.

### **Summary of Prior Working Group Work**

Some of the major accomplishments of this working group include (a) establishing and maintaining an email list-serv which members have used regularly to share and solicit feedback on work and to organize our collective efforts, (b) co-editing a special [journal, description, title, and date of the special issue], (c) publishing an [Publisher] [book title and date], and (d) submitting an AERA grant proposal to support continuing this work (unfunded).

### **Work at PME-NA 44**

During PME-NA 44, 28 people participated in our hybrid working group. Several participants engaged in conversations that were conducted in a hybrid manner using zoom. This design of the conference allowed for participation and engagement by people who would have otherwise been excluded in an in-person only conference and working group. This exclusion can occur for a variety of reasons. For example, people at high risk for COVID-19 are not safe in a crowded indoor environment. Similarly, there are various accessibility concerns about travel and transportation. Additionally, many caregivers are unable to leave town for consecutive days. In our working group, we discussed the various ways that the hybrid access allowed for more inclusivity for disabled and otherwise marginalized scholars. Hybrid access allowed the first three authors of this document to meet and form collaborations across space.

Over the three days we built community, broke into subgroups, and planned for next steps beyond the bounds of our working group meeting. We created multiple folders of collaborative google docs to document our ongoing conversations both as a whole group and in our breakout groups. In order to document the ongoing conversation and notes, we created a collaborative google doc.



## **Day 1: Community, Caring, and Co-Learning**

On Friday, our group started with introductions and sharing access needs. The importance of the invitation to share access needs is discussed by activists and scholars (i.e. Reinholz & Ridgway, 2021; Sins Invalid, 2019). The sharing of access needs was part of our introductions. An example (these are not exact examples, but do use the terminology and wording used by individuals present) of how these introductions may be stated is: “I am [name], and I am at [university], my pronouns are she, her, hers, and it is important that I am able to get up and stretch, and I need to take breaks periodically,” and, “I am [name], and I am a faculty member at [university] my pronouns are he, him, his, and my access needs are being met through having the virtual environment.” Through these introductions, our group immediately built camaraderie as people identified similar needs and ways to collectively meet various needs. The sharing of roles also allowed individuals in similar positions to know each other as well as individuals in need of professional advice from those in other positions to know who they might approach. During these introductions, individuals’ name and university affiliation were recorded on a shared google doc so that group members could reach out to each other.

We discussed areas for discussion, and these were documented in the google doc as well. We then identified themes and grouped the topics into these.

## **Day 2: Generating and Planning**

At our day 2 meeting, our group practiced challenging the view of access as a set of individual needs that create work for others in favor of the view of access as a collective responsibility, thereby moving toward a collaborative actualization of Disability Justice (Piepzna-Samarasinha, 2019). One small concrete example of this occurred as people arrived to the meeting on day 2. Group members immediately began collaboratively rearranging chairs and tables, with members joining in as they arrived, in order to support multiple access needs simultaneously, even if those needs were not shared by those doing the rearranging. They did this because creating greater access meant we all had more understanding of each other’s access needs and we were thus all able to engage together more deeply.

We began our day 2 meeting with (re)introductions and an access check-in followed by a short agenda for the day, which centered around how we incorporate Disability Justice in mathematics education. We then split into subgroups based on the previous day’s conversation. The groups were: (a) Disability Justice in mathematics education special issue, (b) a task force to create a response to the “Science of Math,” and (c) Disability Justice for teachers and pre-service teachers. The “Science of Math” is a website created by special education math educators that had problematic content (in our opinion), such as devaluing problem-solving and inquiry in math. We discussed what each group might work on, and how we would share these conversations moving forward. We identified the fact that we had been narrowing our focus to an ontology of disability and ableism aligned with the neoliberal academy and special education. We discussed how to reach for Disability Justice, we need to acknowledge intersectionalities and the ways in which each of us have been affected by and/or benefitted from racism and ableism. We discussed avoiding a return to an onto-epistemologically limited conceptualization of disability and ableism that is perpetuated by the neoliberal agenda in education. Education research is often done by white, nondisabled people who write about individuals with disabilities and employ racist and ableist methodologies, and they asked us to remember to question who we are speaking *for* whom we should be speaking *with*. The group had an engaging conversation about Disability Justice, *ableism* defined intersectionally (i.e. Lewis, 2022), and disability. We realized that not everyone had the background knowledge of Disability Justice activism,

disability studies, or DisCrit. So, that evening some in the group created a google doc with terminology and resources that was shared with the group in the working group's Google folder. After the conversation, the group split into the sub-groups. Each sub-group created a google doc in the shared folder and maintained notes on their conversations. The group came back together and briefly discussed the small group work.

### **Day 3: Planning for Next Steps**

In our final day we worked towards establishing how these connections and subgroups could continue to make progress throughout the year. We assigned leads to the logistical and intellectual work that we wanted to move forward before our next PMENA working group meeting. In addition to the small groups from the previous day, this work included (a) updating the email Listserv, (b) creating a Google sites repository for group members' work, (c) creating a group position statement, and (d) addressing accessibility at mathematics education conferences. Members of the group debriefed, summarized what we had accomplished, and planned next steps.

### **Continuing the Work**

In January of 2023, three working group members coordinated efforts to submit a symposium proposal to the 12th International Conference of Mathematics Education and Society taking place in Brazil during the summer of 2023. This symposium aims to continue the working group's conversation in a global context. Six members of the working group will be presenting in the proposed symposium. We submitted the proposal on February 10, 2023.

From December of 2022 through February of 2023, three members of the working group began developing a book proposal for Teachers College Press' Disability, Equity, & Culture series. Guided by the slogan used in the disability rights and justice movement, "Nothing about us without is us for us," we aim to use Critical Disability Studies and its branches and offshoots as a way to explore mathematics education policies, pedagogies, and possibilities in dismantling problematic notions of difference and normativity, within and beyond the mathematics classroom. We aim to finalize the proposal to submit for review by 2/28/23.

The subgroup on Accessibility at mathematics education conferences has created a working collaborative google doc where we document and describe various ways that mathematics education conferences can be more accessible and inclusive. We have communicated with each other about how to address these issues by creating concrete actionable items and discussed ways to help conference organizers understand our concerns. One challenge we have had in doing so is balancing our interest in increasing accessibility with our belief that the burden of access should not only be on disabled people. We must be considerate of the labor that we are asking of others (and who we are asking it of) while at the same time advocate for our access. Our discussions have included the ways in which the labor to create access as viewed as extra or added burden, while other forms of labor that is aimed at abled participants is viewed as unquestioned.

### **References**

- Annamma, S., & Morrison, D. (2018). DisCrit classroom ecology: Using praxis to dismantle dysfunctional education ecologies. *Teaching and Teacher Education*, 73, 70–80. <https://doi.org/10.1016/j.tate.2018.03.008>
- Ho, S. (2020). Canfei to Canji: The freedom of being loud. In A. Wong (Ed) *Disability visibility: First-person stories from the twenty-first century* (pp 112–116): Penguin Random House
- Lewis, K. E. & Lynn, D. M. (2018) An insider's view of a mathematics learning disability: Compensating to gain access to fractions, *Investigations in Mathematics Learning*, 10(3), 159-172
- Lewis, T. in community with disabled Black/negatively racialized folk especially @NotThreeFifths (2022). Working definition of ableism- January 2022 update. [bit.ly/ableism2022](http://bit.ly/ableism2022)

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

- Piepzna-Samarasinha, L. L. (2018). Care work: Dreaming disability justice (p. 182). Vancouver: arsenal pulp press.
- Reinholz, D. L., & Ridgway, S. W. (2021). Access needs: Centering students and disrupting ableist norms in STEM. *CBE—Life Sciences Education*, 20(3), es8.
- Sins Invalid. (2019). *Skin, tooth, and bone: The basis of movement is our people* (2nd ed.).
- Tan, P., & Kastberg, S. (2017). Calling for research collaborations and the use of dis/ability studies in mathematics education. *Journal of Urban Mathematics Education*, 10(2). Retrieved from <http://ed-osprey.gsu.edu/ojs/index.php/JUME/article/view/321>

## CRITICAL DISABILITY STUDIES IN MATHEMATICS EDUCATION

Alison Mirin  
University of Arizona  
amirin@math.arizona.edu

Kathryn R. Westby  
Michigan State University  
westbyka@msu.edu

Rachel Lambert  
UC Santa Barbara  
rlambert@ucsb.edu

Paulo Tan  
University of Missouri SL  
paulotan@umsl.edu

Cathery Yeh  
University of Texas Austin  
cathery.yeh@utexas.edu

Katherine E. Lewis  
University of Washington  
kelewis2@uw.edu

Erica N. Mason  
University of Illinois  
enmason@illinois.edu

Annette Bagger  
Örebro University  
Anette.Bagger@oru.se

Amber Candela  
University of Missouri SL  
candela@umsl.edu

Jessica Hunt  
North Carolina State  
Jhunt5@ncsu.edu

Juuso Nieminen  
Hong Kong University  
juuso@hku.hk

Alexis Padilla  
University of New Mexico  
apadilladiv@gmail.com

Kai Rands  
Independent Scholar  
randske@gmail.com

James Sheldon  
University of Arizona  
jsheldon@email.arizona.edu

*The purpose of this group is to move forward towards collective action within the disabled community. Research on mathematics and disabilities traditionally has been conducted within a special education paradigm, which often adopts a deficit model while ignoring the social, discursive, political, and structural context. We must take up equity-oriented approaches to understanding disability and its intersections by utilizing frameworks that honor those most impacted. We welcome all who are interested to join.*

Keywords: students with disabilities; special education; equity, inclusion, and diversity

“Taking up space as a disabled person is always revolutionary” (Ho, 2020, p. 115).

According to the 2020 equity statement, PME-NA is engaging in “recentering equity and criticality.” Unfortunately, disability and its intersections with other marginalizations is often ignored in discussions of equity and diversity, including those in mathematics education (Author date; Author, date). Because disabled people are marginalized in STEM, we ask that mathematics educators enact radical changes to center issues of/with/by the disabled. Our working group draws upon critical theories such as Disability Studies in Education, Critical Race Theory, and DisCrit in order to offer a justice-oriented vision of mathematics education based on conceptualizations of disability and ableism in relation to other forms of oppression and identities.

### History of the Working Group

Our working group formally met from 2016 to 2018, and then again at the 2022 conference.

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

Some of our major accomplishments include (a) establishing and maintaining an active email list-serv; (b) publishing a special issue in the journal *Investigations in Mathematical Learning* called *Critical Approaches for Mathematics Learning of Students with Disabilities* (2018); (c) publishing an NCTM book titled *Humanizing Disability in Mathematics Education: Forging New Paths* (2019); and (d) submitting an AERA grant proposal to support continuing this work (unfunded). Since the 2022 meeting, we have been collaborating in subgroups to (a) write a paper about special education mathematics research; (b) create a second special issue on critical approaches to disability in mathematics; (c) submit a book proposal to Teachers College Press on critical perspectives on disability in mathematics education; (d) work together to create a repository of approaches to integrating critical approaches to disability in mathematics teacher education; (e) advocate to improve teacher resources about mathematics and disability; and (f) advocate for access at mathematical education conferences.

This year, we will maintain and expand collaborations within this working group and work on restructuring research through intersectional disability justice approaches. The space at PME-NA provides opportunities to connect with others and engage in projects with a critical lens. One of the goals of our working group is to find concrete ways to increase the accessibility of the conference so that disabled scholars can fully participate. We cannot ignore ableism, even in our own spaces. Our working group provides a much-needed space for disabled scholars. Disabled students are often denied necessary accommodations in mathematics courses (Author, date) and disabled faculty are denied accommodations (Price, 2021). This working group allows disabled and nondisabled mathematics educators to discuss not only the accessibility of mathematics education for disabled students, but also for disabled scholars. Our working group has resulted in mentorship for junior scholars. This work is particularly important, given the 2023 PME-NA conference theme of engaging all learners.

### **Theoretical Foundations**

We theorize ableism through disability studies and critical special education. We seek to disrupt ideologies of ability and normal (i.e. Annamma et al., 2013; Siebers, 2008) to study the pervasiveness of ableism in education. We apply scholarship on ableism to investigate how this operates in mathematics education (Dolmage, 2017; Price, 2021; Westby, 2021).

We center on Disability Justice framework, which was developed by disabled queer activists and disabled people of color (Sins Invalid, 2019). Given the harm done to disabled communities, Disability Justice allows researchers to recognize the intersecting legacies of white supremacy, colonial capitalism, gender oppression, and ableism to understand how people are labeled “deviant”, “unproductive” and/or “invalid” (Sins Invalid, 2019). Traditionally, researchers complete their studies with no input from disabled individuals of color (Oliver, 1992, 1997). Acknowledging disabled students and communities of color have gifts (Annamma & Morrison, 2018) such as unique mathematical ideas and perspectives and solutions to challenge the deficit narrative and structures of ableism (Author, Date). For example, members of the working group have published an emancipatory research study (Author, Date) where Lewis (disabled researcher) was positioned as the inquirer and Lynn (student with dyscalculia) as the expert.

### **Plan for Active Engagement**

- Session 1: Community, Caring, & Co-learning – Introductions, access needs, and community building (40min), and discussing an agenda for the group for the upcoming year. Describe ongoing projects and welcome new members to join (50min)

- Session 2: Generating and Planning - Identify potential collaborative endeavors and describe ongoing projects (30min), break up into subgroups based on areas of interest (40min), and regroup as a large to share out (20min)
- Session 3: Planning for Next Steps - Subgroup work (40min), large group share out and conversations (20min), and closing and identifying concrete next steps (30min)

### References

- Annamma, S. A., Boelé, A. L., Moore, B. A., & Klingner, J. (2013). Challenging the ideology of normal in schools. *International Journal of Inclusive Education*, 17(12), 1278-1294.
- Annamma, S. A., Connor, D., & Ferri, B. (2013). Dis/ability critical race studies (DisCrit): Theorizing at the intersections of race and dis/ability. *Race Ethnicity and Education*, 16(1), 1–31. <https://doi.org/10.1080/13613324.2012.730511>
- Annamma, S., & Morrison, D. (2018). DisCrit classroom ecology: Using praxis to dismantle dysfunctional education ecologies. *Teaching and Teacher Education*, 73, 70–80. <https://doi.org/10.1016/j.tate.2018.03.008>
- Barnes, C. (2003). What a difference a decade makes: reflections on doing ‘emancipatory’ disability research. *Disability and Society*, 18(1), 3–17.
- Dolmage, J. T. (2017). *Academic ableism: Disability and higher education* (p. 244). University of Michigan Press.
- Ho, S. (2020). Canfei to Canji: The freedom of being loud. In A. Wong (Ed) *Disability visibility: First-person stories from the twenty-first century* (pp 112–116): Penguin Random House
- Lewis, K. E. & Lynn, D. M. (2018) An insider’s view of a mathematics learning disability: Compensating to gain access to fractions, *Investigations in Mathematics Learning*, 10:3, 159-172
- Mirin, A., & Tan, P. (2022). Disability Accommodations in College: Alarming Discrimination in Mathematics. *Proc. of the 24th Annual Conference on Research in Undergraduate Mathematics Education (RUME)*. (pp. 1105-1110).
- Oliver, M. (1992). Changing the social relations of research production?. *Disability, Handicap & Society*, 7 (2),101–114.
- Oliver, M. (1997). Emancipatory research: Realistic goal or impossible dream? In C. Barnes & G. Mercer (Eds.), *Doing Disability Research* (pp. 15–31). Leeds, UK: The Disability Press.
- Price, M. (2021). Time harms: Disabled faculty navigating the accommodations loop. *South Atlantic Quarterly*, 120(2), 257-277.
- Siebers, T. (2008). *Disability theory*. University of Michigan Press.
- Sins Invalid. (2019). *Skin, tooth, and bone: The basis of movement is our people* (2nd ed.).
- Tan, P., & Kastberg, S. (2017). Calling for research collaborations and the use of dis/ability studies in mathematics education. *Journal of Urban Mathematics Education*, 10(2). Retrieved from <http://edosprey.gsu.edu/ojs/index.php/JUME/article/view/321>
- Westby, K.R. (2021). Disability in mathematics education research: A call to action. *Proceedings of the 43rd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*.

## DESCRIBING THE WORK OF MANAGING CLASSROOM DISCUSSIONS: ATTENDING TO THE SUBJECT-SPECIFIC ASPECTS OF TEACHERS' PRACTICES

Keywords: Classroom Discourse, Teacher Knowledge, Problem-Based Learning

Gil Schwarts (Coordinator)  
University of Michigan  
gils@umich.edu

Patricio Herbst  
University of Michigan  
pgherbst@umich.edu

Michelle Cirillo  
University of Delaware  
mcirillo@udel.edu

Alison Castro Superfine  
University of Illinois Chicago  
amcastro@uic.edu

Amy Dunning  
University of Illinois Springfield  
ahewi2@uis.edu

Margaret Walton  
University of Maryland  
mwalton@umd.edu

Hala N. Ghouseini  
University of Wisconsin Madison  
ghousseini@wisc.edu

This new working group is established to foreground discussions about the subject-specific aspects of teachers' facilitation of classroom discussions. Our goal is to lay groundwork for developing a shared understanding of the mathematical aspects of teachers' practices, by: 1) discussing and defining what we mean by subject-specificity; 2) sharing different perspectives of attending to subject-specific aspects; 3) engaging with those different perspectives in a joint analysis of lesson artifacts; 4) discussing research gaps and challenges, and enabling collaborations among interested researchers that may lead to relevant insights and new investigations on this topic.

### Theoretical background

More than three decades ago, Wilson et al. (1987) wondered: "Is pedagogy a set of generic principles that can be applied to any discipline? Is there a subject-specific aspect of pedagogy?" (p. 105). While there has been work on *measuring* instructional quality or knowledge under a subject-specific lens (Hill et al., 2008; Knievel et al., 2015; Schlesinger et al., 2018), we are interested in exploring Wilson and colleagues' (1987) questions with respect to fine-grained *descriptions* of mathematics teachers' practices when they are interacting with student thinking. Following the efforts of educators and policymakers, in the recent two decades there has been an increasing popularity of discussion-based pedagogies in the discourse about mathematics teaching. Despite the wealth of research on instructional practices for discussions (e.g., Ball, 1993; Chapin et al., 2009; Stein et al., 2008), there has not been enough scrutiny into what aspects of the teachers' work in this context are subject-specific vs. generic. Although some research has distinguished between mathematical and non-mathematical teacher practices (e.g., Goldsmith & Seago, 2011), oftentimes the term "mathematical" is not further unpacked or decomposed. A similar concern, focusing on descriptions of responsive teaching, was raised by Robertson et al. (2016) who argued that studies in this area "rarely distinguish among [how teachers respond to] the various facets of disciplinary substance within student thinking" (p. 228). Thus, our goal is to start a conversation that helps us, as a field, to be more explicit about how the mathematical content being taught is present in teachers' practices and represented in researchers' descriptions. The guiding questions for this work are: *How do different researchers define "subject-specificity"? In what ways can these definitions be operationalized to the*

*analysis of management of classroom discussions? When studying mathematics teacher practices, how do we balance the tension between general and idiosyncratic descriptions?*

We see these questions as relevant not only for research purposes but also for informing coaching and professional development activities that aim to promote productive mathematical discussions. These dual purposes are reflected in the design of the working group sessions.

### **Structure of the Sessions**

**Day 1** will begin with introductions, a short presentation of the working group topics and goals, and an attempt to define together what we mean by subject-specificity (30 minutes). This will be followed by three perspectives on *descriptions of subject-specific aspects of teachers' practices*: (a) Amy Dunning will present a framework for selecting strategies for whole-class discussion, highlighting how the first level of the framework could potentially be used across multiple content areas (generic), yet emphasizing the importance of understanding students' mathematical thinking (subject-specific) when selecting strategies for whole-class discussions; (b) Michelle Cirillo's presentation will illustrate the use of subject-specific lenses in the research of geometry instruction, focusing on the importance of attending to logic within proof discourse, and using precise language when working with definitions; (c) Patricio Herbst and Gil Schwartz will present a framework that describes mathematics teachers' practices in problem-based lessons in terms of the mathematical goal of the lesson, the norms of the instructional situation, and the properties of the mathematical problem (30 minutes). This section will be followed by small group discussions where participants jointly analyze lesson artifacts, drawing on the presented perspectives. We will then convene a whole-group discussion to hear the small group insights (30 minutes).

**Day 2** will focus on the question: *What professional development design elements support teachers to learn to lead content-specific discussions?* The session will begin with a summary of the discussions from Day 1 to launch the work of Day 2 (10 minutes). Then, three researchers will present their work (10 minutes each): (a) Alison Castro Superfine will share her perspective on the questions: How do teachers learn to engage students in productive mathematical discussions? What is a productive discussion, under the lens of subject-specificity? How is the facilitation of mathematics discussions different from discussions in other domains?; (b) Margaret Walton will share ideas on how to develop and implement content-specific video clubs that not only help teachers make sense of students' thinking, but also assist them in responding to student thinking and support their learning of particular mathematical concepts; (c) Hala Ghouseini's presentation will showcase two professional learning pedagogies that aim at developing teachers' Mathematical Knowledge for Teaching (MKT) during enactments of leading discussions – Enlisting MKT during *rehearsals*, and real-time collaborative decision-making during *Learning Labs*. Then, small groups will collaboratively discuss and annotate vignettes of classroom and PD discussions (30 minutes). The session will end with small-group presentations to the whole group (20 minutes).

**Day 3** will be devoted to drawing conclusions about what was learned from the discussions in the previous days, discussing research gaps, and developing future plans. We will engage participants in a mind-mapping activity and provide time for participants to work in small groups according to their interests, with the following guiding questions: What are the challenges of subject-specific research, in light of the previous two sessions? What theories and methodologies can inform subject-specific research? What research is needed in this area? (One hour). The final 30 minutes of the session will focus on articulating next steps for the working group.



## References

- Ball, D. L. (1993). With an eye on the mathematical horizon: Dilemmas of teaching elementary school mathematics. *The Elementary School Journal*, 93(4), 373-397. <https://www.jstor.org/stable/1002018>
- Chapin, S. H., O'Connor, M. C., & Anderson, N. C. (2009). *Classroom discussions: Using math talk to help students learn, Grades K-6*. Math Solutions.
- Goldsmith, L. T., & Seago, N. (2011). Using classroom artifacts to focus teachers' noticing: Affordances and opportunities. In M. G. Sherin, V. R. Jacobs, & R. A. Philipp (Eds.), *Mathematics teacher noticing: seeing through teachers' eyes* (pp. 169–187). Routledge.
- Hill, H. C., Ball, D. L., & Schilling, S. G. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education*, 39(4), 372-400. : <https://www.jstor.org/stable/40539304>
- Knievel, I., Lindmeier, A. M., & Heinze, A. (2015). Beyond knowledge: Measuring primary teachers' subject-specific competences in and for teaching mathematics with items based on video vignettes. *International Journal of Science and Mathematics Education*, 13, 309-329. <https://doi.org/10.1007/s10763-014-9608-z>
- Robertson, A. D., Richards, J., Elby, A., & Walkoe, J. (2016). Documenting variability within teacher attention and responsiveness to the substance of student thinking. In A. D. Robertson, R. E. Scherr, & D. Hammer (Eds.), *Responsive teaching in science and mathematics* (pp. 227–248). Taylor & Francis.
- Schlesinger, L., Jentsch, A., Kaiser, G., König, J., & Blömeke, S. (2018). Subject-specific characteristics of instructional quality in mathematics education. *ZDM – Mathematics Education*, 50, 475-490. <https://doi.org/10.1007/s11858-018-0917-5>
- Stein, M. K., Engle, R. A., Smith, M. S., & Hughes, E. K. (2008). Orchestrating productive mathematical discussions: Five practices for helping teachers move beyond show and tell. *Mathematical Thinking and Learning*, 10(4), 313-340. <https://doi.org/10.1080/10986060802229675>
- Wilson, S. M., Shulman, L. S. & Richert, A. (1987). “150 different ways of knowing”: Representations of knowledge in teaching. In J. Calderhead (Ed.), *Exploring Teachers' Thinking* (pp. 104-124). Cassell.

