Proceedings

of the Forty-Third Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education

PME-NA 43

October 14 - 17, 2021

Philadelphia

Productive Struggle

Persevering Through Challenges

Editors: Dana Olanoff, Kim Johnson, & Sandy Spitzer

Proceedings of the Forty-Third Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education

Productive Struggle: Persevering Through Challenges

Philadelphia, Pennsylvania, USA
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Land Acknowledgment

This border around what is colonially known as Pennsylvania represents a tragic and unjust history. We acknowledge the Lenape, Munsee, Susquehannock, Osage, Erie, Massawomeck and Haudenosaunee Tribes, among others, on whose ancient and sacred land we hold this conference. As a PME-NA community we recognize the ever-present systemic inequities that stem directly from past wrongdoings, and we commit ourselves indefinitely to respecting and reconciling this long history of injustice.

PME-NA History and Goals

PME came into existence at the Third International Congress on Mathematical Education (ICME-3) in Karlsrühe, Germany, in 1976. It is affiliated with the International Commission for Mathematical Instruction. PME-NA is the North American Chapter of PME. The first PME-NA conference was held in Evanston, Illinois in 1979. Since their origins, PME and PME-NA have expanded and continue to expand beyond their psychologically-oriented foundations.
The major goals of the International Group and the North American Chapter are:

1. To promote international contacts and the exchange of scientific information in the psychology of mathematics education;
2. To promote and stimulate interdisciplinary research in the aforesaid area, with the cooperation of psychologists, mathematicians, and mathematics teachers; and
3. To further a deeper and better understanding of the psychological aspects of teaching and learning mathematics and the implications thereof.

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Membership is open to people who are involved in active research consistent with PME-NA's aims or who are professionally interested in the results of such research. Membership is open on an annual basis and depends on payment of dues for the current year. Membership fees for PME-NA (but not PME International) are included in the conference fee each year. If you are unable to attend the conference but want to join or renew your membership, go to the PME-NA website at http://pmena.org. For information about membership in PME, go to http://www.igpme.org and visit the "Membership" page.

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Preface

Dear Colleagues,

On behalf of the 2021 PME-NA Steering Committee, the PME-NA 43 Local Organizing Committee, Towson University, Widener University, and the West Chester University of Pennsylvania, we welcome you to Philadelphia, Pennsylvania, USA, for the Forty-Third Annual Meeting of the International Group for the Psychology of Mathematics Education – North American Chapter, held at the Sheraton Philadelphia Downton and virtually.

This year's conference theme is **Productive Struggle: Persevering through Challenges**. The years of 2020-2021 brought a global pandemic and with it, many challenges for mathematics education research. Public schools faced a sudden and prolonged transition to distance education, while higher education experienced a budget crisis as well as the loss of in-person classes and traditional field experiences for teacher education. Many researchers and their communities have encountered unforeseen difficulties including personal or family illness, employment loss, and dramatically increased caregiving responsibilities, all of which fell disproportionately onto already-vulnerable populations. Meanwhile, demonstrations for racial justice highlighted the insidious effects of racism throughout our society. All these challenges reflect long-term issues, while highlighting and uncovering the effects of centuries of unjust structures and systems.

By choosing the theme of *persevering through challenges*, and in Philadelphia, a city which has historically represented an optimistic spirit and a belief in a better tomorrow, we aim to encapsulate an idea of hope towards the future: that through struggle, and through scholarly work, engagement in our community, and sustained effort towards improvement, we can truly make a difference in the lives of teachers and students, and in mathematics education broadly in continent of North America.

We hope this conference serves to provoke learning through productive struggle and to support our field in persevering through these continuing challenges in mathematics education. In particular, we hope that this conference can serve as a model and precedent for implementing a hybrid research conference. Early in the process of planning PME-NA43, we committed to the idea of a fully hybrid conference with the guiding principle that all opportunities should be equally available to both in-person and virtual participants. Each of the 3 plenary talks, 15 working groups or research colloquia, 160 research sessions (presenting a total of 239 papers), and 121 poster presentations are available for live participation and interaction between in-person and virtual participants.

This year's conference will be attended (either in-person or virtually) by more than 640 researchers, faculty members, and graduate students from around the world including Canada, Mexico, Australia, Israel, Cameroon, and across the USA. Each paper was reviewed by multiple referees in an anonymous review process. The result was an overall acceptance rate of 79% of papers accepted in some form (not necessarily in the form in which they were submitted), with 37% of research report submissions accepted as research reports, 48% of brief research report submissions accepted as brief research reports, 83% of poster submission accepted as posters, and 94% of working group submissions accepted. The papers eventually accepted comprised 81 research reports, 158 brief research reports, 121 Posters, and 15 Working Groups or Research Colloquia.

For this conference we created new strands and reframed others. Most notably, we reconfigured the mathematics content strands from being organized by content area (e.g. Geometry, Algebra, Number Concepts) to being organized by grade band (Elementary/Middle Years, comprising early childhood, elementary, and middle-grades mathematics topics; and Later Years, comprising secondary and post-secondary topics.)

We thank the many people who generously volunteered their time over the past year in preparation for this conference. In particular, we thank the three graduate assistants who contributed to these proceedings: Rachael Talbert (Towson University), Kayla Begen (Towson University), and Sarah Gill (West Chester University). Thanks to Carly Sullivan for her invaluable support in planning the in-person events. We would specifically like to highlight the herculean efforts of Kimberly Corum (Towson University) in developing the online conference hub.

We hope that the papers presented within these Proceedings will give you engaging, inspiring, and challenging ideas to transform your practice. And, as we continue to endure a time of challenge and struggle across North America, we hope that this conference can be a learning opportunity for the field to think about what it means to be an active and engaged professional, and how the structure of conferences can support faculty and students across many stages of their lives and careers in *persevering through challenges*.

Thank you,

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Chapter 1:

Plenaries

THURSDAY PLENARY
FROM PRACTICE TO THEORY: LISTENING TO AND LEARNING WITH BLACK MATHEMATICS TEACHERS

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Research on race and racism with respect to Black teachers' experiences is undertheorized in mathematics education. Due to social, historical, political, and cultural forces, Black mathematics teachers at particular social intersections (e.g., racial, socioeconomic, linguistic) experience teaching mathematics in ways that are unique from those in dominant communities. Without a critical and racialized analysis of issues that could potentially influence the attrition of Black mathematics teachers and how they experience mathematics teaching, conversations about the Black mathematics teacher pipeline, and teacher diversity broadly, run the risk of commodifying teachers and reducing their presence to ahistorical notions of diversity solely for the purposes of race matching. In this presentation, I will use data from an NSF-funded mixed-methods research project*, Examining the Trajectories of Black Mathematics Teachers,* to share what our research team has learned from centering Black mathematics teachers' racialized experiences to theorize about race and racism in mathematics teacher education. Additionally, I will share how this work informs research methodology in mathematics education by integrating untapped, yet appropriate, methodologies suitable for challenging issues of recruitment, retention, and praxis of other underrepresented racial and ethnic groups across time periods and school contexts.

FRIDAY PLENARY
PRODUCTIVE PROVOCATIONS

Crystal Kalinec-Craig, *University of Texas - San Antonio*
Pandemics, Scholarship, and Rethinking What Counts

Sam Prough, *University of Delaware*
Parents are Not a Scapegoat for Math Learning Loss

Caro Williams-Pierce, *University of Maryland*
Failure is Not an F-word: If You're Not Failing, You're Not Learning

Rachel Tremaine, *Colorado State University*
Explicit & Expansive: The Importance of (Re)Defining Student Success in Mathematics

Samuel Otten, *University of Missouri*
Diversifying the "Top Tier" of Mathematics Education Journals

Carlos Nicolas Gomez Marchant & Stacy R. Jones, *University of Texas*
Let Us Be the Healing of the Wound/Seamos la Curación de la Herida

PANDEMICS, SCHOLARSHIP, AND RETHINKING WHAT COUNTS

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By now, many of us have readjusted to a new and sadly, a more painful normal. The pandemic left all of us with various scars and lingering emotions. We long hours working in less-than-ideal circumstances at home (especially those working parents who did remote learning while also supporting their children during Zoom classes). We celebrated the first day of kindergarten, graduation, and ceremonies online. We also lost friends and family to the disease. We had socially distanced funerals where we sobbed through masks, unable to grieve and comfort each other. We watched healthcare workers tirelessly go into extreme conditions and beg the world to act with more caution and care towards each other with a highly transmissible disease, only to see so many not take this advice and eventually end up in their hospitals. Many of us had delayed surgeries or medical care. Thousands of workers in the service industry lost their jobs only to find the *same industry* complain that they cannot find enough workers who will come back for less than a living wage and no health benefits. How is any of this humane? We as teachers and teacher educators are not here unscathed as well.

Nearly all of us had to learn quickly about remote learning, even though this went against the core of our teaching philosophies that learning happens in community and in person. For caregivers, they conducted classes on Zoom while helping their children learn remotely, even when learning online was foreign and impersonal. For those who do not have kids, we also faced a crush of increasing workload behind the scenes and added responsibilities as everyone else did. Universities responded by giving early career, tenure-track scholars a pause on their tenure clock to account for the delay in their research and productivity. #ThanksIGuess? Holding each other with grace and humility was a constant battle, even during stressful times, to remember both for ourselves and each other.

But what happens when the university or college calls the "pandemic over" and things should go back to "normal?" Do these struggles and constraints go away just because we can go back to in-person learning? No. The pandemic just illuminated them and made them bigger, more upfront, and more pressing to address. #TheMythOfLearningLoss is creeping into the common language we hear from districts and administration in K-12. Does it mean that schools and districts will welcome researchers back with open arms to engage in scholarship with teachers, children, and families? No. Our work is not the center of their universe (nor should be). Does it mean that suddenly tenure expectations will become more reasonable given a universities' resources, infrastructure, and mission to support and sustain high-intensive research organization? No. What incentive does the university have in doing this?

My provocation begins with a simple question reminiscent of Ball and Forzani's 2007 lecture "What makes educational research "educational"? What does research look like from here on out? How can we reimagine what "counts" as educational research based on our experiences during the pandemic? What can be said of scholarship in the time of caregiving that values our work and does not dismiss it because it is not "a solo author journal article in a top tier journal?" Is that the ONLY work we want to value?

When *Democracy and Education* journal published my interpretation of how I used the Torres' Rights of the Learner with teacher candidates at UTSA, I had mixed feelings because I was not sure if it would have counted for my tenure dossier as a "quality publication." The journal was not JRME. Or JMTE. *Democracy and Education* did not have impact factors or acceptance rates that I could cite in my dossier. But since 2017, I have heard from so many teachers who have been moved by Torres' ideas and how I framed them as a form of divergent formative assessment, that it has been overwhelming. On Twitter, I can see how teachers and other teacher educators can push through their assumptions and elevate children's ideas, voices, and thoughts, without children having to first defend their legitimacy. If I were at another institution, my tenure committee likely would have dismissed that publication and others, without ever considering the content or impact of the work.

Can the work of our colleagues who organize communities also be a part of the valued scholarship field? Can the work of our colleagues who lead protests and create legislative change be valued and elevated as worthy scholarship? Can the work of colleagues who draft ethnic studies standards (especially in states and districts that fear the phrases "Critical race theory" and "colonialism" in Pk-20 classrooms) be a source of valued scholarship? Cathery Yeh, Melissa Corral, Nicole Joseph, and many others should remind us as to how we can create and enact our scholarship in ways that show demonstrable change, especially work that moves and lives beyond our echo chambers in the academe. How can we as a collective begin to advocate for and with each other to reimagine what educational scholarship looks like?

The pandemic is not over and nor can we completely erase the scars left from its impact. But we do have a choice. We can decide to **pivot** as a community to a new vision for what is valued by returning to the humanity of our field, our communities, and our passions. We can also learn to operate in a "new number system for scholarship": reimagine "what counts" as scholarship without simply relying and reifying on traditional models that advanced the careers of some (primarily white scholars), but not nearly all.

For example, we can emphasize and value more teacher-researcher lines of inquiry. Highlight the work of self-study and how this can push us to a more normalized conversation of "what can I do better in my practice through praxis to examine aspects of it?" Emphasize more work that explicitly integrates teaching, research, and service, especially labor that does more than bolster a first author's CV. Find ways to involved full-time faculty like adjuncts and clinical professors of practice and students at all levels (undergrad, grad, doctoral) in research activities… and pay them to do this work. Name and dismantle systems and structures that marginalize, push, and stereotype the scholarship of BIPOC, LGBTQIA, and caregiver scholars… so much that they leave the profession altogether. It is up to the collective to decide what our new normal is going to look like. We shouldn't leave it up to a rubric, committee, or administration.

I invite each of us to take up this question: What do we want the new normal for what "counts" as mathematics educational research to look like? How can we make it more humane and more inclusive for the next generation of mathematics education researchers?

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EXPLICIT AND EXPANSIVE: THE IMPORTANCE OF (RE)DEFINING STUDENT SUCCESS IN MATHEMATICS

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The work of mathematics education often seeks to increase student success, but rarely explicitly defines this concept. When it is, it commonly corresponds with quantitative measures that enable the positioning of students as economic resources within a school or university's institutional structure, providing an incomplete portrait of student success and reinforcing deficit perspectives on student achievement. Fostering critical analysis of how we conceptualize student success within mathematics requires aligning how we define such success with the perspectives of mathematics students. I advocate for centering student voice in the (re)defining of student success, and issue a call to the mathematics education community to (1) make definitions of student success explicit in mathematics education research and policy, and (2) acknowledge and value the expansive nature of students' definitions of their own mathematical success.

Keywords: Systemic Change, Measurement, Equity, Inclusion, and Diversity, Affect, Emotion, Beliefs, and Attitudes

As mathematics educators and mathematics education researchers, we are given a substantive amount of power in determining what is, and thus what isn't, student success within mathematics. Increasing student success is often put forth as a goal and a justification for our work, but rarely is "student success" explicitly defined. When it is, it often corresponds with quantitative measures such as Grade Point Average (GPA), rates at which students receive D's or F's or withdraw from a course (DFW rates), and persistence rates- measures that enable the positioning of students as economic resources within a school or university's institutional structure (Apple, 2006). While these kinds of measures therefore may be useful in arguing for funding and easing student-to-student comparisons, these notions of "student success" do little to serve the student themselves outside of their connection to and status within the university. Further, these definitions are constructed from a top-down perspective (Martin, 2003); those who have already been traditionally successful within the mathematical academic system have the opportunity to maintain their power by defining what is considered successful within that system. This results in entrenched viewpoints regarding what student success can look like in mathematics and fosters an inequitable system in which the values of the system don't align with the values of all of its participants.

As Weatherton and Schlusser (2020) note, the power present in these definitions may be "unknowingly upheld by researchers, faculty, and other institutional-level stakeholders who consider these dominant ideas of success to be 'common sense' or standard" (p. 10). It takes conscious effort by individuals who are considered traditionally successful within a system to critically examine that system. In order to fundamentally shift how we view student success within mathematics, there exists a need to listen to and place value upon the definitions constructed by key stakeholders in student success: the students themselves. We cannot claim, in any context, to be actively involving students in conversations about their own success when we do not allow them a seat at the table to speak on what they believe success embodies. Evidence points to the idea that students define success in multifaceted and complex ways that go beyond traditional quantitative definitions (see O'Shea & Delahunty, 2018; Quiles-Wasserman, 2019; Weatheron & Schlusser, 2020; Yazedijan et al., 2008), and thus if we continue to define student success in traditional quantitative ways, we neglect elements that are crucial to how students themselves are perceiving their own academic experiences.

Attending to student voice in discussions of student success in mathematical contexts provides a valuable perspective to qualify or counter the dominant ideology of those who are considered successful by traditional definitions. Student voice is already utilized in higher education for means of gathering valuable feedback for program evaluation and reorientation (Campbell et al., 2007; Brooman, Darwent, & Pimor, 2015). However, it has been critiqued for providing a one-dimensional view of students in which those students express views without compelling those within power to take action in response to those views (Seale, 2009). To combat this viewpoint of a one-dimensional student, not only does student voice need to be attended to when considering the concept of student success, student voice must also be given weight when determining how student success is conceptualized within the field.

Particular weight must be attributed to the voices of students for whom mathematics education has not been historically oriented to serve. Traditional quantitative definitions of mathematical success do not serve all students equitably, and often serve to reinforce deficit perspectives on who is successful in mathematical spaces (Baldridge, 2014; Jaremus, 2020). Gutiérrez (2017) potently notes that "we cannot claim as our goal to decolonize mathematics for students who are Black, Latinx, and Aboriginal while also seeking to measure their 'achievement' with the very tools that colonized them in the first place" (p. 12). These 'tools' are widespread at all levels of our educational system, and are particularly manifested in national standardized testing as a gatekeeper for funding and student opportunity (Baldridge, 2014; Gasoi, 2009). Such standards were not designed for the achievement of marginalized students, and thus do not necessarily highlight the ways in which they *are* achieving, instead focusing on and easing the process of deficit-oriented gap-gazing (Gutiérrez, 2008).

One way in which such gap-gazing is present in contemporary societal discourse is as "learning loss," a concept highlighting the quantitative effects of the ongoing COVID-19 pandemic in the context of standardized testing and learning. In this context, when we limit student success to quantitative measures, we ignore the myriad of ways in which our students may have experienced success over the past year and a half of mathematical instruction. For many, that included "persevering through challenges"- the theme of this conference! If we consider "persevering through challenges" an important component of student success, then it is crucial that our definitions of student success acknowledge that facet. However, calls to regain "learning loss" experienced during this time have been oriented towards ensuring that quantitative test scores are matched with those of what prior years would call "successful." This does not recognize the inherent mathematical success of persevering - intellectually and emotionally - through mathematics learning during a pandemic, regardless of quantitative outcome.

To take an anti-deficit approach to student success during this time is to shift the focus from quantitative deficits to the ways in which students *have* experienced success in mathematics. My own research and conversations with undergraduate First-Generation, Pell-grant eligible, and/or racially minoritized women students have revealed that students are thinking about success in intricate ways that both build on and extend beyond traditional quantitative definitions. When asked how they would define success within mathematics, these women provided nuanced perspectives, and I highlight several quotes from four of their interviews, attributed to their pseudonyms, in *Table 1*.

Ada	Taylor	Kenzie	Isabel
"I think [success in mathematics] is about learning something and getting something out of it…even if it's not necessarily the content, I need to get something out of the experience, whether that was learning critical thinking or just something positive."	"I think grades are important, of course, but I think more important than that is actually understanding what you're learning… I think success in a college setting is being able to understand what you're learning, especially with math, because it can be really hard."	"I think [success means] being a level ahead, feeling proud after all that work, you know? Every summer I took a math course, so finally being where I wanted to be… that's what I was reaching for."	"For math in general, I would define success as you actually being able to practically apply your math skills… there's being able to do well in school and then there's being able to actually use what you've learned in school."

Table 1: Quotes from four undergraduate women students speaking on their views of what it means to be successful in mathematics.

The direct quotes from these women students are but a fraction of richer conversations in which they expanded on their definitions of success in mathematics, as well as moments in which they themselves had felt successful within mathematics. By presenting their quotes here, I intend to engage the reader reflectively: in what ways are these quotes aligned and/or unaligned with the ways in which student success is traditionally defined in mathematics education? Student perceptions of their own success in mathematics exist beyond the constraints of quantitative systems that were designed to measure their success. Without acknowledging the expansive nature by which individual students define their own success in mathematics, we risk overlooking dimensions of mathematical success that are immensely impactful and influential to how students discuss and experience their own mathematical journey.

Scholars in other disciplines have put forth research regarding how definitions of student success might be expanded on (i.e. Atwood & Childress, 2018, in School Social and Emotional Learning; Beilin, 2016, in Library Science; Ulrich & Strong, 2019, in Engineering; Weatherton & Schlusser, 2020, in Biology Education). Because mathematics often is often perceived as an indicator of student intelligence (Adiredja, 2019; Gutiérrez, 2018; Roth et al., 2015) and holds a privileged place within capitalistic systems (Andrade-Melina, 2017; Valero, 2018; Woodrow, 2003), ensuring that student success within mathematics is critically examined can have broader implications for ways in which student success is conceptualized within STEM. **With this in mind, I challenge the mathematics education community to move toward centering students in the discussion of "student success" through two actionable items: (1) make definitions of student success explicit in mathematics education research and policy and (2) acknowledge and value the expansive nature of students' definitions of their own mathematical success.**

Make Definitions of Student Success Explicit in Mathematics Education Research and Policy

Using the term "student success" without definition creates an assumption of the term as having universal meaning, which establishes success as a "privileged ideal, partially reliant on the possession of certain cultural or academic capitals" (O'Shea & Delahunty, 2018, p. 1069). Assuming universality of perceptions of success restricts who has an entry point into the conversation about student success. In addition, making explicit definitions of student success "before beginning their projects will allow researchers to clearly ground their work and accurately describe what they intend to study" (Weatherton & Schlusser, 2020, p. 6), fostering clarity for all involved in the project- researchers, participants, and eventual readers. Regardless of how student success is defined, making its definition explicit counters the idea of student success within mathematics as a privileged ideal known only to a select few, and allows for more diverse entry points into the conversation surrounding work that focuses on increasing a specifically-defined component of student success.

Acknowledge and Value the Expansive Nature of Students' Definitions of Their Own Mathematical Success

Students, as key stakeholders in their own success, are and ought to be treated as the authority concerning how they experience success in mathematics. Reconceptualizing what we consider student success in mathematics necessitates seeking out and intentionally placing weight on the perspectives of students with identities that are traditionally marginalized in mathematical spaces. Doing so can counter the ways in which traditional quantitative definitions of student success have reinforced deficit perspectives and systemically minimized the achievements of individuals who hold these identities. Acknowledging the varied way in which student success can be defined also enables better alignment among student-level, faculty-level, and university-level priorities, and reduces the cognitive dissonance felt by students whose definitions of their own success contradict that which they see messaged by their institutions and instructors (Ulrich & Strong, 2019).

In Conclusion

As a concept that underlies so much of what the mathematics education community works toward as a field, the notion of student success deserves our attention and intentionality in assuring that we are framing it in a way that is reflective of how our students see their own success within mathematics. Moving beyond deficit-oriented quantitative measures of student success necessitates exploring and valuing student voice regarding what it means to be successful in mathematics, and that we apply those definitions in our work to critically shift how student success is conceptualized and measured in this field. I implore the mathematics education community to both make definitions of student success explicit in our work and acknowledge that traditional quantitative definitions of success are only a fragment of the expansive ways in which students frame their own success. Students are a key stakeholder in their own mathematical success; their perspectives deserve to be heard, and we are privileged with the opportunity to listen and foster change.

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FAILURE IS NOT AN F-WORD: IF YOU'RE NOT FAILING, YOU'RE NOT LEARNING

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I often use the word 'failure' in my work, and the audience I'm talking to dictates their response to my use of that word. I remember back in grad school, saying the word 'failure' in a mathematics education class, and everyone gasped in horror at the f-word! But just down the hall, where I studied games and learning with other folks, we talked about our experiences of failure in games all the time – no gasping required. Because good games always have useful feedback paired directly with any moment of dramatic failure – that is not always (maybe even *not often*) true in classrooms (or indeed, academia in general).

My provocation is: we need to normalize joyful failure at every step in the mathematics learning process. And to do that, we need to make sure that our feedback is consistent and useful – and provokes reflection on the mathematics and our learning, instead of frustration or rote memorization. There are many ways to design such failure and feedback experiences, and I present one such approach here.

Provocative Objects

In examining failure and learning, I've pretty much become obsessed with mathematical play in all sorts of different contexts. In particular, I think of mathematical play as being particularly likely to emerge when failure paired with feedback are regularly present for the player/learner. I developed the idea of a *provocative object* as a particular tool to designing to support mathematical play and learning (e.g., Williams-Pierce, 2019; Williams-Pierce & Thevenow-Harrison, 2021). Initially, provocative objects emerged as a way of thinking about and designing for mathematical play in videogames; however, we recently extended this construct to *provocative environments* in order to better examine and understand mathematical play in informal makerspaces (Shokeen et al., 2020). Consequently, while I focus primarily on provocative *objects* below, I also describe some of the more recent insights we have developed.

Provocative objects are digital environments that provoke mathematical play and learning by using five key characteristics that can be used for both design and analysis purposes.

#1: Consistent and Useful Feedback

First, provocative objects have consistent and useful feedback. Often, we engage with mathematics in environments that do not provide immediate feedback. For example, if I hand you an expression on a piece of paper (Figure 1), you can simplify it any number of ways, both correct and incorrect.

$$44+56+a+72+28$$

Figure 1: an expression written on paper.

If you decide that you can add 56 and *a* together and you write down 56*a*, the paper will not revolt. You can write it down, hand it in, and wander off to recess – and if you're lucky, you'll get feedback on your failure the following day or week. But with a digital context, like Ottmar and team's *From Here to There* (FH2T; Figure 2), you get feedback about that impossibility immediately.

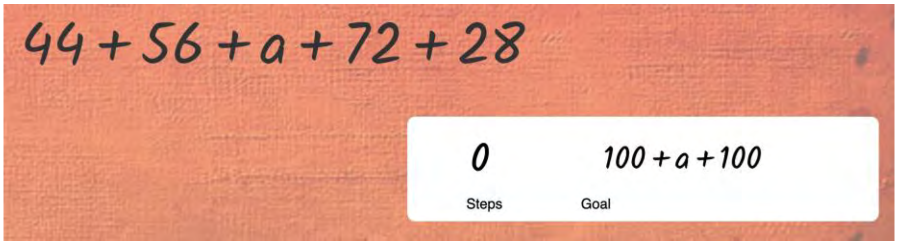

Figure 2: World 1, Puzzle 6. In FH2T, the goal is to modify the top expression to duplicate the target expression in white.

If you tap the + symbol between the 44 and 56, the game will enact the addition, giving you a total of 100 – and feedback that you are engaging in a mathematically possible action. But if you tap the + symbol between the 56 and *a*, the + will shake back and forth – feedback that indicates a mathematically impossible action. In other words, unlike paper, FH2T gives immediate and useful feedback to learners.

#2: Failure Paired with Feedback

The full name of this second characteristic is *high enough levels of difficulty and ambiguity that players experience frequent failure that is closely paired with the feedback.* Failure is so important that when I was designing my dissertation game, *Rolly's Adventure* (RA), one of my first design decisions was how to indicate failure, and how to closely pair that failure directly to feedback without just *telling* the player what they did wrong. That is, how do you tell someone playfully that they are failing, and pair it with feedback that – instead of telling them the answer - helps them reflect upon what they did, so they can learn from it?

This often contrasts with traditional learning contexts – where failure *is* an F-word - and I rely upon game scholars who study failure to understand how wonderful failure can be (e.g., Juul, 2009; Ramirez, 2017). In particular, because the feedback is consistent and useful, every instance of failure comes immediately in the moment: players can see the results of their actions, reflect upon the relationship between their action and the failure/feedback their action evoked, and learn (Williams-Pierce, Dogan, & Ellis, 2021, in revision). For example, in RA, one player was surprised and excited when he experienced failure for the first time (Figure 3).

Figure 3: Emmett (left) experiences fiery death as an indicator of failure in RA (right).

Immediately after this moment of fiery failure, a huge slow grin spread across Emmett's face: "Wh- wh- what?! What the— heck? I don't even know—what just happened?"

So, the first point is that he did not mind failing – shocked surprise was followed immediately by curiosity and interest. And that curiosity and interest led Emmett to discover the feedback that he had not noticed during this first failure: he had not achieved the puzzle goal of perfectly filling the hole (Figure 4).

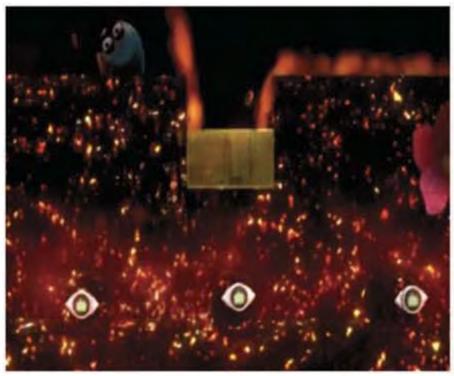

Figure 4: Visual feedback in RA, paired with the fiery failure. (A different puzzle than the previous Figure, as this puzzle is lighter and easier to see in screenshots.)

Note that the fiery death of his adorable avatar (see the rainbow faced minion in Figure 3, right) was the failure indicator, and the feedback indicator (the golden block in Figure 4 that only partially fills the whole) is *not* on fire. The eye of the player is drawn to that golden block, as one of the very few objects not on fire, which helps emphasize the feedback indicator in an otherwise novel (and thus bewildering) context.

However, failure and feedback can be more complex, especially in provocative environments. For example, in ongoing research with Amber Simpson in our mmPlay lab, we've been identifying indicators of failure and feedback in collaborative non-formal robotics makerspaces in elementary school. In this context, failure and feedback are often social in nature, as the students negotiate together to achieve their common goal (Shokeen et al., this volume), rather than a reaction from a designed object.

#3: Non-Standard Mathematical Representations and Interactions

Somehow, our field often seems to settle for ossified representations of and interactions with mathematics for our learners. We point at an expression like $56 + a$, and are deeply confused by our learners trying to add two unlike terms. Much like Emmett not yet seeing the feedback paired with the failure indicator, our learners don't look at a and think that it could be 17, or the color blue, or 80 degrees Fahrenheit – in other words, a does not yet represent *unlike-ness* to them. These learners may instead memorize the rule against adding unlike terms, and try to apply it whenever it seems appropriate. As a consequence, when learners encounter standard representations or interactions, they may immediately try to unthinkingly use a plethora of inappropriate memorized rules. This third characteristic is about avoiding evoking such rules (or a player's mathophobia!), in order to better support learners in joyfully engaging in the mathematics that undergirds both standard and non-standard representations and interactions.

However, this leads to another issue: mathematics learning contexts often over-rely upon written and spoken language as feedback and evidence of learning, instead of letting the mathematics speak directly to the player as happens in RA, or letting the learner use complex modalities to share their understanding (such as gesture; e.g., Williams-Pierce et al., 2017; see also Ng, 2016, for an excellent example of learners using gesture to supplement their mathematics communication in their non-primary langauge). As a result, we struggle as a field to *see* mathematics learning in action because we rely upon written or spoken versions of our ossified school-based field. In our mmPlay lab research, we have found that there are multiple layers of mathematical activity occurring that do not directly manifest in written or spoken language, or manifest as a trivial change in such forms. For the former, as mentioned above, there are gestures that act to uniquely complement spoken language. For the latter, for example, when programming the robot Dash to traverse a path, a student may program in a distance in specific units, only to see Dash go too far. The student then must perceptually compare Dash's location with the desired distance, estimate how much *too far* is, and revise their input to test out their new hypothesis. Here, there is failure (Dash did not stop where the student wanted it to), feedback (the student can visually perceive that Dash went too far), and mathematical activity that involves mentally simulating the distance between the lengths (Williams-Pierce et al., this volume). This mathematical activity may manifest simply in a student revising the code in the application that controls Dash, but that small change represents considerable (and easily missed) mathematical activity on the part of the student.

#4: Mathematical Notation Introduced Late or Not at All

This fourth characteristic is one of my favorites, but it also may be very wrong! Essentially, a provocative object does not start with notation, and only introduces it when – or if – it becomes a deeply useful tool to the learner in their gameplay. Earlier, I used FH2T as an example of the first characteristic because I believe it is a provocative object, but FH2T very clearly violates *this* characteristic as formal notation is the primary representation type in FH2T! Consequently, I'm really psyched that one of my doctoral students, Nihal Katirci, is examining FH2T, mathematical play, and learning for her dissertation. So she'll be the one to tell me if this characteristic is less important than I originally believed, and that I need to dial it down to allow for games like FH2T. But I want to emphasize this crucial fact: I am not *anti-notation*, but I am anti-*notation being mistaken as the math itself*. We have the unfortunate habit of taking representations and teaching learners that those representations *are* the mathematics, instead of merely another *representation* of the mathematics. Stay tuned for Nihal's findings!

#5: The Legitimate Possibility of Alternative Conceptual Paths

Last but certainly not least! RA is a linear puzzle game: you complete one puzzle, then you get to go to the next, and so on – you cannot go from puzzle 1 to puzzle 7 and back to puzzle 2. From one perspective, this can be seen as a violation of this fifth characteristic, but as long as your provocative object is ambiguous and challenging (and rife with failure and feedback), a linear product does *not* dictate a linear conceptual path. In the case of RA, players develop their own conceptual understanding of the mathematics underlying the game, using the non-standard representations and interactions in order to craft their own unique mathematical narrative, and emerge from their gameplay with different experiences. I suspect that FH2T is the same – as I said above, stay tuned for more about that!

Conclusion

In summary, I believe that in mathematics education, with ourselves AND with our students or learners, we need to embrace failure as a beautiful thing – an incredibly wonderful opportunity to learn more about the math or the tool or the people. The five characteristics of provocative objects can help us understand how to *design* such lovely failure, as well as how to *understand* the amazing failure that happens within tools or in our lives.

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SATURDAY PLENARY
CRITICAL RACE THEORY AND MATHEMATICS EDUCATION

Panelists
Cathery Yeh*, Chapman University*
Robert Berry*, University of Virginia*
Christopher Jett*, University of West Georgia*
Maria Zavala*, San Francisco State University*
Moderator: Linda Fulmore*, TODOS: Mathematics for ALL*

Critical Race Theory has made headlines with proponents and opponents grappling about its place in education. Many states, school boards, and localities are passing or proposing legislation banning or limiting the teaching of principles attributed to CRT in public schools. Much of what is being passed or proposed under the guise of CRT is not, in reality, CRT. The panel will discuss: What is Critical Race Theory? What is the significance and potential of Critical Race Theory in mathematics education research? How do we navigate through the noise? And, what can we do from an action standpoint?

Chapter 2:

Curriculum & Assessment

IMPACT OF TEACHERS' IDENTIFICATION OF WRITTEN MATHEMATICAL POINTS ON STUDENTS' LEARNING

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I examined the relationship between teachers' identification of mathematical points in written lessons and students' mathematical learning opportunities. Lessons in teachers' guides and classroom instruction were analyzed for written mathematical points and those articulated by teachers during instruction. Teachers who appropriately identified written mathematical points together with suggested curricular resources to realize them had a positive impact on students' mathematical learning opportunities. Positive impact was influenced by the teacher's ability to appropriately identify the role of available curricular resources in supporting the achievements of written mathematical points, recognize relationships between suggested activities and curricular resources toward written mathematical points, and develop a productive mathematical storyline.

Keywords: Elementary School Education, Curriculum, Assessment

Quality teaching begins with teachers clearly identifying what students need to learn from their teacher's guide and then designing activities or tasks to support them to reach the intended learning goal. Many research studies (e.g., Morris et al., 2009) have investigated teachers' abilities to clearly identify learning goals from students' written work. For example, Morris et al. (2009) investigated the ability of preservice teachers to identify subconcepts in mathematical ideas students are to learn and found that they could accurately identify at least one subconcept of a learning goal but not all when correct student work was presented to them. This result suggests that unpacking learning goals from resources available to teachers or additional resources they may want to include in their lessons might be a challenging skill to develop.

Hiebert et al. (2007) emphasized the need for teachers to clearly state what students should learn from a lesson. According to Hiebert et al., learning goals help teachers determine whether or not students have arrived at the intended learning. Hiebert et al. further argued that breaking general learning goals into sub-goals provides better guidance to examine the link between teaching and learning. This implies that teachers should carefully evaluate and interpret written lessons together with suggested resources to identify what mathematics students ought to learn. After identification of the lesson goals, teachers look forward to planning moves that will enable them have student learning reach these goals.

Sleep (2009) unpacked ways teachers steer lessons toward mathematical points. Sleep defined mathematical point (MP) as "the mathematical learning goal for an activity as well as the connection between an activity and its goal" (p. 13). Sleep's definition focused on the learning goals of an activity and describes the work involved in steering the lesson towards these goals. Van Zoest et al. (2016) defined MP with respect to student thinking in-the-moment during whole class discussion. According to them, "an MP is a mathematical statement of what could be gained from considering a particular instance of student thinking" (p. 323). From these, we see that Sleep (2009) focused on the activities students are to engage in while Van Zoest et al. (2016) focused on student thinking in-the-moment. Despite this interest in MPs, however, little is known about teachers' ability to identify MPs embedded in suggested curricular resources (CRs)—"valuable support provided to teachers within each lesson in the teacher's guide" (Atanga, 2014, p. 3). By mathematical point (MP), I mean important mathematical ideas students ought to learn for each lesson as communicated in a teacher's guide, which may be different from, or the same as, lesson goals. My definition of MP is similar to those of Sleep (2009) and Van Zoest et al. (2016) in that the emphasis is on the mathematical idea students are to learn, but different in that while Sleep looks at activities students are to engage with and Van Zoest et al. (2016) looks at student thinking, I focus on CRs embedded in teachers' guides.

Specifically, this study examined teachers' ability to identify stated MPs for written lessons in teachers' guides and the impact on the mathematics students ought to learn. I also investigated whether the MPs teachers seem to pursue during instruction as they make use of suggested CRs are the same or different from those communicated in the teacher's guide. This study focused on the impact of teachers' identification of MPs embedded in suggested CRs on the mathematical content students have the opportunity to learn. As such, I asked the research question, *How does teachers' identification of written MPs impact students' mathematical learning?*

Theoretical Perspectives

Sleep (2012) defined the work of steering instruction as involving three mutually dependent actions: "(a) articulating the mathematical point, (b) orienting the instructional activity, and (c) steering the instruction" (p. 937). She described the first two as "mathematical purposing" (p. 938) to involve stating learning goals for students. Morris et al. (2009) found that some preservice teachers either correctly (exactly) or partially identified learning subgoals when presented with student work. Steering instruction towards identified learning goals involves teacher moves deployed during planning or enactment of lessons to support students in achieving the mathematics of the lesson.

Sleep (2012) identified such moves and elaborated on them. The moves shown in Figure 1 are relevant for this study as they have direct impact on reaching the MPs. These moves provide a framework for the analysis in my study.

Work of Steering Instruction	Teacher Moves
1. Making sure students are doing the mathematical work	*a. Asking questions that engage students in mathematical reasoning* *b. Getting students into the work without doing it for them* *c. Distributing the mathematical talk and the kinds of mathematical talk*
2. Developing and maintaining a mathematical storyline	*a. Developing a coherent within-lesson storyline by making mathematical connections across a lesson's activities* *b. Progressing the mathematical storyline by engaging with new ideas/practices or engaging with ideas/practices in new (more challenging) ways* *c. Developing an across-lesson mathematical storyline by looking for mathematical coherence across students' prior and future work* *d. Conveying the mathematical storyline to students by framing, narrating, and summarizing the mathematical work*
3. Opening up and emphasizing key mathematical ideas	*a. Using intentional redundancy* *b. Pointing out the use of a focal concept or skill* *c. Providing definitions* *d. Spending more time on key ideas* *e. Using a combination of teacher and student talk*

4. Keeping a focus on meaning	*a. Deploying representations in ways that highlight intended meaning* *b. Explicitly connecting the activity to the intended mathematics*

Figure 1: Work of Steering Instruction and Corresponding Teacher Moves (Sleep, 2012)

Method

Curriculum programs and teacher participants. Data were gathered from teachers in grades 3-5 using two different curriculum programs: *Investigations in Number, Data, and Space* (*Investigations*) and *Scott Foresman Addison Wesley-Mathematics* (*SFAW-Mathematics*), both published in 2008. The former is an NSF-funded program, while the latter was commercially developed. Six teachers participated in this study; Lisa, Maria, and Jennifer used *Investigations*, while Caroline, Dan, and John used *SFAW-Mathematics*. These teachers have teaching experience ranging from 8 to 11 years, from Head Start to grade 8. They have also taught a variety of subjects for the different grades and have been exposed to both NSF-funded and commercially developed CMs.

Data sources. Data used in this study included classroom observations and post-observation teacher interviews. Each teacher was observed teaching three consecutive lessons in spring 2012 and all enacted lessons were videotaped and transcribed. Each of these six teachers was interviewed to determine whether or not they identified MPs of the written lessons embedded in recommended CRs in their teacher's guide.

Data analysis. I determined written or implied MPs by identifying important mathematical ideas students ought to learn for each written lesson based on the suggested activities in each teacher's guide. Suggested CRs used in written lessons were also analyzed to identify what MPs, written or implied, they are intended to foster. As classroom videos and transcripts were analyzed, issues about teachers' identification of MPs in suggested CRs emerged. I then observed each classroom video to identify the MPs each teacher articulated to students. I compared the MPs articulated by teachers to those written or implied in the teacher's guide using the codes (1) *exactly* (*when the articulated MP is exactly the same as that written or implied*), (2) *differently* (*when the articulated MP is exactly different as that written or implied*), (3) *partially* (*when part of the articulated MP is partially the same as that written or implied and part is different*). This enabled me to determine whether MPs articulated by teachers were similar or different from those written in the teacher's guide or implied by the researcher. In addition, from the classroom videos, I coded teacher moves using Figure 1. This enabled me to determine which MPs are pursued in the lesson to determine which MPs students are effectively exposed to and the kind of learning each teacher likely promoted, by the opportunities available to students. I compared what students effectively learned to intended learning to determine whether teachers' identification of MPs impacted student learning fully positively (*when students actually learned what was intended for them by the opportunities to learn available and are able to demonstrate that with accurate execution of assigned task*), partially positively (*when students partially learned what was intended for them and the other part not encountered and are only partially able to execute assigned task*), or fully negatively (*when students actually did not encounter or learn what was intended for them and are not able to execute assigned task*). Also, I looked at all lessons by teacher to determine the overall impact on student learning and categorize each teacher based on the MPs they articulated as measured by the mathematical content students had the opportunity to learn. Lastly, I searched for patterns across all teachers in each category to describe possible reasons for such impact on student learning.

Results

Because the written or implied MPs I identified and those articulated by the teacher are highly related to the possible impact on student learning, I present the results together in order to illustrate their connectedness. Lisa's and Maria's identification of written MPs was classified as negatively impacting student learning, while Caroline's, Dan's, Jennifer's, and John's identification of written MPs was classified as positively impacting student learning. The difference between these two impacts on student learning can be attributed to the MPs teachers articulated and the way teachers in the different categories emphasized key mathematical ideas, developed meaning, and developed and maintained a mathematical storyline—"following a deliberate progression and making connections among mathematical ideas toward the mathematical points over a course of lessons" (Atanga, 2014, p. 154). Teachers' identification of written MPs is explained below with examples from Maria's and John's lessons to illustrate possible impact on student learning.

Negative Impact on Student Learning

The MPs in the CRs for the three lessons Maria taught are "(1) using the inverse relationship between multiplication and division to solve problems, (2) identifying characteristics of these problems, and (3) write multiplication and division story problems" (Wittenberg et al., 2008, Grade 3, Unit 5, pp. 122-136). The MPs Maria articulated to the students and pursued for the three lessons were "to identify key words to determine whether a problem is multiplication or division, to solve problems, and then to write story problems." The MPs identified and stated by Maria are similar to those written in the curriculum in that students ought to solve problems and write their own story problems. Maria's articulated MPs are different from those written in the teacher's guide in two ways. First, Maria did not specify the suggested methods in the teacher's guide students ought to learn to solve the problems, while written MPs indicated solution strategies students should learn. Second, Maria introduced the identification of "key words" to provide a clue for students to determine whether assigned problems use multiplication or division. Maria hoped that these "key words" would support students in writing their own story problems.

Investigations provides a set of six problems (three pairs) for students to solve. Each pair of problems uses the same numbers, one of them being a multiplication problem and the other division. The curriculum particularly suggests that teachers highlight problems 2 and 3 for discussion with students to achieve the written MPs mentioned above. After this discussion, it is expected that students notice the other pairs of problems, use the inverse relationship between these operations to solve them, and subsequently write their own division and multiplication story problems.

During enactment, Maria led a classroom discussion of each of the six problems. She began by reading each problem and consistently asked, "What's my key word on this problem?" Maria underlined the key word and asked the students what notation could be used, and she wrote either or beside the problem as appropriately determined. Afterwards, Maria asked students for the number sentence, which she wrote when correctly provided. In addition, Maria always asked, "How do I solve this one?" and together with students, a correct solution was provided.

As Maria led students through the solution of the six problems one after another, she focused students on problems 2 and 3 as recommended and orchestrated the following interaction.

Maria**:** …look at question number 2 and question number 3...Do you notice anything special about question number 2 and question number 3? …What do you notice?

Student**:** The top one's 5 and the bottom one's 20 because 20 divided by 4 is 5 and then 5 times 4 is 20.

Maria**:** Good. Adding and subtracting are exact opposites, right? So are multiplication and division so these have the same set of numbers in them it's just that this one is the inverse or opposite of the one right above it. Kind of cool. So if you solve this one and you solve the same numbers you automatically know the answer without having to even solve them. They're part of the same…?

Student**:** Fact family.

Maria**:** Fact family, absolutely.

In this interaction, Maria ended up using problems 2 and 3 to get students to see those numbers as members of a "fact family," an MP not intended for this lesson, which neither highlighted attributes of multiplication and division problems nor the potential of using the inverse relationship between these operations to solve the problems. Although the idea of inverse relationships surfaced in the above excerpt, Maria did not pursue it beyond making a comparison with addition and subtraction. Maria also did not push her students to see how the inverse relationship between multiplication and division could be used to solve problems 2 and 3. Figure 2 shows a suggested representation in the teacher's guide, which Maria did not use.

Number of Groups	Number in Each group	Product	Equation
?	4 muffins	20	20 4 = or ____ 4 = 20
5 packs	4 yogurt cups	?	5 4 = 20

Figure 2. Visual to Illustrate Inverse Relationship Between Multiplication and Division and their Attributes (Wittenberg et al., 2008, Grade 3, Unit 5, p. 124)

This suggested representation is to support teachers in accomplishing the written MPs, but when asked during the post-observation interview about what this representation communicated, Maria said, "I don't always use that table…so we talk about the differences in those notations, rather than relying so much on this chart. I don't know that I feel like the chart aids a whole lot." According to Maria, the chart is basically focused on differentiating the notations for both operations and hence is not particularly helpful.

After solving all six problems, Maria and her students identified and created a list of key words list for each operation; those for multiplication problems included *in all, altogether, how many, total* and those for division problems included *how many equally, share equally, how many groups, how many in each group, divide, put in each*. Following this summary of key words for each operation, Maria asked students to create their own multiplication and division story problems as required by the CM. After three days of teaching, students in Maria's class neither wrote correct multiplication and division story problems nor used the inverse relationship between multiplication and division to solve problems. Hence, it can be concluded that Maria's articulated MPs had a negative impact on students' opportunities for mathematical learning.

Positive Impact on Student Learning

Written MPs for a lesson John taught are "(1) a plane figure has two dimensions: length and width; (2) a solid figure has three dimensions: length, width, and height; (3) there is a unique relationship between solid figures and flat shapes; (4) definitions of mathematical terms" (Charles et al., 2008, Gr. 4, Vol. 3, p. 434). John articulated what students ought to learn during enactment as "today we're going to relate two different types of figures together. What we call plane figures and what we call solid figures." John's MP is the same as the third MP in the teacher's guide, but he did not explicitly articulate to students the first two and the fourth MPs.

In the lesson John taught, *SFAW-Mathematics* suggests that teachers distribute copies of the net of a cube to students, cut out the net, and construct a cube. The teacher's guide for this lesson suggests teachers introduce and illustrate the terms *face, edge,* and *vertex* to students and ask questions to determine the number of each. During enactment, John led students through the construction of the net of a cube, using graph paper. He asked questions such as, "The lines that you have on your graph paper are all making what type of shape?" and students answered, "Squares." John added and illustrated that, "A square is an example of a plane figure. Meaning it's flat. It's one surface. It has basically what we call two-dimensions. It has length and it has width, now the square."

In addition, John asked, "How many squares make up this shape [the net of a cube]?" This question focused students on the constructed net and students could see that there are six squares, to answer correctly. John continued to direct students on what has to be done to create a cube from the net. Students followed John's guidance and experienced the transformation from a net to a cube. John explained, "Now, you have six squares that made up the cube. So we have turned six plane figures, in other words flat figures, into a solid figure that has now three dimensions. We have length, width and height," pointing at each dimension to concretize it. Two things about John's actions are noteworthy here. First, a relationship between plane and solid figures was established using a cube. Second, John established a one-to-one correspondence between a solid figure and its dimensions. Therefore, John used suggested representations and guidance provided in the teacher's guide to develop and maintain a storyline from the net of a cube to the cube together with its dimensions and established a relationship between plane and solid figures.

John used the constructed cube to define the other terms—a *face*, an *edge*, and a *vertex*—students were to learn. He held the cube and said,

> Squares. So it's 6, the 6 faces of your cube are all squares. So a flat...so in flat surfaced figures, which is what we're going to be talking about today for the most part, flat surface is a face. Your cubes have six faces those six faces are all squares. Yes?

John called the six squares of the cube faces. He emphasized that because this solid figure is formed from a plane figure, the faces must be flat. Furthermore, John defined an edge and a vertex as below:

> Ok, so an edge, look at the next highlighted part, it says: An edge is a line segment where two faces meet. Everyone hold up your cube. Run your finger along an edge. Very good, that is an edge. Notice where two faces come together is an edge. Any place where you folded them and those faces came together you created an edge. The last one is a vertex, a vertex is where three or more edges meet, the plural is vertices. So, point on your cube to a vertex.

John accurately defined a face, an edge, and a vertex, mapping the terms to a surface, line, and point, respectively, to illustrate what they represent. Also, John asked for the number of faces, edges, and vertices of the constructed cube. As students provided the correct number, John counted the distinct faces, edges, and vertices to concretely justify students' responses. Students in John's class proceeded with assigned problems from the text with minor difficulties, suggesting that his actions had provided them the opportunity to learn the mathematics.

When John was asked during the post-observation interview about the mathematical significance of the activity from the net of a cube to constructing a cube, he said,

> Well, what it's showing us is the faces of a cube are made up of six squares, and it, and it teaches us that, it teaches students that the faces of solid figures are plane figures, …knowing what the definitions are, knowing that, what a face is, what a vertex is, what an edge is…

This revealed that John had an understanding of the mathematical ideas of the lesson students ought to learn. Hence, I concluded that John's identification of MPs likely has a positive impact on students' mathematical learning.

Discussion/Implications

The findings indicate that a big responsibility in teaching is to accurately identify MPs for the lesson as well as how suggested CRs support their realizations. This study revealed three interdependent aspects of teaching that teachers need to attend to with care in order to expose their students to intended mathematical concepts they ought to learn.

First, teachers must appropriately identify MPs that suggested problems are intended to communicate. Understanding the rich mathematical concepts embedded in suggested problems together with solution strategies is a key in supporting students in reaching the intended learning target of the lesson. Second, teachers should be able to identify written MPs that suggested representations are designed to foster. Kilpatrick, Swafford, and Findell (2001) found that use of representations have significant positive influence on student understanding of the mathematics they ought to learn. Maria failed to make use of Figure 2 that could have helped her communicate to her students the attributes of multiplication and division problems as well as the use of inverse relationship between the operations to solve problems. In contrast, John used the representations available in the teacher's guide to communicate and establish a relationship between plane and solid figures. Third, teachers should identify relationships among CRs toward written MPs of the lesson. Understanding and identifying these relationships require deep knowledge of the representations (Castro Superfine, Canty, & Marshall, 2009) and knowledge about how to translate between the different representations while preserving the structural information presented in each of them (Novick, 2004). Maria seemed not to understand the information conveyed by problems 2 and 3 and Figure 2, making it hard for her to make meaningful connections between them. This resulted in her omitting the use of Figure 2 and attempting to use the suggested problems in isolation, and opportunities to learn important mathematical ideas and solution strategies were missed. In contrast, John seemed to understand each representation suggested in the teacher's guide and translated between the net of a cube and the constructed cube, calling the faces of the cube squares and preserving their structural information.

These aspects of teaching extend our understanding of "mathematical purposing" (Sleep, 2012, p. 938). Identifying written MPs of the lesson and those embedded in CRs, identifying and establishing relationships among CRs toward written MPs, and mapping out and developing a productive mathematical storyline from one MP to another provide us with additional fine-grain details of the work of mathematical purposing in classrooms. Mathematics educators might include into their methods and content courses for preservice teachers activities such as identifying MPs of written lessons and CRs, and discussing relationships among CRs in achieving the stated learning goals. This might help improve teachers' mathematical knowledge for teaching and can ultimately add value to teacher training programs. In addition, results of this study has potential of being used by educators to develop teachers' specialized content knowledge (SCK). Morris et al. (2009) argued that an aspect of SCK is focused on what type of representations teachers might use to effectively communicate a particular mathematical idea to students. So, focusing on teachers' ability to identify MPs in suggested CRs might support the development of needed SCK and hence teachers' subject matter knowledge, because the former is a subset of the latter (see Ball et al.'s 2008 model).

Although this study suggested important skills teachers need in order to promote student learning, the absence of student data to substantiate further the benefits of identifying MPs and developing a "productive" storyline and the small number of teachers and lessons involved limit its wide applicability. Therefore, further studies involving student data and a greater number of lessons and teachers over an extended period of time are needed to investigate what it means for teachers to identify written MPs and develop a potentially productive mathematical storyline toward them.

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THREE STEPS ForWARD: VALIDITY EVIDENCE FOR THE PSM3

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This paper's purpose is to discuss validity evidence related to a third-grade problem-solving measure (PSM3). PSM3 is connected to a series of tests designed to measure students' problem-solving performance aligned with the Common Core State Standards for Mathematics. Multiple validity sources are drawn together to support the PSM3's interpretations and uses.

Keywords: Assessment; Elementary School Education, Problem Solving

Problem solving is central to mathematical work (National Council of Teachers of Mathematics [NCTM], 2000, 2014) and is a core part of the Common Core State Standards, which were adopted by 42 of 50 states (Common Core State Standards Initiative [CCSSI], 2010). Problem solving is found in every grade-level across the Standards for Mathematics Content (SMCs) and is described in the first Standard for Mathematical Practice (SMP; CCSSI, 2010). The notions of problem and problem solving are pervasive across the Standards for Mathematical Practice (e.g., "Make sense of problems and persevere in solving them, CCSSI, 2010, p. 6) as well as the Standards for Mathematics Content (e.g., "Solve two-step word problems using the four operations", CCSSI, 2010, p. 23) and therefore should be a part of mathematics assessments. Bostic and colleagues (2015; 2017) reported that problem-solving tests used in scholarly studies tend to fall into three categories: large-scale assessments, measures of mathematical problem-solving distinct from curricular standards, and problem-solving assessments focusing on nonmathematical elements. Unfortunately, few mathematical quantitative instruments used with elementary students have reported validity evidence supporting their uses (Bostic et al., 2019). This study fills a gap in the literature by providing validity evidence for a problem-solving measure connected to curricular standards within elementary settings.

Related Literature

Multiple definitions and frames for mathematical problem solving exist. This study is guided by Lesh and Zawojewski's (2007) modeling-influenced perspective on problem solving: "several iterative cycles of expressing, testing and revising mathematical interpretations – and of sorting out, integrating, modifying, revising, or refining clusters of mathematical concepts from various topics within and beyond mathematics" (p. 782). Such a problem-solving perspective requires tasks that encourage students to engage in productive, reflective, goal-oriented problem solving. While there are multiple frames and definitions for what counts as a problem, this study draws upon Schoenfeld's (2011) features of a problem: (a) it is unknown whether a solution exists, (b) a solution pathway is not readily determined, and (c) there exists more than one way to answer the task. Problem solving happens when a task is a problem, not an exercise, for an individual (Polya, 1945/2004; Schoenfeld, 2011); hence a key component to problem solving is a problem.

The PSMs contain word problems and were designed using Verschaffel et al.'s (1999) characterization of word problems: *Open* word problems can be solved in different ways and offer learners multiple entry points. *Realistic* word problems draw on a problem solver's experiential knowledge and engage the student in a real-world task. *Complex* word problems require an individual to employ sustained reasoning. Communicating definitions is important to this study because developing summary (aka purpose) statements within validation work is derived from purposeful choices and in turn, informs users what the instrument can and cannot do (Carney et al., accepted). These statements are like an abstract for an assessment in that they convey essential information for potential measure users and administrators.

This study draws upon the Standards (AERA et al., 2014) to communicate evidence and connect it to interpretations and use. Aspects of a test's interpretation and use include articulating a construct, describing test administration, and scoring (Carney et al., accepted). The research question for this study is: What validity evidence exists for the PSM3? This examination of the PSM3 builds upon work on past PSMs for grades 4-8 (see Bostic et al., 2015; 2017; 2020).

Method

A design science framework (see Middleton, et al., 2008) guides this study to explore five sources of validity (see AERA et al., 2014): test content, response process, relations to other variables, internal structure, and consequences from testing. Only test content, response processes, and internal structure will be highlighted in this paper due to page limitations. Test content evidence provides a connection between content described in items on a test and the intended construct (AERA et al., 2014; Sireci & Faulkner-Bond, 2014). Reviews from an expert panel are a common and appropriate approach for discerning the degree to which there is a match (AERA et al., 2014). Response process evidence explores if respondents behave in ways that are intended or desirable (Padilla & Benitez, 2014). Think alouds are typical approaches to gather response process evidence for problem-solving tests (Leighton, 2017). Internal structure evidence suggests the degree to which items conform to a desired construct (AERA et al., 2014). Rasch techniques as well as classical test theory approaches are both adequate, yet each approach is beholden to differing assumptions (Rios & Wells, 2014). Qualitative data and analyses were used with test content and response process evidence. Quantitative data and analyses were employed to explore internal structure evidence.

Measure

The PSM3 is composed of 15 word problems with three items coming from each of the five SMC content domains: Operations and Algebraic Thinking, Numbers in Base Ten, Number and Fractions, Measurement and Data analysis, and Geometry. A sample PSM3 item reads "Beth is coloring a picture using crayons. The box of crayons has 6 blue crayons, 4 yellow crayons, 8 green crayons, and 6 red crayons. What fraction of the box of crayons is green?" The PSM3 is designed to measure mathematical problem-solving in relation to third-grade mathematics standards.

Data Collection

To address test content, expert panels were conducted with three grade-level mathematics teachers, two terminally-degreed mathematics educators with expertise in elementary mathematics (grades K-6), and one terminally-degree mathematician. Mathematics teachers were current grade three mathematics teachers who had at least four years teaching experience and at least two years teaching third grade. The mathematics educators have elementary teaching experience and have published and presented peer-reviewed work on elementary mathematics teaching. The mathematicians has experience working with elementary teachers and communicated having read and discussed the Common Core State Standards with their university students. Mathematics teachers and teacher educators responded to the following questions: (1) Is the task a problem? (2) Is the task open? (3) Is the task realistic? (4) What Standard(s) for Mathematics Content are primarily addressed by this task? (5) What Standard(s) for Mathematical Content are primarily addressed by this task? The mathematician responded to questions #1-3 as well and additionally, (6) Describe the mathematics addressed by this task. What are two appropriate, grade-level problem-solving strategies? (7) Is the mathematics in the problem correct? (8) Is there a well-defined solution for the task? Items were reviewed once by the expert panel, revised, and then subjected to a second review when necessary. Each expert panel member submitted responses to these questions.

To address response processes, both 1-1 think alouds and whole-class think alouds were used. 1-1 think alouds were performed with a purposeful sample of 12 students consisting of varying mathematical abilities as report by their mathematics teachers (i.e., above average, average, and below average ability), male and female students, as well as white and non-white students. Ability-level judgments were gathered from teachers' views about students' classwork and prior assessment data. Whole-class think alouds (see Bostic et al., 2021) were conducted one year later with two unique sets of students (n=32). Think alouds were videotaped and student work was collected. Combining think-aloud formats allowed for greater and more diverse information about students' responses.

To address internal structure, third-grade students (n=290) across four Midwest districts completed the PSM3 in the last month of the academic year. Districts represent urban, suburban, and rural schools and each has unique populations consisting of different ethnic backgrounds, socio-economic status, and locations. Students with and without an identified disability completed the PSM3 per any Individualized Education Plan requirements. Based upon prior pilot administrations, teachers gave students approximately 90 minutes to answer the questions.

Data analysis

Expert panel reports and student think aloud data were analyzed using inductive analysis (Creswell, 2012) across three researchers, which maintains a parallel structure from previous peer-reviewed work (Bostic et al., 2015; 2017; 2020). The inductive analysis started with re-reading (or re-watching) to materials (e.g., written work and recorded statements from the conference). Next, we made memos consisting of initial ideas stemming from this examination of the data and later reflected on those memos to synthesize them into support (or not). Then, we sought evidence and counter evidence within the data sets to support our burgeoning themes. Impressions with a paucity of counter evidence and a large set of evidence were retained. Finally, we crafted a thematic statement representing the supporting data. Related to test content evidence, an intended goal was to discern the degree to which items were connected to the intended standards and addressed our selected framework for word problems. Related to response process evidence, an intended goal was to explore ways that students' responses aligned with our a priori conjectures in students' problem solving. Psychometric data analysis for internal structure used Rasch modeling (Rasch 1960/1980). PSM3 items were scored dichotomously by three scorers using a scoring key. Generally, it is important to look multiple components from Rasch analysis. First, separation and reliability values of 2.0 and 0.8 are considered good while 3.0 and 0.9 are excellent (Duncan et al., 2003). Rasch infit and outfit statistics (mean square values between 0.5 and 2.0) are considered acceptable and there should be no negative point-biserial statistics (Linacre, 2002).

Findings

Themes for test content evidence were tasks were: complex enough to be considered problems for third-grade students, open, and solvable in multiple ways using grade-level strategies, and based upon realistic contexts that led to realistic solutions. Mathematicians confirmed three and sometimes four developmentally appropriate strategies that students might use to solve the word problems. Expert panel feedback consistently conveyed that tasks aligned with third grade content standards. One teacher shared a sentiment that others echoed: "These are appropriately difficult word problems that will make students think about the math they learn. These problems require more than just using a procedure." Finally, the expert panel conveyed that word problems met developmentally appropriate reading levels. A Flesch-Kincaid reading analysis confirmed (3.4 grade level). In sum, there was majority agreement between expert panel members and researchers' hypothesized content standards.

A theme about response process evidence was that students responded in anticipated ways. Average- and above-average performing students tended to provide more correct answers than below-average students. It was common for lower-performing students to combine numbers using symbolic notation without making sense of the quantities. In the crayon problem described earlier, there were many students who wrote a fraction that did not answer the question. When pressed to explain their thinking, we heard comments like Natasha's: "I made a fraction with the numbers like it says in the problem." All students were able to read the problems, which supported our finding that the PSM3 met grade-level reading expectations.

Psychometric findings support robust internal structure evidence. All items had acceptable infit (MNSQ Range 0.82-1.29) and outfit (MNSQ Range 0.68-2.00) measures, and no items had negative point biserial values. Rasch item reliability (0.93) and separation (6.43) were strong. Collectively, psychometric data suggest a unidimensional variable of problem solving has been created from items on the PSM3.

Discussion & Limitations

The central aim for this study was to report test content, response process, and internal structure validity evidence for the *PSM3*. Synthesized findings suggest the validity evidence as being supportive of the following claims: (a) Mathematics content found on *PSM3* tasks addresses mathematics content described in grade-level standards; (b) Respondents solved *PSM3* tasks in anticipated ways; and (c) The *PSM3* appears to fit a unidimensional construct, which we characterize as mathematical problem solving. These findings connect back to three desired sources of validity: test content, response process, and internal structure (see AERA et al., 2014). Taken collectively, the *PSM3* is an instrument that may be useful for scholars interested in studying third-grade students' mathematical problem solving within instructional contexts using the Common Core State Standards for mathematics. Evidence for relations to other variables as well as consequences of testing/bias will be further investigated. The findings for this study are limited to native English speakers, which should be explored in subsequent studies.

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ASSESSING THE QUALITY OF MATHEMATICS IN CAMEROON PRIMARY SCHOOL TEXTBOOKS AND ITS IMPLICATIONS TO LEARNING

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Mathematics textbooks for upper primary classes in the English Subsystem of Education in Cameroon were examined to determine the quality of mathematics in them and possible teacher knowledge fostered. The quality of mathematics in these textbooks is classified as medium and the dominant teacher knowledge fostered is common content knowledge. This is because the textbooks are full of accurate standard algorithms and mathematical definitions, yet lack the use of multiple strategies and representations. They also contain high proportion of mathematical explanations that are either partially accurate or accurate but incomplete. Textbooks with medium mathematical quality have high potentials of causing learners and teachers to be mathematically malnourished.

Keywords: Assessment, Mathematical Representations, Curriculum

In 2018, the Ministry of Basic Education in Cameroon introduced reforms in the Primary School curriculum for the English Subsystem of Education. Following this curriculum reform, Cameroon promulgated into law, for the first time, the one textbook policy, meaning only one textbook would be approved by the National Council for the Approval of Textbooks and Didactic Materials (NCATDM) for use in each class for each subject for a period of six years before the selection is reviewed. Following this policy, publishers of textbooks went into writing to submit materials for approval by the NCATDM so that primary school learners and teachers throughout Cameroon would use them for teaching and learning. A goal of the NCATDM is to select the textbook that covers the curriculum in the best possible way to ensure that learners learn appropriate content. This paper focuses on mathematics textbooks only.

Shulman (1986) argued that teachers need more than facts to adequately teach mathematics. A possible point where teachers could obtain knowledge for teaching is during pre-service teacher training programs. Ball, Thames and Phelps (2008) noted that subject matter courses in many teacher preparation programs fail to provide the much needed mathematics content for teaching as the emphasis seems to be on higher mathematics. Therefore, my hypothesis is that in such a case teachers, after being trained, actually encounter the mathematics they are to teach when exposed to textbooks designed for learners. Hence, mathematical knowledge for teaching seems to be encountered and developed as teachers use textbooks to teach.

A number of studies have investigated teachers' mathematical knowledge and its impact on student achievement as well as the quality of mathematics in classroom instruction. Hill, Rowan and Ball (2005) found that the stronger a teacher's knowledge of mathematics, the greater the learning exhibited by learners. Ball, Thames and Phelps (2008) identified the components of mathematical knowledge that the work of teaching demands on teachers. Hill, Blunk, Charalambous, Lewis, Phelps, Sleep and Ball (2008) investigated the quality of mathematics that teachers display in classrooms during instruction and found that there is a strong positive correlation between teacher knowledge and quality of mathematics exhibited in instruction. However, little has been investigated about the quality of mathematics provided in textbooks for

Cameroon schools. This study investigates the quality of mathematics provided in primary school textbooks selected by NCATDM and attempts to answer two research questions. *What is the quality of mathematics in primary school textbooks approved for use by the English Subsystem of Education in Cameroon from 2020-2026? What types of mathematical knowledge for teaching might be promoted for teachers using these textbooks?*

This study has potentials to influence policy on textbook selection, focusing on the high quality of mathematics and the type of mathematical knowledge for teaching promoted. It may also be helpful to mathematics educators to examine the gap between what training of teachers offers and what teachers encounter in textbooks and fill in the space so as to adequately prepare teachers for teaching. In addition, this study can inform professional development on areas to focus so as to strengthen teacher learning.

Theoretical Perspective

Hill, Blunk, Charalambous, Lewis, Phelps, Sleep and Ball (2008) identified key aspects of high quality mathematics in classrooms including accurate mathematical explanations, mathematically accurate and intelligible definitions, accurate summary of mathematical ideas, reflection on explanations, conceptual discussion of procedures, accurate mathematical language, careful use of real world contexts, knowledge and use of multiple solution strategies, use of multiple representations and sequential construction of mathematics from one topic to another. Marshall, Superfine and Canty (2010) argued that multiple representations improve on the quality of mathematics taught in classrooms. Marshall, Superfine and Canty (2010) further argue that just using multiple representations is not enough but called for connections between or among the representations to ensure greater visibility to learners and therefore raise the quality of mathematics in instruction through reflection of the representations, create opportunities for learners to translate among representations. Connections should also be fostered between or among units in a textbook (Ball & Cohen, 1996) as this can help learners see mathematics as a connected subject and be able to pull learning from one unit to another to boost their understanding and sense making in the subject. Teacher's knowledge can also be supported as they use curriculum materials to teach. Ball, Thames and Phelps (2008) identified Common Content Knowledge (CCK), Specialized Content knowledge (SCK), Knowledge of Content and Teaching (KCT) and Knowledge of Content and Students (KCS) as knowledge teachers need to teach.

Methodology

This study is part of a larger study investigating the quality of mathematics in textbooks approved by NCATDM for use in Primary Schools (classes one to six) of the English Subsystem of Education in Cameroon. Learners' textbooks for classes five and six were analyzed for this particular study.

Textbooks for this study. Textbooks approved by NCATDM for classes five and six are published by ASVA Education with titles *Foundation Primary Mathematics 5* and *Foundation Primary Mathematics 6*. Throughout these textbooks, each unit has sections for *let's observe, let's find out, let's retain* and *let's practice. Let's observe* contains demonstration of some methods pupils are expected to learn, *let's find out* contains questions that are presented for learners to reflect on the methods just observed, *let's retain* contains mathematical explanations or definitions of concepts learners are expected to understand as well as examples used to illustrate mathematical concepts learners are to learn and *let's practice* contains problems learners are supposed to engage with in order to reinforce the concepts learned.