## **GENDER AND SEXUALITY IN MATHEMATICS EDUCATION WORKING GROUP: SUPPORTING LEARNERS AND SCHOLARS THROUGH OUR STORIES**

Weverton Ataide Pinheiro  
Texas Tech University  
wataidep@ttu.edu

Jennifer Hall  
Monash University  
Jennifer.hall@monash.edu

Katrina Piatek-Jimenez  
Central Michigan University  
Piate1kl@cmich.edu

Amanda Provost  
Montclair State University  
provosta1@montclair.edu

Ana Lúcia Braz Dias  
Central Michigan University  
Dias1al@cmich.edu

Brent Jackson  
WestEd  
bjackso@wested.org

*Research on gender and sexuality continues to be of great importance to support all students in the teaching and learning of mathematics. The goal of this year's Gender and Sexuality in Mathematics Education Working Group is to build a repertoire of how its members became involved in doing gender and sexuality research, what challenges and supports we have experienced, and what stories we have told through our research. To achieve our goal, we will collect data before, during, and after the conference. During PME-NA 2023, we will collect data through focus group interviews. We aim to learn from one another about our own personal experiences and the work that has been done in gender and sexuality research. We will continue this work after PME-NA 2023 and present our findings at PME-NA 2024.*

Keywords: Gender; LGBTQIA+; Equity, Inclusion, and Diversity; Social Justice

In mathematics education, the focus of gender and sexuality research has been to dismantle inequities experienced by members of underrepresented groups in the teaching and learning of mathematics (Rubel, 2016; Voigt, 2020; Waid, 2020). This type of research is essential for equity and justice so that all learners are supported in their formal schooling (Ataide Pinheiro, 2022). Throughout the years, gender and sexuality research has been conceptualized in multiple ways. Over the past two decades, there has been an increased emphasis among gender and/or sexuality researchers in mathematics education on the theorization of gender. This is in response to the observation that initially many mathematics educators did not theorize gender in their studies, often using "gender" and "sex" interchangeably (Damarin & Erchick, 2010; Esmonde, 2011). In more recent work, many scholars who study gender in mathematics education have been careful in defining gender and distinguishing between gender and sex (Ataide Pinheiro, 2021; Neto & Ataide Pinheiro, 2021; Hall et al., 2022; Lubienski & Ataide Pinheiro, 2020). Gender can be defined as a performative and discursive social construct that varies according to different social contexts and individuals (Butler, 1990; Leyva, 2017).

Sexuality research in mathematics education is still an emerging field, and much of this work that has been done involved queer theory (Dubbs, 2016; Rands, 2009; Voigt, 2020). Through this perspective, sexuality is a fluid social construct relating to how individuals are emotionally, physically, and/or romantically attracted to others (Gillis & Jacobs, 2020). Findings that come from sexuality research in mathematics education comprise the understanding that mathematics teaching and learning propagate sexual and gender identity normativity and heteronormativity (Leyva, 2016; Moore, 2021; Rubel, 2016; Waid, 2020). Furthermore, people belonging to non-normative sexuality groups are excluded from mathematical fields (Ataide Pinheiro, 2022; Kersey & Voigt, 2021).

Building on these theoretical underpinnings and findings, this year's working group will focus on understanding the experiences of those who research gender and sexuality in mathematics education and the stories that these researchers have told through their research.

### **History and Goals**

During previous years, this working group has had the opportunity to engage with research in gender and sexuality in mathematics education in a variety of ways, including (a) exploring gendered mathematical experiences in different cultures around the world, (b) engaging with multiple theories and methodologies, and (c) analyzing data. This year, we aim to (a) collect additional data from participants in this year's working group, (b) analyze the data as a group between PME-NA 2023 and PME-NA 2024, and (c) disseminate the findings of our analysis through a journal article and PME-NA 2024 presentation. In particular, we are interested in answering the following research questions:

1. What are gender and sexuality researchers' stories regarding how they became involved in gender and sexuality research?
2. What challenges and supports have gender and sexuality researchers faced while undertaking their research?
3. What stories have gender and sexuality researchers told through their research?

Prior to PME-NA 2023, the working group leaders will obtain IRB approval for this project. We will also each write our narratives independently regarding how we came to be gender and sexuality researchers, and reflect on how our social identities and years of experience with this kind of research have affected these stories. Then, during the working group sessions, we will collect data through focus group interviews regarding our supports and challenges, and the stories we have told through our data. In the following sections, we discuss how we will engage participants in the three days of the working group sessions.

**On Day 1**, there will be a short presentation (30 minutes) about the study that we are conducting and the data collection process. Then, two leadership team members will conduct focus group interviews with working group participants in groups of five to six people (60 minutes). The Day 1 focus group will be about "our stories." Participants will be asked questions about their experiences as gender and sexuality researchers, in particular, what supports they received and what challenges they faced. The focus group interview will be audio recorded and later transcribed. If anyone chooses not to participate in the focus group interview, they would still be invited to listen and reflect on the discussion.

**On Day 2**, there will be a short presentation on theories and methodologies in gender and sexuality research (30 minutes). A second focus group will be conducted and led by two leadership team members (60 minutes). The Day 2 focus group interview will be about the stories that we have told from our gender and sexuality research. This interview will also be audio recorded and later transcribed.

**On Day 3**, we will organize ourselves to move forward with additional data collection, analyses, and dissemination of findings. New working group participants will be asked to write their narratives and to submit them within a month of the conclusion of the conference. We will also determine which leadership team will analyze the data collected, as well as determine our analysis process. The final product will be a manuscript for publication where we share our stories as gender and sexuality researchers. Furthermore, we will share our findings at the PME-NA 2024 conference.

## References

- Ataide Pinheiro, W. (2021). Dismantling the ‘all-boys club’ a narrative of contradictions women experiences in PhD mathematics programs: A Freirean approach. *International Electronic Journal of Mathematics Education*, 16(3), em0652. <https://doi.org/10.29333/iejme/11090>
- Ataide Pinheiro, W. (2022). *At the intersections: Queer high school students’ experiences with the teaching of mathematics for social justice* (Publication No. 29320623) [Doctoral dissertation, Indiana University]. ProQuest Dissertations & Theses Global.
- Butler, J. (1999). *Gender trouble: Feminism and the subversion of identity*. Routledge.
- Damarin, S. K., & Erchick, D. B. (2010). Toward clarifying the meanings of gender in mathematics education research. *Journal for Research in Mathematics Education*, 41(4), 310–323. <https://www.nctm.org/Publications/journal-for-research-in-mathematics-education/2010/Vol41/Issue4/?ref=1>
- Dubbs, C. (2016). A queer turn in mathematics education research: Centering the experience of marginalized queer students. In M. B. Wood, E. E. Turner, M. Civil, & J. A. Eli (Eds.), *Proceedings of the 38th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1041–1048). The University of Arizona Press.
- Esmonde, I. (2011). Snips and snails and puppy dogs’ tails: Genderism and mathematics education. *For the Learning of Mathematics*, 31(2), 27–31. <https://flm-journal.org/index.php?do=show&lang=en&showMenu=31%2C2>
- Gillis, M., & Jacobs, A. (2020). *Introduction to women’s and gender studies: An interdisciplinary approach*. Oxford University Press.
- Hall, J., Robinson, T., Flegg, J., & Wilkinson, J. (2022). First-year and final-year undergraduate students’ perceptions of university mathematics departments. *Mathematics Education Research Journal*, 34(2), 189–214. <https://doi.org/10.1007/s13394-020-00340-z>
- Kersey, E., & Voigt, M. (2021). Finding community and overcoming barriers: Experiences of queer and transgender postsecondary students in mathematics and other STEM fields. *Mathematics Education Research Journal*, 33(4), 733–756. <https://doi.org/10.1007/s13394-020-00356-5>
- Leyva, L. A. (2016). “Mapping the margins [in mathematics]”: Examining the gendered and racialized intersectionality of mathematics experiences among African American and Latina/o undergraduate students in a large, predominantly White university (Publication No. 10291929). [Doctoral dissertation, Rutgers University]. ProQuest Dissertations & Theses Global.
- Leyva, L. A. (2017). Unpacking the male superiority myth and masculinization of mathematics at the intersections: A review of research on gender in mathematics education. *Journal for Research in Mathematics Education*, 48(4), 397–433. <https://doi.org/10.5951/jresmetheduc.48.4.0397>
- Lubienski, S. T., & Ataide Pinheiro, W. (2020). Gender and mathematics: What can other disciplines tell us? What is our role? *Journal of Urban Mathematics Education*, 13(1), 1–14. <https://jume-ojs-tamu.tdl.org/JUME/issue/view/28>
- Moore, A. S. (2021). Queer identity and theory intersections in mathematics education: A theoretical literature review. *Mathematics Education Research Journal*, 33(4), 651–687. <https://doi.org/10.1007/s13394-020-00354-7>
- Neto, V., & Ataide Pinheiro, W. (2021). Análise comparativa entre Brasil e os Estados Unidos: O problema de gênero em livros didáticos de matemática [Comparative analysis between Brazil and the United States: The problem of gender in mathematics textbooks]. *Revista de Investigação e Divulgação em Educação Matemática*, 5(1), 1–20. <https://doi.org/10.34019/2594-4673.2021.v5.33216>
- Rands, K. (2009). Mathematical inqueery: Beyond “add-queers-and-stir” elementary mathematics education. *Sex Education*, 9(2), 181–191. <https://doi.org/10.1080/14681810902829646>
- Rubel, L. H. (2016). Speaking up and speaking out about gender in mathematics. *The Mathematics Teacher*, 109(6), 434–439. <https://doi.org/10.5951/mathteacher.109.6.0434>
- Voigt, M. (2020). *Queer-spectrum student experiences and resources in undergraduate mathematics* (Publication No. 28264669) [Doctoral dissertation, San Diego State University]. ProQuest Dissertations & Theses Global.
- Waid, B. E. (2020). Supporting LGBTQ+ students in K–12 mathematics. *Mathematics Teacher: Learning and Teaching PK–12*, 113(11), 874–884. <https://doi.org/10.5951/mtlt.2019.0403>

## WORKING GROUP REPORT: APPLYING THEORY TO DATA IN GENDER AND SEXUALITY MATHEMATICS EDUCATION RESEARCH

Weverton Ataide Pinheiro  
Texas Tech University  
wataidep@ttu.edu

Katrina Piatek-Jimenez  
Central Michigan University  
Piate1kl@cmich.edu

Jennifer Hall  
Monash University  
Jennifer.hall@monash.edu

Brent Jackson  
WestEd  
bjackso@wested.org

Julia Przybyla-Kuchek  
University of Georgia  
juliapk@uga.edu

Ana Lúcia Braz Dias  
Central Michigan University  
Dias1al@cmich.edu

*The PME-NA Gender and Sexuality in Mathematics Education Working Group was conceptualized at PME-NA 2017. Since 2018, the working group has re-convened annually to engage participants in current discussions on research in gender and sexuality in mathematics education. In this report, we describe the history of the working group, the activities that took place during PME-NA 2022, and plans for PME-NA 2023. At PME-NA 2022, working group participants engaged with data and analysis regarding the gendered and/or sexualized aspects of K–16 students' experiences. At PME-NA 2023, working group participants will engage in collecting and analyzing data about how mathematics education scholars became gender and sexuality researchers, the supports and challenges that they faced while conducting this type of research, and the stories told by gender and sexuality researchers in their research.*

Keywords: Gender; LGBTQIA+; Equity, Inclusion, and Diversity; Social Justice

### History of the Working Group

The Gender and Sexuality in Mathematics Education Working Group first convened during the PME-NA conference in Greenville, South Carolina in 2018. The impetus behind developing this working group occurred during the PME-NA 2017 conference, when multiple scholars within this sub-field of mathematics education met and discussed the desire and need to expand on the research conducted in this area of study. This working group has met every year since 2018 and has had more than a dozen different scholars from three different countries participate in planning and leading the working group.

During the inaugural working group sessions in 2018, those in attendance shared their research interests and previous scholarly work about gender and/or sexuality in mathematics education. An important discussion that emerged during those meetings was the complexity and ever-changing nature of (a) the language used in our sub-field and (b) society's understanding of gender and sexuality, as a whole. Furthermore, discussions arose surrounding the implications of methodological choices on the findings reported from such work (e.g., closed vs. open-ended gender demographic questions). As a result, the focus for many of the successive working group meetings has surrounded the wealth of theory that can be applied to this sub-field of mathematics education and the application of such theories to study design and data analysis.

Although this working group has many strengths, there are a particular few worth noting here. First, as mathematics has a long and continued history of being a gendered discipline (Ataide Pinheiro, 2021; Leder, 2019; Leyva, 2017), research in this area continues to be of importance. It is critical for gender and sexuality research in mathematics education to reflect

society's continually expanding understandings of gender and sexuality. Hence, this working group is a space for researchers to explore these rapid, ongoing developments. Secondly, this working group has served as an interest, resource, and professional opportunity for many graduate students over the past few years. Generally, approximately half of the attendees of this working group have been graduate students, and five of these graduate students have gone on to help lead the working group. This level of involvement from graduate students demonstrates a strong interest by the next generation of scholars in this area of research. Furthermore, the participants of the working group have been quite diverse with respect to many different social identities, such as gender, sexuality, race/ethnicity, country of origin, and stage of career.

### **Overview of the Conference Activities**

Because the previous two years of this working group focused its attention on theory, we decided to shift our focus in 2022 to applying these theories to data analysis. As such, for the 2022 conference, we decided to bring to the working group research data (transcripts and videos) and share those data with participants to discuss and analyze. Day 1 and Day 2 followed a similar manner in that on those two days, participants formed small groups and had the opportunity to analyze and interpret data. On Day 3, we had the opportunity to reflect on gender and sexuality research and expand on the discussions from Days 1 and 2. Furthermore, we also provided participants time to contribute ideas for the theme for PME-NA 2023. After they filled a Padlet, the participants discussed what they suggested and why. Interesting ideas arose, such as how to bring counter stories into gender and sexuality research and the need to rehumanize the teaching and learning of STEM. Participants also suggested we, as a group, think about data collection mechanisms through the working group. We have chosen to elaborate on this suggestion for PME-NA 2023 and will delve into it further below.

### **2022 Conference Activities**

#### **Day 1**

On Day 1, Brent Jackson and Julia Przybyla-Kuchek shared data that they had independently collected from different middle school mathematics classrooms. Jackson began by sharing data involving seventh-grade boys working on a mathematics task in a small group. Multimodal transcripts (Norris, 2004) were shared to encourage the working group members to consider how genders in a mathematics classroom might be analyzed and interpreted by looking beyond spoken language. In this way, Jackson used multimodal transcripts to attend to students' use of gesture, gaze, body comportment, and proximity, as well as spoken utterances. This method of analysis was combined with Francis's (2012) theory of *gender monoglossia* and *gender heteroglossia*. Francis draws on Bakhtin's (1981) theories, where monoglossic language conveys unitary ideological accounts of socially dominant groups and heteroglossic language reveals contradictions, resistance, and instability. To view gender from a heteroglossic account, any body may perform a constellation of genders; at some times, masculinity may be pronounced, whereas at other times, femininity may be pronounced (as opposed to a monoglossic account where only male bodies perform masculinity and female bodies perform femininity). After presenting the context of data, guiding theories, and analysis, working group members briefly analyzed the multimodal transcripts, by discussing the following questions: (a) Can research involving solely a group of boys be considered 'gender research'?, (b) In what ways can you see these students' interactions as gendered?, (c) How can multimodal transcripts reveal what it means to 'mobilize genders' in a mathematics classroom during small group work?, and (d) What can researchers learn about gender and sexuality in mathematics education through the analysis of multimodal data?

The analysis presented by Jackson and the following discussion resulted in the surfacing of many dilemmas. The largest theme of this discussion was about identifying and classifying acts as either masculine or feminine, such as: What counts as masculine and/or feminine, and who gets to decide? Since notions of masculinity and femininity change over time, how might a theory of *gender heteroglossia* account for the local (e.g., school cultures) and temporal interpretations (e.g., 20<sup>th</sup> versus 21<sup>st</sup> century notions/representations) of masculinity and/or femininity? What are the consequences of employing a binary sense of gender (e.g., masculine versus feminine)? The working group participants discussed these dilemmas.

After Jackson shared his transcripts from the small group work, Przybyla-Kuchek shared a video clip from an extended whole-class interaction in a seventh-grade classroom. The video clip was collected as data from Przybyla-Kuchek's dissertation study, which was designed through a feminist poststructural perspective (Baxter, 2003; Walkerdine, 1998; Walshaw, 2005). In the clip, Tyler, a Hispanic female student shared her strategy for solving a probability problem. Wes, a White male student, had proposed earlier in the lesson that for repeated events, you can multiply the number of outcomes for the single event by itself the number of times the event is repeated.

After some discussion, Tyler asked if her strategy was correct. There was long debate (of about 10 minutes) as to whether or not Tyler's strategy was valid. Although the teacher approved her strategy and other female students fought to claim that Tyler's method was also correct, Tyrone, a White male student in the classroom insisted that she is wrong repeatedly and never ended up agreeing that her reasoning was also correct. Przybyla-Kuchek shared that Tyrone was often seen by other students as super smart, and they even told him that he should be a math professor one day. Meanwhile, Tyler was often positioned in the classroom by both her peers and her teacher as having more social obligations than mathematical ones.

The working group participants discussed the following guiding questions: (a) Is this interaction gendered? (b) What does it mean to perform gender in a mathematics classroom through mathematical discussions? (c) What can we learn about gender and sexuality in mathematics education through analyzing such data? The clip raised questions about whether the video clip, or more generally, classroom video data that does not directly discuss or address gender could be used to analyze gendered interactions in the classroom. The discussion was rich with different perspectives, with some participants agreeing that gender was undoubtedly at play in the interaction. On the other hand, other participants contended that while gender may have been present, it was challenging to foreground its influence without better understanding how students perceived their gendered interactions. The guiding questions posed were designed to highlight ideas of how working group participants might "see" gender based on the different theoretical and methodological perspectives that we take as researchers and how those impact how we study gender in mathematics education research. In the conversation, participants also drew on prior discussions from past Gender and Sexuality in Mathematics Education Working Group meetings, considering the theoretical perspective a researcher takes and how gender is defined in their research.

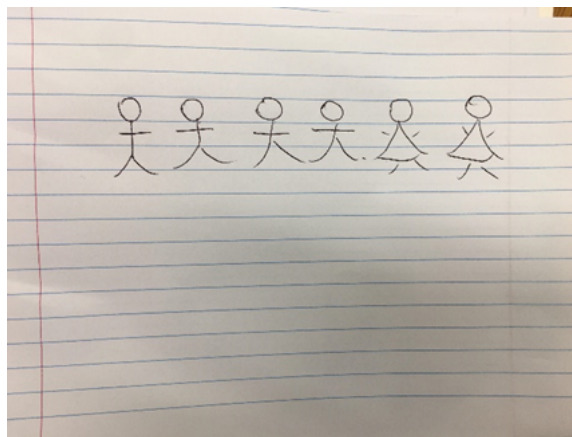
## **Day 2**

On Day 2, Jennifer Hall and Katrina Piatek-Jimenez each shared excerpts of data from interview-based studies (including photographs, in Hall's case) that they have recently conducted with university students that included a focus on gender. Hence, these studies were in contrast to those shared on Day 1, where interactions arose that were perceived as gendered, but the focus of the studies themselves was not gender.

During the session, each of the two data excerpts (interview transcripts) were acted out by the leadership team of the working group to provide attendees with an audio rendition of each transcript. The attendees were provided with written copies of each transcript (including the photograph, in the case of Hall's study) and then formed small groups to analyze the data. Participants discussed their interpretations, drawing on frameworks that they typically use in their own research. Upon completing this task, attendees returned to the large group and shared what their small groups noted from the data.

The first data excerpt was from Hall's research, a multi-phase study that she conducted with colleagues in Australia. The participants in the study were undergraduate and graduate students, as well as faculty members, in mathematics departments at two Australian universities. The goal of the study was to better understand the supports and challenges faced by students as they completed their mathematics studies, with a particular focus on gender. Participants took part in individual interviews, took photographs to represent their experiences, and shared these photographs during focus group interviews (i.e., photovoice; Wang & Burris, 1997). During the working group session, Hall shared an excerpt from a focus group interview transcript, and included a photograph taken by a participant to represent the gender imbalance in her undergraduate classes.

The excerpt that Hall shared was from a focus group interview involving three women and a man who were third-year (final-year) mathematics majors at an Australian university, specifically a portion of the interview wherein one of the women, Eloise, shared the following photograph that she had taken to represent a challenge in her experience of completing her mathematics degree (i.e., the low proportion of women in her mathematics classes).



**Figure 1: Photograph Shared by an Undergraduate Student, Showing the Low Proportion of Women in her Mathematics Classes**

The focus group participants brought up several themes that often arise during discussions of gender equity in STEM fields, such as representation, role modelling, advertising campaigns, and confidence. At one point, Eloise asked Fletcher, the only man in the focus group, for his opinion, and he demurred, stating that he did not think that it was really his place, as a man, to comment. Eloise encouraged him to share his thoughts, which led to a discussion about the potential for brothers to encourage their younger sisters to continue with mathematics, as one way of addressing pipeline, confidence, and other issues.



The second data excerpt was from Piatek-Jimenez's research, a study that she conducted in conjunction with two undergraduate physics students at one U.S. university. Similar to Hall's study, the focus in this study was the supports and challenges faced by university students in a STEM field; however, Piatek-Jimenez's participants were undergraduate physics majors rather than mathematics students. Furthermore, the focus of this study was on the intersectionality of a variety of underrepresented identities (race/ethnicity, sexuality, disability, etc.), not only gender. The data for this study were collected through two individual interviews with each of the participants.

The excerpt that Piatek-Jimenez shared was from the second interview conducted with one participant, Eleanor. Eleanor was a graduating senior who had originally been a double major in mathematics and physics, but had recently decided to drop her physics major. Although Eleanor only has one underrepresented identity in physics, being a woman, Eleanor believed that gender greatly influenced her experiences within the physics department.