Data sources. Data for this study were drawn from the *let's observe* and *let's retain* sections of each unit. This is because these are the sections where representations and mathematical explanations or definitions for concepts learners are expected to learn are provided. Simple random sampling was done and fifty percent of the units in each textbook was selected for analysis. This was to ensure greater coverage to adequately represent each of the textbooks. The following six units out of twelve were selected for analysis in *Foundation Primary Mathematics 5*: Unit 2-basic number operations, Unit 4-number and numeration, Unit 6-modulo arithmetic and number bases, Unit 8-money and shopping, Unit 10-speed, distance and time and Unit 12-graphs and statistics. For *Foundation Primary Mathematics 6*, six units out of twelve selected were: Unit 2-numbers and numeration, Unit 3-basic number operations, Unit 4-base system, Unit 5-fractions and decimals, Unit 6-modular arithmetic, and Unit 7-Rate, ratio and proportion.

Data analysis. In this analysis, mathematical explanations, solutions to examples, representations and definitions were coded. Mathematical sentence were coded using Figure 1.

Codes		**Descriptions**
1A	Accuracy	When all parts of the explanation are correct.
1B	Partially accurate	When some parts of the explanation are correct and other parts are not correct.
1C	Inaccurate	When all parts of the explanation are not correct.
1D	No explanation	When no explanation is provided.
1E	Incomplete explanation	When an incomplete accurate explanation is provided.
2A	Single method	Just one method is used in solving an example.
2B	Multiple methods	In more than one methods used in solving an example.
3A	Connections	Connections made between or among the methods.
3B	No connections	No connections made between or among multiple methods.
3C	Reference made in text	Reference is made about the solution in the text.
3D	No reference	No reference about the solution in the text.
4A	Single representation	Whether a single representation is used employed.
4B	Multiple representation	The use of more than one representation to explain a concept.
5A	Accurate representation	When all parts of the representations are correct, conveying conceptual aspects of the key mathematical ideas to be learned.
5B	Partially accurate representation	When some parts of the representations are correct, conveying conceptual aspects of the key mathematical ideas to be learned while other parts are not correct.
5C	Inaccurate representation	When all parts of the representations are not correct, conveying incorrect conceptual aspects of the key mathematical ideas to be learned.
5D	Reference to representation in the text	When explicit reference is made in the text to explain the representation used.
5E	No reference to representation in the text	When no reference is made in the text to explain the representation used.
5F	Connections between or among representations used	When explicit connections are made in the text to show relationships between or among representations used.
6A	Mathematically accurate and intelligible definitions	All components of the definitions are accurate with no limitations or ambiguity.
6B	Mathematically partially accurate definitions	Some parts of the definitions are accurate while others have limitations or ambiguity.
6C	Mathematically inaccurate definitions	All parts of the definition are not correct.
6D	Accurate but incomplete definitions	Definitions are accurate but incomplete.

Figure 1: Codes and Descriptions

Lastly, I deduced the quality of mathematics learners are likely to learn from using these textbooks as high (*accurate mathematics and representations used and connections among the representations*), medium (*mostly partially accurate mathematics and representations and sometimes connections among them*) and low (*mostly inaccurate mathematics and representations used and no connections among them*). Finally, from the mathematics embedded in these two textbooks, I inferred the dominant kind of mathematical knowledge teachers using them might possibly acquire over time. I coded the knowledge type as CCK (*mathematical knowledge common to other users of mathematics*), SCK (*mathematical knowledge specific to the teaching of mathematics*), KCS (*anticipating what students might think, the confusion/difficulties they might have*) and KCT (*knowledge of teaching and about the mathematics they are to teach, understanding the sequencing of topics, the design rationale of tasks or representations used*).

Results

		UNITS IN CLASS FIVE TEXTBOOK						**UNITS IN CLASS SIX TEXTBOOK**							
		2	**4**	**6**	**8**	**10**	**12**	**2**	**3**	**4**	**5**	**6**	**7**	TOTAL	%
	CODES														
1	1A	50	109	63	4	7	10	55	22	38	101	23	51	533	77.8
	1B	7	6	24	0	0	1	7	1	10	9	0	0	65	9.5
	1C	0	12	11	0	0	0	0	3	1	2	2	0	31	4.5
	1D	0	18	4	1	0	0	0	0	0	0	0	0	23	3.4
	1E	0	4	0	0	2	0	0	3	4	20	0	0	33	4.8
	TOTAL	**57**	**149**	**102**	**5**	**9**	**11**	**62**	**29**	**53**	**132**	**25**	**51**	**685**	**100**
2	2A	5	1	17	2	3	1	14	5	9	38	7	9	111	90.2
	2B	5	1	1	0	0	0	1	1	1	0	1	1	12	9.8
	TOTAL	**10**	**2**	**18**	**2**	**3**	**1**	**15**	**6**	**10**	**38**	**8**	**10**	**123**	**100**
3	3A	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	3B	5	1	1	0	0	0	2	0	1	0	2	0	12	100
	3C	5	0	0	0	0	0	1	0	0	0	0	0	6	50
	3D	0	1	1	0	0	0	1	0	1	0	2	0	6	50
	TOTAL	**10**	**2**	**2**	**0**	**0**	**0**	**4**	**0**	**2**	**0**	**4**	**0**	**24**	
4	4A	6	16	21	1	0	2	0	0	0	3	4	0	53	76.8
	4B	4	8	0	0	0	1	0	0	0	0	3	0	16	23.2
	TOTAL	**10**	**24**	**21**	**1**	**0**	**3**	**0**	**0**	**0**	**3**	**7**	**0**	**69**	**100**
5	5A	1	4	9	1	0	3	0	0	1	3	2	0	24	32.9
	5B	2	0	4	0	0	0	0	0	2	4	0	0	12	16.4
	5C	0	5	0	0	0	0	0	0	0	3	2	0	10	13.7
	5D	2	7	0	0	0	2	0	0	0	5	0	0	16	21.9
	5E	0	7	0	1	0	0	0	0	0	0	3	0	11	15.1
	5F	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	TOTAL	**5**	**23**	**13**	**2**	**0**	**5**	**0**	**0**	**3**	**15**	**7**	**0**	**73**	**100**
6	6A	11	5	0	8	5	7	8	0	0	3	0	3	50	92.6
	6B	0	2	0	0	1	1	0	0	0	0	0	0	4	7.4

	6C	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	6D	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	TOTAL	**11**	**7**	**0**	**8**	**6**	**8**	**8**	**0**	**0**	**3**	**0**	**3**	**54**	**100**

Figure 2: Coding results

Figure 2 shows that of the 685 mathematical explanations provided, 77.8% of them were accurate in all its parts. The accurate mathematical explanations were all standard procedures including steps that have to be executed by learners. For example, in the unit for *Base System*, conversion from one base to another is required. This textbook explains the procedure as follows "to convert numbers from one base to another other than base 10, we first change them to base 10, then, we change to the indicated base" (class 6 textbook, p. 39). Some of these mathematical explanations were simply facts that are to be learned, memorized and reproduced such as "not all prime numbers are odd" (class 5 textbook, p. 15) and "2 is a prime number but is also an even number" (class 5 textbook, p. 15).

Also, of all the 685 sentences providing mathematical explanations, 9.5% of them are partially accurate. In the class 6 textbook, it is explained that "to look for the cube root, first divide the number by all possible factors" (p. 25). This explanation is partially accurate in that we find the cube root of any number by dividing it by possible prime factors only not "all possible factors." The absence of "prime factors" in the textbook's explanation makes it partially accurate.

Of the mathematical explanations provided, 4.5% are inaccurate in all of its parts. For example, in expressing fractions as decimals, $\frac{1}{2}$ is used in the textbook and written as

$$\begin{array}{r|l} & 0.5 \\ \hline 2 & 10 \\ - & \underline{10} \\ & 0 \end{array}$$

together with the following explanation "1 cannot divide 2 so, we put a point above 1 and affix a zero behind 1 to make it 10, 10 divided by 2 is 5" (class 6 textbook, p. 66). This explanation is not correct in all its parts as the point is not put on 1. Note that every whole number has a decimal point after it. So, 1 can be written as 1.0. Now, since 2 cannot go into 1, we put a 0 above 1 and then put the decimal point above the decimal point and insert a zero (0) after the decimal point. Now 5 tenth multiplied by 2 gives 1.0 as shown to the right.

$$\begin{array}{r|l} & 0.5 \\ \hline 2 & 1.0 \\ - & \underline{1.0} \\ & 0 \end{array}$$

In the textbook's explanation, one wonders how we started with the dividend as 1 and ended up with it as being 10. Of the 685 mathematical sentences, 3.4% had no explanations.

In subtracting fractions, the textbook provides a problem as $\frac{3}{3}-\frac{2}{9}$. Then goes ahead to solve the problem as follows $\frac{3}{3}-\frac{2}{9}=\frac{3\times3}{3\times3}-\frac{2\times1}{9\times1}$, then $\frac{3}{3}-\frac{2}{9}=\frac{9}{9}-\frac{2}{9}$ and finally $\frac{3}{3}-\frac{2}{9}=\frac{7}{9}$ (class 5 textbook, p. 44). In this solution, the authors did not explain why the numerator and denominator of the fraction $\frac{3}{3}$ are multiplied by 3 and why that of $\frac{2}{9}$ is multiplied by 1. Without explaining why the multiplications were done, the learners and teachers are left with a thinking that the numbers were chosen arbitrarily, making their understanding flawed. Of the 685 mathematical sentences, 4.8% had incomplete accurate explanations. In explaining a mixed fraction, the textbook said "a mixed fraction is a fraction which has a whole number attached to it to the left side" (class 5 textbook, p. 41). This explanation is accurate but incomplete as the whole number is the quotient when a number is divided by another number. So, the complete accurate explanation could have been, "a fraction represented with its quotient and remainder is called a mixed fraction." In addition, learners are often confused about the operation between the whole number and the fractional part of the mixed fraction. Learners often see that operation as multiplication because ab means $a\ '\ b$. Therefore, emphasis could have been laid by the authors that the mixed fraction $a\frac{b}{c} = a + \frac{b}{c}$ to dispel this confusion and curb misconceptions that learners often have. This accurate complete explanation provided might cause a smooth transition between improper fractions and mixed numbers and fully explain the idea of mixed fractions.

Of the 123 solutions provided, 111 of them have just one strategy while 12 of them have at least two strategies. When solved using more than one method, no connections are made between or among the methods. This is a missed opportunity to have learners decide which approach or strategy they understand best and will be able to use. In 50% of the time, when more than one solution strategies are used, these are referenced in the text while in another 50% there is no reference about the solution in the text. When no reference is made about the solution in the text, learners are left with the option of struggling to understand what they actually mean. In 76.8% of the time, the authors used single representation to solve problems or demonstrate a concept while in 23.2% of the time, multiple representations are used.

The representations revealed that 32.9% of them were accurate in all parts, conveying conceptual aspects of the key mathematical ideas to be learned.

$$
\begin{array}{rrcrrr}
 & 6 & . & 3 & 4 & 6 \\
 & 0 & . & 0 & 3 & 5 \\
+ & 8 & . & 5 & & \\
 & 0 & . & 7 & & \\
\hline
1 & 5 & . & 5 & 8 & 1 \\
\hline
\end{array}
$$

In adding decimals, the authors accurately lined up all the decimal numbers using the place value table to the left and then calculating the sum (class 5 textbook, p. 53).

Of the representations used, 16.4 % are partially accurate, some parts of the representations are correct, conveying conceptual aspects of the key mathematical ideas to be learned while other parts are not correct.

Fractions	**Decimals**	**Percentages**
$\frac{1}{2}$	0.5	50%
$\frac{1}{4}$	0.25	25%
$\frac{1}{3}$	0.33	33%

In changing fractions to decimals and then percentages, the authors presented the table to the left which is not accurate in all its parts (class 5 textbook, p. 52).

The first two rows are both correct and accurate but the third row is not correct as $\frac{1}{3}$ is not exactly 0.33 as a decimal and $\frac{1}{3}$ is not exactly 33% as a percentage. This inaccurate representation of the third row can be very misleading to teachers and learners.

Of the representations, 13.7% are inaccurate in all parts, conveying incorrect conceptual aspects of the key mathematical ideas to be learned. The representation of equivalent fractions is incorrect, conveying misconceptions of the key mathematical idea. For example, $\frac{1}{2}$ is represented as equivalent to $\frac{2}{4}$ and also $\frac{3}{6}$ on two separate diagrams (class 6 textbook, p. 46). The emphasis in this textbook is on the generation and not on the understanding/meaning of equivalent fractions.

As such the multiplication of numerator and denominator by the same whole number to generate equivalent fractions is emphasized and reinforced. Of the representations used, 21.9% of them are referenced in the text explicitly to explain teachers and learners the concept embedded in them, 15.1% of the representations are not referenced in the text and this has the potential of teachers and learners ignoring them for lack of understanding and none (0%) of the representations are connected explicitly or implicitly to show relationships between or among them and rationale for why they were used.

For mathematical definitions provided throughout the textbooks, 92.6% of them were accurate and having no ambiguity. For example, "proper fractions are fractions whose numerators are smaller than the denominators" (class 6 textbook, p. 45). Of the definitions provided, 7.4% are partially accurate. For example, "when an object is divided into equal parts, each part is a fraction of that object" (class 6 textbook, p. 44). This definition offered by the textbook is partially correct as it is not only when the parts are equal that it is a fraction of the whole. A part of a whole is a fraction whether they are equal or unequal. Also, fractions are formed by dividing *n* units into *m* equal parts $\left(\frac{n}{m}\right)$ and then collecting *n* of those equal parts. In addition, the book defines the calculation of speed or average speed as $\frac{\text{Distance}}{\text{Time Taken}}$ (class 5 textbook, p. 101). This definition is true and accurate for speed but not always for average speed. Average speed is calculated using $\frac{\text{Distance covered in an interval of time}}{\text{interval of time}}$ or $\frac{\text{increase in displacement in that interval of time}}{\text{interval of time}}$. Although speed and average speed might be the same at some point, this is usually not the case and should be clearly distinguished to the teacher and learner. Furthermore, none of the definitions are completely inaccurate or completely accurate; they are incomplete.

The results of this study revealed that the dominant kind of teacher knowledge that might be highly promoted is *Common Content knowledge* (CCK). Figure 2 indicate that majority of the mathematical explanations provided are accurate (77.8%). These explanations are mainly those that could be offered by mathematicians as well as other users of mathematics. Also, in the examples provided inside the textbooks, 90.2% of them were solved using a single method and when representations were used, only a single representation is used to explain a mathematical idea. The single solution methods provided are mainly standard algorithms. In addition, when definitions are provided, 92.6% of them are accurate and often these are standard mathematical definitions.

Discussion/Significance

Overall, the quality of mathematics presented in official textbooks for primary 5 and 6 of the English Subsystem of Education in Cameroon can be classified as medium. This is because in these textbooks, the proportion of partially accurate mathematics is significantly high; multiple solution strategies/representations are rarely used; when multiple solution strategies / representations are used, connections between or among them are rarely established; proportion of mathematical definitions that are inaccurate is significantly high. As such, these textbooks fall short of research recommendations for curriculum materials from which teachers can learn.

Davis and Krajcik (2005) recommended that curriculum materials should contain features to support teacher learning. These features include multiple ways learners might respond to a task or problem and together provide mathematical explanations embedded in these responses and representations that might be employed. In addition, connections between and among the strategies/representations used can be helpful in providing multiple access into the mathematical ideas learners are to learn. Marshall, Superfine and Canty (2010) have argued that when multiple representations are used, connections between or among them should be established in order to raise the quality of mathematics being taught. Ball and Cohen (1996) emphasized that these connections should be fostered by textbook authors. Therefore, establishing connections between or among multiple solution strategies/representations used can help to improve on the quality of mathematics learners learn. In addition to improving the quality of mathematics in textbooks, intentionally making connections in the mathematics textbook might enable learners to see the subject as connected and might be induced into making such connections so as to improve on the quality of their learning. The absence of these features in official textbooks selected for use in the English Subsystem in Cameroon seems to project these curriculum materials as creating very little opportunities for teachers and learners to learn appropriate mathematics and hence being mathematically malnourished.

Teachers are often mathematically malnourished when their learning is limited to a unique form of mathematical knowledge for teaching. The dominant teacher knowledge propagated in these textbooks is common content knowledge (CCK). This is because the percentage of mathematical explanations, single solution methods, single representation and mathematical definitions used in these textbooks are very high. In addition, their focus is laid on standard algorithms. The absence of other forms of teacher knowledge in these textbooks is a clear indication that the teachers using them might be limited in their mathematical knowledge for teaching as a whole and as such limited in teaching this subject to learners.

These findings reveal that mathematics textbooks approved for use in class 5 and 6 in the English Subsystem of Education in Cameroon are not fully providing and developing the needed mathematical proficiency in teachers and learners. The National Research Council (NRC, 2001) characterized mathematical proficiency as having five strands namely *conceptual understanding*, *procedural fluency*, *strategic competence, adaptive reasoning* and *productive disposition*. From the emphasis in these textbooks, one could deduce that only one strand, procedural fluency is promoted because of the heavy emphasis on standard methods and definitions. This study identified that mathematics textbooks selected for use by our learners and teachers fall short of the standard to support and develop their mathematical proficiency. Therefore, textbook authors can use the results of this study to develop materials that will support and develop the needed mathematical proficiency for both teachers and learners in Cameroon. The results will also help the NCATDM review their selection criteria for textbooks and focus on aspects that promote learning of both teachers and learners. The outcome of this study will also help professional development experts and teacher educators in Cameroon to focus on building teachers' capacities in areas identified as limited in these textbooks.

Although this study investigated textbooks in Cameroon, the quality of mathematics in many textbooks around the globe might not be promoting desired mathematical proficiency because the features to support improve this quality are highly limited. As such, the following question need further investigation: What combination of the features to develop mathematical proficiency in both teachers and learners is needed in textbooks to yield optimum learning outcomes? Answers to this question will enable textbook developers focus on using only those features whose interactions produce greatest learning outcomes rather than attempt to include all features that might be overwhelming to teachers.

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SLOPE ACROSS THE CURRICULUM: A TEXTBOOK CASE ANALYSIS

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This study reviews how slope is developed in expository materials across a seven-textbook series. Slope development is analyzed using a framework of five slope components to describe which components are used and connected, and by investigating accompanying levels of covariational reasoning. Findings suggest that the series describes slope from multiple components, and this development is grounded in various levels of covariational reasoning. While many connections were found between components, occurrences of both visual and nonvisual approaches within components were not prevalent. Suggestions include building connections between Behavior Indicator and Determining Property components through descriptions of covariation as well as more connections to the Steepness component.

Keywords: Algebra and Algebraic Thinking, Curriculum, High School Education, Middle School Education

Slope describes the constant rate of change of a linear function, a notion that can be understood using a variety of representations and applied for different purposes. Even though it is a “universal topic in every country’s mathematics curricula,” slope has been called *elusive* (Lingefjärd & Farahani, 2018, p. 1188) because a deep understanding of slope is difficult to acquire (Hoban, 2021). Not only does slope involve deeply understanding ratios (Lobato, Ellis & Muñoz, 2003; Walter & Gerson, 2007), students must also develop an understanding of a “function as a process” (Wilkie, 2020, p. 317) that involves covariation (Thompson & Carlson 2017). Students need multiple ways to view situations involving slope (Thacker, 2020); yet research (Styers, Nagle, & Moore-Russo, 2020) suggests that teachers themselves need more experiences with tasks that allow them to build rich, flexible, robust notions of slope.

Slope spans the mathematics curriculum. In algebra, slope is used when considering the covariational contrasts between basic linear and more advanced nonlinear functions (Carlson, Jacobs, Coe, Larsen & Hsu, 2002; Ellis, Ely, Singleton & Tasova, 2018). In statistics, slope impacts linear regression and lines of best fit (Nagle, Casey & Moore-Russo, 2017). In single variable calculus, slope is involved in understanding both average and instantaneous rates of change, as well as working with other key ideas, such as relative extrema and the Mean Value Theorem (Bateman, LaForest & Moore-Russo, 2021). Without a solid understanding of slope, it is difficult to make meaning of derivatives in either single or multivariable calculus (McGee & Moore-Russo, 2015; Zandieh & Knapp, 2006). However, students often struggle to grasp more than rote procedures or mnemonics, such as “rise over run” (Walter & Gerson, 2007). Therefore, it is important to understand how slope is developed in curricular materials.

Framework

This study seeks to describe how slope is developed across a textbook series. The study is informed by past work on textbooks, slope, and covariational reasoning.

Textbooks

Textbooks reflect "significant views of what mathematics is…and the ways that mathematics can be taught and learnt" (Pepin & Haggerty, 2001, p. 166). Textbooks play an influential role in mathematics education (Fan, Zhu, & Miao, 2013; Pepin, Gueudet & Trouche, 2013), especially in how teachers shape and sequence their instruction (Davis, 2009). Fan and Kaeley (2000) suggest that textbooks send "pedagogical messages" to teachers, since teachers using different textbooks display differences in their teaching strategies. While teachers may have access to a variety of resources, the textbook is typically the only common resource for students (Lepik, Grevholm, & Viholainen, 2015). Textbooks influence how students learn and how they consider and solve problems (Massey & Riley, 2013).

Slope

Stump's (1999, 2001a, 2001b) seminal work brought to light that slope is a multifaceted notion that can be conceptualized in many ways. Moore-Russo, Connor and Rugg (2011) introduced conceptualizations of slope as the ways that people think about and make meaning of the topic. Their 2011 conceptualization categorization has been used in studies of curriculum and standards conducted in Mexico, South Africa, and the U.S. (Nagle & Moore-Russo, 2014b; Stanton & Moore-Russo, 2012; Dolores Flores, Rivera López, & Moore-Russo, 2020). Since then, the 11 categories have been revisited and revised in research that bridges secondary to postsecondary mathematics (Nagle, Martinez-Planell, Moore-Russo, 2019; Nagle & Moore-Russo, 2014a; Nagle, Moore-Russo, Viglietti & Martin, 2013) resulting in a more nuanced conceptual framework using five connected components, each with visual and nonvisual approaches (Nagle & Moore-Russo, 2013b). In Table 1, we adopt a revised framework omitting the *Calculus* component since our study focuses on the development of slope in a precalculus context. Furthermore, we include both the *Ratio* and *Constant Parameter* components of slope to more completely delineate the nuances of slope development around these two closely connected components.

Table 1: Slope Component Coding (adapted from Nagle and Moore-Russo, 2013b)

Slope Code	Approach	Description
Constant Parameter	Visual (CP-V)	Defining parameter of linear graph (with a *y*-intercept) that indicates a uniform "straightness" of the line's entire graph; no matter which segment of the line is considered the "straightness" is constant due to similar triangles
	Nonvisual (CP-N)	Defining parameter of linear relationship (with a *y*-intercept) indicating constant rate of change between two covarying quantities; slope calculations remain constant between any two points or on any increment of change in independent variable
Ratio	Visual (R-V)	Ratio calculated by rise/run or vertical change divided by the horizontal change between any two graphed points
	Nonvisual (R-N)	Ratio calculated for any two ordered pair points (x_1, y_1) and (x_2, y_2) using the difference quotient $(y_2 - y_1)/(x_2 - x_1)$
Behavior Indicator of line or linear relationship	Visual (BI-V)	Indicator of (increasing, decreasing, horizontal, or vertical) behavior of linear graph; correlates sign of slope to directions of rise and run to determine graphical behavior
	Nonvisual (BI-N)	Indicator of increasing, decreasing, or constant behavior of linear relationship; correlates sign of slope to relationships between *change in y* and *change in x*
Steepness	Visual (S-V)	Measure of steepness of linear graph (how inclined, tilted, slanted, or pitched a line is seen as being); relates slope to angle of elevation of linear graph

of line's angle of inclination with horizontal	Nonvisual (S-N)	Measure of how extreme a linear rate of change is calculated as being (e.g., relates magnitude of $\lvert y_2 - y_1 \rvert$ with corresponding magnitude of $\lvert x_2 - x_1 \rvert$); relates slope to calculation of tan q
Determining Property between lines	Visual (DP-V)	Property that determines if linear graphs will intersect and how (e.g., if slopes are negative reciprocals, the lines intersect at right angles)
	Nonvisual (DP-N)	Property that determines whether two linear relationships that form a system of equations will have solutions and how many solutions will result

Covariational Reasoning

Covariational reasoning relates to the "mental coordination of two varying quantities while attending to the ways in which they change in relation to one another" (Carlson et al., 2002, p. 354). Slope is a topic that describes the covariational relationship between the dependent and independent variables in a linear relationship. To understand the development of slope reasoning across the curriculum, it is vital to consider how these components are built from an underlying conception of covariational reasoning. Carlson and colleagues (2002) describe five hierarchical levels of covariational reasoning, outlined in Table 2. Within the context of this study, which focuses on the development of slope prior to calculus, we do not code for L5 reasoning.

Table 2: Levels of Covariational Reasoning (Carlson et al., 2002)

Level	Description
L1: Coordination	Coordinate change in one variable with change in second variable
L2: Direction	Coordinate direction of change in one variable with change in second variable
L3: Quantitative Coordination	Coordinate amount of change in one variable with change in second variable
L4: Average Rate	Coordinate average rate of change of function uniform changes in input variable
L5: Instantaneous Rate	Coordinate instantaneous rate of change of function with continuous changes in independent variable

Methods

Data Source

The textbook series for this study was developed by the University of Chicago School Mathematics Project (UCSMP, 2021). This series of textbooks was written to correlate with the Common Core State Standards by emphasizing applications, digital resources, and mastery learning. The seven textbooks that comprise the grade 6-12 series were analyzed. In sequential order, they include: Pre-Transition Mathematics (PTM); Transition Mathematics (TM); Algebra (A); Geometry (G); Advanced Algebra (AA); Functions, Statistics, and Trigonometry (FST); and Precalculus and Discrete Mathematics (PC). The parenthetical letters denote the textbooks abbreviations used in the tables and figures below. Since this study specifically focused on slope of a line or linear function, all textbook coding excluded examples of variable or instantaneous slope, unless explicit connections to linear slope were also made.

Research Questions

This study seeks to answer the following questions:

1) Which components of slope are emphasized within each textbook and across the series?
2) What connections are made between components of slope across the series?

3) How is covariational reasoning developed in relation to slope?

Data Coding and Analysis

Data for this study included all the expository material (i.e., the components of the textbook that conveyed information through explanations and descriptions) within the textbook series. The different types of expository material analyzed included: chapter overviews, explanatory dialogue, examples, and activities. Each chapter began with a two-page overview intended to motivate the topics that followed. Within chapters, each section typically followed a similar format of explanatory dialogue punctuated with examples. The explanatory dialogue was text that introduced new terminology and definitions, reviewed foundational ideas, and provided general explanations. The examples were used to illustrate, clarify, and extend the ideas and relationships provided in the explanatory dialogue. They were either fully complete or mostly complete with a few missing details to prompt student thinking. Some sections included activities, often utilizing digital resources, which guided students through a series of steps with embedded explanations and guided questions. The unit of analysis was easily defined for examples and activities, with each example or activity being a single unit of analysis. For the chapter overview and the explanatory dialogue, a unit of analysis was distinguished as all the content included within a single heading or separated by examples or activities. While most units of analysis included one to two paragraphs of mathematical expository content, some were as short as two sentences and others extended to three or more paragraphs.

Two categories were used to code the data: a) slope conceptualization components (distinguishing between visual and nonvisual approaches) and b) covariational reasoning level. Details for the two coding categories are in Tables 1 and 2, respectively. Each unit of analysis was coded for all slope components noted and for the highest level of covariational reasoning present. Therefore, each unit was coded for up to ten possible slope conceptualization-approach pairs and at most one covariational reasoning level. The lead author was the primary coder, meeting weekly for eight weeks with the second author to review coding. Each section of every textbook was coded for all expository material related to slope. Once coding was complete, the data were sorted and prepared for analysis. The sorting was used to study each of the seven textbooks individually as well as to consider longitudinal trends across the entire series.

Results

Across the entire series, 201 units were identified and coded as addressing slope (see Table 3). All seven textbooks in the series addressed slope, even if not explicitly using the term when first introduced. As anticipated by the research team, the number of slope occurrences was highest in the Algebra and Advanced Algebra textbooks.

Table 3: Relative Frequency of Slope Occurrences across the Textbook Series (n = 201)

Percentage of All Slope Occurrences Across Entire Series	Textbook						
	PT	TM	A	G	AA	FST	PC
	5%	14%	35%	10%	20%	7%	8%

Slope Components

Table 4 displays data related to the slope components identified in each textbook, including the number of occurrences with only visual, only nonvisual, or both visual and nonvisual approaches. Across the series, more than two-thirds of all slope occurrences included the

Constant Parameter or *Ratio* component. Moreover, one of these two was the most prominent component identified for each textbook. Table 4 indicates that at least one of these two components was assigned to 50% of the slope occurrences in each textbook, with both components assigned to 50% or more of the slope occurrences in four of the seven textbooks. All other components were assigned to less than 50% of the occurrences in each textbook. Even though textbooks provided a consistent, heavy emphasis on nonvisual approaches of the *Constant Parameter* and *Ratio* components, there were relatively few occurrences linking the visual and nonvisual approaches within either component.

Overall, visual (V) and nonvisual (N) approaches of the slope conceptualization components tended to vary greatly with strikingly few occurrences incorporating both aspects of a slope component. Occurrences linking visual and nonvisual approaches of the *Behavior Indicator* (BI) component were more prevalent than the other components. Connections between BI-V and BI-N were often facilitated by explanations that incorporated multiple representations of linear functions (e.g., the equation $y = 3x+5$ and the corresponding linear graph) when analyzing what the slope indicates both about the rate of change of y with respect to x (i.e., as x increases by 1, y increases by 3) and about the graphical representation of that relationship (i.e., an increasing line that goes over 1 unit and up 3 units). Note that in situations such as this, the BI-N code was assigned since the corresponding directions of change of the two covarying quantities were linked (L2 covariational reasoning) and connected to the increasing or decreasing behavior of the linear graph (BI-V). However, these occurrences often stopped short of explicitly relating the direction of change to the increasing or decreasing nature of the function itself (e.g., if $x_1 < x_2$, then $f(x_1) < f(x_2)$).

Table 4: Frequency of Slope Components by Approach within Occurrences by Textbook

Textbook (number of slope occurrences)	Slope Components (by Visual, Nonvisual, or Both Approaches)														
	CP			R			BI			S			DP		
	V	N	Both	V	N	both	V	N	both	V	N	both	V	N	both
PTM (n=11)	0	0	2	0	9	0	1	4	0	0	0	1	0	0	0
TM (n=28)	0	14	3	0	19	0	2	6	5	3	0	0	1	0	0
A (n=70)	5	45	8	4	40	5	7	14	10	4	1	1	2	1	4
G (n=21)	0	11	0	2	10	2	3	0	0	3	0	0	10	0	0
AA (n=40)	3	27	0	0	21	4	1	8	4	1	1	1	8	1	0
FST (n=14)	0	9	1	2	3	0	0	2	2	0	0	0	0	0	0
PC (n=17)	0	6	0	0	9	2	1	2	5	0	0	1	1	0	0
Series (n=201)	8	112	14	8	111	13	15	36	26	11	2	4	22	2	4

Figure 1 illustrates the emphasis of each slope component by textbook. In each cluster, the first bar represents the percentage of total slope occurrences across the series attributed to a textbook. The five subsequent bars represent the corresponding percentage of all slope occurrences where a particular slope component was identified in the textbook. For instance, the first cluster shows that the Pre-Transition Mathematics textbook included 5% of all identified slope occurrences across the series, which included roughly 1% of all *Constant Parameter* occurrences, 7% of all *Ratio* occurrences, 6% of all *Behavior Indicator,* 6% of all *Steepness* occurrences, and 0% of the *Determining Property* occurrences. Uniform distribution of slope components across the textbook series would result in approximately equal percentages of each component for a particular textbook. For the most part, this is seen in the relatively equal height of bars within each textbook cluster. However, the *Determining Property* component appears to be heavily emphasized in the Geometry and Advanced Algebra texts (the right most bar in each cluster). The Geometry and Advanced Algebra textbooks included 10% and 20%, respectively, of all slope occurrences but included 36% and 32%, respectively, of the *Determining Property* occurrences. Two-thirds of all *Determining Property* occurrences were identified in these two textbooks even though less than one-third of all slope occurrences occurred in them. Figure 1 also reveals a heavy focus on the *Steepness* component in the Geometry textbook, which might be expected from a geometric (versus algebraic) consideration of lines.

We also considered which slope components were developed together within a single occurrence to determine common component connections. Of the 201 occurrences, 146 included combinations of two more components, while 55 occurrences were assigned a single code. A total of 21 unique coding assignments were made (e.g., *Constant Parameter* only*; Constant Parameter* and *Ratio;* and *Constant Parameter, Ratio, and Behavior Indictor*). Table 5 provides information about each of the coding assignments that were identified in at least 2% of all slope occurrences in the textbooks. Overall, many slope occurrences across the series made connections with the *Constant Parameter*, *Ratio*, and *Behavior Indicator* components. Given the complimentary nature of slope used as a *Behavior Indicator* and *Determining Property* (e.g., recognizing a line perpendicular to an increasing line must decrease), it is also interesting that these two components were linked in only one occurrence and, therefore, were not included in Table 5. *Steepness* was linked with all other slope components at least once throughout the textbook series. However, it had few occurrences across the series, even in the last two textbooks in the series when angles and trigonometry play major roles, and it did not appear in any of the frequent slope component combinations. This is noteworthy since *Steepness*, which can be tied to the tangent of an angle of inclination, is often disconnected from other slope components (Nagle & Moore-Russo, 2013a).

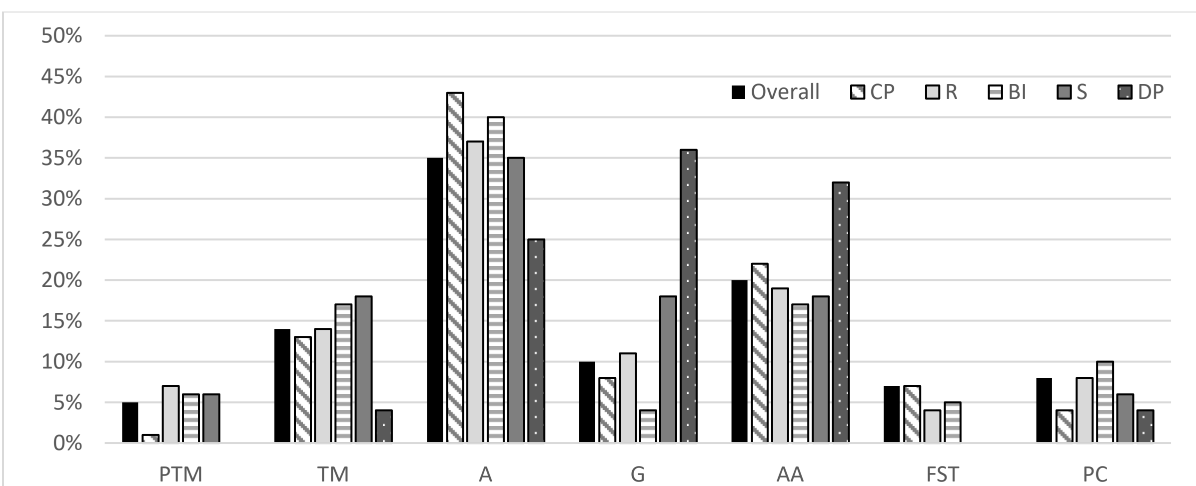

Figure 1. **Relative frequency of occurrences with slope component clusters by textbook.**

Table 5: Prevalent Slope Component Coding Assignments

Individual Component and Component Combinations Codes	% of occurrences (n=201)
CP-R	26.4%
CP-R-BI	12.4%
CP	10.4%
R-BI	10.0%
R	8.5%
CP-DP	6.5%
BI	6.0%
CP-BI	5.0%
R-DP	3.5%

Covariational Reasoning

Table 6 provides the percentage per individual textbook for each of the four levels of covariational reasoning. Nearly two-thirds of occurrences incorporated some level of covariational reasoning. As might be expected based on the definition of slope in terms of quantifying the ratio $\frac{change\ in\ y}{change\ in\ x}$, the majority of the explanations incorporated L3 covariational reasoning coordinating the amount of change in one variable with the amount of change in the other variable. Table 5 illustrates a shift from L1 reasoning in the earlier books in the sequence to L2 and L3 reasoning in the later books in the sequence. The results reveal a shift to a larger percentage of occurrences that include no covariational reasoning at later stages of the curriculum. Early curriculum explanations relied heavily on describing the covariational relationship between two quantities, even with simple L1 acknowledgment that those changes do in fact correspond. In the series, this led to defining slope as a topic that provides the quantification for this rate of change. Later curriculum explanations then frequently used slope as a tool without recounting its interpretation in terms of covarying quantities. Once slope has been formally defined, it seems as though it is often assumed that the covariation exists, but when covariation is acknowledged in later textbooks, it was at higher levels, as would be appropriate.

Table 6. Relative Frequency of Covariational Levels for Slope Occurrence by Textbook

Textbook	None	L1	L2	L3	L4
PTM (n=11)	9%	82%	9%	0%	0%
TM (n=28)	4%	11%	18%	68%	0%
A (n=70)	33%	1%	13%	51%	1%
G (n=21)	81%	0%	0%	19%	0%
AA (n=40)	45%	5 %	3%	40%	8%
FST (n=14)	50%	0%	29%	21%	0%
PC (n=17)	35%	0%	18%	18%	29%
Series (n=201)	36%	8%	11%	40%	5%

In Table 7, the percentage of slope occurrences assigned a level of covariational reasoning are listed by slope component. The results highlight that *Determining Property* was rarely developed using covariational reasoning. Recall that *Determining Property* and *Behavior Indicator* were rarely combined in occurrences, and that L2 reasoning could provide a foundation on which to build this connection. The lack of *Determining Property* occurrences with L2 reasoning further support this observation. Interestingly, *Constant Parameter* had the next highest percentage of occurrences with no covariational reasoning, because slope is often identified as the leading coefficient in a linear equation without discussion of what that represents and often reported in general terms as what makes a line straight without describing the covariation of rise and run on the line's graph. However, *Constant Parameter* also included a high percentage of L3 covariational reasoning when such as description was present. We do not view this as an indicator that the *Constant Parameter* component was developed without covariational reasoning, but that its applications supported many occurrences that did not explicitly denote the covariational relationship it represents.

Table 7. Relative Frequency of Covariational Levels by Slope Component

Component	None	L1	L2	L3	L4
CP (n=134)	36%	4%	7%	51%	2%
R (n=132)	23%	8%	4%	58%	6%
BI (n=77)	17%	5%	27%	44%	6%
S (n=17)	29%	6%	24%	29%	12%
DP (n=28)	93%	0%	0%	7%	0%

Conclusions and Future Work

This study reports the development of slope in a textbook series' expository content, considering slope components and accompanying covariational reasoning. Results suggest that this textbook series provides consistent opportunities for students to develop the various slope components across the series. As expected, slope receives the most attention in the Algebra and Advanced Algebra textbooks, but the previous and subsequent texts in this series carefully build and extend a foundation including all five of the slope components. Furthermore, covariational reasoning frequently accompanied the development of slope components, particularly in the earliest stages when the notion of slope is first being developed from students' intuitive knowledge of covarying quantities. These approaches align with recommendations from the Common Core Standards (Nagle & Moore-Russo, 2014b).

Slope was, for the most part, richly developed as a notion related to the covariational change between two quantities in a linear relationship. One exception is the lack of *Steepness* component occurrences; this is of concern especially in textbooks where angles and trigonometry are emphasized. Meaningful connections to *Steepness* could be created through covariational descriptions of the severity of change in the output variable relative to change in the input variable in contextual situations. Another exception is the *Determining Property* component, which occurred mostly in the Advanced Algebra and Geometry textbooks. The emphasis was on using the previously developed notion of slope as a tool to describe the parallel or perpendicular relationship of lines (often visually in the Geometry textbook). However, comparisons between slopes were seldom interpreted in relation to how the quantities represented by the linear graphs covaried (e.g., equal slopes suggest the same constant rate of change, so lines don't intersect). Connections to the *Behavior Indicator* component of slope utilizing L2 covariational reasoning might facilitate a more connected view of slope from these lenses.

Although slope was developed in terms of covariational reasoning and connections of various slope components, visual and nonvisual approaches within the slope components were only explicitly connected in a few instances. Nagle and Moore-Russo (2013b) describe the importance of developing a robust, flexible understanding of slope consisting of all five slope components with meaningful connections within approaches and between components. The analysis of this textbook series suggests that while the links between components were developed, the links between visual and nonvisual representations within a single slope component were often underdeveloped. In particular, the *Ratio* and *Constant Parameter* components were built heavily from nonvisual perspectives and seldom included links between visual and nonvisual approaches. Since this analysis only considered the expository material, it is quite possible that some of the additional connections between these components may come from exercises or other features of the textbook. Future analysis should explore additional elements of the textbooks to see whether opportunities for making connections between the visual and nonvisual approaches to these components might be fostered in the exercises.

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TRACING BEGINNING TEACHERS' MATHEMATICS CURRICULUM USE IN THEIR FIRST THREE YEARS OF TEACHING

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This study explores how five beginning elementary teachers used their mathematics curriculum materials in their first three years of teaching. Prior research suggests that teachers' curriculum material use in their earlier careers may not change significantly from year to year (e.g., Valencia, Place, Martin, & Grossman, 2006). Our investigation builds on this prior research with a focus on elementary mathematics curriculum use. We analyzed interview transcripts from five teachers' first three years drawing on a framework developed by Forbes and Davis (2010). Our analysis indicates that these five teachers used their mathematics curriculum in different ways from Year1 through Year 3. They followed their mathematics curriculum with few adaptations in their first year, and then began to modify their curriculum more extensively in their second and third years.

Keywords: Curriculum, Elementary School Education

Purpose

The purpose of this paper is to explore how novice elementary teachers use mathematics curriculum materials in their first three years of teaching. Understanding novice teachers' use of mathematics curriculum materials across the first three years contributes to a body of research related to teacher-curriculum interactions (e.g., Remillard, 2005). Prior research on teacher-curriculum interactions has tended to explore teachers' curriculum use in the short term (e.g., Pak & Drake, 2021; Brown, 2009; Forbes & Davis, 2010; Remillard, 2005). One exception is Valencia, Place, Martin, and Grossman (2006). They investigated beginning teachers' curriculum use related to reading in the first three years because "early teaching experiences lay the foundation for future success in the classroom" (Valencia et al., 2006, p. 99). Building particularly on Forbes and Davis (2010) and Valencia et al. (2006), we focus in this paper on how beginning elementary teachers "mobilize and adapt" (Forbes & Davis, 2010, p. 821) mathematics curriculum materials in the first three years of teaching. In doing so, we aim to contribute to the field's understanding of the beginning teachers' mathematics curriculum use in the early years of teaching. Increasing our understanding of novice teachers' curriculum use, particularly in the current quickly changing and unpredictable curriculum landscape, is important not only in theory, but also in practice as teacher educators and school and district leaders seek new approaches to supporting teachers' curriculum use.

Perspectives

We draw on the two perspectives in this paper, which are related to curriculum use in the contexts of reading and science. In relation to the first perspective, we drew on a finding by a group of researchers who have explored how beginning teachers use curriculum materials in their early careers, including during teacher preparation programs. In a longitudinal study, Valencia and colleagues (2006) conceptualized teaching practices as being shaped by interactions among the curriculum materials, teachers' knowledge, and the contexts. They followed three beginning elementary teachers from a teacher preparation program to their third years of teaching to understand how they used curriculum materials for teaching reading. They found that “over the first three years of full-time teaching, there was relatively little change” concerning “how they used them” (Valencia et al., 2006, p. 111). For example, Dorothy, a beginning teacher in the study, tended to closely follow a reading curriculum in her first three years of teaching reading. Since this study involves novice teachers’ reading curriculum use, this specific finding raises a question as to whether or not beginning elementary teachers closely follow mathematics curriculum in their early years. Further, because the curriculum landscape has changed significantly in recent years, including the availability of a much wider range of resources than in previous years (Aguirre et al., 2019), we were interested in novice teachers’ curriculum use in this new context.

We began to answer this question in another paper (Pak & Drake, 2020) in which we analyzed one beginning teacher’s mathematics curriculum use. Our finding was different from the finding of Valencia and colleagues (2006). In our paper, we found that the beginning teacher followed the mathematics curriculum closely in her first year and began to modify the curriculum in her second and third year of teaching. The finding provided us with a start to understand the patterns of the trajectory related to beginning teachers’ mathematical curriculum use. As such, we investigate in this current paper how five beginning teachers used their mathematics curriculum materials in their first three years of mathematics instruction.

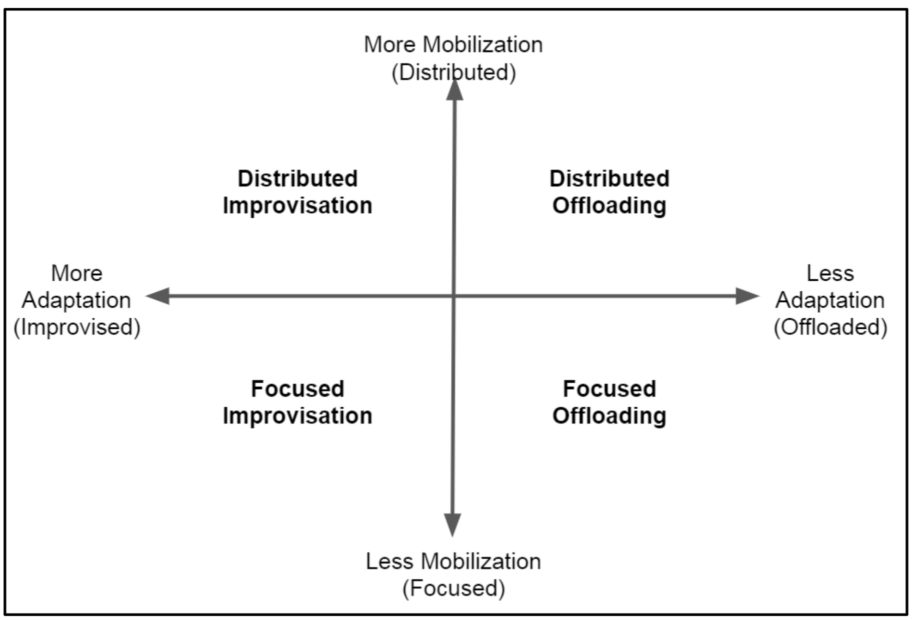

Figure 1. Framework for teacher-curriculum interactions (adapted from Forbes and Davis, 2010, p. 823)

The second perspective is related to a framework from Forbes and Davis (2010). Particularly, our analysis for this current paper draws on the framework and also utilized in our prior work analyzing the curriculum use patterns of beginning teachers in their first year (Pak & Drake, 2021). As we noted in the prior work, the framework is built on the work of Brown (2009) and Remillard (2005) in relation to teacher-curriculum interactions. The framework conceptualizes how prospective teachers mobilized and adapted existing curriculum materials (e.g., lesson plans and students' worksheets) for their science instruction by creating a two-dimensional space (Figure 1). Forbes and Davis (2010) also included a third dimension focused on the "inquiry orientation" of the lesson, but, here, we are focusing only on the dimensions of mobilization and adaptation. According to Forbes and Davis (2010), mobilization refers to the number of different curriculum materials used and adaptation means the degree to which teachers modified curriculum materials.

Building on Forbes and Davis's framework, Figure 1 shows how teachers' use of curriculum materials is conceptualized in this paper. In Quadrant 1, teachers adapt multiple curriculum materials (DI: distributed improvisation). In Quadrant 2, teachers offload teaching responsibility (i.e., make few adaptations) using multiple curriculum materials (DO: distributed offloading). In Quadrant 3, teachers offload teaching responsibility using a single curriculum material (FO: focused offloading). In Quadrant 4, teachers adapt a single curriculum material (FI: focused improvisation). It is important to note that this framework categorizes curriculum use in terms of the quadrants defined by the axes and not in terms of the endpoints of those axes. In other words, we recognize, and also found empirically, that these quadrants represent continua along the dimensions of mobilization and adaptation and that most teachers' curriculum use is not found in the extreme corners of the framework.

We use this framework as our second perspective in this current paper because it guides us to make sense of how beginning teachers mobilized and adapted mathematics curriculum materials in their first three years of teaching. In this paper, we are building on the analysis we did of beginning teachers' curriculum use in their early years (Pak & Drake, 2020) by following five beginning teachers in depth over three years to understand teachers' mathematics curriculum use.

Methods

This study was part of a four-year longitudinal large-scale research project, Developing Ambitious Instruction. The purpose of the larger project was to explore the relationships between teacher preparation and ambitious instruction in elementary mathematics and English Language Arts (ELA). This study recruited 175 participants from five teacher preparation programs in three states. There were two cohorts of participants who completed pre-service teacher preparation in 2016-2017 and 2017-2018, respectively. The project team collected data including online surveys, observations, and interviews. For this paper, we focused on analyzing the interview data from the first cohort (N=69). The project team conducted three, two, and two interviews with each of the teachers in their first, second, and third years if they taught both subjects (ELA and mathematics). Interview questions included questions related to classroom/school/local community contexts, as well as lesson planning and enactment, which led the novices in some cases to talk about ways they used curriculum materials in ELA and mathematics.

Data Sources and Analysis

For this particular study, first, we analyzed interview excerpts from 27 beginning teachers who talked explicitly about how they mobilized and adapted mathematics curriculum materials in their first year of teaching. We identified 84 excerpts from these 27 beginning teachers. The second step was individual coding of each excerpt. In coding, we used codes (DI, DO, FI, and FO) based on the framework (Forbes & Davis, 2010). After individual coding, both authors compared codings to find and resolve disagreement. Building on these codes and excerpts, we then selected five teachers to explore mathematics curriculum use in the first three years of teaching. We chose these five teachers because they demonstrated a clear curriculum use pattern in Year 1 through Year 3, which means we had the full three years of their interview transcripts. We identified 15, 22, and 21 excerpts from these teachers' interview transcripts in the first, second, and third year, respectively.

The last step was looking for common patterns across the three years of all five teachers. The patterns we identified in this step included transitions the teachers made from Year 1, to Year 2, and to Year 3. To visualize the patterns, we created a line chart that showed the numbers of the four codes (FO, DO, FI, and DI) by year (Figure 2). To understand the patterns on the part of individual teachers, we also visualized a coordinate graph for each teacher as described below (Figure 3).

To obtain the final coordinate of each teacher by year, we counted the numbers of I (Improvised), O (Offloaded), D (Distributed), and F (Focused), and found the final coordinate by cancelling different codes on the same axis (Improvised-Offloaded on X axis and Distributed-Focused on Y axis). For example, we coded four excerpts in Year 1 of one beginning teacher as two FOs, one DO, and one FI, respectively. We separated these codes into one I, two Os, two Fs, and one D. By cancelling one I and two Os on X axis, and one D and two Fs on Y axis, we got a coordinate at (1, -1) as the final coordinate.

As shown in Figure 3, some teachers have the same final coordinates, either a given year or in a different year. For example, the final coordinate of three teachers (310.050, 310.115, and 310.076) in Year 1 are placed together at (2, -2), and the final coordinate of one teacher (210.055) in Year 1 and another teacher (310.050) in Year 2 are placed together at (-1, 1). Distance from the origin at the coordinate graph suggests a magnitude of codes of each teacher. If a teacher has similar codes, then the teacher's final coordinate is distant from the origin. For example, since a teacher (310.076) has many similar codes (FI) in Year 3 (Figure 3), the final coordinate (-5, -7) is relatively distant from the origin. By creating this coordinate graph, we attempted to present a way to detail the general patterns (Figure 2) on the part of these individual teachers.

Results

Although the specifics differed for each teacher, we were able to identify some general patterns in relation to how the five beginning teachers used their mathematics curriculum in their first three years (Figure 2).

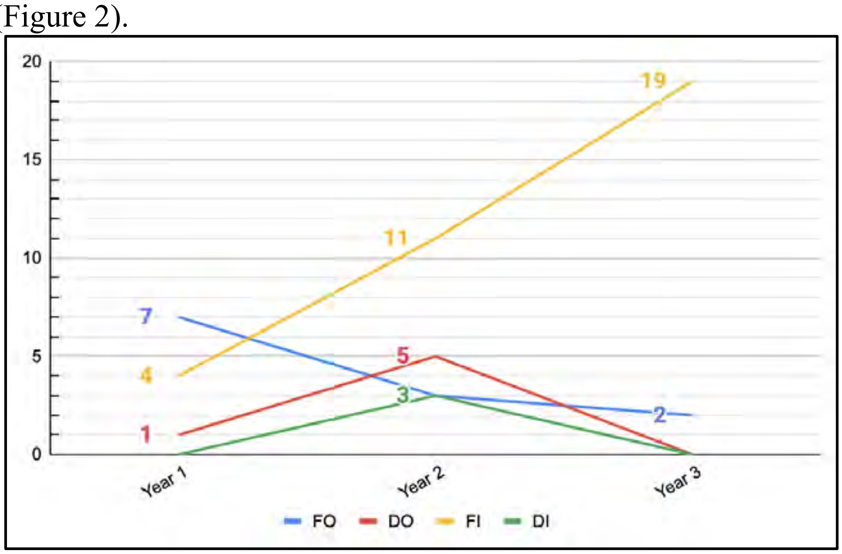

Figure 2. Pattern related to beginning teachers' mathematics curriculum use by year

Year 1 shows more FOs than FIs or DIs, suggesting that these five beginning teachers typically followed their mathematics curriculum materials closely, with few adaptations of the curriculum to meet their students' needs. Year 2 shows a change in mathematics curriculum use on the part of these five teachers. Unlike Year 1, there are higher numbers of FIs in Year 2, indicating that these beginning teachers began to modify their mathematics curriculum in different ways. Year 3 illustrates predominantly FIs, which suggests that these beginning teachers made more consistent adaptations of their mathematics curriculum materials.
To illustrate these patterns, we present an excerpt from Year 3 interview with one beginning teacher who reflected on how she used her mathematics curriculum in her first three years of her teaching. This excerpt describes many of the patterns shown in Figure 2.

> Yes. So my very first year it was, "You're doing this program with fidelity. You are not doing anything else but this program." And so even something as little as like fact fluency, it was like, "No. You are doing this program." Now no one was knocking on our door asking us like, "What do you do for Math?" Or watching our math instruction. But, it was like required for us to do it... Like we have a pacing guide that our instructional coaches came up with, and we were to follow, "OK, this lesson's one day. This lesson's two days. This lesson's one day." You know? And pace it out through there and where we should be by the end of the year and in November and everything. My second year, I moved grade levels, and everything was new to me. So I was taking recommendations from my teammates and how long to spend on certain lessons or like how to do reteaching. But still it was like... People, I think, started to supplement a little more. I know like, for example, I still did the pacing guide with fidelity last year, but I would supplement with like different hands-on games and not use all the materials from *Envision*. Instead I would find different ways to like hit the same standard, but to practice in a different way. And then this year (her third year), it has been kind of like loosey-goosey, like it's like they redid the (inaudible) binder this summer.

As described in the excerpt, this teacher followed her mathematics curriculum material "with fidelity" (FO) in Year One. The fidelity was required and expected by the school and school district through a pacing guide reinforced by her instructional coaches. In another part of the same interview, she mentioned "And supplementing isn't like... I wasn't told not to do it, but it's not like told to do it. Like, really we should be sticking to this curriculum." In Year 2, moving grade levels, she began to modify her curriculum (FI) even though she used the pacing guide with fidelity. Taking advice from her grade teammates, she supplemented *Envision* curriculum materials with "different hands-on games" and she did "not use all the materials from *Envision*." In Year 3, she began to modify Envision (FI) in a "loosey-goosey" way. Another excerpt in the same interview suggests why she became more comfortable at modifying the mathematics curriculum material. She said, "A lot of the reason I'm trying new things this year (third year) is that I'm comfortable at this grade level."

In Figure 3 (below), we present the differences among individual teachers in terms of how they used their mathematics curriculum materials in their first three years. These differences indicate that there are different trajectories among these teachers, each taking their own paths within the general pattern seen in Figure 2. For example, in her first year of teaching, one beginning teacher (210.055) began by using and adapting multiple mathematics curriculum materials using a DI (Distributed Improvisation) approach. In her second year, the teacher tended to follow the multiple curriculum materials more closely. More details about the similarities and differences across the five trajectories will be shared during our presentation.

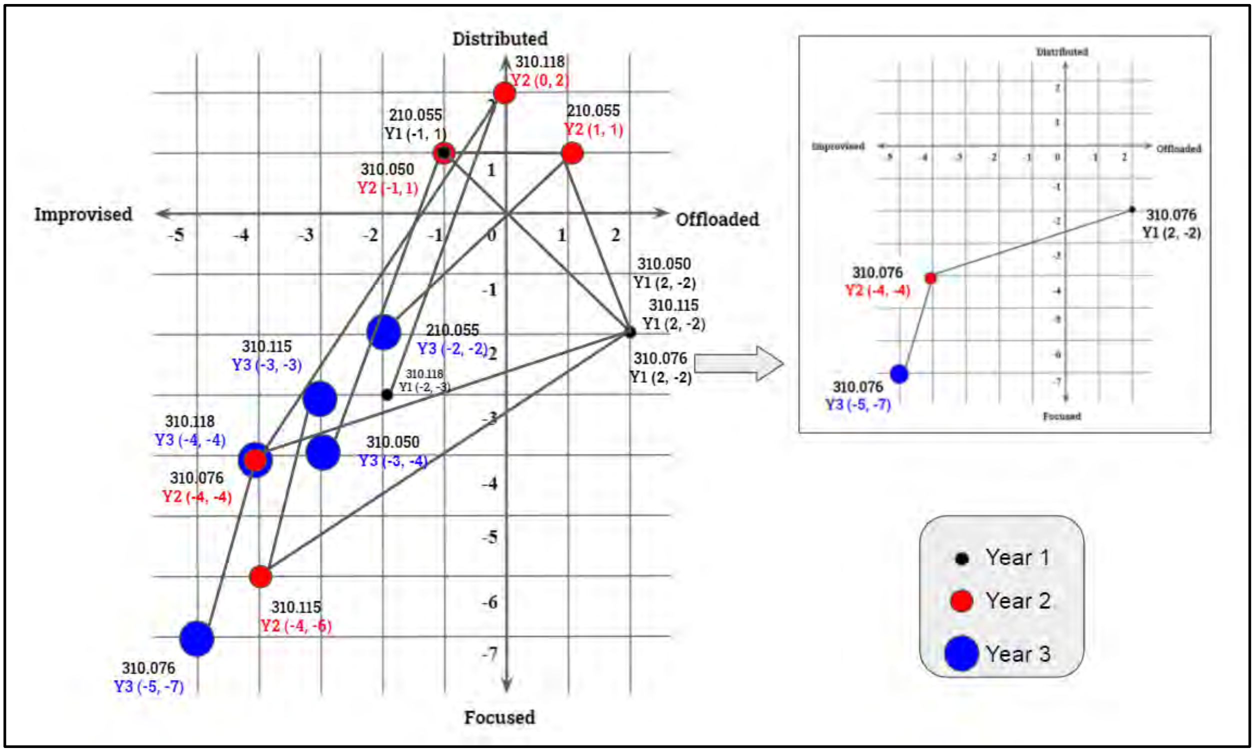

Figure 3. Yearly trajectories of five individual beginning teachers' mathematics curriculum use (each teacher is represented by code number, such as 310.076)

We found that each of the teacher's patterns of curriculum use involved interactions among the teachers' own beliefs, knowledge, and orientations toward mathematics curriculum and the school contexts in which they worked, including school and district policies related to curriculum use and the curriculum use patterns of their colleagues. Nonetheless, as depicted in Figure 3, all five of the teachers' curriculum trajectories arrived by Year 3 in the Focused Improvisation quadrant. Each third-year teacher could provide a clear rationale for their FI approach to curriculum use at that point. For example, one teacher (210.055) said, "And if it's not enduring curriculum, we have the freedom to not spend as much time on it. So that's been good." In other words, in this teacher's case, they were in a school context that required the use of a core curriculum to address the big mathematical ideas of the grade level, but also allowed them the agency to drop or modify less essential curricular components. A different third-year teacher, also in the FI quadrant, described how, through experience at the grade level, she was learning to use and adapt (improvise) her core curriculum materials in a way that was more responsive to her students versus strictly following the curriculum, as she had in her first year (when she was in the FO quadrant):

> So, the benefit of teaching the same grade two years in a row is that I... Like last year I would say I was more like, "OK, what does the book tell me to do?" Where this year, I'm more like, "OK, what do the kids need?" And that has helped a lot to not sound robotic, I guess. And make sure that what the kids are saying are actually what they're saying instead of like, "Oh, the book says that they have to say this, and I think I heard that." Or rephrasing it into like what the book wants them to say. And so the structure of the lesson was very similar to what

> the series structure is, where they start with a collaborative solve and share and then move into some independent practice.

In this excerpt, the teacher describes the elements of the curriculum series she still follows closely (e.g., the lesson structure), but also identifies ways in which both her planning and her classroom interactions are now focused on listening and responding to her students instead of doing what the curriculum "tells" her to do and expecting her students to always say what the book indicates they will say.

Significance

We found that these five beginning teachers changed how they used mathematics curriculum materials over the first three years of teaching. These teachers tended to make a transition from FO (Focused Offloading) or DI (Distributed Improvisation) to FI (Focused Improvisation) in the second and third years. These findings can contribute to the research on teachers' mathematics curriculum use in several ways.

First, this paper offers the field ways to integrate perspectives from research conducted in the context of English Language Arts (Valencia et al., 2006) and science education (Forbes & Davis, 2010) to understand beginning teachers' mathematics curriculum use in their first three years. As mentioned in the introduction section, the literature on teacher-curriculum interactions has explored (novice) teachers' curriculum use in the short term. This current paper allowed us to understand teacher-curriculum interactions in mathematics curriculum in the longer term (their first three years). Our findings differed from those of Valencia and colleagues (2006), which found little change in their first three years. Besides the different nature of mathematics and English Language Arts, one reason for this difference might be that since Valencia and colleagues' study (2006), mathematics teacher educators have been improving ways to help beginning teachers modify and adapt their mathematics curriculum materials, for example, by supporting prospective teachers in using educative curriculum materials in mathematics (e.g., Drake, Land, & Tyminski, 2014). In addition, as mentioned above, the curriculum landscape has changed substantially in the past 15 years. Building on this finding, further research can use the perspectives to understand m in mathematics curriculum use of larger numbers of beginning teachers in their early years.

Second, this paper contributes to the field's conceptual understanding of what a yearly trajectory of individual beginning teachers can look like in their early years. In another paper (Pak & Drake, 2020), we investigated a case of this trajectory with one beginning teacher in her first three years. In this current paper, we extended the number of beginning teachers to five teachers and analyzed excerpts we obtained from these beginning teachers in Year 1 through Year 3. The finding related to the general patterns (Figure 2) provides the field with an initial understanding of beginning teachers' mathematics curriculum use in their first three years. Our finding related to individual teachers' yearly trajectories (Figure 3) also suggests that, within the general patterns, each teacher illustrated a unique story from Year 1 through Year 3. Nonetheless, all five teachers modified and adapted their mathematics curriculum in their third years, suggesting a potential area for the future research in relation to a deeper understanding of individual teachers' mathematics curriculum use.

Lastly, this paper also contributes to developing how to analyze ways beginning teachers use their mathematical curriculum in their first three years. As an analytic tool, we drew on the framework Forbes and Davis (2010). In particular, we analyzed the teachers' movement along the quadrants in the framework (Figure 1). In our prior work (Pak & Drake, 2020; 2021), this framework illustrated well how beginning teachers used their mathematics curriculum. In this current paper, we conceptualized the yearly trajectory in terms of a coordinate graph where we could see each beginning teacher's movement from one quadrant to another (e.g., FO → DI → FO). We think that analyzing and visualizing data in this way shows a way to understand how beginning teachers use curriculum materials in relation to mathematics as well as other content areas (e.g., science and ELA), which would be a contribution to teacher education in general.

In addition to theoretical, conceptual, and methodological implications above, this paper contributes to mathematics teacher education. By providing a detailed understanding of the range of ways in which beginning teachers use curriculum materials in their early years, this paper can support mathematics teacher educators to better prepare prospective teachers to mobilize and adapt their mathematics curriculum. On the whole, we hope that this study can contribute to a deeper understanding of how teachers, including novice teachers, interact with curriculum materials in mathematics (e.g., Remillard, 2005) in their early years of teaching.

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MATHEMATICAL MODELING COMPETENCY IN UPPER-ELEMENTARY: VALIDITY EVIDENCE AT THE ITEM LEVEL

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Mathematical modeling is a high-leverage topic that involves connecting real-world situations, phenomenon, and/or data with mathematical models. Despite broad consensus of its importance in K-12 mathematics curriculum, instruments measuring mathematical modeling in elementary grades have remained scarce. We addressed this need by designing the Mathematical Modeling Student Assessment (MMSA) for students in grades 3 through 5. In this brief report, we investigate evidence supporting the use of the MMSA for these targeted grades by performing differential item functioning (DIF; Penfield & Camilli, 2006) to investigate potential bias which may impact score interpretations. We describe results and discuss implications for the use of the MMSA, as well as future measurement of elementary mathematical modeling.

Keywords: Assessment, Research Methods, Modeling

Mathematical modeling is a high-leverage topic, involving connecting real-world situations, phenomenon, and/or data with mathematical models (Council of Chief State School Officers & National Governors Association Center for Best Practices, 2010). To date, most of the measurement development and subsequent analyses have focused on upper grade levels. Zöttle and colleagues (2011), for instance, designed and administered an assessment of modeling competency to grade 8 students. To provide a modeling assessment focused on elementary grades, we developed the Mathematical Modeling Student Assessment (MMSA). Previously we have reported on the development (Turner et al., accepted) and presented initial validation results (Turner et al., 2021) based on the five forms of validity evidence (AERA et al., 2014).

Validation, however, is an ongoing process and is only as strong as the evidence supporting the inferences and assumptions foundational to the use of an instrument (Lavery et al., 2019; Zumbo, 2006). As the design of the MMSA is meant to capture mathematical modeling competency for upper elementary students (grades 3, 4, and 5), further validity evidence is needed to support the assumption that the scores are uniformly interpretable (i.e., lack bias). Performing a differential item functioning (DIF) analysis, we focused on the following research questions: (1) Which items, if any, are identified as meaningfully functioning differentially across the grade bands? (2) For items identified as demonstrating meaningful DIF, does the effect reflect relevant or irrelevant differences in mathematical modeling competency?

Perspective

Validity has received increased attention in recent years to ensure high quality measures are being used in mathematics education research (Krupa et al., 2019). A key point in this effort is that validity is not a function of in the instrument itself (Kane, 2013; Lavery et al., 2019; Zumbo, 2006); the underlying assumptions and inferences about the instrument require evidence to support the intended use and interpretation of scores. One approach for providing evidence, DIF analysis, investigates the impact of construct-irrelevant variance due to group membership on the measurement of a targeted construct (Penfield & Camilli, 2006). In short, DIF investigates the impact group association (such as different grade levels) has on score interpretation. DIF has evolved to include statistical methods for identify potential violations which are then examined to understand why particular items may or may not demonstrate DIF (Zumbo, 2007). After identifying stratifying variables (i.e., grouping variables), researchers select appropriate statistical methods for analysis. If DIF is detected, the analysis is rerun, omitting problematic items to ensure the remaining items do not include DIF that was masked by other items (Penfield & Camilli, 2006).

Crucially, these results are not taken at face value. Items deemed to be understandably differential are not removed if the difference in scores reflects the underlying trait (i.e., construct-relevant variance). For instance, grade 3 students might simply have less modeling experience that makes certain items more difficult. On the other hand, if an item reflects a difference outside the construct of interest (i.e., construct-irrelevant variance), the item will need to be changed or removed as it would bias scores and affect interpretations. For example, a complex word problem may benefit grade 5 students because of their higher vocabulary knowledge, not their modeling competency. In this case, the mathematical modeling measurement might be biased, reflecting vocabulary ability. Although preventing such issues was a focus during the MMSA development, further analysis provides stronger validity evidence.

Methods

The data used for the analysis came from the second administration of the MMSA to students in grades 3, 4, and 5. After removing students whose assessments included more than four items with no response which indicated students were unable to complete the assessment for extraneous reasons (e.g., pulled to another classroom or dismissed), our final data included 737 tests collected from two states, including: 320 in grade 3, 250 in grade 4, and 167 in grade 5. Student participants mirrored the demographics of our participating urban schools (with racially, culturally, and linguistically diverse populations) and were roughly consistent across grades.

The MMSA consists of 9 dichotomously-scored (i.e., 0 or 1) multiple-choice (MC) items and 4 polytomous-scored (i.e., 0, 1, 2, etc.) constructed-response (CR) items. The test form was consistent for students in each grade[1]. To calculate a mathematical competency trait score (represented by θ), we used item response theory (IRT) resulting in a best fitting model combination of a two-parameter and generalized partial credit model (for more information about the MMSA see Turner et al., 2021, accepted). The resulting modeling competency estimates can be roughly interpreted as standard deviations (e.g., $\theta = .5$ is roughly .5 standard deviations above average).

For the statistical analysis, we used the *lordif* package (Choi et al., 2011) in *R* version 4.03 (R Core Team, 2020). To identify DIF, this package uses likelihood ratio (LR) tests to first test for uniform DIF by comparing two regression models: one with ability (M1) as a covariate and another with both ability and group membership as covariates (M2). Uniform DIF reflects a consistent difference in scores for all ability levels (e.g., an item is predicted to be more difficult for everyone in grade 3 relative to grade 4). To test for non-uniform DIF, an interaction term of ability and group is included (M3) and compared to M2. Non-uniform DIF indicates differentiation relative to competence. For instance, students estimated to have low competency in grade 3 may have a relative advantage for getting the item right but then high competency students would have a relative disadvantage for the same item. For the LR tests, the null hypothesis states there is no DIF. At a critical level of .05, a statistically significant comparison would identify an item as demonstrating DIF. We then used pseudo-R^2 as a measure of effect size to determine if the statically significant result is meaningful.

Results

Table 1 shows the statistical results for the four MMSA items flagged for DIF. Each of these items had very small effect sizes (Sawilowsky, 2009) meaning that the individual impact on the final scores has little to no practical effect on the final modeling competency estimates. We also found the collective impact had little to no meaningful impact on final competency estimates. The mean difference between the estimates of the original model and a model account for DIF was essentially 0 (sd < .001) with no difference greater than the median standard error of competency estimates (.378). Thus, we concluded that there was no meaningful association between group membership and modeling competency estimates which supports the use of the measure across the target grade levels.

Although we found no practical effect, we proceeded on to analyze the items further to better understand why these statistical results might have occurred to inform future development. CR items 4 and 13 demonstrated uniform DIF. The DIF in item 4 reflected the lower modeling competency required by grade 3 students ($\theta = -.87$) to have a .5 probability to transition from a score of 1 to 2 compared to students in grades 4 ($\theta = -.27$) and 5 ($\theta = -.13$). We reasoned that the DIF likely reflects grade 4 and 5 students embedding more difficult calculation (e.g., trying to directly apply the long division algorithm) into their modeling approach which caused more errors compared to grade 3 students who applied concrete approaches such as visual models.

Table 1. Items Flagged for Statistically Significant DIF

Item	Question Type	χ^2 M1 vs. M2	R^2	χ^2 M2 vs. M3	R^2
3	MC	.004*	.012	.022*	.009
4	CR	.005*	.007	.805	<.001
9	CR	.990	<.001	<.001*	.012
13	CR	<.001*	.013	.254	.001

Note. Only items demonstrating DIF are included in the table. * indicates a significant χ^2 value $p < .05$.

The uniform DIF of item 13 was the result of a higher difficulty in transitioning from a score of 3 to a score of 4. Grade 3 students are estimated to need modeling competency of around 2.95 for a .50 chance of scoring a 4 compared to grade 4 ($\theta = 1.60$) and grade 5 ($\theta = 1.95$) students. Although the score required slightly lower competency estimates for grade 4 than grade 5, grade 3 required a full standard deviation more. We reasoned that this item captures a true difference in modeling sophistication by asking students to complete multiple steps in the modeling process (i.e., identifying relevant quantities, constructing and operating on models, and interpreting results), making this item reasonably more difficult for grade 3 students and roughly equivalent for grade 4 and 5 students.

CR item 9 demonstrated non-uniform DIF indicating that grade 4 students matched to students with equal modeling ability do not always have the same probability answering the item either partially or completely correct. This interaction effect is reflected in the differential item information (*a)*, or slope, estimates for the grade 4 ($a = 3.12$) compared to grade 3 ($a = 2.35$) and grade 5 ($a = 2.44$). Concretely, grade 4 students are predicted to have a lower probability of scoring a 1 and then a higher probability of scoring a 2 compared to grade 3 and 5 students with the same competency estimate. It is difficult to determine what led to the differential probabilities for grade 4 students. The very small effect size makes this item unproblematic overall, but further analysis may be informative for developing future CR items.

Finally, MC item 3 demonstrated uniform and non-unform DIF. While the difficulty for this item was roughly the same for grade 4 (θ = -.98) and grade 5 (θ = -.94) students, the item was more difficult for grade 3 students (θ = -.64). The non-unform DIF resulted from students in low modeling competency grade 5 students having a lower probability to answer the item correctly compared to grade 3 and grade 4 students. However, grade 5 students with average to high modeling competency have higher chances of answering correctly. Both the uniform and non-uniform DIF pattern reflected our expectation for this item. Grade 3 students have a familiarity with the money concepts required in this problem, but decimals as a mathematical concept are not introduced until grade 4 with more opportunities for learning in grade 5. The DIF pattern likely reflects the increasing familiarity students have with the content of this item.

Discussion

Overall, our analysis identified 4 items with DIF: items 4 and 13 for uniform DIF, item 9 for non-uniform DIF, and item 3 for both uniform and non-uniform DIF. The effect sizes for each of these items were negligible with pseudo-R^2 ranging from .007 to .013 suggesting these flags will not affect modeling competency estimates (de Ayala, 2009). The collective effect of these items was also not meaningful. Total score estimates had an average change of 0 with no scores impacted beyond the median standard error of measurement. Taken together, this provides further validity evidence to support the use of the MMSA in grades 3, 4, and 5.

Despite not finding DIF with large effect sizes, further analysis of items 3, 4, and 13 demonstrated the differences likely reflect differences in modeling. We are unsure, however, about item 9. Although it is possible this difference has no explanation, it is also likely that the item captures something tangential, such as understanding of division, which is important to consider in designing future items of this type.

At least two limitations are worth considering in these results. First, a larger sample size would provide more robustness to the results. Given the small effect size estimates, it is possible one or more of the flags were false positives. Second, this analysis assumes homogeneity of students within the grades. Further analysis is needed to investigate any impact or bias that may result from the racial, cultural, and linguistic diversity of the students within each grade.

Conclusion

As underlying populations become more diverse, DIF analyses become an increasingly important component of the ongoing validation process (Gómez-Benito et al., 2018). Understanding how consistent or inconsistent items perform as a function of this diversity provides a more robust collection of evidence for appropriate uses and limitations of instruments. Here, our DIF analysis provided further evidence for a uniformly interpretable modeling competency scores for students in grades 3, 4, and 5. We additionally gained insight into how item designs reflected our assumptions about modeling competencies. By performing similar DIF analysis, a more robust collection of validation evidence can be collected for instruments like the MMSA and provide insights into future directions for instrument development.

Note

[1] Due to a printing error, an item was omitted for 48 grade 3, 83 grade 4, and 64 grade 5 students. IRT is able to handle missing data and we were able to produce estimates for these students (de Ayala, 2009).

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VALIDATING AN ASSESSMENT OF STUDENTS' COVARIATIONAL REASONING

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In this mixed methods study, we validated a fully online assessment of students' covariational reasoning. We combined qualitative and quantitative methods to analyze 30 responses from undergraduate college algebra students during individual task based interviews. Our findings were statistically significant; students' total number of items correct could be explained by their evidence of covariational reasoning. We conclude with discussion of our work moving forward.

Keywords: Assessment, Technology, Algebra and Algebraic Thinking

Covariational reasoning is a high leverage form of reasoning, which can foster students' understanding of key mathematical concepts, such as rate and function (Carlson, 2002; Thompson & Carlson, 2017). Function is central to the undergraduate college algebra curriculum; hence, promoting students' covariational reasoning dovetails with the goals of the course (Olson & Johnson, 2021). The mixed methods study we report is part of a larger research project designed to promote college algebra students' engagement in covariational reasoning.

We explain how we validated a fully online assessment of students' covariational reasoning. Combining qualitative and quantitative methods, we analyzed 30 responses from undergraduate college algebra students taking part in individual, task-based interviews. Our findings were statistically significant; students' total number of items correct could be explained by evidence of their covariational reasoning. We conclude with discussion of our work moving forward.

Theoretical and Conceptual Background

We ground the construct of covariational reasoning in Thompson's theory of quantitative reasoning (Thompson, 1994), which explains how individuals can conceive of attributes of objects as being possible to measure. For example, consider a toy car moving around a square track. There are a variety of attributes to which students might attend, such as the toy car's total distance traveled around the track or the toy car's distance from a center point. A student engaging in quantitative reasoning could conceive of the attributes as possible to measure. Thompson (1994) calls such a conception a "quantity." For instance, a student may mark off lengths of string to measure one of the distances. To engage in quantitative reasoning, it is sufficient for students to conceive of the possibility of such measurement; they may or may not do the measuring itself.

By variational reasoning, we mean students' conceptions of a single attribute that is not only possible to measure, but also capable of varying (Thompson & Carlson, 2017). For example, a student could conceive of the toy car's distance from a center point as increasing and decreasing, as the car moves around the track. By covariational reasoning, we mean students' conceptions of relationships between attributes that are capable of varying and possible to measure (Carlson et al., 2002; Thompson & Carlson, 2017). For example, a student could conceive of a relationship between the different distances; for instance, the toy car's distance around the track continues to increase while the toy car's distance from the center point increases and decreases.

Students' variational and covariational reasoning intertwine with their conceptions of what Cartesian graphs represent (Johnson et al., 2020). For example, a student engaging in covariational reasoning could interpret a graph as representing a relationship between quantities, such as toy car's total distance traveled and distance from the center. Yet students may conceive of graphs as representing the motion of an object (Kerslake, 1997) or a literal depiction of an object (Leinhardt et al., 1990). For example, students may conceive of a graph of the toy car situation as representing the motion of the toy car around a track or as the literal track itself.

Drawing on Piaget's theory, Moore et al. (2019) distinguished between individuals' figurative and operative thinking when sketching and interpreting Cartesian graphs, as well as graphs with unconventional coordinate systems. Individuals who conceived of graphs as representing literal motion or depictions of objects would engage in figurative thinking. In contrast, individuals who conceived of graphs in terms of quantities and relationships would engage in operative thinking.

Johnson et al. (2020) developed a four-item coding framework in which they made distinctions between students' conceptions of what graphs represent: Covariation (COV), Variation (VAR), Motion (MO), and Iconic (IC). Rather than making fine grained distinctions within covariational and variational reasoning, Johnson et al. (2020) targeted particular levels posited by Thompson and Carlson (2017): gross variation and gross coordination of values. The first marked a student's conception of a quantity as being capable of varying; Johnson and McClintock (2018) called this type of reasoning quantitative variational reasoning. The second marked a student's shift from conceiving of variation in individual quantities (e.g., this one, then that other one) to forming a relationship between quantities (both quantities vary together). Put another way, the COV and VAR codes would evidence operative thinking, while the MO or IC codes would evidence figurative thinking, per the constructs of Moore et al. (2019).

Methods

Assessment Design

To design the covariation assessment, we have integrated different theoretical perspectives and consulted experts in the field (Johnson et al., 2018). The assessment, developed in Qualtrics, is fully online; students can complete it on smartphones, tablets, or computers. There are four items, appearing in random order. Each item incorporates a situation involving changing attributes, with two question groups per item (Table 1). The situations include a turning Ferris wheel (Ferris Wheel item), a person (Nat) walking on a path to and from a tree (Nat + Tree item), a fish bowl filling with water (Fish Bowl item), and a toy car moving around a square track (Toy Car item). We have incorporated innovative elements, including items containing unconventional graphs that do not pass the vertical line test (e.g., Moore et al., 2014).

Table 1: The Covariation Assessment: Item Question Groups

Question Groups	Description
1: Comprehension Check	Play video animation of the situation. State if you understand the situation. If yes, move to question group 2. If no, explain why.
2: Select Graph and Explain	Select a graph (ABCD) that represents a relationship between attributes in the situation. Explain your choice.

Task Based Interviews: Description and Rationale

We hypothesized that students' covariational reasoning could explain their graph selections. Through task based interviews (Goldin, 2000), we were able to gather evidence of students' reasons for their graph selection, and make inferences about their covariational reasoning. If students were selecting correct graphs for reasons not associated with covariational reasoning, for instance, by appealing only to physical features of graphs rather than quantities being represented, we could use that information to revise assessment items.

Data Collection

Participants were undergraduate students enrolled in college algebra at a university located in the metro area of a large U.S. city, serving large percentages of students of color and first generation to college. We selected three groups of 10 participants, interviewed in consecutive spring, summer, and fall semesters. The interviews took place midway during the semester. Student volunteers left for part of class to go to a nearby office for the interview. Our selection method resulted in a not entirely random sample; students who participated in interviews may have been those more invested in the course or more willing to talk about their thinking. Johnson conducted 30 individual interviews, which were video and audio recorded. One graduate research assistant (GRA), either Gardner or Smith, observed each interview. The GRA engaged in two activities: monitoring the video camera and writing field notes. Johnson used a semi-structured protocol for the interviews, to gather evidence of three areas: students' comprehension of assessment questions; students' engagement in covariational reasoning; and students' experiences with the technology. GRAs used the protocol to organize their field notes. Because we designed the covariation assessment to work on smartphones, tablets, and computers, we wanted students to work across a range of devices. Students could bring a device of their choosing. If students did not bring a device, or if students wanted a different device, we had a tablet and a laptop computer available for use. We allowed students to choose a device, because that is what students would do during the actual assessment.

Data Analysis

Qualitative analysis. Johnson led the qualitative analysis, adapting the four-item coding framework from Johnson et al. (2020). Table 2 shows the codes, descriptions, and examples. The COV and VAR codes are bolded, because they represented conceptions of graphs in terms of quantities and relationships. When students reasoning was coded as COV or VAR, they provided evidence of engaging in reasoning consistent with at least the levels of gross variation (VAR) or gross coordination of values (COV), per Thompson and Carlson (2017). When students' reasoning was coded as MO or IC, they provided evidence of a conception of the object's literal motion (Kerslake, 1977) or an object's physical appearance (Leinhardt et al., 1990).

Table 2: Four Item Coding Framework, Adapted from Johnson et al. (2020)

Code	Description	Example
COV	**Relationship between two attributes capable of varying and possible to measure**	**As the toy car's total distance traveled increases, the toy car's distance from the center decreases then increases.**
VAR	**A single attribute capable of varying and possible to measure**	**The Ferris wheel cart's distance from the center gets larger, then smaller.**
MO	Literal motion of an object	Nat walked back and forth, so the graph goes back and forth
IC	The shape of an object	The graph is shaped like a fishbowl.

Johnson watched each student's interview video, drawing on what students said and did as sources of data. For each item, in the order they appeared for each student, Johnson assigned students a single reasoning code. If students demonstrated multiple forms of reasoning, Johnson assigned the most sophisticated one as the overall code, to indicate if students were conceiving of quantities and/or their relationships. For example, if a student said that the graphs showed how the toy car moved around the track (MO), then went on to explain how each of the distances were changing together (COV), Johnson coded the reasoning COV. Prior to our quantitative analysis, Johnson shared the codes and notes with another researcher as a member check.

Quantitative analysis. Analysis of variance (ANOVA) was used to analyze device effects. A simple linear regression model, at 95% confidence, was used to analyze the relationships between students' total number correct and evidence of their covariational reasoning.

Results

Students used a range of devices on the assessment: 13 used a computer, 10 a tablet, and 7 a smartphone. The choice of device did not impact students' reasoning. Students who selected a computer or tablet expressed a preference for a larger screen size or touch screen capability. Students who selected a smartphone preferred to use their own device, which they had with them. None of the students expressed dissatisfaction with the interface on their device.

Across the four items, students selected a correct graph 48% of the time; 55% of student responses provided evidence of covariational reasoning (COV), 23% variational reasoning (VAR), 25% motion reasoning (MO), and 1% iconic reasoning (IC). Across the 30 students, 10% got all four items correct; 23.33% three items correct, 23.33% two items correct, 40% one item correct, and 0.34% zero items correct. The correlation between the total number correct and students' evidence of covariational reasoning was .709, which indicated a significantly high degree of correlation ($p < .01$). The regression analysis also showed that 50.2% of the total variation in the dependent variable, students' total number correct, can be explained by their covariational reasoning, which is also statistically significant ($p < .001$).

Discussion

This fully online assessment, containing built-in video animations, and designed for the undergraduate college algebra population, is the first of its kind. Our results demonstrate its validity, for assessing covariational reasoning, at least at the level of gross coordination of values, per Thompson and Carlson (2017). Furthermore, distinguishing between students engaging in variational or covariational reasoning, rather than motion or iconic reasoning, could be useful for diagnosing students' figurative or operative thinking (Moore et al., 2019) on graphing tasks.

We are encouraged that students' choice of device did not impact their reasoning. Given the prevalence of mobile phones, the feasibility of the assessment for use on this type of device can increase its potential for usability.

Continued validation work with a larger sample size is a vital next step. We are expanding the assessment to include six items and examining the qualitative coding scale via Rasch modeling to quantitatively corroborate its hierarchical nature.

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"I FAILED TO REMEMBER WHAT I SHOULD HAVE LEARNT." RESOLUTION OF MISCONCEPTIONS DURING COMPARATIVE JUDGEMENT

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Learning from worked examples is a highly effective pedagogical approach. In this study, we explore a new way of learning from worked examples: comparative judgement. This involves students assessing other students' work in pairs and deciding which of the two they regard as 'better'. We discuss the experience of one student, Rhys, with low prior knowledge as he evaluated a mathematics problem on rational inequalities. Results suggest that Rhys was unable to notice key features of the work he assessed that might have helped resolve his existing knowledge gaps, despite appearing to understand the underlying concepts. We suggest that Rhys was 'metacognitively blind' to the features that might have helped resolve his knowledge gaps and that providing him with problems to practice in between comparisons might have been beneficial.

Introduction

Comparative judgment is a relatively new assessment approach. It involves assessing students' work in pairs from which assessors judge which of the two solutions is 'better'. A reliable rank of students' work from 'best' to 'worst' is then formed after several assessors complete multiple judgements. Most research has focused on the use of comparative judgement as an assessment tool showing that it can be used to form a reliable ranking of students' work that is comparable with traditional marking methods (e.g., Jones et al., 2014, 2019). Recent studies have advocated for its use as a learning tool, typically in the context of peer-review (Bartholomew & Jones, 2021; Strimel et al., 2020). Students review their peers' work comparatively, often giving and receiving feedback that they can then apply to improve their own work. It is argued that the process of comparing makes important features more noticeable for the learner (Holyoak, 2012) thereby making it more likely they will apply such features to their own work. Such an approach seems promising, with many studies reporting that students find comparative judgement to be valuable and worthwhile (e.g., Potter et al., 2017). To support the assumption that comparative judgement might improve student outcomes, we draw upon the literature on learning from worked examples.

Learning from worked examples

Instructional designs where students learn by practicing unfamiliar problems place heavy demands on working memory. When practicing problems learners often must rely on their own ability to find a set of steps that can lead them to the desired goal, because no previous models are likely to be available to them. This is known as a means-end analysis. The learner is required to consider the current problem state, the final goal state, evaluate differences between the two states, and find a set of steps that can be used to move from the current state to the goal state. This process imposes a heavy load on working memory and will generally not lead to learning.

On the other hand, learning from worked examples involves providing students with the following: the problem itself, the steps taken to reach a solution, and the final solution (Renkl, 2014). Students then study and reflect on multiple worked examples before they are expected to try answering similar questions on their own (Große & Renkl, 2007).

Studying worked examples removes the means-end approach. Learners no longer need to search their own prior knowledge for solution methods and instead can focus their attention on current problem states. This reduces working memory and leaves cognitive resources available for self-explanation (Sweller et al., 1998). Only when self-explanation is encouraged, that is, when students explain the reasoning behind a solution, do worked examples appear beneficial (Chi et al., 1989; Renkl, 2014).

One of the key features of comparative judgement is that two solutions must be *compared*. Studies on learning from worked examples strongly suggest that presenting worked examples simultaneously to encourage comparisons between them leads to greater learning outcomes than presenting worked examples one-by-one (Rittle-Johnson & Star, 2007; Star & Rittle-Johnson, 2009). By making all solutions available at the same time, cognitive load is reduced as students no longer need to hold representations from the previous example active in their working memory to compare with the next example (Begolli & Richland, 2016).

One reservation with comparative judgement is the exposure to incorrect examples. Studies have shown that providing students with both correct and incorrect examples is more beneficial to student performance than showing correct examples only (Adams et al., 2014; Booth et al., 2013; Durkin & Rittle-Johnson, 2012). Incorrect solutions can help students recognise incorrect strategy choice by drawing attention to the feature of the problem that makes the strategy inappropriate (Booth et al., 2013; Siegler, 2002), and improve students' error detection skills which is not possible from correct examples alone (Tsovaltzi et al., 2010). Additionally, drawing attention to errors may help students replace incorrect knowledge with correct knowledge (Chi et al., 1981).

Research design

Based on our argument from above, we assumed that comparative judgement would be useful for learning. We were also mindful of the fact that, in our study, because we did not prompt students to self-explain, it may have reduced the likelihood of positive learning outcomes.

Students completed a pre- and post-task where they solved the inequalities $\frac{x+1}{x-7} > 3$ and $\frac{5x-2}{x+5} > 6$. No feedback was provided to students following the pre-task nor were they provided with the correct answer. Following the pre-task, students were shown six pairs, one pair at a time, of other students' solutions of the same problem. Students were asked to choose which of the two solutions in each pair they thought was 'better'. What 'better' meant was left to each student to decide. As each pair of solutions was shown, students were asked to think aloud through their assessments (Ericsson & Simon, 1993). A short semi-structured interview followed once students had completed their six comparisons.

Eight first-year undergraduate students studying an introductory calculus subject in Australia participated in the study. Here we present the case of one student, Rhys, one of four students in the study who reported not finding comparative judgement helpful. We chose to focus on Rhys because he clearly articulated his thinking and provided insight into how comparative judgement could be improved in the future.

Rhys' experience of comparative judgement

Rhys was unable to complete the pre-task. His intended approach was to sketch the graphs of $y_1 = \frac{x+1}{x-7}$ and $y_2 = 3$ on the same axis and visually interpret when $y_1 > y_2$. Rhys did not know how to sketch the graph of y_1 and he did not think to (or know how to) rearrange $\frac{x+1}{x-7}$ as $1 + \frac{8}{x-7}$. Here we discuss Rhys' experience as he evaluated a comparative judgement solution that had used this same graphical approach.

Rhys appeared confused while evaluating this solution and was initially unsure how the student had used their graph to obtain the required interval of solution, $7 < x < 11$.

Rhys: But that doesn't quite… make sense. How did they… what have they done?

He slowly re-read the solution, explaining the solution to himself. This led to an Aha! moment.

Rhys: So, it's… oh. Oh! That's genius. Ok I like this solution then.

Following his moment of realisation, he then provided a full explanation of what the student has done:

Rhys: By drawing the hyperbola, they've managed to figure out that once you draw in a line at $y = 3$, you can make that inequality equal to three and so you know the lower bound. So, you know that's out of the range already [Points at section of graph where $y_1 < y_2$]. So, then you've got 7, to the intersection at 11.

Rhys' explanation suggested that he understood that solving for $y_1 = 3$ rather than $y_1 > 3$ gave the point of intersection between y_1 and y_2, allowing the student to know exactly when $y_1 > y_2$.

Comparing Rhys' pre- and post-tasks, his post-task showed little improvement. During the post-task, Rhys tackled the problem using the same graphical approach he had used during the pre-task but was once again unsure how to sketch the hyperbola. This contradicted what we would have expected from the literature. We anticipated that the combination of an Aha! Moment followed by a detailed explanation of the reasoning behind the solution would have left Rhys primed for learning (Chi et al., 1989). We suspect the lack of improvement was because Rhys overlooked a key part of the worked solution that might have helped him. While he had explained how the student had used the graph to find the solution, he did not make any comments regarding how the student had drawn the graph in the first place.

When asked about any changes he had made between his pre- and post-tasks, Rhys was forthcoming in discussing why comparative judgement may not have been helpful in improving his second solution.

Rhys: I can follow instructions. But they don't stick around long enough in my brain when the instructions are no longer in front of me. So, I literally looked at someone's answers and I still couldn't apply it to another question.

Int: So, you don't think you really changed your method here?

Rhys: No. Cause it's more like my actual mathematical ability rather than memory being tested overall and it didn't change at all between questions. I failed to remember what I should have learnt from the other questions, I think.

Rhys' perception was that he was able to understand and follow the pairwise solutions but was unable to apply what he felt he should have learnt to his own solution. We argue that Rhys overlooked key parts of the solutions that would have facilitated his understanding. This draws on the notion of *abstraction* which Hershkowitz, Schwarz, and Dreyfus (2001) define as "a process in which students vertically reorganize previously constructed mathematics into a new mathematical structure" (p. 195). Abstraction involves three processes: 1) *Constructing* involves integrating abstracted knowledge together to form new knowledge; 2) *Recognising* involves identifying the mathematics that is relevant to the problem; and 3) *Building-with* involves using the mathematical procedures in a new context. We suggest Rhys was able to *recognise* (some but not all) rather than *construct* relevant underlying concepts as he engaged with the graphical solution but was not given the opportunity to immediately use these techniques in a new context. If unable to *build-with*, it is likely that Rhys was unable to reconstruct short-term existing knowledge structures. Subsequently, the process of abstraction did not occur ("I failed to remember what I should have learnt"). For Rhys, comparative judgement provided a space to recognise, but not to build-with, ultimately limiting his ability to construct new insights. Given Rhys was aware of some of his own pre-existing knowledge gaps, we expected Rhys to actively look for queues from the presented solutions to help resolve these gaps. Rather, Rhys failed to notice specific parts of the solution that would have been helpful to him (Chi et al., 1989). In short, Rhys was *metacognitively blind* (Goos, 2002).

Implications for teaching

Comparative judgement may need to be paired with practice problems students can work on immediately following the activity to allow time to build-with. We wonder whether digital-based tutoring systems used when learning from worked examples, such as those described by Adams et al. (2014) and Booth et al. (2013), could be embedded within comparative judgement. Such systems typically encourage self-explanation with the use of drop-down menus. Students complete a sentence about a given worked example by selecting from a number of options from a pull-down menu which are designed to support students' construction of self-explanations. In our context of comparative judgement, drop-down menu prompts could be designed to direct students' attention to aspects of solutions that might be helpful for common misconceptions. Of course, designing scaffolding of this nature may be impractical for a comparative judgement setting. If using students' work for authentic peer-assessment, generating drop-down options specifically tailored for each pairing is unrealistic. Rather, educators might include two or three pre-designed solution pairs with self-explanation prompts as part of the overall comparative judgement process.

Conclusion

Rhys' case-study offers insight into the limitations of comparative judgement for the purposes of knowledge acquisition. Rhys not only generated self-explanations but generated correct explanations. However, because Rhys was unable to immediately practice and apply any skills or concepts he had recognised (build-with), he was unable to apply what he should have noticed from the comparative judgement activity to the post-task.

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THE INTRODUCTION AND DEVELOPMENT OF TRIANGLE CONGRUENCY IN CHINESE AND US TEXTBOOKS

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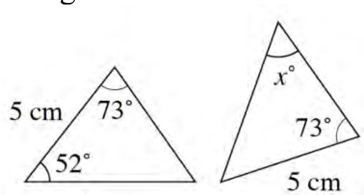
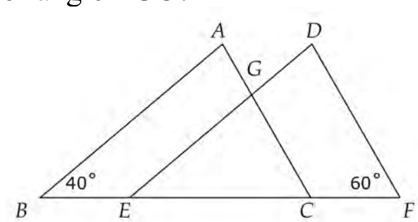
In this paper, we report findings from a comparative analysis of the introduction and development of triangle congruency from two countries: an eighth-grade mathematics textbook from China and a high school geometry textbook from the U.S. While both textbooks considered triangle congruency as a fertile setting for the development of reasoning and proving, there are both similarities and differences in their approaches to achieve this goal. Building from the transformation conception of congruency, the U.S. textbook provides students with more opportunities to practice writing proofs. Drawing from multiple conceptions of corresponding conceptions of congruency, the Chinese textbook provides students with more varied opportunities to write proofs and make connections to a real-life context.

Keywords: curriculum, geometry and spatial reasoning, triangle congruency

The concept of congruency is an important topic in geometry. Euclid's Elements included three conditions for triangle congruency: side-angle-side (SAS), side-side-side (SSS), and side and two angles (SAA) as propositions 4, 8, and 26. These conditions were then used to prove many subsequent propositions. Ever since, triangle congruency has become a constant feature in upper-level geometry lessons worldwide (Jones & Fujita, 2013). In addition, triangle congruence provides a rich setting for developing students' geometric intuition and encouraging them to form conjectures that create the necessity for the proofs associated with these conjectures (Wang, Wang & An, 2018).

However, students worldwide have found triangle congruence a challenging topic (e.g., Wang et al., 2018). For example, only 35% of the eighth graders participating in the TIMSS 1995 study were able to answer the question in Figure 1a correctly. About 80% of the TIMSS 2003 participating eighth graders from high-achieving Asian countries such as Korea answered the question in Figure 1b correctly, while only 36% of their counterparts from the U.S. did so. The difference in the performance data might be the result of the different approaches mathematics textbooks have adopted to present triangle congruence theorems.

These triangles are congruent. The measures of some of the sides and angles of the triangles are shown. What is the value of x? 	In this figure, triangles ABC and DEF are congruent with $BC = EF$. What is the measure of angle EGC? 
Figure 1a: A TIMSS 1995 released item (Beaton et al., 1996, p. 32)	**Figure 1b: A TIMSS 2003 released item (IEA, 2005, p. 71)**

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An in-depth analysis of the introduction and development of triangle congruency theorems in the U.S. and China may provide helpful insights that are critical in promoting student learning of triangle congruence theorems.

Theoretical Perspectives and Prior Studies

González and Herbst (2009) identified four different student conceptions of congruency while using dynamic geometry software. The perceptual conception of congruency determined two figures were congruent by their look, whereas the measure-preserving conception of congruency checked whether the two objects had the same measures to determine the congruency. The transformation conception of congruency concludes that two objects are congruent if there exists a sequence of rigid transformations that can map from one object to the other object. Finally, there is the correspondence conception of congruency, which is supported by traditional teaching practice. González and Herbst asserted that the correspondence conception of congruence is a special manifestation of the transformation conception of congruency.

Jones and Fujita (2013) developed a framework based on these four conceptions of congruency to analyze the introduction and development of triangle congruency in an eighth-grade Japanese textbook. They found that while the Japanese textbook expected students to use various conceptions of congruency to explore ideas of congruent triangles experientially, the student experiences focused almost exclusively on the correspondence conceptions when constructing formal proofs. This approach, shared by many East Asian countries, was different from the approach adopted by the Common Core State Standards for Mathematics, which used the transformation conception as the basis for defining congruency (Usiskin, 2014). However, little is known about how textbooks introduce and develop congruent triangle theorems to their students based on these different approaches. This study will fill the gap by analyzing both the content and problems from the congruent triangle lessons from a Chinese and a U.S. textbook.

Methodology

This is a comparative case study (Stake, 2000) of a set of lessons on congruent triangles from both a U.S. and a Chinese textbook. This study intends to produce a dense narrative and interpretations that answer the central research question, "*How are the criteria of triangle congruence introduced and developed in the U.S. and Chinese textbooks*?"

The content of congruent triangle theorems was listed as a geometry topic for the secondary school, while the same content was found in the eighth-grade standards in the Chinese curriculum guidelines. The *Eureka Math Geometry* (Great Minds, 2015), a Common Core-aligned open resource U.S. textbook, and an eighth-grade mathematics textbook published by People's Education Press (PEP, 2013) were selected for this analysis because of the prominent roles they play in their respective countries. Each textbook devoted four lessons to the five congruent triangle theorems: SSS, SAS, ASA, AAS, and HL. They are lessons 22-25 in Eureka Math and section 12.2 in PEP Math. Focusing on the same content and the same number of lessons in each textbook made the comparison more compatible despite the grade-level difference.

The unit of analysis for this study was a textbook instance, as described by Teuscher et al. (2016), "as the way in which textbook content is delimited or sectioned by authors for the purpose of communicating ideas or providing students opportunities to engage with the mathematics content" (p. 3). The analysis was done in three stages. In the first stage, we identified the textbook instance in each lesson. We then went through each textbook instance focusing on the intended mathematical objectives as well as the conceptions of congruency. Finally, we went through the textbook instances again, searching for additional themes that were not captured by our previous coding. All the identified themes illuminated the introduction and development of congruent triangles in each textbook.

Findings and Discussion

Both textbooks were consistent in their individual lesson structures. Each lesson of Eureka Math opens with some opening exercises, followed by one or two explorations that address the main objectives of the lesson, and ends with a set of exercises that students can practice in class. Each lesson of PEP Math continues to have one or two explorations, work-out examples, and "think about it," although not always in this particular order. It always ends with a set of two exercises. Starting with different criteria for triangle congruence (SAS vs. SSS) naturally led to the different content sequence for each textbook, as can be seen in Table 1. However, the difference went beyond the logical necessity. In the remaining parts of this analysis, we will share three themes we have learned that set these two textbooks apart.

Table 1: Overall Content in Eureka Math and PEP Math

	Number of Lessons	Sequences of Criteria	Number of Exercise
Eureka Math	4	1. SAS	4
		2. Base Angles of Isosceles Triangles	5
		3. ASA & SSS	5
		4. AAS & HL, SSA* and AAA*	2
PEP Math	4	1. SSS	2
		2. SAS, SSA*	2
		3. ASA & AAS, AAA*	2
		4. HL	2

*Criteria that does not lead to congruency

Logical consideration. Both textbooks provide students opportunities to apply the triangle congruent theorems they just learned to prove other theorems. For example, both SAS and SSS were used to prove some properties of isosceles triangles. However, they were applied in a different manner. In PEP Math, SSS was used to prove that the two triangles created by connecting the vertex A and the midpoint of BC are congruent. This is an equivalent conclusion of "two base angles of an isosceles triangle are congruent," the focus of Lesson 23 in Eureka Math, which was proved by applying transformation and SAS. There is one distinct difference in the location of the criteria that did not lead to congruency. PEP Math examined the nature of a set of criteria immediately after a set of similar criteria being proved to be valid in proving congruency, for example, SSA right after SAS and AAS, and AAA right after ASA. Eureka Math, on the other hand, examined SSA and AAA after all five criteria for congruent triangles were established.

Conceptions of congruency. Transformation continued to play a significant role in proving congruency throughout the four lessons in Eureka Math. For example, in Lesson 23, Eureka Math provided two proofs for the "Base angles of an isosceles triangle are congruent" statement; one used transformation and one used SAS criteria to prove that the base angles of an isosceles triangle are congruent. Later in Lesson 24, a rigid transformation, the drawing of an auxiliary line, and the SAS were used to prove SSS. In PEP Math, compass-and-straightedge construction, which was based on the measure-preserving conceptions of congruency, played a similar role as that of transformation for Eureka Math throughout the four lessons. Both Eureka and PEP used previously proved congruent triangle criteria to prove the new ones, which was an approach based on the correspondence conceptions of congruency.

Exercises embedded in real-life context in PEP Math. In Eureka Math, all exercise problems are context-free problems. This characteristic is in sharp contrast to the many exercises embedded in real-life context in PEP Math. The contextual exercises enrich students' understanding of congruent triangles in the following areas: (a) understanding the mathematical rationales behind common measuring tools, (b) solving challenging indirect measurement problems and c) making sense of the distance relationship between angles and distances. Figure 2 below provide examples of two such exercises.

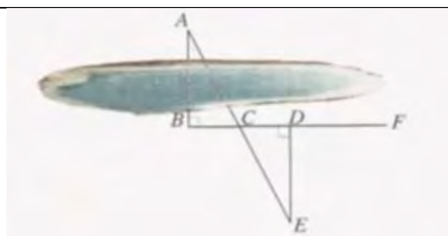
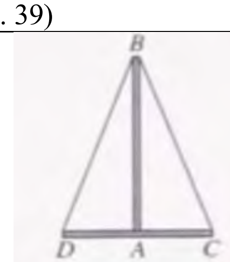
See the diagram below. To measure the distance between two points *A* and *B* from two sides of the pound. You can draw a line segment BF perpendicular to line AB. Identify two points *C* and *D* from BF such that *BC=CD*. Then draw a line segment *DE* perpendicular to BF such that points *A, C* and *E* are on the same line. Then DE and AB have the same length. Why? (PEP Math, p. 38)	See the diagram below. A north-side street connects points *A* and *B*. From point A, walk the same distance to arrive at points *C* and *D*. Will the distance from *C* to *B* equal to the distance from *D* to *B*? Why? (PEP Math, P. 39)
	

Figure 2: Examples of exercises in real-life context from PEP Math

Conclusions and Implications

In this paper, we investigated how two textbooks help their students to develop their concepts of triangle congruency. The analyses provided the main support for the following main conclusion: both textbooks were very consistent within their own approach but were also very different from each other. The difference went beyond whether they use transformation or correspondence conceptions of congruency. It also reflected on how they sequenced different topics, organized the in-class activities, and designed their end-of-lesson exercises.

Such information makes it possible to develop assessment items that are more sensitive to the country-level factor, such as the mathematics textbook, that might contribute to the difference in student performances. For example, Fan and colleagues (2017) found that the integration of transformation instruction, while created no significant difference in students' ability to solve general proof questions, students in the experimental group performed much better on challenging problems involving constructing auxiliary lines. A similar study with students using PEP and Eureka Math might shed more light on both the role of transformation in constructing geometric proof as well as establishing a better link between the textbook and student learning, as suggested by Fan et al. (2013) as a research area that needs more work.

This study also identified a few other areas that warrant further investigation, for example, the effect of presenting SSA right after SAS vs. after all the true congruent triangle criteria have been established on students' views of the necessity principle of proof. In addition, further investigation should be conducted on the possible effect of using exercises with real-life context on developing students' perseverance in mathematics learning with challenging mathematical topics similar to congruent triangles, which is the theme of the 2021 PME-NA conference.

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INVESTIGATION OPPORTUNITIES OF COMMON CORE THEOREMS PRIOR TO PROOF IN HIGH SCHOOL GEOMETRY TEXTBOOKS

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Even though prior research has found that textbooks provide limited opportunities for students to engage in proof in high school geometry, there is little research on the opportunities that textbooks provide for students to investigate and make conjectures prior to proof. This study addresses that gap in analyzing investigation opportunities for 17 theorems in the CCSSM across five textbooks. Overall, the five textbooks provide opportunities for investigation about half of the time, although there is a great deal of variation among the number of opportunities offered for each textbook. Theorems about parallel lines and triangles are more likely to have investigations than theorems about parallelograms and converses of theorems.

Keywords: Curriculum, Geometry and Spatial Reasoning, Reasoning and Proof, Standards

Purpose of the Study

The *Common Core State Standards for Mathematics* (*CCSSM*) includes proof as a significant part of school mathematics (NGA & CCSSO, 2010). Within high school geometry, the *CCSSM* states many theorems for students to prove about lines, angles, triangles, and parallelograms, and *CCSSM*'s Standards for Mathematical Practice contain many elements of proof.

The release of the *CCSSM* was a watershed moment in mathematics education in the United States. Because initially 45 states adopted them, there was nearly a de facto national intended curriculum defining "what students should be able to do" (p. 4). But the NGA and CCSSO (2010) were clear that the *CCSSM* "do not dictate curriculum or teaching methods" (p. 5). Therefore, a central question is how the *CCSSM* would be brought to "life" in the classroom. Textbooks, as the written curriculum, often become the implemented curriculum when teaching proof (Bieda, 2010; McCrone et al., 2002). Teachers also follow the teacher's edition closely for planning and pacing on proof (Sears & Chavez, 2014). Thus, research on textbooks is an important part of understanding student opportunities to engage with and ultimately learn proof.

Many studies have examined high school geometry textbooks (e.g., Hummer, 2016; Otten et al., 2013; Otten et al., 2014; Sears & Chavez, 2014). Despite the importance of proof, research consistently has provided evidence that textbooks offer limited opportunities for students to write proofs (Otten et al., 2014; Thompson et al., 2012). Despite studying chapters that were more likely to involve proof in geometry, less than 5% of all exercises coded involved asking students to construct a proof (Otten et al., 2014).

Even when textbooks asked students to write a proof, research revealed a limited exposure to the nature of proof—focusing on particular statements instead of general ones (i.e., proving specific cases instead of proving theorems). Hummer (2016) looked at two geometry textbooks for how proofs aligned with the *CCSSM*. Most proofs in the exercises required writing a proof of specific cases instead of general ones. Otten et al. (2013) found that there were more statements involving general proof statements in the exposition, but in the exercises, textbooks expected students to prove statements about a particular statement instead of a general statement.

Perspectives

Absent from the body of research on proof in textbooks is the opportunities students have for exploration and investigation prior to writing a proof. This type of reasoning that students might do leading up to proof is captured by Stylianides' (2008) framework for reasoning-and-proving. Stylianides (2008) proposed that the mathematical component of this framework could be a useful tool for research on textbook analysis because it views proof from a holistic standpoint rather than isolated from other activities in mathematics. Furthermore, prior research has shown that this treatment of proof has led to abysmal results for students learning proof (Harel & Sowder, 2007; Senk, 1985). Specifically, the mathematical component of the framework delineates four activities associated with proof: (a) identifying patterns, (b) making conjectures, (c) providing non-proof arguments, and (d) constructing proofs. Using the first three activities of the reasoning-and-proving framework as a lens for addressing this gap in the research on proof and textbooks, we report on the results that answers the question: How do high school geometry textbooks engage students with specific theorems listed in the *CCSSM* prior to formal proof? We report the results pertaining to the fourth activity of the framework of how textbooks provide opportunities to construct proofs elsewhere (Nirode & Boyd, in press).

Methods

Although there are a few additional theorems listed for students to prove in the Common Core High School Geometry Standards, we focused on the three standards listed in the *Prove Geometric Theorems Cluster* in the *Congruence Domain* for the basis of this research. These standards focus on lines and angles; triangles, and quadrilaterals G.CO.9, G.CO.10, and G.CO.11 (NGA & CCSSO, 2010, p. 76). These three standards include a total of 17 theorems.

We denote the five textbooks in our study with an asterisk (*) in the references. We refer to them using an acronym based on the publisher's name (e.g., BI, HMH, KH, MHG, and PE). Prior research on geometry textbooks and proof have used earlier editions of these five textbooks (i.e., Hummer, 2016; Otten et al., 2013; Otten et al., 2014; Sears & Chavez, 2014). We also chose textbooks from 2014 or 2015 so publishers had 4–5 years to align to the *CCSSM*.

We conducted both a priori coding and opening coding of the student editions of these five textbooks. We based our a priori codes on the first three parts of Stylianides' (2008) framework. We coded for (a) whether the textbook had students investigate the theorem prior to proof, (b) materials used in the investigation, and (c) whether the conjecture students made was on their own or already partially completely by the textbook (i.e., a fill-in-the-blank conjecture). For the open coding, we coded each student investigation for aspects that were not captured by our a priori codes. To locate opportunities for student investigation, we used the publisher-provided correlation documents.

Results

Of the 85 opportunities to introduce the theorems (17 theorems across 5 textbooks), textbooks stated the theorems without any prior investigation 39 (45.9%) times. Table 1 shows a two-letter code for each of the 46 (54.1%) times where students investigated the theorem prior to proof. The first letter indicates whether the investigation used Dynamic Geometry Software (DGS) or physical materials (e.g., patty paper, compass, protractor, straightedge, or ruler). The second letter uses a "F" if students had to fill in one or more missing words in the provided conjecture and a "L" if the conjecture was left completely to the students. All but one of the 46 investigations were specific about which tools students should use, with 22 using DGS and 23 using physical materials. Thirteen times the conjecture was a fill-in for the students. This usually involved one word left to fill in, for example, “congruent” or “equidistant.” The textbook left the conjecture for the students 33 times, although in 25 of these, the textbook stated the conjecture immediately following the investigation or within the next two pages.

Table 1: Investigation Opportunities across 17 Theorems for Five Textbooks

	Textbook					
Brief theorem description	BI	HMH	KH	MHG	PE	Investigation opportunities
Vertical angles		PL	PF			2
Alternate interior angles	DL		PF	DL	DL	4
Corresponding angles	DL		PF	DL	DL	4
Perpendicular bisector	DL		PF			2
Perpendicular bisector converse			PF			1
Triangle angles	DL	PL	PF	PL	PL	5
Base angles	DL	PL	PF			3
Midsegment	DL	PL	PF		DL	4
Medians	DL		PF	PL	DL	4
Parallelogram sides	DL	DL	PF			3
Parallelogram angles	DL	DL	PF			3
Parallelogram diagonals	DL	DL	PF			3
Parallelogram sides converse	DL					1
Parallelogram angles converse	DL					1
Parallelogram diagonals converse			PL			1
Rectangle diagonals		PL	PF			2
Rectangle diagonals converse	DL	PL	D/PL[a]			3
Total investigation opportunities	13	9	15	4	5	46

Note. D = Dynamic Geometry Software; P = physical materials; L = conjecture left to students; F = student fills in a partially completed conjecture provide by textbook. [a]KH did not specify whether to use DGS or physical materials to investigate.

When focusing on specific theorems, Table 1 shows that only the Triangle Angles theorem was investigated in all textbooks. Students investigated four other theorems in all but one of the textbooks: Alternate Interior Angles, Corresponding Angles, Midsegment, and Medians. Although students investigated each of the 17 theorems in at least one textbook, there were some theorems less likely to have investigations. There was only one investigation for each of the following theorems: Perpendicular Bisector Converse, Parallelogram Sides Converse, Parallelogram Angle Converse, and Parallelogram Diagonals Converse—all converses.

KH and BI both invested heavily into students investigating prior to proof. KH was the textbook most likely to have students investigate a theorem prior to proof. Students investigated 15 (88.2%) of the 17 theorems with 14 using physical materials—typically patty paper. The investigations frequently had four or five numbered steps for students to complete before being instructed to compare their results with others in their group. Then, students were asked to make a conjecture by filling in missing words in an already partially complete conjecture. KH was the only textbook that used partially completed conjectures—occurring in 13 out of the 15 investigations. In BI, students investigated 13 (76.5%) of 17 theorems prior to proof. Students were instructed to work with a partner and use DGS to complete four or five steps before writing a conjecture. Although all the investigations used DGS, BI did not capitalize on the dynamic nature of the software. Ten of the 13 investigations directed students to repeat steps for constructing and measuring a new triangle or quadrilateral as opposed to dragging their constructions to see multiple versions of the same figure. BI left all 13 conjectures for students to make, but BI also stated the theorems for 12 of those within the next two pages.

In HMH, students investigated nine (52.9%) of the 17 theorems: six with physical materials and three with DGS. All nine investigations left the conjectures for the students to make; however, they all subsequently appeared either immediately following the investigation or within the next two pages. Finally, both PE and MHG rarely had student investigate theorems prior to proof—just five (29.4%) and four (23.5%) of the 17 theorems, respectively. Neither textbook provided students investigations for any of the perpendicular bisector or parallelogram theorems (including their converses). PE used one with physical materials and four with DGS. PE left all the conjectures to students, although PE stated four of them within the next two pages. MHG had two investigations with DGS and two with paper. It left all four of the conjectures for students to make, with two of them stated for students within the next two pages.

Discussion

Although textbooks included 46 investigations out of 85 opportunities, and each theorem was investigated in at least one textbook, we had concerns with the meaningfulness of these opportunities. Of the 46, 13 of these were in KH where the conjecture was already partially completed, and students only filled in one or more words. In another 25 of the investigations, the textbook stated the theorem within the next two pages. This means that textbooks only had eight investigations where they truly left something to students to determine a conjecture, with one or two of these investigations in each textbook. Although BI and KH seemed to value students investigating theorems prior to proof, those investigations were overly prescriptive with limited opportunities to identify patterns and make conjectures in significant ways.

A plausible explanation for why textbooks use partially completed conjectures or stated the conjecture as a theorem a few pages after the investigation is that one purpose of textbooks is to be used as a resource outside of class. Not having at least partially completed conjectures or theorems would make it more difficult for the textbook to be used as a resource outside of class. Thus, we posit that there is considerable tension between textbook publishers and authors providing students opportunities for open-ended investigations of theorems prior to proof while also remaining true to textbooks being a resource for use outside of the geometry classroom.

Conclusion

Although there is prior research on proof and high school geometry textbooks, we undertook this study because there is a gap in the research on the opportunities that textbooks provide students to engage in reasoning-and-proving as part of the proof process. We also conducted this research to understand how the intended curriculum of the *CCSSM* becomes the written curriculum in textbooks with respect to investigating theorems prior to proof in high school geometry. Our results show that only a little over 50% of the time do textbooks provide student investigations. Further, there are more opportunities for students to investigate theorems about parallel lines and triangles, then there are about parallelograms and converses. There also is much variation across the five textbooks with respect to frequency of student investigations in each textbook. We suggest further research on whether teacher support materials beyond the student edition provide alternative opportunities for student investigations prior to proof. We also recommend research on if and how teachers implement student investigation prior to proof.

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A THEORETICAL ANALYSIS OF TANGRAM PUZZLES

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Although Tangram puzzles are an important tool used in early education to promote students' spatial visualization and composing and decomposing abilities, classification of the puzzles' difficulty is lacking. Current classifications rely on the extent to which particular shapes are distinct within the puzzles. Using a theoretical framework that highlights characteristics of puzzles along two continua (integrating to decomposing and elements to structures), we analyzed 114 Tangram puzzles. This framework can help researchers and educators identify puzzles that target specific spatial visualization strategies.

Keywords: Geometry and Spatial Reasoning, Elementary School Education, Assessment, Curriculum

Working with pattern block and tangram puzzles can support students' developing spatial visualization abilities (National Research Council, 2009; van den Heuvel-Panhuizen & Buys, 2008). Spatial visualization, in particular shape composition and decomposition, underlies students' later work in upper-level geometry (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010), chemistry, engineering, and other STEM fields (National Research Council, 2006). However, while puzzle-solving is a regular activity of preschool classrooms and advocated as key to supporting students' development on the shape composition and decomposition learning trajectory (Clements & Sarama, 2009), similar activities are not as prominent in early elementary classrooms in the United States (National Research Council, 2009).

The Shape Composition and Decomposition (SCD) learning trajectory suggests that kindergarteners, in general, are moving from the picture maker to shape composer level, where they can solve tangram puzzles with some internal lines to puzzles with no internal lines (Clements & Sarama, 2009). Tangram puzzles belong to a specific class of puzzles made with seven pieces (or tans): two small, right isosceles triangles; two large, right isosceles triangles; one square; one parallelogram; and one medium, right isosceles triangle. As students progress through the levels, they become more intentional about turning and flipping shapes, as well as combining shapes (Clements & Sarama, 2009). Puzzle tasks they suggest using to encourage students to advance from one level to the next include removing more of the internal lines and increasing the size of the puzzles (https://www.learningtrajectories.org/; see also Clements & Sarama, 2009). To move beyond the shape composer level, they suggest having students solve puzzles in multiple ways (Clements & Sarama, 2009). However, there is no standard classification for what makes one tangram puzzle different from another (and therefore potentially more difficult), other than the distinctness of each piece within the puzzle (van den Heuvel-Panhuizen & Buys, 2008). A more detailed classification system could greatly aid researchers and educators in systematically varying what children experience in their puzzle solving to improve their thinking. In this theoretical paper, we present an analysis of tangram puzzles and argue that this classification can support more targeted investigations of students' spatial visualization and development of interventions to support their spatial visualization.

Theoretical Framework

Martin and Schwartz's (2014) creative thinking framework articulates four visual representation strategies involved in creating visuals from data. These strategies involve two dimensions: an elements versus structures dimension and an integrating (or composing) versus decomposing dimension (Martin & Schwartz, 2014). These dimensions form a larger framework that subsumes ideas presented in the SCD learning trajectory and spatial visualization and imagery trajectory (Clements & Sarama, 2009). Choosing puzzles that emphasize certain dimensions could encourage students to use varied spatial thinking strategies. The four quadrants resulting from the intersecting dimensions represent the four strategies.

Flexible abstraction, at the intersection of elements and decomposing, refers to removing details in order to look at data in new ways (Martin & Schwartz, 2014). Interpreted through the lens of classifying tangram puzzles, tangram puzzles range in terms of how much they employ flexible abstraction based on the number of sides they have or the number of sections they have. Puzzles that have more sides, generally result in puzzles in which individual pieces are more defined (Baran et al., 2007), which are considered easier—one exception are puzzles with holes on the inside, which are considered difficult (van den Heuvel-Panhuizen & Buys, 2008). Combinations, at the intersection of elements and integrating, refers to multiple ways to put data together to tell different stories (Martin & Schwartz, 2014). In terms of tangram puzzles, combinations play a key role. Some of the individual tans can be composed to make the larger tans, and pairs of tans can be composed in multiple ways. For example, the two small triangles could form a medium triangle, a parallelogram, or other shapes (see Table 1). Furthermore, combinations of the smaller tans can be combined to combinations of the other pieces (see Table 2). The extent to which tangram puzzles can be solved in multiple ways, or with multiple combinations of tans, could influence their difficulty.

Table 1: Common Combinations Made from the Tans in Tangram Puzzles

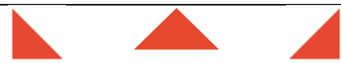
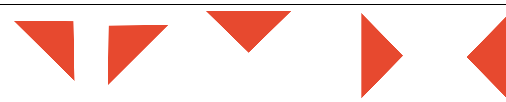
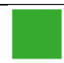
Elements	Integration
	

Table 2: Examples of Combinations of Tans That Have a Common Structure

Equivalent Combinations

Reinterpretation, at the intersection of structures and decomposing, involves changing how the data looks in order to think about it in new ways (Martin & Schwartz, 2009). Likewise, tangram puzzles differ in the extent to which students might need to reinterpret how the individual shapes are used. Compared to their standard orientations--where the bottom side of the shapes are horizontal (see Table 3)—some tans may need to be turned (or flipped in the case of the parallelogram), requiring them to be used in nonstandard orientations (Baran et al., 2007). Because students are exposed to limited orientations of shapes (e.g., always seeing a triangle with one point on top, Clements, 2004; Nurnberger-Haag, 2017), the extent to which puzzles have shapes in standard versus nonstandard orientations could influence their difficulty.

Table 3: Standard and Nonstandard Classification of the Tangram Pieces

Shape	Standard Orientation	Nonstandard Orientation
Triangle		
Square		
Parallelogram		

Borrowing structure, at the intersection of structures and integrating, involves finding similarities among structures (Martin & Schwartz, 2009). Across tangram puzzles, some puzzles share similar compositions of shapes; therefore, solving puzzles with recognizable, common structures across a series of puzzles may support students in completing the puzzles.

Methods

Materials

We analyzed a series of 114 Tangram puzzles, including all example puzzles from the Mathagon site (https://mathigon.org/tangram), the majority of freely available puzzles from the Tangram channel website (https://www.tangram-channel.com/tangram-puzzles/), and a few from Pinterest when we needed additional puzzles that included particular structures of shapes. We recreated the puzzles using the manipulatives on the Mathagon site, identifying all possible ways to solve each puzzle. For puzzles with shapes that continued to look like recognizable shapes when turned, we also analyzed the turned versions of the puzzles.

Analysis

In terms of flexible abstraction, we analyzed each puzzle's outline for its number of sides and whether the sides were internal (for shapes with holes) or external. For combinations, we identified the number of different ways to solve the puzzle and whether the puzzles had multiple solutions due to symmetry of the shape or multiple solutions due to different combinations of the pieces. For reinterpretation, we identified which pieces required a standard versus nonstandard orientation for each puzzle and its variants. Finally, we identified sets of puzzles that "borrowed" common structures or combinations of triangles and squares.

Results

Flexible Abstraction

Across the puzzles we analyzed, nine of the puzzles had holes on the inside. The other 105 puzzles ranged from having three sides to twenty-five sides. The majority of puzzles, 54 of the 105, had between 10-16 sides; we classify these as medium-sided puzzles. There were 28 small-sided puzzles (3-9 sides), and 22 large-sided puzzles (17-25 sides).

Combinations

Out of the 114 puzzles, 43 (38%) of them had one solution and 71 (62%) of them had multiple solutions. Of these, 12 only had multiple solutions because the puzzle had symmetry. An additional 35 puzzles did not have symmetry but had multiple solutions because the pieces could be combined in multiple ways. Finally, 24 puzzles had additional solutions because they had both symmetry and multiple possible combinations. Overall, for puzzles with multiple solutions, having two solutions was most common (30 puzzles, 26% of puzzles).

Reinterpretation

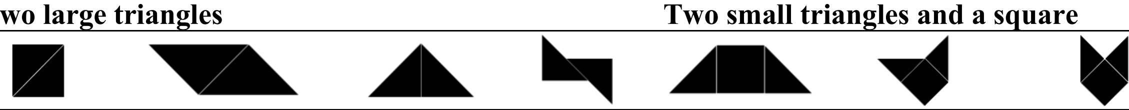
Over half (56%) of the puzzles and their variants involved using three to four of the pieces in standard and three to four pieces in nonstandard orientations. Only 2% had its pieces in only nonstandard or only standard positions. The square and parallelogram were almost equally likely to be in standard versus nonstandard position, and the other three shapes were in the standard position about 40% of the time and nonstandard position about 60% of the time.

Borrowing Structure

Across puzzles, 13% of puzzles shared at least one of three combinations involving two large triangles; these included two triangles forming a larger square, a larger parallelogram, and a larger triangle. The next most likely combination (offset triangles) occurred in 8% of puzzles. Of the combinations with the square and two smaller triangles, the most common was forming a trapezoid (11% of puzzles) and what was often used as a head with two ears where one ear was tilted (10% of puzzles) versus both ears upright (7% of puzzles) (see Table 4).

Table 4: Common Structures "Borrowed" Across Puzzles

Two large triangles				Two small triangles and a square		
Square	Parallelogram	Triangle	Offset	Trapezoid	Tilted ear	Upright ears



Theoretical Discussion

Our in-depth analysis highlights additional aspects of puzzles that may contribute to students' ease or difficulty in solving them, beyond just how distinct the pieces are within them (e.g., van den Heuvel-Panhuizen & Buys, 2008). The number of sides to the puzzles varied greatly, and this information could be helpful when aiming to help students see more or less of the structure. However, very few of the puzzles had pieces only in standard orientations, so even if the puzzles show a lot of the structure (i.e., have a greater number of sides), students may have difficulty solving them if they are not yet intentional in their flipping and turning of shapes (e.g., Clements & Sarama, 2009). Our puzzle classification could help teachers select more appropriate puzzles. For example, teachers could select puzzles not only with more sides but also with more shapes in standard orientation for beginning puzzle solvers. Likewise, puzzles with symmetry or multiple solutions might be easier for students since students are more likely to place a shape somewhere where it would work in the puzzle.

In future work, we plan to test how puzzles' composition (more or fewer shapes in standard orientation or whether there are more solutions), influences their difficulty when students place pieces with and without outlines. Similarly, knowing which puzzles have common structure can be useful for supporting students to borrow structure as a strategy for solving puzzles. If they are given several puzzles that contain similar structure, they may start to look for larger compositions of shapes in new puzzles, rather than only looking for elements. Continued explorations using this framework will help us better understand the roles of flexible abstraction, combinations, reinterpretation, and borrowing structure in students' puzzle solving so that we can create instruction that helps them build upon these spatial thinking strategies as well as add to our theoretical understanding of how these spatial strategies support students' puzzle solving.

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ITEMS WITH POTENTIAL FOR COVARIATIONAL REASONING ACROSS MATHEMATICS AND SCIENCE TIMSS ASSESSMENTS

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Constructing quantities and coordinating covarying quantities is a central component of STEM learning and development. We examined 323 publicly released fourth- and eighth-grade mathematics and science TIMSS 2011 assessment items to explore the extent to which such items could potentially elicit such reasoning. We found items potentially eliciting covariational reasoning across all four assessments, as well as an increase in number and complexity of such questions from fourth to eighth grade regardless of content domain. Further, we highlight the diverse content domains that leveraged graphical representations in such items. We discuss implications of our findings for the teaching and learning of middle-school STEM and for future study of covariational reasoning through international assessments.

Keywords: Middle School Education, Assessment, Integrated STEM / STEAM

Constructing quantities and representing relationships between simultaneously varying quantities (i.e., quantitative and covariational reasoning; Thompson & Carlson, 2017) is a central component of STEM learning and development. For example, numerous researchers have shown that covariational reasoning can be productive for students' and teachers' mathematics development generally (e.g., Carlson et al., 2003; Johnson, 2015; Moore, 2014) and in middle school particularly (Ellis, 2011; Ellis et al., 2015; Lobato et al., 2012; Paoletti, 2019). Further, there is a growing body of evidence that covariational reasoning can be productive for individuals developing key ideas in statistics (e.g., Gil & Gibbs, 2017) and various areas of science, including climate change (Basu & Panorkou, 2019; González, 2021), gravity (Panorkou & Germia, 2020), chemical kinetics (Rodriguez et al., 2019), and physics (Sokolowski, 2020).

Although covariational reasoning is critical across STEM fields, there is evidence that covariational reasoning is more salient in at least some East Asian countries' curricula (e.g., Thompson & Carlson, 2017) and teachers' knowledge (e.g., Thompson et al., 2017). Hence, as part of a larger project, we are interested in exploring if there are differences between countries in student performance on questions that may involve covariational reasoning; such differences could help researchers identify curricula or instructional activities that support such reasoning. In this report, we address a critical first step in this endeavor by describing our process of coding released TIMSS 2011 items as potentially eliciting covariational reasoning. Specifically, we examined 323 released fourth- and eighth-grade mathematics and science TIMSS items to explore the research questions: (RQ1) To what extent could a students' covariational reasoning be productive for addressing TIMSS items across science and mathematics? (RQ2) How does the frequency of such questions vary according to grade and content domain? (RQ3) What representations are used in such items (e.g., tables, graphs, pictures)?

TIMSS Assessment, Covariational Reasoning, and Coding

The TIMSS assessment, a project of the International Association for the Evaluation of Educational Achievement (IEA), aims at measuring trends in mathematics and science achievement in fourth- and eighth-grade students in 63 countries. TIMSS data is often used for international comparisons, which have important implications for educational practice and policy both in the U.S. and internationally (Carnoy et al., 2015; Hiebert et al., 2005). In this study, we examined all of the 323 publicly released TIMSS 2011 mathematics and science items across the fourth- and eighth-grade assessments (available at https://nces.ed.gov/timss/released-questions.asp) to explore if these items could potentially elicit covariational reasoning.

Although there are several descriptions of covariational reasoning in mathematics education literature (see Thompson and Carlson, 2017, for a review), we adopted Carlson et al.'s (2002) definition for this analysis due to its broad nature. Specifically, Carlson et al. (2002) defined covariational reasoning "to be the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (p. 354). Hence, we were interested in examining which items could potentially support students in considering how two quantities vary in relation to each other. Further, as graphs are a common way to represent covarying quantities across STEM fields (e.g., Glazer, 2011; Paoletti et al., 2020), we were particularly interested in the extent to which TIMSS items leveraged graphical representations when potentially eliciting students covariational reasoning.

Coding

To support our analysis, we operationalized the features of an item that we believed could elicit students' strategies entailing covariational reasoning, which we identified as a *potential covariational reasoning item (PCR)*. Specifically, if the researcher was able to 1) identify a way a student might imagine or identify two changing quantities and 2) determine some solution strategy that could reasonably entail covariational reasoning, we coded the item as a PCR item.

For example, consider the fourth-grade mathematics item in Figure 1. Although a student may be able to determine the correct value by using memorized rules, another strategy could entail the student coordinating one-centimeter changes in the map corresponding to four-kilometer changes on land to determine that eight such centimeter changes would result in 32 kilometers traveled, thereby reasoning covariationally to address the item. Given that the researchers were able to identify a way a student might envision two changing quantities and a solution strategy that involves covariational reasoning, we identified this as a PCR item.

The scale on a map indicates that 1 centimeter on the map represents 4 kilometers on the land. The distance between two towns on the map is 8 centimeters. How many kilometers apart are the two towns?

A. 2

B. 8

C. 16

D. 32

Figure 1: A fourth-grade TIMSS 2011 mathematics item we categorized as PCR (IEA, 2013, p. 12). Copyright © 2013 International Association for the Evaluation of Educational Achievement (IEA).

To develop this coding scheme for a potential covariational reasoning item, the lead author first engaged in semi-open coding (Braun & Clarke, 2006). He initially examined each item against the Carlson et al.'s (2002) definition of covariational reasoning to explore whether an item did, did not, or might elicit such reasoning. He then met with the research team, presenting several examples of each type of item to develop their criteria for a PCR item. The researcher then applied this operationalization to all items across the four TIMSS assessments.

During a second round of semi-open coding, the researcher examined all items coded as PCR to look for commonalities amongst such items. Although many items did not fit within a category (e.g., considering how coarseness of salt influences dissolution speed in fourth-grade science), the researcher did identify two common types of items in the PCR items: 1) items that could potentially require *one-step proportional reasoning* (e.g., Figure 1) and 2) *pattern-finding items* (e.g., tasks like the Tiling a Patio task, Stein & Smith, 2011). The researcher re-examined all items coded as PCR to determine the frequency of such items.

Due to our particular interest in the extent to which TIMSS items leveraged graphical representations, during this process, the researcher also identified the representation(s) that the item included to describe the covarying quantities (e.g., graph, table, equation, written description). If a question used two representations (e.g., a table and a pictorial representation), he counted each representation.

Results

In this section, we highlight particular findings relevant to addressing our research questions. We first describe general trends in the fourth- and eighth-grade items across mathematics and science and compare the prevalence of items across content domains. We then describe the representations used across domains and grades.

Tables 1 and 2 provide an overview of the released mathematics and science items in the fourth- and eighth-grade assessments. We note there were appreciably more questions in the eighth-grade science items (39 out of 90 items, 43%) that could potentially elicit a strategy entailing covariational reasoning (PCR) compared to the fourth-grade science items (23 out of 72 items, 32%). At first blush, there is a comparable percentage of PCR mathematics items across grades: 27 out of 73 items, or 37%, in fourth grade and 31 out of 88 items, or 35%, in eighth grade. However, during the second coding round, we found that of the 27 fourth-grade mathematics items coded as PCR, five were coded as ‘pattern finding’ and nine were coded as ‘proportional reasoning’ items. In contrast, from the 31 eighth-grade mathematics PCR items, only four were coded as ‘pattern finding’ and four as ‘proportional reasoning.’ In total, 23 of 31 (74%) eighth-grade mathematics PCR items did not entail pattern finding or proportional reasoning, compared to 13 of 27 PCR fourth-grade items (48%). Hence, although the percentage of PCR items appears similar, there was greater diversity in the types of PCR items in the eighth-grade mathematics assessment.

Table 1: Number of fourth-grade PCR questions and representations used in them.

Content Domain	*Total*	*PCR*	*Written*	*Pictorial*	*Graph*	*Table*
Number	40	20 (50%)	10	11	0	3
Geometric Shapes and Measures	24	1 (4%)	0	1	0	0
Data Display	9	6 (67%)	0	3	4	3
TOTAL (Mathematics)	73	27 (37%)	10	15	4	6
Life Science	30	7 (23%)	7	2	0	0
Physical Science	28	11 (39%)	9	9	0	1
Earth Science	14	5 (36%)	5	2	0	0
TOTAL (Science)	72	23 (32%)	21	13	0	1

Table 2: Number of eighth-grade PCR questions and representations used in them.

Content Domain	*Total*	*PCR*	*Written*	*Pictorial*	*Graph*	*Table*	*Equation*
Number	20	2 (10%)	1	0	0	0	1
Algebra	30	15 (50%)	1	6	0	5	5
Geometry	20	2 (10%)	2	1	0	0	0
Data and Chance	18	12 (67%)	5	1	5	1	0
TOTAL (Math)	88	31 (35%)	9	8	5	6	6
Biology	32	12 (38%)	2	2	5	3	0
Chemistry	18	6 (33%)	5	1	1	0	0
Physics	24	13 (54%)	12	8	1	0	0
Earth Science	16	8 (50%)	8	2	0	0	0
TOTAL (Science)	90	39 (43%)	27	13	7	3	0

Looking across content domains, we highlight the Data content strand had the highest percentage of PCR questions in both the fourth- and eighth-grade mathematics items, and Geometry had the lowest percentage in both. Further, we highlight that across all eighth-grade science content domains, over 30% of questions were coded as PCR.

Finally, we note that across both fourth-grade mathematics and science PCR items, written and pictorial representations were common, with graphs and tables also being observed but with less frequency. Graphs were not observed in any science fourth-grade PCR items and were rare in the mathematics items, occurring only in the Data Display content domain. Although the eighth-grade science items continued to show a preponderance of written and pictorial representations, the eighth-grade mathematics representations were more evenly distributed (i.e., between 5 and 9 occurrences across all PCR items). Within the eighth-grade mathematics PCR items, we observed tables and equations almost exclusively in the Algebra items and graphs only in the Data and Chance items.

Discussion and Conclusion

We first note, addressing RQ1, that from fourth to eighth grade, the assessment items seem to have an implicit expectation that students will be able to engage in more covariational reasoning in science, and more varied forms of covariational reasoning in mathematics (e.g., more items that move beyond pattern finding or proportional reasoning). Such findings underscore the importance of supporting middle school students in developing their abilities to reason about covarying quantities throughout middle school.

Second, addressing RQ2 and RQ3, when examining the types of representations used, it is perhaps unsurprising that a majority of items across all four assessments relied on written descriptions and pictorial representations. However, in the eighth-grade mathematics assessment, we were intrigued by the limited number of items that used equations (six items) and graphs (five items), and we were surprised that all five items that involved a graph were in the Data and Chance content strand. Further, we note in the eighth-grade science assessment that there were seven items across three content domains that included graphs. This finding, in conjunction with the fact that all four content strands in science and Data and Chance had substantial percentages of items coded PCR, allows us to echo other researchers' calls (e.g., Glazer, 2011) indicating the importance of covariational reasoning and graphical interpretation across STEM fields.