Throughout the interview, Eleanor discussed her disappointing experiences as a physics major. Although Eleanor believed that the physics faculty may have had good intentions, she argued that the tone within the department is one of indifference towards the students. She reported that the physics faculty showed very little understanding of what it means to be a contemporary college student and demonstrated a lack of caring regarding the success of their students. Furthermore, she found most of her physics professors to be unrelatable and awkward, and believed that they had no interest in getting to know their students. Eleanor contrasted this with her experiences within the mathematics department at her university which, as she put it, has more "female energy."

One major discussion point of the working group participants surrounding this transcript was on how STEM fields are often dehumanized in academia and what this means for the experiences of our students, especially for those with under-represented identities. Furthermore, the group discussed essentialism and some challenged what it would mean to have "female energy."

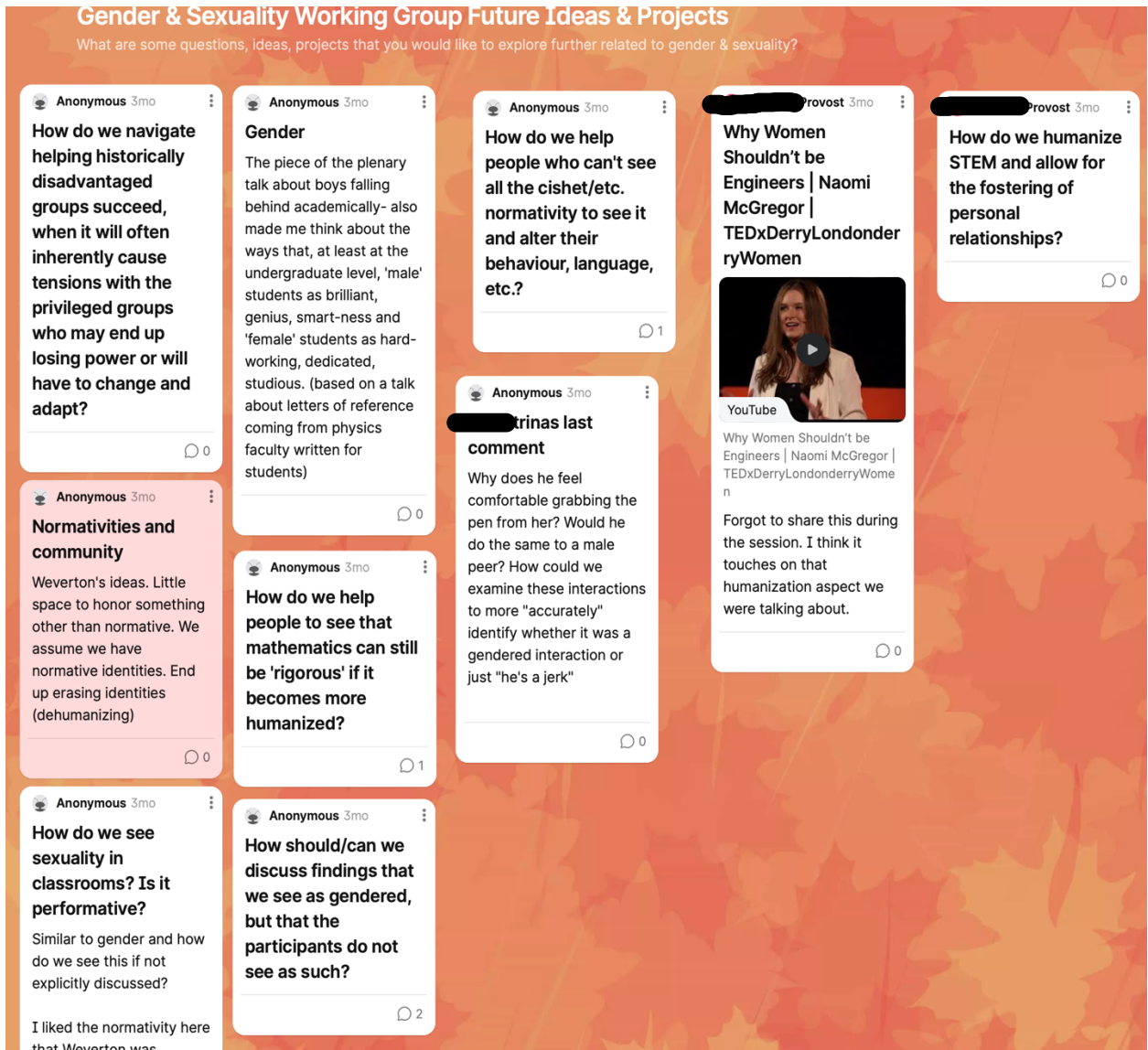
### **Day 3**

Day 3 began with working group participants summarizing for new attendees the topics that were addressed and the discussions that occurred during Days 1 and 2 of the working group. Furthermore, Piatek-Jimenez shared another short data excerpt from her study with undergraduate physics majors, this time relating to sexuality. In this data excerpt, a student shared a story of how physics majors who were also members of the LGBTQ+ community kept their LGBTQ+ identities out of the physics classroom. This prompt led to a rich discussion regarding the culture of STEM fields often preventing students from "bringing their whole self to class" and about visible versus hidden identities.

Other themes that arose for discussion on Day 3 surrounded two of the plenary talks. In particular, participants connected our discussions about data analysis in gender and sexuality research with one of the central questions of the plenary on the first day of the conference (*Remembering Place: Mathematical Actions for Innovative, Resilient, and Culturally Rich Communities* by Florence Glanfield, Cynthia Nicol, and Jennifer S. Thom): How can we honor students as participants of our research? Another critical discussion that arose concerned the views presented in the plenary on the second day of the conference (*Whose Voices are Not Centered in my Work?* by Maria Blanton). As communicated post-conference by the local organizing committee of PME-NA 2022, the discussions about gender in the plenary were harmful to many participants at PME-NA 2022. For example, working group participants pointed

out that, despite boys' and girls' performance differences in school mathematics, there continues to be a reverse gender gap in the mathematics workforce. Furthermore, participants raised the point that discourses still shape the ways that students are seen through a gendered lens, which affects students' success in mathematics at the university level.

Working group participants also discussed how hard it is sometimes to ensure that we, as researchers, are not essentializing gender and seeing gender performativity in data when perhaps the observable behavior is just an individual attribute that is not particularly grounded in the person's gender. Also, based on the analysis of these short data snippets from the previous days of the working group, we discussed that gender performativity should not be equated with gender performance. Although some of us saw certain behaviors as gender being performed (in other words, an individual performing something that we could expect a person of a certain gender to do, because of their socialization), others thought that we could not classify behaviors as performing one gender or another, because that would be essentializing (Fuss, 1989). Furthermore, we would have to further look for gender performativity, instead, in the data, which could be speech acts explicitly referring to what people from a particular gender are expected societally to do, thus constructing that very notion of gender (Butler et al., 1994). It was suggested that one way to avoid seeing gendered actions where there might be none is by validating the analysis with research participants or involving them in the analysis. The discussion then moved on to more general topics, elaborating on ways to include the participants in all phases of the research process, including data analysis.



**Figure 2: Padlet of Thoughts around Future Ideas and Projects for the Gender and Sexuality in Mathematics Education Working Group**

The last portion of our time on Day 3 was open for participants to write on a Padlet (Figure 2) their questions, ideas, and projects that they would like to discuss and explore further related to gender and sexuality research. After participants shared their ideas, we had many interesting discussions about the questions and suggestions posted. The central questions to our final discussions were:

1. How do we navigate helping historically-disadvantaged groups succeed in mathematics when it will often inherently cause tensions with the privileged groups who may end up losing power and/or have to change and adapt?
2. How do we see sexuality in mathematics classrooms? Is it as performative as gender?

3. How do we humanize STEM and allow personal relationships to be fostered? How do we help people to see that mathematics can still be rigorous even if it becomes more humanized?
4. How do we help students to build critical consciousness and awareness of how gender and sexuality normativities influence their lives? What is the role of language in assisting the students in building critical consciousness and awareness of these issues?
5. How can we discuss findings that we see as gendered but that the participants may not see as such?

### **Progress Made Between the 2022 and 2023 Conferences**

Our goal for the 2023 working group is to build a repertoire of how the working group participants became involved in gender and sexuality research, the challenges and supports that we have experienced, and what stories we have told through our data. To achieve this goal, between the 2022 and 2023 conferences, we will collect data regarding these questions. We will continue collecting data through focus group interviews during the PME-NA 2023 conference. Through our stories, we aim to learn with and from one another about our own personal experiences and the work that has been done in gender and sexuality research.

In particular, last year's participants from the working group will be invited to write narratives about becoming and being gender and sexuality researchers in mathematics education. These narratives will be initial data that will be analyzed after PME-NA 2023 by a subset of the working group participants. We will also develop a questionnaire to inquire about the demographics regarding our social identities and research experience. Furthermore, prior to PME-NA 2023, we will develop focus group interview protocols to be used during the conference.

One of the reasons that we will start collecting data prior to PME-NA 2023 is that the leadership team of the working group wants to ensure that the questions we will ask to guide in writing our narratives, the questionnaire, and the focus group interviews are piloted and refined before we distribute the questions to the PME-NA 2023 participants. The 2022 leadership team will submit an IRB application to the first author's institution prior to collecting any data.

### **Questions to Address in the Future**

As the Gender and Sexuality in Mathematics Education Working Group moves forward, we have decided to document our stories as gender and sexuality researchers. The idea of story sharing will serve as a mechanism to inform newer scholars about the methodologies, theories, and types of research that have been done concerning gender and sexuality in mathematics education. Also, by sharing our stories, we all will benefit by learning about the challenges that we have faced as researchers in a critical field, and also the support that we received as researchers in this field. Furthermore, participants of the working group will be able to continue learning with and from one another and better understand the work with which we have been engaged, for possible future research collaborations.

The collaborative effort to comprehend the narratives of gender and/or sexuality in mathematics education research will result in a presentation at PME-NA 2024. Additionally, we will propose an edited book for gender and/or sexuality researchers in mathematics education to write chapters about their research in this field.

## References

- Ataide Pinheiro, W. (2021). Dismantling the 'all-boys club' a narrative of contradictions women experience in PhD mathematics programs: A Freirean approach. *International Electronic Journal of Mathematics Education*, 16(3), em0652. <https://doi.org/10.29333/iejme/11090>
- Bakhtin, M. M. (1981). *The dialogic imagination: Four essays*. University of Texas Press.
- Baxter, J. (2003). *Positing gender in discourse: A feminist research methodology*. Palgrave Macmillan.
- Butler, J., Osborne, P., & Segal, L. (1994). Gender as performance: An interview with Judith Butler. *Radical Philosophy*, 67(2), 32–39. <https://www.radicalphilosophy.com/issues/067>
- Francis, B. (2012). Gender monoglossia, gender heteroglossia: The potential of Bakhtin's work for re-conceptualising gender. *Journal for Gender Studies*, 21(1), 1–15. <https://doi.org/10.1080/09589236.2012.639174>
- Fuss, D. (1989). *Essentially speaking: feminism, nature and difference*. Routledge.
- Leder, G. C. (2019). Gender and mathematics education: An overview. In G. Kaiser & N. Presmeg (Eds.), *Compendium for early career researchers in mathematics education* (pp. 289–308). Springer. [https://doi.org/10.1007/978-3-030-15636-7\\_13](https://doi.org/10.1007/978-3-030-15636-7_13)
- Leyva, L. A. (2017). Unpacking the male superiority myth and masculinization of mathematics at the intersections: A review of research on gender in mathematics education. *Journal for Research in Mathematics Education*, 48(4), 397–433. <https://www.nctm.org/Publications/Journal-for-Research-in-Mathematics-Education/2017/Vol48/Issue4/?ref=1>
- Norris, S. (2004). *Analyzing multimodal interaction: A methodological framework*. Routledge.
- Walkerdine, V. (1998) *Counting girls out: Girls and mathematics*. Routledge Falmer.
- Walshaw, M. (2001). A Foucauldian gaze on gender research: What do you do when confronted with the tunnel at the end of the light? *Journal of Research in Mathematics Education*, 32(5), 471–492. <https://doi.org/10.2307/749802>
- Wang, C., & Burris, M. A. (1997). Photovoice: Concept, methodology, and use for participatory needs assessment. *Health Education & Behavior*, 24(3), 369–387. <https://doi.org/10.1177/109019819702400309>

## INNOVATIONS TO SUPPORT ELEMENTARY MATHEMATICS TEACHERS NAVIGATING DIGITAL CURRICULA

Steve Rhine  
Pacific University  
[steverhine@pacificu.edu](mailto:steverhine@pacificu.edu)

Shannon Driskell  
University of Dayton  
[sdriskell1@udayton.edu](mailto:sdriskell1@udayton.edu)

Rachel Harrington  
Western Oregon University  
[harringtonr@mail.wou.edu](mailto:harringtonr@mail.wou.edu)

Ann Wheeler  
Texas Woman's University  
[AWheeler2@twu.edu](mailto:AWheeler2@twu.edu)

*Digital curricula (DC) are gaining ubiquity in elementary mathematics classrooms and dramatically impacting the nature of mathematics teaching. Teacher educators and professional development providers are not keeping pace with the changing learning environments. This Working Group will examine video case studies of teachers implementing DC in varied ways, identify unique features of instruction with DC, brainstorm innovations to teacher education and professional development, and determine a research agenda. Pepin et al.'s (2017) theoretical framework of Learning Spaces will guide our discussion as well as four thematic questions addressing students' procedural fluency versus conceptual understanding, Common Core Mathematical Practices, affordances and constraints of DC, and teachers' use of DC generated data to engage all learners.*

Keywords: digital curricula, mathematics, teacher education, professional development

The amount schools are spending on digital curricula (DC) is rising fast: from \$1.8 billion in 2013 to \$11 billion in 2019 and \$13.1 billion in 2020 (Cauthen, 2020). Further, the transition to online schooling during the COVID pandemic dramatically increased the use of DC (Bradley, 2021). DC are rapidly becoming ubiquitous in the educational environment with companies claiming broad adoption. For example, iReady is used by over 10 million K-8 students in 14,000 schools (Braatz, 2021), Dreambox claims it is used by nearly 6 million students in all 50 states (Garrard, 2022), and Prodigy boasts that "One million teachers use Prodigy Math Game in the United States" (Peters & Mahimker, 2021). Perhaps the most concerning is that companies are "rolling out [DC] programs in America's public schools with relatively few checks and balances" (Singer, 2017, n.p.). EdSurge ([www.edsurge.com/products](http://www.edsurge.com/products)) lists approximately 150 different programs for elementary mathematics DC or supplemental curricular resources. DC have the potential to have a dramatic impact on the nature of math instruction, particularly with some companies' efforts to 'teacher proof' elementary mathematics instruction. *What are the implications for mathematics teacher education and inservice professional development?*

The goals of this Working Group are to consider how elementary mathematics instruction is changing due to the advancement of DC, begin to develop recommendations for innovations to teacher education and professional development to respond to this new environment for learning, and establish a research agenda to further understand the impact that DC is having on learning and teaching. Teacher education, in particular, has fallen behind in responding to this new environment that elementary teachers face and may be inadequately preparing future teachers to engage all learners of mathematics effectively with DC.

We base and organize our discussion around Pepin et al.'s (2017) theoretical framework of Learning Spaces that can potentially be transformed by DC: (a) the presentation space (how the DC is used to present material and topics to students); (b) the problem space (types of problems and possible solution paths); (c) the work space (resources available to help students solve problems); and (d) the navigation space (how learners logically proceed through each topic). The presentation space varies widely with DC, ranging from rigid, 'teacher proof' videos, slides, or scripts for teachers to use to teacher freedom to develop instruction with multiple digital resources.

### Organization of the Working Group on Digital Curricula

In **Session 1**, we will begin with sharing our case study research on teachers implementing DC in order to provide participants with a sense of the range of ways teachers enact DC in the classroom (Author et al., 2023, Author et al., 2019). Participants will also share their experiences. Through our case study research, we have created a video database of teacher instruction with DC and interviews for participants to use. So, following the presentation of research, we will divide into pairs for participants to analyze videos of teachers making use of DC with the Learning Spaces framework as a guide. After sharing and discussing the insights gained from the analysis, we will form four small groups with each group focusing on one thematic question listed below. Participants will be instructed to find resources (e.g., research and examples of DC) to bring to the second session that might help them with the discussion.

- *How do we prepare teachers to balance the development of students' procedural fluency and conceptual understanding while using DC?*
- *How do we prepare teachers to develop students' Common Core Mathematical Practices while using DC?*
- *What are the affordances and constraints of DC that can influence teachers' instruction?*
- *How do we prepare teachers to use DC generated data to inform their instruction?*

In **Session 2**, we will focus the majority of time on answering the four questions in small groups, using the Learning Spaces as a guide for discussion. Participants will examine multiple DC and create a spreadsheet organized by range of curricula and Learning Spaces and how each DC environment impacts potential responses to their question. Groups will present their spreadsheets to the whole community for discussion and feedback.

In **Session 3**, we will continue to use the small groups and begin brainstorming ideas of how to modify teacher education and inservice teacher professional development based on the findings in Session 2. In the whole group discussion, each group will have 10 minutes to present their ideas and receive feedback. We will also discuss potential areas of further research. We will conclude with a brief discussion of next steps for the participants: to continue the development of innovations to teacher education and professional development that will engage all learners in a future working group and pursue avenues of research in small groups.

**Table 1: Working Group Plan**

	Activity	Format	Time
SESSION 1	Introduction	Presentation	20 min
<i>Baselin</i>	Video Analysis	Pairs	30

<i>e</i>	Sharing of Insights Next Steps/Identify Resources	Whole group discussion Presentation/Small groups	min 20 min 20 min
SESSIO N 2 <i>Questio ns</i>	Outline of Session Group Work Report Progress/Next Steps	Presentation Small groups Whole Group Discussion	10 min 60 min 20 min
SESSIO N 3 <i>Action</i>	Outline of Session Group Work Report Progress Next Steps	Presentation Small groups Whole Group Discussion Presentation	5 min 40 min 40 min 5 min

## References

- Braatz, K. (2022, September 1). Is i-Ready dangerous? Curriculum Associates. <https://www.curriculumassociates.com/blog/is-iready-dangerous#:~:text=The%20i%2DReady%20system%20is,safe%20and%20reliable%20learning%20assessments>
- Bradley, B. (2021, July 29). Spending on tech-based curriculum jumps during the pandemic, new survey of IT leaders finds. EdWeek Market Brief. <https://marketbrief.edweek.org/marketplace-k-12/spending-tech-based-curriculum-jumps-pandemic-new-survey-leaders-finds/>
- Cauthen, L. (2020). The 2020 digital transition survey results are in: Just how large is the K12 ed-tech market? The Learning Counsel. <https://thelearningcounsel.com/article/2020-digital-transition-survey-results-are-just-how-large-k12-ed-tech-market#:~:text=Digital%20curriculum%20spend%20in%202020,spend%20does%20not%20include%20staffing>
- Driskell, S. O., Rhine, S., Wheeler, A. & Harrington, R. (2019). Enactment of Digital Curricula in Elementary Classrooms and Impact on Mathematical Practices. In Proceedings of the 41st Annual Conference of the North American Chapter of the International Group for the Psychology of Mathematics Education. St. Louis, MO.
- Garrard, S. (2022, September 20). Dreambox. School districts nationwide accelerate student math and reading achievement with dreamBox learning. <https://www.dreambox.com/press/school-districts-nationwide-accelerate-student-math-and-reading-achievement-with-dreambox-learning>.
- Pepin, B., Choppin, J., Ruthven, K. B., & Sinclair, N. (2017). Digital curriculum resources in mathematics education: Foundations for change. *ZDM*, 49 (5), 645–661.
- Peters, A., & Mahimker, R. (2021, June 3). 10 Years of helping every student love learning! Prodigy. <https://www.prodigygame.com/main-en/blog/prodigy-education-turns-10/>.
- Authors. (Submitted 2023). Elementary mathematics digital curricula and the future of teacher education. In S. Oraki (Ed.) *Redesigning the future of education in the light of new theories, teaching methods, learning, and researches*. Information Age Publishing.
- Singer, J. (2017, June 6). The Silicon Valley billionaires remaking America’s schools. *New York Times*. <https://www.nytimes.com/2017/06/06/technology/tech-billionaires-education-zuckerberg-facebook-hastings.html>.



## WORKING GROUP ON MATHEMATICS CURRICULUM RECOMMENDATIONS FOR ELEMENTARY TEACHER PREPARATION: SECOND REPORT

Julien Corven  
Illinois State University  
jccorve@ilstu.edu

Joseph DiNapoli  
Montclair State University  
dinapolij@montclair.edu

Kim Johnson  
West Chester University  
KJohnson2@wcupa.edu

Valerie Long  
Indiana University of Penn.  
vlong@iup.edu

Dana Olanoff  
Widener University  
dolanoff@widener.edu

Jennifer Tobias  
Illinois State University  
jtobias@ilstu.edu

Tuyin An  
Georgia Southern University  
tan@georgiasouthern.edu

Dan Clark  
Western Kentucky University  
daniel.clark@wku.edu

Richard Velasco  
University of Oklahoma  
richard.velasco@ou.edu

Kristy Litster  
Valdosta State University  
klitster@valdosta.edu

Bona Kang  
Ohio Wesleyan University  
bkang@owu.edu

*This report summarizes the work and conversations of the second meeting of the working group on Mathematics Curriculum Recommendations for Elementary Teacher Preparation at PME-NA 44. We share the eight areas of research that we believe encompass the goals of the group and provide a summary of the current structure of the working group. We also share our research plans for the future.*

As the Mathematics Curriculum Recommendations for Elementary Teacher Preparation (MCRETP) Working Group entered its second phase, the organizers intended for our meeting at PME-NA 44 to focus on brainstorming and developing potential pilot research studies that could address questions relevant to producing a research-based set of recommendations for elementary teacher preparation programs. Although our conversation went in a slightly different direction, we are still pleased with the progress made. In this report, we will summarize the outcome of our meeting at PME-NA 44 in Nashville, TN. We will then detail our post-conference activity that provided additional structure to the working group. Next, we will describe the eight research subgroups formed to meet the goals of the group. We provide descriptions of each research area and the names of each subgroup's leader(s); we emphasize that we welcome new participants to join the working group at any time. Whether you are conducting a research study that fits into one of these areas or you are interested in learning more about a particular research area and how you can contribute, we encourage you to reach out to the appropriate group leader to get plugged in to this work. We will end this report with our revised long-term plan for the working group and discuss what questions we hope to address in the future.

### Our Meeting at PME-NA 44

#### Summary of Day 1

On Day 1 of the conference, we reviewed what this MCRETP working group had previously accomplished during past meetings, as well as established what we hope to accomplish in the future. We began Day 1 with each attendee introducing themselves and describing their interest in the working group. We had new participants join our group in Nashville, so we wanted to make sure everyone in attendance, new and existing working group members, were properly

introduced. Next, we reviewed the long-term plan for the MCRETP working group. This review included how we spent 2021 understanding the concerns about elementary teacher preparation; our plans for 2022 to design research studies that can inform curricular recommendations; and looking ahead to 2023-2026 in which we intend to conduct original research, disseminate this research, and draft curricular recommendations for elementary teacher preparation.

Also during Day 1, we shared research presentations and frameworks that guide our vision. We included research presentations and guiding frameworks from our first meeting of the MCRETP working group (see Corven et al., 2021), as well as added additional guiding perspectives from our new group organizers. The purpose of sharing these presentations was to (re)orient our new and existing participants to the ideas of the working group. First, we reviewed survey literature on mathematics content courses for elementary teachers, in part to show the high variability of teacher preparation programs in the United States (Masingila & Olanoff, 2022; Masingila et al., 2012). Next, we discussed results from a longitudinal study investigating the relationship between the number of instructional minutes on mathematics topics in teacher preparation and the specialized content knowledge (SCK; Ball et al., 2008) program graduates demonstrated for those topics (Corven et al., 2022). We emphasized a finding that implies the importance of spending ample instructional time on mathematics topics during elementary teacher preparation in order for teachers to remember and apply their knowledge of those topics years later, in their teaching. In essence, this finding suggests the futility of survey courses during which prospective teachers (PTs) spend a small amount of instructional time on many mathematics topics.

Additionally, we explained curriculum design principles from the Elementary Mathematics Project (EMP; Chapin et al., 2021), a curriculum that focused on high-leverage mathematics topics and instructional practices. We emphasized results from research on EMP that showed that resources within the curriculum aided the development of agency in PTs (Gibbons et al., 2018) and supported the pedagogy of mathematics teacher educators (MTEs; Gibbons et al., accepted). Then, we presented an overview of Li and Howe's (2021) theoretical framework about knowledge-oriented vs. thinking-oriented approaches to teacher preparation that will help guide our work in the MCRETP working group. Knowledge-oriented approaches prioritize teaching content that PTs need to know to effectively teach mathematics. In contrast, thinking-oriented approaches prioritize learning about reasoning, explaining, and making sense of the mathematics content needed for teaching. We posited that teacher preparation programs should generate outcomes that are aligned with both teacher knowledge and teacher thinking.

Next, we shared about ongoing work investigating how engaging PTs in the development of a task analysis framework may support their understanding of high-quality tasks (Long et al., 2020). Analysis of this project is still in progress; however, we emphasized the need for PTs to develop prowess in selecting high-quality mathematics curriculum materials to help create robust learning experiences for their future students. Lastly, we presented about the development of training programs to improve fieldwork in elementary methods classes. We emphasized the importance of partnering PTs with mentor teachers that support student-centered mathematics pedagogies (e.g., Ronfeldt et al., 2018). We also previewed a forthcoming grant proposal led by a MCRETP working group organizer that will help identify mentor teachers' needs to best support prospective teachers and their developing mathematical pedagogical content knowledge.

During the latter half of our Day 1 meeting, we focused on specifying the obstacles within elementary teacher preparation programs. We engaged our participants in a discussion about their perspectives on the research literature, as well as their experiences as mathematics teacher

educators. We reviewed the results of a similar discussion we held at PME-NA 43, and, as a group, we started to synthesize these two sources of information to posit research areas that the MCRETP working group could pursue moving forward.

### **Summary of Days 2 and 3**

On Day 2 of PME-NA 44, we began designing small-scale research studies that MCRETP working group members could conduct that would align to one or more of the posited research areas of need. We spent most of our time in small groups brainstorming about research designs, discussing the different perspectives afforded by each working group member's context, and workshopping drafts of research questions that could guide such work. We also encouraged participants to try to leverage synergy from potential collaborations and to consider research studies that could be designed across institutions. Members of small groups were encouraged to create an artifact of their ideas in a shared Google Document. Near the end of the Day 2 meeting, each small group shared their ideas for potential research studies.

Day 3 was dedicated to planning our work over the next year before we meet again at PME-NA 45. First, we spent some additional time brainstorming and sharing our different ideas for potential research studies. Then, as a large group, we engaged in a discussion about the most important next steps to keep the MCRETP working group engaged beyond PME-NA 44 and productively moving forward toward our collective goals. We outlined an action plan that included creating a survey of working group members' interest in each research area. Based on this survey, we planned to create subgroups for each research area. These subgroups will meet throughout the next year to design and begin conducting small-scale studies. We created a listserv email communication plan to keep members of the working group informed between conferences and Zoom meetings. We also discussed a plan to convene periodically as a full working group, via Zoom, to provide full-group updates about progress and to support one another. In all, Day 3 proved to be an important meeting because every attending member of the MCRETP working group's voice was heard about the best ways to stay connected and to support each other as we transition into the phase of our long-term plan during which we will conduct original research studies.

### **Developing Areas for Future Research**

Our work from PME-NA 44 identified eight areas for research for us to focus on moving forward. These areas include (in no particular order): (1) Literature Review and Critique of Current Recommendations, (2) Survey of Current Program Structures and Local Requirements, (3) Preparation Program Alumni Survey and Interviews, (4) Selection of Topics in Content Courses, (5) Pedagogical Practices, Tasks, and Thinking-Oriented Approaches in Content Courses, (6) Pedagogical Practices in Methods Courses, (7) Integration of Content and Methods Courses, and (8) Field Experiences and Clinical Work. Each of these research areas will be described below in the Our Current Research Subgroups section.

#### **Post-Conference Activity**

Between November 2022 and February 2023, we have done the following:

7. We sent out a survey to working group members to identify research areas that they were interested in and their level of interest in each area (i.e., group leader, group participant, interested in results). We had over 25 responses, and we were able to identify one or more leaders for each subgroup.
8. The organizing committee for the working group met four times to develop the survey and discuss the results, as well as our plans for moving forward. For each research

subgroup, we identified a leader or co-leaders and provided them with contact information for working group members who said they were interested in being a researcher or collaborator.

9. We tasked the leader/co-leaders of each subgroup to organize a research subgroup meeting and to contribute information about their research area for this report.

Below, we briefly describe each of the subgroups and their plans for moving the research forward. See the author list of this report for contact information for each group leader.

### **Our Current Research Subgroups**

#### **Literature Review and Critique of Current Recommendations**

*Group co-leaders: Dana Olanoff, Kim Johnson, Valerie Long, and Jennifer Tobias*

Although many members of our working group have studied previous recommendations for the mathematical preparation of teachers, this subgroup aims to compile and synthesize the literature related to this topic. We plan to identify current recommendations from groups such as the Conference Board of Mathematics Sciences (CBMS), the Association of Mathematics Teacher Educators (AMTE), and the National Council of Teachers of Mathematics (NCTM) to get an overview of the state of recommendations for teacher preparation. We feel that this is important because, while numerous groups have made recommendations for the mathematics preparation of teachers, in practice, the recommendations are not always feasible for many teacher preparation programs. In the future, we plan to identify aspects that are crucial for teacher preparation, while also attempting to make recommendations that are usable by different types of programs that are often left out of current recommendations. Additionally, the literature that we compile and synthesize will be used by other subgroups in grounding their studies.

#### **Survey of Current Program Structures and Local Requirements**

*Group co-leaders: Tuyin An and Dan Clark*

This research area aims to understand how elementary teacher preparation programs are structured across the nation through a national survey. The survey will focus on models of mathematical content and pedagogy integration and also address social justice-related topics. The elementary teacher preparation programs in the US are highly varied, in terms of the varying coverage, sequencing, and integration of the mathematical content and pedagogy, as well as mathematics teacher educators' varying beliefs about the goals of the teacher preparation programs (An et al., 2021), which could lead to inconsistent and insufficient preparation of elementary teachers' mathematics content knowledge for teaching (MKT; Ball et al., 2008). For example, according to two national surveys about mathematics content courses for elementary PTs conducted in the past two decades, most teacher preparation programs in the US do not meet recommendations to require 12 credits of mathematics content courses (e.g., AMTE, 2017; CBMS, 2012), and most content course instructors lack elementary teaching experience (Masingila & Olanoff, 2022; Masingila et al., 2012). Bertolone-Smith et al. (2023) show similar findings. To improve the effectiveness of teacher preparation in developing PTs' MKT, it is crucial to understand how elementary teacher preparation programs are structured (e.g., how the content and methods courses are offered and taught). Community-based endeavors will be undertaken based on the knowledge learned from the proposed national survey.

The ultimate aim of the survey of programs would be to create a clearinghouse of elementary teacher preparation program structures with respect to mathematics education. The clearinghouse would be a well-known, publicly available website where programs looking to make improvements regarding the mathematical preparation of their PTs could look to see how other

programs accomplish their goals, possibly leading to collaboration. Research will be needed at several steps. First, the survey will be undertaken to create the first version of the clearinghouse. Further effort will be needed to maintain it. Then, research can be conducted on how programs are using the available information to make decisions about their program design. As an example, there is a current push in mathematics education to include more education for PTs on infusing social justice with their mathematics teaching. As teacher preparation programs determine how to best do that, they could use the clearinghouse as a resource to learn from others who are already doing that well.

### **Preparation Program Alumni Survey and Interviews**

*Group leader: Dana Olanoff*

One of the ideas that came out of our meeting at PME-NA 44 was that alumni from our current programs are some of the people best equipped to identify which aspects of their teacher preparation were most valuable for them in their work as teachers. Therefore, we are planning on developing a survey to ask alumni about their experiences as PTs. We will ask questions about teachers' experiences in their classes and field experiences and how they are using what they have learned in their current jobs. Additionally, we will aim to identify aspects that may have been missing from their teacher preparation programs that they wished had been included. This subgroup will first develop survey questions. Then all working group members who are able will send the survey to alumni from their programs. We will compile and analyze the data in order to help make recommendations for future mathematics teacher preparation programs.

### **Selection of Topics in Content Courses**

*Group co-leaders: Joseph DiNapoli, Valerie Long, and Jennifer Tobias*

This research area investigates the affordances and constraints related to the instructional time spent (or not spent) on specific mathematics topics included in content courses for prospective elementary teachers. Recent research has shown that instructional time on specific topics in teacher preparation mathematics content courses matters for the development of SCK that is retained and demonstrated post-graduation, as former PTs begin careers as elementary educators in real classrooms (Corven et al., 2022). Further, this longitudinal study suggests that survey courses, which spend small amounts of instructional time on many mathematical topics, may have little or no impact on future teachers' retention and use of SCK. As a result, mathematics teacher educators must devote ample time to less topics in their content courses, and thus, make difficult decisions about which topics to select and which to omit.

Several ideas for research studies in this area were discussed during and after PME-NA 44. For instance, one study seeks to investigate the relationship between instructional time and the development of mathematical practices: do elementary PTs learn to persevere more with challenging tasks about topics for which more instructional time was spent? As this study and others unfold, and as we synthesize new findings with existing ones, the future of this research area will focus on ways to invest instructional time to support elementary PTs in developing content knowledge for specific mathematics topics, as well as developing the mathematical practices that teachers should seek to develop in their students.

### **Pedagogical Practices, Tasks, and Thinking-Oriented Approaches in Content Courses**

*Group co-leaders: Julien Corven and Jennifer Tobias*

The purpose of this research subgroup is to consider recommendations for how to teach mathematics content within elementary content courses. Although recommendations for what to teach are important, without parallel recommendations for how to teach the content, the recommendations may end up ineffective. Li and Howe (2021) assert that it is critical for teacher

preparation programs to focus not just on PTs' knowledge of the mathematical content (a knowledge-oriented approach), but more so on how to apply this knowledge to the craft of teaching by, for example, using such knowledge to understand and base instruction on students' thinking (a thinking-oriented approach). However, Li and Howe (2021) acknowledge that this assertion is their opinion, and they echo a call from Hiebert and Berk (2020) for more research that would support a professional knowledge base for MTEs on thinking-oriented approaches.

Subgroup members are currently planning specific research projects in the content courses they teach to assess the effectiveness of particular pedagogical approaches and tasks. For example, one member is currently collecting data on whether and how teaching PTs an American Sign Language number tale (e.g., Zernovoj, 2005) for order of operations affects their conceptual understanding of the topic and/or their conceptions and perceptions of Universal Design for Learning (e.g., Hitchcock et al., 2002). Another member is designing a study on how PTs attend to cultural relevance when examining specific mathematical tasks, extending work by Jackson and Jong (2017) to generate professional knowledge on how MTEs can prepare PTs to teach equitably (e.g., Koestler, 2012).

We see this research subgroup's work as supporting our future recommendations by giving examples of how to enact them effectively. We anticipate creating a clearinghouse of tasks, ideas, and pedagogical principles that have been shown to be effective for teaching particular content. MTEs can then modify and iterate on these activities (Hiebert & Morris, 2009). We would also suggest activities in this repository include rationales for instructional decisions and information about how PTs may respond to the activities as a way to support novice MTEs (e.g., Superfine & Pitvorec, 2021; Suppa et al., 2020). This approach generates useful and usable professional knowledge for improving teacher preparation, as suggested by Hiebert et al. (2003).

### **Pedagogical Practices in Methods Courses**

*Group co-leaders: Valerie Long and Richard Velasco*

This research area includes examining the learning experiences within a mathematics methods course that provide opportunities for preservice teachers to develop a deep understanding of how students think and learn mathematics. Examining this research area is important since the purposes of "a mathematics methods course is to develop preservice teachers' understanding of how students think and learn about mathematics" and also "develop strategies for understanding and build students' meaningful mathematical sense making and use of mathematical practices" (AMTE, 2017, p. 34). An area of research discussed during our working group sessions was a semester-long project where preservice teachers created a rubric to determine if tasks were high quality. Future research for this area may include examining multiple learning experiences (e.g., writing mathematics lesson plans, conducting teaching experiments, analyzing videos, examining misconceptions for a topic, etc.).

### **Integration of Content and Methods Courses**

*Group co-leaders: Kristy Litster and Bona Kang*

This research area seeks to develop recommendations for integrating both mathematical content and mathematical teaching methods that will assist MTEs in supporting and assessing PTs' competencies. Specifically, it is looking at strategies to help PTs develop the skills necessary to independently learn both the content and methods to teach that content for any standard or grade level when they become future teachers. This research area is important as MTEs strive to provide guidance in key structures, assignments, and methods that connect in order to meet the broader goals of preparing high-quality elementary teachers to teach mathematics (AMTE, 2017). Teachers need to understand both the content they teach and

methods to teach, evaluate, and adapt their teaching strategies to support a diverse population. Teaching PTs to develop deep knowledge of children's mathematical thinking as well as ambitious teaching practices is a complex process that takes time, but many elementary teacher preparation programs experience tensions in how to structure their courses due to the limited time available (Berry, 2004; Bertolone-Smith et al., 2023; Cochran-Smith et al., 2015; Saclarides et al., 2022). With limited allotments of mathematics courses in elementary teacher preparation programs, integrating content and teaching methods helps to build a cohesive understanding of what mathematics to teach and how to teach it. Such integration could also support thinking-oriented approaches to teacher education curricula (Li & Howe, 2021).

Members of this research group met at PME-NA 44 in Nashville, TN and again at the AMTE 2023 conference in New Orleans, LA. Several ideas were discussed at these meetings, such as selecting key content areas with which to practice methods. We discussed three specific questions: a) What are the specific teaching methods that are most important to include in methods courses when splitting time between content and methods? b) How do we integrate content into methods courses and assess competencies of both content and methods in methods assignments? c) What are the specific teaching methods that are most important to be modeled or used within the content areas to link content with future methods or uses of content knowledge?

We also discussed three potential future areas of research relating to these questions and discussions. They include: a) looking at processes of PTs' adaptive teaching (e.g., exploring content via learning trajectories or research papers, preparing lessons, activities, or clinical interviews for students at a variety of levels, and adapting questions or scaffolds based on student responses); b) using surveys of past PTs who are now teaching to determine what content and what methods were the most influential or helpful in helping them be prepared to teach the grade level they were hired to teach or what they wish they had learned; and c) trying out specific strategies to integrate content and methods and assessing qualitative and quantitative outcomes.

### **Field Experiences and Clinical Work**

*Group leader: Kim Johnson*

One of the ideas that came out of discussions at PME-NA 44 involved the importance of mentor teachers in field experiences for prospective teachers. These field experiences can include both student teaching as well as time in the field during classes prior to student teaching throughout teacher preparation. Clarke et al. (2014) report that mentor teachers in field-placement schools are found to play the most influential role in elementary PTs' learning to teach in their program. Unfortunately, both elementary teachers and elementary PTs lack confidence in teaching mathematics (Bursal & Paznokas, 2006). This subgroup aims to develop programs to mentor teachers to use and implement effective mathematical teaching practices (NCTM, 2014) and for PTs to both observe and experience these models in actual classrooms. We will explore PTs' learning gains from both observing and implementing NCTM's (2014) effective mathematical teaching practices in their field placements through systematic data collection and analysis. Additionally, we talked about how early field experiences have been beneficial in retention of students in education majors (Philipp et al., 2007). We discussed ways to incorporate either video or direct work with children into early math content courses. This group plans to explore the best possible experiences for introductory mathematics courses.

### **Our Long-Term Plan**

Our long-term plan for the MCRETP working group consists of five phases, the first two of which have occurred already. Phase I occurred in 2021 as the working group began and focused on understanding the concerns about elementary teacher preparation. Phase II occurred in 2022,

culminating at PME-NA 44 in Nashville, TN, and focused on designing our research subgroups and planning research studies that can inform curricular recommendations. Phases III, IV, and V will encompass the next several years of the working group and will focus on conducting research to inform our curricular recommendations.

In Phase III of the long-term plan for the MCRETP working group, we will conduct our small-scale research studies in 2023 and 2024, aiming to answer the research questions that guide each research subgroup. During Phase III, we will also begin to disseminate and synthesize our findings. Each research subgroup will meet periodically to offer support and feedback as members of the subgroup carry out their planned research. The full working group will also convene periodically to allow subgroups to share about their progress and seek additional feedback. We will encourage working group members to write proposals to present research findings at PME-NA or other relevant professional conferences and to publish reports in whatever outlets they think are most appropriate.

In Phase IV, we will propose a PME-NA 47 (2025) research colloquium during which we will cohesively present the research findings from each subgroup of the MCRETP working group. This colloquium will provide an opportunity for our working group to formally synthesize our collective findings and answer questions about the implications of our work. Related to this work, we expect to organize a special issue of a journal that will contain a synthesis article from each of the research subgroups and a concluding article that will discuss the policy implications of the research findings.

In Phase V of the long-term plan for the MCRETP working group, we first expect to publish the special issue that we organized in Phase IV. Additionally, we will seek a grant to host a conference with the goal of writing and debating a research-supported set of recommendations for elementary teacher preparation. We anticipate hosting such a conference in 2026.

In closing, our MCRETP working group has developed a clear vision of producing a research-based set of recommendations for elementary teacher preparation programs. We are encouraged by the productive beginnings of the working group experienced at PME-NA 43 and PME-NA 44, and we are excited to pursue the next steps in our long-term plan. We reiterate that we welcome new participants to join the working group in its current phase to contribute to a research subgroup and help us achieve our long-term goals.



## References

- An, T., Clark, D. L., Lee, H. Y., Miller, E. K., & Weiland, T. (2021). A discussion of programmatic differences within mathematics content courses for prospective elementary teachers. *The Mathematics Educator*, 30(1), 52–70. <https://eric.ed.gov/?id=EJ1315097>
- Association of Mathematics Teacher Educators [AMTE]. (2017). Standards for preparing teachers of mathematics. <https://amte.net/standards>
- Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, 59(5), 389–407. <https://doi.org/10.1177/0022487108324554>
- Berry, A. (2004). Self-study in teaching about teaching. In J. J. Loughran, M. L. Hamilton, V. K. LaBoskey, & T. Russell (Eds.), *International handbook of self-study of teaching and teacher education practices* (pp. 1295–1332). Kluwer. [https://doi.org/10.1007/978-1-4020-6545-3\\_34](https://doi.org/10.1007/978-1-4020-6545-3_34)
- Bertolone-Smith, C. M., Munson, J., Krause, G., Garner, B., & Saclarides, E. S. (2023, February 2-4). The landscape of US elementary mathematics teacher education: Course requirements for mathematics content and methods [Conference Session]. Association of Mathematics Teacher Education 27th Annual Conference, New Orleans, LA, United States.
- Bursal, M., & Paznokas, L. (2006). Mathematics anxiety and preservice elementary teachers' confidence to teach mathematics and science. *School Science and Mathematics*, 106, 173–179. <https://doi.org/10.1111/j.1949-8594.2006.tb18073.x>
- Chapin, S. H., Gibbons, L. K., Feldman, Z., Callis, L. K., & Salinas, A. (2021). The Elementary Mathematics Project: Supporting preservice teachers' content knowledge for teaching mathematics. In Y. Li, R. E. Howe, W. J. Lewis, & J. J. Madden (Eds.), *Developing mathematical proficiency for elementary instruction* (pp. 89–113). Springer. [https://doi.org/10.1007/978-3-030-68956-8\\_4](https://doi.org/10.1007/978-3-030-68956-8_4)
- Clarke, A., Triggs, V., & Nielsen, W. (2014). Cooperating teacher participation in teacher education: A review of the literature. *Review of Educational Research*, 84, 163–202. <https://doi.org/10.3102/0034654313499618>
- Cochran-Smith, M., Villegas, A. M., Abrams, L., Chavez-Moreno, L., Mills, T., & Stern, R. (2015). Critiquing teacher preparation research: An overview of the field, Part II. *Journal of Teacher Education*, 66(2), 109–121. <https://doi.org/10.1177/0022487114558268>
- Conference Board of the Mathematical Sciences [CBMS]. (2012). *The mathematical education of teachers II*. American Mathematical Society and the Mathematical Association of America.
- Corven, J., DiNapoli, J., Gibbons, L., Long, V., & Starks, R. (2021). New working group: Mathematics curriculum recommendations for elementary teacher preparation. In D. Olanoff, K. Johnson, K., & S. Spitzer, (Eds.) *Proceedings of the 43rd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1941–1943). Philadelphia, PA.
- Corven, J., DiNapoli, J., Willoughby, L., & Hiebert, J. (2022). Long-term relationships between mathematics instructional time during teacher preparation and specialized content knowledge. *Journal for Research in Mathematics Education*, 53(4), 277–306. <https://doi.org/10.5951/jresmetheduc-2020-0036>
- Gibbons, L., Feldman, Z., Chapin, S., Batista, L. N., Starks, R., & Vasquez-Aguilar, M. (2018). Facilitation practices in mathematics teacher education and the mathematical identities of preservice elementary teachers. *Mathematics Teacher Education and Development*, 20(3), 20–40. <https://eric.ed.gov/?id=EJ1195989>
- Gibbons, L. K., Chapin, S., Feldman, Z., & Starks, R. (accepted). Preparing for ambitious mathematics teaching: Examining the role of video in supporting mathematics teacher educators. *Educational Studies in Mathematics*.
- Hiebert, J., & Berk, D. (2020). Foreword: Building a profession of mathematics teacher education. *The Mathematics Enthusiast*, 17(2–3), 325–336. <https://doi.org/10.54870/1551-3440.1490>
- Hiebert, J., & Morris, A. K. (2009). Building a knowledge base for teacher education: An experience in K-8 mathematics teacher preparation. *The Elementary School Journal*, 109(5), 475–490. <https://doi.org/10.1086/596997>
- Hiebert, J., Morris, A. K., & Glass, B. (2003). Learning to learn to teach: An “experiment” model for teaching and teacher preparation in mathematics. *Journal of Mathematics Teacher Education*, 6, 201–222. <https://doi.org/10.1023/A:1025162108648>
- Hitchcock, C., Meyer, A., Rose, D., & Jackson, R. (2002). Providing new access to the general curriculum: Universal design for learning. *Teaching Exceptional Children*, 35(2), 8–17. <https://doi.org/10.1177/004005990203500201>
- Jackson, C., & Jong, C. (2017). Reading and reflecting: Elementary preservice teachers' conceptions about teaching mathematics for equity. *Mathematics Teacher Education and Development*, 19(1), 66–81. <https://eric.ed.gov/?id=EJ1152727>
- Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