By providing an initial set of items coded as potentially eliciting covariational reasoning, we open avenues for future research exploring middle school students' covariational reasoning in the U.S. and internationally. Such research can add to the body of research examining middle school students' covariational reasoning in mathematics (e.g., Ellis, 2011) and science (e.g., Panorkou & Germia, 2020). We call for continued research in this area, as such experiences could be foundational to students' future STEM courses and careers.

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PRODUCTIVE MATHEMATICAL MEANINGS AS A GUIDE TO ANALYZING ALGEBRA TEXTBOOKS

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Thompson (2015) proposed that a productive mathematical meaning is one that prepares students for future learning and lends coherence to extant meanings. We propose that a productive mathematical meaning related to expanding algebraic expressions and factoring polynomials revolves around developing an understanding of structural relationships between (binomial) factors and polynomials. We briefly outline a combinatorial approach to support a productive mathematical meaning for expanding and factoring relationships. We then report on our analysis of four Algebra 1 and Algebra 2 textbooks as we investigated their approaches to these topics. Our findings suggest that significant work on curricular development is needed to support productive mathematical meanings in this domain.

Keywords: Curriculum, Algebra & Algebraic Thinking, High School Education, Combinatorics

Introduction

Expanding algebraic expressions and factoring polynomials make up a significant portion of the content in algebra textbooks (Sherman et al., 2016). Moreover, practitioners routinely identify expanding and factoring as an area of struggle for students (Clinch, 2018; Frank, 2019; O'Neil, 2006; Steckroth, 2015). Despite these two considerations, a relatively small amount of research has been conducted on how students engage in this work (Kieran, 2007; Warren et al., 2016) especially for situations involving more advanced, non-linear, algebraic relationships (Stacey & Chick, 2004). The majority of research on expanding and factoring has focused either on how students use computer-assisted software (CAS) to engage in this work (e.g., Jankvist et al., 2019; Kieran & Drijvers, 2006) or how they engage with it in the context of reasoning about functions (e.g., Heyd-Metzuyanim et al., 2018; Mourao, 2002). We posit that one reason for the limited research is researchers' lack of articulation of productive mathematical meanings that could serve to bring coherence to students' mathematical reasoning in this area (Thompson, 2015). Our purposes for this paper are two-fold. First, we use this paper to articulate what we see as components of a productive mathematical meaning for this domain (Tillema & Gatza, 2016; Tillema & Burch, 2020). Second, we analyze four Algebra 1 and four Algebra 2 textbooks to determine how expanding and factoring are typically treated in curricular materials. The following research questions guide this paper:

1. What themes related to expanding algebraic expressions and factoring polynomials emerged from an analysis of Algebra 1 and 2 textbooks?
2. To what extent, is the textbook development of expanding and factoring aligned with developing the productive mathematical meaning as we have described it?

After presenting the textbook analysis, we discuss the ways in which current curricular materials either do or do not support the development of the productive mathematical meaning we outline.

Conceptual Framework

Thompson (2015) described productive mathematical meanings as "propaedeutic (preparing the student for future learning) and [lending] coherence to the meanings students already have" (p. 444). For expanding and factoring, we contend that a productive mathematical meaning is one that allows students and teachers to anticipate structural relationships between (binomial) factors and the corresponding polynomial. Elsewhere, we have outlined various ways to develop such relationships (Tillema & Burch, 2020). Here, we focus on an approach that is rooted in combinatorial and quantitative reasoning—arguing that it has the most promise for supporting productive mathematical meanings. In developing this approach, we have reported on studies that: (a) supported students to develop quantitative and combinatorial reasoning using 2-D and 3-D arrays (Tillema & Gatza, 2016; Tillema, 2018); (b) supported students and teachers to generalize this reasoning to establish algebraic identities like $(x+1)^3 = x^3 + 3x^2 + 3x + 1$ (Tillema & Gatza, 2017); and (c) supported teachers to apply their combinatorial reasoning in purely symbolic problems (Burch & Tillema, 2020; Tillema & Burch, under review). Given space constraints, we focus our conceptual analysis on applying combinatorial reasoning within purely symbolic problems. We use the expansion of $(x+a)(x+b)(x+c)$ to discuss structural relationships between binomial factors and resulting polynomials.

Combinatorially, determining the product of *n* binomial factors can be interpreted as a counting problem where each binomial represents an independent binary event with the resulting polynomial representing the set of outcomes. In the cubic case, $(x+a)(x+b)(x+c)$ involves selecting either the variable, *x*, or the constant term (*a*, *b*, or *c*, respectively) from each binomial. For this reason, each partial product in the final polynomial will have exactly three factors because there are three binomials from which to select. In addition, the total number of partial products in the final polynomial will be 2^3 because each of three binomial factors represents a choice between two possibilities. The partial products can then be organized by the number of times *x* is selected to produce them. Organizing the 2^3 partial products in this way corresponds to the binomial coefficients: for the cubic case, $2^3 = 1 + 3 + 3 + 1$. That is, 1 partial product (of the 8) contains exactly three x's (i.e., $x \cdot x \cdot x$), 3 partial products contain exactly two x's (i.e., $a \cdot x \cdot x$, $x \cdot b \cdot x$, $x \cdot x \cdot c$), 3 partial products contain exactly one *x* (i.e., $a \cdot b \cdot x$, $a \cdot x \cdot c$, $x \cdot b \cdot c$), and 1 partial product contains zero x's (i.e., $a \cdot b \cdot c$). These partial products can be re-written in the final polynomial as $x^3 + (a+b+c)x^2 + (a \cdot b + a \cdot c + b \cdot c)x + (a \cdot b \cdot c)$. Re-writing the polynomial in this way highlights that the coefficients of the resulting polynomial are connected to the roots of the polynomial. For example, the coefficient for the quadratic term in the cubic expansion is the sum of the constant terms of the binomials, which means they are the negative sum of the roots of the polynomial.

Generally, a combinatorial perspective on expansion of an algebraic expression establishes four relationships: (a) relationships between the number of initial factors and the number of factors in each partial product of the final polynomial (in our case three initial factors and three factors in each partial product); (b) relationships among the number of initial factors, the number of terms per initial factor, and the total number of partial products in the final polynomial (in our case 2^3); (c) relationships between the number of partial products of a particular kind (e.g., cubic, quadratic, linear, or constant) and the binomial (or multinomial) coefficients (in our case

$1 + 3 + 3 + 1$); and (d) relationships between the roots of the polynomial and the coefficients of the polynomial (in our case, for example, the coefficient of the quadratic term being the negative sum of the roots of the polynomial). We consider these relationships to be key components of the productive mathematical meaning in the domain of expanding and factoring that we would like to engender in secondary students and teachers. We now analyze how secondary algebra textbooks support the development of expanding and factoring and of connections between these two processes.

Methods

Our sample included one secondary algebra textbook series (Algebra 1 and Algebra 2 texts) from the three predominant publishers in the U.S. secondary mathematics textbook market—Houghton Mifflin Harcourt *Into Algebra* series; McGraw Hill *Algebra 1*, *Algebra 2*; and Pearson Prentice Hall *enVision Algebra* (Banilower et al., 2018; Meaney, 2016). Additionally, we included one reform-oriented series, namely the *Algebra* and *Advanced Algebra* sequence from the University of Chicago School Mathematics Project (UCSMP). We worked from the presumption that textbook authors intended to build ideas across their respective Algebra 1-2 series. Initial analysis focused on the first research question: What themes emerged related to expanding algebraic expressions and factoring polynomials? To answer this question, we coded and discussed the narrative portions of textbook sections involving polynomial multiplication introduced. We draw examples from the textbooks in our sample to illustrate emergent themes regarding how students might understand these concepts given these presentations. We then consider the implications of these themes for developing the productive mathematical meaning as we have described it.

Emergent Themes Related to Expanding and Factoring

Across textbooks, themes emerged related to the development of expanding and factoring and the equivalence of corresponding expressions (i.e., factored and expanded). Here, we present data related to expanding and briefly outline themes related to factoring and algebraic identities. In our presentation, we will present data across all themes.

Themes Related to *Expanding* Algebraic Expressions

We found that all textbooks present the Distributive Property as *the* fundamental mechanism behind expanding algebraic expressions—regardless of the number of factors. Initial development of expanding two binomial factors occurs in Algebra 1 texts as a precursor to working on quadratic functions (Figure 1a). We note that in contrast to initial treatments of the distributive property (e.g., $7 \cdot 13 = 7 \cdot (10 + 3) = 7 \cdot 10 + 7 \cdot 3$), only two of the four books (Pearson *enVision*, UCSMP *Algebra*) gave any numeric examples to motivate the process of two factor binomial expansion (e.g., $13 \cdot 13 = (10 + 3)(10 + 3) = 10 \cdot 10 + 10 \cdot 3 + 3 \cdot 10 + 3 \cdot 3$). Additionally, Algebra 1 textbooks consistently offer at least one alternative strategy for two-factor binomial expansion (e.g., FOIL, table [box method], area model). However, these strategies are presented either as shortcuts for (e.g., FOIL, table) or visual representations (e.g., area model) of the Distributive Property (Figure 1b). Expansion is then revisited in Algebra 2 in the context of determining a polynomial given its roots (Figure 1c). Within presenting these methods, the main focal point is a step-by-step narration (see Figure 1a) of repeatable procedural steps that can be used to guide computational activity.

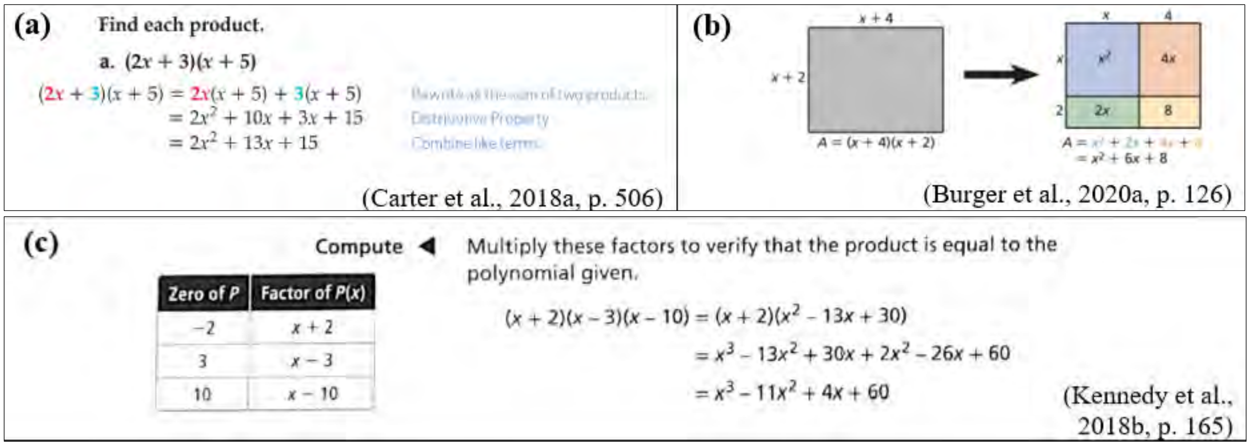

Figure 1: Expanding Algebraic Expressions

Notably absent from all Algebra 1 textbook sections on expanding are prompts to reflect on the makeup of terms in the final polynomial; instead, the focus is on combining like terms and simplifying when possible. For example, in Figure 1b, the coefficient of the linear term is written as $6x$ rather than as $(4 + 2)x$ and the constant term as 8 rather than $(4 \cdot 2)$. We do note, and will discuss in our presentation, that all textbooks present this relationship in the section on factoring monic quadratic polynomials. However, only one of the eight textbooks highlighted such relationships in factoring sections for any other polynomials—HMH *Into Algebra 1* textbook for non-monic quadratics. Taken together, we interpret these choices as missed opportunities in sections on *expanding* to help readers anticipate relationships that would be presented later in sections on factoring, and overall an inconsistent rather than clear focus on these relationships.

Across all four Algebra 2 textbooks, the number of intermediate steps commensurately with the number of factors in the expansion. The UCSMP *Algebra* text set up three-factor expansion by telling students to "multiply any two of its factors. Then, multiply the product of those factors by the third factor" (Brown et al., 2016, p. 755). Using the example in Figure 1c, $(x - 3)$ and $(x - 10)$ are multiplied to produce $x^2 - 13x + 30$, which is then multiplied by $(x + 2)$ to produce $x^3 - 11x^2 + 4x + 60$. We interpret this choice, beginning with two factors and enfolding additional binomials one-by-one, as positioning expansion as a recursive process that naturally terminates when all binomials have been included. The recursive process, coupled with the push to combine like terms between iterations, reinforces equivalence as resulting from a one-directional computation and further obscures relationships between the initial binomial factors and final polynomial.

Discussion & Conclusion

We consider supporting students to develop an anticipation of relationships between factors and expanded polynomials to be a worthy aim of instruction on expanding and factoring. In our textbook analysis, we could not infer intentionality from textbook authors toward developing relationships between initial binomial factors and resulting polynomials. Rather, when expanding binomials, textbooks focused on activating and carrying out sequential procedure with little consideration about what initial factors might mean for the final polynomial expansion. One exception was the UCSMP *Algebra* text that explicitly referenced the Multiplication Principle as useful for anticipating the number of partial products (Brown et al., 2016, p. 681). We acknowledge that textbook authors may rely on teachers to support these connections. However, this presumes that these relationships are inherent to teachers' meanings. Further, we suggest that textbooks miss opportunities to support developing productive mathematical meanings by not explicitly supporting these relationships.

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USE OF TEACHER-CREATED CURRICULAR RESOURCES BY ELEMENTARY MATHEMATICS TEACHERS: BEFORE AND DURING THE COVID-19 PANDEMIC

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This study explores elementary teachers' use of mathematics curricular resources before and during COVID-19. We administered a survey to a national sample of third through fifth grade teachers. The findings show the prevalence and increased use of teacher-created materials during the pandemic. This has implications for researchers and administrators as they consider how to best support teachers in designing their own curricular materials, especially for diverse learning contexts.

Keywords: Curriculum, Teacher Beliefs, Instructional Activities and Practices, Elementary School Education

Even before COVID-19 upended classrooms, teachers' use of mathematics curriculum was shifting. Many teachers no longer relied on or had access to sets of materials from textbook companies. Instead, pre-pandemic anecdotal evidence suggests an increasing popularity of online, teacher-created curricular materials (e.g., Gewertz, 2014; Monahan, 2015; Ross, 2015). Research has documented teachers' use and modification of published curricular materials (e.g., Choppin, 2011; Remillard, 2005; Sherin & Drake, 2009). These researchers report modifications such as changing physical materials, omitting parts, reorganizing features (such as ungrouping sets of problems), and adding transitional activities to lessons. However, the shift to online resources and teacher-created curriculum is a more dramatic change in teachers' use of curricular materials and one that we know little about.

In this study, we seek to describe the curricular landscape for upper elementary mathematics curricula before and during the pandemic. In particular, we are interested in making sense of what materials teachers are using, including teacher-created materials, and how they are making their curricular decisions.

Online Resources

Recent studies have started to track teachers' use of online resources. Sawyer et al. (2020) found that elementary teachers, regardless of years of experience, were turning to online mathematics resources weekly; 89% reported using Teachers Pay Teachers (TPT) and 74% reported using Pinterest. The limited research literature on teachers' uses of sites such as TPT and Pinterest suggests substantial disagreement about the benefits and limitations of teachers sharing, choosing, and using teacher-created curricular materials from these sites. For example, some research (Shelton & Archambault, 2019; Torphy et al., 2020) highlighted the positive aspects of networks of sharing across teachers without the need to involve administrators and publishers, as well as teachers' capacity to curate the wide range of materials available on these sites and to have immediate access to the materials in order to be maximally responsive to their students' needs. Other research, however, has focused more on the limitations of the activities available on TPT and Pinterest, including the predominantly low levels of cognitive demand of the tasks shared on these sites and the tendency of both sellers and buyers to choose fun, colorful activities (Sawyer et al., 2019). Most existing work concludes with the need for further research to understand how and why teachers are sharing and using teacher-created curricular materials.

Methods

We utilized the services of MDR to assist us with survey design, administration, and analysis. MDR administered the survey in September 2020. In the survey, teachers were asked about their curriculum before the pandemic, during remote teaching in Spring 2020, and their plans for Fall 2020. In the survey, we defined "Mathematics Curriculum Materials" as any materials used by teachers for the purposes of planning, teaching, and/or assessment. We asked teachers questions about the mathematics curricular materials they used in September 2019-February 2020, March 2020-June 2020, and September 2020-December 2020. In addition to the specific curricular options we provided, respondents could also select "I designed my own materials," "Other (please specify)" or "None of the above." Once the curricular materials used were established during each time frame, we asked questions about changes in curriculum reported.

We received survey responses from 524 third, fourth, and fifth grade teachers from across the U.S. Most taught in public schools (90%) located in suburban (54%), urban (28%), and rural districts (17%). The majority of the teachers we surveyed (62%) work in schools with at least half of the students eligible for *free or reduced lunch* (FRL), and many (43%) of the teachers work in schools with at least 75% of the students eligible for FRL. We distinguish between four categories: high FRL level (75%-100% of students qualify), medium-high (50%-74%), medium (15%-49%), and low (0%-14%). Descriptive frequency data for the full sample and for groups based on school FRL levels are reported. In addition, we examined and coded responses to the open-ended questions to further understand the reasons for teachers turning away supplemental and core curricula in the context of COVID-19.

Findings

Teacher Autonomy

We asked teachers how much control they had over curricular decisions during the pandemic. Teachers reported that curricular decisions were primarily made by district leaders (60%), principals (41%), grade-level teams (39%), and school boards (17%), with few teachers reporting that they were completely in control of their curricular decisions (11%).

Teachers from rural, suburban, and urban communities reported a range of control over their curriculum. For example, 16% of rural, 19% of suburban, and 29% of urban teachers reported they had no control over their curriculum. At the other end of the spectrum, 13% of rural, 12% of suburban, and 7% of urban teachers reported they had full control over their curricular decisions. Most teachers reported either "a bit" (35% rural, 40% suburban, 42% of urban) or "a lot" of control (35% rural, 29% suburban, 23% urban).

Shifts Away from Published Core Curriculum

Less than half of teachers (41%) surveyed were using at least one core curriculum prior to the pandemic; this dropped to 37% in Spring 2020 and Fall 2020. The most popular core curricula were Envision Math, Engage NY/Eureka, and Go Math. Many teachers used TPT, Pinterest, and other online teacher-created resources. TPT was the most widely used curricular resource by teachers before and during the pandemic, reportedly used by nearly half of surveyed teachers. Other online curricular supplements such as BrainPOP and IXL were used by approximately one-fourth of the teachers, decreasing slightly in their use during the pandemic.

Interestingly, at the onset of the pandemic, the frequency with which teachers designed their own materials increased. Before the pandemic, 27% of teachers designed their own materials. This increased to 35% in the Spring and 32% in the Fall. We were not surprised by this finding considering that most existing curricular materials were created for in-person contexts.

Digging Deeper into TPT Usage

TPT was the most popular curricular resource across community and economic contexts. As seen in Table 1, in rural and urban settings, the data show a linear relationship between use of TPT and economic status of students they were serving, with more teachers using TPT as the percentage of students receiving FRL increases. Around a quarter of teachers used this resource in suburban settings, regardless of the economic status of students. We have not yet been able to account for the difference in use of TPT across suburban and urban/rural contexts. These patterns do not correlate with core curricula or patterns of teacher autonomy across community contexts.

Table 1: Location and FRL Status of Teachers Using TPT prior to COVID

	Pre-Pandemic				Early Pandemic (Spring 2020)				During Pandemic (Fall 2020)			
FRL	Low	Med	MH	High	Low	Med	MH	High	Low	Med	MH	High
Rural	2%	8%	30%	60%	0%	7%	32%	57%	2%	11%	32%	52%
Urban	7%	15%	8%	66%	6%	14%	9%	67%	6%	15%	10%	64%
Suburb	24%	26%	20%	24%	25%	26%	19%	25%	26%	26%	21%	22%

Interestingly, the use of TPT and Pinterest did not neatly match teachers' reports about their curricular autonomy; in fact, 57% of teachers who reported "a bit" or no curricular control used these resources compared to only 43% of those reporting "a lot" or complete control who reported no control over curricular decisions. It may be that teachers were turning to TPT and Pinterest to supplement their mandated curriculum while those with more control over their curriculum were more satisfied with curriculum from educational publishers.

Teachers' Reasons for Turning Away from Online, Supplemental, and Core Curriculum

We also asked teachers to explain the reasons they stopped using online teacher-created curricula (i.e., TPT, Pinterest), supplemental curricula (i.e., IXL, BrainPop) and core curricula (e.g., Go Math, Envision Math, and Engage NY/Eureka). Teachers' reasons for stopping supplemental curriculum include: there was not enough class time to use supplements, there was not enough time for the teacher to find material, their school used specific curriculum/had enough resources, the resources were not available in electronic or easy to use online format, the teachers did not want to spend the money on resources, their district did not allow use of these resources during remote learning, teachers wanted to limit resources children needed to manage at home, students did not have access to sufficient technology at home/or the resources were not easy for all students to access, the supplemented did not meet the content needs, the resources were not offered by the school anymore, it was not rigorous enough, and they would rather use their own materials or other curricula.

When teachers were asked about the reasons they stopped using core curricula, they explained that they stopped using these resources because they only used physical book version and did not have textbooks at home, the core curriculum was not digital learner friendly, the district got rid of it/switched curriculum, they wanted to limit online platforms to make things easier for kids/parents, there was lack of rigor/declining test scores, and they would rather use their own materials.

Table 2: Reasons for Turning Away from Curriculum

Reasons	Teacher	Supplement	Core
Not enough class time	X	X	
School/district provided curricular resources	X	X	
Materials were in print form/could not be used remotely.	X	X	X
Cost/end of school subscription	X	X	
Needed to limit resources	X	X	X
Not enough time to find material	X		
Materials did not meet the instructional needs		X	X

Discussion

These data point to a novel, potentially pandemic-related trend towards teachers needing to design their own curriculum: teachers were making more of their own materials in contrast to a previously reported general trend toward more online supplemental and teacher-created materials (e.g., Sawyer et al., 2020). Rather than looking online for materials created by other teachers, teachers were inventing their own materials, not to sell, but rather because what they had available was not meeting students' needs during online/remote instruction.

Before the pandemic, our research goals were to learn more about how the curricular landscape had changed as a result of the internet and CCSSM. So many of the changes imposed upon classrooms require that teachers take up the heavy lift of managing the implementation and impact of the change. For CCSSM, teachers became the front-line workers, pulling together new curricula because their classrooms lacked the necessary resources to match the new standards (Pittard, 2017). While principals and others provided important support, the slow pace of infusion of new published curricula meant that teachers were necessarily the ultimate bridge between school shifts and children.

This was just as true during the pandemic. As the pandemic hit and teaching and learning entered entirely new territory, teachers were the ones who were in the best position to keep students learning (and feeling connected to something stable) during the new and changing notion of schooling. The existing curricular resources, including those available online, were not adequately attuned to students' new realities and needs, realities and needs that were best understood by the teachers who were connecting online with students and their families. As a result, teachers found increasing needs to create their own materials.

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THE STORY OF DEFINITE INTEGRALS: A CALCULUS TEXTBOOK NARRATIVE ANALYSIS

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Keywords: Written Curriculum, Teaching and Classroom Practice, Narrative Framework, Definite Integral

Teachers have to choose appropriate textbooks from a plethora available for every course. While most textbooks incorporate the same content and organize the content in a similar way, it is important for teachers to understand the textbook they have chosen to support their students' learning, as textbooks can have a significant impact on student learning and teacher instruction (e.g., Remillard et al., 2014). Additionally, teachers' understanding of how students negotiate the curriculum can improve student achievement (Larson et al., 2017). This study aims to understand how teachers can best use their resources and read curriculum through a different lens. I was guided by the research question: "How does an analysis of definite integrals through a narrative framework contribute to the understanding of the written calculus curriculum?" I am responding to the call for research on the relationships between stories and knowledge construction (Healy & Sinclair, 2007). Dietiker's (2015) narrative framework was adapted to analyze written calculus curricula introducing the definite integral in working to understand the story of definite integrals and students' knowledge construction of definite integrals when read as stories.

Dietiker's (2015) narrative framework provided a conceptual tool for understanding the story of the written mathematics curriculum. A *mathematical story* as defined by Dietiker (2015) is "the interpretation of the chronological sequence of mathematical changes in a mathematics textbook by a reader" (p. 288). I split the lessons of the definite integral from the five calculus textbooks (Dietiker et al., 2017; Goldstein et al., 2009; Hughes-Hallet et al., 2009; Rogawski et al., 2015; Weir et al., 2008) into "acts," denoted by changes in focus in the lesson. I, then, coded the acts into story arcs, defined as the transitions from asking to answering a question, and I subcoded the story arcs into elements of proposal, explicit question, partial answer, disclosure, and other literary elements to write the story of the lesson. Specifically, the sequencing of each story became a focal point of the study as the sequencing plays into development of the character, the definite integral, and sets the context for future understanding.

Analyzing the mathematical story of the definite integral provided an in-depth comparison of one individual lesson from five different calculus textbooks. Findings suggested key similarities and differences in the five stories. The findings revealed an overarching theme of working towards defining the definite integral. The key question, *What is the definite integral?,* was introduced early and developed throughout the lessons analyzed. However, the path to answering this question varied from lesson to lesson as some focused on the area model and others utilized the Fundamental Theorem of Calculus or introduced properties and formulas to compute definite integrals. Even the lessons focused on using the area model to define the definite integral had different structures. Hughes-Hallet et al.'s (2009) lesson introduced Riemann sums in the lesson, while other lessons revisited Riemann sums and worked toward a more generalized understanding while introducing and working to define the definite integral. These insights provide teachers and researchers a new perspective and a deeper understanding of these textbook lessons and how the variances in a story can alter student learning.

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PROMOTING INTEREST IN STATISTICS WITH SOCIAL JUSTICE DATA INVESTIGATIONS

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Today's data-driven world demands a data-literate citizenry and a workforce with strong statistical thinking skills. Studies show, however, that schools are neither preparing students adequately nor drawing enough students to data science fields (Henke et al., 2016). The Strengthening Data Literacy across the Curriculum (SDLC) project has developed and is studying two curriculum modules for non-AP high school statistics to promote interest and skills in statistical thinking and data analysis among high school students—particularly those from historically marginalized groups. The modules *Investigating Income Inequality in the U.S.* and *Investigating Immigration to the U.S.* pose social justice questions that students investigate using large-scale socioeconomic data from the U.S. Census Bureau and the Common Online Data Analysis Platform (CODAP). Students enact a four-step data investigation cycle (Bargagliotti et al., 2020) to explore questions like "How much income inequality exists between males and females in the U.S.?" and "How do labor force participation rates for immigrants and the U.S. born compare, before and after controlling for education?" Statistical concepts of focus include measures of center and variability, conditional proportions, and multivariable thinking.

The SDLC modules have been iteratively developed and tested with over 500 students in Boston-area high schools. Design-based, mixed-methods research efforts examine these questions: 1) In what ways do SDLC modules support students' interests in and learning of statistical concepts and practices? 2) To what extent do students who use SDLC modules show improved understandings of important statistical concepts and greater interest in statistics and data analysis? Research data include classroom observations, student work, teacher logs, student and teacher interviews, and pre-/post-assessments. The project team has analyzed qualitative data with *a priori* and open codes to explore participants' views of module activities and student outcomes, and it has examined growth in students' pre- vs. post-module interests and learning with paired *t*-tests. Interest scales were adapted from instruments by Linnenbrink-Garcia et al. (2010) and Sproesser, Engel, and Kuntze (2016), and statistics assessment items were drawn from the LOCUS (Jacobbe et al., 2014) and CAOS (Garfield et al., 2006) assessments.

Findings from the income inequality module show growth in student interest and learning. Students' individual interest in statistics and data analysis was significantly higher after completion of the module, and students' overall understanding of assessed statistical concepts improved significantly between the start and end of the module. These preliminary results suggest that these modules may hold promise for promoting student interest and learning in statistics and data analysis. In a world that urgently needs larger numbers and a greater diversity of students to develop interests and practices in data science, the SDLC project is building

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insights about a curriculum strategy that may make a difference.

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A COMPARATIVE STUDY OF TRIGONOMETRY STANDARDS IN TURKEY, ZAMBIA, AND THE UNITED STATES

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Keywords: Standards, Curriculum, Policy, Comparative

There has been a growing interest in how students from different nations learn mathematics (Delice, 2003). Results of international comparative efforts such as Trends in International Mathematics and Science Study and Program for International Student Assessment show differences and similarities in the educational systems across the world (Cai, 2001). We analyzed a secondary mathematics topic, trigonometry, in the U.S., Turkey, and Zambia's mathematics curriculum to assess the similarities and differences in learning expectations for students in three economically, socially, geographically different countries. Engaging with trigonometry enhances students' problem solving, reasoning, and visual representation skills (Fi, 2003; Tuna, 2013). It is perceived as an essential topic to comprehend college-level courses, i.e., "understand periodicity and recognize graphs of periodic functions, especially the trigonometric functions" (Conley, 2003, p. 34).

Only 22% of the students who took the composite test score (ACT) reached the college-readiness level in mathematics, which includes trigonometry standards. A possible reason for the low readiness could be the intended and implemented curriculum (Schmidt et al., 2001). Due to its low achieving, researchers were more likely to compare the United States' mathematical standards with high-achieving countries' educational systems (e.g., Porter et al., 2011; Schmidt et al., 2005). Comparison of developed countries' mathematics curriculum for instance the U.S., with the developing countries' mathematics curriculum such as Turkey and Zambia would add to existing literature. However, there is a dearth of research to inform this kind of work.

We independently analyzed and later together compared cognitive expectations by focusing on verbs of the trigonometry standards based on Webb's (2007) Depth of Knowledge (DOK) framework. This framework has four levels namely Level 1 measuring at recall and reproduction level, Level 2 measuring at skills and concepts level, Level 3 measuring at strategic thinking and reasoning, and finally Level 4 measuring extended thinking, which reflect the complexity of the analysis. We used Hess's (2013) guide to decide the level of verbs in Webb's framework. For example, *working* with special angles refers to Level 2, and *drawing* graphs for functions refers to Level 3. Then, we used a direct analytic approach to compare the three sets of standards side by side (Tran et al., 2016).

Preliminary findings indicated that all three countries' standards have similar trigonometric topics; however, the DOK levels varied across countries. For example, all three countries have standards related to drawing trigonometric functions. Turkey and Zambia have standards with a lower-level verb (to draw) while *CCSSM* used a higher-level verb (to model). *CCSSM* did not have any standards at Level 1 (Recall and Reproduction) while Turkey had [14%] and Zambia [19%]. Likewise, compared with Turkey and Zambia, *CCSSM* [69%] was more likely to emphasize higher-order thinking skills at Level 3 and Level 4. Turkey [43%] as less likely to support higher-order thinking skills than Zambia [51%]. Future studies could extend this study by analyzing additional curriculum resources such as textbooks.

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TEACHERS' INTERPRETATIONS OF ASSESSMENT RESULTS

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A way to evaluate an assessment's worth is in its contributions to student learning (Cronbach, 1988). "Classrooms are complex social environments. Economic, language, cultural, and mental health issues are just some of the key variables that need to be considered in relation to students [learning]" (Leighton, 2020, p. 27). Teachers provide a unique influence on their students' learning through their beliefs, content knowledge, and pedagogical content knowledge (Brookhart, 2003). A classroom's social context is an area where teacher-created assessments differ from externally-developed standardized assessments. One goal of standardized testing is to eliminate psychometric noise like social contexts by attempting to account for factors not related to the construct being measured (AERA et al., 2014). On the other hand, teacher-created assessments are contextually relevant as they are developed with certain students in mind (Brookhart, 2003). Teacher-created assessments are more likely to align with a unique social context of a classroom. The purpose of this study is to explore middle grades math teachers assessment practices and impact on student learning.

We address the question: What are middle grades math teachers' perceptions and uses of teacher-created and standardized assessment results when making inferences about student learning? Data were collected for this qualitative study through semi-structured interviews with seven purposefully selected inservice teachers. Interviews were transcribed and themes were identified through open and axial coding (Saldaña, 2015). One finding was that teachers perceived results from teacher-created assessments to be more useful than standardized assessment data when making inferences about student learning. Figure 1 shows participant support of this theme. Teacher-created assessments provided evidence of student thinking like how students solved problems, which standardized assessment results lacked.

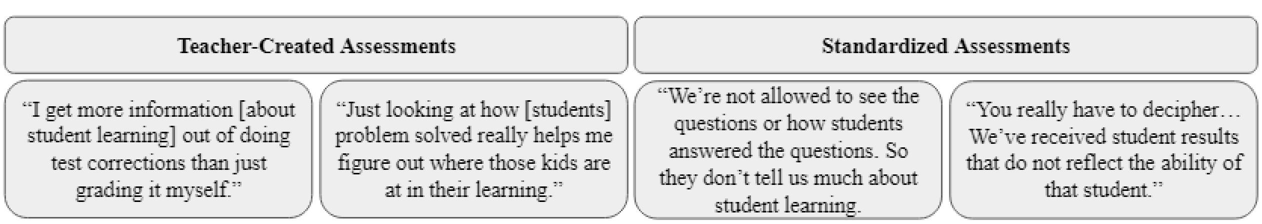

Figure 1. Participating teachers statements about assessment results

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THE ROLE OF HUMAN BEINGS IN MATHEMATICS CURRICULUM: CONNECTING UNDERGRADUATE MATHEMATICS WITH TEACHING SECONDARY MATHEMATICS

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Keywords: Curriculum, Mathematical Knowledge for Teaching, Preservice Teacher Education, Undergraduate Education

Mathematics and teaching are human activities: mathematics is taught and learned by people in a social and cultural context. Teachers are called on to engage in interpersonal interactions that require both mathematical expertise and skills for probing student thinking or finding meaning in learners' perspectives (e.g., Ball et al., 2008; NCTM, 2014). As such, mathematics teachers' interactions with learners interweave their content knowledge with their capacity to respond to and guide student thinking, and future teachers need opportunities to engage in these practices throughout their teacher preparation programs (AMTE, 2017). Yet, secondary teachers can have difficulty in recognizing mathematics as part of a social space (Parker et al., 2017). Because prospective secondary teachers are among the population of undergraduates typically enrolled in mathematics major courses, we posit that these content courses can play a large role in providing prospective teachers with opportunities to engage in such practices and to grow in their understanding of mathematics as a human endeavor.

The META Math project has focused on designing and researching the use of tasks (specifically applications to teaching secondary mathematics) in content courses. The tasks highlight connections between undergraduate mathematics and teaching secondary mathematics (see Arnold et al., 2020) and have been designed in a way to emphasize the human element of mathematics so that the prospective teachers who engage with our materials will see that the human context of mathematics is held on par with the mathematics content (see Álvarez et al., 2020). We report on a qualitative case study in which we investigated the following research question: *What is the nature of undergraduate students' experiences with tasks designed to include the human context of mathematics?* Our tasks were implemented in four content courses: abstract algebra, calculus, discrete mathematics, and statistics. We collected all undergraduates' written responses to our tasks and invited a subset of them to participate in interviews.

We found that undergraduates, including those who did not plan to teach secondary mathematics, were thinking about human beings while doing the tasks. Further, the inclusion of these tasks into the curriculum did not detract from their learning of core mathematical content. Although challenging, the undergraduates found value working on these tasks as they made undergraduates "think more critically about the problem." We view our results as confirmation that creating and using tasks that emphasize the human context of doing mathematics is a useful approach to give prospective teachers opportunities in their content courses to engage in practices central to teaching and to see that mathematics and teaching are human endeavors.

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RELATIONSHIPS IN MATHEMATICS EDUCATION FOR INDIGENOUS PRE-COLLEGIATE STUDENTS

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Mathematics curricula are often treated as neutral or infused with contexts that point back to stereotypes about cultural groups. Instead, the conceptual framework for curricular design in this program is derived from core principles of Indigenous perspectives. One of the core principles guiding the design of this project is relationality (Nicol, et al., 2020).

By curriculum, I am referring to not only the content students will be learning with and from, but also the design of the learning environment, including how students interact with each other, their teachers, and their communities both in- and outside of the learning space. This perspective on curriculum calls for mathematics to be utilized as a lens through which to gain perspective and information from the world. This perspective contrasts with that of the traditional, Eurocentric classroom, in which mathematics education is isolated to math classrooms, or the math time of the school day.

Relationships exist at a multitude of levels - human to human, human to more-than-human (e.g., plants, animals, stars, natural phenomena), human to land, etc. (Bang & Medin, 2010; Cajete, 2016; Gutiérrez, 2017; Kimmerer, 2013; McGinty & Bang, 2016; Simpson, 2017; Wilson, 2008). Relationships are also often utilized to describe things mathematically, and mathematical relationships are inherent in the world. In this learning environment, the intelligence of geometry will be embraced, connecting geometry to the students physically, both in body and geographical location, as well as to their cultures (Gutierrez, 2017). The connection of geometrical concepts from standards-based curriculum to stories and significant relationships from students' own cultures provide the college preparatory content required, while also centering Indigenous epistemologies. Students from many tribal nations gather together to participate in the precollegiate program, which means multiple epistemologies are represented in the classroom at any given time.

This qualitative project will collect information from student interviews, reflections, and class work, as well as communications and guidance from Indigenous educators and individuals working with Indigenous students. The connection of geometrical concepts from standards-based curriculum to stories and significant relationships from students' own cultures provide the college preparatory content required, while also centering Indigenous epistemologies.

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CLASSROOM ASSESSMENTS BUILDING TOWARDS STANDARDIZED ASSESSMENTS

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Administrators, educators, and stakeholders have faced the dilemma of determining the most effective type of data for informing instruction for quite some time (Pella, 2015). While the type of standardized assessment a teacher gives during instruction is often set at the district or state level, teachers often have autonomy in the formative and summative assessments that serve as the day-to-day tools in assessing a student's progress (Abrams et al., 2016). Choices about in-class assessment and instruction are building blocks towards a student's success on standardized assessments. The purpose of this phenomenological qualitative study is to explore how 4th-8th grade math teachers' preparation and instructional practices are influenced by the types of assessments administered to their students in one school. Research questions are as follows: (a) How do 4th-8th grade math teachers describe the math assessments they use? (b) How do 4th-8th grade math teachers adjust their instructional practices as a result of their students completing formative, summative, and standardized math assessments?

Five rural 4th-8th grade teachers are participants in this case study and took part in a 20-minute semi-structured interview via Zoom that was transcribed verbatim. Two of the researchers conducted open and axial coding to determine emerging themes and consolidated the themes into categories (Saldaña, 2015). The researchers worked to address the four tenets of trustworthiness according to Lincoln and Guba (1985).

Findings showed teachers shared a progression of the ways they described their mathematics assessments. Formative math assessments were described as influencing moment to moment and day-to-day instruction, and the success of students was a clear indication on the difficulty of content and delivery by the teacher. As teachers described summative math assessments, they shared how they viewed these as more of a balance in the success of their students between instruction and student preparedness. Finally, standardized assessments were described as guiding a teacher's planning by addressing standards but not considered in their day to day activities. Two teachers mentioned how it was important for students to have a "blank slate" coming into their classes because each year is different.

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CONTRASTING CASES IN GEOMETRY: THINK ALOUDS WITH STUDENTS ABOUT TRANSFORMATIONS

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Theoretical Framework & Design of Materials

There is strong empirical evidence in support of learning from comparisons in mathematics education research (Rittle-Johnson & Star, 2007; Star, Pollack, et al., 2015; Star et al., 2016). Comparisons have produced gains in students' procedural knowledge, flexibility, and conceptual knowledge of algebra (Lynch & Star, 2014; Star, Newton, et al., 2015; Star, Pollack, et al., 2015). The *Animated Contrasting Cases in Geometry* project seeks to extend this research and transform the learning of geometry for middle school students by designing a supplementary digital animated curriculum.

The curriculum materials for each lesson are organized into Worked Example Pairs (WEPs), which include five unique features: a page for the first student's solution strategy on a given geometry task, a page for the second student's solution to a geometry task (which could be the same or different task as was shown on first student's page), a page with both students' strategies side-by-side, a discussion sheet with four questions for the students to answer, and a thought bubble page summarizing the key mathematical concepts in the problem. The discussion sheet and thought bubble page are designed to make the instructional goal of each WEP more explicit and to scaffold discussions among students as they summarize their work from the WEPs (Star, Pollack, et al., 2015). This paper focuses on the Transformations unit, which is one of four units.

Methods

After fully developing the 8th grade geometry materials, we conducted 56 hour-long think aloud interviews (Piaget, 1976) with individual students ($n = 42$ students). There were 18 think alouds for the Transformations unit conducted with 13 unique students. We transcribed each interview and began *a priori* (Saldaña, 2013) coding based on our key design features. We then added emergent (Saldaña, 2013) Level 1 codes for the students' geometric thinking and curricular form and Level 2 codes as appropriate. In all, there were 556 turns coded.

Findings

We observed 96 (17.27%) turns where students were making comparisons between the WEP characters. Most often they were discussing differences between the characters ($n = 58$), but they also noted similarities ($n = 35$) and used both WEP characters' strategies to verify a mathematical idea ($n = 3$). We observed 119 (21.40%) turns where students were discussing the geometric thinking of the WEP characters. When discussing the thinking displayed by the WEP characters ($n = 44$), students most often provided insight into their personal beliefs about the characters' thinking. Students' geometric thinking accounted for 203 (36.51%) turns of the Level 1 coding. A majority of the codes regarding students' geometric thinking indicated that the student was making sense of the mathematics in the WEP ($n = 105$).

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TITLE OF PAPER: USING FIELD-TEST DATA TO INFORM DECISIONS ABOUT USING CONSTRUCTED-RESPONSE AND SELECTED-RESPONSE ITEMS

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Keywords: Assessment, Elementary School Education, Research Methods

Selected-response (i.e., multiple-choice) item formats are commonly used in mathematics assessments in the United States. The selected-response format invokes concerns about convergent thinking, low levels of cognitive demand, and limitations to the inference that can be drawn with respect to student abilities or understanding. Constructed-response item formats also have limitations. They require extensive resources to score, can yield ambiguous responses, and can result in low levels of scoring reliability. Constructed-response items can be especially difficult to score when they are administered to young children whose handwriting and other communication abilities are in nascent stages.

Methods

We present an approach to decision making for item format used in the Elementary Mathematics Student Assessment (Schoen et al., 2016a, 2016b; 2017). The approach involves a multistep process where items are drafted and reviewed by experts. The resulting items are then used in a one-on-one interview setting to enable a think-aloud/cognitive interview protocol. Items are then used in a constructed-response format with a large sample of students in the target population. Final answers are then reviewed. When the four most common answers represent more than about 85% of the responses, we convert the item to a selected-response (i.e., multiple-choice) format, using the responses from the field-test for the response options. When responses to items are more varied, the constructed-response format continues to be used for those items.

Discussion

Selected-response format have several benefits and limitations. We find that examinees from the target population will often respond in predictable ways to items that are used in a constructed-response format, but a large number of examinees will sometimes provide responses that were not expected during item development. Field testing items in a constructed-response format with a sample of examinees from the target population before creating the selected-response options yields response options that align with the answers that would have been provided in a more open-ended format. We assert that this approach to developing selected-response items through empirical response validation improves the testing experience for the examinees and the overall quality of measurement. We recommend that test developers use similar empirical approaches to determining the item format and the response options for selected-response items. We think the process serves to increase efficiency and reliability of the test without sacrificing the quality of the inference that may be drawn from tests using selected-response item formats with respect to student performance or understanding.

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Chapter 3:

Equity & Justice

Olanoff, D., Johnson, K., & Spitzer, S. (2021). *Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Philadelphia, PA.

STORYLINES IN NEWS MEDIA TEXTS: A FOCUS ON MATHEMATICS EDUCATION AND MINORITIZED GROUPS

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We identify storylines about youth from minoritized cultures and/or languages in Norwegian news media to identify positionings made available to migrated and Indigenous mathematics students in this public discourse. Our search from 2003-2020 in a Norwegian media database including newspapers, journals, tabloids, etc, identified 1896 articles, reduced to 96 for relevancy. These storylines were identified: 'The majority language and culture are keys to learning and knowing mathematics', 'Mathematics is language- and culture-neutral', 'Minoritized mathematics students are motivated by gratitude', 'Extraordinary measures are needed to teach students from minoritized groups mathematics', 'Students from minoritized groups must put in extraordinary effort to learn mathematics', 'Students from minoritized groups' mathematics achievements are linked to culture and gender', and 'Students from minoritized groups underachieve'

Keywords: Communication, First nations and Indigenous cultures, Social Justice

Our investigation of news media launches the beginning of a longitudinal, participatory research project with school leaders, teachers, youth, community members, and families to understand the storylines at play and enact the ones that would position youth in asset-based ways. Mathematics education scholars have shown for decades the inequities in students' opportunities to learn mathematics, and we are driven by this fact and focusing on the Scandinavian context because this issue has become more pressing with recent migrations (Källberg, 2018; Ryan, 2019, Udir, 2018). We also know that the positionings presented in news media may affect individual students' and groups of students' identities (Mendick, 2005; Wagner, 2019), the relations with and expectations of mathematics education and thereby opportunities for mathematics learning and life choices. These are the reasons for us to investigate storylines about youth from minoritized cultures and/or languages in news media with the goal to identify positionings made available in this public discourse. Although we focus here on storylines in Norway, the methods and findings are relevant elsewhere. That is, news media concurrently reflects and influences public opinion on mathematics and mathematics education, and about migrated and Indigenous youth everywhere.

We here use the word *minoritized* for the groups and youth we are interested in. We note here that we are aware that any wording contains possibilities for misunderstanding, possibly stigmatizing or are unfamiliar for the people themselves. The alternatives—*nondominant, minority, othered, non-Norwegian, multicultural*, etc.—all rest on attributions that are not always consensual and they imply problematic power relationships.

Storylines

According to positioning theory, people interpret their experiences through storylines – through "lived stories for which told stories already exist" (Harré, 2012, p. 198) such as for instance a coach/athlete storyline. Berman(1999) pointed out that the multiple storylines at play "are organized through conversations around various poles, such as events, characters, and moral dilemmas. Cultural stereotypes like nurse/patient, conductor/orchestra, mother/son may be called on as a resource" (p. 39).

Storylines make positions available, which could be either accepted or resisted—e.g., a parent helping a child with homework could position himself as a teacher in a teacher/student storyline. The child could resist and try to interact within a different storyline. Hence storylines are negotiable; they are reciprocal and contingent (Wagner & Herbel-Eisenmann, 2009).

Storylines about Mathematics and Mathematics Education in Public Media

Recent scholarship has begun to identify storylines present in news media on mathematics and mathematics education. One such storyline is *mathematics equips society* identified by Herbel-Eisenmann et al. (2016). This storyline connects mathematics with the pursuit of economic growth and national prosperity and positions students and their mathematical achievements as national commodities valued by means of global ranking systems such as PISA and TIMMS (Lange & Meaney, 2018). Yasukawa (2019) described how these rankings translate into national pride or shame. In contrast to the storyline that positions countries as competitors, the storyline *mathematics equips the individual* positions individuals as combatants in pursuit of social and economic advancement (Wagner, 2019) or as citizens equipped for collective action (Jablonka, 2003; Rodney, Rouleau & Sinclair, 2016).

Storylines in Norway

Although the media storylines discussed in the previous section apply to many contexts, different countries also have storylines that might be more particular to that context. In Scandinavian societies, it takes a 'mathematics for all' approach (Nortvedt, 2018) as for example the Norwegian national curriculum where qualities such as social justice, equity, and equal opportunities are emphasized. As stated in the introduction, however, such ideals in education and mathematics education, may not have been realized for minoritized groups in Scandinavia.

In Norwegian contexts, several groups with a different origin than Norwegian are mentioned, often with a reference to a connection to a (former) nation state. These groups are mainly seen mentioned from the 1970s and onwards when migrant workers from Pakistan, Turkey, and Morocco started to arrive in Norway. More recently immigration also comprises people from other European countries and from conflict areas in, for example, Asia and Africa (Reisel, Hermansen & Kindt, 2019). In addition to these, there are six peoples/nations of Norway (without their own nation state) that appear in the contexts we are interested in; the Kven and Sami peoples belonging to the northern part of Norway, the Forrest Finns in the South, and the non-territorial Romani, Rom and Jews. These minorities have suffered from various injustices and assimilation policies over the centuries—e.g., for the Romani people the most important measure for assimilation was that children were taken away from their families and placed in Norwegian families, later supplemented with forced sterilisation of Romani women (Kommunal- og moderniseringsdepartementet, 2015).

For our research, such context is part of the data because it provides sources of potential storylines. What people say about a country and its inhabitants are storylines. For minoritized groups in Norway there are (historically shifting) storylines, some of which several of the groups have in common. Some might be seen as positive: e.g., they are good for the labour force; are necessary for the growth of Norway; they produce high quality trade and handicrafts. Others are not positive: e.g., the culture and languages of the peoples are unwanted; the groups do not share with others; the groups do not comply to norms of the society.

Methodology

This research is part of the MIM-project, that in collaboration with partners in the U.S. and Canada investigates educational possibilities and desires, here in Norwegian contexts, particularly focused on mathematics education in times of societal changes and movements. Although we focus on the Norwegian context, we recognize that these kinds of societal changes and movements impact many countries throughout the world. With these changes and movements of people, language diversity may be the most obvious challenge in mathematics classrooms, but they connect to cultural differences and conventional characteristics of the discipline. Indigenous communities have experienced linguistic and other challenges for decades as a result of colonization. Such tensions are now appearing in "ordinary" Norwegian classrooms because tensions in education are intensified by language and cultural differences in times of large migration (Cenoz & Gorter, 2010). These tensions are reflected in public news media. They are local but reflect global trends. News media reflects these trends and thus reifies them as public storylines, which impacts students' potential positionings.

We drew on text-based mass media sources that acknowledge Redaktørplakaten, an ethical codex for publishers in Norway: including daily newspapers, weekly or monthly journals, tabloids, etc. We focused on articles published from January 2003 to September 2020 to include the time in which a new national syllabus was launched in 2004 and the discussion leading to the launch. A librarian supported our search of the Norwegian database Atekst (http://retriever.no) to identify articles that included words from each of three groups that represent the categories shown below with their groups of words (these are English translations of the actual words):

A) Indigenous and migrational contexts: Indigenous, monitories, migration, immigration, Sami, Kven, Forest Finns, Romani, Jews, multilingual, multicultural, diversity

B) Education: education, school, upper elementary, high school, teaching, pedagogy, didactics, class, classroom, teacher, student, assessment, grades

C) Mathematics: mathematics, math, mathematics didactics, science (2 different words), economy, statistics, coding, geometry

We use positioning theory and ask: What storylines about minoritized youth and their relationship with mathematics education are portrayed in the news media articles?

To answer our question, we read the identified 1896 articles and narrowed them to 501 after removing articles deemed irrelevant to our research focus. In this reading, positioning theory focused our attention towards how a) mathematics and b) minoritized students were portrayed and positioned and how those positionings were enacted as interconnected in and across the articles. While reading the newspaper articles we noted 25 concepts which roughly expressed the positionings we were paying attention to and hence could provide preliminary grounds for storyline identification. Based the presence of 25 concepts in the 501 newspaper article titles we selected 96 articles for further analysis. We operationalised some of the concepts into words that could be applied as search words in the freeware AntConc's concordance tool to identify excerpts in which the words appeared. This generated 319 excerpts which we read several times and preliminarily coded in an iterative process based on the positionings we found in relation to the search words. To identify storylines of minoritized students required us to look at how students from the dominant group were positioned too, due to the reciprocity of positioning. The process was conducted both jointly among us and individually which allowed us to compare and refine our coding. Finally, we grouped the excerpts according to the coding and re-read the excerpts to articulate the broader storylines about minoritized youth (See Table 1).

Table 1: Storylines identified in Norwegian media and their coding.

Final coding of excerpts	Storyline
Excerpts that explicitly or implicitly position students from minoritized groups by explaining their (lack of) opportunities in mathematics education in relation to the majority society.	*The majority language and culture are keys to learning and knowing mathematics*
Excerpts that position mathematics in relation to students from minoritized groups.	*Mathematics is language- and culture-neutral*
Excerpts that explicitly or implicitly evaluate students from minoritized groups' mathematics achievements.	*Minoritized mathematics students are motivated by gratitude*
Excerpts that explicitly or implicitly position students from a minoritized system/curriculum/teaching.	*Extraordinary measures are needed to teach students from minoritized groups mathematics*
Excerpts that position minoritized students as the ones who need to give extra effort.	*Students from minoritized groups must put in extraordinary effort to learn mathematics*
Excerpts that explicitly or implicitly explain students' (lack of) opportunities in mathematics education in relation to aspects of them being students from minoritized groups.	*Students from minoritized groups mathematics achievements are linked to culture and gender*
Excerpts that point out students from minoritized groups' (lack of) achievements in mathematics education without giving reasons.	*Students from minoritized groups underachieve*

Results

The storylines that we identified are entangled and sometimes overlapping. Other well-known storylines in mathematics education that do not solely relate to students from minoritized groups such as *mathematics is a gatekeeper to success* were also present in the data material, either as connected to or separated from the seven storylines we identified. Some of the excerpts of texts seemed contradictory or resistant to the storylines that we identified, but they were still positioned in relation to the storylines of the article. We provide examples of each of these storylines in the next sections. All the quotes are our translations to English.

The majority language and culture are keys to learning and knowing mathematics

The majority language and culture are keys to learning and knowing mathematics is the most commonly occurring storyline in our data. We find it referenced by students, educators, policy makers and everyday citizens. The student, NN[1], is quoted: "Norwegian is the key. If you know Norwegian, you can learn math and science as well, says NN, who will take up health subjects this autumn" (*Aftenposten*, 5 August, 2014). An educator is quoted: "They spend a year here learning different subjects such as Norwegian, English, science, social studies and mathematics. We focus mostly on Norwegian because they are immigrants" (*Arendals Tidende,* 6 December, 2016). And policy-maker "HH in the Education Association believes much of the explanation lies in the language. Those who do not master Norwegian well enough also have difficulty keeping up with the math lessons, she believes" (*Aftenposten,* 19 September, 2007). When text in the media is not attributed to individuals in the education system, we see it as a representation of common citizen views—for example, "With poor knowledge of Norwegian, this also leads to weaker results in other core subjects in upper secondary school. Today, parents themselves must buy a Norwegian mathematics book in order to be able to assist their children which shows that we are 'astray' in Sami school policy" (*Finnmark Dagblad*, 19 November, 2008)

There is, however, some recognition of complexity in this storyline. The fact that this complexity needs to be explained also reminds us that the simplistic storyline is dominant. For example, a teacher is referenced here: "[EL] has experienced that foreign language students have had great difficulty understanding professional language, even though they cope/do well with the [majority] language in daily life" (*Kommunal Rapport*, 26 October, 2005). Both complex and simplistic versions of the storyline suggest a competition that is set in a storyline of limited resources. For example, the questioning of the value of the Sami language: "There has been too much focus on Sami textbooks in all subjects which has harmed the quality of language subjects and mathematics for the students" (*Finnmark Dagblad,* 6 November, 2008). This storyline positions minoritized groups' languages and cultures as irrelevant or unwanted. Despite the evidence that minoritized groups' languages and cultures are resources for mathematics learning (Huru, et al., 2018; Planas & Setati-Phakeng, 2014), this is still a prevailing storyline.

Mathematics is language- and culture-neutral

Mathematics is language- and culture-neutral is a storyline that is well-known in mathematics education (e.g., Wagner & Herbel-Eisenmann, 2009) but has not been analyzed in news media. The storyline appears contradictory to other storylines we identified, including *the majority language and culture are keys to learning and knowing mathematics* and *students from minoritized groups' mathematics achievements are linked to culture and gender*. The storyline appears explicitly, as in "mathematics and physics are the subjects where the cultural barriers are least/smallest. They are universal subjects" (*Aftenposten,* 3 May, 2010). The storyline also often appears tacitly, for example, visible by pointing to the fact that other subjects are language and culturally rich: "You do not need to learn much more than 'open up' in a new language, smiles dental student [AAY] (35) who originally comes from a Kurdish area in Turkey" (*Osloby*, 10 April, 2014). Another example of this storyline being represented comes in the newsworthiness of culturally-based mathematics programs, which would not be newsworthy if the public recognized the cultural aspects of mathematics: "They pointed out that Sami culture must be the starting point for teaching and not just a supplement. … They first arranged a culture-based mathematics day at several grade levels, which was so successful that they also arranged a culture-based oral exam in mathematics" (*Finnmark Dagblad,* 5 March, 2014).

Minoritized mathematics students are motivated by gratitude

The storyline *minoritized mathematics students are motivated by gratitude* is a tacit storyline that appears in positionings of obligation, gratitude and benevolence. This storyline is closely entangled with a storyline about the 'grateful immigrant' which imposes certain societal behaviours, expectations and obligations such as willingness to work hard, gratitude to the host nation and unwillingness to be a burden to the state resources (Schwöbel-Patel & Ozkaramanli, 2017). It suggests that the model student from a minoritized group must excel in education, contribute to labour, and display vulnerability and weakness to honour the benevolence and superiority of the host county's culture (Thiruselvam, 2019).

Gratitude and benevolence materialize, for example, on the 17th of May, the Norwegian National Day, when it is common for immigrant students to express their gratitude in speeches at their schools. This newspaper article reports on how an immigrant girl exclaims gratitude for her life and the opportunity to go to school in Norway as part of such celebrations. Her exclamation makes available a benevolent position for the society to inhabit, as the reciprocity of positioning requires an analogous positioning for majority culture: "'Dear all,' she begins, 'I am so grateful! Thankful that I get to go to school here in Norway'" (*Haugesunds Avis*, 9 December, 2019).

This debt of gratitude implies an expectation of loyalty: "Norway has given us a safe place to be. Then we must show Norway respect back. I cannot sit still and wait. Only I can help my children to become good people, to get good jobs that will help Norway" (*VestNytt,* 28 June, 2019). To succeed and pay their debt students from minoritized groups must work harder and be more ambitious than students from the dominant group—an effort that is expected of them as part of the storyline about the grateful immigrant (Schwöbel-Patel & Ozkaramanli, 2017): "Immigrant youth have higher ambitions than the rest of the students. But they struggle at school" (*Aftenposten,* 19 September, 2007). Herbel-Eisenman et al. (2016) identified a storyline present in the public realm which imply that the main goal of mathematics education is to produce a STEM workforce. This is how mathematics equip society (Wagner, 2019) and how students from minoritized groups' mathematical knowledge can materialize the "Norwegian dream" as exclaimed by a former Norwegian prime minister.

Extraordinary measures are needed to teach students from minoritized groups mathematics

The storyline *extraordinary measures are needed to teach students from minoritized groups mathematics* is entangled with the storyline *the majority language and culture are keys to learning and knowing mathematics* because they each position students as lacking the majority language and culture. Therefore, this storyline connects to work about in(ex)clusiveness and positioning of students from minoritized groups as the deficit other. This positioning has been comprehensively discussed in mathematics education research (e.g., Gutiérrez, 2008; Källberg, 2018). Here, the reporter notes that teaching in multilingual classrooms requires extraordinary measures: "The math teacher speaks clearly. He paced off and showed with his whole being how they can calculate the area and volume of the classroom. The large differences in knowledge in the class require a little extra from the teachers" (*Bergens Tidende,* 26 November, 2018). To a professional teacher these two teaching strategies may appear mundane, but the reporter communicates them to the public as newsworthy and thus extraordinary. The positioning of the "large differences in knowledge" is not about valuing knowledge as a resource but indicates that it is extra work for a teacher because dominant knowledge is what matters most.

This storyline also intersects with the *gratitude* storyline as the benevolence of the society materializes in extraordinary measures on policy levels: e.g., introduction classes, summer schools, special language programs, and national syllabuses for Sami students. These special measures position additional languages (additional to the two Norwegian standard written varieties *Nynorsk* and *Bokmål*) as problems rather than resources, which is a phenomenon of interest in multiple contexts of mathematics education (e.g., Planas & Setati-Phakeng, 2014).

Students from minoritized groups must put in extraordinary effort to learn mathematics

In contrast to the storyline above, *students from minoritized groups must put in extraordinary effort to learn mathematics* says that students from minoritized groups work extra hard and put in more effort to learn and perform well in mathematics. One student is quoted "I go for the best possible grades. In the other subjects, I can read most things, but in mathematics I have to understand all the concepts" (*Aftenposten,* 19 September, 2007).

Some migrant students go to community-governed extracurricular Saturday schools to do better at the compulsory school: "Principal DD says the Saturday school did not come about because the children learn too little at the regular Norwegian school. But they need someone who pushes them further, and here they learn a culture to work, he says. *(Bergens Tidende,* 27 December, 2015). Some migrants volunteer to help fellow migrant students to pass compulsory school courses in mathematics: "MM (16), who is very good at math, helps the less experienced SS. The talk goes. In both Somali and Norwegian." *(Bergens Tidende,* 26 November, 2018). Additionally, the schools offer extra courses: "Many of the students at TT spent the last holiday week closing knowledge gaps in mathematics in the transition between lower secondary school and upper secondary school. (*Arendals Tidende,* 14 August, 2018).

Lastly, a migrated student offers advice: "If you have to choose; should the kids master math, or should they master Nynorsk [the non-dominant Norwegian standard variation]? Make a priority, because there is a real need to spend more time on mathematics, Bokmål [the dominant Norwegian standard variation] and so on. Subjects we actually need later in life." *(Altaposten*, 23 May, 2018). This storyline intersects with the gratitude storyline: students themselves recognize the extra work that is expected (Schwöbel-Patel & Ozkaramanli, 2017).

Minoritized students' mathematics achievements are linked to culture and gender

The storyline *minoritized students' mathematics achievements are linked to culture and gender* is often present in communication acts in the media which refer to statistical surveys. Roughly this storyline suggests that some immigrant boys fail at school and consequently in the society at large. Immigrant girls, on the other hand, are usually positioned as more successful than students from the dominant group. "Less than half of those who start upper secondary school complete on time, i.e., three years. This gloomy statistic is extra gloomy for boys with immigrant backgrounds. Only one in three with such a background gets through in three years" (*Nordlys,* 29 May, 2008). "About half of the girls with non-western backgrounds in [school name] take higher education. It is far above average and shows what a resource the students from minoritized groups are to the Norwegian society" (*Romerikes Blad*, 27 June, 2012.).

This storyline is connected to other storylines involving cultural superiority and inferiority stereotypes: "[NN, MM and PP] won the math competition. 16 of 38 finalists had a background as students from minoritized groups, as did six of nine winners. […] Researchers believe this is due to the fact that math has a higher status in Asian countries" (*Aftenposten,* 3 May, 2010). "Many are positively surprised that a boy from Eritrea can do so well. The fact that people are surprised motivates me a lot to continue to work hard" (*Innherred,* 18 August, 2018). This storyline intersects with racial narratives about academic ability (Shah, 2017).

Students from minoritized groups underachieve

The storyline *students from minoritized groups underachieve* intersects with the storyline *students from minoritized groups' mathematics achievements are linked to culture and gender* but differs because it makes no distinctions among genders and cultures and offers no causes for the underachievement. For example, "In Norway, Sweden, Belgium and France, more than 40 percent of first-generation students lack elementary math skills. This also applies to a third of the students with a background as students from minoritized groups who were born in Norway" (*Aftenposten,* 19 September, 2007).

Mathematics is a subject that stands out in students from minoritized groups' underachievement: "The most visible is the difference in subjects such as main/chosen

Norwegian standard variety and mathematics, where immigrants got more than half a grade lower than other students. Norwegian-born students with immigrant parents have somewhat higher grades than immigrants, but on average somewhat lower than other students" (*Østlendingen*, 18 November, 2011. This storyline is closely entangled with and perhaps even an inevitable consequence of the storyline *the majority language and culture are keys to learning and knowing mathematics* since the majority language is a necessity for being positioned by the dominant group as knowledgeable in mathematics. Further, it relates to well-known storylines about achievement gaps among different groups (e.g., Gutiérrez, 2008)

Discussion

Our motivation to investigate the storylines about youth from minoritized groups in the Norwegian text-based mass media sources was initiated by an urgency to understand some of the storylines and positionings that might be available in this context. This investigation is the beginning of a longitudinal project in which we are collaborating with teachers, administrators, community members, youth and families to understand what storylines they would like to have made available to them in the teaching and learning of mathematics. Drawing on a participatory design, these storylines can then be used to imagine new positionings and practices in mathematics classrooms and in schools. We started with an analysis of media to sensitize ourselves to some of the existing storylines and, thus, to recognize storylines already potentially available and shaping positionings. Our hope that our sensitization to existing societal storylines will help us, our collaborators, and hopefully others to (re)imagine and to enact storylines that position minoritized youth in asset-based ways.

Our investigation shows that minoritized youth are positioned in relation to an array of storylines that sometimes overlap, intersect or contradict each other. What strikes us about our findings is how the storylines (once again) show that the burden is put upon minoritized youth with no recognition of history, systems or structures that contribute to inequities. According to some of the storylines they are expected to work hard at learning the language and practices of school academic mathematics and also at learning the majority language. While working hard through this double learning burden, minoritized youth also carry the burden of being thankful and expressing gratitude towards the benevolence of the majority society for undertaking extraordinary measures on their behalf. Taking the storyline *students from minoritized groups underachieve* into account here suggests that minoritized students carry the burden of the majority society's disappointment—despite the extraordinary measures, minoritized youth do not seem to meet the expected learning outcomes. Positioning theory reminds us that positionings and storylines are negotiable. This means that minoritized students' available positionings can be renegotiated, for instance by actions that remove burdens and deficit-based storylines.

We are also struck by how contradictory language appears in the storylines. Language and culture appear to be keys to learning and knowing mathematics. Concurrently mathematics appears to be language- and culture-free. We are intrigued to further investigate how this contradiction makes its way into mathematics classrooms as one tension that becomes intensified in times of national and international migration (Cenoz & Gorter, 2010). Dealing with this tension influences how minoritized students might be positioned in asset-based ways and consequently involves actions that can remove burden.

Note

[1] Two cap letters are used to pseudonym people in the data.

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NARRATIVES OF RAZA IN MATHEMATICS EDUCATION RESEARCH LITERATURE

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We present a finding from a literature analysis of Raza populations published in top-tiered peer reviewed mathematics education journals. We look at how narratives are perpetuated and resisted at the intersections of Raza, mathematics education, and research. The findings reveal the field of mathematics education research is perpetuating deficit narratives of Raza through 1) simplistic descriptions of Raza which perpetuate a racial hierarchy; 2) white institutional spaces group, order, and Americanize Raza populations; and 3) counter-stories of La Raza; however, we will only concentrate on the first finding for this manuscript. The examined literature continues to center Anglos' narratives and values while maintaining a social hierarchy and the assimilation and Americanization of La Raza. Finally, we provide implications for disseminating our research to go beyond simplistic demographics of social constructs.

Keywords: Equity, Inclusion, Diversity; Research Methods; Social Justice; Systemic Change

There is a long history of racism in the making of a capitalist society by Anglos in the United States (Haney López, 2006; Molina, 2014; Peller, 2012; Yancey, 2004). Westward expansion and building the economy is a romanticized story passed from one generation to the next when in reality "the massive extermination of indigenous people provided our land base; the enslavement of African labor made our economic growth possible; and the seizure of half of Mexico by war…extended this nation's boundaries" (Martínez, 2017, p. 43). Furthermore, dominant narratives describe Raza as lazy, dirty, ignorant, untrustworthy, and unambitious (see Gonzalez, 1990; Muñoz, 2007; San Miguel, 2001) allowing Anglos to justify the brutal violence and lynching of La Raza (Martinez, 2020). We use the term Raza and La Raza as a political move to disrupt Anglos' oppressive strategies of naming, defining, and centering Eurocentricity in order to maintain dominance (see Anzaldúa, 1987; Martínez, 2017; Gutiérrez, 2001). La Raza is a term derived from the community, loosely translating to "The People" or "The Race". This term has been used previously in political uprisings, such as El Movimiento, in phrases such as "Viva La Raza" to energize and empower La Raza to work towards social justice (see Gutiérrez, 2001).

An Anglified history has omitted the lived experiences, knowledges, stories, and contributions of Asian, Black, and Raza communities. Anglos have always been positioned as the heroes, saving others; this is the consequence of telling history from only the perspectives of Anglos (i.e., Texas Rangers, see Swanson, 2020). Dominant narratives strategically erase the contributions, voices, and lived experiences of Raza populations. Raza have been responsible for the development of land and building the U.S. economy, active in politics and government affairs, enlisted in the U.S. military and gave their lives in wars, and persistent in acquiring better and more just educational experiences for themselves and future generations (Muñoz, 2007; San Miguel, 2001). Further, Raza continue to be active in the previously mentioned ways in spite of being treated as second class citizens, where they are oppressed, segregated, manipulated, de-humanized, and lynched by Anglos (Martinez, 2020).

The previously mentioned deficit narratives, histories, and ways of acting are embedded within our ways of knowing and doing in the present (racism as permanent and endemic in our society; Delgado & Stefancic, 2017). In order to make changes to the system, we must reveal, interrogate, and dismantle deficit storylines of La Raza. Oftentimes, we justify actions and dialogue because 'that's the way it's always been done' or by calling actions and dialogues 'norms'. There is also an apprenticeship mentality in academia where doctoral students and early career faculty are expected to follow the ways of their predecessors. Just as society has found ways to perpetuate racism without (always) being explicitly racist, academia perpetuates racism without (always) being explicitly racist as well. These implicit ways of perpetuating racism also permeate mathematics education research. In this paper, we share one of our findings from a critical literature analysis on the narratives being written in mathematics education research on Raza populations. We use a critical lens to uncover how racism is perpetuated in mathematics education scholarship.

Theoretical Framework

In this section, we describe Critical race theory and Latinx critical theory as a framework guiding our critical analysis of the literature in mathematics education research on Raza populations.

Critical Race Theory and Latinx Critical Theory

Thus far, we have provided a history of racism in regard to Raza as well as an understanding of racism as permanent and endemic in our society. The belief that racism is permanent and endemic within our society is the first tenet of critical race theory (Bell, 2018) and consistent with other critical race scholars (e.g., Delgado & Stefancic, 2017; Solórzano & Yosso, 2002). Bell, along with other legal scholars, activists, and lawyers, developed critical race theory in legal scholarship to "combat the subtler forms of racism that were gaining ground" when the Black Power and Chicano Movements of the 1960's began to subside (Delgado & Stefancic, 2017, p. 4). Latinx critical theory extends critical race theory to highlight the racialized experiences of Raza communities. Raza have unique experiences being racialized based on language, citizenship, immigrant status, phenotypes, and surname. Therefore, Latinx critical theory's specificity to the Raza community will provide a more targeted lens for this literature analysis.

Other tenets of critical race theory and Latinx critical theory vary, however, the tenets guiding our work are the social construction of race, challenging the dominant ideology, and centering experiential knowledge of Raza (Delgado & Stefancic, 2017; Solórzano & Yosso, 2002). The social construction of race "holds that race and races are products of social thought and relations…Not objective, inherent, or fixed, they correspond to no biological or genetic reality; rather, races are categories that society invents, manipulates, or retires when convenient" (Delgado & Stefancic, 2017, p. 7). Anglos have constructed narratives in order to place races in a hierarchy, always with Anglos at the top of the hierarchy as a superior race. Furthermore, critical race scholars challenge dominant ideologies (i.e., colorblindness, neutrality, meritocracy, equal opportunities) and "argue that these claims act as a camouflage for the self-interest, power, and privilege of dominant groups in U.S. society" (Solórzano & Yosso, 2002, p. 26). We highlight this tenet through our critical literature analysis by revealing how dominant ideologies have been camouflaged in mathematics education research and perpetuate racial hierarchies.

The final tenet of critical race theory and Latinx critical theory used to frame the critical analysis is centering experiential knowledge. Delgado and Stefancic (2017) assert that:

> The voice-of-color thesis holds that because of their different histories and experiences with oppression, [B]lack, [Indigenous], Asian, and [Raza] writers and thinkers may be able to communicate to their white counterparts matters that the whites are unlikely to know (p. 11).

Therefore, we take a critical look at the narratives being perpetuated about Raza communities. In centering our own voices, as Raza, we are able to reveal the racism being perpetuated in mathematics education research.

Using critical race theory and Latinx critical theory, we conducted a critical literature analysis on the narratives of Raza in mathematics education research. The purpose of this literature analysis is to understand the narratives of Raza students in mathematics education research literature. An interrogation of the narratives being told will provide insight into how racism is embedded within our practices, how dominant ideologies are perpetuating racism, and how the field of mathematics education research can be more equitable.

The research questions guiding the overall literature analysis are: 1) How are Raza populations discussed in mathematics education research literature? 2) How might mathematics education research perpetuate deficit narratives and/or racial hierarchies of La Raza? 3) How are mathematics education researchers challenging dominant narratives of La Raza? For this paper, we focus on the second research question.

Methods

Data collection involved conducting a comprehensive search of relevant literature and identifying specific criteria for selecting and appraising appropriate primary research, which served as the data set for the critical literature analysis.

Phase 1: Journal and Article Identification Process

To locate relevant articles, we began with Williams and Leatham (2017) list of top-tiered journals in mathematics education research. The purpose for choosing top-tier journals in mathematics education research is the impact these journals have on the field, therefore, the narratives constructed in these journals are more likely to be representative of acceptable narratives within mathematics education research. Williams and Leatham (2017) compiled a list of journals based on opinion- and citation-based criteria. We used journals which appeared on both lists (opinion and citation) and could access. The journals, in alphabetical order, include: *Educational Studies in Mathematics*, *For the Learning of Mathematics*, *International Journal of Mathematical Education in Science and Technology*, *Journal of Mathematical Behavior*, *Journal of Mathematics Teacher Education*, *Journal of Research in Mathematics Education*, *Journal of Urban Mathematics Education*, *Mathematics Education Research Journal*, *Mathematics Teaching and Learning*, *School Science and Mathematics*, and *ZDM*.

Once the journals were identified, a list of keywords were compiled which captured how Raza are described as a population. The following keywords were used to search in each of the 11 journals using their home database search engines (e.g., JSTOR, Wiley): Chicana, Chicano, Chicanx, Chican@, Hispanic, Latina, Latino, Latinx, Latin@, Latino/a, and Latina/o. Using the keyword search and journal selection criteria, 447 articles were identified.

We did not use terms around language which are often used to describe Raza populations, such as Emergent Bilingual, Limited English Proficiency, English Language Learner, English as a Second Language because these terms are not specific to Raza populations. We argue in order to make claims about Raza populations in the circumstances of language, then these identity markers would need to be supplemented by the use of the above keywords used in our search.

Furthermore, we analyzed articles which only took place in the United States. We chose this criterion in order to be consistent with the ways Raza are oppressed and dehumanized within the same context as well as there to be a familiarity of this oppression and dehumanization with our audience. Finally, the ages of Raza in the data range from pre-K to adults: children ages 3 or 4 to parents, teachers, and preservice teachers.

Phase 2: Identifying Relevant Articles

During the second phase, we searched each article for the keyword term(s) to determine the capacity in which they were being used within the publication. The articles which only used keyword(s) in the references were not considered for analysis. Further, articles were discarded that were book reviews, research commentaries, or conference proceedings. Additionally, we excluded articles published prior to 2000 in order to provide a more contemporary analysis of the literature. At this point, there were 256 articles.

Next, we determined our inclusion criteria for data synthesis to include empirical studies where the Raza population was at least 50% of the population being studied or there were specific claims being made about Raza populations within their study. For example, in Jackson et al. (2015) students self-identified as 10% and 2.4% Raza in each of the middle schools, however, the authors make specific claims to Raza populations. They state, "As we examine the data by race, the largest average gain from pre to post of the experimental group was by the African American students…whereas the lowest average gain were [Raza] (2.44%), and whites had a gain of 5.44%" (p. 339). While the Raza population is below 50%, the authors make claims specific to La Raza, which help to understand the narratives of Raza in mathematics education research (research question 2). This criterion was set because making an argument about the narratives specific to Raza from these findings would be challenging. Even with a Raza population of 40%, unless the author(s) make specific claims about the Raza population somewhere else in the article (i.e., results, discussion), we were not convinced a strong enough argument could be made about their findings in regard to Raza. Therefore, the articles that only mentioned the keyword(s) as a percentage of their population, and the percentage is below 50% do not help in answering our research questions. After applying each of the criteria to determine relevant articles, we had 52 articles to analyze.

Synthesizing Articles

We analyzed all of the articles using a grounded theory approach to allow themes to emerge from the data. In order to more fully understand how narratives of Raza are discussed in mathematics education research literature it was important to allow the themes to emerge as opposed to applying an initial coding scheme to the data. We felt this would provide the opportunity to uncover instances that a coding scheme may be blinded to. For each article, we identified the background literature, research questions, description of the population, results, and implications. We also kept track of any comparisons of populations based on race/ethnicity, how Raza populations were being positioned, and if being Raza was necessary to the research questions, framing, results, or implications. Analyzing the data from the description of the population being studied provided the foundation for our first finding, which we will concentrate on in this paper. Further, keeping track of comparisons helped to understand how racial hierarchies play a role in the construction of narratives of Raza populations. We also needed to understand the different ways Raza populations were being positioned within the article, this allowed us to categorize articles which perpetuated a deficit narrative of Raza populations in comparison to articles which provided a counter to dominant narratives.

In the second iteration of analyzing the data, each aspect of an article, background literature, methods (including the description of the population as a separate category), results, and implications, were each coded as either providing a dominant narrative, countering dominant narratives, or neither. This allowed the authors to analyze each section of an article independently to understand how aspects of an article may perpetuate or counter dominant narratives, as well as see the article as a whole to understand how narratives of Raza are being perpetuated or countered. In doing so, we were able to categorize articles and sections of articles as perpetuating or countering a dominant narrative. For example, we categorized Roy and Rousseau (2005) as countering a dominant narrative because they describe a teacher with high expectations and his success in working with Raza populations. In the description of the population studied, however, they also perpetuate a racial hierarchy of ability stating,

> Matthew was a mathematics teacher in an urban high school in the Midwestern United States. The school served a large [Raza] population (approximately 60%) with smaller populations of [African American], Asian, and white students. The student population of the high school was predominantly of low socioeconomic status, with 65% of the student body eligible for free or reduced lunch…Matthew was assigned to teach some of the lowest-level mathematics classes offered at the school (p. 16).

Our analysis allowed us to see Roy and Rousseau (2005) as a counter to dominant narratives because of the entire paper; however, when looking specifically at the description of the population being studied, they perpetuate a deficit narrative of Raza. Therefore, in order to more fully understand how mathematics education research literature is perpetuating or countering dominant narratives, we needed to look at each section of the article as well as the article as a whole with a critical lens. In doing so, we were able to tease out any nuances in how the field works to perpetuate dominant narratives as well as how some scholars are disrupting these norms in order to provide a more complete understanding of Raza populations in mathematics education research.

Upon dissecting each article, we were able to form themes on the narratives of Raza populations. The initial themes include 1) Raza described as low income, low mathematics ability, and disabled; 2) white institutional spaces group, order, and Americanize Raza populations; and 3) counter-stories of Raza populations. We will concentrate on the findings for the first theme.

Descriptions of Raza Populations Perpetuate a Dominant Narrative

In this section, we present findings related to the second research question: How might mathematics education research perpetuate deficit narratives and/or racial hierarchies of La Raza? We provide representative examples of the articles within this theme as opposed to an exhaustive review of each article due to the conceptual nature of this critical literature analysis (see Harper, 2019). Our analysis produced three major findings; however, we are only concentrating on the first finding for this paper.

When considering the context of the study within an article, the field of mathematics education research perpetuates the use of nominal social constructs, such as race/ethnicity, class, gender, and ability to be included in order to provide the reader with an understanding of where the study is taking place. After critically analyzing the literature, we found it necessary to think more deeply about how norms have been constructed for the purposes of creating and maintaining a social and racial hierarchy. Of the 52 papers analyzed, two-thirds of them used deficit narratives of social constructs to provide context to the study. We, the authors, have also been guilty of perpetuating deficit narratives of Raza populations in the context of our work (see Gomez et al., 2020). It is important, however, to critically analyze how we frame participants within the storylines we write because social constructs are attached to white superiority ideology and used and created to other certain communities (see Haney López, 2006). In our critical analysis, we found the format of perpetuating deficit narratives of Raza varies from one piece of data to the next, however, when interrogating dominant narratives about Raza populations, deficit framings were consistent in perpetuating dominant narratives: Raza populations are low income, do not succeed academically, and are disabled.

We begin with an example from equity focused scholars to highlight how perpetuating norms are not divided along an equity lens, but stem from white ideology of what it means and looks like to do research. Therefore, such norms go unnoticed even with well-intentioned equity scholars. This speaks to the pervasiveness of white supremacy and how we have all been socialized in the academy, a white institutional space, to perform certain acts even when our research works to dismantle systemic racism. Battey and colleagues (2016) state, "[I]t was one of the lowest performing districts in California...100% of the students served were students of color (73% Latin@, 27% African American), 58% were classified as English language learners, and 93% received free or reduced-cost lunch" (p. 6). The authors relate low academic performance to social constructs of race/ethnicity, language, and income status; perpetuating a narrative of what it means to be Raza. However, it is not necessary for the authors to make this connection because we have been socialized to believe large populations of Raza students relate to low income, language inefficiency, and low academic performance (Gándara & Contreras, 2010). The above quote is not an anomaly in academia, but a confirmation of the storylines we perpetuate of the Raza community. Furthermore, Battey and colleagues perpetuate a norm of telling a partial story of their participants through the nominal demographics of race/ethnicity, language, and socioeconomic status as determined by students receiving free and or reduced-cost lunch.

As previously stated, the format of perpetuating deficit narratives varies and can be less discernible through the use of charts and tables. In the following representative example, Hunt (2015) used a table to provide context to the study, naming students' ethnicities, gender, age, and ability. While these social constructs seem harmless, social constructs carry meaning and provide a racial and social hierarchy of ability. Hunt described the three third graders she worked with in the context of the study:

> This exploratory case study sought to uncover how one third-grade child with LD (i.e., "Bill"), one third-grade child deemed as low achieving (i.e., "Carl"), and one third-grade child deemed typically achieving (i.e., "Albert") ...The researcher classified children by their performance on three standard mathematics tests as typically achieving (25th percentile or above on all tests), low achieving (15–25th percentile on all tests), or LD (below 15th percentile on all tests). (Hunt, 2015, p. 96)

Hunt (2015) provides a hierarchy of ability in her positioning of children based on their mathematical performance. This positioning further maintains a racial hierarchy of ability with the following table where the students' ethnicities become part of the story:

Table 1
Student characteristics.

Name[a]	Age	Gender	Ethnicity	LD status and IEP goals	Performance, Woodcock Johnson III (Percentile)	Key Math-R, mental calculation (Percentile)	Key Math-R, numeration (Percentile)
Albert (TA)	9	Male	Caucasian	None	37th	35th	45th
Carl (LA)	9	Male	Hispanic	None	16th	15th	20th
Bill (MLD)	9	Male	Hispanic	LD, math-based	9th	4th	6th

[a] Names have been changed to protect student identities.

Figure 1. Table from Hunt (2015).

In this example, students' ethnicities are not relevant to the rest of the study. Hunt does not mention any more about how their ethnicities play a role in the context of her study, being Caucasian or Hispanic as she described, is not influential in any other parts of the article. Therefore, it is important to take a critical look at the reasons we are including nominal characteristics when providing context to our research.

Furthermore, some authors perpetuated dominant narratives of a racial hierarchy without the use of numbers to describe the population. For example, Selling (2016), described the participants in her study as follows:

> Jorge, Luke, and Carlos had all finished 7th grade in their various schools, but they differed noticeably in their prior academic achievement in math. Luke had previously found success in school math, achieving an A in the final quarter before the summer. In contrast, Jorge arrived at summer school an F, and Carlos had arrived with a D. Jorge and Carlos are both [Raza]. Luke is White (p. 193).

Selling's discourse did not use numbers or percentages to present a racial and social hierarchy based on letter grades. Selling makes it a point to identify the ethnicity of each participant after discussing their academic abilities, positioning the Anglo student as successful and the Raza students as unsuccessful. While Selling does argue Jorge and Carlos find success and contribute to the mathematical discussions in positive ways in the findings of the article, the storyline in the description of the population being studied perpetuates a racial hierarchy of ability with Anglo students at the top and Raza students at the bottom.

Finally, we would like to consider the following example where Gutstein (2016) provides a detailed history of the school where his study took place. Gutstein states:

> Lawndale has suffered disinvestment and neglect for years but has a strong history of activism and efforts toward community betterment. White flight occurred in the 1960s and 1970s, and Lawndale became overwhelmingly populated by people of color, many working in nearby factories. Deindustrialization seriously injured Lawndale: Chicago lost 330,000 manufacturing jobs from 1967–1990 (Betancur & Gills, 2000). Concurrently, the city cut services, property tax revenue fell, and the area suffered. Nevertheless, individuals, community and civic organizations, and churches throughout the neighborhood have worked hard to improve conditions, secure more city support, and develop cultural programs and social services (p. 462–463).

Gutstein, however, goes on to perpetuate the norms of mathematics education researchers by including nominal social constructs to provide context to his study, stating, "Each of the four schools on campus has roughly 375 students, originally 70% Latin@, mainly from Little Village, and 30% Black, from North Lawndale…Sojo has overwhelmingly low-income (approximately

98%) students of color (100%). Students' ACT scores averaged 16.0" (p. 463). The history included in this study provides the reader with an understanding of how racism and capitalism concurrently impacted the community and schools. The history situated Gutstein's work within historical, racial, and social contexts as opposed to being neutral or blind to how these factors impact the community, schools, and students. Gutstein also provides us the opportunity to consider and compare how the nominal social constructs tell a colorblind and race neutral story of the same context, sans historical and social influence on the community.

As the field continues to provide research from only one perspective, that of Raza who are low income, low performing, English language learners, we argue, using nominal social constructs, such as race/ethnicity, class, gender, and ability is problematic when these demographics are not considered in historical contexts. The U.S. has a long history of defining citizenship, ability, and success from an Anglo, male, middle- and upper-class perspective (Martinez, 2020; Muñoz, 2007; Yancey, 2004). Furthermore, the demographics we use to provide context to studies are socially constructed in order to justify privileges or withhold privileges from certain groups (i.e., racism, sexism, classism). Therefore, we have been socialized into bringing in assumptions about each of these nominal social constructs; using demographics to provide context to our work perpetuates dominant narratives about equal opportunity, who can do mathematics, and what it means to be Raza. By including historical and racial contexts within our work, we not only acknowledge how history and racism influence our communities, but we also rehumanize our work and the communities with whom we work.

Discussion

Feagin & Cobas (2014) assert that "[m]ost [Raza] regularly face social environments where whites have the power to racially characterize who they are, including their racial identities" (p. 24). In this article, we use critical race theory and Latinx critical theory as a lens to critically analyze the narratives specific to Raza in mathematics education research. We find it necessary to take a critical look into the literature and stories mathematics education research values and perpetuates of Raza populations based on its choices in publications. Our findings reveal, question, and disrupt the ways we position Raza, and more broadly our participants in general, through our publications. We argue for a more critical look at the field of mathematics education research and the norms that have been set for conducting, disseminating, and reflecting on research endeavors.

Our findings highlight the deficit narratives being perpetuated of Raza populations through the descriptions of La Raza in mathematics education research. The purpose in providing details about race, ethnicity, gender, class, and ability are justified in order to give context to the study. These demographics are included to tell the reader important information, but we cannot determine what information these percentages and descriptions are providing to our audience. We argue the need to be more critical of how we are positioning the populations with whom we are working; they are the ones providing information to us, not the other way around. We need to be respectful of the narratives we are providing for others to read and make sense of. Instead of providing surface level characteristics, which are stereotypes, perpetuating deficit narratives, and complying with and continuing the narratives of a racial hierarchy of ability and belongingness, we need to dig a little deeper with the context of our research. Providing information on the history of the place, the organizational logic, laws and policies in place which impact the humans with whom we work, and other factors which place our human participants in their current situations provides pertinent information for the reader to understand the research and findings.

Furthermore, it also prevents the reader from having to make assumptions about meritocracy or a long history of discrimination, policies, and social hierarchies about who belongs and who is capable.

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COMMUNITY MATH STORIES: INFORMAL ADULT EDUCATORS EXPLORING MATHEMATICS IDENTITY THROUGH DIGITAL MATHEMATICS STORYTELLING

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During the COVID-19 pandemic, many urban schools relied on community centers with existing computer labs and high-speed internet that could provide in-person support for a small number of children engaging in online learning. Using a digital storytelling approach, this research report analyzes the mathematics identities of 14 informal adult educators. Situating the stories shared though Critical Race Theory counternarratives, this study enables participants to ground their narratives within their own spaces of power–to tell and forge their own digital mathematics story. Because informal adult educators are not family members nor school-based educators, they often are invisible variables in conceptualizing a child's mathematics learning. This research seeks to elicit their mathematics stories and understand how to enactment digital mathematics storytelling through listening to how the community positions and visions math.

Keywords: Equity, Inclusion, and Diversity; Design Experiments; Informal Education; Technology

Perspectives

The Shift in Education during COVID-19

During the COVID-19 pandemic, many urban emergent school districts[1] in the United States of America, already dealing with racial and socioeconomic segregation, had to switch to a completely remote model of learning, meaning that children were expected to engage in all their school interactions online (TODOS, 2020). This forced adoption of virtual and online learning exasperated already stark divides in technology access, requiring children to have access to high-speed internet, a stable tablet or computer, and a regular work space just to attend school. This was an almost impossible ask of families and communities who have historically been marginalized in multiple ways, resulting in many of the largest urban and urban emergent school districts in the U.S.A. reporting that as many as 70% of their students were not attending online schools (Christakis et al., 2020).

One community-based approach that rose to address the needs of families struggling to support their children's education is the Learning Extension Center (LEC), community centers with existing computer labs and high-speed internet that could provide in-person support for a small number of children engaging in online learning. These LECs are partnerships between the school district and "community, faith based, and other public sites that provide physical space within the facility to allow workstations for students. The purpose of the LECs is to provide educational and social-emotional support to Columbus City School students in a safe environment and welcoming environment during our prolonged period of virtual learning" (*Columbus City Schools 2020-21 Reopening Archive / Learning Extension Centers*, 2020).

These types of partnership sprang up in urban emergent communities across the country (Greenberg et al., 2020), filling in the gaps in technology access by providing a meeting space for a child to use a sanitized computer, an adult educator who provided nominal technology and instructional support, snacks and meals, and a feeling of community for children. While these LECs were not able to, nor designed to, replicate the affordances of a school classroom, they did provide a necessary space for children in urban emergent communities to continue their school-based learning during the COVID-19 pandemic. The LECs, thereby, became a place where masked children could attend in order to connect with other children, complete online school lessons, and work with an informal adult educator. For many, the LECs played a crucial role in supporting urban emergent families and communities.

Informal Adult Educators

The LECs are staffed by informal adult educators, usually community members dedicated to supporting children, but who are not formally trained or certified teachers. These informal adult educators bring many backgrounds with them. Some might be college students studying to eventually become teachers, local sports coaches who regularly work with community youth, or community members whose other employment was disrupted by the COVID-19 pandemic. Because these informal adult educators are not formally prepared teachers, they often bring more mainstream perspectives to mathematics learning. For instance, informal adult educators may express anxieties and fears about mathematics based on prior experiences. Additionally, informal adult educators may also have limited experiences with inquiry-oriented mathematics pedagogy and therefore position mathematics learning as heavily procedural and focusing on memorization because of their own experiences. Finally, while informal adult educators might be comfortable with using technology in their own lives, they often may have limited experiences with instructional technology (e.g., Google Classroom, Flipgrid).

Informal adult educators are rarely studied within the mathematics education research, partially because their limited role often exists entirely outside of the classroom. However, due to COVID-19 pandemic, informal adult educators now found themselves as the only *adults* outside of a child's household that were regularly physically working with children. Informal adult educators were no longer positioned as supplementary workers or after-school helpers. Due to community-based restructuring of our nation's schools during the COVID-19 pandemic, new roles and possibilities involving the nature of informal adult educators and their impact on children's mathematical learning became apparent. And therefore, as the informal adult educators' roles in urban emergent communities grew, the question we as mathematics education researchers ask is: How do informal adult educators narrate and conceptualize their own relationship to math–their own mathematics identities?

Community and Family Mathematics vs. School Mathematics

These informal adult educators, suddenly tasked with supporting students as they engaged in learning traditional school mathematics, hoped to find ways to explore their own mathematical knowledge in juxtaposition the mathematics they were seeing in online and hybrid learning models they were supporting their children with. One approach to address this issue to connect out-of-school mathematics with in-school mathematics is through recognizing the mathematics knowledge shared in community spaces (Aguirre et al., 2013). For instance, when someone shares about the daily mathematics checks they do to make sure that each of their animals is fed, juggling the feeding schedules of lizards who eat every other day, fish that eat multiple times a day, and birds that eat every day, the mathematics of finding common factors emerges. Mathematics that exists within the community is often dismissed as not "real" mathematics because family and community mathematics often operates socially, involving storytelling and group problem solving, while school mathematics is often positioned as internal knowledge measured individually (D'Ambrosio, 1985; González et al., 2001; Powell & Frankenstein, 1997).

Mathematics Identity

Mathematics identity, often only explored as a students' mathematical identity or as a mathematics teachers' identity, has been shown to be positional, performative, and socially constructed (Esmonde & Langer-Osuna, 2013; Holland et al., 1998; Martin, 2000; Wenger, 1998). For teachers, mathematics identity involves the negotiating of one's provisional self with regard to various sociopolitical positions taken when teaching mathematics (e.g., sexuality, ethnicity, and economic status) (de Freitas, 2008a, 2008b). Additionally, mathematics teacher identity can involve the position a mathematics teacher takes to either confront or reify the oppressive practices within traditional mathematics teaching and learning (Gutiérrez, 2013; Herbel-Eisenmann et al., 2013). Mathematics teacher identity, therefore can "consists of knowledge and lived experiences, interweaving to inform teaching views, dispositions, and practices to help children learn mathematics." (Aguirre et al., 2013, p. 27).

Intersectionality, a lens with roots in Critical Race Theory, Feminist Theory, and Poststructuralism, focuses on examining the intersection of various social spaces of oppression (Crenshaw, 1991). Research examining issues of equity and identity can sometimes position social identities as static variables (Darragh, 2016), leading to generalizations that erase voices and position all members of a category as similar. An intersectionality approach allows participants to present the nuances within their various social identities, showing ways in which they are individually unique and human (Leyva, 2017). Each of these social identities carries with it some artifact of oppression and opportunity. An intersectionality approach, therefore, examines the stories where these multiple social identities overlap. The stories revolve around lived experiences, not second-hand evaluations or judgments (Solorzano & Bernal, 2001), valuing the varying narratives that the adults tell about their backgrounds, pedagogies, beliefs, and positions (Walshaw, 2013).

Work that conceptualizes identities from a *Critical Race Theory* perspective, and *LatCrit* specifically, draws attention to the multiple layers of subordination based upon race, class, gender, language, immigration status, accent, and phenotype experienced by Latinx communities (Delgado Bernal, 2002; Johnson & Martinez, 1998; Lopez, 1997; Solorzano & Bernal, 2001). This LatCrit perspective draws a cultural connection between participants' stories and the cultural construct of *testimonios*–first-person narratives one tells about one's self to others (Gutiérrez, 2013). These narratives often reveal accounts of systemic oppression hidden from more structural modes of inquiry (Yosso et al., 2009). Mathematics identity therefore, from a LatCrit perspective, can be understood as these specific narratives that directly expose or confront oppression (Solórzano & Yosso, 2002; Zavala, 2014).

An intersectionality and LatCrit approach moves away from characterizing identity as a categorical adjective or noun; instead identity becomes a verb, something in-flux, being made and remade, as a sociopolitical act. Identity through storytelling is agentive–an emancipatory act in which adults construct counter narratives that confront and claim power within the oppressive worlds they live in (Aguirre et al., 2013; Gutiérrez, 2013). Since identity involves the telling of one's stories (Sfard & Prusak, 2005), the counter narratives themselves become a theoretical, methodological, and pedagogical means of understanding a mathematics identity (Solorzano & Bernal, 2001).

Storytelling

The most active way to explore mathematical identities involves storytelling. When members of urban emergent communities tell narratives about their out-of-school mathematical experiences, they position themselves and their communities as mathematical (Aguirre et al., 2013; Love, 2014). Given the popularity of video and image-sharing platforms like TikTok, YouTube, and Instagram for sharing personal stories (Auxier et al., 2020; Rideout, 2017), there is a strong case for utilizing these emerging digital literacies to connect out-of-school mathematics to in-school mathematics (Ozpinar et al., 2017) for exploring mathematics identity. In this study, we explore Digital Mathematics Storytelling (DMST), a mechanism for using videos, photographs, and audio to craft and share mathematically-rich narratives from communities, to potentially connect out-of-school mathematics with in-school mathematics in informal education settings and explore identity.

Activity Theory

Cultural-historical activity theory (CHAT) positions learning as occurring when individuals participate in culturally organized activities that are constituted by six essential entities: human subjects (individuals or groups), objects (artifacts and motivations), instruments, rules, community, and division of labor (Engestr m, 1987). This activity system can be depicted as an "activity triangle" (see Figure 1) that is useful in examining and interpreting artifact-based interactions in social learning contexts.

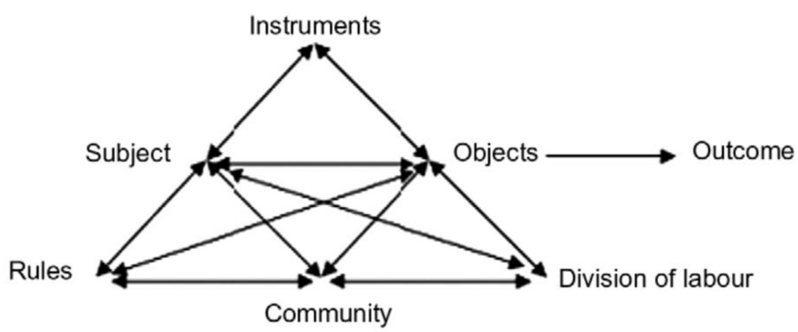

Figure 1. The structure of a human activity system in CHAT. (Engestr m, 1987, P.78)

CHAT attempts to explain the distributed and situated nature of knowing in human activities mediated by artifacts and culture (Engestr m, 1987). When achieving objects in an activity, subjects not merely produce outcomes but also produce or reproduce themselves (Wenger, 1998). It is this dialectic relation between subject and object that prompts the transformations of learners' identities and their learning community.

Digital mathematics storytelling can be situated as a CHAT-oriented learning activity that involves informal adult educators (subjects) using digital technologies (instruments) to create mathematically-rich personal narratives (objects) in the online storytelling circles (community). Through participating in a digital mathematics storytelling experience, adult educators' craft and hear each other's narratives around mathematics, impacting their evolving mathematics identities.

Objectives

This research study focuses on 3 objectives: (1) To develop a protocol for Digital Mathematics Storytelling through Participant Design Research in partnership with adult educators who work in urban emergent community centers, (2) To listen to the stories that informal adult educators tell when given the opportunity to create Digital Mathematics Stories to understand their narrative-based mathematics identity, and (3) To explore the impact of Digital Mathematics Storytelling on their mathematics knowledge, digital literacies, and feelings of connection with the communities they serve. Simply put, this research report analyzes the role of adult informal adult educators within two community learning extension centers during the COVID-19 pandemic. Through the lens of mathematics identity through storytelling from a Critical Race Theory Counternarrative perspective, we explore a framework of mathematics identity specifically involving informal adult educators who work within marginalized urban emergent communities.

Research Modes of Inquiry

Researching Identity

Studying something as personal and complex as identity, particularly when working with communities that are often oppressed and marginalized, requires research strategies and tactics that avoid deficit colonizing perspectives (Patel, 2016). This requires eliciting narratives, and recognizing that these narratives are the participants' identities (Sfard & Prusak, 2005), thereby avoiding unnecessary evaluation and judgment from the researcher. This narrative-biographical approach to studying identity also enables participants to ground their narratives within their own spaces of power–simply, participants tell their own story (Kelchtermans, 1993).

Community-based Participatory Design Research

The community-based participatory design research methodology builds upon design-based research by positioning all participants as integral to the designing of research goals (Bang & Vossoughi, 2016). The first objective of this research project involved developing a model digital mathematics storytelling protocol for the informal adult educators to adapt and use with the children at their LEC. From there, feedback from participants shaped the refinement and modification of the protocol for use with other students in the local community. Within the workshop itself, informal adult educators had the opportunity to question and reshape the research team's plans, while also engaging in an iterative process of telling, refining, sharing and imagining to refine their own digital math stories.

Participants.

This project involved 14 adult educators across two community LECs who work directly with over 100 K-8 children in an urban emergent community. Of the 14 adult educators, all 14 identified as a Black or a Latinx person of color, and had lived in or had substantial roots to the communities they worked in.

The Data Collection Phases

The data collection took place over four phases, with at least one week passing between each phase. In the first session, the participants engaged in a workshop to explore their own feelings towards mathematics and to brainstorm the ways mathematics exists in their lives. Between the first and second session, participants created a draft video in which they shared the ways math might exists in their live. In the second session, the participants engaged in a facilitated reflective protocol called the Storycircle, in which each participant shared their initial story, received feedback from their peers, and thought out loud about how to refine their mathematics story

(Lambert, 2013). Between the second and the third session, the participants worked on crafting their Digital Mathematics Stories, a 3-5-minute video in which they shared a story about mathematics and how it related to them. These videos focused not on a mathematics problem, but on the ways that mathematics occurs in a story that only the participant could tell. In the third session, the participants presented their finished Digital Mathematics Stories and also give feedback and commentary to their peers' stories. After this "screening" session, each informal adult educator engaged in a 1-on-1 interview with one of the authors to discuss how this process connected to their mathematics identity, how it affected the way they helped their children with mathematics in the LEC, and how they might alter the protocol to for their own community.

Measurement/Instrumentation

We focus on three measures in this report.

Digital Math Stories. The informal adult educators created, workshopped, and refined their own short videos stories. Both the draft and the final stories were analyzed.

Weekly Comments. The adult educators also added significant online comments, feedback, and questions to shared, online documents during and after every weekly session and also in the chat feature within the online conferencing platform. These comments and questions involved questions to the research team, feedback to the other informal adult educators, and advice or even further stories and anecdotes about implementing Digital Math Storytelling in their LEC.

Interviews. After the informal adult educators finished their workshops, they each engaged in a 1-on-1 interview to discuss the effect of this experience, how they thought about implementing digital storytelling in their own work at the LEC, and how this experience affected their feelings of connectedness to the community.

Analysis

In order to analyze this data, the research team used constant comparison analysis (Corbin & Strauss, 2008) and narrative inquiry (Clandinin & Connelly, 2000) to compare data from the informal adult educators' digital stories and their written comments during workshops in order to understand the changing aspects of the educators' mathematics identities (Aguirre, et al., 2013; Langer Osuna & Nasir, 2016) and their ability to employ digital tools to meet their needs as storytellers, educators, and mathematical actors. In the analysis, we focused specifically on connecting the ways that the informal adult educators (1) expressed their sense of agency and comfort with mathematics; (2) imagined the process of engaging children in telling their math stories; and (3) gave one another feedback on the stories they shared. The written contributions were analyzed alongside the digital video stories that adult educators created.

Results

Informal Adult Educators' Mathematics Identity

Generally, each informal adult educator expressed trepidation about mathematics in their initial story, revealing fears and anxieties about mathematics based on their own experiences in school. These stories focused on feeling "dumb" or "stupid" in their mathematics class, racial and gender stereotyped reinforced by educators and other adults in the community, and a focus on school mathematics as being about following directions and preparing for tests. Overall, we found that the first sessions of working with the informal adult educators required listening to stories of frustration coupled with empathy for the violent and oppressive mathematics experiences that most of the informal mathematics educators had experienced.

However, in the short time between the initial and the final mathematics stories, many of the informal adult educators had quickly shifted in the way they positioned themselves towards mathematics. Based on the sharing of the stories, the feedback and the commentary of shared negative experience in mathematics, and the shepherding of the research team that mathematics is not about judgement or evaluation but about problem-solving and creativity, the informal adult educators generally showed major shifts in their stories.

For instance, one educator, who was also a college student, shared about how this experience gave her the confidence to sign up for a mathematics course, even though she had been avoiding taking mathematics courses throughout her matriculation. Another educator shared a story about how she felt engaged in the validating the mathematics she did each morning to calculate whether or not she had time to stop for a cup of coffee. And another student shared a story about connecting to the indigenous mathematics knowledge he was curious about in ways to mix specific herbs and plants together to create healthy teas and potions. Overall, the stories showcased the ways that the adult informal educators had found aspects of mathematics in their own lives, but more importantly were not able to recognize that they themselves were mathematically powerful.

The Types of Digital Math Stories

In analyzing the stories, we found that in terms of mathematics content, the stories could be categorized as either *a quantity-based mathematics story* (e.g., how much money do I have in my budget?) or a story involving a *basic arithmetic scenario* (e.g., how do I figure out how much food I need to feed my pets?). In terms of the context, we found that the stories fit within four specific types that each illuminated aspects of the storyteller's life: (1) Work or time management (e.g., how much time do I need to commute to work?), (2) Money or budget management (e.g., will I have enough money to pay for a *Disney+* subscription if I move to a different apartment?), (3) Hobbies or sports (e.g., how can I mathematize my improving basketball playing ability?), and (4) Caregiving (e.g., How do I create a system for taking care of my plants?).

Insight into Supporting Children's Emerging Mathematics Identities

The stories and the commentary surrounding the stories showed how the informal adult educators felt about their math self-efficacy (e.g., negative personal experiences with timed test), how they were starting to grow in their beliefs about mathematics learning (e.g., recognizing the need to listening to children's mathematical thinking), and their approach to enacting mathematics storytelling either through engaging children in a math problem or eliciting mathematical routines or stories directly from children. Additionally, the informal adult educators delved into discussions around the ways they would talk about and position mathematics, particularly paying attention to the mathematics terminology they used in their stories (e.g., tallying vs. counting; being clear about units). These emerging insights show that, while these informal adult educators had little formal mathematics teacher preparation, just going through this exercise engaged them in conversations that dominate most inquiry-oriented university-based mathematics methods courses.

Digital Literacies

Finally, the informal adult educators showed a wide-range of comfort in their "digital literacies" enacted as they created and crafted their videos. They often commented to each other about the "crispy"-ness of each other's videos, focusing on the attention-grabbing transitions and inviting visual techniques used in the videos. Yet, many of the comments also expressed frustration with the availability and usability of the video editing tools they had access to, focusing on what it even means to be "digitally literate." For instance, while we as a research group assumed from the initial conversations that the informal adult educators were well-versed in creating videos for Instagram or TikTok, the commentary around the digital stories showed that many of the informal digital educators used this experience as an opportunity to create their first ever Instagram or TikTok video. We are still unpacking how this might challenging the definitions and forms of what it means to be "digitally literate", that it is less about technical skill or familiarity, but more about opportunities to tell "your" story.

Discussion

While we are still analyzing the stories, the commentary, and the interviews for deeper insights, we will share that this experience of working with informal adult educators has helped us conceptualize the importance of engaging and listening to community members that impact our children's mathematical learning. The informal adult educators live in a unique space; they are not family members nor are they school-based educators. They often are ignored in the conceptualization of the many variables affecting a child's mathematics learning. Yet, community centers and the informal adult educators who work within them are integral members of the community, and their stories and enactments of mathematics has a significant impact onto how our children thereby position and see mathematics in their own world.

Note

[1] The term urban emergent signifies communities in large cities, but not as large as metropolitan areas such as New York or Los Angeles. These urban emergent communities, however, encounter the same scarcity of resources and historical issues of segregation (Milner, 2012).

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CUENTOS MATEMÁTICOS EN COMUNIDAD: EDUCADORES INFORMALES EXPLORANDO IDENTIDADES MATEMÁTICAS A TRAVÉS DE CONTAR CUENTOS MATEMÁTICOS DIGITALES

BECOMING AWARE: AN EQUITY NOTICING FRAMEWORK

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To develop an equity-centered orientation in teacher education programs, it is essential teacher educators recognize what prospective teachers attend to in classroom events and how they relate these events to mathematics instruction. In this study, we examine what prospective teachers attend to in a classroom vignette focused on cultural, racial, and economic biases. Using the Equity Noticing Framework, we identify what prospective teachers attend to, how they critically examine hidden biases, and what actions they would take to be change agents. The results indicate the importance of providing opportunities for prospective teachers to become aware of systems of oppression and ways to foster change in the mathematics classroom.

Keywords: Equity, Inclusion, and Diversity; Preservice Teacher Education; Teacher Noticing

Teaching mathematics from an equity stance requires teachers to understand that students from diverse backgrounds come into mathematics classrooms with different worldviews. Equitable mathematics instruction requires each and every learner has the opportunity to participate in high-quality instruction as a valued member of the mathematics community. However, within the United States K-12 educational system, students from non-dominant backgrounds are often denied equitable opportunities to learn (Wager, 2014) especially when their teachers are different from their own culture and background (Futrell et al., 2003; Turner et al., 2012). Black and Latinx students are more likely to face deficit perspectives from teachers in the mathematics classroom (Berry, 2018). Although mathematics education researchers have examined ways to teach mathematics to diverse populations of students (e.g., Vomvoridi-Ivanovic & Chval, 2014), documented disparities exist for minoritized students (Berry, 2018; Lambert, 2018). Without explicitly recognizing the barriers minoritized populations face from systemic oppression, the vision for equitable mathematics instruction is nullified (Shah & Coles, 2020). Knowledge of these disparities emphasizes the need to better prepare prospective teachers (PTs) to effectively teach each and every learner by developing PTs' orientations toward equity in teacher education programs. Furthermore, "teachers' perceptions of cultural and linguistic competency as they relate to helping children achieve academic and social potential play a very critical role in the type of educational services provided to culturally and linguistically diverse children" (McSwain, 2001, p. 54). Mathematics teacher educators must recognize what PTs attend to in classroom events and how they relate these events to mathematics instruction because teaching expertise requires the ability to notice and interpret classroom events (Mason, 2008). The purpose of this study is to examine what PTs attend to in a classroom vignette focused on cultural, racial, and economic biases.

Equity Noticing Framework

Teacher noticing is an integrated process comprised of three distinct, yet interrelated, levels: (a) attention, (b) interpretation, and (c) instructional decisions (van Es & Sherin, 2002). The process begins when a specific event captures the teacher's attention and heightens their awareness of specific phenomena (Shah & Coles, 2020). Once a teacher focuses their attention on specific components of a classroom episode, they immediately interpret the event and make pedagogical decisions in response to the classroom phenomena (Jacobs et al., 2010; Shah & Coles, 2020; Wager, 2014).

Traditionally, researchers focused teacher noticing on students' thinking; yet the field has expanded the locus of attention to include equity and equitable practices (Hand, 2012; Louie, 2018; Thomas et al., 2020; Turner et al., 2012; Wager, 2014). Because individual preferences and biases impact teachers' locus of attention when examining classroom events (Patterson et al., 2019; Star & Strickland, 2008) and their pedagogical decisions based on the interpretations of the events, teachers' noticing can be productive or unproductive. Consequently, mathematics teacher educators must provide opportunities for PTs to recognize and respond to perceptions that lead to deficit views that negatively impact students' access to high-quality mathematics, academic achievement (I et al., 2020), and identities as mathematics learners. Our Equity Noticing Framework (see Figure 1) is designed to identify what teachers' notice and position them as agents of change who critically examine how bias permeates educational systems, structures, methods, and pedagogies and then strengthens their resolve to act against oppression.

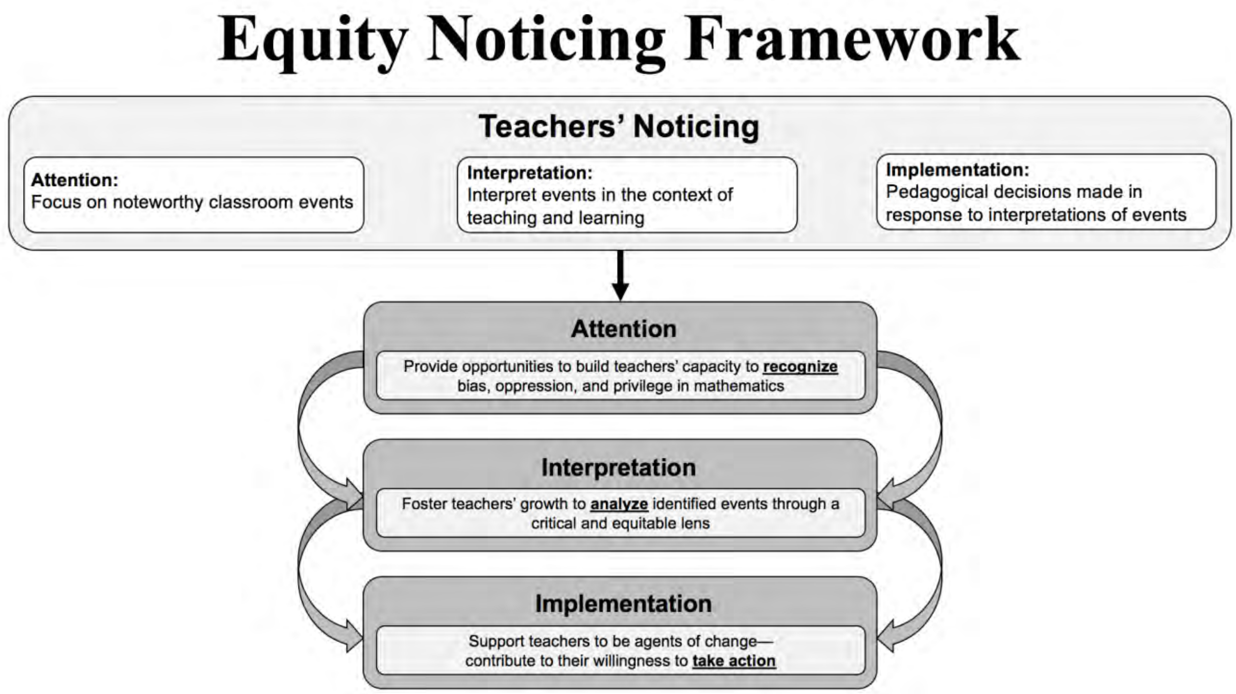

Figure 1: Equity Noticing Framework

The Equity Noticing Framework begins by utilizing the three levels of noticing to engage PTs in explicit analyses regarding salient features in mathematics classrooms and identifying how pedagogical decisions result from teachers' interpretations of the noteworthy events. From the teachers' noticing, the framework identifies the next steps in the role and responsibilities of mathematics teacher educators as they provide insight into how observations and subsequent instructional decisions reveal biases that future teachers must confront.

At the attention level, mathematics teacher educators provide opportunities where PTs attend to what was noticed in vignettes or classroom videos and subsequently build their capacity to recognize oppression and privilege. Interpretation in our framework fosters PTs' growth by engaging them in explicit discussions that situate their analysis of the noticed events through a critical lens such as colorblindness, microaggressions, or privilege. In the third level, implementation expands beyond pedagogical decisions and also involves tasks that contribute to PTs willingness to take action against systemic bias and become supportive agents of change.

Methods

Prospective early childhood and elementary teachers (n=209) from four different universities in the United States participated in the study. Data consisted of PTs' individual responses to prompts from a classroom vignette (see Figure 1) on an authentic topic focused on an elementary teacher who did not take time to develop relationships with her racially diverse students. Since elementary teachers are primarily white women and K-12 classrooms are becoming more diverse, the first author created a vignette, to mirror a white woman teaching diverse students, to engage PTs in analyzing systems of oppression and privilege. We intentionally asked our PTs to respond to a written vignette instead of watching a video clip to ascertain what they noticed. This way, PTs had time to process the scenario and consider how they would respond. Furthermore, by providing a written vignette, we narrowed the focus of what PTs could attend to by excluding extraneous classroom details (Wilkerson et al., 2018). In van Es and Sherin's (2006) study on mathematics teachers' noticing in a video club, they found teachers tended to focus on details that were not specific to the lens they were asked to view the video. The open-ended nature of the prompts allowed PTs to choose what they attended to in the vignette, how they interpreted those events, and how they positioned students of color in the mathematics classroom.

Case: Ms. Roberts

Ms. Roberts, a Caucasian elementary teacher grew up in a rural farming community, where her interactions with students of color had been limited to migrant farm workers who rotated in and out with the growing seasons. Before she entered the teaching profession, Ms. Roberts had limited experiences with people from other cultures. In her school, eighty-five percent of the students are students of color and their socio-economic status is at or below the poverty level. She believes that "people are people, no matter what color they are." Early in the school year she had to confront her true beliefs. During mathematics instruction, one of her students started crying. She went over to comfort the student, but she felt herself hesitate for a moment. This haunted her for weeks and weeks.

1. What are your thoughts on the above scenario?
2. How do we "get over" our perceived doubts and fears so we can positively work with students who are "different"?
3. How do you view your role as an elementary mathematics teacher who has to deal with "hidden" biases?

Figure 1: Ms. Roberts Vignette

Data were analyzed using the Equity Noticing Framework for a priori coding. We then used pattern coding to appropriately group and label similarly coded data as a way to attribute meaning (Saldaña, 2016) and identified common them1es (Delamont, 1992). Finally, we discussed the appropriateness of the themes. All discrepancies were resolved.

Results

After providing an opportunity for our PTs to respond to the Ms. Roberts vignette, we discovered some PTs were aware of their own biases. For example, one PT expressed, "Even though we don't want to and don't like it, we all hold some kind of judgment in our head and that at times it can come out and make us uncomfortable." Some PTs identified with Ms. Roberts as noted by one PT,

> I also grew up in a rural farming community, so com[ing] into the city, while not huge, was somewhat of a culture shock to me. I ran into people on campus of all sorts of culture, and I realized that I often automatically stereotype them without meaning to.

Although the PTs were able to identify with Ms. Roberts, there were some who denied having any biases. For instance, one PT stated, "I honestly have not had a problem with it. I have grown up around many brown people, so I do not think this affects me as much. It is our job to model the proper behavior." Moreover, PTs expressed it was their responsibility to treat all students equally. A PT commented, "All students are students. They are there to learn and you are there to facilitate that learning. Colors, sexuality, religion doesn't mean anything." In these instances, it appears the PTs equate being around other people of color eliminate any bias they may have, and students' identities do not play a role in their learning. However, some PTs recognized the importance to identify their own privilege, and it would be challenging to "change (positive or negative) [because it] is hard for most people." But, they articulated change was necessary.

> I feel it's extremely vital that we push the views, thoughts, and actions facing American society today, out of our minds. There is so much hate and discrimination about hate and discrimination that as teachers, we have to be the ones to stand tall and not be affected or swayed by the negativity in the nation.

In order to take action, the PTs expressed the importance of educating themselves and investing time in their students' community. A PT noted, "I think for me the biggest help has been becoming educated on topics surrounding these issues. Having open discussions has been very beneficial as well. As a teacher…we can use that to make positive social change." Another PT commented,

> As an elementary teacher who has to deal with hidden biases, I hope to spend time learning about the community I will be teaching in, getting to know a little more about the traditions and expectations held, as well as the cultures present in my own classroom.

One PT succinctly summarized the findings stating, "A hidden bias cannot be hidden anymore if you face it [recognize], learn about it [analyze], and respond in a positive way [take action]."

Conclusion

Teacher preparation programs must provide opportunities for PTs to move beyond teaching mathematics from a Eurocentric viewpoint. The results from this study continue to reinforce the importance of devoting time to helping PTs become more cognizant of their own beliefs about equity and equitable mathematical practices and providing opportunities to analyze these beliefs through research-proven attitudes and practices (Jackson & Delaney, 2017). The implications of this work expand both research and practice. Enhanced teacher preparation means emphasizing strategies that meet the needs of each and every learner, but also requires PTs have the opportunity to develop their orientations toward equity with respect to students' social identities in mathematics content and methods courses. Discussions generated from vignettes, like Ms. Roberts, not only enrich PTs' pedagogical noticing, but these conversations also challenge existing stereotypes, hidden biases, and unproductive beliefs about students from diverse backgrounds (Jackson & Delaney, 2017). Thus, PTs' pedagogical noticing evolves to also emphasize equity noticing: (a) developing an awareness of equity, (b) defining and interpreting what equity means in classroom instruction, and (c) implementing equitable practices within the mathematics classroom. Moreover, researchers must investigate how teacher educators structure experiences that encourage PTs to begin facing existing—and often hidden—biases in order to broaden PTs' ways of seeing (Jackson & Delaney, 2017; Jackson, et al., 2018) and foster positive, equitable change in the mathematics classroom.

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CLIMATE JUSTICE ALGEBRA: WHAT ARE THE TENSIONS BETWEEN THE MATHEMATICS AND CULTURALLY RELEVANT PEDAGOGY?

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In this article I examine a new course offering at an urban high school, Climate Justice Algebra, to determine if the course is Culturally Relevant Pedagogy (CRP). I examine the tensions between academic success (grade level mathematics learning) and climate justice learning found during two tasks administered to the class. I consider the antagonism between covering grade level mathematics standards and engaging students in Culturally Relevant tasks when melding the two practices together.

Keywords: High school Education, Culturally Relevant Pedagogy, Social Justice, Curriculum

In this article, I examined two Advanced Algebra tasks designed for a high school Climate Justice Mathematics class. By utilizing components of Culturally Relevant Pedagogy (CRP; Ladson-Billings, 1995), I analyzed the tensions between focusing on *academic success* and focusing on *cultural competence* and *critical consciousness.* In the Climate Justice Algebra class, I found that an antagonism exists between meeting required mathematical standards and engaging students in culturally relevant tasks for the social/social justice portion of class. The two topics—mathematics & climate justice—compete for time, engagement, and buy-in from students. To explore these tensions that arise while designing social justice tasks, I examined the design and implementation of two tasks with respect to the three elements of CRP. My research question is: *What tensions arise between academic success, critical competences, and critical consciousness when enacting Advanced Algebra tasks with a climate justice focus?*

Background

This study took place in an urban, comprehensive high school in the Pacific Northwest. At this school, the mathematics department follows a traditional course offering along with optional International Baccalaureate (IB) advanced courses.

Theoretical Framework

Ladson-Billings (1995) defines Culturally Relevant Pedagogy using three elements: *student achievement*, *critical(cultural) competence*, and *critical(cultural) consciousness*. First, student achievement can be described as computing, posing, and solving problems while participating in peer review of proposed and enacted activities. Then, in order to attend to cultural competence, students are supported in maintaining their cultural identity while succeeding academically. Lastly, Ladson-Billings introduced cultural consciousness (critique), in which teachers are expected to support students to "recognize, understand, and critique current social inequities" (Ladson-Billings, 1995, p. 476-7).

Effective classrooms can be thought of as ones in which students are learning content under CRP including all three components simultaneously (Ladson-Billings, 1995). As shown in Figure 1, CRP is achieved when the components of CRP are balanced.

Figure 1: Balanced CRP

The Climate Justice Class

To offer students a more holistic approach to mathematics, I created Climate Justice Algebra, which is an algebra course that satisfies the graduation requirement (three years of math are required in this state).

Climate Justice Algebra covers the advanced algebra mathematics standards. The scope is broad, and the sequence does not provide much time to delve deeper into all the standards. The climate justice topics include different energy types (coal, wind, solar, etc.) and their community impacts. Discussing climate justice is an essential part of Climate Justice Algebra. There is a shared understanding that we will talk about social inequities as a necessity for learning math in this course. This eliminates the question from students, *"Why are we talking about this in math class?"* and addresses the notion that mathematics is never politically neutral (Gutiérrez, 2013).

One goal of Climate Justice Algebra is for students to be actively engaged in solving problems. This goal aligns the concept that math should be inquiry based, interactive, and incorporate student interests (Thomas & Berry, 2019). By approaching the algebra class through a climate justice lens, students can approach problems in a more culturally conscious and relevant way. However, the disconnect between standards and applications causes tension between CRP and mathematics pedagogy (Enyedy & Mukhopadhyay 2007, Rubel 2017).

Climate Justice Algebra is bound by the scope and sequence of mathematical content created by the school district, including a suggested time frame and correlating priority standards. Therefore, as a teacher, I still have to "play the game" (Gutiérrez, 2013) while attempting to find equity through playing and then changing the game (Rubel, 2017, p. 78). Harper (2019) states in a metasynthesis that teaching math for social justice did not necessarily incorporate grade level mathematics content (Harper, 2019), which was evident in my course.

Methods

Students enrolled in the Climate Justice Algebra class were recommended by previous teachers or were seniors who previously failed a math course and needed the course to graduate. Section 1 had 23 students and Section 2 had 17. Student demographics are presented in Table 1. The student demographics in Climate Justice Algebra did not reflect the school population, which punctuates the importance of CRP.

Table 1: Student Racial Demographics, Climate Justice Algebra vs. School Wide

	White	Latinx	Black/African American	Multiple	Asian	Pacific Islander
Class	57.5%	17.5%	10%	7.5%	5%	2.5%
School-wide	69%	8%	2%	10%	10%	<1%

The Tasks

The first task was designed as a hands-on lab to support our climate justice learning target relating to coal energy. Creating and implementing a task that is hands-on is one identifier for students of the innate difference between Climate Justice Algebra and Advanced Algebra. The second task is a follow up of Task 1 and is an exploration of a GIS map. Task 2 depicts a more "normal" lesson in our Climate Justice Algebra class. The task swings between climate learning targets (coal) and grade level mathematics learning targets (parent graphs).

Task Evaluation

Tasks were evaluated by coding segments of two video recordings of 2020/21 class periods with respect to the three components of CRP. For relevance, I only considered the mathematics and climate justice segments. I also considered how the mathematics and climate justice were paired and if there was any overlap of mathematics standards.

Task One: Cookie Mining

The Cookie Mine (2017) was planned as a lab activity and was our first hands-on task of the course. I chose to analyze this task because I felt that the elements within the triangle of CRP (Figure 1) were not equally attended to during the class lesson. Task 1 was coded "no" for segmented due to the fact that the lesson was continuous for the entire class period. Further, there was also no distinct transition to "math" or cuing for notes as seen in Task 2. Observations on Task 1 include:

- **Academic success** was not met in terms of mathematical standards. The lab covered addition, subtraction, and multiplication with integers. Students created an area model and a sketch of their "topography" from their cookie mine. See Figure 2.
- **Cultural competence** was met. Within the Ladson-Billings framework (1995), students were able to be authentic during this task and able to code switch between academic language (reclamation form) and non-academic ("Do you just put your cookie in the middle?").
- **Critical consciousness** was met during this task as students started to calculate their profit/loss. Students who expanded their mines were then made aware of the cost that a mining company incurs.

Task Two: GIS Map Exploration with Notes

In task two, students explored various topography ideas, a GIS map looking at coal mining in Kentucky, and a lecture on exact equations. Task 2 was segmented by topic, math and climate justice. Observations include:

- **Academic success** was met in terms of the standards covered in the unit. Students worked on writing exact equations in vertex form from a given graph. See Figure 2.
- **Cultural competence** was lacking as students did not demonstrate high engagement, as measured by student voice & text input, students did not demonstrate code-switching
- **Critical consciousness** was not met during this task. While students did consider a GIS map of Kentucky and the proximity of mines to towns, there was no further exploration of this topic. Students were not critiquing cultural norms or institutions that might be responsible for the proximity of the mines near the towns, for example. See Figure 2 for this representation.

Results: CRP Components Unbalanced

In my findings**,** there were two different ways the tasks were unbalanced:

- Low *academic success* (see Figure 2 left) with the other two components showing up (*critical competence* and *critical consciousness*)
- High *academic success* thus affecting the other two components to be marked as "low" (Figure 2 right)
-

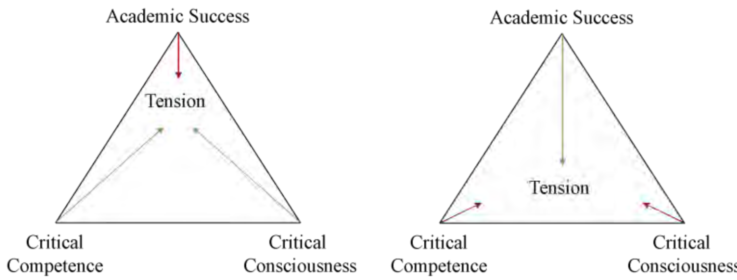

Figure 2: Unbalanced CRP; Low Academic Success vs. Low Critical Competence & Critical Consciousness

If we consider that the three components of CRP balance an activity, a class, or a task, then when one component is lacking, the balance is altered, and strain is created. For Task 1, tension was created by low *academic success* and the balance was altered as shown in Figure 2 left. The math in the activity did not necessarily pertain to the math standards that were relevant to the course, despite the activity pertaining to the climate justice lens. With increased *critical competence* and *critical consciousness,* I found a higher level of student engagement and the lesson ran more cohesively (not segmented).

In Task 2, we see the balance is altered again as shown in Figure 2 right. Tension appears with a lower amount of *critical **c**ompetence* and *critical **c**onsciousness*, despite having high *academic success*. The math standards of the activity met those in the Advanced Algebra ¾ standards. However, with high *academic success*, students were less engaged, used less student voice, and did not readily critique cultural norms or attend to potential social justice inequities.

When examining my two tasks, I first consider time as a tension followed by rigor. In considering tasks, lesson planning, and assessment, CRP is ultimately my goal. After I examined Task 1 and Task 2, I noticed that they were not balanced among the CRP components.

Conclusion

In teaching the Climate Justice Algebra course, I acknowledge that tensions are present. Regarding my goal in considering what the tensions are when one tries to meld advanced algebra with climate justice, I've found *academic success* in balance with *critical competence* and *critical consciousness* to be the main tension. Strain also manifests in time management, which refers to the first tension: how much time is spent on mathematics compared to other elements of Climate Justice Algebra? While I have no immediate solution for these tensions, I believe the first step is to be aware of them as they happen, and to reflect on how the blend of mathematics and social justice components might become more fluid.

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CHALLENGING AN AFFECTIVE BINARY IN MATHEMATICS EDUCATION

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A long-standing focus on quantitatively measuring affective responses in mathematics education research has created a binary perspective of seeing affect as either positive or negative. In this paper, I examine this through a preliminary analysis of research on affective responses in the latter half of the 20th century in the United States. Subsequently, I problematize this affective binary, not just for its limited theorization of affect, but also for the exclusionary mechanisms it enables in the form of comparison, surveillance, and intervention.

Keywords: Affect, Emotion, Beliefs, and Attitudes. Equity, Inclusion, and Diversity

Getting students to feel *positively* about mathematics has been a long-standing goal within mathematics education, not just in the United States, but globally. Boaler (2009) mentions:

> There are two versions of math in the lives of many Americans: the strange and boring subject that they encountered in classrooms, and an interesting set of ideas that is the math of the world and is curiously different and surprisingly engaging. Our task is to introduce this second version to today's students, get them excited about math, and prepare them for the future. (p. 12)

The desire of transforming students' affective responses (for e.g. – beliefs, attitudes, emotions, confidence, etc.) towards mathematics has produced numerous books, articles, and research studies that address and attempt to 'fix' students who feel or think negatively about mathematics. These attempts often share a common goal of getting students to develop certain pre-determined affective responses that are considered desirable. Studies on the affective domain in mathematics education have historically provided ways of classification of students through the creation of a *positive-negative* binary (Zan and Di Martino, 2007). Such classifications have established a normative learner of mathematics with specific beliefs, emotions and feelings about mathematics – towards which all students are expected to converge. In this paper, based on preliminary analyses of studies on affective responses in the United States in the second half of the 20th century, I briefly show that quantitative studies of affective responses using measurement scales reinforced a binary view of affect. I, then, problematize this dichotomy, not just for its limited theorization of affective responses, but also for the mechanisms of comparison and surveillance that it enables. Engagement with the affective in mathematics education is often reduced to the identification, classification, and 'correction' of students who do not have desirable affective responses – an approach that requires scrutiny and rethinking towards a more holistic perspective.

Conceptual and Methodological Notes

McLeod (1992) refers to the affective domain in mathematics education as comprising "a wide range of beliefs, feelings, and moods that are generally regarded as going beyond the domain of cognition" (p. 576). I derive my arguments in this paper from the idea that there exists an implicit classification of affective responses as either *positive* or *negative.* This binary is closely related to the *positive-negative* dichotomy of 'attitudes – an idea that became a naturalized assumption in most lines of mathematics education research (Zan and Di Martino, 2007). According to Zan and Di Martino (2003), even though the notion of attitudes towards mathematics is significantly complex, this dichotomy prevailed due to the fact that "most studies have concentrated on the creation of measurement instruments, rather than on the development of a theoretical base" (p.2). An ambiguous understanding of attitude and other affective responses such as beliefs and emotions has led to their conflation. For example, a *positive* belief about mathematics (such as seeing it as useful) might be significantly different and exclusive to a *positive* emotion (such as joy while solving a problem) – but the two are often assumed to be related under a bigger umbrella of *positive* affective responses. Furthermore, the tendency of "classic studies" to only "investigate the correlation between positive attitude and success" (Zan and Di Martino, 2007, p.158) has resulted in positing certain ways of thinking and feeling – unrelated and yet tied together by statistical correlations – as *positive*, and all else as *negative*.

What this binary view of affective responses then provides are ways of classifying not just children's emotions, feelings, and mental processes, but children themselves. This is tied to the idea of governing children – where governing is the "the visualizing and inscribing of distinctions that classify and order a child's conduct, action, and participation" (Popkewitz, 2004, p. 4). The standardization of what counts as affectively *positive* enables comparative logics based on imaginary empirical scales (Popkewitz, 2004) – resulting in new categories of children in the mathematics classroom such as the *mathematically anxious*, the *underconfident*, the *disinterested*, etc. Such classifications earmark these children for intervention by "informed rescuers'' (p.13), and encourage a new objective of changing the affective responses of these children towards characteristics deemed desirable by the *positive-negative* dichotomy.

As a part of a larger historical study of how perceptions of affective responses towards mathematics have changed over time in the United States, I looked closely at mathematics-related educational and psychological research in the second half of the 20th century. This period was marked by an increased volume of studies on the affective in education – the ideas and methodologies of which continue to persist even today in educational research. Besides searching for individual journal articles and book chapters, I also used reviews of research (published in journals, handbooks, NCTM Yearbooks, etc.) on the topic from this period, such as Feierabend (1960), Aiken (1970), Kulm (1980), and McLeod (1992), to understand the landscape of affect-related research from the 1950s to the turn of the century. Viewing previous research done on affective responses in mathematics as empirical 'artifacts', I use this paper to report my initial findings and arguments based on my preliminary analyses of them.

Initial Findings and Analysis

Conversations about mathematics-related affect in the first half of the 20th century viewed affective responses as 'non-cognitive' and often clubbed them under the general idea of *love* or *liking* for the subject (Reeve, 1925; Butler, 1930; Cook, 1931). However, in the post-World War II period, relationships between the affective and the cognitive began getting highlighted due to the increased uptake of psychological tools and methods in mathematics education research (McLeod, 1992; Pais and Valero, 2012). The methodological approaches of most of these studies were aimed at quantifying affective responses, particularly attitudes, for the purposes of statistical correlations. In this era marked by process-product research to identify mediating factors towards student achievement, the studies of attitudes towards mathematics were initially triggered by an interest in making sense of differences in problem solving ability on the basis of sex (Feierabend, 1960). Attitudes and motivation towards mathematics, thus, started getting used as justifications for differences in performance when 'ability' was held constant. Most of the research on attitudes during this period was "characterized by its emphasis on definition of terms, its preoccupation with measurement issues, and its reliance on questionnaires and quantitative methods" (McLeod, 1992, p. 577). Given the efforts to establish correlations between attitudes and achievement, student attitudes were 'measured' in most studies using Thurstone and Likert type scales (Feierabend, 1960) – a fairly common practice even today.

Commonly used attitude scales – such as Dutton's (1954) – were developed by recording student responses and creating statements from those corresponding to equally spaced scores on a number line – for example, 1 indicating a strong dislike and 10 showing a strong liking for mathematics. The use of such scales to study affective responses enabled two things. First, the assignment for scores for each item on the scales facilitated a quantified *positive-negative* evaluation for them. Associating certain attitudes with higher numbers and others with lower numbers on this scale acted towards strengthening a binary understanding of attitudes (Zan & Di Martino, 2003; 2007). Second, most attitudes towards the higher end of such scales were already correlated with high test scores – often due to the very process of designing these scales. This rendered those attitudes as desirable, making others on the lower end of the scale as *negative* or undesirable. Research conducted in the 1970s-1990s would go on to adopt these naturalized assumptions about an affective binary, extending the perspective of seeing certain affective responses as desirable based on their correlation with test scores (Kulm, 1980; Aiken, 1970). This approach was applied to research on other affective responses too such as beliefs, emotions, and self-efficacy (McLeod, 1992; Hackett and Betz, 1989; Reyes, 1984; Fennema, 1989). Despite significant differences in how various affective responses were theorized, the extension of a similar quantitative thinking across them caused the various meanings of *positive* for each to overlap and collapse into a singular *positive* affective response. This broad categorization of affect often urged individuals to place others or themselves on either the *positive* or the *negative* side – reifying a polarized notion of affective responses towards mathematics.

This binary notion of affective responses posited specific ways of feeling and thinking about mathematics as desirable – a view which percolated well into the 21st century. This 'standardization' of affective responses towards mathematics can be seen as a direct consequence of the quantitative reasoning in earlier studies. For example, the National Research Council (NRC) (2001) describes a 'productive disposition' as a "habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy" (p. 116). Comparing this description to an earlier attitude scale used by Dutton (1962), some of the attitude statements with the highest assigned scores – the '*positive* attitudes' – align with the elements of a productive disposition, such as – 'Arithmetic is very interesting', 'Working with numbers is fun', 'I like arithmetic because it is practical', 'I think about arithmetic problems outside school and like to work them out', and 'I never get tired of working with numbers', among others. By holding up productive dispositions as necessary for mathematical proficiency, NRC's description reinforces the *positive* end of attitude scales as desirable in a student of mathematics. Students are then expected to necessarily possess this 'productive disposition' – and are pushed towards developing the same if they already do not.