- Koestler, C. (2012). Beyond apples, puppy dogs, and ice cream: Preparing teachers to teach mathematics for equity and social justice. In A. A. Wager & D. W. Stinson (Eds.), *Teaching mathematics for social justice: Conversations with educators* (pp. 81–97). National Council of Teachers of Mathematics.
- Li, Y., & Howe, R. E. (2021). Toward a thinking-oriented training in mathematics for elementary school teachers. In Y. Li, R. E. Howe, W. J. Lewis, & J. J. Madden (Eds.), *Developing mathematical proficiency for elementary instruction* (pp. 13–49). Springer. [https://doi.org/10.1007/978-3-030-68956-8\\_2](https://doi.org/10.1007/978-3-030-68956-8_2)
- Long, V., Raymond, K., & Gearing, N. (2020, November 5-7). Using task analysis as a framework for elementary mathematics methods courses [Conference session]. *School Science and Mathematics 2020 Convention, Virtual*.
- Masingila, J. O., & Olanoff, D. (2022). Who teaches mathematics content courses for prospective elementary teachers in the USA? Results of a second national survey. *Journal of Mathematics Teacher Education*, 25, 385–401. <https://doi.org/10.1007/s10857-021-09496-2>
- Masingila, J. O., Olanoff, D. E., & Kwaka, D. K. (2012). Who teaches mathematics content courses for prospective elementary teachers in the United States? Results of a national survey. *Journal of Mathematics Teacher Education*, 15, 347–358. <https://doi.org/10.1007/s10857-012-9215-2>
- National Council of Teachers of Mathematics [NCTM]. (2014). *Principles to actions: Ensuring mathematical success for all*. Author.
- Philipp, R. A., Ambrose, R., Lamb, L. L. C., Sowder, J. T., Schappelle, B. P., Sowder, L., Thanheiser, E., & Chauvot, J. (2007). Effects of early field experiences on the mathematical content knowledge and beliefs of prospective elementary school teachers: An experimental study. *Journal for Research in Mathematics Education*, 38(5), 438–476. <https://doi.org/10.2307/30034961>
- Ronfeldt, M., Brockman, S. L., & Campbell, S. L. (2018). Does cooperating teachers' instructional effectiveness improve preservice teachers' future performance? *Educational Researcher*, 47(7), 405–418. <https://doi.org/10.3102/0013189X18782906>
- Saclarides, E. S., Garner, B., Krause, G., Bertolone-Smith, C., & Munson, J. (2022). Design principles that support course design innovation for elementary mathematics methods courses. *Mathematics Teacher Educator*, 11(1), 9–25. <https://doi.org/10.5951/MTE.2020.0081>
- Superfine, A. C., & Pitvorec, K. (2021). Using community artifacts to support novice math teacher educators in teaching prospective teachers. *International Journal of Science and Mathematics Education*, 19(Supplement 1), 59–75. <https://doi.org/10.1007/s10763-021-10152-7>
- Suppa, S., DiNapoli, J., Thanheiser, E., Tobias, J. M., & Yeo, S. (2020). Supporting novice mathematics teacher educators teaching elementary mathematics content courses for the first time. *The Mathematics Enthusiast*, 17(2–3), 493–536. <https://doi.org/10.54870/1551-3440.1496>
- Zernovoj, A. (2005). Telling, writing and reading number tales in ASL and English academic languages: Acquisition and maintenance of mathematical word problem solving skills [Unpublished master's thesis]. University of California, San Diego. <https://eric.ed.gov/?id=ED495068>

## MATHEMATICS CURRICULUM RECOMMENDATIONS FOR ELEMENTARY TEACHER PREPARATION: ESTABLISHING A RESEARCH BASE

Julien Corven  
Illinois State University  
jccorve@ilstu.edu

Joseph DiNapoli  
Montclair State University  
dinapolij@montclair.edu

Kim Johnson  
West Chester University  
KJohnson2@wcupa.edu

Valerie Long  
Indiana University of Penn.  
vlong@iup.edu

Dana Olanoff  
Widener University  
dolanoff@widener.edu

Jennifer Tobias  
Illinois State University  
jtobias@ilstu.edu

*In our second year of meeting at PME-NA, we worked to create a stronger organizational structure for the group and began to brainstorm research studies that could support a future set of recommendations for the curriculum of elementary teacher preparation in mathematics. Our present goal is to support current and new working group members in conducting this research.*

Keywords: Preservice Teacher Education, Teacher Educators, Mathematical Knowledge for Teaching, Elementary School Education

Elementary teacher preparation programs in North America vary substantially on what content and methods are taught, to what degree topics are taught, how topics are taught, amount and types of clinical experiences for prospective teachers (PTs), and how many credit hours of mathematics PTs must take (e.g., Hrusa et al., 2020; Malzahn, 2020; National Center on Education and the Economy, 2016). Bertolone-Smith et al. (2023) found over 50% of a representative sample of elementary teacher preparation programs in the United States failed to meet the Association of Mathematics Teacher Educators's (2017) recommendations for credit hours in both mathematics content and methods courses. We therefore need to convince teacher preparation programs, accreditation bodies, and state teacher licensing agencies to change their current practices if we want to improve the mathematics preparation of elementary teachers. To effect changes to teacher preparation, we need research-based evidence supporting our claims that (1) the amount of time to effectively prepare teachers of mathematics we will recommend is needed and (2) instructional practices in preparation courses make a difference in PTs' future classrooms. To make convincing arguments for these points, we must be specific regarding what features of teacher preparation programs are associated with teacher effectiveness. This working group's primary goal is to collaborate to produce research-based, specific recommendations for elementary teacher preparation programs regarding their mathematics curriculum.

### **Progress of Current Work**

We acknowledge the efforts of the National Council on Teacher Quality (NCTQ) in collecting and synthesizing specific data on elementary teacher preparation programs (Walsh et al., 2022) as part of the process of revising their standards of what constitutes a high-quality teacher preparation program. Averaging the amount of time teacher preparation programs spend on different content areas can serve as a starting point for more specific recommendations. However, we believe calculating the average number of hours spent teaching individual topics across programs does not necessarily yield the ideal amount of time for each topic. We believe that the activities that PTs spend time on in teacher preparation matter substantially (e.g., Corven et al., 2022), and that recommendations for teacher preparation must be research-based. Additionally, NCTQ's rules for rating teacher preparation programs state that "programs

[cannot] compensate for a missing topic by allocating more time to another topic” (Walsh et al., 2022, p. 6). This perspective runs counter to research from Corven et al. (2022) that suggests that elementary PTs require substantial instructional time to acquire lasting specialized content knowledge (Ball et al., 2008) for mathematical topics. Thus, we believe it is neither feasible nor effective for teacher preparation programs to teach elementary PTs every Common Core mathematics content standard. In our opinion, the question of what works in teacher preparation remains unsettled, and our working group is poised to contribute substantially to resolving it.

### **Organization and Presentation Plan**

At PME-NA 45, we expect members to collaborate, present, and receive feedback on planned or in-progress research studies. The first meeting will orient everyone to the structure of the working group. We will give each subgroup leader time to explain the purpose of their subgroup so that new members can make an informed decision about which subgroups best align with their research interests. Each leader will also present theoretical frameworks that inform their subgroup’s work. We will then give each subgroup time to meet and plan out what kind of research studies they would like to conduct and what they will present to the whole group.

We will allocate each research subgroup time during the second and third sessions to present a preliminary outline of one research study they expect to conduct (or that may be in progress). The presentations must include questions or items for feedback from the group. The other members of the working group will ask questions and provide suggestions to support the progress or development of the study presented. After all presentations are concluded, we will provide subgroups with time to meet and revise their research plans. The goal is for each member of the working group to leave PME-NA 45 with concrete next steps for conducting research that will support our eventual recommendations.

### **Opportunities for New Members to Contribute and Future Plans**

With the establishment of the eight research subgroups (see our working group’s report), we see many opportunities for new members to contribute to our goals before, during, and after PME-NA 45. We suggest that new members select one or two subgroups that interest them. New members can contribute ideas for research studies to a subgroup, or they can collaborate on an existing research study by helping with data collection and/or analysis. We encourage interested new members to read our working group’s report and reach out to subgroup leaders to be added to their email distribution lists. We expect the research studies of our group to proceed through 2025, so new members will have a role to play in the next two years of the working group.

We expect to transition to a Colloquium for PME-NA 47 in 2025, as the research subgroups will then be presenting complete study results. We will always welcome new members’ perspectives, but we recognize the lift required of new participants entering at that stage. We intend to organize a special issue of a journal with articles that summarize and synthesize the work of each research subgroup, along with a concluding article containing policy recommendations and consequences gleaned from each individual article. We will also apply for a grant to host a conference where a draft of the recommendations can be discussed and debated.

We recognize changes in requirements for teacher preparation could exacerbate existing systemic barriers to teacher recruitment and retention, such as low pay and lack of professional respect (e.g., Steele, 2020). However, changing a complex system requires a multi-faceted approach. We see our work as one productive step towards improving mathematics instruction for elementary PTs (and, ultimately, their future students) across North America.

## References

- Association of Mathematics Teacher Educators [AMTE]. (2017). Standards for preparing teachers of mathematics. <https://amte.net/standards>
- Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, 59(5), 389–407. <https://doi.org/10.1177/0022487108324554>
- Bertolone-Smith, C. M., Munson, J., Krause, G., Garner, B., & Saclarides, E. S. (2023, February 2-4). The landscape of US elementary mathematics teacher education: Course requirements for mathematics content and methods [Discussion session]. 27th Annual Conference of the Association of Mathematics Teacher Educators, New Orleans, LA.
- Corven, J., DiNapoli, J., Willoughby, L., & Hiebert, J. (2022). Long-term relationships between mathematics instructional time during teacher preparation and specialized content knowledge. *Journal for Research in Mathematics Education*, 53(4), 277–306. <https://doi.org/10.5951/jresmetheduc-2020-0036>
- Hrusa, N. A., Much Islas, P., Schneider, J. A., & Vega, I. J. (2020). Policies for teacher professionalization in Mexico's education reform. In F. M. Reimers (Ed.), *Empowering teachers to build a better world: How six nations support teachers for 21st century education* (pp. 63–85). Springer. [https://doi.org/10.1007/978-981-15-2137-9\\_4](https://doi.org/10.1007/978-981-15-2137-9_4)
- Malzahn, K. A. (2020). 2018 NSSME+: Trends in U.S. mathematics education from 2012 to 2018. Horizon Research, Inc. Retrieved January 20, 2023, from <http://horizon-research.com/NSSME/wp-content/uploads/2020/11/Math-Trend-Report-Final.pdf>
- National Center on Education and the Economy. (2016). Canada: Diversity and decentralization. Author. Retrieved January 20, 2023, from <http://ncee.org/wp-content/uploads/2017/02/CanadaCountryBrief.pdf>
- Steele, M. (2020). Foreword. In W. G. Martin, B. R. Lawler, A. E. Lischka, & W. M. Smith (Eds.), *The mathematics teacher education partnership: The power of a networked improvement community to transform secondary mathematics teacher preparation* (pp. xi–xiv). Information Age Publishing.
- Walsh, K., Drake, G., & Holston, S. (2022). Teacher prep review: Preparation for teaching elementary mathematics. National Council on Teacher Quality. [https://www.nctq.org/dmsView/NCTQ\\_Teacher\\_Prep\\_Review\\_Preparation\\_for\\_Teaching\\_Elementary\\_Mathematics\\_2022](https://www.nctq.org/dmsView/NCTQ_Teacher_Prep_Review_Preparation_for_Teaching_Elementary_Mathematics_2022)

## OPEN SCIENCE WORKING GROUP: DATA MANAGEMENT AND SHARING

Gizem Solmaz-Ratzlaff  
Learning Systems Institute and School of  
Teacher Education  
Florida State University  
gsolmaz@fsu.edu

Charity Buntin  
Learning Systems Institute and School of  
Teacher Education  
Florida State University  
cbauduin@lsi.fsu.edu

Robert C. Schoen  
Learning Systems Institute and School of  
Teacher Education  
Florida State University  
rschoen@lsi.fsu.edu

### Introduction

The White House Office of Science and Technology Policy has declared 2023 as the The “*Year of Open Science*” (The White House, 2023). National Aeronautics and Space Administration (NASA; 2023a) declared that 2023 will be a year to “*celebrate the benefits and successes of open science and to inspire more scientists to adopt open science practices.*” NASA also introduced the Transform to Open Science initiative (NASA, 2023b), a 5-year plan with the goal of accelerating open science practices. Open Science is an international movement in research, not just something that is occurring in the US. The European Commission (2019; 2021) stated that they are adopting Findable, Accessible, Interoperable, and Re-usable data practices (FAIR; O’Carroll et al., 2017), and that shared data should become an expected output of all EU-funded scientific research. So, how do FAIR principles of open science affect mathematical education, and educational research as a whole? And how might mathematics education research change in response to the open-science movement? Come spend three hours with us to answer these questions and let us share some of our lessons with you. You are encouraged to bring your own data and use this time open science journey.

### Description of the Working Group

This working group will discuss what open science is, why should we utilize open science practices in educational research, and some worked open science examples that researchers can utilize in educational research. Our research lab has been working on utilizing open science practices in our research for several years now. We have published more than 20 data sets, including replication code, data dictionaries, protocols, and technical reports. One of the struggles we had through this process was to find a template for sharing educational data. We have created our own protocols and working models based on our emerging understanding of sound data-management principles. We will share our story and introduce the model that we have developed to share data. Our working model for preparing and organizing the data for sharing is still work-in-progress, but we think it is useful, and we look forward to presenting it to attendees and discussing ways to improve it.

We have now used the model that we have developed to share data from four waves of data collection (Schoen et al. 2019; 2020; 2021; 2022) over a four-year period for the Mathematics Anxiety and Attitudes Scales for Students (MAASS). MAASS is a short questionnaire designed to measure student mathematics attitudes (confidence, enjoyment, and satisfaction) and anxiety.

MAASS is designed to be short and efficient and consists of a small number of questions, each with a corresponding Likert-type response scale with four categories. We share the items on the questionnaire, response categories, data, data dictionary, replication code, output files, and a brief summary finding based on our use of the MAASS with tens of thousands of grade K–5 students. Each wave includes data from thousands of students.

In this working group, we will start by describing the work we have done so far and our guiding principles. We will share links to data we have shared through osf.io and introduce our working model for preparing and sharing those data. We will invite attendees to take a close look at the material we have shared. After providing the opportunity for attendees to analyze and discuss what we have shared (and how we are sharing it), we will invite attendees to share examples of other models to review and discuss. We will then lead a discussion on ways that we can modify and improve the model/template. At the end of the working group, participants will have experience creating a project page in Open Science Framework and drafting a protocol for sharing the data they have (or will have).

### **Plan for the Working Group**

Through this series of working groups, we will have presentations related to open science, preparing data for sharing, worked examples from our data-sharing journey, and learned lessons. Presentations will include active involvement of the participants. Participants will have a chance to bring their own materials to the working groups.

Day 1: Principles of Open Science

Introduction to the daily schedule and establishing norms (10 minutes)

Planned presentation (15 to 20 minutes)

Work time & discussion (45 to 50 minutes)

Closing (10 minutes)

Day 2: Principles of Data Management

Introduction to daily schedule and establishing norms (10 minutes)

Planned presentation (15 to 20 minutes)

Work time & discussion (45 to 50 minutes)

Closing (10 minutes)

Day 3: Principles of Data Sharing

Introduction to daily schedule and establishing norms (10 minutes)

Planned presentation (15 to 20 minutes)

Work time & discussion (45 to 50 minutes)

Closing (10 minutes)

Acknowledgments

The research and development reported here were supported in part by the Institute of Education Sciences and the United States Department of Education through Award Numbers R305A180429 and U423A180115 to Florida State University. The opinions expressed are those of the authors and do not represent the views of the U.S. Department of Education.

### **References**

European Commission, Directorate-General for Research and Innovation, (2019). Cost-benefit analysis for FAIR research data: cost of not having FAIR research data. Publications Office.  
<https://data.europa.eu/doi/10.2777/02999>

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

- European Commission, Directorate-General for Research and Innovation, (2021). Towards a reform of the research assessment system: scoping report. Publications Office. <https://data.europa.eu/doi/10.2777/707440>
- National Aeronautics and Space Administration. (2023a, January 11). Guide to a Year of Open Science. [https://nasa.github.io/Transform-to-Open-Science-Book/Year\\_of\\_Open\\_Science\\_Guide/readme.html](https://nasa.github.io/Transform-to-Open-Science-Book/Year_of_Open_Science_Guide/readme.html)
- National Aeronautics and Space Administration. (2023b, January 11). Transform to Open Science (TOPS). <https://science.nasa.gov/open-science/transform-to-open-science>
- O'Carroll, C., Vandavelde, K., McAllister, D., Metcalfe, J., Maas, K., Rentier, B., Kaunismaa, E., Esposito, F., and Cabello Valdes, C. (2017). Evaluation of research careers fully acknowledging Open Science practices: rewards, incentives and/or recognition for researchers practicing Open Science. Publication Office of the European Union.
- Schoen, R. C., Solmaz, G., Guven, A., & Buntin, C. (2022). Psychometric Report for the Mathematics Anxiety and Attitudes Scales for Students (MAASS) in Fall 2019. [Data set]. Open Science Framework. <https://doi.org/10.17605/OSF.IO/9UJNS>
- Schoen, R. C., Solmaz, G., Guven, A., & Buntin, C. (2022). Psychometric Report for the Mathematics Anxiety and Attitudes Scales for Students (MAASS) in Fall 2020. [Data set]. Open Science Framework. <https://doi.org/10.17605/OSF.IO/VWZE4>
- Schoen, R. C., Solmaz, G., Guven, A., & Buntin, C. (2022). Psychometric Report for the Mathematics Anxiety and Attitudes Scales for Students (MAASS) in Spring 2021. [Data set]. Open Science Framework. <https://doi.org/10.17605/OSF.IO/5VDPF>
- Schoen, R. C., Solmaz, G., & Buntin, C. (2022). Psychometric Report for the Mathematics Anxiety and Attitudes Scales for Students (MAASS) in Spring 2022. [Data set]. Open Science Framework. <https://doi.org/10.17605/OSF.IO/KQJBF>
- The White House. (2023, January 11). FACT SHEET: Biden-Harris Administration Announces New Actions to Advance Open and Equitable Research [Press release]. <https://www.whitehouse.gov/ostp/news-updates/2023/01/11/fact-sheet-biden-harris-administration-announces-new-actions-to-advance-open-and-equitable-research/>



**EMBODIED MATHEMATICAL IMAGINATION AND COGNITION (EMIC)  
RESEARCH COLLOQUIUM**

**COLOQUIO DE INVESTIGACIÓN: COGNICIÓN E IMAGINACIÓN MATEMÁTICA  
INCORPORADA**

**COLLOQUE DE RECHERCHE : IMAGINATION ET COGNITION MATHÉMATIQUES  
INTÉGRÉES**

<u>Mitchell J. Nathan</u> U. of Wisconsin - Madison mnathan@wisc.edu	<u>Candace Walkington</u> Southern Methodist University cwalkington@mail.smu.edu	<u>Dor Abrahamson</u> Univ. of California, Berkeley dor@berkeley.edu
<u>Erin Ottmar</u> Worcester Polytechnic Inst. erottmar@wpi.edu	<u>Martha Wagner Alibali</u> U. of Wisconsin - Madison Martha.alibali@wisc.edu	<u>Michael Swart</u> U. of Wisconsin - Madison mswart@wisc.edu

*The Embodied Mathematical Imagination and Cognition (EMIC) Research Colloquium offers hands-on individual and collaborative mathematics activities; technological, curricular, and pedagogical demonstrations; and open spaces for exploration and discussion regarding the embodied nature of mathematics education. At PME-NA 45 we will use the conference theme of “Engaging all learners” to invite the community to directly experience how embodied perspectives on teaching and learning expand notions of mathematical activity, effective teaching, and assessment. We will explore some current contributions, future potential, and challenges for more inclusive and effective educational experiences for learners and teachers.*

Keywords: Cognition; Learning Theory; Teaching; Social Justice; Systemic Change; Technology

### **Theoretical Background**

It is increasingly apparent that effective mathematics education systems need to address the needs of embodied learners by designing educational experiences that: 1) are grounded in people’s natural propensities for movement, sensation, perception, and collaborative interactions with cultural and scientific tools; 2) support “multiple linguistic repertoires,” (McKinney de Royston, et al., 2020, p. 10) including verbal, nonverbal, and multimodal ways of knowing, communicating, and assessment; and 3) facilitate learners’ participation in systems that support the distributed nature of knowledge and reasoning (e.g., Abrahamson 2014; Alibali & Nathan, 2012; Arzarello et al., 2009; De Freitas & Sinclair, 2014; Edwards et al., 2014; Radford 2009; Walkington et al., 2021).

### **A Brief History and Motivation of the EMIC Working Group**

The EMIC working group officially started in East Lansing, MI during PME-NA 2015. However, it emerged from a variety of earlier initiatives, including a 2007 AERA symposium, “Mathematics Learning and Embodied Cognition,” a 6-year NSF-REESE grant (Nemirovsky et al., 2008), special issues of *Educational Studies in Mathematics* (2009), *The Journal of the Learning Sciences* (2012), and an NCTM 2013 research pre-session keynote panel, “Embodied cognition: What it means to know and do mathematics.” A group of PME-NA members recognized a “critical mass” of investigators involved in related projects, and contributed findings and conceptual frameworks to form a stable and growing mathematics education

research community to discuss, explore, and advance this scholarly program. We have convened annually at PME-NA ever since, organizing events that investigate the embodied nature of mathematics through physical and digital games, art and craft, individual and collaborative dance and movement, language, perception, immersive experiences, and more. EMIC members continue to explore new technologies (e.g., Harrison et al., 2018; Walkington et al., 2021) and research methods (e.g., Abrahamson et al., 2021; Closser et al., 2021). They go beyond scholarly publications and presentations to offer: novel embodied technologies, including some that have gone to scale, for classroom use, special education, and emerging bilinguals; teacher learning programs for instructional uses of gestures; and webinars for teachers and parents.