Discussion

Quantitative measurement and reasoning used by researchers in studies of affective responses in the past strengthened binaries of *positive* and *negative* affective responses. While an individual analyzed in such studies might have had different *positive* and *negative* scores on scales used to measure different kinds of affective responses, these studies often took a sum of all these scores to generate a single *positive* or *negative* evaluation (Zan and Di Martino, 2007). A *positive* affective response then became an umbrella term that implicitly assumed certain ways of thinking and feeling about mathematics as desirable – without theorizing about the different kinds of affect or considering the multiplicity of affective states. Moreover, the quantification of affective responses facilitated an ordered classification of students – often in the attempts to correlate affect with achievement. Establishing the normative with respect to affective responses, thus, became a way to standardize 'mathematical minds'. Such a standardization often creates an imaginary empirical scale (Popkewitz, 2004) that enables discourses of difference such as *negative* attitudes and beliefs, 'low' confidence and motivation, and *mathematically anxious* by measuring affective responses against it. These discourses produce new empirically-identifiable categories of students in the mathematics classroom who are unlike the normative learner – resulting in closer examinations and further surveillance of children's affective responses.

The focus on desirable affective responses sets in motion a subjectification that is carried out in school mathematics, with the purpose of "providing an effective governmentalization of the learners into a reduced form of identity as a mathematics learner that has to converge towards the social norms of a mathematical culture" (Pais & Valero, 2012, p. 17). I problematize the governmentalization done by the *positive-negative* affective dichotomy for three primary reasons. First, seeing a student as having *positive* or *negative* affective responses towards mathematics assumes 'mathematics' as a singular entity. A student who specifically 'dislikes' fractions might be assumed to dislike other topics in school mathematics too. At the same time, this dichotomy also clubs together different affective responses under each category, despite possible areas of exclusivity. For example, a student with a *positive* self-efficacy in relation to solving mathematical problems might be inaccurately assumed to also have *positive* beliefs about the usefulness of mathematics. Both such assumptions are inhibitory of a holistic theorization of distinct affective responses in mathematics education.

Second, the binarization of affective responses due to quantitative measurements enables a 'logic of comparison' – practices of division that construct different kinds of students in need of different strategies of intervention (Kirchgasler, 2019). Making students' affective states converge towards certain ways of thinking and feeling about mathematics is highly reductive of the myriad complex ways in which students learn and experience mathematics. Any effort to make 'all' students feel a certain way about mathematics demands a prerequisite of identifying students who do not already do so – a mechanism of exclusion that is central to such efforts. This mechanism is not only harmful due to its exclusionary logics of marking students for intervention, but also for its limited perception of the complex multiplicity of affective responses and ways of learning. Third, the subjectification that gets set in motion with an aim to identify and correct children without desired affective responses enables surveillance practices in education. While these practices have existed earlier in the form of observing a student's demeanor, body language or discourse to categorize them 'affectively', recent times have seen the use of digital technologies such as student sensor bracelets to determine stress, fear or engagement (Williamson, 2016) in classrooms. Such practices of laying bare a student's embodied processes for examination and analysis are deeply dehumanizing – and bring to the fore deeper ethical questions about research on student affect.

The consideration that has been given to students' affective responses in mathematics education is certainly important – because it offers meaningful ways to understand the complex ways in which students learn, feel about, and experience mathematics. However, a narrow and binary theorization of affect based on quantitative reasoning is not only reductive of this complexity, but also damaging due its exclusionary tendencies. Conversations about students' affective responses towards mathematics, thus, need to take these limitations into consideration, and scrutinize their own aims and orientations.

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"DEAR MATH, I'M NOT A FAN OF YOU": SHIFTING MIDDLE SCHOOL GIRLS' PERCEPTIONS OF MATHEMATICS

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In this study, we collected data from 48 middle school girls who attended a five-day residential summer mathematics program. At the program beginning and end, the girls wrote a brief "Dear Math" letter to share their thoughts and feelings about mathematics, and they were asked to draw a picture of themselves doing math and to explain it. Participant data were analyzed into themes, and pre- and post-program data were compared to look for evidence of change. The data show favorable movement, particularly in viewing mathematics as being more collaborative and less procedural.

Keywords: Affect, Emotion, Beliefs, and Attitudes; Gender; Informal Education; Middle School Education

Women are underrepresented in mathematics and other STEM disciplines. Interventions often target the middle grades as a way to "reach" girls to support their continuation in the school-to-career pipeline. To gauge the influence of a summer program for girls, we administered an instrument at the beginning and end and here reflect on the results of our preliminary analysis.

Purpose of the Study

The purpose of this study is to investigate the difference in middle-school girls' apparent perceptions of mathematics as a discipline and of doing mathematics through writing a letter directly to "Math" and creating a self-portrait of engaging with mathematics. The letters and portraits were collected at the beginning and end of a five-day residential summer camp for young women entering the seventh or the eighth grade. The program aims to engage participants in mathematics using contemporary teaching strategies that lean heavily on problem solving and social construction of mathematics understanding. The central research questions for this study were:

- What influence does participation in a five-day math program have on middle school girls' perception of mathematics?
- What influence does participation in a five-day math program have on middle school girls' perceptions of themselves as doers and thinkers of mathematics?

Conceptual and Theoretical Framework

Women continue to be underrepresented in mathematics-based fields (e.g., Makarova et al., 2019). Various performance-related and social, psychological, and emotional factors can influence women's pursuit of and perseverance in mathematics. Unfavorable gender stereotypes in STEM (science, technology, engineering, and mathematics) are a major culprit (Cheryan et al., 2015; Makarova et al., 2019; Zhao et al., 2018). Girls and women thus tend to hold lower self-concepts, expectations for themselves, and career intentions in relation to STEM compared with boys/men (Robnett, 2016; Robnett & Thoman, 2017; Song et al., 2017).

One promising approach to supporting girls/women in STEM is out-of-school-time (OST) learning. OST programs have been shown to increase participants' STEM knowledge and performance, engagement, and career aspirations (Allen et al., 2019; Author2 et al., 2021; Demetry & Sontegrath, 2020). Importantly, these programs have also been found to improve girls' STEM identities (Allen et al., 2019; Hughes et al., 2013) and their dispositions toward STEM (Author2 et al., 2021).

Although societal change is necessary to address gender-STEM stereotypes, classroom teaching can also influence students' relationship with STEM in general and along gendered lines. OST programs tend to use contemporary methods that are more conducive to favorable outcomes. Whereas traditional mathematics teaching tends to be teacher-oriented with passive students who are expected to acquire prescribed knowledge and procedures, often through memorization, contemporary methods emphasize active student involvement that employs critical and creative problem-solving and sense-making approaches that make use of written and oral communication, interaction with others, hands-on approaches, productive struggle, and real-world applications (Li & Schoenfeld, 2019; Noreen & Rana, 2019; Yuanita et al., 2018).

To frame our work, we align with the social constructivist view that cultural background and social interactions affect cognitive development and influence how one thinks and what one thinks about (Vygotsky, 1978). Vygotsky (1978) argued that "every function in the child's cultural development appears twice, first on the social level and later on the individual level" (p. 57). This phenomenon includes learning events that happen at home, in classrooms, in the community, in OST experiences, and in any other formal or informal settings where learning might occur. Boaler and Greeno (2000) found that different math learning environments (social level) where one focused on group problem solving and the other on lecture developed different mathematical identities (individual level). Young women encounter an array of events while learning mathematics. A learning event can be considered "leading" or critical, not because of its dominance in the present, but due to its role in shaping a person's thinking processes and a person's development in relation to such activities (Leont'ev, 1981, as cited in Black et al., 2010). We specifically chose to personify math by having study participants write a letter directly to "Math" and draw a picture of themselves doing mathematics to potentially elicit both social and individual perceptions of mathematics as a discipline, as well as themselves as doers and thinkers of mathematics.

Methods

Participants and Context

The Northern Nevada Girls Math and Technology Program was founded in 1998 to increase girls' knowledge, skills, and confidence in mathematics and technology in order to enhance mathematical and technological competence in girls' personal, academic, and occupational lives. A five-day residential summer camp is the signature event for the program and serves as the context for this study. Content chosen for the camp includes topics considered particularly important for students in general or for girls in particular (e.g., areas in which they tend to demonstrate weak performance and/or dispositions). All students do geometry/measurement, problem solving, and spatial skills and are exposed to female role models in mathematics and computer science. The rising eighth graders also learn algebra, whereas the rising seventh graders have lessons on data analysis/probability. Instructors use the following approaches: collaborative group work, student communication, technology use, and hands-on learning.

Participants in this study were 48 Nevada girls ages 12-13 who attended the camp in the summer of 2019. Only data from girls who attended the full camp and completed both the pre- and post-assessment were included. Of the study sample, 26 were rising seventh-grade students (i.e., would attend the seventh grade the following fall) and 22 were rising eighth graders. Because participants in the summer program were drawn randomly from those who applied, the girls had varied ability levels, background experiences, community types, and other background characteristics.

Data Collection Procedures

Participants completed the same prompt at the beginning and the end of the five-day residential camp. They were asked to write a letter directly to "Math" and create a self-portrait of themselves doing mathematics accompanied by a description of what appears in the drawing. Autobiographies as pre/post "allow a broader opportunity for descriptions of experiences" and the researchers are able to record "a transition in attitudes based on further experiences" (Ellsworth & Buss, 2010, p. 357). A self-portrait might also provide an opportunity for metacognition and self-assessment by removing the barrier of language and instead relying on a visual and emotional perception of impactful mathematics learning experiences (Mukhopadhyay, 1996). By collecting self-perceptions through multiple, mediums it might also be possible to identify contradictions in reported self-perceptions (Hall et al., 2018).

Data Analysis

Student responses were anonymized by a program intern and given to the researchers with and a code that corresponded to demographic information. Participant responses were compiled, and all data for each participant was analyzed concurrently. Data were first analyzed by the research team using line-by-line open coding. Through this process the researchers generated a list of codes. The research team then used the codes to analyze participant responses. They then jointly read, re-read, coded, and re-coded the data to refine prior codes (Emerson et al., 2011). The codes were revised, eliminated, or added based on discussion by the research team. During team discussion, the lead researcher kept notes on the meanings of each code and discussion points highlighted by the team. The research team used frequency counts of codes to identify major or "grand" themes across the analyzed data. At the time of this proposal, 25% of the data had been analyzed.

Results

Codes constructed by analyzing the data were categorized as more favorable (e.g., math as utilitarian, positive dispositions, communicative approaches) or less favorable (e.g., math as procedural, negative dispositions, teacher-oriented) in terms of participants' views of mathematics as a discipline or of themselves as mathematical doers and thinkers and with support by the mathematics education literature. In looking across the codes from pre-test to post-test in relation to the subset of data analyzed thus far, the most favorable change noted was from participants viewing mathematics as a solitary endeavor to that which involves student interaction in the form of group/peer work. (See Figure 1.) Somewhat less salient, but still notable, was a view of mathematics and mathematics teaching/learning as less traditional (e.g., computational and teacher-oriented) to more a contemporary characterization (e.g., use of multiple paths for solving problems and active student involvement). Overall, responses moved from less favorable to more favorable–or at least neutral–perspectives.

Figure 1: Pre (left) and Post (right) Self-Portraits of Learning Mathematics

Discussion

Our preliminary analysis of participants' letters to mathematics and self-portraits of doing mathematics shows that a five-day OST STEM program for middle school girls can favorably influence participants' perceptions of 1) mathematics as a discipline, and 2) themselves as thinkers and doers of mathematics. For example, in many of the self-portraits on the pre-assessment, students drew themselves alone and completing computations as a critical learning experience. A clear shift took place in the girls' perceptions of learning mathematics in the post portraits, with participants drawing themselves working with other students and communicating about mathematics. This change is, of course, dependent on the type of learning experience provided in the intervention, which was designed to align with dominant current thinking in the field about teaching and learning mathematics.

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THEORIZING COMPLEX EMBODIMENT IN MATHEMATICS

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Cognizant of educational debts (Ladson-Billings 2006) for Latinas in math, as well as for students with Learning Disabilities (LD), we seek to develop understanding of the experiences of Latinas with LD in math classrooms. To do so, we need theoretical and methodological tools to analyze the emotional, embodied experience of learning mathematics within figured worlds of mathematics classrooms. Scholarship on identity in math currently lacks a systematic theoretical and methodological grounding that would allow for critical analysis of both intersectionality, relationships and emotion. Using analysis of a how a Latina with LD was described by her teachers, this paper describes a collaborative, interdisciplinary theoretical and methodological project to develop these theoretical tools.

Keywords: Equity and Diversity; Critical Special Education; Disability Studies in Education

Educational equity demands an increased focus on mathematical learning for students from historically multiply marginalized non-dominant backgrounds. For too long, mathematics education has not done enough to develop understandings of the mathematical experiences of students of color, students with disabilities, and girls, among other overlooked identity and power categories. Additional attention is needed to develop understanding of students' intersectional identities particularly for students of color with disabilities, as very little research exists for this group of students (Lambert & Tan 2020). Bringing the theory of *complex embodiment* from Disability Studies into mathematics education (Lambert 2019; Siebers 2008), we map embodied differences, as well as social construction and feeling-meaning making (Lemke 2013) around those differences creating a new theory of Complex Embodiment in Mathematics. Embodiment matters for how math class is experienced, and how individual and collective feeling-meaning-making happens in mathematics. To better understand intersectionality, we map circulating master narratives and how students are positioned in terms of race, disability, gender and emotionality. To illustrate our arguments, we present data from an ethnographic and interview study of Latinx students with learning disabilities (Lambert 2015).

RQ1: How can we theoretically and methodologically expand complex embodiment to mathematics, including emotion, intersectionality, relationships and the embodied experiences of learners with LD of color in mathematics classrooms?

RQ2: What cultural practices and discourses circulate in the figured world of the mathematics classrooms that position Latinas with LD?

Conceptual Framework

To understand the processes through which individual students come to identify, or not, with mathematics, we use a *practice theory of identity* (Holland et al., 1998). We understand mathematics classrooms as *figured worlds*, activity systems where participants create meaning around through shared participation in cultural practices and discourse. Learning mathematics occurs within racialized spaces (Nasir & Royston, 2013). As individuals participate in figured worlds, they internalize some circulating discourses and reject others, rearranging self-understandings around mathematics (Bakhtin 1981). Contradiction between multiple discourses creates space for change. Circulating discourses can include master narratives (Bamberg & Andrews, 2004) about ability and disability in mathematics (Lambert 2015), as well as stereotypes about race, disability and gender (Lambert 2018; Lambert, Hernández-Saca & Mireles Rios, 2020). Research on Latinx identity processes must account for how students understand master narratives of achievement (Zavala & Hand, 2018). Undergraduate Latinas negotiated a complex set of discourses to be successful in math, some of which were specific to Latina experience such as pressure to be a mother (Leyva 2016). Additionally, research has shown that for Latina elementary school girls showed that more communication where their mom was associated with higher grades in math and liking math more (Mireles Rios & Romo, 2010).

Understanding how important relationships with teachers and peers are for learning mathematics (Black et al., 2009), we seek to understand how relationships with peers in the mathematics classroom and relationships, particularly with mothers, matter for mathematical identity of Latinas with and without LD. We know that while peer support has been found to be important for students of both genders, it may be especially beneficial for adolescent girls' engagement with STEM subjects, given their greater interest in and emphasis on relationships with peers (Fredricks et al., 2018). A study conducted by Riegle-Crumb et al. (2006) found that high school girls who reported having female friends that excel in math and science courses were more likely to subsequently take advanced courses in those subjects themselves. Perceived peer support has also been linked to an increase in adolescent girls' cognitive and social engagement with science and math, and several studies have found that greater perceived support from peers is associated with more positive emotionality towards science and math, as well as greater self-efficacy in these subjects (Fredricks et al., 2018). Peers also play a role in adolescent girls' development of future identities. Thus, a number of studies have demonstrated a link between adolescent girls' relationship with math and science and peer relationships/support. This may be even more important for Latina girls whose teachers may hold lower expectations for their academic achievement (Mireles Rios & Romo, 2014).

Methodology

Using a longitudinal ethnographic case study design, this paper comes from a larger study which analyzes the identification processes of Latino/a focal students and teachers as they participate in a seventh-grade mathematics classroom (Lambert 2015; 2017; 2019). The research was designed to provide multiple data sources, both of the classroom and individuals. At the classroom level, data were collected through participant-observation, video recordings, and classroom artifacts. Interviews were the primary source of data at the individual level. Trustworthiness was established through triangulation of these multiple data sources, concurrent data analysis and member checks.

Participants

The student, Rita, identified herself as a girl, as "Dominican", and as "always good at math." From formal school records, Rita was also a long-term English Language Learner and received special education services for a Specific Learning Disability. She attended a school in which 98% of students were identified as Hispanic, and 85% were classified as low-income. Her school had features that supported the growth of students: stable leadership, teacher autonomy, and a vibrant arts program. The school used a co-teaching model for students in special education, in which a general educator and a special educator shared teaching responsibilities for a class that included both students with and without disabilities. The sixth-grade special education teacher, Ms. Emerson, was white and in her third year of teaching. Her co-teacher, Mr. Pierce, was also white and in his third year of classroom teaching.

Data Collection and Analysis

Multiple sources of data were necessary to understand the relationships between the construction of ability and disability through cultural practices and individual student identity processes including field notes from participant-observation, classroom artifacts (worksheets and photographs of student journals), and video recordings. Teacher level data included field notes, formal and informal interviews, some during class time and others held separately. Student level data included all the aforementioned, with the addition of student interviews. All data was collected by the first author, who also did the initial data analysis. Our collective analysis sessions focused on reanalyzing data through our new theoretical lens.

Findings

We focus in this paper on the relational aspects of our work. Her 6th grade special education teacher, Ms. Emerson, described Rita's entry into sixth grade dramatically:

Ms. Emerson

> [Rita] came in as like, hmmm, for a lack of another way of putting it, the uber special ed kid, cried on her way to school, really was always upset, very, seemingly struggled a lot,
>
> she also seemed like she used to a lot more be between Spanish and English, and I don't know whether it was like . . . fear of speaking in English.

Rita was named as the "uber special ed kid," and the immediate evidence was emotional: "cried on her way to school." This description was painful for the first author to hear, and was very different from observations of Rita, who was a focused and engaged student by the time the first author entered her classroom. The sixth-grade teachers painted a portrait of Rita that I could not understand: "crying," "struggled a lot," "not a lot of basic skills." Rita is positioned here not only as "the uber special ed kid" but also as an emergent bilingual. This highly negative, emotionalized portrayal of a student, however, cannot be understood in isolation, because both teachers used this picture to immediately emphasize Rita's transformation into the confident student observed in class in the second half of the year. Her transformation was understood as relational and embodied, based on her relationship with a girl named Shaundra. As Mr. Pierce narrated it,

Mr. Pierce

> [Rita] took on the persona almost of Shaundra, people just grouped them together [puts hands out, hands come together], *oh there is Shaundra and Rita they are both smart*, and Rita kind of grew into being,[hands together come up] a pretty good student, when at the beginning of the year she seemed not so good.

Rita brought this new set of behaviors, a new role as a serious, smart student into seventh grade, and was easily recognizable as a serious student. Additional analysis of Rita includes her own emerging narrative of herself in relationship to math, which particularly connect to her relationship with her mother, a part of the data that is critically important because it portrays Rita's agentic self-authoring and resistance to being framed as an educational problem.

Discussion

We return to the themes that we believe will deepen scholarship in math identity development: emotional, relational, and intersectional, focusing on the relational. Rita's transformation is relational, through her friendship with Shaundra. This was not the only instance in the data from this study in which students learned mathematics in close relations with friends. Another pair of girls always worked together, one girl counting on her fingers under the table and then whispering her answers to her friend. Friendships and relationships seem critical in the process through which students learn to identify with math, or not. As we noted, interviews with Rita in her 7th grade year focus on her relationship with both math and her mother, which is closely intertwined. We see female relationships at the forefront of Rita's construction of mathematical identity. Her relationships with her friends mediate her learning of mathematics and her relationship with mathematics. Her relationship with her teachers, particularly her Latina seventh grade teacher, mediates her learning of mathematics and her relationship with mathematics. Her relationship with her mother mediates her learning of mathematics and her relationship with mathematics.

We present this work as our emergent theorizing of how complex embodiment in mathematics could deepen current analysis of identification processes in math. Absent from this data is student authoring, which we also include in our data. How does Rita self-author in the face of her positioning? How does she craft a math identify for herself? Again, this analysis of her self-authoring will analyze emotion, relationships, and intersectionality.

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MATHEMATICAL ACTIVITY AS A SITE FOR TEACHERS TO DEVELOP A CRITICAL MATHEMATICAL CONSCIOUSNESS

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Preparing teachers to create more just mathematics classrooms entails supporting disciplinary aspects of teaching while also helping teachers attend to power and privilege. While these aspects of the work of teaching are often considered separately, we are interested in working at their intersection. In this paper, we argue that teachers need opportunities to make sense of ideologies at the interactional level, in mathematical activity. We propose the conception of critical mathematical consciousness, as a framework to support teachers to investigate the ways individual math learning is nested within specific math contexts, ideologies about mathematics, and the larger sociopolitical context. We propose principles to support the development of critical mathematical consciousness, which we illustrate through our respective projects which center mathematical activity as a site to disrupt existing mathematical ideologies.

Keywords: Equity and Justice, Pre-Service Teacher Education

Disciplinary aspects of teaching are often addressed in teacher preparation programs in content and methods courses that are separated from critical and justice-oriented aspects of education. Both aspects matter for preparing teachers; teaching mathematics well depends on thinking deeply about how to support students to make sense of content (Hill & Ball, 2009, Lampert et al, 2013) and understanding the ways injustice is built into systems (Freire, 1996) and shaped and reshaped in mathematics classrooms (Gutierrez, 2013; Gutstein 2006; Martin, 2013). Drawing on the observation of Philip and Gupta (2020) that "it is in interactions that the cultural and historical traces of power are rendered visible, remade, and/or challenged", our work starts from the assumption that preparing teachers to create more just mathematics classrooms entails learning to analyze the ways reproduction of social hierarchies occurs at the interactional level.

To support teacher learning toward this aim, we reconceptualize the role of doing mathematics in teacher preparation as a site to investigate the intersection of mathematical learning and ideology. We view mathematical participation as socially constructed and connected to human activity (Esmonde & Langer-Osuna, 2013; Gresalfi et al., 2009; Nasir & Hand, 2008) and we consider that activity to be shaped by ideologies and the ways they are embedded in discourse, practice, and institutions (Hall, 1981; Philip & Gupta, 2020). As people engage in new forms of activity, disruptions to those ideologies can be created and new ways of being become possible. Doing mathematics with preservice teachers can thus become a productive space to make visible the ways that opportunities for learning are afforded and constrained in moment-to-moment interaction and connected to larger systems of oppression. The approach we outline in this paper aims to bridge disciplinary and justice-oriented aspects of teaching to foster what we call *critical mathematical consciousness*.

Background

To help us conceptualize the intersection of justice and disciplinary learning in the work of teaching, we draw on scholarship that treats learning and ideology as intrinsically linked (Philip, 2011; Philip et al., 2018; Philip & Gupta, 2020). We define ideology as "socially shared systems of representation" that guide how human beings experience and make sense of the world (Philip et al., 2018, p. 4). As people interact with their world, they develop different ideological chains of meaning such as what it means to be mathematically proficient. These systems of representation are intertwined with learning in that "the interactional forging and working out of ideological convergence can either afford or constrain learning as an activity of heterogeneous meaning making" (Philip et al., 2018, p. 6). This conception of ideology as dynamic, constructed in interaction, and shaping learning underlies our approach to supporting teacher learning toward justice-oriented mathematics teaching.

The notion of ideology has been explored in mathematics education in various ways. The sociopolitical turn (Gutiérrez, 2013a) has led to the emergence and understanding of mathematics and mathematics education as a racial project (Martin, 2013). This work helps situate constructions of mathematics and mathematical competence in a cultural and historical context that is designed to uphold racialized hierarchies of mathematical ability (Gutiérrez, 2013b). Building on these ideas, research on racial narratives and storylines has helped to illuminate how ideologies of mathematics and of race intertwine and shape meaning-making in the mathematics classroom (Nasir et al., 2012; Nasir & Shah, 2011; Shah, 2017). Bringing this into the work of teaching, recent work on teacher noticing in mathematics education has begun to elaborate the social and cultural construction of what teachers attend to and interpret in the mathematics classroom and the implications for reproducing or disrupting dominant hierarchies of power and privilege (Louie, 2018; Shah & Coles, 2020; Wager, 2014).

In addition to these approaches to thinking about mathematics and ideology, there is a growing body of scholarship that examines how positioning and power shape identity and participation in mathematical interaction among students in the classroom (Esmonde & Langer-Osuna, 2013; Langer-Osuna, 2011, 2015). While this work does not directly address ideology, these lines of inquiry have helped to elaborate the social construction of engagement in mathematical activity, and the ways this construction is shaped by understandings of social categories and hierarchies such as those related to race and gender. This body of work has informed our thinking about social interactions in mathematics classrooms as functioning to challenge, perpetuate, or create inequities in participation.

The insights we have described above related to learning and ideology and their specific intertwining within the mathematics classroom have led us to reexamine the role of doing mathematics in teacher preparation as a site to begin to develop critical mathematical consciousness. We define critical mathematical consciousness as reasoning and making connections across the different nested layers shown in Figure 1.

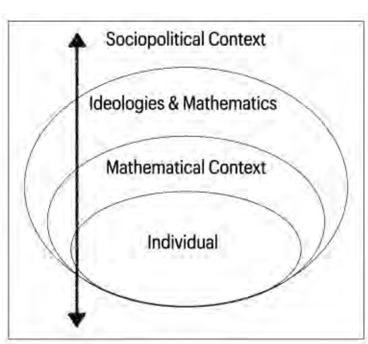

Figure 1: Critical mathematics Consciousness

We define critical mathematical consciousness as reasoning and making connections across the different nested layers shown in Figure 1. This image depicts the ways individual learners are nested within a specific mathematical context that affords and constrains mathematical participation. This context is shaped by narratives and ideologies related to mathematics and mathematical competence, and these in turn are part of a larger sociopolitical context that makes available particular ideological chains of meaning. We consider the work of creating more just mathematics classrooms to entail thinking across these different layers to make sense of the ways mathematical learning is continually shaped and reshaped in interaction. As working toward justice is an ongoing project, in our conception, critical mathematical consciousness is always in progress rather than an outcome or destination.

Fostering a Critical Mathematical Consciousness with Pre-Service Teachers

To support pre-service teachers to make connections across these nested layers, we have developed a set of principles that guide our work: 1) Begin with personal mathematical histories, 2) Situate teachers as mathematical learners, 3) Attend to mathematics as a human activity that involves emotion and relationships, 4) Use personal experiences as math learners to explore and interrogate ideologies toward creating more just mathematics learning environments.

We start with personal mathematical histories as a way to help novice teachers reflect on their own experiences, and begin to consider mathematics classrooms as racialized, gendered, and classed environments that can be experienced quite differently by individuals (Marshall & Chao, 2017). We view testimonios as a productive tool for this principle since it comes with the premise that an injustice exists and demands for a call to collective action (Delgado Bernal et. Al, 2012). For elementary teachers who have had a wide range of experiences during their own math journeys, sharing and discussing personal math histories can also be a place to heal from any anxiety or trauma they may have experienced and to learn from each other's different journeys. Personal mathematical journeys thus serve as a resource for pre-service teachers to begin to connect with young people in the human experience of doing mathematics.

In tandem with reflecting on past experiences, we also seek to expand possibilities for pre-service teachers by providing opportunities to engage with mathematics as learners. Our purposes for doing mathematics differ somewhat from the purposes that have been written about more extensively in the literature of developing content knowledge or mathematical knowledge for teaching (Ball, 2017; Hill & Ball, 2009) and/or learning specific teaching practices (Lampert et al., 2010; Lampert & Graziani, 2009). While we recognize and value these purposes, we also consider mathematical activity to be a generative site for investigating how teaching choices and learning environments function to reproduce or to disrupt existing social hierarchies from a more personal, learner-centered perspective. With this shift in purpose, we hope to support new ways of relating to mathematics, and more expansive ways of thinking about how to design classroom math communities that build from the diverse resources each learner brings to mathematics.

In supporting teachers' as mathematical learners, we treat mathematics as an embodied activity that involves human relationships and emotion. We seek to design mathematical experiences that support learners to bring their full selves as they engage with mathematics (Gutierrez, 2012; Takeuchi & Dadkhahfard, 2019). This requires choosing tasks that are meaningful and invite diverse ways of knowing, fostering productive norms of interaction, and structuring participation so that learners have opportunities to authentically engage with each other's ideas (Bartell et al., 2017). We also encourage reflection as an ongoing component of mathematical activity to help pre-service teachers connect their embodied experience as learners to aspects of mathematical learning environments as well as to ideologies about mathematics and to the larger sociopolitical context.

By centering and attending to mathematical activity in this way, teachers' personal histories and ongoing experiences can then become resources for them to interrogate the ways mathematics classrooms function to reproduce or disrupt injustice at the level of interaction. By offering ongoing opportunities to connect the nested layers of individual, math context, math ideologies, and sociopolitical and sociohistorical contexts, we aim to support pre-service teachers to construct new ideological chains of meaning that challenge injustice as enacted in day-to-day teaching.

Methods and Research Approach

In bringing these principles into our work with pre-service teachers, we draw on social design-based approaches that are founded in Cultural-Historical Activity Theory. Social design-based research focuses on designing at the level of the activity system and thus attends to the ways different intertwined aspects of design, such as the tools and artifacts provided, the roles participants occupy, and the norms that guide interaction, work together to mediate learning toward a shared goal (Engeström, 2011; Gutiérrez & Jurow, 2016; Gutiérrez & Vossoughi, 2010). Across our different research contexts, doing mathematics is a central part of the design of the activity system, so that teachers have ongoing opportunities to think about how mathematical learning is afforded and constrained within nested layers. To design for this shared goal, we attend to the mathematical tasks used, the roles participants are able to take on, the artifacts and tools made available for engaging with mathematics and with ideology, and the implicit and explicit norms that guide interaction. The projects we are working on include the (re)design of math methods courses and the co-design of learning communities with preservice teachers that take place outside of institutional contexts.

Moving Forward

Across our two very different contexts, we treat mathematical activity as a site for beginning to develop a critical mathematical consciousness. Mallika works within the constraints of teacher preparation as it currently exists by infusing an attention to identity and ideology throughout mathematics methods coursework. Sandra works with women of color outside of a formal institution with an explicit attention to rearticulate new meanings with and of mathematics. In both contexts, opportunities for participants to engage in mathematics and then reflect on the experience are treated as central to a rearticulation of ideologies. We acknowledge that the work is complicated and messy and thus the relational aspect of how we engage with one another is important and foundational to productive discussions. In our projects, creating spaces of trust and vulnerability supported engagement with the real tensions and contradictions of trying to engage in critical work in institutions. Such an approach avoids the "rhetoric of 'us' and 'them'" that can sometimes dominate in critical educational spaces (Philip & Zavala, 2016, p.660). We offer a framework and principles to foster critical mathematical consciousness as a way of supporting preservice teachers to grapple with the complexities and nuances of working to create more just mathematics classrooms.

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CONCEPTUALIZING MATHEMATICS MODULES THAT INTEGRATE PROFESSIONAL NOTICING AND EQUITY

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In this paper, we describe the theory guiding the development of microlearning modules connecting noticing and equity in mathematics. Gutiérrez's (2009) four dimensions of equity framework is used to inform the modules. The professional noticing of children's mathematical thinking (Jacobs, Lamb, & Philipp, 2010) is also woven into the module development. We analyze data from preservice elementary teachers' ideas about equity and responses to a video to inform our project and discuss the importance of making equity explicit in mathematics methods courses. Results indicate that preservice elementary teachers' ideas of equity primarily fall into the dominant axes of access and achievement, but also show evidence of the critical axes of identity and power in responses to the classroom video.

Keywords: Equity, Inclusion, and Diversity; Teacher Noticing; Preservice Teacher Education; Instructional Activities and Practices

Culture permeates every aspect of life by definition; thus, it has some impact on how students think about mathematics (Civil, 2018), and should be recognized when considering equitable contexts for the teaching and learning of mathematics. Students from non-dominant cultures should be encouraged to draw upon their experiences to think critically in mathematics. Projects like Funds of Knowledge (Civil, 2007; Moll, Amanti, Neff, & Gonzalez, 1992) propose teaching structures that include and use individuals' unique cultural experiences and knowledge. While culture may be a positive and powerful classroom dynamic, teachers' assumptions about mathematical ability based on any student attributes are an inequity that comes from student culture (Gutiérrez, 2002). Students' understanding of dominant mathematics, or what counts as mathematics, can also hinder equity (Civil, 2014; Gee, 2002) through the reinforcement of their positioning within a cultural outgroup (Gutiérrez, 2008). This research responds to the call for equity to be used as a lens to enhance the collective mathematics education research enterprise by challenging "the false dichotomy between equity and mathematics education research" via the fundamental conjoining of equity concerns with responsive mathematics teaching practice (Aguirre et al, 2017, p. 130). In this paper, we describe the theory that guides the development of microlearning modules integrating noticing and equity in mathematics. We include baseline results from preservice elementary teachers' (PSET) ideas about equity that inform our project and discuss the critical importance of making equity explicit in mathematics methods courses.

Theoretical Frameworks

There is emerging interest in connecting and studying aspects of equity in conjunction with professional noticing (Jong, 2017). For example, both Kalinec-Craig (2017) and Hand (2012) have examined student positioning and power in the context of professional noticing. Such connections are consistent with portrayals of professional noticing as contested and political space (Lefstein & Snell, 2011; Louie, 2018). Regarding the pedagogical activity that might productively influence such spaces, van Es et al. (2017) posited several practices and associated foci for professional noticing that they describe as *noticing for equity*.

Gutiérrez (2009) describes four dimensions of equity as a way to frame the complex ways in which equity plays a role in mathematics education. The dimensions include access (resources children have available), achievement (student outcomes), identity (connecting to students' backgrounds), and power (voices in the classroom and challenging structural inequities). She notes that access and achievement are part of the dominant axis, yielding to the status quo, whereas identity and power comprise the critical axis, confronting or challenging the status quo. Gutiérrez (2002) makes it clear that the two axes are essential, and may even act symbiotically.

Teacher noticing literature frequently emphasizes a cognitive perspective, concerned with the cognition needed to recognize and act on mathematical thinking. Jacobs, Lamb, and Philipp's (2010) work expressly addressed professional noticing of children's mathematical thinking defined as three interrelated components, attending, interpreting, and deciding. Recent studies consider the three components in the context of the broader construct of teacher competence and address a situational aspect of noticing under the perception, interpretation, and deciding, or PID, model (Blömeke, Gustafsson, and Shavelson, 2015). Jong et al.'s (2021) empirical study demonstrated support for this proposed relationship between dispositional resources and noticing (i.e. PID). Santagata and Yeh (2016) further proposed that the continuum is not linear, but that practice, too, influences PID; this idea is evident in Louie, Adiredja and Jessup's (2021) work. These situational, or ecological, approaches to noticing are relevant when Gutiérrez' dimensions of equity (2002). Teacher noticing along both axes has the potential to broaden equity through the dynamic interplay of cognition, attitudes, beliefs and practice with noticing. The microlearning modules incorporate situational noticing to increase PSETs' knowledge and shift their beliefs regarding equity and support the development of equitable noticing practices.

Methodology

Project Design

For this project, we focus on microlearning experiences as an avenue for the construction of equitable professional noticing practices which will enhance learning compared to more traditional modalities across a range of subjects (Mohammed, Wakil, & Nawroly, 2018). The titles of the modules indicate the mathematical and some of the equity-based concepts they contain: Intro to Professional Noticing and Equity, Fractions and Productive Struggle, Number Talks and Smartness, Patterns and Student Work, Functions and Inclusive Story Problems, Ratio and Language, Fractions and Representations, and Social Justice Applications. Instructors implementing these modules start with the introduction but can select which other modules are included based on preference and instructional goals. Each module after the introduction contains a review of the components of professional noticing and the dimensions of equity highlighted in that module, a segment which focuses on the language of teachers and how chosen words can be harmful to students in marginalized populations, and a wrap-up of the theories explored with an opportunity for discussion of the highlighted components and dimensions.

Data Collection and Analysis

PSETs in three sections of mathematics methods courses completed a survey at the beginning and end of the semester in fall 2020. Using random selection, 200 responses from pre- and post-surveys of 50 PSETs inform the initial stage of analysis. The first item inquired about PSETs' own ideas of equity (Q1, Table 1). Then, the PSETs viewed a brief (74-second) clip of a diverse group of second grade students engaging in a number talk. After the video, they were asked one question each concerning the professional noticing components about the video followed by a prompt relating the video to equity (Q5, Table 1). Preservice teacher responses to the two equity items were analyzed through the lens of Gutiérrez's (2009) four dimensions of equity; a codebook was created to determine which phrases would code a PSET response with a specific equity dimension. (e.g., "differentiation, opportunity to learn, teaching in a fair way", were coded as *access*, and "gender, race, connecting to students' lives, or representation" were coded as *identity*). Four members of the research team individually coded each response for the four dimensions to include any/all that applied, and inconsistencies were negotiated by pairs of coders to achieve complete agreement (Campbell et al., 2013).

Results and Discussion

Table 1: Frequency of Responses by Equity Dimensions

	Q1. What does equity in teaching mean to you?			Q5. Describe how equity relates to this classroom scenario.		
Equity Dimensions	Pre	Post	Total	Pre	Post	Total
Access	45	50	95	32	28	60
Achievement	17	23	40	5	7	12
Identity	13	6	19	15	19	34
Power	1	2	3	33	35	68

The Dominant Axis: Access and Achievement

The most common views in all responses were about giving all students the same opportunities or some notion of fairness. While both ideas are basic and could be categorized into the access dimension of equity, there is a distinction between ideas of equality and equity. While we coded access responses with notions of equality or mentions of being "fair" to all students, we also looked for other key indicators, such as differentiation, use of multiple strategies, mentions of manipulatives, and responses that mentioned the use of resources. In a broad sense, the PSETs think equity in teaching does mean some level of fairness; however, those responses of fairness come with varying sophistication. For example, in a pre-response to question 1, a PSET stated "…I will be fair when it comes to all my students. I will try my best to meet the needs of all my students when teaching," which indicates a baseline approach to the notion of fairness. A post-response that addresses and interprets multiple dimensions of equity from a different PSET was,

> "Equity in teaching means that all students are presented with what they need to be successful in the classroom. Despite a child's backgrounds or specific needs, they are able to reach their goals by whatever means possible in the classroom. This means that the teacher provides the necessary components for each student to be successful. This could mean that instruction is differentiated to fit individual students' needs. Each student with an IEP is accounted for and is given the appropriate help. Students with language and cultural differences are given what they need specifically to succeed."

When identifying responses as "achievement", we used codes such as a mention of student success, expectations, standards, quality of education or learning, and references to "correctness" of the mathematics. Substantial increases in the pre to post responses in the achievement dimension were not evident in either question. Perhaps the most interesting outcome was that only twelve of the 100 responses from question 5 contained a reference to achievement, eleven of which mention "correctness" of mathematics. This is likely linked to the specificity of the video and many of the codes from achievement were links to broader scale observations that are not necessarily evident in a short video clip. Those broader scale responses would be evident in question 1 that prompts for their overall understanding of equity. Most of those indicated the teacher let students share responses, regardless of whether it was correct or incorrect.

The Critical Axis: Identity and Power

Limited responses to the first question touched on the dimension of identity by stating that regardless of students' backgrounds, they should have access. One part of a response that was coded as identity stated, "Every student has different life experiences and backgrounds and we have to be aware of these things." It was somewhat surprising that identity responses decreased for the item. While many more included the word "differentiate" in post-responses, this was not enough for the identity code. However, many more responses to question 5 were coded as identity, which increased slightly from pre to post. The limited responses informed the project team of the need to focus more teacher education instruction on students' identity through more interactions that involve connections to students' culture, language, and experiences.

It was somewhat surprising that preservice teachers' responses to question 1 did not include more characteristics of power at the end of the mathematics methods course. In the few instances, there was mention of listening to students and allowing them to share their ideas. Thus, it was not a connection to broader power structures for question 1. However, there was a dramatic increase in the responses coded as the power dimension for question 5. This was likely due to the video including three students who were selected to share their three unique answers to the open-number sentence and an explanation of their thinking. Thus, student voice, sharing strategies, and eliciting multiple strategies were all aspects of the power dimension. The brief video limits connections to the power dimension; however, our goal is for PSETs who participate in the social justice module to empower students to use mathematics to change the world.

Conclusion

Preliminary results indicate a need to engage preservice elementary teachers in more critical dimensions of equity (Gutiérrez, 2009). Students' backgrounds and identities can serve as an asset to teaching, and power can be incorporated as a way for their students to have a voice in their mathematics learning and use mathematics as a tool to analyze the world around them.

Our results show varying operationalizations of complex ideas such as fairness and opportunity. For some, these are signified by the equalizing of instructional practices across students, others see them as the product of differentiation, and for others, these ideas are taken as self-evident and left unclearly defined. Our next analytical steps will be to consider varying levels of each dimension and examine anti-deficit language in responses as we establish a more complex process in understanding PSETs ideas and patterns (Jacobs, 2017). We are encouraged by PSETs' capacity to perceive different dimensions of equity. Across both presented items, we observed each of the dimensions within our response set. Interestingly, although PSETs tended not to personally connect to the equity dimension of power (Q1), when presented with an instructional vignette, PSETs perceived aspects of the power dimension therein. This suggests a relatively broad formation of equity ideas within our sample of PSETs that might connect an instructional moment to broader sociopolitical contexts (Louie et al., 2021).

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CONCEPTUALIZAR MÓDULOS DE MATEMÁTICAS QUE INTEGRAN LA MIRADA PROFESIONAL Y LA EQUIDAD

GENDER DIFFERENCES IN STUDENTS' SENSE OF BELONGING IN MATHEMATICS: THE ROLE OF THE INSTITUTION

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A student's sense of belonging can strongly influence their decision to stay in STEM. This influence is especially strong for women, who often report a lower sense of belonging in math (Good et al., 2012; Rainey et al., 2018). In this study, we utilize sociopolitical theory to analyze select statements about students' perceived frequency of classroom activities related to sense of belonging from the Progress Through Calculus census survey. Results show that this perception differs between genders, with the majority showing women constituting a higher proportion of low frequency responses and a lower proportion of high responses. We also see this relationship vary across surveyed universities, with further exploration of two particular institutions. We discuss how these results fit within previous literature in this area as well as expand on the sociopolitical perspective around institutional norms about inclusion and exclusion.

Keywords: Equity, Inclusion, and Diversity, Gender, Calculus, Undergraduate Education

In many STEM fields, women remain underrepresented and are significantly more likely than men to switch out before obtaining the degree (Ellis, Fosdick, & Rasmussen, 2016). Women receive less than half of mathematics and statistics bachelor's degrees and this proportion continues to decline (National Science Foundation and National Center for Science and Engineering Statistics, 2017). One factor shown to influence students' decision to stay in STEM is their sense of belonging, referring to "students' sense of being accepted, valued, included, and encouraged by others (teachers and peers) in the academic classroom setting and of feeling oneself to be an important part of the life and activity of the class" (Goodenow, 1993, p.80). This association is particularly important in mathematics and in relation to gender, whereby women in mathematics often report a lower sense of belonging which can mediate leaving STEM (Good, Rattan, & Dweck, 2012; Rainey, Dancy, Mickelson, Stearns, & Moller, 2018). Rainey et al. suggest four areas as especially important for students' sense of belonging: interpersonal relationships, competence, personal interest, and science identity. The combination of these factors all affect a students' sense of belonging in mathematics and could be particularly influential in retaining women in mathematics. Thus we ask in this study (1) What is the relationship between gender and the perceived frequency of certain classroom activities related to a student's sense of belonging? And (2) In what ways does this relationship change, if any, when considering the institution as the bounds of the mathematics culture system?

Theoretical Perspective

The aforementioned characteristics of a sense of belonging interact within the system of the mathematics environment, as informed by sociopolitical theory (Adiredja & Andrews-Larson, 2017). Sociopolitical theory emphasizes the connections between knowledge, power, and identity within the mathematics culture system, encircled by the math classroom, the department, and/or the institution. Knowledge, power, and identity within the system emerge and are often constrained by the norms of the environment, constructed from the words, actions, and relationships therein. The sociopolitical viewpoint questions the currently "accepted" norms within the culture of mathematics, which often inequitably affect students' perceptions of their sense of belonging in the field. Sociopolitical theory rejects student assimilation into the current dominate culture and instead promotes fostering students' sense of belonging within a system that considers, supports, and affords space to various backgrounds and identities.

Methods

The data for this project originate from the Progress Through Calculus census survey from Fall 2017. This survey was administered to introductory mathematics students from 12 different institutions in the US approximately ¾ths into the semester. This survey aimed to explore different aspects of these institutions' mathematics courses, programs, and departments. Included in this survey were statements about students' perceived frequency of classroom experiences on a scale from (1) Does not occur to (5) Very descriptive. This analysis focuses on 8 of these statements as informed by Rainey et al.'s (2018) four categories related to students' sense of belonging in mathematics and Good et al.'s (2012) Math Sense of Belonging Scale. The selected 8 statements are: (1) The class activities connect course content to my life and future work; (2) A wide range of students respond to the instructor's questions in class; (3) The instructor knows my name; (4) Class is structured to encourage peer-to-peer support among students; (5) There is a sense of community among the students in my class; (6) I share my ideas (or my group's ideas) during whole class discussions; (7) A wide range of students participate in class; and (8) My instructor uses strategies to encourage participation from a wide range of students.

The survey asked students to describe their gender identity from selecting all that applied from the following options: Man, Woman, Transgender, and Not listed (please specify), or Prefer not to disclose. To define gender, we grouped respondents into the categories Man and Woman to include any student who selected *at least* Man or Woman, respectively. Students who selected both Man and Woman were placed into both categories. We recognize the limitation that "the process of converting life experience into data always necessarily entails a reduction of that experience – along with the historical and conceptual burdens of the term" (D'Ignazio & Klein, 2020, p. 10). We hope that our gender definition mitigates this reductionism and emphasizes the shared experience of women in an inequitable environment such as in the field of mathematics.

With this gender categorization, our data set consists of 18,061 responses with 10,069 Men and 7,992 Women. We first use descriptive statistics and univariate analysis to examine the overall relationship between gender and students' perceived frequency of the 8 statements related to sense of belonging. We consolidated the answer choices into two bins: Low to represent frequencies of Does not occur and Minimally Descriptive and High to represent frequencies of Mostly Descriptive and Very Descriptive. This removes the middle answer choice to better compare non-neutral responses. After reviewing the overall trends, multivariable analysis assesses these trends across the 12 universities. For each statement, we compare the patterns seen across the universities to the Overall Trend as determined by the univariate analysis. Multivariable analysis then informs our exploration of two institutions as particularly interesting cases with additional qualitative data to supplement and understand their unique results.

Results

Looking at the overall relationship between gender and the perceived frequency of the 8 selected classroom activities shows differences within this perception. Two main patterns emerge. The first is that women constitute higher proportions of Low responses and lower proportions of High responses. This pattern arose to varying degrees, with the largest gender gap occurring in statement (1) (Low gap = 8.6%, High gap = 6.7%) while the smallest gap appears in statement (5) (Low gap=3.3%, High gap=1%). The second pattern shows women constituting the larger proportion of Low responses and being marginally higher but practically equal in High responses. This happens in both statements (3) and (4). Figure 1 shows an example of each of these patterns (the percentages do not equal 100 due to removing the middle response). The results of each statement within this analysis is called the Overall Trend.

Figure 1: Examples of the Two Patterns for Overall Trends

The multivariable analysis of these data adds in the additional component of institution. For each institution, we use bar charts and contingency tables to compare the same relationship between gender and perceived frequency in each of the 8 selected statements. We found that for each statement, variation emerges between the institutions. Each statement has at least two institutions that do not match the Overall Trend. Out of the 12 institutions, two did not fit the Overall Trend in statements (1), (6), (7), and (8), four did not fit in statements (2) and (5), five did not fit in statement (3), and 8 did not fit in statement (4). Figure 2 below showcases different trends across three institutions for statement (7).

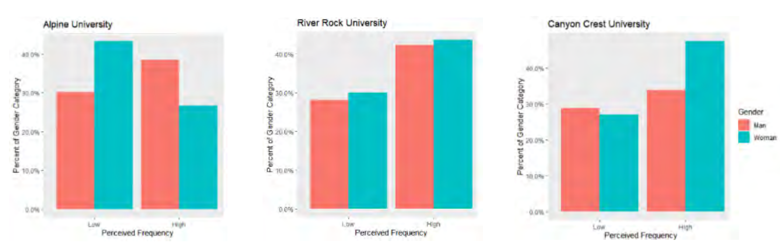

Figure 2: Gender versus Participation across Three Institutions

The Overall Trend for this statement has women constituting a higher proportion of Low responses and a lower proportion of High responses, which matches Alpine University. River Rock University has women producing just slightly higher proportions in both Low and High responses. Canyon Crest University portrayed opposite to the Overall Trend such that women constitute a lower proportion of Low responses and a higher proportion of High responses.

Comparisons of trends within each institution led to noticing two in particular that consistently produced noticeable characteristics across the 8 statements. For all 8 statements, Alpine University follows the trend by which women constitute a higher proportion of Low responses and a lower proportion of High responses (even when the Overall Trend did not represent that pattern) and each statement portrayed the relatively largest gender gap. In four of the statements, Canyon Crest University follows the trend that women constitute a lower proportion of Low responses and a higher proportion of High responses which is not seen in any Overall Trend. Additionally, they represent one of the relatively smallest gender gaps for statements in which they do follow the Overall Trend. To explore these two unique cases further, we brought in pre-collected qualitative data (Fall 2017) providing a thick description of each institution. Table 1 displays six areas of relevant comparison between the two institutions.

Table 1: Areas of Comparison of Canyon Crest University and Alpine University

Area of comparison	Canyon Crest University	Alpine University
Gender percentages overall	47% men and 53% women	54% men and 46% women
Gender counts of survey respondents by course	PC: 94 men and 105 women C1: 58 men and 81 women C2: 50 men and 75 women	PC: 912 men and 886 women C1: 804 men and 619 women C2: 622 men and 183 women
Class sizes in calculus sequence courses (P2C2)	30-40 students across P2C2 courses	50-90 students in PC and C1 20-40 students in C2
Students enrolled in P2C2	454 students	3996 students
Teaching methods	A variety of pedagogy across courses, including lecture, active learning, and IBL	Mainly lecture or lecture with minimal active learning components
Diversity supports	First Year Academic Success program but relatively few initiatives to support HU students in STEM	Emerging Scholars Program focused on HU students but mainly catered to first generation more generally

Discussion

The overall trends we see within the 8 selected statements related to a sense of belonging in math support previous literature in this area. Both Rattan et al. (2018) and Good et al. (2012) found that perceptions of the math classroom differed based on gender, and men often report higher feelings of belonging. The latter result parallels the first Overall Trend where women constitute higher proportions of Low responses and lower proportions of High responses. This perception can have critical outcomes for women pursuing additional math courses. The second Overall Trend may relate to areas in which women are more attuned, thus showing up more prevalently in both Low and High responses. The two statements with this trend particularly connect to student-instructor and student-student interactions, both of which strongly influence women's further enrollment in math (Gayles & Ampaw, 2014, Kogan & Laursen, 2014).

We see important variability in student perceptions across institutions. This supports the sociopolitical perspective in which each institution carries their own mathematics culture that influences the accepted norms around who belongs in that space. Even with the variability, though, many of the institutional trends overall still portray women making up the higher proportion of Low responses. This may imply that many of these institutions still conform to traditional norms of mathematics culture that continue to exclude women students – Alpine University especially fits within this idea. Canyon Crest University breaks from these expectations and portrays a contrary case. The notion of representation may play a role here, with sense of belonging relating positively with the number of same gender peers within the major (Rainey et al., 2018). Not only does Canyon Crest University overall have a higher percentage of women, but in all three P2C2 courses in the survey, women make up the majority. We see that men dominate all of these spheres for Alpine University. The teaching methods may also promote Canyon Crest University's results, as active learning and methods such as IBL can especially support women students in mathematics, while Alpine University utilizes mainly lecture (Cooper et al., 2015; Laursen et al., 2014). Further exploration of these cases will involve looking into how class size may impact instructor-student relationships and the gender demographics of instructors as potential role models.

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"VERY BRIGHT BUT QUIET": RACIAL, GENDER, AND COMPETENCE NARRATIVES IN MATHEMATICS TEACHER PROFESSIONAL DEVELOPMENT

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This paper examines how mathematics teacher-researchers (TRs) talked about students while involved in equity analytic work in a partnership with university-researchers. We draw on interview transcripts from a professional development setting to understand how ideologies of race, gender, and competence became instantiated as the TRs characterized their students as "shy," "quiet", or "confident". We explore how these characterizations may have been shaped by privileged mathematics teaching narratives and the research environment itself.

Keywords: Equity, Inclusion, and Diversity; Gender; Professional Development

Implicit biases and ideologies shape the everyday work of teaching (Battey & Leyva, 2018; Gilliam et al., 2016; Louie, 2018; Philip, 2011). We investigated these biases and ideologies as revealed during teacher-researchers' (TRs) reflections with analytic summaries of student participation. Although implicit race and gender biases have been investigated in laboratory settings (Amodio & Devine, 2006; Gilliam et al., 2016), few studies have examined the impact of teachers' implicit biases in classrooms. Our analysis echoed the innate complexity of equity work (Gutiérrez, 2009) and the ways ideologies about race, gender, and competence can become instantiated in authentic professional development settings. To do so, we attended to social narratives that were indexed in the words and syntax of TRs in reflective conversations with university-researchers (Urs) discussing equity analytics collected in TR's classrooms. We conclude that (a) TRs instantiated ideologies about race, gender, and competence in complex, uneven ways when discussing their students as shy, quiet, or confident (or their opposites), (b) the instantiation of these ideologies cannot be disentangled from (1) mathematics education narratives that assert discourse as necessary for learning and (2) the research environment itself. We explore the nuances of how racial, gender, and competence narratives, as parts of ideologies that shape mathematics teaching, are not statically innate emanating from the minds of teachers.

Racial, Gender, and Competence Narratives in Mathematics Teaching

Racial and gender narratives are dynamic, relational, and context-dependent connections of traits and characteristics to racial (Nasir et al., 2012; Shah 2017) and gender (Langer-Osuna, 2011; Mendick, 2005) identity markers. These narratives are expressed through discourse that hierarchically restricts available positions of individuals and groups (Wagner et al., 2019). Hall (1986) defines ideology as "the mental framework—the languages, the concepts, categories, imagery of thought, and the system of representation—which different classes and social groups deploy in order to make sense of, define, figure out, and render intelligible the way society works" (p. 29). Social narratives are instantiations of racial and gender ideologies. Mathematics education has addressed racial narratives (see e.g., Lei, 2003; Shah, 2017) and racialized gender narratives (see e.g., Gholson, 2016; Gholson & Martin, 2019; Morris, 2007).

Importantly, these narratives do not play out in isolation and impact how students enact and perceive race and gender (Gholson & Martin, 2014) as well as how teachers perceive students' performance (Lei, 2003). The interplay of racial and gender narratives and context have a complex impact on teaching. Morris (2007) noted that the treatment of Black middle school girls was uniquely "influenced by dominant ideas of race, class, and gender more broadly" (p. 510). Elsewhere, educators explained away the non-participation of Asian-American students in athletics with the racial narrative that Asians are not good at sports yet Asian-American students noted racial tensions and employment constraints as the primary factors of their non-participation in school athletics. Crucially, the evocation of these narratives can powerfully restrict students' participation in classrooms and their development of positive identities as students.

We analyze reflection interviews between TRs and Urs to identify words and word combinations that index social narratives, thereby identifying ways TRs may be taking up racial and gender ideologies as they made sense of and talked about data from their own classrooms. This professional development used Complex Instruction (Cohen & Lotan, 2014), which supported discussions of competence, status, and positioning in terms of participation equity.

Data and Methods

This study involves a nine-year partnership focused on facilitating classroom discourse (Herbel-Eisenmann et al., 2017). Across the district, approximately 40% of the students were from non-dominant communities (primarily African American students) and about 35% of the students qualified for free and reduced lunch. About 15% of students were emergent bilinguals. The TRs were aware that studies had documented boys receiving a disproportionately greater number of high-level questions than girls (Sadker et al., 2009) and similar patterns where Black students are relegated to lower-level aspects of mathematical tasks (McAfee, 2014). In 2017-2018, five of the TRs decided to focus on implicit bias in their classroom discourse. In particular, they all focused on identifying students as girls/boys and as Black, White, Latinx, Middle Eastern, and Asian. The TRs were all White, had 10-30 years of experience, and taught a range of secondary mathematics.

We used the EQUIP (Equity Quantified In Participation) app, a classroom observation tool that helps teachers identify patterns of implicit bias (see Reinholz & Shah, 2018) and provides quantitative information on the distribution of participation and participation opportunities to particular groups and individual students. Lessons were video recorded monthly and coded based on which student participated during a given participation sequence and aspects of the participation such as teacher solicitation method. At three times across the school year, the TRs examined their analytics and then were interviewed by one of the Urs.

To surface prevalent descriptors of students, we used Antconc, a concordance software that was designed to work on large data sets. Based on the literature, one co-author used the software to search for and examine particular descriptive words (e.g., confiden*, quiet, loud). Another co-author looked at a frequency list of words generated by Antconc and grouped words based on the kind of descriptor they were (e.g., mental verbs like "reflective"). We compared our lists and focused on a subset of recurring words (e.g., shy, quiet, dominat*, engage, confiden*). In our findings, we focus on these words and answer: How do mathematics teachers index social narratives about race and gender while talking about their students in the context of engaging in equity work? Further, how does teachers' use of language get mediated by norms shaped by situational factors of the classroom and the research setting itself?

Findings and Discussion

We discuss two situational factors that mediated the ways ideologies of race, gender, and competence were instantiated as the TRs talked about their students during debrief interviews related to the equity analytics partnership work. We situate our discussion in the context of other initial findings, to be discussed at length in a subsequent paper, where we counted the number of times the TRs used the terms shy, quiet, and confident/confidence ("confident") to refer to their students across the entire corpus of interview transcripts. The frequencies, disaggregated by the race and gender of the referenced student, revealed that TRs were not consistently drawing on stereotypical racial and gendered narratives about students. For instance, in contrast to the literature, Black girl students were most frequently described as shy and not confident compared to all other demographic groups and were also often described as quiet. These results motivated us to examine situational factors to explain the differences.

Situational Factors: Quiet with Shy and Quiet with Not Confident

We explored how TRs used combinations of "quiet," "shy," and "confident" to describe their students. For instance, Ms. Snow combined quiet and shy to describe Sapphire, a Black girl student. While looking at a bar graph that plotted the number of contributions each student made during four classroom observations, Ms. Snow described Sapphire as "very bright but quiet" and noted Sapphire was "very excited about meeting with [the UR]. She comes up to me one on one and will talk with me. Maybe it's just that she's just nervous or shy." In the same debrief, Ms. Snow noted that the two questions she directed to Sapphire were what-questions which received what-answers. Ms. Snow said, "[I] need to work on that part 'cuz she's a very strong student. She just is very shy, I think, and quiet in class. I need to work on positioning her to be seen as more of a competent student." In these two conversational turns, Ms. Snow's use of "but" and "just" seem to indicate that a "bright" or "strong student" is expected not to be "quiet." This indexes a social narrative of bright students as participatory. The context here is important as the research partnership included discussions about Complex Instruction. Ms. Snow's expressed need to position Sapphire as competent references competence as a status marker in a community of learners and not a static characteristic of a person; this use of status markers and positioning was shared theoretical ground for the Urs and TRs.

This awareness is also evoked when, later, one of the Urs and Ms. Snow looked at the bar graph for question type by race. Although Sapphire was not explicitly discussed at that point, Ms. Snow's desire to position Black students as competent echoed her desire to position Sapphire as competent. The multiple layers of linking quiet and shy revealed the messiness of implicit bias: something is happening because Ms. Snow thought the equity ratios of question type would be better, yet her language did not consistently evoke racial or gender narratives that would suggest a one-to-one implicit bias.

We also identified instances of collocation of "quiet" and "confident": for all but one of these, quiet was paired with low confidence. In one instance, Ms. Hill, described a White girl, Mary, as quiet and added, "I think she's confident though". The use of the contrasting connective *though* positions confidence as a trait contradictory to the behavior of quiet. This languaging of quiet as contradictory to confidence was a part of the other six conversational turns that contained both confident and quiet. One contrasted quiet in a White boy with confidence through the phrase "doesn't have a lot of confidence". The remaining five positioned students who were quiet as having less confidence. Thus, less participation seemed to also index less confidence.

Situational Factors: Gestures and Bars

Here, we look at how the data analytics became an important situational factor in how the TRs talked about their students. Figure 1 shows a scene from a debrief interview with Mr. Smith during which he talked about a bar graph showing various "equity ratios", defined as the amount of actual student participation divided by expected participation based on demographic characteristics, disaggregated by race.

UR:	Let's look at race. I think out of this—I want us to focus on Black students and Latinx students.
Mr. Smith:	I think the Latinx numbers are up cuz they were down at the beginning of the year.
UR:	That's right. I think you have two Latinx students in your class, Maria and Opal, right? I agree. I've noticed that too. If you look at—
Mr. Smith:	Maria's feeling really good right now. She is really taking the bull by the horns, and she's got a ton of confidence.

Figure 1: Debrief Interview with Mr. Smith

The concept of "equity ratio" acquired social meaning when Mr. Smith talked about the equity ratio for Latinx students. Figure 1 began when the UR prompted Mr. Smith to discuss his Black and Latinx students. Pointing to the data visualization, Mr. Smith swept his finger across the screen from right to left while stating, "I think the Latinx numbers are up cuz they were down at the beginning of the year". The phrases "the numbers are up" and "they were down" are reminiscent of increased efforts to use data for instructional decision making and are associated with desirable and undesirable, respectively. Further, Mr. Smith's statement "because they were down at the beginning of the year" and his comment about Maria "taking the bull by the horns" suggest that he equated more participation with something positive.

Mr. Smith might have used similar language without the data visualization, but we find his physical gestures noteworthy. Moving his finger parallel to the horizontal line on the graph, which represents an equity ratio of one, Mr. Smith divides the graph into two pieces. Parts of the graph above the horizontal line are desirable while parts of the graph below it are undesirable. The bar for Latinx students surpassed this horizontal line, and Mr. Smith conjectured that Maria, who "got a ton of confidence", was likely responsible for this increase.

Looking Forward

In this paper, we highlighted how racial, gender, and competence narratives, as part of larger ideologies that shape mathematics teaching, became operationalized in an equity-focused professional development setting. To do so, we explored how two situational factors potentially shaped teacher discourse during such professional development. Our findings suggest that further research is needed to better understand teachers' conceptualization and interpretation of student behaviors and performed racialized and gendered identities within the context of the teachers' pedagogical choices. Such work would provide an understanding of the inconsistent use of words that traditionally index racial and gender narratives.

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BLACK STUDENTS' MATHEMATICS IDENTITIES IN RURAL APPALACHIA

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The ways in which Black students identify with and experience mathematics is contextual, so it is necessary to explore the peculiarities and complexities of the experiences of Black learners of mathematics in various spaces and geographies while also attending to the intricacies of individual and schoolwide interactions. One context that needs this attention is rural Appalachia. The region is defined by a normative Whiteness and by economic issues, resulting in issues of race being made invisible. This study explores the mathematics identities and experiences of Black students in rural Appalachia. One in-progress counterstory of a Black student in the region is presented here. This highlights the necessity to explore mathematics identity and socialization in the context of rural Appalachia.

Keywords: Equity, Inclusion, and Diversity; Social Justice

Background

There has been a recent sociopolitical turn (Gutierrez, 2013) in mathematics education research that focuses on issues of identity and power in the field. This movement has led mathematics education researchers to bring issues of race to the forefront in their work. Martin (2009) calls for work with an emphasis on the racialized way in which students learn mathematics. It is necessary to examine the ways that Black students and other students of color view their mathematics abilities and develop identities as mathematics learners. While a colorblind lens is popular among some educators, it is essential to recognize that race affects the way that students experience the world and mathematics education (Chapman, 2013). The ways in which Black students are socialized into mathematics is also highly contextual, making it necessary to explore the peculiarities and complexities of the experiences of Black learners of mathematics in various spaces and geographies while also attending to the intricacies of individual and schoolwide interactions (Martin, 2009).

In this work, I apply the focus on racialized mathematics identity and socialization to rural Appalachia. Because of its perceived "rural, normatively white population" (Anglin, 2002, p. 566), little scholarly work has attended to Black lives in central Appalachia and racial issues are often disregarded in the region (Blee & Billings, 2001). Because of the normativity of Whiteness and the monolithic representation of Appalachians as impoverished, class and economic issues are in the forefront of the narrative about Appalachia. This emphasis on "white problems" has created a situation in which racial issues in Appalachia are ignored (Cabell, 1985, p. 3). In fact, some White people see economic concerns as being overshadowed by fights for LGBTQ+ and racial justice (Catte, 2018). The entanglement of race, the economy, and education in rural Appalachia leads me to the following questions: (1) How do Black students construct racialized mathematics identities in rural Appalachia? (2) What experiences lead to this construction?

Theoretical Framework

In order to address these questions, I draw on Martin's (2000) framework for research on mathematics identities and socialization. This tiered framework acknowledges that mathematics learning is affected by the sociopolitical and historical context, the communities and schools in which students live and learn, and students' individual agency and efficacy. This examination of micro-, meso-, and macro-levels (Martin, 2013) of mathematics learning while Black is especially important in rural Appalachia.

Mathematics Identity and Socialization

While there are multiple conceptions of mathematics identities (i.e. Cobb et al., 2013), I draw on Martin's (2000) definition that mathematics identity is students' beliefs about multiple aspects of mathematics learning: their ability to perform, the importance of mathematics, roadblocks and opportunities to learn, and motives and methods for learning. This definition attends to the ability for students to define "what it means to be African-American in the context of mathematics learning" (Martin, 2000, p. 20). Mathematics socialization is closely intertwined with mathematics identity. Martin (2000) defines socialization as the "processes and experiences by which individual and collective mathematics identities are shaped in sociohistorical, community, school, and intrapersonal contexts" (p. 19).

There is an abundant body of extant research examining the construction of the racial and mathematics identities of successful Black males (Berry, 2008; Berry et al., 2011; Noble, 2011; Jett, 2011, 2019). These studies cite a dearth of research on mathematics achievement in Black males and an essentialization of them as lacking persistence and a drive to succeed. A smaller, but growing, field of research uses feminist theory as a frame for studying Black girls and their mathematics identities (Johnson, 2009; Gholson & Martin, 2014; Leonard et al., 2020). The Black feminist viewpoint allows researchers to examine the unique perspective of Black girls in the White male dominated field of mathematics. Some studies on mathematics identities allow for Black males and females to "share theoretical spaces" (Gholson, 2016, p. 297). Many of the studies focused on mathematics socialization also address identity issues (Walker, 2016; Jackson, 2009; Stinson, 2006, 2008, 2011, 2013).

Critical Race Theory (in Mathematics Education)

This study is framed by Critical Race Theory (CRT). Some of the principles that drive CRT have allowed researchers to center educational issues on race and to highlight the way that racial inequities manifest and function in American schools. There is an emerging body of research in mathematics education that uses CRT as a framework for work with Black students (Davis & Jett, 2019). This study draws on the work of various CRT scholars and theorists (Bell, 1980; Harris, 1993; Ladson-Billings & Tate, 1995; Delgado & Stefancic, 2012) and builds on their work to situate it within a context in which race is rarely examined, especially in mathematics education. I draw on tenets of CRT including that racism is endemic in American society and schools, that they are based on property and economics, and that experiential knowledge of Black students is a rich source of knowledge. Other guiding principles are a resistance to neutrality on issues of race and a focus on counterstories from individuals.

Methods

This report is part of a larger, in-progress study being conducted with Black students in West Virginia. I use a critical race methodology incorporating aspects of narrative inquiry. CRT can provide a "methodological instrument for collecting and understanding the perspectives of marginalized groups" (Morris & Parker, 2019, p. 25). Narrative inquiry is the study of human lived experiences in the form of stories (Clandinin, 2013). These stories are embedded in interactions and societal structures, and they relate to power, status, and identity (Atkinson &

Delamont, 2006). This study uses counterstories because "in CRT, narrative is counter-storytelling" (Berry & Cook, 2019, p. 88).

I recruited participants through my work in an after-school science and mathematics program. Through this work, I have a mentorship relationship with the students and have known each of them for multiple years. I first asked the students to write mathematics autobiographies that allowed them to reflect on their experiences learning mathematics throughout their lives. I used these to guide my semi-structured interviews with them relating to mathematics identities and socialization. Autobiographies and interview transcripts were coded according to macro- and meso-level aspects of the context and for examples of mathematics identity and socialization. Critical events (Webster & Mertova, 2007) were also noted, and they guided narrative development. From these codes, I constructed narratives for each student that serve as critical race counterstories (Berry & Cook, 2019). Here I present one in-progress counterstory from the larger study to illustrate the necessity to explore mathematics identity and socialization in the rural Appalachian context. Multiple phases of member checking have happened and will continue to occur, and all quotations in the narrative are LaMarcus' words.

Findings: LaMarcus' Counterstory

Located in the heart of coal country in southern West Virginia is Martin County. LaMarcus has lived in the town of East Branch, "a simple town" with a Family Dollar and a gas station, his entire life. It is a place with little opportunity for recreation or other activities, where you "are more likely to get … a four wheeler or a dirt bike … than a real bike." It is a rural area, and LaMarcus' first descriptor for the town was "poverty." His immediate surrounding community is made up of other Black people, and his school has "a pretty good mix of Black and White" students. This is significant considering the population of West Virginia is less than five percent Black/African American. While already known for its economic issues, Martin County became infamous with the election of Donald Trump and the rise of so-called Trump Country.

> It might not be nothin' like uh compared to what you see on CNN, but yeah. You have your rebel flags all the time especially when Trump came into office and said certain things … I had so many debates with some of the people I know…they be like "He (Trump) saved coal!" and they'll say stuff like that.

LaMarcus attends Hilltop High School which is in a combined building with the middle school of the same name. He refers to Hilltop as a normal high school in West Virginia. However, he believes it is seen as a "black cloud" by surrounding schools because of the relatively large proportion of Black students at the school. This has led to racist comments that end in scuffles in athletic competitions. LaMarcus says that he hears the occasional "N word" at school but that it is not too often. There are occasional fights between Black and White students around racial stereotyping, racism, and racist comments, but "it's not like every month."

LaMarcus' mathematics identity has been consistently defined by his self-professed struggles with the subject which has led to a dislike. He says, "Learning math in elementary and middle school was difficult for me. For most of it I didn't like any of it." It seems that many of his negative feelings towards mathematics come from an embarrassment that has come from being asked to publicly solve procedural problems with speed. He states that, "You had to worry about other people making fun of you for not knowing something." He also cites a mathematics relay race causing anxiety that contributed to his early negative feelings towards the subject.

In middle school, because of lagging grades, LaMarcus was moved to a remedial mathematics class. While not what he wanted, he did believe that his teachers and family had his best interests in mind. However, he worked hard and had positive mathematics experiences to work back to "on-level" courses by his sophomore year of high school. His progress seemingly came undone because of a racist incident at school during that year.

> I had this guy … he said the "N word" and it went from like joking, like I said I don't pay it no mind, but ya know he was joking and ya know and he was playing around and he just end up pushing me a little too hard at one moment and it was just like, I … I snapped. And I pushed him right on back. And that turned into a little fight.

This led to a ten-day suspension from school for LaMarcus. After the suspension, he fell behind in his mathematics class and was placed back in a remedial class in the middle of the year. LaMarcus feels that his time in remedial classes has led to a lack of access to mathematics content that he perceives as important for college, including content that is found on the ACT. About the content on the ACT he says, "I can do certain things. My biggest issue is most of the math … is I've just never seen it before." He believes he has strong skills in real world mathematics and dealing with money, which he perceives as very important. This stems from his time working the cash register at a local store: "But when I had that job working the cash register counting like money and stuff, yeah, it helped me."

While LaMarcus views mathematics as extremely important, mostly from others telling him that it is, he still has developed a frustration from his time in school. It began from a young age when he felt like he did not get the assistance he needed. When asked what could have helped him, he replied, "Maybe got the extra attention that I needed as a kid" in his mathematics classes. LaMarcus is graduating from high school this spring and plans to attend college. Even with his negative mathematics experiences, he is still planning to pursue a STEM career. This pursuit has been aided by his own perseverance, the support of his mother and family, and his participation in an after-school program. LaMarcus would have liked to be a cancer therapist, but he has decided on a career in exercise science because he believes it will require less mathematics. His lack of confidence in mathematics still shows, "Hopefully I can get the proper tutoring if I need it."

Discussion and Implications

LaMarcus' story exemplifies many of the problems facing Black students in rural Appalachia when learning mathematics. First, LaMarcus must deal with implicit and explicit racism on a regular basis. While he often "doesn't pay it no mind," sometimes it leads to a defensive response which in turn affects his academics and mathematics learning. His experiences have also led to a mathematics identity that involves valuing and believing in the importance of mathematics but also feeling like it is a constant struggle. These include a procedural, public, and competitive way of learning and a denial of access to certain concepts or topics. While he has found a way to thrive and to choose a path he is happy with, mathematics is keeping him, at least currently, from what he would really love to do.

LaMarcus' narrative shows the importance of a closer look at the intersection of context and mathematics education, particularly in rural Appalachia. His experiences in Martin County illustrate the endemic racism that is present in the region. While the racism that LaMarcus faces is not unique to Appalachia, its intersection with economics and education is novel and will be examined further in this study. We should also aim for an expanded view of what mathematics teaching and learning can be and what success in mathematics is defined as. Other data in this study will explore the experiences in which LaMarcus and other students have achieved success in mathematics, what assets they bring to mathematics classes that are devalued, and what an ideal mathematics education would look like for them. This study will also look at the nuance of racialized mathematics identities in different locales and schools and in different individuals across the region.

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PERSPECTIVES ON REHUMANIZING UNDERGRADUATE MATHEMATICS: ELEVATING THE VOICES OF LATINA AND MIXED-RACE WOMEN

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Building on the recent sociopolitical turn in mathematics education research (Adiredja & Andrews-Larson, 2017; Aguirre, et al., 2017; Gutiérrez, 2013), this research takes up calls to rehumanize mathematics classes (Gutiérrez, 2018) with a particular focus on Latin (Leyva, et al., 2021) and mixed-race women. This case study explores the experiences and ideas of four STEM majors who took a Calculus 2 class with the first author in order to determine what they find rehumanizing in their STEM learning experiences. Using Gutiérrez's eight rehumanizing dimensions (2018), we find that the central themes emerging from these students' comments aligned with the dimensions emphasizing one's participation and positioning, ownership, and living practice. We explore the specific themes arising in these dimensions and discuss potential teacher moves that could help to create more humane mathematics classes.*

Keywords: Calculus; Undergraduate Education; Gender; Equity, Inclusion and Diversity

Introduction

Many compelling arguments have been made for a sociopolitical turn in mathematics education research (Adiredja & Andrews-Larson, 2017; Aguirre, et al., 2017; Gutiérrez, 2013). Gutiérrez (2018) more specifically calls for mathematics courses to be rehumanized with a primary focus on Black, Indigenous and Latin* (Leyva, et al., 2021) students, who the system has failed, and to find ways to identify and draw on their strengths in our research and teaching. Furthermore, focusing on women who belong to these groups is particularly important given the white patriarchal norms that dominate mathematics education in the United States (e.g., Leyva, 2021). These norms, including an emphasis on competition and individualism, create a system that poses particular challenges for Black, Indigenous, and Latin* women in STEM (e.g., Charleston, et al., 2014; Ong, et al., 2011).

This research seeks to center students as experts in their own experiences and explore what concrete steps are being done or can be done in undergraduate mathematics classrooms to move toward rehumanization, specifically for Latina and mixed-race female STEM major students. (Note: We focus on Latina and mixed-race women students because we did not have any Black female students in the Calculus 2 course from which participants were drawn for this study.) In attempting to fill a gap in the research literature in this regard, we aim to answer the following research question: What classroom practices and structures are perceived as rehumanizing by Latina and mixed-race women calculus students? We then use these findings to suggest implications for undergraduate mathematics teaching, also calling for more research including a wider range of voices.

Rehumanizing Mathematics Framework

There have been many calls for mathematics teaching practices that are inclusive and equitable (e.g., Abell et al., 2017; Bressoud et al., 2015; National Council of Teachers of Mathematics, 2014). However, as Gutiérrez (2018) points out, equity is often perceived as a destination, when in reality it should be seen as a continual process. Thus, she introduces the term *rehumanizing* to highlight this focus on action.

Gutiérrez (2018) outlines eight dimensions of mathematics classes that might count as rehumanizing. These include (1) participation/positioning, (2) cultures/histories, (3) windows/mirrors, (4) living practice, (5) creation, (6) broadening mathematics, (7) body/emotions, and (8) ownership. Of particular interest for this paper are domains 1, 4, and 8. The *participation/positioning* category includes shifting the authority and power dynamics in the classroom between teacher and students, with a focus on student relationships and student interaction. The *living practice* dimension includes rule breaking, divergent thinking, and doing math for one's own purposes; whereas *ownership* includes posing new mathematical questions and grappling with math problems outside of school time. To date, there is not a large body of research indicating what students view as rehumanizing, particularly at the undergraduate level. One question this brings up then is what kinds of learning experiences Latin*, Black, and Indigenous students perceive as rehumanizing, which is where we situate this work.

Researcher Positionality

To align this work with explicit calls for mathematics education researchers to consider how our identities impact our research, we'd like to offer some clarity about our positionality as researchers (Aguirre et al., 2017). Both researchers identify as white, cisgender women. One researcher is a tenure-track professor in an education department (focusing on mathematics education), while the other researcher is a teaching professor in a mathematics department (and was the instructor of the focal course being studied in this research). As women, we have both experienced alienation in various male-dominated mathematics spaces. At the same time, we recognize the privilege we have received due to our race. This has led us both to center and learn from the voices of Black, Latin*, and Indigenous women, whose everyday experiences and identities are often not recognized or elevated in the mathematics classroom.

Research Approach & Data Collection

This study was conducted with a small subset of students who took the first author's Calculus 2 course in fall of 2018 at a large R1, predominantly white institution in the western United States. Over a few-year period, and as part of an effort to rehumanize her calculus courses, the instructor implemented many changes in her Calculus 2 courses, including flipping the course structure, incorporating active learning strategies, sending out a weekly email highlighting mathematicians from institutionally marginalized groups, incorporating a class mission statement, and implementing group exams (see Dobie & MacArthur, 2021 for more details on the group exam structure). During the Fall 2018 semester, survey data was collected from all Calculus 2 students who were willing to participate (approximately 84% of the 223 students who finished the course) to better understand their experiences in the course, and with group exams, in particular. Analyses of two survey questions revealed negative or lukewarm sentiments towards the idea of group exams as rehumanizing mechanisms among six Latina and mixed-race women students–standing in stark contrast to the majority of students who expressed positive feelings (MacArthur, 2021). These women were thus identified as candidates for a case study, as a purposive sample (Devers & Frankel, 2000; Patton, 1990), to better understand both their experiences with group exams and their ideas about rehumanizing mathematics, more broadly. Of the six students who were invited to be interviewed, four agreed to participate via Zoom. Denise and Sofia both identified as Latina; Selena identified as white, Asian, and Pacific

Islander; and Lilly identified as white, Hispanic, and Native Hawaiian/Pacific Islander (note that all names are pseudonyms). The case study approach was chosen specifically to meet the goal of reporting a rich description of the experiences of these students (Creswell & Poth, 2018).

Following up from the survey data, a primary theme explored in the interviews was what makes mathematics, and STEM learning experiences more broadly, rehumanizing versus dehumanizing. Specific interview questions in this area explored what it would look like for a STEM learning experience to feel humane; prior STEM learning experiences at the university that the students would describe as humanizing or dehumanizing; views on whether group exams are humane or inhumane, and why; and suggestions for instructors to rehumanize STEM learning experiences and assessments.

Data Analysis of Interview Data

We utilized a thematic analysis process (Braun & Clarke, 2006) to analyze the data from the interviews for all four students. After first transcribing the interviews and reading through the transcripts at least once each, we extracted all excerpts where students identified rehumanizing, or humane, aspects of learning environments. Then we conducted open coding of those excerpts, and organized them into themes, agreeing upon each theme. We then worked together to determine where each of these themes fit within the eight rehumanizing dimensions, leaving open the possibility that some comments or themes might not fit squarely in one of the eight dimensions. All four interviews were coded by both researchers, and any discrepancies were resolved. To increase trustworthiness and credibility, we use direct quotes when sharing our findings in order to convey the students' thoughts and feelings as authentically as possible.

Findings

Themes emerging from these analyses fell primarily into three of the rehumanizing dimensions: participation/positioning, ownership, and living practice. As such, we focus the remainder of our findings within these domains.

In the participation/positioning dimension, three out of the four interviewees focused on how important interaction, dialogue, and relationships are for creating a humane environment in the mathematics classroom. In this dimension, there were two sub-themes that emerged from analysis of these excerpts: (a) accessibility and opportunities for interactions with the instructor and (b) interaction and dialogue with peers.

The first sub-theme focused on the importance of the student-instructor relationship. When asked about how to create humane STEM classes, Sofia stated that "having a good relationship with your professor" and the "support of professors" made classes feel rehumanized. In particular, Sofia highlighted how that support made her math classes "more approachable and less overwhelming." Similarly, Denise noted that her Calculus 2 instructor "became human to me" because of the instructor's attempts to make students "feel comfortable" in the class and because of how she structured the class. Regarding this last point, one aspect of the structure that Denise noted is how the class "permits for social engagement in different ways," a central part of the next theme.

Second, the interviewees frequently commented on how interaction and dialogue with other students in the class felt humane to them. Denise suggested that to humanize the classroom, we need "that aspect of dialogue with other individuals," referring to other students. One aspect of group exams that Sofia found rehumanizing was that students "could all contribute and [they] could collaborate with things they didn't agree on." Selena also emphasized how interaction and collaboration with classmates during group exams made the exams more humane. She spoke of the value of "shared responsibility...as a team," highlighting how it is humane to succeed and fail together, rather than alone. This notion of the value of working together with others was captured by Denise with the term "two-way learning," which she used to center "interaction" and the need to "supplement" one's "knowledge with someone else's and have dialogue."

At the intersection of living practice and ownership, three out of four women commented on the importance of mathematics being relevant for and useful in their lives. Lilly said that math classes are rehumanized when they address her question of "what are we going to use it for?" Denise voiced a similar sentiment, adding that she wanted "real-life applications" as part of her homework assignments, as well, to make math classes rehumanized. Finally, Selena explained that an example provided by an instructor in one of her math classes felt "really useful" to her and like a "real thing that we can do," which "felt rehumanizing."

Discussion

In this study, we learned that the students interviewed perceived discourse and relationships, both between teacher and student and among peers, and real world relevance as playing a dominant role in rehumanizing their undergraduate mathematics classes. Their comments suggest implications for both how we understand the rehumanizing dimensions and how we as educators structure our courses and implement rehumanizing efforts for our students. Regarding the former, the relatively brief descriptions of the rehumanizing dimensions (Gutiérrez, 2018) and little research thus far using the dimensions as an analytical tool leave room for expanding our understanding of aspects that might fall within each dimension. Based on these students' comments, we propose expanding the participation and positioning dimension to consider not only shifted authority to students and a focus on student interaction but also supportive teacher-student relationships that are characterized by opportunities for meaningful interaction with one's instructor. Additionally, findings highlight that some themes cut across categories, and we should not lose sight of these potential rehumanizing mechanisms that do not fit neatly within one category or another, such as real-world relevance.

In terms of implications for educators, working to shift the power dynamics in mathematics classrooms through relationships, interaction, and dialogue can help to rehumanize the math learning environment for undergraduate students. Such moves might be especially powerful for Latina and mixed-race women, as shifting power dynamics helps to move us away from white patriarchal norms that focus on independence and competition (Leyva, 2021). It is also important to note that while we discuss the relationship with the instructor as a rehumanizing mechanism, this relationship should center students and position the instructor as an approachable, accessible support, rather than the authority in the classroom. Finally, instructors should work to make connections between STEM content and students' everyday lives.

Moving forward, research should focus on collecting data from a wider range of women at different institutions, particularly including Black women's voices since they were missing from this study but are critical for this work. Future interview questions should also investigate undergraduate students' perceptions of some of the rehumanizing dimensions not explored in this study. While we did not seek generalizability in this research, we hope that these findings serve as a productive starting point in elevating the voices of Latina and mixed-race women and learning from their experiences.

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(UN)PRODUCTIVE STRUGGLE IN MATHEMATICAL SPACES: WOMEN'S PERCEPTIONS OF COMPETENCE AND BELONGING

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We present data from a pilot study that took place between Spring 2019 and Spring 2020. The goal of our analysis was to gain a deeper understanding of women and racial/ethnic minority students' experiences during the secondary school-tertiary transition in mathematics. In this brief report, we draw upon a three-dimensional model of attitude (Di Martino & Zan, 2010) to examine students' perceived competence in mathematics and its relation to emotions. We focus on a nascent mathematician's productive struggle in mathematics to discuss associations between perceived competence in mathematics and sense of belonging. We also highlight gendered stereotypes and raise awareness about women's struggles with these during their secondary-tertiary transition.

Keywords: Affect, Emotions, Beliefs, and Attitudes, Equity, Inclusion, and Diversity, Gender, Undergraduate education

Recent studies have documented that women and racial/ethnic minorities persist as underrepresented populations in STEM-related fields (Anderson & Kim, 2006; Nix & Perez-Felkner, 2019). Sax et al. (2015) found that females reported lower mathematical self-concept than their male counterparts despite women's slightly better representation in mathematics discipline compared to other STEM domains. Yet, increases in representation often do not ensure that women and racially/ethnically minoritized students equally participated in public spaces such as classroom discourse. In an examination of classroom discussion, Ernest et al. (2019) found that males tended to dominate public talk even if females similarly contributed to discussions in small groups. While men are typically recognized as full participants of the mathematics community given their stronger beliefs in abilities for success, women must contend with gendered stereotypes to acknowledge their potential in mathematics (Solomon, 2007). Considering the established norms in traditionally male-dominated mathematics communities, women are inclined to question their mathematical competence which has detrimental influences on their mathematician identity construction (Solomon et al., 2011).

Belonging and developing positive feelings are essential social aspects and affective components of meaningful participation, along with communicating mathematically in the process of involvement in a new community (Lave & Wenger, 1992). Women's sense of belonging is prone to decrease when they are exposed to gender stereotyping and environments that view ability as innate and fixed (Good et al., 2012). Sense of belonging to an academic community becomes an important element in understanding and explaining women's representation and perseverance in mathematics (Good et al., 2012). Women's perception of themselves as learners of mathematics is often conflicted with traditionally masculinized mathematical communities, which impacts their belonging and thus limits their participation.

Considering these issues, we address women's perceived competence in mathematics and its potential connection to their sense of belonging within mathematical communities. Next, we highlight certain historical gendered stereotypes and norms that play a role in women's identity construction in becoming a mathematician. Also, we explore how these factors relate to women's participation in mathematical spaces, which can be taken as a component of one's belonging.

Theoretical Perspectives

In our investigation of students' experiences in the transition from school to university mathematics (Uysal & Clark, 2020), we draw upon a three-dimensional model of attitude towards mathematics (Di Martino & Zan, 2011). The model conceptualizes the relationship among and between students' vision of mathematics (i.e., beliefs on the nature of mathematics), perceived competence in mathematics, and emotional dispositions towards mathematics in transition from secondary school to university mathematics Di Martino & Gregorio, 2019). They claimed that first-time experiences in university settings lead students to evaluate their mathematical competence and question the nature of mathematics, accompanied by emotions.

In the study, we investigated affective dimensions of students' experience during their transition from school to university mathematics. However, we also noticed that the social construct of belonging to an academic community is emerged, particularly with respect to the development of women and racial/ethnic minorities' mathematics learner identities (Master & Meltzoff, 2020). Accordingly, we focus on the influence of affective dimensions (i.e., perceived competence and emotions) and sense of belonging on how students perceive of and how they view their participation within the undergraduate mathematics program, especially when considering women's engagement with mathematics as one aspect of their learner identity.

Method

Participants and Settings

The research described in this brief report is part of a pilot study in which we focused on students majoring in mathematics, all of whom were from racially/ethnically minoritized populations (e.g., women, students of color). Twelve students participated in the Women and Underrepresented Minorities in Mathematics (WURMM) pilot study, which took place from Spring 2019 to Spring 2020. Here, we focus on the second seminar, which comprised one component of the WURMM study in Spring 2020. Our overall aim of this seminar was to make sense of students' transition experiences by addressing affective factors (e.g., perceived competence) in mathematics and mathematical identity. Four undergraduate mathematics students participated in the seminar. Three students were pure mathematics majors (Amelia, Dana, and Sunny), and one student (Manuel) was double majoring in secondary mathematics teaching and pure mathematics. We captured students' perceptions of the attributes of mathematicians as a result of addressing notions of mathematics identity and perceived competence in mathematics during the seminar activities. The data sources include videorecording of the four seminar sessions (each session was two hours in duration), post-survey responses, and students' written artifacts from the fourth session's activity. Prior to the seminar, we also conducted interviews with three of the four participants, including Dana.

Session 4 Activity: Exploring Attributes of Mathematicians

We implemented an activity designed to motivate discussion about a mathematician's identity and to capture the prominent attributes of a mathematician from the students' perspectives. The goal of the activity was to prompt students to elaborate on what it means to be a competent mathematician and make sense of students' ideas about their own perceived competence in mathematics. The activity entailed sharing a Google document divided into four quadrants with a guiding question for each:

- Who is a mathematician?
- What does a mathematician do?
- What does a mathematician say?
- What do you consider a mathematician is not?

Each participant populated the document with phrases that captured their response to each question. Additionally, we asked the same questions in the post-survey to provide students with the opportunity to articulate their personal views without the in-the-moment influence of other participants' responses.

Preliminary Findings

In this brief report, we present aspects of Dana's case regarding her perceived mathematical competence, as well as how she described the attributes of a competent mathematician. We found especially remarkable that Dana selected phrases that deviated from those of her peers. In response to "Who is a mathematician?", Dana included attributes of critical and creative thinking. Likewise, in her contribution to "What does a mathematician do?", Dana focused on recognizing relationships and engaging in an action that involves perseverance (e.g., "partake in trial and error"). During discussion on the different attributes, Dana was able to relate her experiences to multiple aspects of interest, such as gender and perceived competence in mathematics, in a way that also manifest affect and sense of belonging in mathematics discipline.

When we asked participants about the salient characteristics of a competent mathematician related to a mathematician's identity, actions, and discourse (i.e., what a mathematician does and says), Dana responded with: "The way in which they communicate automatically provokes thinking critically and stimulates conversation" (Dana, Post-survey 2020). When we probed about how mathematicians can change their confidence, Dana's post-survey response was similar to her previous comment during the seminar: "by 'doing' more mathematics, and by surrounding themselves in a supportive environment that promotes intellectual conversation about mathematics without rejecting the ideas of others" (Dana, Post-survey 2020). Her responses to these two prompts illustrated her perceptions of a competent mathematician that are aligned with participation and engagement in an intellectual mathematical community.

Participants were also asked about their perceived competence in mathematical spaces (e.g., classrooms, study groups). Dana articulated a perception of herself as a mathematics learner conflicted with what she values in a competent and confident mathematician. In the beginning, Dana described a significant aspect of her mathematical confidence related to being able to determine relationships between different abstract concepts as she progressed in her course work:

> I feel as if in a way... it has made me feel more confident, but also less confident for different reasons. Like for example, I feel more confident because of these higher-level math courses. I'm learning about math as this abstract concept and the more I'm just learning about these different concepts and I'm able to connect different relationships that are allowing me to just form different connections and different relationships. And so, in that way, I begin to feel more confident. (Dana, Seminar 2020)

However, Dana also expressed views that seemed to perturb her perception of her mathematical competence and which made her question her mathematical confidence:

> But also [I'm] just less confident. The more I'm surrounded by people that are just really, really brilliant and … I don't know how to explain it. I feel as if, even though I am in the same classes as them and I do have a right to be there, I always just, never built up enough confidence to, like, for example, just ask a question or just be involved and I feel not to turn this into a whole gender thing, but I feel as if … Because … mathematics still is more of a male based subject, it's hard for me to really voice my opinions and voice what I feel. When I am in a group of all guys during a study group or like when I'm just doing partners, you know, I just tend to get less confident. (Dana, Seminar 2020)

Dana's description of her identity as an emerging mathematician evoked emotional responses pertinent to her perceived competence in mathematics and sense of belonging to the mathematics community in which she was participating. Throughout Dana's seminar participation, she presented several examples of tensions in her negotiations of her mathematical identity, particularly in instances of participation as a woman in mathematical spaces. Her previous experiences, such as being one of few young women in her high school mathematics courses, even led her to question her sense of belonging in mathematics as a discipline. Despite these, Dana eloquently described her struggles of rejecting societal norms against women in mathematics as a potential conflict with her participation in the mathematics classroom, despite her outstanding academic success and enthusiasm in mathematics.

Discussion

We sought to demonstrate Dana's ongoing yet productive struggle toward breaking the "glass ceiling" equivalent in the discipline of mathematics, and in the near future, to persist in mathematics as a woman. Dana's significant interest, curiosity, and skills in mathematics are important assets for her to be academically successful in her major. Yet, some of the deficit beliefs on women's perceived competence in mathematics pervasive in a patriarchal society's actions seem to constitute psychological tensions for Dana. Evidence of the lived experience of Dana, a young woman, and a thus-far successful emerging mathematician, demonstrated an example of hidden mathematical competence that takes place when gendered underrepresentation is prevalent (Ernest et al., 2019).

Women, including those in mathematics, are challenged mentally and psychologically which results in an identity negotiation pertinent to what they can or cannot achieve. Challenges related to the masculinity of mathematics might lead women to unproductively question their sense of belonging which has the damaging potential to limit their active participation. Even though Dana believed in her mathematical abilities (i.e., perceived competence), gendered stereotyping (Leyva et al., 2021) appears to have created obstacles preventing her from participating in mathematical discourse—which can be intricately related to sense of belonging.

The secondary-tertiary transition involves many complexities. Consideration of intersectionality in learners' identities such as gender, race, and/or ethnicity can broaden our understanding (Leyva, 2017). The types of obstacles that confront students are also contingent upon a community's culture and systems of power. Individuals' actions are significantly influenced by how they are historically treated and positioned in society according to these values. Our research provides insights into women's experiences with mathematics and illustrates negative impacts of gendered biases on their perceived competence in mathematics. Our pilot study provides a potential new lens from which to examine the complexities of the secondary-tertiary transition and to extend the work of Di Martino and Zan (2011) regarding their three-dimensional model of attitude. Moreover, we highlighted the associations between perceived competence and sense of belonging that inhibit women from participating in mathematics. Our analysis indicates that patriarchal values potentially curtail possibilities for women to demonstrate their mathematical competence and thus perpetuate their marginalization. Therefore, women subtly acquire their secondary role in communities of practice in mathematics, which can have the effect of disregarding their skills and mastery in mathematics.

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THEORIZING DATA SCIENCE EDUCATION: AN INTERSECTIONAL FEMINIST PERSPECTIVE ON DATA, POWER, AND "PLAYING THE GAME"

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Recent growth in attention to data has led to calls to incorporate data science education (DSE) into the school mathematics curriculum. Many calls for reform, however, do not explicitly attend to the central role that data, writ large and through associated social structures, play in historical and ongoing systems of inequality and oppression. This paper offers a theorization of the interplay between data and power relevant for education and concludes with suggestions for reimagining K-12 DSE.

Keywords: Equity, Inclusion, and Diversity; Social Justice; Data Analysis and Statistics

A popular movement is underway to include data science education (DSE) as the centerpiece of high school mathematics (e.g., Boaler & Levitt, 2019). This is in parallel with related calls for efforts to improve mathematical, statistical, and quantitative literacies (e.g., Craig, 2018). While claiming that these literacies provide greater access to participation in workforce arrangements, these calls typically do not explicitly attend to power or the central role that either mathematics or data play in shaping and reinforcing systems of oppression. A pattern of simultaneously advocating for inclusion yet failing to attend to power could be seen as being complicit with those systems (Gutiérrez, 2013; Martin, 2019). We problematize approaches to DSE or statistics education that do not attend to, or downplay, power, like Weiland (2017). As an alternative, we explore an intersectional feminist approach to data science put forward by D'Ignazio and Klein (2020).

Playing Versus Changing the Game

Data science, like mathematics, tends to be positioned as neutral and objective (Benjamin, 2019). Yet both play a gatekeeping role that reinforces current hierarchies, particularly around race and gender (Gutiérrez, 2013; Martin, 2019). Gutiérrez's (2007) model of equity along dominant/critical axes is instructive. Its dominant axis includes access and achievement and relates to resources and participation to impact "how well students can play the game called mathematics" (Gutiérrez, 2011, p. 20). The critical axis includes identity and power, which acknowledges students as historicized, racialized, gendered, and classed selves and seeks to "build critical citizens so that they may change the game" (p. 21). DSE is often positioned as essential to playing the game, but less attention has been paid to the need for a DSE oriented around changing the game (i.e., challenging and dismantling the systems, structures, and institutions that produce those inequities). In this paper, we consider the tension between the dominant (play the game) versus the critical (change the game) with respect to K-12 DSE, through a comparative reading of the *PreK-12 Guidelines for Assessment and Instruction in Statistics Education II* (GAISE II, Bargagliotti et al., 2020) and *Data Feminism* (D'Ignazio & Klein, 2020).

Methodological Approach

We chose GAISE II as a text of analysis because it represents an official position of the National Council of Teachers of Mathematics and is endorsed by the American Statistical Association. GAISE II is an update of an earlier GAISE report (Franklin et al., 2005), which championed data and statistical literacy across PreK-12; the update reflects the growth in data science (as an interdisciplinary field, arguably distinct from statistics) in recent years. We chose *Data Feminism* as a text due to its potential to speak to issues of data and power within DSE. We engaged in a discourse analysis of these texts, drawing inspiration from Gee's (2011) building tasks of language. These building tasks are premised on how language builds or destroys "things in the world" (p. 88). We drew on one area of reality built through language according to Gee: the connections building tool, which focuses on how words and grammar build or destroy connections or relevance between things. We created tables of excerpts in which authors used the word "power" to attend to its explicit use and considered, for example, the grammatical role it played, other nouns (e.g., data, counting) being put in relation to power, the types of metaphors, what was indexed in the use of power, etc. We developed a theorization of data and power organized along thematic categories and informed by Gutiérrez' (2011) distinction between playing and changing the game.

Findings

In this section, we present thematic categories that theorize the relationship between data and power in ways that are useful for statistics and DSE practitioners and researchers and illustrate these themes through a comparative reading of GAISE II and *Data Feminism*.

Securing Access to Participate in Systems of Power

The first thematic category refers to who is permitted to participate in the data pipeline and how cultures of data science view and treat people, institutions, and communities. This theme reflects the perspective that access to skills for navigating and innovating with data is a promising path toward economic upward mobility. We found that GAISE II epitomizes the "play the game" perspective, arguing for the ubiquity of data, which, as "models of reality," "shape us," and are a "means of communication, community building, and discovery" (p. 12). "Power" appears three times in GAISE II, to characterize data tools and reasoning as conferring power to individuals to make decisions, to find meaning, and to compute statistics. This power is portrayed as neutral, focusing on the data processes while ignoring the imbalance of power in the contexts in which these processes exist. For example, one extended example in GAISE II pertains to potential statistical studies of music preferences. Despite this subject's racialized nature, there is only a surface-level nod to multiculturalism through the inclusion of genres commonly associated with African American people. Such a move could further alienate marginalized students and would not allow for developing data literacy about race or, as Philip et al. (2016) note, racial literacy about data. An intersectional feminist alternative, instead, acknowledges oppression as a beginning assumption and replaces cautions about bias with goals of data justice and co-liberation.

Reifying Systems of Power

The second category refers to how historical and contemporary data practices shape and reinforce existing systems of power and privilege. Here we expand on Gutiérrez's (2011) play the game/change the game distinction and add that data and data science are used to establish the metaphorical game's parameters. That is, governments, corporations, other institutions, and individuals use data and data science to sway the game and make it systemically unfair. Yet power as a concept scarcely appears in GAISE II. Rather, we note a thematically related emphasis on the importance of interrogating data as a regular part of the data reasoning process. Forms of interrogating appear 18 times across the text. The emphasis of this interrogation of data is on finding potential biases emerging from the processes rather than from existing structural inequality and oppression. That is, human actors–those who use, benefit, or profit from data or those who generate data but are systematically excluded from its benefits or profits–are not mentioned. Nor does GAISE II mention how distinctions among people who benefit or do not benefit from data typically correspond to gender, race, and their intersections.

GAISE II emphasizes the importance of "knowing how and when to bring a healthy skepticism to information gleaned from data" (p. 3). That is, in this text, the science of playing the game is acknowledging and guarding oneself against the fact that the game itself could be unfair by design. It is framed as individual-level self-protection, in the form of skepticism. The GAISE II report stresses the importance of questions that interrogate data but only briefly cautions that "without this interrogation, biases and misuses might emerge" (p. 12). To instantiate this claim, the report points readers to a data-based investigation of criminal justice systems (Angwin et al., 2016), citing that it "reveals inequities" in those systems (p. 12). Even this small mention of bias, especially in relation to a highly racialized and politicized criminal justice system, could be groundbreaking for a mainstream DSE position statement. Nevertheless, we note the use of the tentative and passive tense "might emerge" frames bias as a stochastic process rather than a likely outcome of systemic oppression. This description of Angwin et al.'s report suggests a location of bias in faulty algorithms, rather than as part and parcel of the oppression that is perpetuated by broader, entrenched systems of domination. There is no discussion of using data or data science to change the game, only cautionary references to bias. Myths that people can use data science (like mathematics) to objectively and reliably approximate past, present, and future reality, free from bias or ethical obligation, allow data science to be used as a tool to consolidate and maintain systems of power (Benjamin, 2019). Being a "healthy skeptic" (p. 67), as recommended in GAISE II, can expose unfairness in a dataset but falls short of seeking accountability or justice.

Transforming Systems of Power

The third category centers the capacity for people, institutions, or communities to leverage data practices to redress and challenge existing power relations. This category refers to the use of data to challenge and change systems of power or the transformation of data practices themselves. Whereas the process described in GAISE II emphasizes reflexive, but supplemental, questioning and acknowledgement of cautionary bias, the intersectional feminist approach described in *Data Feminism* takes as a premise historical and ongoing oppression and inequality. Concepts like ethics, bias, and fairness, D'Ignazio and Klein (2020) argue, are inadequate on their own to challenge existing hierarchies.

D'Ignazio and Klein (2020) rely on Crenshaw's (1991) concept of intersectionality and Collins' (2008) matrix of domination and use the term "power" to "describe the current configuration of structural privilege and structural oppression" (p. 24). The concept of power is explicitly woven throughout the text, appearing 192 times in 121 excerpts, in various forms. D'Ignazio and Klein structure *Data Feminism* around seven core principles/chapters, one of which is to challenge power. "Power" appears 50 times in 27 excerpts in this chapter alone.

After showing how data can be used to examine power, D'Ignazio and Klein name a series of actions to challenge power. A starting point is to use data science to expose inequalities. D'Ignazio and Klein refer to the same Angwin et al. (2016) report cited as an example in GAISE II. D'Ignazio and Klein not only describe how the report exposes inequalities in the criminal justice system but also trace how the report spurred legislation in New York City about algorithmic accountability, meaning legal responsibility for an algorithm's impact. D'Ignazio and Klein describe how Angwin et al. used data science methods to prove systemic racial bias in the algorithms that defend processes in the criminal justice system as raceless and neutral. D'Ignazio and Klein caution, however, that equal outcomes ought not to be the guiding goal. Rather, a goal of co-liberation leads to alternative kinds of investigations and projects and demands different metrics and other kinds of relationships with communities. For example, seeking counterdata with and from communities can generate missing stories and needed conclusions. Underlying the recommended actions is how DSE could be oriented with particular goals of demographic shifts in the field of data professionals to include, for example, women, BIPOC, and nonbinary people.

This third category aligns with critical conceptions of equity akin to Gutiérrez's (2011) notion of *changing the game*, which include using mathematics to critique society, examining alternative notions of knowledge, and rethinking mathematics as a field. In the context of data science and DSE, the third category involves a similar range of transformative work that include traditional uses of data to effect change and innovative uses that challenge data science as a field.

Discussion

How might an intersectional feminist view of data and power guide a reimagining of K-12 DSE so as to extend beyond dominant orientations to equity? We raise this question for reflection, discussion, and empirical investigation. An intersectional data feminist perspective suggests that DSE activities need to explicitly attend to power. DSE designs that center power would ask students to not only engage in discussions or demonstrations of discriminatory bias but also reconsider and envision alternative data metrics or measures for data collection and analysis. Second, an intersectional data feminist approach suggests that DSE should be committed to goals of co-liberation. In addition to encouraging students to use data science practices to expose unfairness in their communities and society at large, co-liberation aims to direct students toward dismantling unjust systems of power by harnessing the power of data and reworking data practices. The goal of co-liberation results in a fundamentally different path for DSE and different metrics for what individual and collective success in DSE looks like. We emphasize that co-liberation is not a goal to be pursued "out there" but rather, requires that students be able to find themselves and their communities in data. This suggests a need for educators to make space for this crucial step as an intentional and early part of any instructional design. Such an approach might support better statistical understandings and foster deeper understandings of the relationship between data and power relative to questions or problems that matter to students, which is a necessary precursor to transformative, co-liberatory change.

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ENTANGLING AND DISENTANGLING INQUIRY AND EQUITY: VOICES OF MATHEMATICS EDUCATION AND MATHEMATICS PROFESSORS

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Keywords: Equity, inclusion, and Diversity; Instructional Activities and Practices; Social Justice; Teacher Educators

Inquiry – asking and investigating answers to meaningful questions (Brown & Walter, 2005) – is promoted for multiple purposes across mathematics education, including developing meaningful understandings of mathematics (Goldin, 1990), fostering productive dispositions among learners such as self-efficacy in mathematics (Cerezo, 2004), or promoting powerful identities (Melville, Bartley, & Fazio, 2013). Teaching approaches aligned with inquiry include guided reinvention (Freudenthal, 1973; Gravemeijer, Cobb, Bowers, & Whitenack, 2000), discovery learning experiences (Goldin, 1990), or problem-based learning (Roh, 2003). Tang and colleagues (2017) asserted that common themes across inquiry-based mathematics courses, such as student ownership of developing mathematics knowledge or collaborating with peers, can align with four dimensions of equity (access, achievement, identity, and power) (Gutiérrez, 2002). However, the enactment of inquiry-oriented teaching alone does not ensure equitable outcomes or equitable experiences for students (Johnson et al., 2020; Lubienski, 2002).

In this poster, we extend Tang and colleagues' (2017) reflections on alignment between inquiry and equity in pursuit of the following research question: *How and in what ways can inquiry and equity be viewed as intersecting?* Data for this study consists of interviews with 24 professors who identify as mathematics education professors and/or mathematics professors. These professors participated in a week-long summer institute, during which they pursued an inquiry project and reflected equity in the experience of inquiry. During the institute, there appeared to be a shared perspective that inquiry and equity could not be separated. We examined this perspective through two interviews with each participant, the second interview being a member check, and by using a co-writing methodology (Manning, 2018).

Results illustrated three possible categories of intersections between inquiry and equity: (a) equity possibilities and dilemmas are always present during inquiry; (b) equitable inquiry is a vision for how collaborative inquiry can operate; and (c) inquiry can be a vehicle that moves us toward greater equity. For example, equity is always present in inquiry because ways of knowing mathematics are culturally and historically situated (de Freitas & Sinclair, 2020), and inquiry becomes more inequitable if ways of knowing mathematics aligned with the dominant culture are privileged. Power and status dynamics are always at play during collaborative inquiry (Cohen, 1994); equitable inquiry includes recognizing and valuing strengths of fellow inquirers. Inquiry questions that address socio-political issues can provide insights in moving toward a more equitable world (Felton-Koestler, 2020), but inquiry is not enough to achieve equity. Understanding these multiple perspectives can allow colleagues to communicate about their efforts to increase opportunities to engage students in inquiry and to promote equity by anticipating various viewpoints as we dialogue with one another, because different perspectives on equity and inquiry may be held to achieve different goals (Gutstein et al., 2005).

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RACE IN MATHEMATICS EDUCATION: WHAT TOPICS APPEAR AND HOW THEY CHANGE OVER TIME?

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Keywords: Equity, Inclusion, and Diversity, Social Justice, Research Methods

A growing body of studies has documented racial issues in the field of mathematics education (Wagner et al., 2020). Even though many scholars conducted literature reviews in this field (e.g., Gholson & Wikes, 2017; Larnell et al., 2016; Martin, 2009; Nasir & Hand, 2006), those studies only focused on a particular field. Thus, we have little information regarding what topics appear and how they change over time. In this study, we examined all articles relevant to race and mathematics all time from a database using topic modeling methods (Blei et al, 2003). We focused on the following questions: 1) What are the research topics of race-math related studies that appear in the field of educational research? How do the research topics change over time? 2) How do topics classify into equity framework? How do research trends of race in mathematics education change over time?

We used the ERIC database to collect relevant research studies. After the screening, a total of 1,600 abstracts were used for topic modeling analysis. Based on two indices (complexity and harmonic mean), we agreed 20 was the most reasonable number of topics for this data set. Then, examining top 10 keywords and relevant articles, we labeled the 20 topics. In addition, we classified each topic into the four dimensions based on Gutiérrez's (2012) equity framework which consists of four dimensions: access, achievement, identity, and power. Last, we examined how each topic and dimension evolve over time.

Of the 20 topics, early 2000s seems a critical turning point for racial issues related to mathematics. Especially between 2010 and 2020, some studies related to 'achievement gap', 'academic success in college', 'culturally responsive teaching', 'Achievement of immigrant students', and 'achievement related to SES' topics showed a high increase compared to other topics. We classified 20 topics into four dimensions of equity framework and calculated the weight of each dimension. The achievement dimension took the highest weight followed by access, identity, and power. Only about 25% of the articles were related to identity or power. The studies related to identity and power were rarely conducted before 2005, whereas the studies related to achievement and access were continuously conducted from the 1980s. We found that a majority of the studies found in the search were mainly related to achievement and access. While we found a gradual increase in the studies related to power and identity, these studies took a relatively small portion compared to the other two dimensions, especially achievement. To answer the patterns that we have identified in this present study, an additional study is required to better understand the research trends within each of the four dimensions. The findings of the current study will inform researchers how race (e.g., racial, cultural aspects) and mathematics have been used in and beyond the field of mathematics education.

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PRODUCTIVE STRUGGLE EVEN IN MATHEMATICS INTERVENTION? YES!

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Keywords: Affect, Emotion, Beliefs, and Attitudes; Middle School Education; Special Education; Problem-Based Learning

Topic Overview

Inspirations & Ideas (*I&I*; Lindemer et al., 2015) is a problem-based intervention course intended to be a companion course for students enrolled in 8th grade mathematics. *I&I* is unique because it provides opportunities for students who have struggled in mathematics to engage in conceptual understanding, collaboration, critical thinking, productive struggle, and mathematical visualization. The course goals of *I&I* include *(1) creating a community of mathematicians, (2) bolstering students' problem-solving skills, and (3) getting students to fall in love with mathematics.*

Conceptual Perspective

Algebra readiness is a marker of success in the mathematics community, and high school readiness is of equal importance. This research is guided by the perspective that students learn more with contextualized instruction as opposed to a traditional skill-based instructional setting (Boaler, 2002) and that those effective teaching practices can also be successful in intervention settings.

Research Design & Questions

In order to examine the effectiveness of curricular features in reaction to the course goals, this poster draws on data from three distinct studies of *I&I* conducted in the academic years between 2018 and 2020. Each study involved 6-9 teacher participants and their *I&I* students. All three studies included electronic pre- and post- teacher surveys, two studies also included student surveys, and one study included teacher interviews and classroom observations using Swivl technology. Two studies during *I&I*'s pilot year focused on curriculum strengths and growth areas as well as students' interactions and experiences throughout the course. The one study the following year centered on teachers' enactment of the curriculum and how those supported or hindered students' mathematical sense making, learning, and dispositions.

Data Collection Techniques & Analysis

The data from the three studies were aggregated into smaller themes following the recommendations of Creswell (2013). Matched pair responses from pre and post surveys were used to note changes over time, and teacher interviews were utilized to better understand the uniqueness of each class in the study. Classroom observations were transcribed and coded for emerging themes and were used for validation of the emerging themes in the surveys and interviews.

Findings & Implications

Researchers were specifically looking to see if/how the three course goals were being met and how teachers and students experienced and reacted to the curriculum. Data suggest that following one year with the program, students' habits of mind, such as motivation, confidence, and problem solving skills were fostered, as evidenced by pre- and post- survey growth and teacher interview reports. Students' academic achievement also improved, as evidenced by various progress monitoring tools. Teachers reported, "*I&I* took seventh grade followers and turned them into eighth grade leaders." And classroom observations showed evidence of a positive learning communities. These findings were validated with classroom observations that showcased positive classroom cultures. The implications of these findings are vast. This curriculum could be the key to providing equitable instruction for at risk students who need it the most, and could be a tool used to get students both academically and dispositionally ready to be successful in high school mathematics.

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DECENTERING WHITENESS IN A SOCIAL JUSTICE MATHEMATICS COURSE

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Keywords: Curriculum, Professional Development, Social Justice, Teacher Educators

While research has moved to investigate whiteness in mathematics education (e.g., Battey, 2013; Harper et al., 2020; Martin, 2009; Stinson, 2011), efforts to decenter whiteness in mathematical spaces, such as equity-oriented pedagogies, can still perpetuate whiteness (Berry et al., 2014; Harper, 2019). Given the need for ongoing efforts to decenter whiteness and produce culturally relevant pedagogies, I detail an action research study (Kemmis et al., 2014) on how a critical investigation of whiteness (Frankenberg, 1993) in a social justice mathematics course I designed (i.e., *Knowing the World Through Mathematics* [*KWM*]*;* Lolkus & Newton, 2020) informs (a) my work as a mathematics teacher educator (MTE), and (b) revisions of *KWM*.

Framing and Methods

Frankenberg (1993) conceptualized whiteness as "the unwillingness to name the contours of racism, the avoidance of identifying with racial experience or group, the minimization of racist legacy, and other similar evasions" (p. 23). While mathematics is often conceptualized as a colorblind discipline (e.g., Stinson, 2011), mathematical spaces often ignore or devalue the values, cultures, and experiences of people of color, and Black and Indigenous communities (Gutiérrez, 2017a, 2017b). To support the examination of racist structures in mathematics education, Battey & Leyva (2016) outlined three dimensions of Whiteness: *institutional, labor,* and *identity.* In this study, I specifically explored the *institutional* dimension, which includes, for instance, how mathematics is often taught as neutral or cultureless, whose voices and histories were privileged, and what forms of mathematical representation were given space in *KWM*. Drawing from action research (Kemmis et al., 2014) and the qualities of critical research in mathematics education (i.e., *current situation, imagined situation, arranged situation*; Skovsmose & Borba, 2004), I engaged in a critical reflection of how I perpetuated whiteness in *KWM*. This action research study foregrounds my efforts, as a white, cisgender, male MTE, to disrupt and challenge the oppressive nature of mathematical spaces that reinforce whiteness.

I used thematic analysis (Braun & Clarke, 2012) to explore all evidence sources. My analysis was guided by a critical perspective (Skovsmose & Borba, 2004), and Battey and Leyva's (2016) *institutional* dimension of whiteness in mathematics education. Following Nowell et al.'s (2017) recommendations for trustworthiness in thematic analysis, I relied on critical friends for triangulation of findings (Flick, 2018), and maintained documentation of the coding and debriefing processes throughout each phase of thematic analysis in reflexive journals.

Summary of Findings and Implications

This action research study informs efforts to decenter whiteness in mathematics education through a critical reflection of implicit connections to whiteness in a social justice mathematics course. I provide preliminary examples of my own complicity in perpetuating whiteness, and how I am taking action to revise the curriculum by centering ethnomathematics (D'Ambrosio & Rosa, 2017) and voices of activists and policy makers central to our mathematical investigations.

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DEVELOPING PRESERVICE MATHEMATICS TEACHERS' POLITICAL CONOCIMIENTO THROUGH DATA SCIENCE

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Keywords: Mathematical Knowledge for Teaching, Social Justice, Preservice Teacher Education, Data Analysis and Statistics

This poster brings together a discussion about data science education, teaching for social justice (Berry et al., 2020; Gutstein, 2006; Lesser, 2007), and preservice mathematics teachers' (PMTs) political conocimiento (Gutiérrez, 2012, 2017). It is crucial that teacher preparation programs provide experiences for pre-service math teachers (PMTs) to engage with statistics as a pathway to data sc2ience (Gould et al., 2017), including developing statistical literacies. Given the impact of data-informed decisions and centrality of contexts in statistics (Cobb & Moore, 1997), statistics provides an avenue to develop critical literacies (Weiland, 2017). To realize the full potential of statistics and data science, PMTs should be provided opportunities to develop understandings of how statistics classrooms can serve as spaces to learn about data science as well as how data can be used to identify, critique, and challenge social injustices.

Teaching Statistics for Social Justice (TSSJ) is a related body of research (Berry et al., 2020; Gutstein 2006; Lesser, 2007). TSSJ invites teachers to design justice-oriented instruction that interweaves content goals that focus on statistical literacies with social justice goals that focus on critical literacies. Combined, the content and social justice goals may help develop teachers' political knowledge. Particularly, Gutiérrez's (2012, 2017) political conocimiento for teaching mathematics framework builds on traditional teacher knowledge models that include content and pedagogical content knowledge (Ball, 2008; Hill et al., 2008; Shulman, 1986). Political conocimiento adds that teachers also need political knowledge about navigating the sociopolitical landscapes of teaching, such as using data can be used to advance social justice. This research is guided by the following research question: What design features support the development of PMTs political conocimiento for teaching mathematics?

This research discusses the findings of a teaching experiment (TE; Prediger et al., 2015) that uses a social justice approach to teach data science for PMTs. The TE takes place at a four-year public Hispanic Institution in the US-Mexico borderlands of Southern California. The TE includes 12 one-hour lessons on study design and regression. All lessons discuss issues of race and racism in educational contexts. The TE includes 16 participants, 14 of which are PMTs.

This work is in progress, but preliminary findings may be used to guide considerations for designing data science, social justice, and teacher preparation learning environments. For example, to avoid gap-gazing (Gutiérrez, 2008), pedagogies of despair (Giroux, 2001), or reifying deficit narratives, PMTs should consider how data and analyses reflect individual or systemic structures (e.g., how the "achievement gap" may inherently place blame on individuals and does not account for systemic racism). Further, there may be challenges with scaffolding when presenting a social justice problem context. For instance, providing too much information without allowing students to create their own knowledge and judgements may disempower students (Brantlinger, 2013), leading to in the banking method (Freire, 2018) that positions students as passive learning and). Thus, it is important for instructors to allow for multiple entry points for PMTs to develop their own critical consciousness (e.g., learning from community members, being aware of their own positionality, reading related news articles and literature).

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DISABILITY IN MATHEMATICS EDUCATION RESEARCH: A CALL TO ACTION

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Keywords: Students with Disabilities; Social Justice; Research Methods; Doctoral Education

Mathematics education research perpetuates structural and epistemological obstacles to the inclusion of disabled people, as disabled bodyminds are not considered knowledge creators and are excluded from our rank and file through the structures of our doctorate degrees. I use the term disabled to mean a complexly embodied sociopolitical location that is dynamically relational (Kafer, 2013; Siebers, 2011), and I use identity-first language as a reclamation of power that socioculturally situates disability as an identity marker (Andrews et al., 2019). I use the term bodymind as a disruption of the assumption of separation of body and mind and as a call to the embodied impact of oppression (Schalk, 2018). I here call our field to action: expand disability epistemologies in our research and disrupt ableist structures in our doctorate programs.

Through Critical Disability Studies, DisCrit, and Disability Studies in Education, mathematics education researchers have begun to expand epistemologies. Our ontology of disability has been challenged (e.g. Lewis, 2014), and our epistemologies within the experiences of students (e.g., Lambert, 2019), pre-service teachers (e.g., Tan & Padilla, 2019), in-service teachers (e.g., Tan & Thorius, 2019), and instructional pedagogies (e.g., Lambert, 2015). Epistemological assumptions of whose knowledge is privileged have been challenged by Lewis and Lynn (2018). As we continue these epistemological and ontological pushes, I call on us to learn *from* Disability Justice Communities, which remind us that interdependence is pivotal to justice (Sins Invalid, 2019). Through interdependence, we can challenge assumptions that prioritize researchers' knowledge and collaborate with disabled communities.

Additionally, access to the power of "researcher" is kept behind the gate of a doctorate degree, therefore the available researcher standpoints are limited to those who have been able to complete such programs within institutions that continue to instantiate ableist academic structures (Dolmage, 2017). I call on us to push against this gatekeeping. We must interrogate how pedagogies and languages in our courses marginalize disabled students. If we seek inclusive pedagogies and instructional design, then we can move away from compliance as sufficient. Also, we must examine and interrogate program trajectories and requirements to understand their role in exclusion.

The mathematics education research community must privilege the knowledge of disabled people by expanding our epistemologies and our doctorate programs. Three ways we can widen our epistemology are: (a) collaborate with disabled communities; (b) increase liberatory and emancipatory research; (c) learn from disability justice communities. Three ways we can address the gatekeeper aspects of doctorate programs are: (a) interrogate the ways disabled students are positioned in courses, (b) resist the idea of accommodation as a singular *post-hoc* fix, and use inclusive course design and instructional pedagogies, and (c) identify and "try out" ways to increase, without capping, the flexibility of program trajectory. Who conducts our research and how they do so does not exist separate from the inequities of our world. The pandemic has reminded many that disabled bodyminds are lower priority for both disease management and educational systems: we can use this reminder to motivate us to increase equity for disabled people in mathematics education.

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Chapter 4:

Math Content — Early Years

INTEGRATION OF MATHEMATICS HISTORY INTO MODEL-ELICITING ACTIVITIES FOR MAKING SENSE OF NEGATIVE INTEGERS

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This study aims to explore seventh-grade students' understanding of negative integers as they engaged in mathematics history integrated model-eliciting activities in small groups. For this educational case study, we designed model-eliciting activities based on six design principles of the models-and-modeling perspective that incorporated history of negative integers. Both written data and video records of students were analyzed to elicit the facets of their models of negative integers. We found that students' thought that either daily life contexts or people's need drove the invention of negative integers. The findings also indicated students' reasoning on the evolvement of mathematics ideas by contribution of different culture, revealing the role of math history integration into the modeling process. In this sense, our study presents a unique approach in modeling literature.

Keywords: History of mathematics; models-and-modeling perspective; model-eliciting activities; negative integers

Negative integers have always been an interesting topic in mathematics education research, and related studies indicated that although students could perform the operations with integers, they struggled in making sense of negative integers (Lyte, 1994; Steiner, 2009). One of the major reasons for this struggle was the difficulty of connecting negative integers with real-life situations (Gallardo, 2002). Therefore, we approached to this phenomenon, making sense of negative integers, from the Models-and-Modeling Perspective that was centered around meaningful situations in developing a mathematical model (Doerr & Lesh, 2003). We, on the one hand, aimed to elicit students' understanding of negative integers through model-eliciting activities; and, on the other hand, incorporated mathematics history into model-eliciting activities. Hence, our study presents a unique approach in modeling literature by addressing the following research question: *What understandings do 7th grade students develop negative integers as they engage in mathematics history integrated model-eliciting activities in small groups?*

In the sections below, we briefly presented our theoretical framework involving history of mathematics and models-and-modeling perspective and presented our findings regarding 7th grade students' understandings of negative integers.

Theoretical Framework

History of Mathematics

The integration of history of mathematics into mathematics education has been on the agenda of many mathematics education researchers (Fenaroli, Furinghetti, & Somaglia, 2014). While some investigated the ways of including historical origins of mathematical concepts in teaching (Tzanakis & Arcavi, 2000), some explored the role of math history in teacher education (Clark, 2012; Fenaroli, Furinghetti & Somaglia 2014).

The use of history of mathematics in mathematics education was analyzed by Jankvist (2009) in terms of reasons (the whys) and integration ways (the hows) of history of mathematics in mathematics teaching and learning. The two main reasons of integration of the history of mathematics are (i) to assist mathematics instruction (i.e., use of history as a tool) and (ii) to learn the history of subject (i.e., use of history as a goal) (Jankvist, 2009). The first reason focuses on improving students' understanding in terms of cognitive and affective aspects of mathematics learning with the help of history of mathematics. The second addresses that history of mathematics encourages students to considering about the evolution of mathematics and role of humanity on the development of mathematics (Jankvist, 2009).

National Council of Teachers of Mathematics (NCTM) pointed out that mathematics is affected by different cultures and inherited to humanity, and students should be allowed to notice and perceive worldwide human effect on the field of mathematics (NCTM, 2000). With this in mind, Jankvist (2009) stated three basic approaches to include history of mathematics in mathematics education: (i) the modules refer to the integration of history of mathematics into a range of mathematics lessons related to topic, (ii) the history-based approach in which mathematics lessons are fully arranged taking the history and evolution of mathematics into account, (iii) the illumination refers to include some historical facts and information in mathematics lessons.

Several researchers mentioned about the benefits of integrating math history in mathematics education (e.g., Fried, 2001; Liu, 2003; Tzanakis & Arcavi, 2000). These benefits can be listed as follows:

- It encourages students to value mathematics as cultural and human product.
- It makes mathematics more interesting, understandable, and attainable for students by helping to perceive mathematical concepts, problems and their solutions.
- It facilitates learning activities by enhancing mathematical thinking ability.
- It affects students' affective dispositions towards mathematics.
- It guides teachers for the learning and teaching activities while asserting that the difficulties mathematicians encountered in the past helps teachers to identify and prevent the problems of students of today.

Regarding the last point, Jankvist (2009) claimed that historical development of a subject provides a parallel path to learn this subject within context revealing relationships between ideas, definitions, and applications: "To really learn and master mathematics, one's mind must go through the same stages that mathematics has gone through during its evolution" (p.239). Similarly, Savizi (2007, p.46) stated: "For students, issues of past real world are more tangible and understandable than today's problems or solving problems from real life by using human approaches may work better than application of complicated methods or offering high amount of information." That also improves students' self-confidence and encourages them to believe in their own abilities as human beings (Savizi, 2007). Moreover, recent studies on this field have indicated that students experiencing mathematical concepts within a meaningful historical context developed more positive attitudes towards concepts (Lim & Chapman, 2010).

When the mathematical concepts are presented as disconnected from real-life, students demonstrate difficulties in understanding the mathematical concepts, and in this vein the integration of history of mathematics enables students to understand the need for the concept (Gulikers & Blom, 2001). Since one of the distinctive features of the modeling perspective was the reality or meaningfulness of the context (Lesh, Hoover, Hole, Kelly, & Post, 2000), we considered the models-and-modeling perspective as a complementary strand of our theoretical frame.

Models-and-Modeling Perspective

The activities with meaningful contexts make students more willing to learn about the subject while they understand the importance of mathematics and real-life relevance of the concept (Lim & Chapman, 2010; NCTM, 2000). The models-and-modeling perspective proposed a problem-solving approach that involves problem-solvers' making sense of the real-life context mathematically, mathematizing the context, and developing a mathematical model that was expressed, tested, and revised iteratively until it provides a sufficient solution for the real-life problem (Lesh & Zawojewski, 2007). The term "mathematical model" refers to the conceptual systems that are built, defined, emphasized mathematically significant products, processes and mathematical reasoning (Doerr & Lesh, 2003). In modeling classrooms, teachers focus on students' understanding and processes of constructing, expressing, reasoning abilities while solving mathematically word problems rather than solely arithmetic computations (Lehrer & Schauble, 2000). However, eliciting students' models was not an easy task, and therefore Lesh and his colleagues proposed a genre of modeling activities called Model-Eliciting Activities (MEAs) (Lesh et al., 2000).

The MEAs involves real life situations in which students make meaningful mathematical explanations (Doerr & Lesh, 2003). To foster students' development of mathematically significant models, Lesh and colleagues (2000) identified six design principles of MEAs: (1) model-construction principle, (2) model-documentation principle, (3) reality (meaningfulness) principle, (4) self-assessment principle, (5) model shareability and reusability principle, and (6) effective prototype principle. Therefore, via MEAs, students produce mathematically significant, shareable and reusable model related to real-life situations. Moreover, these thought-revealing activities allow students assess their thinking and encourage working in groups to produce better models.

With these in mind, we, in this study, integrated history of negative integers into the MEA approach and conjectured that integration of math history would not only take students' interest but also provide them a deeper understanding of negative integers.

Students' Understanding of Integers

There have been many studies investigating how to advance students' understanding of integers by neutralization and number line models (Lyte, 1994). Whilst the neutralization model includes physical objects such as two-colored counters to represent negative and positive integers and operations with integers, the line model focuses on operation with integers considering the position and distance of integers by the direction of movement on the number line (Lyte, 1994).

The concept of negative integers and making sense of the use of negative integers in real-life was difficult for students because it was not as easy to grasp negative integers contextually as natural numbers (Whitacre et al., 2017). There have been several studies arguing that students had difficulty in understanding negative integers as they tried to accommodate their prior knowledge about natural numbers (Gallardo, 2002; Whitacre et al., 2017). This transition between natural numbers and integers led to difficulties in terms of number sense and making sense of the negative integers. In addition, the sense of negative integers and the idea of a number less than zero seemed nonsense for most of the everyday contexts from the viewpoints of students (Whitacre et al., 2017).

Although students meet negative integers in their everyday life, after they encounter and focus on operational procedures in school, they do not make connection between outside-the-school learnings and school instruction (Steiner, 2009). The related studies showed that the real-life contexts, word problems and models including incomes and expenses, assets and debts, elevators, weather temperatures support students' understanding and reasoning about integers (Pettis & Glancy, 2015; Stephan & Akyuz, 2012). However, students might still struggle in comprehending situations involving opposites such as incomes and expenses, weather temperatures and elevators (Pettis & Glancy, 2015). Thus, it is important to encourage students to think within the context to improve their understanding of integers, for which we designed mathematics history integrated MEAs in this study.

Mode of Inquiry

To explore students' understanding of negative integers, we designed mathematics history integrated MEAs and carried out a qualitative educational case study with eight groups of 7th grade students (29 students in total). We explained characteristics of the case participants and the nature of data collection and analysis below.

Participants

The participants of this study were 7th grade students (15 male and 14 female) of a public middle school class in Istanbul, one of the metropolitan cities in Turkey. The students engaged in mathematics history integrated MEAs in small groups and randomly assigned to groups by the second author who was also the mathematics teacher of the classroom. Ten groups were formed in the classroom but only eight of them whose parents provided the consent for their participation in the study were included in the data set. A general view of the mathematics teacher for the participating students was that most of the students' prior knowledge and mathematics backgrounds were similar to each other because they attended the same classes during the primary and middle schools, and their mathematics achievement was average.

Data Collection Procedure

Students' group work was video recorded during the implementation of mathematics history integrated MEAs. There were eight groups containing 3-4 students per group; 29 students in total. The data set involves their written work in activity sheets and video records of their work during the implementation of the activities.

The mathematics history integrated MEAs were implemented with the aim of guiding students to achieve related objectives of middle school mathematics teaching program. The activities covered three dimensions of students' understanding on integers: (i) why negative integers were needed in mathematics, (ii) how to identify positive and negative integers, and (iii) how to use negative integers in real life contexts. In this proposal, we delimited our focus only on one MEA called "The Problem of Diophantus" that addressed the first dimension. The MEAs were designed considering the six design principles and implemented in a one lesson hour. Before the implementation of activities, any prior teaching about integers was not provided to the students. During the implementation of the activities, students were expected to reflect their understanding and making sense of negative integers. The essential principles followed during the implementation sessions were: (i) students should study as small groups and interact with each other, (ii) after they finished their studies, students should be encouraged to share their works and opinions with the guidance of teacher during whole class discussion, (iii) teacher should guide students when they needed without providing any right answer for the questions of the activities, and (iv) students should be allowed to reveal and reflect their own experiences by making connection with everyday life contexts.

The Problem of Diophantus. In this MEA, we aimed to take students' attention to the origins of negative integers. First part of the activity emphasized how people use the negative integers in daily life and why people needed negative integers in the history. The researcher intended to help students to consider and question necessity and need for negative integers not only for mathematical operations but also in everyday problems. The second part of the activity contains information about Diophantus, a mathematician, and is followed by a problem (i.e., 4×?+20=4) which is called as "absurd" by the Diophantus because of its' negative solution (Hettle, 2015). Students were expected to write a letter explaining their rationale for why mathematicians needed negative integers, their solution to the Diophantus's problem, and their reasoning for why Diophantus might have called the solution as absurd. Although not readable, the screenshot of the MEA (in original language) was given below to help readers make sense of the material that students received as a math history integrated MEA.


Tatilde okuduğum bir matematik dergisinde yazanları size anlatacağım. M.Ö. 2. yüzyıl ve 4. yüzyıl arasında Diophantus adında İskenderiyeli Yunan bir matematikçi yaşamış ve bizim g6. Sınıfta öğrendiğimiz cebirsel ifadeler konusunda çalışmış ve 130 problemden oluşan Arithmetica adlı bir kitap yazmış. Diophantus "cebirin yaratıcısı" olarak biliniyormuş. Aslında cebir çok daha önceki zamanlarda bile kullanılıyormuş. Merak edip araştırdım Rhind Papirüsü de M.Ö. 1650 yılında Mısırlı Ahmes tarafından yazılmış ve çarpma, bölme, cebirsel ifadeler gibi konuları içeriyormuş. Birçok ileri matematik konusunun gelişimine katkıda bulunan eski uygarlıkların ve matematikçilerin negatif tam sayıları anlamaları aslında çokta kolay olmamış ve çeşitli zorluklar yaşamışlar. Tıpkı bizim gibi!

Arkadaşlar dergide Diophantus'un bir problemin çözümünde karşısına çıkan

$$4 \times ? + 20 = 4$$

ifadesinin çözümü için "absürt (saçma)" ifadesini kullandığını okudum.

1. Acaba Diophantus neden bu problemin çözümü için saçma demiş olabilir?
2. Arkadaşlar soru işareti yerine gelecek olan tamsayıyı bulmama yardım eder misiniz?
3. Diophantus'un kendinden önceki cebir çalışmalarını bir araya toplayarak geliştirmesi bize matematik bilimi hakkında ne fikir verebilir?

Cevaplarınızı ve açıklamalarınızı bana mektup olarak gönderirseniz okul dergisindeki yazımda paylaşacağım. Mektuplarınızı dört gözle bekliyorum.

Sevgili Ali,

Figure 1: The Problem of Diophantus MEA-Part 2

Data Analysis Procedure

Students' performances on MEAs were recorded in written form on activity sheets and as video and audio records. These written data were coded through two cycles: (1) initial coding and (2) descriptive coding (Saldana, 2009). In the first cycle, the written data of each group first examined holistically, and then open codes were identified to make sense of students' conceptions. In the second cycle, these open codes were revised to create categories that were more descriptive of students' conceptions. Afterwards, the resulted codes were checked with the video and audio data. Specifically, audio and video records were not coded separately but used to make sure about students' conceptions written in the activity sheets.

The codes were then checked by another researcher, the first author, for the interrater reliability (Lincoln & Guba, 1985). Multiple sources of data helped to triangulate the findings. In addition, the second author, implementer of the MEAs, kept a research journal during both data collection and data analysis. Writing each step of the study transparently contributed to the credibility of the interpretations (Lincoln & Guba, 1985).

Findings

In this section, we present the findings including seventh-grade students' models of the negative integers that they developed during their small group engagement in math-history integrated MEAs. Lesh and Harel (2003, p. 150) defined the models as "conceptual systems that generally tend to be expressed using a variety of interacting representational media, which may involve written symbols, spoken language, computer-based graphics, paper-based diagrams or graphs, or experience-based metaphors." Hence, the students' models presented in this section were in the form of verbal descriptions and mathematical symbols and more importantly indicated their conceptual structure of negative integers.

Making Sense of Appropriate Contexts for Negative Integers

Seventh grade students' understanding of negative integers were associated with four contextual situations: (i) representation of weather temperature, (ii) representation of the debt and loss, (iii) representation of elevation, and (iv) an indication of floor numbers in elevators. Although the last two situations were related, students differentiated them. Group #1 and #5 identified a reference point and indicated that the interval below the reference point would be considered as negative. For Group #1 the sea level was a reference point (zero), and below sea level is represented with the negative integers. Similarly, Group #5 accepted the ground floor as a reference point (zero), and they represented the flats under the ground floor with negative integers. On the other hand, the fourth context, indication of floor numbers in elevators, referred to a static position. For instance, Group #2 and #4 stated that the buttons in the elevator included negative integers as symbolic representations of the levels of the floors.

Furthermore, students reasoned about the origins of negative integers and why people needed them. In this regard, Group #2, #4, and #5 stated contextual reasons and Group #1, #6, and #8 indicated that people such as mathematicians, scientists, folks needed and invented negative integers to illustrate the values less than zero. Groups considering contexts such as very cold weather, and debt and loss situations stated that people needed negative integers for their daily life requirements such as trading. For instance, one of the students from Group 5 stated that "one day, when the weather was too cold and snowy, people used negative integers to express the very cold weather." Students mentioning the scientists or mathematicians, on the other hand, stated that people needed to represent numbers less than zero: "A scientist might have invented negative integers to help his calculations with a scientific experiment" (A student from Group #1).

Making Sense of the "Absurd" Problem of Diophantus

In the second part of the MEA, groups were expected to write a letter to help a peer student's school magazine involving their thinking about the problem of Diophantus and possible reasons of why this problem might have called "absurd" by Diophantus. All groups of seventh-grade students thought that it was due to the lack of knowledge of negative integers by then. Regarding this, they had two slightly different aspects:

- A problem is called absurd when one does not have the knowledge of negative integers.
- A problem is called absurd when one cannot find a solution for a mathematics problem.