### **EMIC 2023: Engaging all Learners**

The organizers (including the authors; and coordinator M. Nathan) have several years of experience co-designing and facilitating the PME-NA sessions. In 2023, we will organize hands-on activities and group discussions around the central questions that frame this year's conference theme, "Engaging all Learners." On **Day 1**, we will first do introductions and a brief overview of grounded and embodied mathematical thinking and teaching. We will then form 3 groups for those most interested in (1) student thinking, (2) teaching and teachers, and (3) embodied design. Groups will describe what engagement means to them, ways of documenting engagement, and benefits for math education from high levels of engagement. Each group will identify one math idea (e.g., division by fractions, or teaching with multiple representations) and generate one high- and one low-engagement activity for the same math idea. Members of the other groups will participate in the high- and low-engagement activities in a jigsaw-type design. The whole group will convene to share these ideas and experiences so all participants will have delved into the topic of engagement from each group perspective. Day 1 will conclude with open-ended Q&A on the nature of embodiment theory, research methods, and design principles.

On **Day 2**, we will continue our 3 participant-groups focused on embodied learning, teaching, and technology (membership can be fluid). Each group will co-design a math activity for high engagement. Organizers will facilitate effective practices for group co-design, such as record keeping, and having all voices respected. Collaborative co-design will operate on two levels: (1 Math) What is the mathematics to be learned, taught, or assessed? (2 engagement) What makes this activity engaging? For which learners? How can we know if learners are engaged? Groups will present to the whole group their provisional designs and proposed principles for embodied engagement. Day 2 will conclude with a facilitated discussion identifying principles, practices, and assessment methods for engaging all learners.

**Day 3** People will engage with the designed activities and use Jam Board (or another online group discourse tool) to share impressions of the experiences. Participants will then reconfigure around common themes that arose in topic-specific groups on Days 1 & 2. It will invite participants to identify cross-cutting themes. A facilitated discussion will follow on why embodiment is a powerful lens for designing for engagement.

### **Conclusions and Looking Ahead**

Embodied approaches to design, instruction, and assessment offer natural inroads for engaging all learners. Movement-based and perception-based learning activities can reach students who may otherwise feel left out of highly discursive and symbol based activities. Teachers may explore a greater range of teaching practices by applying effective instructional methods such as using gestures for connection making, and for using nonverbal indicators of students' reasoning, trouble spots, insights, and learning. The emergence of embodied

technologies using movement sensors and AR/VR offer new ways of interacting with mathematical ideas. By working with these ideas with the PME-NA community, we hope to foster a more engaged community committed to transformative education.

### References

- Abrahamson, D. (2014). The monster in the machine, or why educational technology needs embodied design. In V. Lee (Ed.), *Learning technologies and the body: Integration and implementation* (pp. 21-38). New York: Routledge (Taylor & Francis Group).
- Abrahamson, D., Nathan, M. J., Williams-Pierce, C., Walkington, C., Ottmar, E. R., Soto, H., & Alibali, M. W. (2020). The future of embodied design for mathematics teaching and learning [Original Research]. *Frontiers in Education*, 5(147). <https://doi.org/10.3389/educ.2020.00147>
- Abrahamson, D., Worsley, M., Pardos, Z. A., & Ou, L. (2021). Learning analytics of embodied design: Enhancing synergy. *International Journal of Child-Computer Interaction*, 100409.
- Alibali, M. W. & Nathan, M. J. (2012). Embodiment in mathematics teaching and learning: Evidence from learners' and teachers' gestures. *Journal of the Learning Sciences*. (Special Issue on Embodiment in Mathematics.), 21(2), 247-286. DOI: 10.1080/10508406.2011.611446.
- Arzarello, F., Paola, D., Robutti, O., & Sabena, C. (2009). Gestures as semiotic resources in the mathematics classroom. *Educational Studies in Mathematics*, 70(2), 97-109.
- Closser, A. H., Erickson, J. A., Smith, H., Varatharaj, A., & Botelho, A. F. (2021). Blending learning analytics and embodied design to model students' comprehension of measurement using their actions, speech, and gestures. *International Journal of Child-Computer Interaction*, 100391.
- De Freitas, E., & Sinclair, N. (2014). *Mathematics and the body: Material entanglements in the classroom*. Cambridge University Press.
- Edwards, L., Ferrara, F., & Moore-Russo, D. (Eds). (2014). *Emerging perspectives on gesture and embodiment in mathematics*. Charlotte, N.C.: Information Age.
- Harrison A., Ottmar, E., Arroyo, I., Rosenbaum, L., Bakker, A., Abrahamson, D., Hulse, T., Manzo, D., & Landy, D. (2018, April). Embodiment and action in mathematics games. Symposium conducted at the *American Psychological Association's conference on Technology, Mind & Society*, April 5-7, 2018, Washington, DC.
- McKinney de Royston, M., Lee, C., Nasir, N. I. S., & Pea, R. (2020). Rethinking schools, rethinking learning. *Phi Delta Kappan*, 102(3), 8-13.
- Nemirovsky, R., Hall, R., Nathan, M. J., Alibali, M. W., & Leander, K. (2008). Tangibility for the teaching, learning, and communicating of mathematics. Grant proposal to the National Science Foundation Research and Evaluation on Education in Science and Engineering (NSF-REESE; DRL-0816406; 2008-2014).
- Radford, L. (2009). Why do gestures matter? Sensuous cognition and the palpability of mathematical meanings. *Educational Studies in Mathematics*, 70(2), 111-126.
- Walkington, C., Gravell, J., Velazque, J., He, T., Hickey, G., Nathan, M. J., & Cuevas, A. (2021, June). Collaborative virtual learning in the shAR geometry simulation environment. In A. Wichmann, H. U. Hoppe, & N. Rummel (Eds.), *General Proceedings of the 1st Annual Meeting of the International Society of the Learning Sciences 2021* (pp. 17-20). Bochum, Germany: International Society of the Learning Sciences.

## RESEARCH COLLOQUIUM: MODELS AND MODELING PERSPECTIVES

### PERSPECTIVAS HACIA LA MODELACIÓN MATEMÁTICA

Corey Brady  
Southern Methodist  
Univ.  
corey.brady@smu.edu

Hyunyi Jung  
University of Florida  
hyunyi.jung@coe.ufl.edu

Angeles Dominguez  
Tecnologico de Monterrey  
angeles.dominguez@itesm.mx

Jeff McLean  
U. No. Carolina Chapel  
Hill  
mclean@unc.edu

Aran Glancy  
Hill-Murray School  
aglancy@hill-murray.org

*The Models and Modeling Working Group was initiated with PME-NA itself in 1978, and it has consistently convened since its formation. This year, we propose to continue our work within the Research Colloquium format. Expanding upon the discussions of the 2021 Colloquium, we will continue the work to articulate a coherent North-American voice on key topics in the international modeling research community. In particular, we will explore opportunities for collaboration and communication of our perspectives on the topics of Interdisciplinarity in Modeling, Modeling and Citizen Science and Culturally Sustaining approaches to Modeling.*

Keywords: Modeling, Problem Solving, Culturally Relevant Pedagogy, Integrated STEM/STEAM.

#### **History of the Models and Modeling Perspective at PME-NA**

Over the 44 years since it first formed (at the first PME-NA), the Models and Modeling Working Group has offered researchers a forum to share results; to coordinate research collaborations; and to welcome new researchers into the community. In early years, the Group placed great emphasis on designing and analyzing activities that fostered realistic forms of modeling among groups of learners. The goal was to pursue mathematics learning as a design-based research project (before DBR was a widely-accepted methodology). During this phase of the field's evolution, a primary focus was to articulate design principles for such "Model-Eliciting Activities," or *MEAs* (Doerr & English, 2006; Lesh & Doerr, 2003; Lesh, Hoover, Hole, Kelly, & Post 2000; Lesh, Hoover, & Kelly, 1992; Hjalmarson & Lesh, 2007; Zawojewski, Hjalmarson, Bowman, & Lesh, 2008), and to generate images of *idea development* in implementing MEAs (Lesh et al, 2000; Lesh & Doerr, 2003). Quickly, however, MEAs were also acknowledged to be innovative *learning environments* and authentic *performance assessments*. The resulting "turn" to connect more integrally with classroom ecologies has raised important questions that are still being pursued today.

Over the past three years, participants in this community have been engaging with the international modeling community, an effort that has been enhanced by the timing of the ICME conference and the accessibility, via online formats, of that conference and other international conferences such as ICTMA during the pandemic. In 2021, the Group convened under the Research Colloquium format, and we propose to continue in this format this year.

### **Areas for Discussion in the Colloquium**

Our focus at PME-NA this year will be to share findings and strategize ongoing and future work along three dimensions, which we feel have both intrinsic richness and importance as perspectives that need to be shared with the international community. These dimensions are:

1. Interdisciplinarity in Mathematical Modeling
2. Citizen Science and Mathematical Modeling
3. Culturally Sustaining approaches to Mathematical Modeling

#### **Interdisciplinarity**

Authentic modeling problems create the need for constructs that are not restricted to a single discipline (Lesh et al, 2007). In this way, solving modeling problems can create opportunities for students to experience interdisciplinary thinking, and can offer the means for implementing STEM integration (English, 2016). Conversely, realistic problem situations embedded in subject areas other than mathematics (STEM and beyond), can offer opportunities for *mathematizing*. Applying a mathematical lens and identifying mathematical structures in these areas helps learners to forge personal links between mathematics and different disciplines. We will use the lens of interdisciplinary design to raise questions about the mathematical foundations of STEM; and we will consider some of the rich intersections between mathematical modeling and data science, leveraging computational thinking and programming.

#### **Citizen Science**

In citizen science, communities of volunteers participate in creating datasets that can support and shape scientific research into topics important to them. The intersection between Model-Eliciting Activities and citizen science offers a rich terrain for design-based research. This is particularly true given that citizen science is increasingly interacting with school-based learning, creating powerful contexts that can cultivate mathematical practices and identities (McLean et al, 2023).

#### **Culturally Sustaining Approaches**

Culturally sustaining pedagogy (Paris, 2012) builds on Ladson-Billings's (1995) culturally relevant pedagogy, which aims to support the development of students who can “achieve academically, demonstrate cultural competence, and...both understand and critique the existing social order” (p. 474). Paris (2012) emphasized the notions of cultural relevance informed by a concern for sustainability - the fostering of cultural pluralism of diverse learners and ever-changing states of their cultural ways of being. Mathematical modeling can play an important role in connecting students' cultures and the mathematical world by allowing students to represent multiple perspectives and systemic societal issues (Anhalt et al., 2018). We will discuss how modeling can be used to draw on students' cultures and mathematical knowledge and expand their understanding of societal issues.

### **Organization of the Sessions**

As we have done within the former Working Group format, our Research Colloquium will attend to the needs of both “newcomers” and “old hands” to the MMP. Based on the participant group, we will cast our “net” widely.

Our principal facilitators will describe recent work that foregrounds the focal dimensions of

the Colloquium. Each session will begin with a precis of one or more recent studies, followed by structured discussion of issues highlighted therein. These recent efforts will not be presented as “finished products;” rather, we will hold an open discussion of issues of *theory* and *method* that have arisen in recent work, with the goal of opening the conversation. Each of our thematic dimensions will have follow-up opportunities outside of the Colloquium meeting times, and into the upcoming year.

### References

- Anhalt, C. O., Staats, S., Cortez, R., & Civil, M. (2018). Mathematical modeling and culturally relevant pedagogy. In Y. J. Dori, Z. Mevarech, & D. Baker (Eds.), *Cognition, metacognition, and culture in STEM education* (pp. 174-195). Netherlands: Springer.
- Doerr, H. M., & English, L. D. (2003). A modeling perspective on students' mathematical reasoning about data. *Journal for research in mathematics education*, 34(2), 110-136.
- Doerr, H. M., & English, L. D. (2006). Middle grade teachers' learning through students' engagement with modeling tasks. *Journal of Mathematics Teacher Education*, 9(1), 5-32.
- English, L. D. (2016). STEM education K-12: Perspectives on integration. *International Journal of STEM education*, 3(1), 1-8.
- Hjalmarson, M. & Lesh, R. (2007) Design research: Engineering, systems, products, and processes for innovation. In L. D. English (Ed.). *Handbook of international research in mathematics education: Directions for the 21st century* (2nd edn). (pp. 520-534). New York, NY: Routledge.
- Ladson-Billings, G. (1995). Toward a theory of culturally relevant pedagogy. *American Educational Research Journal*, 32(3), 465-491.
- Lesh, R., & Doerr, H. M. (2003). In what ways does a models and modeling perspective move beyond constructivism? In R. Lesh & H. Doerr (Eds.), *Beyond constructivism: A models and modeling perspective* (pp. 519-556). Mahwah, NJ: Lawrence Erlbaum Associates.
- Lesh, R. A., Hamilton, E., & Kaput, J. J. (Eds.). (2007). *Foundations for the future in mathematics education*. New York: Lawrence Erlbaum.
- Lesh, R., Hoover, M., & Kelly, A. E. (1992). Equity, technology, and teacher development. In I. Wirszup & R. Streit (Eds.), *Developments in school mathematics education around the world* (Vol. 3, 104-129). Reston, VA: National Council of Teachers of Mathematics.
- Lesh, R., Hoover, M., Hole, B., Kelly, A. E., & Post, T. (2000). Principles for developing thought-revealing activities for students and teachers. In A. Kelly, R. Lesh (Eds.), *Research Design in Mathematics and Science Education*. (pp. 591-646). Lawrence Erlbaum Associates, Mahwah, New Jersey.
- McLean, J. A., Brady, C., Jung, H., Dominguez, A., & Glancy, A. W. (2023). Seeing the Forest for the Trees: Investigating Students' Data Moves in a Citizen Science Based Model-Eliciting Activity. In *Advancing and Consolidating Mathematical Modelling: Research from ICME-14* (pp. 193-204). Cham: Springer International Publishing.
- Paris, D. (2012). Culturally sustaining pedagogy: A needed change in stance, terminology, and practice. *Educational researcher*, 41(3), 93-97.
- Zawojewski, J. S., Hjalmarson, M. A., Bowman, K., & Lesh, R. (2008) A Modeling Perspective on Learning and Teaching in Engineering Education. In *Models and modeling in engineering education: Designing experiences for all students*. Rotterdam: Sense Publication (pp. 1-16).

## TEACHING AND LEARNING WITH DATA INVESTIGATION: TOWARD A SOCIALLY AND ENVIRONMENTALLY JUST WORLD

Sunghwan Byun  
North Carolina State  
University  
sbyun2@ncsu.edu

Travis Weiland  
University of Houston  
tweiland@uh.edu

Susan Cannon  
University of Georgia  
cannonso@uga.edu

Anthony Fernandes  
University of North Carolina  
- Charlotte  
anthony.fernandes@charlotte.  
edu

Emmanuel Nti-Asante  
University of Massachusetts  
- Dartmouth  
emmaasante2012@  
gmail.com

Franziska Peterson  
University of Maine  
franziska.peterson@  
maine.edu

Karoline Smucker  
Eastern Oregon University  
ksmucker@eou.edu

Kingsley Adamoah  
Middle Tennessee State  
University  
kya2d@mtmail.mtsu.edu

Christopher Engledowl  
University of Florida  
c.engledowl@coe.ufl.edu

*We continue the past four years of the working group on teaching and learning data investigation by 1) sustaining the ongoing cross-institutional collaboration to develop frameworks and resources for supporting mathematics educators in facilitating data investigations and 2) advancing our discussions on designing data investigations with an eye towards taking action to promote socially and environmentally just outcomes. Based on the role of context in teaching and learning with data, we organize this working group into three themes: the context of mathematics teacher preparation, the context of cross-disciplinary work, and the context of research and collaboration.*

Keywords: statistics education, data science education, data investigation

Existing recommendations for teaching statistics and data science in K-12 education, such as GAISE II, and NCTM Principles and Standards, are extensive and longstanding (Bargagliotti et al., 2020; NCTM, 2000). In the past working group sessions, we built a consensus that the primary challenge in statistics and data science education is in turning these recommendations into action. Mathematics teachers need support in navigating the messiness of facilitating data investigations, especially in light of disciplinary differences between mathematics and statistics (Cobb & Moore, 1997). Statistics education is embedded in mathematics curricula and standards documents that do not always attend to these differences (Shaughnessy, 2007). Therefore, we suggest that mathematics and statistics educational researchers work together to investigate the teaching and learning of statistics and data science through data investigations in mathematics classrooms and to establish shared discourses and practices that support such efforts. It is in this interdisciplinary space of collaboration that we propose this working group continue its efforts to create community and collaboration between mathematics and statistics educational researchers.

### Theoretical Background

This year's working group will sustain collaboration on developing frameworks and resources that can support facilitating data investigations in the mathematics curriculum. Due to

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

significant disciplinary distinctions between mathematics and statistics (Lee et al., 2022; Cobb & Moore, 1997; Wild & Pfannkuch, 1999), existing frameworks and resources for teaching mathematics need adaptation to adequately address the unique needs of designing and facilitating data investigations. The working group is currently working on adapting the 5 Practices framework (Smith & Stein, 2011), and there are other frameworks used in mathematics education that could be adapted for the specific goal of facilitating productive and equitable data investigations.

We view data investigation as a vehicle to make sense of and act upon issues of social and environmental justice around the world. For instance, systemic racism, operating through policies and institutional practices (e.g., traffic stops, school discipline), has a persistent impact on the current sociopolitical landscape. Moreover, in the era of global climate change, investigating data related to environmental change and its impact on respective communities is timely (Bhattacharya et al., 2021; Kjølvik & Schultheis, 2019; Monroe et al., 2019). We will discuss the prominent opportunity for mathematics educators to contribute to developing data literacy that is situated in their social and material environments.

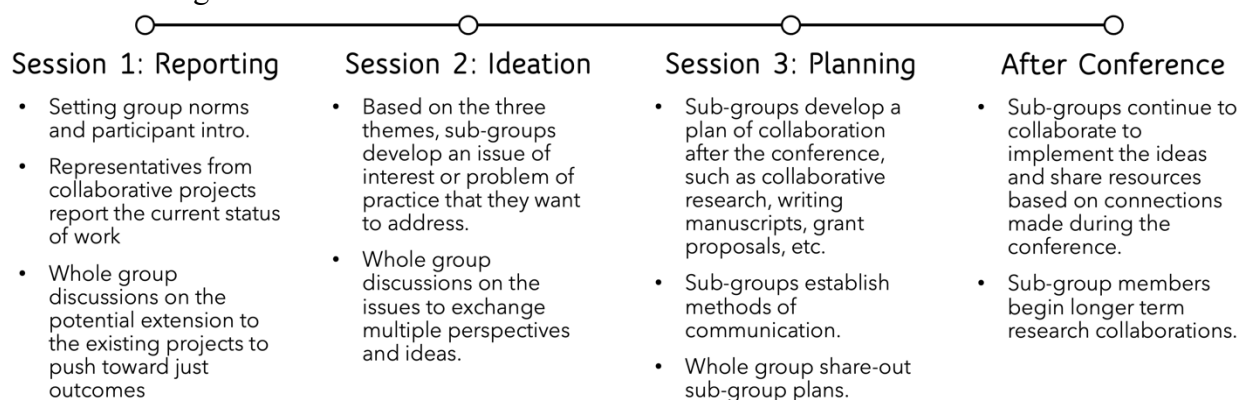
This view also calls for further exploration of methodological perspectives on research and collaboration so that data investigations can serve broader purposes of social and environmental justice beyond K-12 classrooms. For instance, scholars called for social design-based experiments (Gutiérrez & Jurow, 2016; Nasir et al., 2021) to better align education with the axiological and epistemic orientation of the community, which is most impacted by the social and environmental outcomes. Applying participatory approaches could shed new light on how co-designing of and co-participating in data investigation could be a medium to embrace participatory consciousness (Osibodu et al., 2023; Heshusius, 1994). We will venture into different methodologies to make data investigation authentic for learners and true to its purpose in multiple disciplinary contexts.

### **Organization of Working Group Activities**

In last year's working group, we successfully cultivated cross-institutional collaborations by devoting much of the session time to ideation and planning among participants. We will maintain this emergent approach while highlighting the need for taking action toward social and environmental justice in data investigation processes. The first session will focus on reporting the existing collaborations and identifying ways to extend these efforts. This way, we can share the ongoing projects with new participants and invite them to the collective efforts. The second and third sessions will be devoted to opening new opportunities for collaboration toward the goal of social and environmental justice through data investigation. Attending to multiple features of this complex work, we will form sub-groups of interest that will outlast the working group sessions



with identified goals and methods of collaboration.



**Figure 1: Session Plans and Goals**

### References

- Bargagliotti, A., Franklin, C., Arnold, P., Johnson, S., Perez, L., & Spangler, D. A. (2020). Pre-K-12 guidelines for assessment and instruction in statistics education II: (GAISE II): A framework for statistics and data science education. American Statistical Association.
- Bhattacharya, D., Carroll Steward, K., & Forbes, C. T. (2021). Empirical research on K-16 climate education: A systematic review of the literature. *Journal of Geoscience Education*, 69(3), 223–247. <https://doi.org/10.1080/10899995.2020.1838848>
- Cobb, G. W., & Moore, D. S. (1997). Mathematics, statistics, and teaching. *The American Mathematical Monthly*, 104(9), 801–823. <https://doi.org/10.1080/00029890.1997.11990723>
- Gutiérrez, K. D., & Jurow, A. S. (2016). Social design experiments: Toward equity by design. *Journal of the Learning Sciences*, 25(4), 565–598. <https://doi.org/10.1080/10508406.2016.1204548>
- Heshusius, L. (1994). Freeing ourselves from objectivity: Managing subjectivity or turning toward a participatory mode of consciousness? *Educational Researcher*, 23(3), 15–22. <https://doi.org/10.2307/1177221>
- Kjelvik, M. K., & Schultheis, E. H. (2019). Getting messy with authentic data: Exploring the potential of using data from scientific research to support student data literacy. *CBE Life Sciences Education*, 18(2), 1–8. <https://doi.org/10.1187/cbe.18-02-0023>
- Lee, H., Mojica, G., Thrasher, E., & Baumgartner, P. (2022). Investigating data like a data scientist: Key practices and processes. *Statistics Education Research Journal*, 21(2), 1–23. <https://doi.org/10.52041/serj.v21i2.41>
- Monroe, M. C., Plate, R. R., Oxarart, A., Bowers, A., & Chaves, W. A. (2019). Identifying effective climate change education strategies: a systematic review of the research. *Environmental Education Research*, 25(6), 791–812. <https://doi.org/10.1080/13504622.2017.1360842>
- Nasir, N. I. S., Lee, C. D., Pea, R., & McKinney de Royston, M. (2021). Rethinking learning: What the interdisciplinary science tells us. *Educational Researcher*, 50(8), 557–565. <https://doi.org/10.3102/0013189X211047251>
- National Council of Teachers of Mathematics (NCTM). (2000). *Principles and Standards for School Mathematics*. NCTM.
- Osibodu, O., Byun, S., Hand, V., & LópezLeiva, C. (2023). A participatory turn in mathematics education research: Possibilities and tensions. *Journal for Research in Mathematics Education*, 54(3), 225–232. <https://doi.org/10.5951/jresmetheduc-2021-0147>
- Shaughnessy, J. M. (2007). Research on statistics learning and reasoning. In F. K. Jr. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 957–1010). Information Age Publishing, Inc.
- Smith, M., & Stein, M. K. (2011). *5 Practices for Orchestrating Productive Mathematics Discussions*. NCTM.
- Wild, C. J., & Pfannkuch, M. (1999). Statistical thinking in empirical enquiry. *International Statistical Review*, 67(3), 223–248. <https://doi.org/10.1111/j.1751-5823.1999.tb00442.x>

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

## TEACHING AND LEARNING WITH DATA INVESTIGATION: WORKING GROUP REPORT FROM 2022

Sunghwan Byun  
North Carolina State  
University  
sbyun2@ncsu.edu

Travis Weiland  
University of Houston  
tweiland@uh.edu

Susan Cannon  
University of Georgia  
cannonso@uga.edu

Anthony Fernandes  
University of North Carolina  
- Charlotte  
anthony.fernandes@charlotte.  
edu

Emmanuel Nti-Asante  
University of Massachusetts  
- Dartmouth  
emmaasante2012@  
gmail.com

Franziska Peterson  
University of Maine  
franziska.peterson@  
maine.edu

Karoline Smucker  
Eastern Oregon University  
kismucker@eou.edu

Christopher Engledowl  
University of Florida  
c.engledowl@coe.ufl.edu

*In the working group this year, we built on past statistics education working groups to co-develop a syncretic space in which new knowledge can emerge while embracing the tension between critical and dominant views. We took a renewed focus on data investigation and its relationship to issues of equity and justice to invite a wide range of discussions and collaborations within the domain of statistics and data science education. The primary goal of the working group continued to be initiating cross-institutional collaborative efforts based on overlapping research interests. Several subgroups were formed, and currently, a group of collaborators is working on a research project envisioned during the working group.*

Keywords: statistics education, data science education, data investigation

The 2022 conference marks the fourth meeting of the working group focused on statistics and data science education. Beginning at the 2019 conference, we explored a framework for meaningful statistical learning environments (Ben-Zvi et al., 2018). Then in the 2020 conference, the focus was on considering statistics education across boundaries in collaboration between scholars in the US and Mexico held in Spanish and English. In 2021 we began to consider how data science education might play a role in statistics education and the mathematics curriculum. We also began to discuss issues of social justice in data investigations and methodological issues related to measurement in the teaching and learning of statistics. In this most recent working group, we seriously considered the growing interest and movement toward incorporating data science education into K-12 schools (EDC, 2015; Lee et al., 2022). In particular, some states in the United States have been developing standards for data science, including data science as a part of the official curriculum. Against this backdrop, the prior working group on statistics education widened its scope to include data investigation. This allowed the working group to be more inclusive of the growing focus on data science while drawing from the prior work in K-12 statistics education. Based on this renewed focus on the teaching and learning of data investigation, the 2022 working group started discussions on what data investigation is and how we could use its conceptualizations as means to further theorize and develop students' experiences with data in multiple disciplinary and informal learning contexts.

The working group attendees numbered approximately 20 and included returning and new members. The group included participation from research and teaching institutions in multiple states across the United States: California, Georgia, Louisiana, Maine, Michigan, Missouri, New Mexico, North Carolina, Oregon, Tennessee, and Texas. There were participants from Ontario, Canada and Mexico City, Mexico. This geographically diverse representation demonstrates the broad interest in the topic, and this allowed participants to make connections across the various research agendas and interests. While leveraging these diverse perspectives and representations, we were pleased to build inter-institutional collaborations during the working group at PME-NA. Based on our conversations, a core group of interested people is moving forward, with clear goals and objectives for their work. This has always been a goal in past years, but this is the first time a specific collaboration project has materialized. Furthermore, though the participation of the group changed from day to day we were able to maintain a core group and also invite others in. Much of the time during the sessions was dedicated to participants sharing their ideas and discussing connections across the research questions we are pursuing. The working group took advantage of the hybrid setting, with two facilitators in the room and two other facilitators interacting with participants in the virtual setting. In this report, we offer an overview of the working group activities, our ongoing collaborative research, and future questions that the working group may pursue.

### **Overview of the Working Group Activities**

The PME-NA 44 marked the fourth iteration of our working group since 2019, and we are continuing to build and expand the community for mathematics educators who share a keen interest in integrating data investigation into the mathematics curriculum and studying the teaching and learning of data literacies. This year's working group centered on a new framework for data investigation. This focus aligns with international and national research trends and policies related to K-12 statistics and data science education.

#### **Day 1: Framing Data Investigation and Finding Common Interests**

We began the first day by inviting participants to share a question that they are investigating related to teaching and learning with data. These initial interests among participants are further detailed in a later part of the report. We then had a presentation by an invited guest, Dr. Hollylynn Lee. Lee and colleagues recently published an article outlining a framework for data investigation (Lee et al., 2022). Lee shared how the team conceptualized this framework, named the data investigation process. Lee highlighted how their framework was informed by prior frameworks, including the Guidelines for Assessment and Instruction in Statistics Education (GAISE II) Framework (Bargaliotti et al., 2020) and the Problem, Plan, Data, Analysis, Conclusions (PPDAC) Framework (Wild & Pfannkuch, 1999). Lee also emphasized that the different parts of the data investigation process (i.e., Frame Problem, Consider & Gather Data, Process Data, Explore & Visualize Data, Consider Models, Communicate & Propose Action) should be seen as a holistic, dynamic, and nonlinear process. Lee's presentation was followed by group discussions on how the participants' interest in data investigation may be situated within (and beyond) the shared framework. The group used this framework as a way to organize discussions around data investigations. Initial interest groups were formed and participants spent time discussing their thoughts about the framework and also common interests in statistics education and data investigations more broadly. We concluded the first session by sharing highlights from small group discussions.

A clear indicator of the growing interest in the working group is that there were 23 participants in the first session. During small group discussions, participants shared a wide range of interests in the working group that included: the role of simulated data; qualitative and quantitative data; commonalities and differences between data science and statistics education; role of statistical literacy in data investigation; professional development and teacher preparation; and how to engage with sociopolitical issues (e.g., systemic racism) using data. From discussions of these interests, we began to look for larger themes to create groups for further discussion. The key themes that emerged from those discussions included:

### ***Data Investigation and Mathematics Teacher Preparation***

This group discussed teachers' challenges with facilitating data investigation and the lack of existing resources to support prospective and practicing teachers in this domain. Stephanie Casey shared a teacher preparation module in statistics, developed by the MODULE(S2) project (see also, Casey & Ross, 2022). Facilitating productive classroom discourse during data investigation emerged as a key area for collaboration. To this goal, the group discussed the utility of adapting the 5 Practices (Smith & Stein, 2011) in the context of data investigation.

### ***Data Investigation in Social Justice Contexts***

Anthony Fernandes and Travis Weiland shared aspects of their work grounding discussions around systemic racism in data investigations using real data (e.g., traffic stops and school discipline). They discussed the framing and design of their modules that could be used in content courses for prospective teachers. The goal of the curriculum is to normalize conversations about race and racism and to develop prospective teachers' critical consciousness and statistical content knowledge. Attendees expressed interest in using the modules in the content courses that they taught. This work could lead to further collaborations for implementation and research.

### ***Data Investigation in Environmental Science Contexts***

In the era of global climate change, investigating environmental science data has become essential and is the focus of many science units (Bhattacharya et al., 2021; Monroe et al., 2019). Analysis of climate and other environmental data is also a focus in the Next Generation Science Standards (NRC, 2013). This group discussed the need to support science teachers in the area of statistics education and data literacy, along with the potential for mathematics and science teachers to collaborate on curricula in these areas. Opportunities to bring environmental data into classrooms through authentic scientific research explorations were also considered. Franziska Peterson shared a current project (INSPIRES) focusing on forestry data to investigate environmental challenges through big data acquisition. A sub-team of the INSPIRES project directly involved science teachers in the data of forestry work. There was interest in continuing the discussions around bringing authentic environmental data to classrooms in accessible and meaningful ways for future collaborations and research.

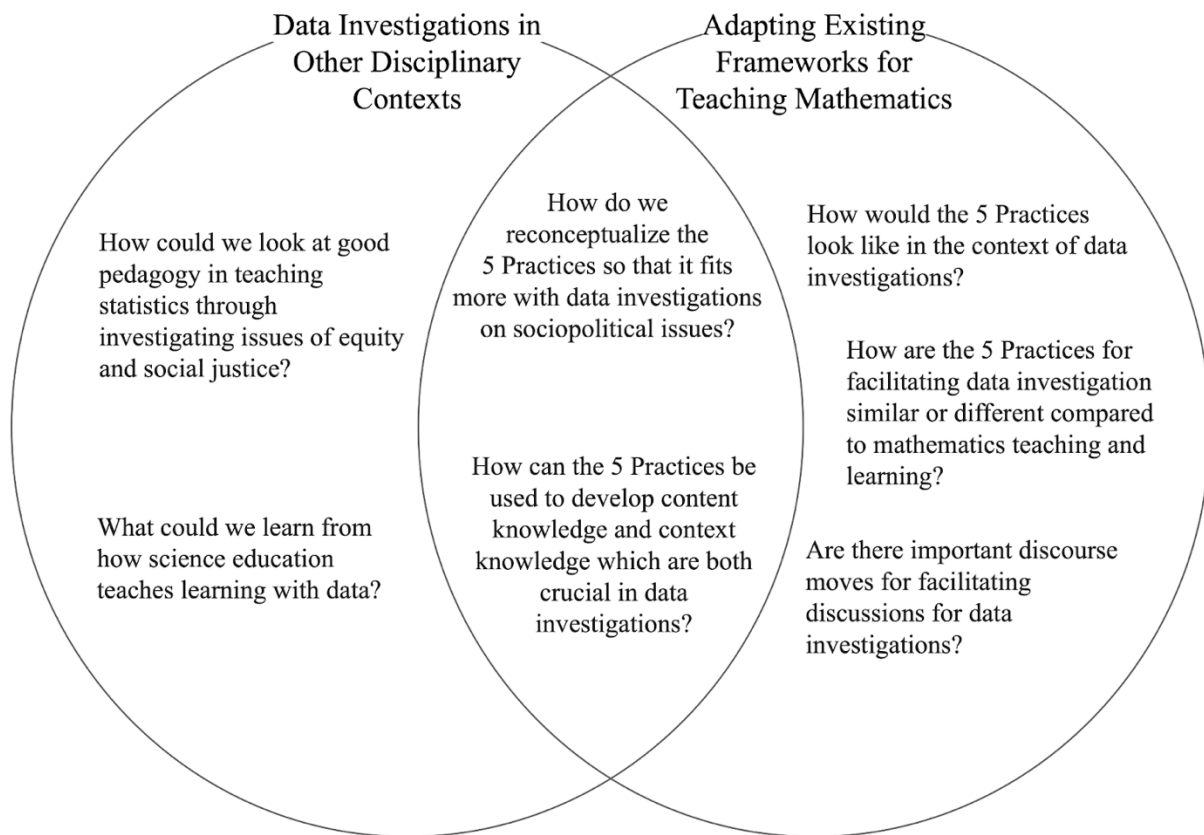
### **Day 2: Other Considerations of Data Science**

On the second day, we began with an invited speaker, Dr. Margret Hjalmanson, an NSF program officer. Hjalmanson shared some of the trends in funding for statistics and data science education projects. She highlighted how multiple projects are integrated with different disciplines, such as epidemiology and arts, in contexts including both formal and informal learning settings. Next, Chris Engledowl shared his recent experience working as a quantitative user experience researcher in a corporate setting. These discussions guided the working group to view data investigations in collaborative, interdisciplinary settings beyond the scope of mathematics classrooms and to consider important ethical issues in the data investigation process.

To start forming small groups with a more refined focus of collaboration, the working group engaged in roundtable conventions about 1) lingering questions, stuck places, problems of practices, and tension related to shared interests in statistics and data science education and 2) types of new collaborations that may help the group better engage with those questions and problems. Building on these conversations, substantial time was devoted to small groups' ideations that could lead to further collaboration after the conference. Participants joined small groups that aligned with their interests; these groups focused on teaching for social justice, integrating environmental science, and developing resources for teacher preparation. Some of these groups further developed their collaboration plans in the following session.

**Day 3: Continuing Conversations and Charting Next Steps**

The group of participants had decreased by the last session, but the core group remained engaged in discussions. Much of the discussion during this day of the working group was focused on thinking about tangible ways of working together and producing deliverables beyond just the discussions held at the conference. Instead of groups taking on different projects, those attending the last session banded together to think of a tangible project with a clear goal of producing at least one journal article. During our discussion of what project to tackle, the participants brainstormed questions for future inquiry. Figure 1 highlights the topics we discussed in two main areas: data investigation in disciplinary contexts outside mathematics and supporting statistics education within mathematics classrooms by adapting existing frameworks for teaching mathematics to account for the complexities of teaching data investigation.



**Figure 1: Questions for Future Inquiry**

## Next Steps of Working Group

Over this year, we cultivated and further refined the purpose of the working group by drawing from a variety of expertise and interests of the growing number of participants. The overarching goal has been building theories and developing resources to support teachers in teaching with authentic data investigation in relevant contexts for learners. A promising indicator of our success toward this goal is the ongoing collaboration that started after the conference. Some participants decided to think about the 5 Practices (Smith & Stein, 2011) within the context of teaching statistics. The group is regularly meeting and working toward a manuscript for the *Mathematics Teacher: Learning and Teaching PK-12* journal. This continuing collaborative effort shows the promise of the working group, informing both teachers and teacher educators about teaching and learning through data investigation.

The lingering questions, stuck places, problems of practices, and tensions shared by the working group participants also point to important areas of further inquiry and future directions of the working group. These questions include: How could the longstanding recommendations in statistics and data science education inform this goal of adapting existing frameworks in mathematics education? How should the interdisciplinary nature of statistics and data science inform other discussions in mathematics teacher education? How could teaching and learning with data investigations directly contribute to socially and environmentally just outcomes and changes?

## References

- Bargagliotti, A., Franklin, C., Arnold, P., Johnson, S., Perez, L., & Spangler, D. A. (2020). *Pre-K-12 guidelines for assessment and instruction in statistics education II: (GAISE II): A framework for statistics and data science education*. American Statistical Association.
- Ben-Zvi, D., Gravemeijer, K., & Ainley, J. (2018). Design of statistics learning environments. In D. Ben-Zvi, K. Makar, & J. B. Garfield (Eds.), *International Handbook of Research in Statistics Education* (pp. 473–502). Springer International Publishing.
- Bhattacharya, D., Carroll Steward, K., & Forbes, C. T. (2021). Empirical research on K-16 climate education: A systematic review of the literature. *Journal of Geoscience Education*, 69(3), 223–247. <https://doi.org/10.1080/10899995.2020.1838848>
- Casey, S., & Ross, A. (2022). Developing equity literacy and critical statistical literacy in secondary mathematics pre-service teachers. *Mathematics Teacher Educator*, 11(1), 40-56. <https://doi.org/10.5951/MTE.2021.0015>
- Education Development Center (EDC). (2015). *Call for action to promote data literacy*. EDC Oceans of Data Institute. <http://oceansofdata.org/call-action-promote-data-literacy>
- Lee, H., Mojica, G., Thrasher, E., & Baumgartner, P. (2022). Investigating data like a data scientist: Key practices and processes. *Statistics Education Research Journal*, 21(2), 1–23. <https://doi.org/10.52041/serj.v21i2.41>
- Monroe, M. C., Plate, R. R., Oxarart, A., Bowers, A., & Chaves, W. A. (2019). Identifying effective climate change education strategies: a systematic review of the research. *Environmental Education Research*, 25(6), 791–812. <https://doi.org/10.1080/13504622.2017.1360842>
- National Research Council (NRC). (2013). *Next generation science standards: For states, by states*. The National Academies Press. <https://doi.org/10.17226/18290>
- Smith, M., & Stein, M. K. (2011). *5 Practices for Orchestrating Productive Mathematics Discussions*. National Council of Teachers of Mathematics.
- Wild, C. J., & Pfannkuch, M. (1999). Statistical thinking in empirical enquiry. *International Statistical Review*, 67(3), 223–248. <https://doi.org/10.1111/j.1751-5823.1999.tb00442.x>

## THE POWER OF COMPUTATIONAL THINKING IN MATHEMATICS AND DATA SCIENCE EDUCATION

Zuhal Yilmaz  
Louisiana State University  
zyilmaz@lsu.edu

Terrie M. Galanti  
University of North Florida  
terrie.galanti@unf.edu

Brian R. Lawler  
Kennesaw State University  
blawler4@kennesaw.edu

Fernando Alegre  
Louisiana State University  
falegre@lsu.edu

Keywords: Computational Thinking, Mathematical Thinking, Data Science

With the increasing emphasis on computational thinking (CT) as a critical skill in K–12 teaching and learning (Committee on STEM Education, 2018), the mathematics education community has an exciting opportunity to broaden its conceptualization of mathematics as a school subject. Although there are various definitions of CT in the existing literature, we refer to Wing’s (2006) article that elaborated on what CT is and sparked the conversation on the importance of integration of CT in different subject areas. She defined CT as “the thought processes involved in formulating problems and their solutions so that the solutions are represented in a form that can be efficiently carried out by an information-processing agent.” (Wing, 2011, p. 1). CT and mathematics are naturally and historically connected (Gadanidis, 2017) because both emphasize pattern seeking and generalization in quantitative and symbolic relationships. Integrating CT and mathematics encourages students to reason using abstraction, decomposition, pattern recognition, and algorithms and to better comprehend and contextualize mathematical concepts and skills. We also see a natural connection to the emerging ideas for K–12 instruction in Data Science, an inherently integrated field.

As CT continues to attract more attention in K–12 STEM education (Román-González et al., 2017; Yadav et al., 2018), there is an increasing number of studies exploring the synergies between computing and mathematics. These studies (e.g., Brating & Kilham, 2021; Hickmott et al., 2021; Rundel & Rundel, 2017) suggest that integration of computation in mathematics and data science (DS) has the potential to foster deep understanding of ideas such as functions, variables, modeling, and data manipulation. But only a few recent studies make explicit connections between mathematical thinking (MT) and CT. Hickmott et al. (2018) reviewed 393 studies and found that the main reason numerous projects only incidentally draw a connection between MT and CT is lack of expertise in mathematics education. Furthermore, Horton & Hardin (2021) described the need for more research on the convergence of mathematics skills and computation within the context of DS. Thus, building upon the emergent interest within the PME-NA community on integration of CT in mathematics learning (e.g., Alegre et al., 2022; Brady et al., 2021; Galanti, 2022; Kocabas et al., 2021) and its examination in a culturally relevant context (Alegre et al., 2022), this new working group will advance conversations about synergies between CT in mathematics and DS education with aims to launch new collaborations.

### Session Plans and Focus

#### Session 1 – Synergies Between CT, MT and DS

As we honor diverse experiences of the working group’s participants across CT, mathematics, and DS education contexts, we will start with a short CT activity to launch a conversation about conceptual overlaps between CT, MT, and DS in our problem-solving

approaches. Building upon our shared language and perspectives, we will collaboratively critique an existing framework for CT-MT Integration (Sneider et al., 2014). Then, small groups will create their own representations that show synergies between MT, CT, and DS using Google Jamboard. Small groups will share their representations, and we will discuss each representation. The following questions will prompt the analysis of these representations: “What are common and different aspects across all representations?” “Why do we need to explore the synergies between CT, MT, and DS?” and “What are the implications of these synergies for student learning?”

### **Session 2 – CT Integration in K–12 Settings, Equity, and Access**

Envisioning the use of CT in mathematics and DS education presents a variety of challenges. Many teachers do not have the knowledge, skills, and materials needed to successfully implement CT in their practice (Yadav et al., 2016), and the lack of access to resources to encourage CT integration is particularly pervasive for historically underserved populations (Gilbert et al., 2008). There is a pressing need for research-based pedagogical strategies to foster rigorous and inclusive practices in the CT integrated classrooms (Ni et al., 2021). This session will start with two presentations on equity and access in CS integration in K–12 STEM classes and efforts to support in-service and pre-service teachers’ professional learning for CT integration in mathematics and DS. Next, we will form sub-thematic small groups: Synergies among CT, MT, and DS; Integration of CT in K-12 settings; CT Professional Learning; and Access and Equity in CT Integration. The small group discussions may include how to equip mathematics classrooms to enable CT integration, how to support in-service and pre-service teachers’ understanding of CT integration into their instructional practice, and what possible challenges teachers and schools could experience in this integration. These small group discussions will strategize research agendas to address some of the challenges and opportunities in integrating CT in K–12 settings.

We will also discuss as a whole group the importance of culturally responsive pedagogy (CRP) and how we can adapt our existing knowledge of CRP practices in mathematics education to the integration of CT in K–12 settings and professional learning settings. We will share resources with participants on a few projects on culturally responsive computing that have targeted particular underrepresented communities, such as African American (Gilbert et al., 2008) and Latino girls (Scott & White, 2013).

### **Session 3 – Promoting Collaboration Among Disciplines and Outcomes**

The final session will emphasize that the discussions and engagement in the activities together during the prior sessions are the first step for initiating productive and open conversations and collaboration among people from interdisciplinary backgrounds such as mathematics, CS, and DS Education. CT can empower teachers to leverage abstraction, automation, modeling, and simulations as their students investigate relationships in mathematics and data science. The working group will reflect on the ways in which mathematics, DS, and CS educators can collaborate to realize this goal. Ideas for continuing collaboration will be grounded in the research questions, challenges, and big ideas discussed during the first two sessions of the working group. Time will also be given to thematic sub-groups to continue to work on and enrich their plans for collaboration. In addition, we will encourage collaboration beyond the conference and share means to stay connected after the conference.

### **Acknowledgement**



This work was partially supported by Gordon A. Cain Center for STEM Literacy, NSF awards 2031490 and CNS- 1923573, and the U.S. DOE awards U411C190287 and S423A20001.

### References

- Alegre, F., Yilmaz, Z., Moreno, J., & Kendrick, R. (2022, November). Culturally relevant computing tasks: Evidence of synergies between students' mathematical and computational thinking (pp. 1891 – 1895). In A. E. Lischka, E. B. Dyer, R. S. Jones, J. Lovett, J., J. Strayer, J., & S. Drown, S. (2022). *Proceedings of the 44th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1977 – 1981). Nashville, TN: Middle Tennessee State University.
- Brady, C. Vogelstein, L., Gresalfi, M. & Knowe, M. (2021). Circular reasoning: Shifting epistemological frames across mathematics and coding activities. In Olanoff, D., Johnson, K., & Spitzer, S. *Proceedings of the 43rd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1182 - 1190). Philadelphia, PA.
- Bråting, K., & Kilhamn, C. (2021). Exploring the intersection of algebraic and computational thinking. *Mathematical Thinking and Learning*, 23(2), 170–185.
- Çetinkaya-Rundel, M., & Rundel, C. (2018). Infrastructure and tools for teaching computing throughout the statistical curriculum. *The American Statistician*, 72(1), 58–65.
- Gadanidis, G. (2017). Artificial intelligence, computational thinking, and mathematics education. *The International Journal of Information and Learning Technology*, 34(2), 133–139.
- Galanti, T. M. (2022). Thinking outside the box: Preparing elementary teachers to integrate computational thinking and mathematics. In A. E. Lischka, E. B. Dyer, R. S. Jones, J. Lovett, J., J. Strayer, J., & S. Drown, S. (2022). *Proceedings of the 44th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1977 – 1981). Nashville, TN: Middle Tennessee State University.
- Hickmott, D., Prieto-Rodriguez, E., & Holmes, K. (2018). A scoping review of studies on computational thinking in K–12 mathematics classrooms. *Digital Experiences in Mathematics Education*, 4(1), 48–69.
- Horton, N. J., & Hardin, J. S. (2021). Integrating computing in the statistics and data science curriculum: Creative structures, novel skills and habits, and ways to teach computational thinking. *Journal of Statistics and Data Science Education*, 29(sup1), S1-S3.
- Kocabas, S., Chen, L., Bofferding, L., Aqazade, M., & Haiduc, A. (2021). Identifying and fixing double counting errors in mathematics and programming. In Olanoff, D., Johnson, K., & Spitzer, S. *Proceedings of the 43rd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 302-307). Philadelphia, PA.
- Ni, L., Bausch, G., & Benjamin, R. (2021). Computer science teacher professional development and professional learning communities: a review of the research literature. *Computer Science Education*, 1-32.
- Prensky, M. (2008). Turning on the lights. *Educational leadership*, 65(6), 40–45.
- Scott, K. A., & White, M. A. (2013). COMPUGIRLS' standpoint: Culturally responsive computing and its effect on girls of color. *Urban Education*, 48(5), 657–681.
- Sneider, C., Stephenson, C., Schafer, B., & Flick, L. (2014). Exploring the science framework and NGSS: Computational thinking in the science classroom. *The Science Teacher*, 38(3), 10–15.
- Wing, J. M. (2006). Computational thinking. *Communications of the ACM*, 49(3), 33–35.
- Wing, J. M. (2011). Computational thinking: What and why? *The link: The Magazine of the Carnegie Mellon University School of Computer Science*.
- Yadav, A., Gretter, S., Hambrusch, S., & Sands, P. (2016). Expanding computer science education in schools: Understanding teacher experiences and challenges. *Computer Science Education*, 26(4), 235–254.

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

## WORKING GROUP REPORT: CONCEPTIONS AND CONSEQUENCES OF WHAT WE CALL ARGUMENTATION, JUSTIFICATION, AND PROOF

AnnaMarie Conner  
University of Georgia  
aconner@uga.edu

Megan Staples  
University of Connecticut  
megan.staples@uconn.edu

Michelle Cirillo  
University of Delaware  
mcirillo@udel.edu

Karl W. Kosko  
Kent State University  
kkosko1@kent.edu

Kristen Bieda  
Michigan State University  
kbieda@msu.edu

*The Conceptions and Consequences of What We Call Argumentation, Justification, and Proof* working group met in 2022 to focus on frameworks for studying argumentation, justification, and proof. Working group participants engaged in conversations interrogating past uses of frameworks and considered potential directions for making progress on both new and long-standing questions involving argumentation, justification, and proof. Discussions revolved around the following topics: the teacher's role in joint sensemaking, student agency, defining validity in proof and proving, the role of emerging technologies, and research and practice related to different audiences and argumentation, justification, and proof.

Keywords: Reasoning and Proof; Research Methods; Advanced Mathematical Thinking.

### History of Working Group

The *Conceptions and Consequences of What We Call Argumentation, Justification, and Proof* Working Group (AJP-WG) has met nearly every year since the 37<sup>th</sup> Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (PME-NA). AJP-WG sessions have been well-attended, drawing groups of scholars with interests in various aspects of AJP each year. The group has resulted in an edited book (Bieda et al., 2022), several conference papers, and two white papers (Cirillo et al., 2016; Staples et al., 2017). The group has focused on a variety of aspects of AJP over the years. In 2015, the group focused on definitions and relationships among argumentation, justification, and proof (Cirillo et al., 2015). In 2016, the group focused on the lenses implied by definitions and the implications for analysis of shared artifacts (Staples et al., 2016). In 2017, participants focused on examining data from differing perspectives on argumentation, justification, and proof (Conner et al., 2017). In 2018, the group focused on how mathematics teacher educators prepare K-16 teachers for enacting argumentation, proof, and justification in their own classrooms and the consequences of teachers' conceptualizations of these constructs (Conner et al., 2018). In 2020, the group led a research colloquium (Cirillo & Bieda, 2020) in which selected aspects of the edited book were discussed. The focus of the 2021 AJP-WG was looking at argumentation, justification, and proof through an equity and inclusion lens (Cirillo et al., 2021). All of these previous iterations of the AJP-WG contributed to the ideas discussed in the 2022 AJP-WG.

### Goals and Focus

The 2022 AJP-WG examined the frameworks currently used and potentially needed for research related to argumentation, justification, and proof. The focus originated from Kosko and Bieda's observations in the concluding chapter of *Conceptions and Consequences of Mathematical Argumentation, Justification, and Proof* (Bieda et al., 2022), in which they concluded that both definitions and analytic frameworks are crucial for understanding how data

is interpreted by researchers. As reported in our Working Group proposal, our 2022 sessions were guided by the following questions:

(1) What frameworks for AJP have been used, how have they been used, and what have those frameworks allowed us to accomplish in past research? (2) How does analyzing an artifact (e.g., classroom episode or interview response) from the lens of argumentation, justification, or proof through a particular framework influence claims we can make? and (3) What frameworks for analyzing AJP are still needed to help move the field forward with respect to teaching and learning AJP, classroom research, fostering reasoning and sensemaking, promoting equitable participation, classroom practice, and other aspects of researching AJP? (Conner et al., 2022, p. 2107)

### **Theoretical Background**

Argumentation, justification, and proof have been studied from a wide range of theoretical and analytic perspectives; these perspectives include frameworks that include social or cognitive factors, discursive aspects, and content or task analyses. As previous iterations of this working group have confirmed, even the definitions of argumentation, justification, and proof used by researchers in this area vary widely (see Cirillo et al., 2016). Despite the range of frameworks already in use, emerging questions in the field suggest that new or revised frameworks that center aspects of equity and student autonomy may need to be developed.

### **Working Group 2022 Sessions**

The AJP Working Group convened at the 2022 conference in Nashville, TN, where we supported both online and in-person participation. We had three days of activities and discussion. We summarize here the focus and key points from each day. Overall, we had a total of approximately 25 people participating, spanning from graduate students to seasoned faculty.

#### **Day 1**

In our first session, we focused on examining the frameworks that have been used to study argumentation, justification, and proof (AJP). We began by brainstorming the common topics studied involving AJP. Next, we turned to the common frameworks used for studying AJP. We offered, as examples, cognitive, sociocultural, and discursive frameworks, and we elicited additional frameworks from participants.

Some of the research topic ideas brainstormed by AJP-WG participants related to AJP included:

- Tools to equip teachers to facilitate argumentation,
- Teaching proof vs teaching through proof,
- How to create a classroom culture where students feel safe to take risks in AJP, and
- Building from informal reasoning and play toward proving.

We began the discussion of frameworks by describing a few familiar frameworks in AJP research. We offered Toulmin's scheme (Toulmin, 2003), and systemic functional linguistics as examples of discursive approaches to argumentation. For sociocultural perspectives, we offered practical rationality (Herbst & Chazen, 2003) and the construct of sociomathematical norms (Yackel & Cobb, 1996). And as an example of cognitive approaches, we discussed Harel and Sowder's (2007) proof schemes. Participants brainstormed additional frameworks, which included positioning theory and embodied cognition.

As a next step, participants associated the research topics related to AJP with different frameworks. As one example, a participant who focused on “How to leverage AJP to support student agency” identified sociocultural perspectives as useful, specifically looking at sociomathematical norms. We wrapped up Day 1 with a gallery walk that provided opportunities for participants to familiarize themselves with the range of questions and frameworks.

After the first session, the AJP-WG team reviewed the brainstormed research topics and created themed groups. This resulted in the following five groups, which were presented as options for discussion groups on Day 2:

- Teachers’ Role in Joint Sensemaking
- Student Agency in AJP
- Who defines validity in proof/proving?
- Role of Emerging Technologies in AJP
- Research & Practice and AJP

## Day 2

As we began Day 2, we hoped to build on the initial Day 1 conversation and offer some level of synthesis around the frameworks and research topics that have guided the past work on AJP. We then hoped to turn forward, identifying areas of need and approaches that were promising. We shared with the participants the following two goals for Day 2:

10. Get clearer on the frames used, what we’ve gained, where we need to go next, and how to get there.
11. Begin to develop our networks and find common interests.

Each group focused on their theme/area, and considered the following questions:

- To what extent have others studied this topic and what has been the focus?
- What analytic frames have been used to study this topic?
- What **affordances** do these frames offer?
- What **other frameworks** could be useful to advance our knowledge?

We had productive discussions, many of which are summarized in the Day 3 section. We concluded with a jigsaw, in which participants heard from group members about the other groups’ discussions.

## Day 3

A main goal of Day 3 was to synthesize our two days of discussions and consider individual and collective next steps. Each group continued their discussions for a little while and then used those discussions to construct a summary of their work—to different degrees and levels of detail. We share the summaries constructed by each group in the following paragraphs. Names are included to credit those who constructed the summaries on Day 3.

**Small group: The teacher’s role in joint argumentation.** (Keith Leatham, Calli Shekell, AnnaMarie Conner). Our discussion was about what we do and do not know about teachers facilitating joint argumentation. We mentioned some frameworks (e.g., Conner et al., 2014; Leatham et al., 2015; Mikeska, Howell & Straub, 2019), but our conversation was about how the teacher actually scaffolds the argumentation. With respect to scaffolding, we feel like one way to think about that teacher work is that somehow the teacher has to construct a model of the

argument, both the current argument and the potential argument that exists based on the ideas the students are putting forth. How does a teacher engage in that process “Given what’s been said, what are the ways students could proceed in the argument?” in order to make decisions about how to move forward?

**Small group: Student agency and autonomy in AJP.** (Kristin Doherty, Salvador Huitzilopochtli, Karl Kosko). Our group focused on the following areas: How do we create opportunities for students to pursue and justify their own representations and paths to justify? How can teachers create opportunities for students to engage in AJP via their own representations and reasoning practices? How can teachers do this in ways that incorporate students’ multiple identities and cultural norms?

**Small group: Who defines validity in proof/proving?** (Jordan Kirby, Matthew Hardee, Annie Selden). This group discussed a variety of frameworks and lenses for studying proof. These lenses included Stylianides’s (2007) definition that encompasses elementary children, Selden and Selden’s (2003) study of proof validations, ideas of authority (Otten et al., 2017; Bleiler-Baxter et al., 2023), Harel and Sowder’s (1998; 2007) proof schemes, and de Villiers’ (1990) roles of proof. Other potential frameworks that could be useful to advance our knowledge included some sort of reflective framework that addresses (with students) “Is this valid? Why or why not? What makes it valid? Have your opinions changed over the course of a semester?” and investigating how textbooks or fields determine validity of a text (similar to Otten et al., 2014). Another question discussed was instigated by Jones (2000): Does what constitutes a proof change over time?

**Small group: Role of emerging technologies in AJP.** (Hyejin Park, Maria Alejandra Calderón González, and Megan Staples). We had three main areas where we saw emerging technologies as relevant:

- Emerging technologies for research: These technologies allow us to investigate questions in new ways, or investigate questions that we couldn’t investigate before, for example, eye tracking technologies. Similarly, the existence of technologies can prompt new questions (e.g., how do students make sense of examples in a dynamic environment?)
- Emerging technologies in professional development and teacher learning: These technologies offer new learning opportunities for teachers, and new opportunities to analyze teaching (overlap with above) and provide feedback to teachers. A goal being pursued is using machine learning techniques to offer feedback to teachers.
- Emerging technologies that are part of classroom instruction; this was the bulk of our conversation.

We are suspicious that current frameworks may not help us see the important aspects of teaching and learning and argumentation (justification/proof) when analyzing technology-embedded learning experiences. Hyejin’s work and Maria’s work provide some evidence for this suspicion! Different types of reasoning emerge because there are different tools to support reasoning, different mathematical ideas come to the fore and others are backgrounded. Generalization might be a particularly challenging area. We wonder how students are making sense of the representations and mathematical objects that are offered to them in the dynamic environments (e.g., a dynamic, generic example—or—1000 different cases). Maria notes—in one of her analyses—see “two objects.” The move creates a new triangle. Students are using these tools—and the knowledge about geometry (or other subject) is being developed, but do we

understand how this is really being developed and the nature of their knowledge. How we come to know is tightly connected to what we know and how we produce new knowledge. What tools do we learn to use to develop proof? How does the fact that this is different from pen-and-pencil impact what students know and how they come to know in the future?

Concerning equity and studying equitable instruction or outcomes, what are the implications of the use of technology within classrooms for teaching and learning for position, power, mathematical authority?

- Does technology change the power dynamic in the math classroom—implications for AJP?
- Does technology impact the role of the teacher—teacher as determiner of validity of claims?

How might the mathematics community navigate new spaces of technology supporting reasoning, thinking, conjecturing, proving?

What is the impact of this technology on math education's thinking about AJP, and perhaps particularly proof? Does technology—and its ubiquitous nature, its changing role in professional circles (e.g., 4-color theorem) —impact the goals we have for justification, argumentation and proof? What if AI learns how to prove? How might that impact our thinking or the goals of mathematics education?

As a final comment on analytic frameworks, there was some sense that we would need to expand our analytic frameworks. We felt we would likely need to look more at classrooms as systems, and consider instructional systems, and less focus on individual components. We thought activity theory might be helpful, as that focuses on tools and accounts for the system.

**Research and Practice and AJP.** (Orly Buchbinder, Sam Otten, Merav Weingarden, Rachel Rupnow). We discussed various ways in which research and practice on AJP can inform each other. We raised a few existing lines of research that is directly related to practice (like how proof is enacted in classrooms, how to prepare teachers to enact proof in classrooms, research on learning and teaching proof at the undergraduate level, etc.) We also mentioned several frameworks used by these research projects. Then we switched to talking about what else is “out there” and identified classroom enactment of AJP as a gap in the research and the need to frame the research agenda on working collaboratively with teachers on this. That is, rather than researchers coming to classrooms with their own agenda, working with teachers collaboratively. Another topic we identified as an important direction for the research / practice agenda is connecting proof in mathematics with proof in science and social sciences. With respect to that, we discussed ideas related to “border crossing” between the world of mathematics and the world of other disciplines.

### Conclusions

The 2022 AJP-WG discussions revealed ample room for new questions and new frameworks in future research involving argumentation, justification, and proof. Many discussions revolved around considering the actions and autonomy of students and teachers as they engage in these important mathematical activities. The continued importance of considering students' cultures and identities was discussed in several of the groups. Across groups, the importance of connecting to what teachers and students are doing in classrooms remained paramount.

## Acknowledgements

The authors would like to thank all of the participants in the 2022 AJP-WG for their thoughts and discussions, which pushed all of us to think more deeply about research in argumentation, justification, and proof.

## References

- Bieda, K. N., Conner, A., Kosko, K. W., & Staples, M. (Eds.). (2022). *Conceptions and consequences of mathematical argumentation, justification, and proof*. Springer.
- Bleiler-Baxter, S. K., Kirby, J. E., & Reed, S. D. (2023). Understanding authority in small-group co-constructions of mathematical proof. *The Journal of Mathematical Behavior*, 70, 101015.
- Cirillo, M. & Bieda, M. (2020). Conceptions and consequences of what we call argumentation, justification, and proof. Research Colloquium in A. I. Sacristán, J. C. Cortés-Zavala, & P. M. Ruiz-Arias (Eds.), *Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. (pp. 156–158). Mazatlán, Sinaloa, Mexico: Cinvestav/AMIUTEM/PME-NA.
- Cirillo, M., Kosko, K. W., Newton, J., Staples, M., & Weber, K. (2015). Conceptions and consequences of what we call argumentation, justification, and proof. Working Group Paper in Bartell, T.G., Bieda, K.N., Putnam, R.T., Putnam, R.T., Bradfield, K., & Dominguez, H. (Eds.). *Proceedings of the 37th Annual Meeting of the North American Chapter of the International Group for the Psychology Mathematics Education*. (pp. 1343–1351). East Lansing, MI: Michigan State University.
- Cirillo, M., Kosko, K. W., Newton, J., Staples, M., Weber, K., Bieda, K., Conner, A., Mejía-Ramos, P. Otten, S., Creager, M., Hummer, J., Singh, R., & Strachota, S. (2016). Conceptions and consequences of what we call argumentation, justification, and proof. [White paper]  
[https://www.researchgate.net/publication/300088457\\_White\\_Paper\\_2016\\_Conceptions\\_and\\_Consequences\\_of\\_What\\_We\\_Call\\_Argumentation\\_Justification\\_and\\_Proof](https://www.researchgate.net/publication/300088457_White_Paper_2016_Conceptions_and_Consequences_of_What_We_Call_Argumentation_Justification_and_Proof)
- Cirillo, M., Staples, M., Kosko, K.W., Newton, J., Conner, A., & Gomez, C.N. (2021). Conceptions and consequences of what we call argumentation, justification, and proof. In D. Olanoff, K. Johnson, & S. M. Spitzer, *Proceedings of the 43rd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. (pp. 1922–1925). Philadelphia, PA.
- Conner, A., Gomez, N., Park, H., Foster, J. K., Zhuang, Y., Staples, M., Cirillo, M., Bieda, K., & Newton, J. (2018). Conceptions and consequences of what we call argumentation, justification, and proof. In T. E. Hodges, G. J. Roy, & A. M. Tyminski, (Eds.), *Proceedings of the 40th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1402–1411). Greenville, SC: University of South Carolina & Clemson University.
- Conner, A., Kosko, K. W., Cirillo, M., Staples, M., & Bieda, K. (2022). Conceptions and consequences of what we call argumentation, justification, and proof: Interrogating our frameworks. In A. E. Lischka, E. B. Dyer, R. S. Jones, J. N. Lovett, J. Strayer, & S. Drown (Eds.). *Proceedings of the forty-fourth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 2106-2108). Middle Tennessee State University.
- Conner, A., Kosko, K. W., Staples, M., Cirillo, M., Bieda, K., & Newton, J. (2017). Conceptions and consequences of what we call argumentation, justification, and proof. In E. Galindo & J. Newton, (Eds.), *Proceedings of the 39th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1464–1473). Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.
- Conner, A., Singletary, L. M., Smith, R. C., Wagner, P. A., & Francisco, R. T. (2014). Teacher support for collective argumentation: A framework for examining how teachers support students' engagement in mathematical activities. *Educational Studies in Mathematics*, 86(3), 401–429. doi: 10.1007/s10649-014-9532-8
- De Villiers, M. (1990). The role and function of proof on mathematics. *Pythagoras*, 24, 17-24.
- Harel, G., & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. *American Mathematical Society*, 7, 234-283.
- Harel, G., & Sowder, L. (2007). Toward comprehensive perspectives on the learning and teaching of proof. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning*, (pp. 805–842). Information Age Publishing.
- Herbst, P., & Chazan, D. (2003). Exploring the practical rationality of mathematics teaching through conversations about videotaped episodes: The case of engaging students in proving. *For the Learning of Mathematics*, 23(1), 2-14.
- Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2). University of Nevada, Reno.

- Jones, K. (2000). The student experience of mathematical proof at university level. *International Journal of Mathematical Education in Science and Technology*, 31(1), 53-60.
- Kosko, K. K. & Bieda, K. N. (2022). Conclusion: Considering the consequences of our conceptions of argumentation, justification, and proof. In K. N. Bieda, A. Conner, K. K. Kosko, & M. Staples (Eds.), *Conceptions and consequences of mathematical argumentation, justification, and proof* (pp. 313–323). Springer.
- Leatham, K. R., Peterson, B. E., Stockero, S. L., & Van Zoest, L. R. (2015). Conceptualizing mathematically significant pedagogical opportunities to build on student thinking. *Journal for Research in Mathematics Education*, 46(1), 88-124.
- Mikeska, J. N., Howell, H., & Straub, C. (2019). Using performance tasks within simulated environments to assess teachers' ability to engage in coordinated, accumulated, and dynamic (CAD) competencies. *International Journal of Testing*, 19(2), 128-147.
- Otten, S., Bleiler-Baxter, S. K., & Engledowl, C. (2017). Authority and whole-class proving in high school geometry: The case of Ms. Finley. *The Journal of Mathematical Behavior*, 46, 112-127.
- Otten, S., Gilbertson, N. J., Males, L. M., & Clark, D. L. (2014). The mathematical nature of reasoning-and-proving opportunities in geometry textbooks. *Mathematical Thinking and Learning*, 16(1), 51-79.
- Selden, A., & Selden, J. (2003). Validations of proofs considered as texts: Can undergraduates tell whether an argument proves a theorem? *Journal for Research in Mathematics Education*, 34(1), 4-36.
- Staples, M., Newton, J., Kosko, K., Conner, A., Cirillo, M., & Bieda, K. (2016) Conceptions and consequences of what we call argumentation, justification, and proof. In Wood, M. B., Turner, E. E., Civil, M., & Eli, J. A. *Proceedings of the 38th Annual Meeting of the North American Chapter of the International Group for the Psychology Mathematics Education*. (pp. 1704–1712). Tucson, AZ: University of Arizona.
- Staples, M., Newton, J., Kosko, K. W., Conner, A., Cirillo, M., Bieda, K. N., Yopp, D., Zaslavsky, O., Hummer, J., Strachota, S., Singh, R., An, T., Going, T., Zhuang, Y. (2017). Using artifacts to explore conceptions and consequences of argumentation, justification, and proof. [White paper] Retrieved June 22, 2020, from Research Gate at:  
[https://www.researchgate.net/publication/317267228\\_White\\_Paper\\_Using\\_Artifacts\\_to\\_Explore\\_Conceptions\\_and\\_Consequences\\_of\\_Argumentation\\_Justification\\_and\\_Proof](https://www.researchgate.net/publication/317267228_White_Paper_Using_Artifacts_to_Explore_Conceptions_and_Consequences_of_Argumentation_Justification_and_Proof)
- Stylianides, A. L. (2007). Proof and proving in school mathematics. *Journal for Research in Mathematics Education*, 38(3), 289-321.
- Toulmin, S. E. (2003). *The uses of argument* (updated ed.). Cambridge University Press.  
<https://doi.org/10.1017/CBO9780511840005>.
- Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education*, 27(4), 458-477.