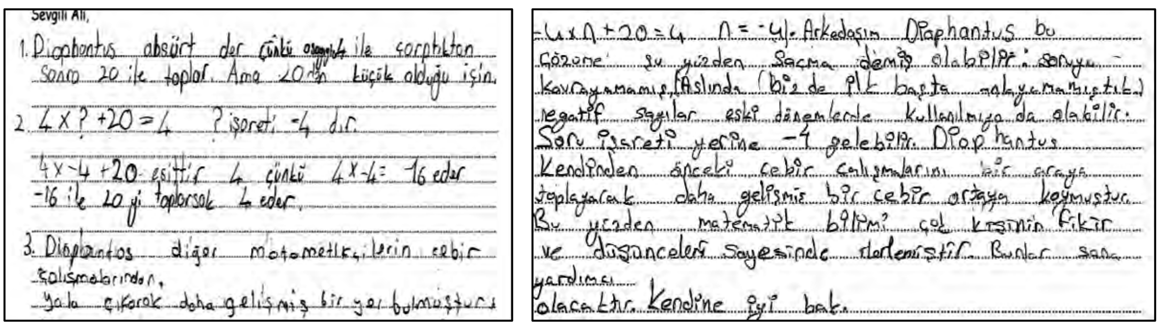
When students engaged in finding the value of the unknown shown with a question mark on the given problem (i.e., 4x?+20=4), the students used trial and error method in two different ways:

- Some groups stated that the unknown cannot be a positive integer, so they accepted the unknown as a negative integer. They tried negative integers respectively to find the value of unknown.
- Some groups tried zero and positive integers at first, but they didn't find a correct solution. Therefore, they tried negative integers to find the correct value for the unknown.

Moreover, groups discussed their interpretations about how mathematics evolved, considering the given information that Diophantus gathered the algebra studies before his era and developed his studies based on the prior work of other mathematicians. Students' discussion revealed three aspects:

- Mathematics is a continuously evolving field.
- A single math idea was developed by contribution of many peoples thinking and studies.
- Various mathematicians and cultures contributed to the field with their work evolved one after another.

These aspects along with the two ways of using trial and error method that were associated with how students made sense with the absurdity of the answer indicated the facets of students' models of negative integers. These facets were illustrated in the letters of the two groups given in Figure 2a and 2b below.


Sevgili Ali,
1. Diophantus absürt der çünkü 4 ile çarptıktan sonra 20 ile toplar. Ama 20'den küçük olduğu için.
2. 4 x ? +20 = 4 ? işareti -4 dir.
4 x -4 +20 eşittir 4 çünkü 4 x -4 = -16 eder
-16 ile 20 yi toplarsak 4 eder.
3. Diophantos diğer matematikçilerin cebir çalışmalarından,
yola çıkarak daha gelişmiş bir yol bulmuştur.

-4 x n +20 = 4 n = -4 Arkadaşım Diophantus bu
çözüme şu yüzden saçma demiş olabilir: soruyu
kavrayamamış. (Aslında (biz de ilk başta anlayamamıştık.)
negatif sayılar eski dönemlerde kullanılmıyor da olabilir.
Soru işareti yerine -4 gelebilir. Diophantus
kendinden önceki cebir çalışmalarını bir araya
toplayarak daha gelişmiş bir cebir ortaya koymuştur.
Bu yüzden matematik bilimi çok kişinin fikir
ve düşünceleri sayesinde ilerlemiştir. Bunlar sana
yardımcı
olacaktır. Kendine iyi bak.

Figure 2a and b: The letter of Group #2 (on the left) and Group #7 (on the right)

As the letter of Group #2 stated, the problem was called absurd since the unknown number multiplied by 4 and added to 20 and somehow the result would be less than 20. Similarly Group #7 expressed that negative integers were not known in the past, and so the question did not make sense; that's why the given problem was called absurd. Students in Group #7 also stated that mathematics developed with the help of many people's opinions and Diophantus collected and improved the algebra studies based on the prior work, which was also observed in other groups' letters. Hence, with help of the math history integrated MEA, students could develop a rationale about the historical development of mathematics and improved their understanding of why people needed negative integers in daily life.

Conclusion and Discussion

We observed that seventh-grade students who encountered the integers formally for the first time with math history integrated MEAs found the topic interesting. The MEAs not only took their attention but also motivated them to understand why people needed integers in the history and what kind of mathematics equation would lead to a negative integer answer. Developing models of negative integers in small groups could reveal students' understanding. More specifically, although they used symbolic representation of a negative integer in their explanations, they also supported their statements with the contextual illustrations. Real-life contexts identified by the participating seventh-grade students included incomes and expenses, assets and debts, elevators, and weather temperatures, which were also observed in the related literature about integers (e.g., Stephan & Akyuz, 2012; Pettis & Glancy, 2015).

One of the major contributions of this study was integrating mathematics history into the modeling perspective, which has not been present in the related literature yet. This integration increased the motivational and attitudinal effect of MEAs as Savizi (2007, p.46) stated: "For students, issues of past real world are more tangible and understandable than today's problems or solving problems from real life by using human approaches may work better than application of complicated methods or offering high amount of information." To illustrate, in the Problem of Diophantus MEA, students studied on a problem called "absurd" by the Diophantus because of its' negative solution and noticed that Diophantus also did not make sense with the problems, likewise the students who sometimes do not make sense with math problems.

In this study, math history integrated MEAs activities brought real-life related mathematics problems from the past and the present together. After the implementation of the MEAs, most of the students expressed their wishes to continue mathematics lessons by working on similar modeling activities, which confirmed other researchers' claim that including history of mathematics could help students overcome their math anxiety (Liu, 2003; Tzanakis & Arcavi, 2000). Similar arguments regarding the affective benefits of modeling experiences were also exist in the models-and-modeling literature (English, Lesh & Zawojewski, 2003). In this sense, mathematics history integrated MEAs were beneficial tools to create a meaningful and real-life related learning environment in which modeling is significant not only for computing, but also for constructing, describing, mathematical reasoning and understanding (Doerr & Lesh, 2003). Comprehending the situations involving opposite directions such as incomes and expenses and cold and hot weather was not easy for students (Pettis & Glancy, 2015), but possible with encouraging students to think within the context (Whitacre et al., 2017). Furthermore, focusing on only operations with integers in school hindered students' understanding of situations involving negative integers in real-life problems (Gallardo, 2002). Our study showed that with the help of mathematics history integrated MEAs, students made sense of negative integers in historical situations.

This study involved three mathematics history integrated MEAs and in this proposal we focused only on one of them. Although the implementation of the activities was arranged considering the middle school mathematics teaching program and limited with the annual plan of mathematics lessons, more meaningful data about the students' understanding of negative integers might reveal if more time was spent and more activities were implemented. Another suggestion could be expanding the use of math history integration into different mathematics topics. In other words, we recommend a future research considering different mathematics topics in different grade levels for the integration of the mathematics history into the modeling perspective. Although the present study investigated the role of mathematics history integration into MEAs on students' understanding, these activities can also be used to improve mathematics teachers' education for their teaching repertoire. Thus, this study also suggests a professional development aiming to train teachers how mathematics history and modeling perspective can be used to enhance students' mathematical understanding.

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WHEN IS A GUESS MORE THAN JUST A GUESS? MIDDLE-GRADES STUDENTS' GUESS AND CHECK STRATEGIES

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The appropriateness of guess and check as a problem-solving strategy has been debated. This qualitative study examines the use of guess and check by middle-grades students to solve linear systems of equations. Students' reasoning is examined within the number sequences framework, which is based in part on students' units coordination. Only students at the fourth and fifth stages (out of five) correctly solved systems of equations algebraically; this is attributed to their operations on two- and three-level unit structures, and to a disembedding operation. Students at the third stage applied strategic guess and check methods, which is attributed to assimilating with composite units (i.e., units of units), but these students could not correctly use an algebraic method. For students at the second stage, guess and check was non-strategic, which is attributed to their construction of composite units in activity. Implications for instruction are discussed.

Keywords: Algebra and Algebraic Thinking; Middle School Education; Number Concepts and Operations

Literature Review

Guess and check is a common strategy for students to apply in problem solving situations (Johanning, 2004). *Systematic guess and check* is form of reasoning in which a student "works with the situational context and applies relational reasoning to solve the problem" (Johanning, 2010, p. 123). Thus, students operate within the problem-solving context while simultaneously reasoning about the quantitative relationships to arrive at increasingly better approximations of the solution. More general definitions of guess and check range from trial-and-error (Gallagher et al., 2000), which may or may not be systematic, to "random guess and try" (Capraro et al., 2012, p. 112).

Guess and check is particularly relevant to solving algebra problems. While Knuth and colleagues (2006) define guess and check strategies as pre-algebraic, Kieran (1996) describes global meta-level activity as an algebraic activity that aligns closely with systematic guess and check. Therefore, it is unclear to what extent guess and check, and particularly systematic guess and check, is a productive algebraic strategy.

Johanning (2010) asked middle-grades students to solve linear systems of equations word problems and found that systematic guess and check was the most common method applied. She argued that guess and check is algebraic in nature and supports students in developing more sophisticated algebraic methods. By this rationale, systematic guess and check is a worthwhile skill with the potential to improve students' reasoning about systems of equations. In contrast, Malloy and Jones (1998) found that eighth-grade students who applied guess and check to linear systems of equations problems often failed to find a solution and did not initiate the use of alternative methods when guess and check failed. As these studies demonstrate, the conclusions surrounding the productive nature of guess and check are inconsistent. Furthermore, the research does not offer a theoretical rationale for students' widespread dependence on guess and check. This study asks, in what ways do the number sequences account for students' guess and check solutions to linear systems of equations? And, are students' strategies for solving systems of equations more closely tied to their number sequence or course enrollment?

Theoretical Framework

Olive and Çaglayan (2008) framed middle-grades students' algebraic solutions to linear systems of equations within their units coordination. The number sequences are based on units coordination (Steffe, 2010; Ulrich, 2015; 2016a), but also take into account mental operations such as iterating and disembedding (Steffe, 2010). This allows the number sequences to distinguish among three groups of students with varying stages of fluency operating with composite units (Ulrich, 2016b). Students who have constructed the tacitly nested number sequence (TNS) construct composite units in activity (Steffe, 2010); students who have constructed an advanced tacitly nested number sequence (aTNS) assimilate with composite units (Ulrich, 2016b); and students who have constructed an explicitly nested number sequence (ENS) assimilate with composite units, iterate units of one, and disembed (Steffe, 2010). Zwanch (2019, in review) demonstrated that the distinction among these three stages can be used to model their representations of multiplicative algebraic relationships. As such, the number sequences will be used to analyze students' use of guess and check to solve linear systems of equations.

Tacitly Nested Number Sequence (TNS)

TNS students assimilate with one level of units and construct a second level, or *composite unit*, in mental activity (Steffe, 2010; Ulrich, 2015). The operations of a TNS support double counting because TNS students can monitor the number of times that they count on. Consider the problem asking, what is seven more than 24? To a TNS student, the number word "seven" stands for a counting sequence from one through seven, but in mental activity can be chunked into one composite unit containing a counting sequence of seven units. Thus, TNS students can transpose the counting sequence to monitor their counting beginning at 24 and increasing to 31.

TNS students' algebraic reasoning. TNS students do not disembed, but Hackenberg (2013) found that disembedding is critical to algebraic reasoning. *Disembedding* is a mental operation that allows students to think about removing one unit from another without destroying either unit, and to reflect on the relationship between the two units (Steffe, 2010). For instance, to abstract the relationship between quantities such as 10 and 8 or 6 and 4 as x and $x - 2$, requires the student to disembed the smaller quantity from the larger and reflect on the relationship (Hackenberg, 2013). This reflection supports the algebraic representation of the two related quantities. As TNS students do not disembed, Hackenberg's (2013) findings suggest that TNS students will be limited in their symbolic representations of related unknowns.

Advanced Tacitly Nested Number Sequence (aTNS)

aTNS students assimilate with composite units and construct or coordinate a third level of units in activity, but do not disembed (Ulrich, 2016b). To *assimilate* with composite units implies that aTNS students can immediately perceive of a number word, like "seven," as one unit containing seven units of one. This allows aTNS students to reason strategically by operating on embedded composite units (Ulrich, 2016b). For example, an aTNS student may find the difference between 39 and 62 is 23 by reasoning that 40 plus 22 is 62, and 39 is one less than 40, so the difference is one more than 22. aTNS students are only tacitly aware of the nesting of the subsequences, 39 and 23, within 62. This makes explaining their thought process challenging (Ulrich, 2016b).

aTNS students' algebraic reasoning. Zwanch (2019, in review) found that aTNS students can write algebraic equations to represent additive and multiplicative relationships, but they do so inconsistently. Their algebraic reasoning is supported by assimilatory composite units, which

Hackenberg et al. (2017) find support operations on unknowns. However, aTNS students' algebraic representations are inconsistent due to not disembedding (Zwanch, in review). This research demonstrates that aTNS students can write symbolic equations representing one-step additive and multiplicative relationships because they can operate on composite units in activity, thereby forming a third level of units. Following mental activity, however, the third level of units decays. As aTNS students cannot disembed one quantity from the other to reflect on the relationship following this mental decay, they have no material for reflection (Zwanch, 2019, in review).

Explicitly Nested Number Sequence (ENS)

ENS students also assimilate with composite units, but in addition can disembed and iterate units of one (Steffe, 2010; Ulrich, 2016a). Iterable units of one and disembedding support multiplicative reasoning (Steffe, 2010) because ENS students can, for instance, think about removing a unit of one from a composite unit of seven and repeating the unit seven times to fill the whole – seven is seven times the size of one.

ENS students' algebraic reasoning. Olive and Çaglayan (2008) utilized the coin problem (Figure 1) to examine how units coordination was related to students' algebraic solutions to a linear system of equations problem. One participant, Ben, who assimilated with composite units, wrote the equations $(.05N) + (.1D) + (.25Q) = \$5.40$, $D = N + 3$, and $Q = N - 2$. Although Ben explained that N, D, and Q represented the numbers of nickels, dimes, and quarters, respectively, he struggled to substitute $n + 3$ and $n - 2$ in place of D and Q. When he was pressed to do so, he conflated the numbers of dimes and nickels with their values. This was a limitation of his units coordination because he could not operate on the initial equation, which represents a three-level unit structure (i.e., the value of a single coin, within the number of a type of coin, within the total value, $5.40; Olive & Çaglayan, 2008).

Generalized Number Sequence (GNS)

A GNS is the most sophisticated number sequence, and GNS students assimilate with three levels of units and can construct four or even five in activity (Steffe, 2010; Ulrich, 2016a). One mental operation of a GNS is iterable composite units. This implies that GNS students can "collapse" a composite unit to form a "singleton unit" (Steffe, 2010, p. 42), and conceive of composite units as identical, which allows them to be iterated to solve problems (Steffe, 2010).

GNS students' algebraic reasoning. In response to the coin problem (Figure 1), Maria, who assimilated with three levels of units wrote the equation $.05N + .1(N + 3) + .25(N - 2) = 5.40$ with "ease" (Olive & Çaglayan, 2008, p. 280). Assimilating with three levels of units allowed Maria to operate on the initial equation, a three-level unit structure, by substituting expressions for D and Q without the same difficulty as Ben.

Research Questions

The literature demonstrates that students' algebraic reasoning can be modeled by their number sequences. Additionally, differences in students' fluency with composite units and the construction of a disembedding operation are critical to their algebraic reasoning. Therefore, this study asks, in what ways do the number sequences account for students' guess and check solutions to linear systems of equations? Furthermore, the literature is unclear as to the appropriateness of guess and check strategies. This study will also ask, are middle-grades students' solution methods for linear systems of equations more closely related to their number sequence or course enrollment?

Methods

This study included 18 students in grades six through nine at a rural middle and high school in the southeastern United States. Students are listed in Table 1 by math class and number sequence. The first letter of each pseudonym matches the first letter of their number sequence attribution. According to the state standards and the teachers of these students, Math 6, Math 7, and Pre-Algebra did not include any instruction on solving systems of equations. Algebra 1, Algebra 1 Parts, and Algebra 2 did include instruction on algebraic methods for solving systems of equations. Students with an asterisk are students who had received instruction on algebraic methods for solving systems of equations in their math class. Students' number sequence was determined by a survey (Ulrich & Wilkins, 2017) and confirmed by screening questions during semi-structured clinical interviews. Clinical interviews were conducted with each student on two occasions, for approximately 45 minutes each, and in addition to confirming their number sequence attribution also included algebra tasks. The tasks reported here are the coin problem and the modified coin problem (Figure 1). Students were given time to solve each problem with any method they chose but were prompted to try an algebraic method if they did not do so independently.

Table 1: Participants by Math Course and Number Sequence

	Math 6	Math 7	Pre-Alg	Alg1	Alg1 Parts	Alg2
TNS	Tabitha				Travis*	
aTNS	Aaron	Alyssa		Amanda*	Alex*	
	Abby	Andy				
	Ann	Ava				
ENS	Elle		Emily	Erin*	Elizabeth*	Emma*
	Evan					
GNS			Greg	Gavin*		

*Denotes students who received instruction on algebraic methods for solving systems

<u>The Coin Problem (Problem 1; from Olive & Çaglayan, 2008)</u>: Ms. Speedy keeps coins for paying the toll crossing on her commute to and from work. She presently has 3 more dimes than nickels and 2 fewer quarters than nickels. The total value of the coins is $5.40. Assuming that she does not have any pennies, find the number of each type of coin she has.
<u>The Modified Coin Problem (Problem 2)</u>: I have 17 coins – some quarters, some dimes, and some nickels. I have 6 more dimes than nickels and 1 fewer quarter than nickels. Find the number of each type of coin that I have.

Figure 1: The Coin Problem and Modified Coin Problem

Results and Analysis

This study asked whether students' methods for solving systems of equations were more closely tied to their math class or number sequence. Table 2 shows that the two TNS students correctly solved problem 2 using guess and check, although one had taken algebra and the other had not. All six aTNS students who attempted problem 1 used guess and check, and seven of eight aTNS students used guess and check on problem 2. This was also regardless of whether they had taken an algebra course. Thus, students who had constructed only a TNS or an aTNS tended to use guess and check, regardless of whether they had received instruction on algebraic methods to solve systems of equations. GNS students always used algebraic methods on problems 1 and 2, regardless of whether they had received algebra instruction. ENS students' methods varied. On the coin problem (1), all ENS students attempted an algebraic method, although only one ENS student was successful. The other five ENS students did not arrive at an answer algebraically and did not guess and check when the interviewer suggested it, presumably due to the quantitative complexity of problem 1. In contrast, on the modified coin problem (2), which involves less quantitative complexity, all three ENS students who had not received algebra instruction used guess and check, and all three ENS students who had received algebra instruction solved problem 2 using an algebraic method. Middle-grades students' solution methods to linear systems of equations were more closely tied to their number sequence than their course enrollment, with the exception of ENS students. ENS students' solutions were more closely tied to their course enrollment and the quantitative complexity of the problem. This pattern is indicated in Table 2 by the cluster of grayed cells among all TNS and aTNS students, and those ENS students who had not received algebra instruction, as well as the second cluster of grayed cells among all GNS students and the ENS students who had received algebra instruction.

Table 2: Results of the Coin and Modified Coin Problems by Number Sequence, Solution Method, and Math Course

	Coin Problem (Problem 1)				Modified Coin Problem (Problem 2)			
Method	Guess and Check		Algebraic Method		Guess and Check		Algebraic Method	
Course	<Alg	Alg	<Alg	Alg	<Alg	Alg	<Alg	Alg
TNS					1/1	1/1		
aTNS	¼	½			5/5	2/2	0/1	
ENS			0/3	1/3	3/3			3/3
GNS			1/1	1/1			1/1	1/1

Each numerator represents the number of students who correctly solved the problem with that method in that number sequence stage, compared to the number who attempted it (denominator). Grayed cells indicate 50% or more of solutions were correct. Neither TNS student attempted problem 1 due to their perceived frustration level. Two aTNS students did not complete problem 1 due to time. <Alg indicates a math class that did not offer algebra instruction. Alg indicates a math class that did offer algebra instruction (see Table 1).

This study also asked to what extent students' number sequences could be used to model their guess and check solutions to linear systems of equations. For brevity, this analysis is limited to the modified coin problem, and GNS students' solutions are not presented, as they did not guess and check. One response from each number sequence was selected to be representative.

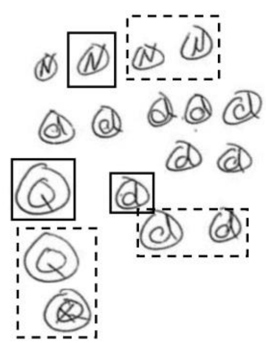
TNS Students' Solutions

Travis guessed on the modified coin problem by saying, "I'm trying to get a number … [of] dimes that have six more than nickels so that I can see how many quarters…" This shows that he was thinking about each type of coin sequentially. His first guess was 13 dimes, 7 nickels, and 6 quarters. He was satisfied that this was the answer until the interviewer asked if there were 17 coins total. This is evidence that Travis did not keep track of the dual goals of utilizing all 17 coins and maintaining the relationships between the numbers of coins. Once he finished the problem, he summarized his solution: "I would pick a number [for dimes] and … see what would be 6 less than that, and one less than that. Then I would add them all up and see if they would equal 17." His summary shows the sequential nature of Travis's guess and check process.

Building on Olive and Çaglayan's (2008) analysis, Travis's sequential determination of the numbers of dimes, nickels, and quarters is due to a limitation of the units coordination defined by his TNS. Travis could assimilate the task with one level of units (e.g., a number of dimes) and construct a second level in activity (e.g., a number of nickels in relation to a number of dimes). This facilitated his double counting and supported the sequential determination of the numbers of each type of coin. Following mental activity, the relationship between the numbers of coins decayed and Travis could only reflect on his answer. Travis's need to construct composite units in activity also limited his reflection on the relationship between his guess and the total. His first guess of 13 dimes, 7 nickels, and 6 quarters was, from the interviewer's perspective, implausibly large. To Travis, the guess was not concerning because he could not conceptualize the number of each type of coin embedded within the total number, so he worked through the problem by sequentially calculating the numbers of coins, and retrospectively checking the relationship to the total.

aTNS Students' Solutions

Abby solved the modified coin problem by drawing 17 circles to represent the coins. This shows that she anticipated the need to exhaust all 17 coins. Then she filled one circle with an N to represent one nickel, seven circles with ds to represent seven dimes, and no circles with Qs. Because she did not fill all of the circles, she knew her answer was not correct (Figure 3). Next, she made incremental adjustments to the coins by adding one nickel, dime, and quarter to her drawing, and then two nickels, dimes, and quarters. These incremental adjustments are evidence that Abby understood adding the same number of each type of coin would maintain the necessary relationships. Finally, she concluded that the solution was 4 nickels, 3 quarters, and 10 dimes.



Unboxed coins show Abby's initial guess.
Solid rectangles show Abby's first incremental adjustment.
Dashed rectangles show her second incremental adjustment.

Figure 3: Abby's Representation of the Modified Coin Problem

Abby's guess and check included two key components – dual awareness of the goals of finding 17 coins total and of the relationships between the numbers of coins. This was supported by an assimilatory composite unit. Prior to activity, Abby could conceive of the situation holistically as a composite unit of 17 coins, containing 3 tacitly embedded composite units representing the numbers of dimes, nickels, and quarters. As aTNS students cannot disembed, Abby relied on figurative materials to support her reasoning. However, her operations on embedded composite units supported her understanding that she could make incremental adjustments to each guess; this is a form of strategic reasoning. In total, seven aTNS students solved the modified coin problem using guess and check, six made incremental adjustments to their guesses, and they all used figurative material to support their reasoning.

ENS Students' Solutions

Emily guessed the correct solution on the first try. She drew 4 nickels, then directly above the nickels she drew 10 dimes, and below the nickels drew 3 quarters (Figure 4). While drawing, she said, "Well, if there's 4 nickels then that would be 6 more dimes is 10, and 1 less quarter is 3. So that works." When asked how she generated that guess, she responded, "I just thought if it's 4 nickels, then 6 more is 10 dimes so that's 14 right there, and so the 3 [quarters] just worked out." Emily's explanation does not indicate how she arrived at four nickels as an initial guess, but there is no indication that she guessed other combinations of coins prior. Similarly, Evan guessed the correct solution on the first try "by accident."

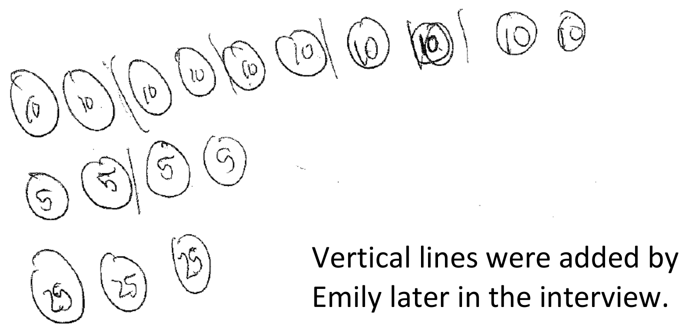


Figure 4: Emily's Representation of the Modified Coin Problem

ENS students' guesses were supported by an assimilatory composite unit, similar to aTNS students, which allowed them to conceive of the situation as a composite unit of 17 coins containing three embedded composite units representing the numbers of dimes, nickels, and quarters. However, the ease with which Emily and Evan guessed the solution indicates, however, that their reasoning was more sophisticated than that of the aTNS students. Based on the limited evidence provided by these two ENS students, it is difficult to attribute this sophistication to any particular mental operation. Elle's solution will be presented next because her work provides more clear evidence that a disembedding operation supported the accuracy of their guesses.

Elle used an unwinding strategy to solve the modified coin problem. This is another pre-algebraic strategy in which students solve a problem "by working backward through the constraints provided in the problem… by inverting operations and performing arithmetic operations rather than using algebraic manipulation" (Knuth et al., 2006, pp. 301–302). Elle applied an unwinding strategy when she said,

> I'm subtracting the amount of dimes from that that we already have [writes 17 minus 6 equals 11]. And I'm just trying to figure out, like, how many nickels and dimes. … So that [subtracting six] sort of equalizes the number of dimes and nickels, doesn't it, but we have one fewer quarter than nickels. … Well, we already have one less quarter than nickels, so that's one more [writes 11 plus 1 equals 12]. [Adding one] sort of balances it, quarters with the nickels.

Elle divided 12 by 3 to find that the solution was 4 nickels, 10 dimes, and 3 quarters.

Elle's unwinding strategy is evidence that she applied both an assimilatory composite unit and a disembedding operation. As with the ENS students who guessed and checked, Elle assimilated the situation as a composite unit of 17 containing 3 embedded composite units. Disembedding allowed Elle to simultaneously conceive of the relationship between the numbers of dimes and nickels to the total, which supported her reasoning that equating the number of dimes and nickels would reduce the total to 11. She then applied the same disembedding operation to consider the relationship between the numbers of nickels and quarters to 11. Thus, Elle leveraged her reflection on the relationships between the numbers of each type of coin to simply the problem context. Although neither Emily nor Evan seemed overtly aware of this process, it is possible that similar reasoning supported their "accidental" guesses.

Discussion

Studying algebraic solutions to linear systems of equations is typical of middle- and high-school Algebra 1 curricula (e.g., CCSSI, 2010), but guess and check remains prevalent (Johanning, 2004). This study offers a lens to interpret these difficulties and implications for students' preparedness to receive instruction on solving linear systems of equations. TNS students did not solve systems of equations algebraically, regardless of their course enrollment. However, they did correctly solve a linear system of equations with limited quantitative complexity using guess and check. This was supported by their construction of composite units in activity. Although it is unlikely that these students are prepared to accept instruction on algebraic methods to solve systems of equations, they may benefit from instruction that promotes systematic guess and check. Knuth et al. (2006) maintain that guess and check is pre-algebraic. While this may be so, supporting TNS students' use of systematic guess and check may be more productive than attempting to teach them to apply algebraic methods without understanding.

aTNS students also did not use algebraic methods, regardless of their course enrollment, which implies that they are also unlikely prepared to accept instruction on algebraic methods for solving systems of equations and apply those methods in novel problem-solving situations. However, aTNS students had access to more sophisticated solutions than TNS students, including systematic guess and check. Thus, aTNS students may benefit from instruction that includes active reflection on the relationships between the unknown quantities in systems of equations to support that readiness for instruction. Additional longitudinal research is necessary to assess an instructional trajectory that may engender aTNS students' construction of disembedding. However, Zwanch and Wilkins (2021) found that students who have constructed an aTNS by sixth or seventh grade are more likely to construct an ENS and a disembedding operation by the time they enter eighth grade, when compared to students in sixth and seventh grades who have not yet constructed an aTNS. This implies that the early middle grades are a critical time in students' construction of number, particularly in supporting the construction of an ENS. In combination with the present study, these findings suggest that supporting aTNS students' construction of disembedding to support their preparedness to learn algebraic methods of solving systems of equations is likely more productive in the early middle-grades.

ENS students capitalized on the algebraic methods taught in their algebra classes, if they had taken one. This indicates that their assimilatory composite unit and disembedding operation prepare ENS students to accept instruction on algebraic methods for solving linear systems of equations. However, their success with these methods was limited to situations such as the modified coin problem, which had limited quantitative complexity. Longitudinal research should consider how to support ENS students' solutions to systems of equations with greater quantitative complexity, such as the coin problem, in novel problem-solving situations. Overall, these results demonstrate that providing instruction on linear systems of equations in a middle- or high-school Algebra 1 course is most likely to be productive if instruction is differentiated to support solution strategies that students are prepared to accept.

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ELEMENTARY PATTERNING PROBLEMS: VISUAL AND NUMERICAL STRUCTURING

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Research on elementary students' reasoning on patterning problems with pictorial representations has illustrated that students can visualize structure in patterns in different ways. In this paper, we offer a characterization of students' spatial structures and numerical structures and explain how the link between these two structures can support students' generalization of a pattern or prediction of a future value.

Keywords: Elementary School Education, Algebra and Algebraic Thinking

Reasoning about and with functions is a foundational topic in K-12 mathematics. Functional thinking in algebra can be defined as "representational thinking that focuses on the relationship between two (or more) varying quantities, specifically the kinds of thinking that lead from specific relationships (individual incidences) to generalizations of that relationship across instances" (Smith, 2008, p. 143). Functional thinking builds from patterning in elementary grades to generalized algebraic equations in secondary mathematics. In grades 3-5, elementary students are expected to "describe, extend, and make generalizations about geometric and numeric patterns; represent and analyze patterns and functions, using words, tables, and graphs" and use equations to express mathematical relationships, inherently linking patterns, relations, and functions (NCTM, 2000, p. 158). To prepare students for functional thinking in later grades, Blanton and Kaput (2004) propose that elementary students should move beyond simple patterns in one variable to focus on problems in which two or more quantities vary simultaneously. Indeed, such complex patterning problems are often included in research studies with elementary students (e.g., Stephens et al., 2017; Wilkie & Clark, 2016) and on standardized assessments for elementary students, such as the National Assessment of Educational Progress (NAEP) mathematics assessment and Trends in International Mathematics and Science Study (TIMSS).

The mental activities used by students to generalize a pattern from a table, graph, or pictorial representation are of particular interest in studying students' functional thinking (Smith, 2008). Recent research on student thinking about patterning problems considers both functional thinking and spatial visualization. By analyzing student work on patterning problems from both an analytic and visualization perspective, researchers can understand the ways students reason with and about different function representations, including figures, tables, and generalized rules. While other studies have reported on students' spatial visualization when solving patterning problems with pictorial representations (Hershkowitz et al., 2001; Wilkie & Clark, 2016), in this paper, we identify both the spatial and numerical structures students use when solving a patterning problem and describe how linking a spatial structure with a numerical pattern structure can support a student's generalization of a pattern or prediction of a future value. "Spatial structuring is the mental act of constructing a spatial organization or form for an object or set of objects. Numerical structuring is the mental act of constructing an organization or form for a set of computations" (Battista et al., 2018, p. 211). *Spatial numerically-linked structuring* is a coordinated process in which numerical operations are performed based on a linked spatial structuring (Battista et al., 2018).

Literature Review and Framework

Three modes by which researchers analyze student reasoning about functions and patterning problems with two varying quantities are recursive, covariation, and correspondence approaches (Blanton & Kaput, 2011; Stephens et al., 2017). The *recursive* approach describes the change within a sequence of values (Blanton & Kaput, 2011). It indicates how to obtain the next value in a sequence given the current sequence value. In a two-column table with two varying quantities, a student using a recursive approach would identify the change in one column independent of the other column and use this change to move from one value to the next within a column. The *covariation* approach describes how the change in two quantities is related (e.g., as x increases by one, y increases by 2) (Confrey & Smith, 1991). The *correspondence* approach describes a rule or mapping that relates any given x-value to a unique y-value (e.g., y = 2x + 3, indicating y-values are 3 more than twice the x-values) (Confrey & Smith, 1991).

While there are multiple learning progressions in the literature describing the ways elementary students may develop these different types of functional thinking (Blanton et al., 2015; Stephens et al., 2017; Wilkie & Clark, 2016), the progressions all provide evidence that students typically begin with recursive or covariational approaches and move toward more sophisticated correspondence approaches. In studies with students in elementary grades, researchers often present patterning problems by providing a series of figures or manipulatives that show a growing pattern in two variables (Stephens et al., 2017; Wilkie & Clark, 2016). This offers an opportunity for students to recognize the relationship between two variables. At times, tables are used to organize or display patterns and data (Schliemann et al., 2001). Standard questions include "far-prediction" problems or tables with a break in the sequence of values which have been used to encourage students to shift their approach from a recursive strategy to either a covariational or a correspondence approach or from a specific relationship between two items to a generalization for the whole set (Blanton et al., 2015; Blanton & Kaput, 2004; Schliemann et al., 2001; Stephens et al., 2017). In general, these studies have shown that young children are capable of functional thinking.

When functional relationships are represented pictorially, spatial thinking becomes an important part of students' reasoning with functions. Students identify and visualize changes from figure to figure in a pictorial representation in many different ways (Hershkowitz et al., 2001). Visualization is the process involved in constructing and transforming visual mental images (Presmeg, 1997) and impacts the resulting spatial mental image that encodes properties such as location, size, and orientation (Sima et al., 2013). Battista (1999) defines spatial structuring as the mental process by which a person constructs an organization for a set of objects. The process of spatial structuring includes identifying the spatial components of the figure and organizing the components into composites with certain relationships between them. This is of particular interest for patterning problems with pictorial representations because the way a student sees the figure components, figure composites, and interrelationships between figures becomes a part of the student's reasoning process. Visualization and the resulting spatial structures have the potential to enhance a student's understanding of algebraic and function concepts (Boaler et al., 2016) and can sometimes influence the way in which a student generalizes a visual pattern or predicts future values. Wilkie and Clark (2016) found that students sometimes transition between multiple visualizations of a pattern while solving a single patterning problem and report that these visualizations likely lead to specific types of generalizations of the numerical pattern.

Method

To explore students' thinking with pattern and relationship problems, clinical interviews (Ginsburg, 1981) were conducted with a convenience sample of three 4th grade students at a public elementary school in the United States. The students were all in the same mathematics class. According to the teacher, the three students represented the typical range of mathematical abilities in her classroom. The problem chosen for this study was the Pattern of Circles Item (**Figure 1**) of the 4th Grade 2011 TIMSS Questionnaire for which 75% of U.S. students (International 68%) correctly answered Part B, while only 47% of U.S. students (International 39%) correctly answered Part C (IEA, 2013). The problem provides opportunities for students to reason with both pictorial and table representations while predicting a future value. Students were asked to solve the problem while the researcher (second author) observed and asked the students to clarify their thinking. The video and audio recorded interviews were transcribed and reviewed by both authors, examining for evidence of the spatial structure students used when working with the pictorial representation, how they interacted with the table, and how they predicted the number of circles in future figures in the pattern. (Note: Figures in bold refer to the inserted figures in the paper. Figures not in bold refer to the Figures in the Pattern of Circles item of the 4th Grade 2011 TIMSS Questionnaire).

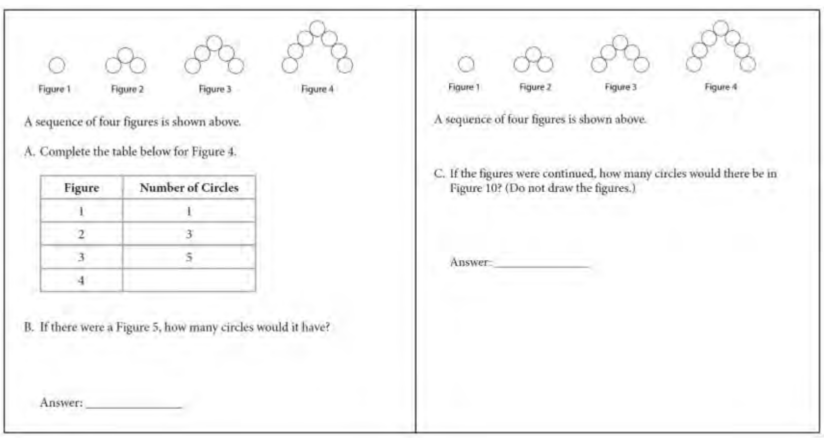
Figure 1 Figure 2 Figure 3 Figure 4

A sequence of four figures is shown above.

A. Complete the table below for Figure 4.

Figure	Number of Circles
1	1
2	3
3	5
4	

B. If there were a Figure 5, how many circles would it have?

Answer: ____________

Figure 1 Figure 2 Figure 3 Figure 4

A sequence of four figures is shown above.

C. If the figures were continued, how many circles would there be in Figure 10? (Do not draw the figures.)

Answer: ____________

Figure 1: Pattern of Circles Problem TIMSS 2011 Assessment (IEA, 2013)

Findings

Student 1: Dennis

Two students in the study, Dennis and Miles, used the same spatial structure (Battista, 1999) when describing the pictorial representation provided in the Pattern of Circles Problem (**Figure 1**). In Part A, Dennis stated, "I know the sequence, it's just adding on two [points to the circles at the bottom of each 'leg' as highlighted in **Figure 2**]." Dennis identified the way he saw the two additional circles in successive figures as the bottom two circles on each 'leg.'

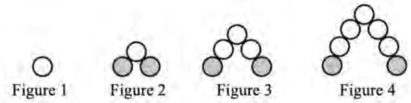

Figure 2. Dennis' Spatial Structure for the Pictorial Representation

While Dennis recognized the pattern in the pictorial representation of the problem, he stated "I don't really get this table." When the researcher explicitly asked Dennis what the numbers in the table might mean, he correctly related the first column to the figure number. The researcher further prompted him by asking what the second column in the table refers to and Dennis replied, "Oh, okay, … The numbers of circles that are in each triangle shaped thing." He then filled in the missing value in the table with the numeral "7".

To solve Part B, Dennis drew Figure 5 consistent with how he spatially structured the two additional circles in successive figures and counted to correctly conclude that there were nine circles in Figure 5. The order in which circles were drawn for Figure 5 is shown in **Figure 3**.

Figure 3: Dennis's Drawing of Pattern of Circles Figure 5

For Part C, Dennis attempted to count the number of circles in Figure 10 by tapping his pencil from left to right under each 'leg' of Figure 4 while counting aloud from circle seven: "8, 9; 10, 11; 12, 13; 14, 15; 16, 17 [see **Figure 4**]. So, I think it's 17." However, this is the correct number of circles for Figure 9, rather than Figure 10.

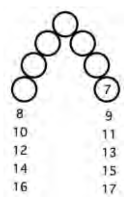

Figure 4: Dennis's Visualization and Counting of Additional Circles

When prompted to further explain his thinking, Dennis recounted the number of circles in Figure 10 using the same spatially structured counting method but was more explicit about the way he kept track of the figure numbers and the number of circles. Starting from circle seven in Figure 4 he stated, "So 8, 9, that would be one [figure more]; 10, 11, that would be two [figures more]; 13, 14 that would be three [figures more]; 15, 16 that would be four [figures more]; 17, 18, that would be five [figures more]." Two errors occurred when Dennis counted the second time. The first error was that he counted five figures from Figure 4, rather than six just as he did the first time he counted. The second error was skipping the number 12 when counting the circles. Coordinating the number of figures and the number of circles at the same time was challenging.

When asked how he knew when to stop adding circles, Dennis stated, "You only need to do five times two. Just need to do two five times. That's how you get your answer." While further explaining his thinking, Dennis corrected his counting error: "Because it says 10. Wait, six times [not five]. If you work with [Figure] four. Yeah, it's six times." Dennis again recounted the number of circles, using the same spatially structured counts illustrated in **Figure 4** and reached the correct number of circles, 19. He then generalized the counting process. "So yeah, you do two six times. Two times six, plus seven…Because there's only Figure 4, not Figure 5. If there was a Figure 5, then you only need to do five times, but there's no Figure 5."

Dennis correctly concluded he would need to add six sets of two circles to build Figure 10 from Figure 4, and extended his thinking in a way that would have facilitated starting with a different figure number. Dennis developed his generalization by imagining the changes from Figure 4 to Figure 10 using the spatial structure he described when looking at the figures (**Figure 2**), while explicitly stating the relationship between the addition of two circles and the successive figure in the sequence.

Student 2: Miles

In contrast to Dennis for Part A, our second student, Miles immediately wrote "7" in the table for the number of circles in Figure 4, making an unprompted connection between the pictorial and table representations.

> Miles: So, I saw one here [points to the one circle in Figure 1 and to the "1" in the output column], three here [points to the three circles in Figure 2 and to the "3" in the output column], five here [points to the five circles in Figure 3 and to the "5" in the output column]. So, I counted these [the circles in Figure 4], and I got an answer of seven, so I put that in the box.

When asked how he determined the number seven, Miles described the same spatial structure as Dennis (**Figure 2**).

> Miles: So, I saw figure one, and then I saw figure two, and right away I saw that it added two more circles [points to the bottom two circles on each 'leg' of Figure 2]. So then in figure three, I saw it add two more circles [points to the bottom two circles on each 'leg' in Figure 3]. And again, in figure four, I saw it add two more circles [points to the bottom two circles on each 'leg' in Figure 4]. So, I thought there was an addition of two from one going up to seven.

For Part B, Miles added two plus seven to correctly conclude that Figure 5 would have nine circles without producing a drawing. In explaining his reasoning, Miles stated, "I knew that there was a pattern of adding two [gestures from left to right over the figures]. So, I just add two to seven, if there was a figure five, and I got nine."

For Part C, Miles generalized from Figure 5 and stated that the answer would be 19 circles, because, "I knew that after each figure, two [circles] would be added. So, if there were five figures [from Figure 5 to Figure 10] and two were being added each time, I knew that it would be 10. So, I add 10 plus nine to get my answer of 19." Miles made this generalization without drawing or explicitly visualizing additional figures like Dennis did; rather Miles used the difference in figure numbers from five to ten to generalize the pattern.

Student 3: Margot

The third student, Margot, recognized the addition of two circles for each successive figure, but she saw the additional two circles in a different spatial structure than Dennis and Miles. After reading Parts A and B of the question, Margot initially analyzed the figures, saying, "Um, first you do like—so two [moves pencil across Figure 2 as shown in **Figure 5]**, two [taps Figure 3 as shown in **Figure 5**]. Um, two [moves pencil across Figure 4 as shown in **Figure 5**]. It would be like, 1, 2, 3, 4, 5, 6, 7 [counts Figure 4 as shown in **Figure 5**]." Rather than seeing the additional two circles in each figure added to the bottom 'legs' of the previous figure as Dennis and Miles did, Margot saw the two circles on one 'leg' of the figure. From her comments in later dialogue, we infer that she saw the number of remaining circles in the figure as equal to the number of circles in the previous figure, indicating that she may have recognized the recursive nature of the pattern.

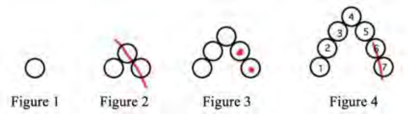

Figure 5: Margot's Spatial Structure for the Pictorial Representation

Initially when answering Part A, Margot incorrectly wrote "6" in the table as the Number of Circles in Figure 4. When the researcher asked her to explain her thinking, Margot responded, "Um—I just think—oh, now I know [erases the "6" and puts "7"]. Seven, because like, … I think it just matches. I don't know." Though Margot's second answer of seven was correct, she had difficulty explaining her reasoning within the table representation. The researcher then asked if she knew what the table was referring to, and Margot was prompted to relate the table to the figures. She verified her answer of seven in the table by recounting the number of circles in Figure 4, "Yeah, 1, 2, 3, 4, 5, 6, 7," (as illustrated in **Figure 5**). While Margot's reasoning with the numerical values in the table was imprecise, she ultimately relied on the pictorial representation to definitively and correctly state the number of circles in Figure 4.

When asked to solve Part B, Margot, like Miles, correctly predicted the number of circles in Figure 5 by simply adding two to the number of circles in Figure 4 without producing a drawing. However, she still indicated the additional two circles in each figure as shown below.

> Margot: Pretty sure I know it's nine…It's nine…Because there's—so one [points to the one circle in Figure 1], three [points to the three circles in Figure 2], because these are—and there's two more [gestures to the right 'leg' of Figure 2 as indicated in **Figure 6**] than each of them. Two [gestures to right 'leg' of Figure 3 as indicated in **Figure 6**], Two [gestures to right 'leg' bottom two circles in Figure 4 as indicated in **Figure 6**] more than each of these other ones, so I'm pretty sure it's nine. Because seven plus nine is, wait, seven plus two is nine.

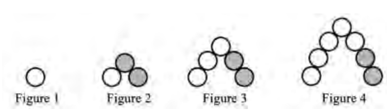

Figure 6: Margot's Gesturing of the Two Additional Circles in Each Figure

For Part C, Margot attempted to draw the figures up through Figure 10, but did not continue the pattern of circles following the spatial structure of adding two illustrated in **Figure 6**. Instead, she drew long, straight chains of circles to represent each figure but did not consistently draw the straight chains with an accurate number of circles (**Figure 7**). At times she added two circles to the next figure and at times she added three, ultimately leading to a series of figures that produced an incorrect answer.

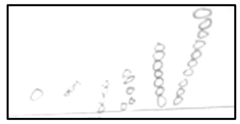

Figure 7: Margot's Figure Growth Drawings for Part C

Discussion

Two of the three participants in our study, Dennis and Margot, faced challenges when determining the number of circles in Figure 10 in Part C. By characterizing the spatial structures used by the students as spatial numerically-linked structures (Battista et al., 2018) or non-spatial numerically-linked structures, we provide insight for when a spatial structure may support students' reasoning about patterning problems.

The spatial structure utilized by Dennis and Miles (**Figure 2**) can be classified as a spatial numerically-linked structure (Battista et al., 2018) which has the potential to support student reasoning about far-prediction problems. By seeing the two additional circles at the bottom of the figure, Dennis and Miles were using a spatial structure that is aligned with a recursive numerical process. Numerically, a recursive pattern adds a value to a previous value; spatially, this can be thought of as adding objects to a previous congruent figure. Dennis' and Miles' spatial structuring organized the components of the figures, the circles, into composites: one part is the previous figure and one part is the two additional circles for each successive figure (**Figure 8**). This organization includes the geometric properties of symmetry within the figures and congruence between figure components. Even though neither Dennis nor Miles stated these geometric properties, these visually salient qualities may support imagining or visualizing future shapes.

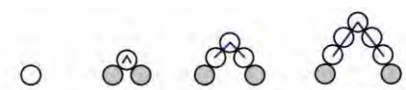

Figure 8: A Spatial Numerically-linked Structuring for the Pictorial Representation

A spatial numerically-linked structuring can provide a way to coordinate the two varying quantities in a patterning problem. The way in which Dennis saw the additional two circles added to each figure provided a way for him to coordinate the figure number and number of circles resulting in a numerical structure that provided an organization for his set of computations (**Figure 9**). Each time he imagined a new pair of circles being added, he moved to a new row and tapped his pencil adding the additional circles in an organized way. Even when he made two counting errors, he was able to recognize and correct those errors and generalize his process because his spatial structure and numerical structure were linked.

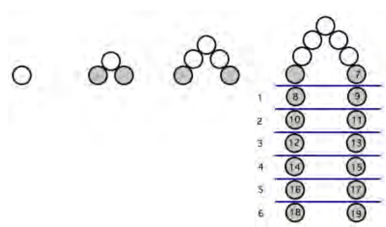

Figure 9: Dennis' Spatial Numerically-linked Counting

In comparison, Margot, did not reach the correct solution for Part C. One explanation for why Margot's visual structure did not support predicting the number of circles in Figure 10 is that the spatial structure was not linked to a numerical structure for adding on two circles in an organized way. Like Miles and Dennis, Margot also recognized that two circles were added in each successive figure. However, the way in which she saw the two additional circles within the figure (**Figure 6**) did not show the addition of the circles to a congruent previous figure or maintain the symmetry of the figures. Because the spatial structure was not linked to the recursive numerical pattern she verbalized, it was very difficult for Margot to imagine or draw the next figure even though she could explain the provided figures using her spatial structure. By observing her gestures and descriptions of the pattern, we hypothesize that she verified the pattern by recognizing that the collection of three white circles in Figure 3 were the three circles from Figure 2 in a different spatial arrangement (**Figure 10**). She used similar gesturing to verify that the collection of five white circles in Figure 4 is just a different spatial arrangement of the five total circles in Figure 3. But without a symmetric, congruent spatial structuring, creating a new figure, such as Figure 5, using the recursive relationship of adding two circles is very challenging, making a resulting numerical structure for computations to determine the number of circles in Figure 10 very difficult to coordinate with the pictorial representation.

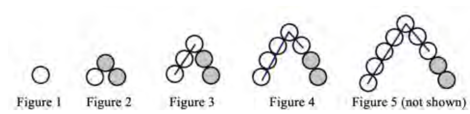

Figure 10: Margot's Spatial Structure

Indeed, for Part C, Margot did not attempt to draw using the same spatial structure. Instead, she drew long chains of circles to represent each figure (**Figure 7**). However, Margot's second spatial structuring was also not a spatial numerically-linked structure because it was not connected to the numerical recursive pattern of adding two. We hypothesize that Margot's re-arrangement of the figures into long strings of circles could be spatially linked to the numerical structure if the straight lines were maintained and the circles were congruent (**Figure 11**). This would offer the same spatial-numerical link as Miles' and Dennis' structure because the additional two circles would be added to a previously congruent figure. While symmetry within the figures is not as visually salient, the equal "heights" of the strings could have helped coordinate the additional two circles added to congruent strings. However, because Margot's drawings did not incorporate these features, it became very difficult for her to keep track of the total number of circles in each figure and to consistently add two circles to each string ultimately causing her to reach an incorrect solution.

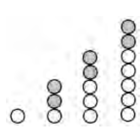

Figure 11: A Spatial Numerically-linked Structure Example

While spatial numerically-linked structures can help students reason about pattern problems with pictorial representations, it is certainly not required if other representations are utilized. A student could have been successful on the Pattern of Circles problem without such a structure by using the table representation to add two to the previous output value, extending the table to determine the number of circles in Figure 10. However, none of the students in the study explicitly reasoned with the table. Miles was the only student who did not draw additional figures while predicting future values. While it is possible that he used information from the table to support the development of his generalization, he also used the same spatial numerically-linked structuring as Dennis while reasoning about the problem.

Conclusion

Research has shown that students can think about pictorial representations with different visual structures, and we have offered evidence that these visual structures can support student thinking about patterning problems when the spatial structure and the numerical structure are adequately linked. By identifying features of spatial structures and numerical structures that are helpful for students when solving patterning problems, we can better understand how students' visualizations can facilitate the development of numerical generalizations and predictions.

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LEARNING TO MAKE SENSE OF DATA IN A CODAP-ENABLED LEARNING ENVIRONMENT: INTERACTIONS MATTER

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In this study, we investigated how sixth and seventh grade students used CODAP to make sense of roller coaster data while engaged in Exploratory Data Analysis (EDA). Using instrumentation theory, we examined students' instrumentation approaches, as well as the types of instrumental orchestration utilized by teachers as they interacted with student pairs during EDA.

Keywords: Data Analysis and Statistics, Technology, Instructional Activities and Practices

Introduction

Statistics has gained prominence in school curricula in the US (Franklin et al., 2007; National Council of Teachers of Mathematics, 2000; National Governors Association Center for Best Practice & Council of Chief State School Officers, 2010), which includes a focus on reasoning about data. One way to encourage students to reason about data is providing opportunities to engage in Exploratory Data Analysis (EDA). EDA first developed by Tukey (1977), involves exploring data to summarize main characteristics. EDA is the "art of making sense of data by organizing, describing, representing, and analyzing data, with a heavy reliance on informal analysis methods, visual displays" (Ben-Zvi & Ben-Arush, 2014, p. 197). While approaches often use visual methods, such as graphs and other representations, statistical measures are sometimes calculated to make sense of data. Ben-Zvi (2004) points out that exploring data involves examining features such as shape, center, and spread; it involves considering graphs and looking for other characteristics of data like clusters, gaps, and outliers. Cobb and McClain (2004) recommend that EDA should be the focus of early experiences with instruction because of the emphasis on finding trends and patterns.

EDA often involves the use of technology, and there is evidence that innovative technology tools aide students in developing statistical thinking (e.g., Biehler et al., 2013). We are interested in understanding students' engagement with the Common Online Data Analysis Platform (CODAP), (https://codap.concord.org/), which has many powerful dynamic visualization and calculating capabilities that make it an ideal tool for engaging in EDA. Specifically, we investigated the following research questions:

- *RQ1: How do students use CODAP to make sense of data while engaged in EDA?*
- *RQ2: What types of orchestration emerge as teachers interact with students as they engage in EDA using CODAP?*

Theoretical Perspectives

Our study draws on two theoretical perspectives from instrumental theory: instrumental genesis and instrumental orchestration. To understand students' learning processes as they made sense of data during EDA using CODAP, we used Ben-Zvi and Ben-Arush's (2014) types of instrumentation. Instrumental orchestration was used to understand teachers' interactions with students as they explored 157 US roller coasters using CODAP (Drijvers et al., 2010).

Instrumental Genesis

Five components comprise instrumental genesis (IG) (Ben-Zvi & Ben-Arush, 2014). The *subject* is a learner who accomplishes a task using an instrument. An *object* is a specific task. An *artifact* (a component of a tool) is a physical or virtual device that is used by the subject, which has no meaning for the learner in isolation. A *utilization scheme* is a cognitive scheme that the subject uses to accomplish a task using one or more artifacts. When the subject has successfully used the utilization scheme to accomplish a task, the artifact becomes an *instrument* for the learner to use. The authors indicate that IG occurs when a subject uses utilization schemes to transform an artifact into an instrument that can be used as a meaningful tool to achieve a particular goal.

There are two components of IG, *instrumentalization*, the ways in which the subject's prior knowledge acts on the tool, and *instrumentation*, the way the instrument influences the subject's learning process. In this work, we are interested in instrumentation. Ben-Zvi and Ben-Arush (2014) identify three processes of instrumentation that learners use to investigate data: unsystematic, systematic, and expanding. An *unsystematic* approach to investigating data involves actions that are not intentional or systematic, where learners make sense of a few basic artifacts and associated actions. *Systematic* instrumentation involves intentional and somewhat organized exploration, occurring after the learner has become familiar with artifacts, and may be more focused on the tool rather than the task. The third process involves *expanding* emerging instrumentalization (i.e., ways in which students' prior knowledge acts on the tool) of an artifact and associated actions that transform into a more usable and powerful instrument that can be used in a meaningful way in new contexts and situations.

Instrumental Orchestration

Instrumental orchestration is the teacher's intentional and systematic organization and use of various artifacts in a learning environment to guide the learners' instrumental genesis in relation to a mathematical task (Drijvers et al., 2010; Trouche, 2004), or in our case a statistical task. The three elements within instrumental orchestration include the following: a) *didactical configuration*, referring to the design of the teaching setting and artifacts, b) *exploitation mode*, referring to the ways the teacher makes decisions to exploit the didactical configuration to achieve the learning goals, and c) *didactical performance*, referring to the in the moment decisions made by the teacher on how to act on the didactical configuration and enact the exploitation mode. While Drijvers et al. and Mojica et al. (2019) identified orchestration types of teachers' purposeful use of technological tools during whole class instruction, we are interested in teachers' orchestration as they interact with pairs of students.

Participants and Context

Participants in this study were 19 sixth grade and 25 seventh grade students between the ages of 11 and 12-years old from a small urban school in the southeastern US. The school is racially diverse, and 48.6% of the students receive free/reduced lunch. Less than half of the students are proficient in mathematics (40.2%), as compared to 63.2% in their district.

We report on the same 72-minute mathematics lesson, implemented in both a sixth and seventh grade classroom, where students engaged in EDA using CODAP to make sense of roller coaster data. This lesson took place during the second week of the school year, prior to any formal instruction on statistics. This was students' first experience with CODAP. Both classes were taught by an experienced researcher, from a large research university in the southeastern US, with expertise in the teaching and learning of statistics, as well as using technology tools. The regular mathematics classroom teachers were also present during the lesson and interacted with students while they engaged in EDA. Since this paper focuses on how the teachers interacted with students only as they worked in pairs (not as a whole class) during EDA, we refer to all as teachers.

Each lesson consisted of four parts: 1) teacher launching the investigation (whole class); 2) teacher introducing CODAP as a tool using a small data set (whole class); 3) student pairs investigating larger data set using CODAP (pairs); and, 4) teacher facilitating discussion as student pairs present interesting noticings (whole class), the results of their EDA. The teacher launched the lesson by asking students to consider aspects of roller coasters that might make the ride thrilling or scary and then showed a video of a wooden roller coaster from the data set, from the point of view (POV) of a rider, to introduce the context of the data. Students discussed attributes of coasters they thought might be thrilling or scary, and then the teacher introduced students to CODAP by facilitating the exploration of a small data set of 31 US roller coasters using a CODAP document. Our analysis focuses on part 3 of the lesson where students worked in pairs to explore a larger data set of 157 US roller coasters, with 15 numerical and categorical attributes (e.g., name, location, design, top speed, maximum height, etc.). Students were encouraged to ask their own questions and find interesting things they could share about the coasters using features in CODAP, such as graphs. While student pairs engaged in EDA, all teachers monitored student work and interacted with students.

Methods

Data collected for this study is part of a larger project. Classes were video recorded using three cameras from multiple perspectives. While student pairs used CODAP to investigate the roller coaster data, all cameras recorded the teachers' interactions with student pairs or focused on student pairs as they worked. Six student pairs' laptop screens were recorded as screencasts throughout the entire class. The regular mathematics classroom teachers selected pairs to represent divergent student thinking. We used a deductive approach to selecting video for analysis (Derry et al., 2010). To examine how students used CODAP to make sense of data, we selected video recordings from the screencasts of students' laptops while they were engaged in EDA with the 157 roller coaster data set using CODAP, as well as video recordings from cameras that showed students' and teachers' interactions. All selected video was initially viewed to identify episodes, our unit of analysis. Episodes were defined as an action or group of closely related actions that resulted in a process of instrumentation. After multiple researchers had viewed the video, episodes were established after arbitration and agreement was reached.

Once episodes were identified, we created content logs to provide a time-indexed description of the events on the video (Derry et al., 2010). Each episode was coded by two different researchers. Episodes of student pairs' screencasts were coded to identify the processes of instrumentation that learners used to investigate data (Ben-Zvi & Ben-Arush, 2014): unsystematic, systematic, and expanding. To identify the types of instrumental orchestration that emerged, we first identified all questions and interactions between the teachers and students as they worked in pairs. We used open coding until themes emerged to identify orchestration types. When disagreements between coders occurred, the authors arbitrated until consensus was reached, and in some instances a third researcher made the final decision.

Results

Students' Use of CODAP to Make Sense of Data

To investigate how students used CODAP to make sense of data, we identified the instrumentation processes that six student pairs used to reason about 157 US roller coasters. The number of episodes, where an action or group of closely related actions resulted in a process of instrumentation, varied across pairs. Table 1 shows the instrumentation process identified for each pair as they engaged in EDA. Unsurprisingly, all pairs initially engaged in unsystematic instrumentation. While pairs 1, 2, 5, and 6 moved between unsystematic and systematic instrumentation, only pairs 3 and 4 engaged in unsystematic, systematic, and expanded instrumentation. Pair 2 is the only pair that worked unsystematically for most of their EDA.

Table 1: Instrumentation Processes of Student Pairs

Pair	Ep 1	Ep 2	Ep 3	Ep 4	Ep 5	Ep 6	Ep 7	Ep 8	Ep 9
1	Unsys	Unsys	Sys	Sys	Sys				
2	Unsys	Sys	Unsys	Unsys					
3	Unsys	Unsys	Sys	Exp					
4	Unsys	Unsys	Unsys	Sys	Sys	Exp	Unsys	Unsys	Unsys
5	Unsys	Sys	Unsys	Sys	Sys	Sys	Sys		
6	Unsys	Sys	Sys	Sys	Sys	Unsys	Sys	Unsys	

Example of pair that used unsystematic and systematic instrumentation. Pair 6 is an example of a pair that engaged only in unsystematic and systematic instrumentation. They began their exploration by clicking on different features of CODAP, including the map feature, slider, and opening graphs. It is important to note that opening a new graph window results in cases being displayed as a configuration of randomly scattered data points. Additionally, the map and slider features of this CODAP document were not linked to the data. This unsystematic approach enabled them to identify CODAP features that were available to them that could be potentially used to make sense of data. The pair quickly took a systematic approach by adding different attributes to a graph. Figure 1a shows the graph that was created after one student asks if the *maximum drop* is affected by the *number of inversions*. After answering a few questions about the data, the pair is then curious about how many attributes they can include on the graph, which leads them to unsystematic instrumentation as they create a new graph, see Figure 1b. Using a trial and error approach, they add three attributes, *state, year opened,* and *top speed* to the graph to conclude that at most three attributes can be added.

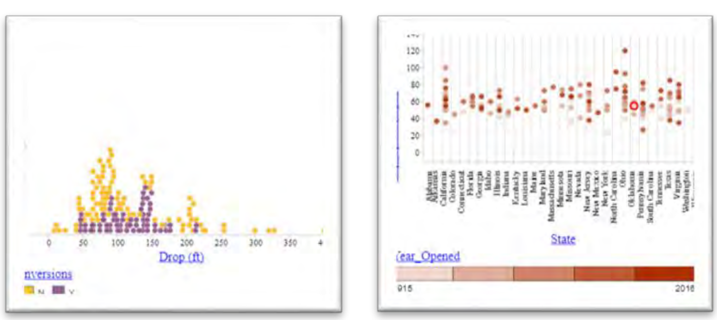

Figure 1a and b: Examples of Systematic (1a) and Unsystematic Instrumentation (1b)

Example of pair that used unsystematic, systematic and expanded instrumentation. Pair 3 not only used features in CODAP in an intentional way to make sense of data, they engaged in expanded instrumentation. For example, they created a scatter plot comparing the *maximum drop* to the *top speed*, and then overlaid *type* on the graph to investigate if the material a roller coaster is made of affects the relationship between *top speed* and the *maximum drop* (see Figure 2). This made the use of the graph more powerful for them by allowing them to pose and answer a new question while using more features of CODAP. One of the students concluded that a lot of wooden coasters are slower and have a "shorter" drop, and the fastest ones are steel.

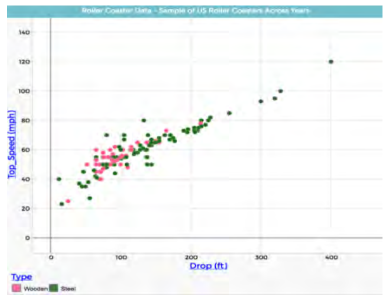

Figure 2: Student Created Scatterplot

Types of Teachers' Orchestration that Emerged

We identified 42 instances of orchestration by the teachers that were categorized into eleven different types. Table 2 illustrates the types, provides a definition and example, as well as indicates the percent of time each type occurred. Several of these orchestration types seem applicable to contexts beyond statistics and data analysis and using technologies other than CODAP, such as inserting terminology (2.38%) and providing technical assistance (4.76%). However, most of the orchestration types related specifically to teaching statistics, such as noticing trends or relationships in data and suggesting data moves. Suggesting a data move (28.57%) and assessing students' progress in their EDA (21.43%) accounted for a majority of the orchestration types. Four of the orchestration types occurred only one time (2.38%): insert terminology, clarify, focus on a case, and link multiple representations. Noticing trends and/or relationships (11.90%), making a claim or inference (9.52%), and explaining statistical reasoning or supporting a claim (7.14%) made up 28.56% of the orchestration, which are all significant in designing learning environments to support students in developing productive statistical thinking. It is beyond the scope of this paper to provide examples of every orchestration type. Therefore, we will focus on suggesting a data move and inserting terminology.

Table 2: Orchestration Types for Interactions with Student Pairs

Type	Percent	Definition	Example
Assess Progress	21.43	Assess where students are in their exploration or statistical investigation cycle (pose, collect, analyze, interpret).	What are you exploring? What are you looking at in your graph?
Relate to Context	7.15	Discuss own experiences or students' experiences related to the context.	I've never been to Carowinds, but I go to Busch Gardens a lot.
Insert Terminology	2.38	Introduce statistical terminology.	Officially that is called a scatterplot. You'll learn a little bit more about those later. It is where you are looking at two variables at the same time.

Provide Technical Assistance	4.76	Provide technical assistance – help student perform specific data move or use features of CODAP.	Click on that and remove it.
Clarify	2.38	Clarify directions or information	In case you don’t know, you’re supposed to be answering question number 8.
Focus on Case	2.38	Focus students’ attention on specific case.	Who is that?
Suggest Data Move	28.57	Suggest a data move.	I’m gonna throw a twist into your graph, and see if you guys can make sense of this. Ok? I want you to grab wood vs. steel. I think it is type. Grab type. Put it in the middle of your graph.
Link Multiple Dynamic Representations	2.38	Draws students’ attention to the dynamic nature of multi-linked representations in CODAP.	Did you notice that when you click on the case here in the other graph it shows up and in the table it will show up, too.
Notice Trends and/or Relationships	11.90	Encourage students to notice trends or patterns, which may include relationships between multiple attributes.	Ok, and do you think that it matters whether you are inverted or not and how fast you go?
Make Claim or Inference	9.52	Encourage students to make a claim or inference.	What states tend to have coasters that go really fast?
Explain Statistical Reasoning/Support Claim	7.14	Provide opportunity for students to explain their reasoning and/or support a claim/inference with evidence	What does the graph tell you?

Suggest a Data Move

Almost 30% of the orchestration types were identified as a suggest a data move. Within this type, we noticed two distinct themes, which resulted in different learning opportunities for students. An example of each kind will be illustrated below. The first shows the way a teacher interacted with Pair 3, a seventh-grade pair, whose scatterplot was previously shown in Figure 2.

Teacher 1: So, I’m gonna throw a twist into your graph, and see if you guys can make sense of this. Ok.
Student 1: Yeah.
Teacher 1: So, I want you to grab wood versus steel. I think it’s type. Here we go. Grab type. Put it in the middle of your graph.
Student 1: Yeah.
Teacher 1: Yep. What did it do?
Student 1: That’s pretty cool. It’s telling us right now which parts are wooden and which coasters are steel.
Teacher 1: There we go. Take a look at that, and see in a little bit if you could tell the class anything that you might notice that’s interesting.
Student 2: How about …
Student 1: A lot of wooden ones are slower and have a shorter drop, and the fastest ones are steel.

Pair 3 had already constructed a scatterplot while exploring the relationship between *drop* and *top speed*, and the teacher suggested that they drag and drop the attribute *type* in the center of the graph, coloring wooden coasters pink and steel coasters green. After the teacher suggested adding the categorical attribute to the graph, she followed up with a question that encouraged students to notice a relationship. Almost immediately, Student 1 was able to reason about the relationship between three attributes. When this teacher suggested a data move, she almost always followed it by a question encouraging students to notice a trend in the data or a relationship when exploring multi-variate data.

The following shows the way another teacher interacted with a different seventh-grade pair.

Teacher 2: Here, drag this a little bit so you can see. Where's your graph?
Student 3: Our graph is down here.
Teacher 2: [Takes control of the mouse.] Oh, ok, so what you can actually do is drag a category here and one right here so you can compare two things. So, compare, like, the max height to the max speed, see if they correlate.
Student 3: You have to create another graph though.
Teacher 2: No, you don't. Let me show you. Now you can see there's a trend, that has the height increases the speed increases.

In this instance, the teacher also suggested a data move. In contrast to the previous example, the teacher did not merely make a suggestion but took control of the mouse and created a scatterplot to show the relationship between *maximum height* and *top speed*. Rather than encouraging the students to notice a relationship between these two attributes, the teacher describes a positive relationship. We argue that this type of orchestration limited students' opportunity to reason statistically. It is plausible to infer that the students thought the teacher was suggesting that *maximum height* and *top speed* be graphed as dotplots on two separate graphs, since Student 3 indicated she thought they needed to create another graph. Perhaps these students may not have been ready to reason about the relationship in the way that the teacher suggested and ultimately constructed for them.

Insert Terminology

After the seventh-grade Pair 3, described earlier, had created a scatterplot (Figure 2), comparing *drop* and *top speed*, the teacher asked students what kind of graph they created. One student responded that it is a "spaceship", and the other student responded that it is an "aurora". The teacher then explained, "Officially that is called a scatterplot. You'll learn a little bit more about those later. It is where you are looking at two variables at the same time. So, what does that graph tell you?" As indicated in the section above, the students were able to reason about the relationship. While this only occurred one time, it provides an example of an appropriate way to introduce statistical terminology. Students were able to reason without knowing the name of the graph and learned new vocabulary. We conjecture that the second example in the previous section shows a way that using new terminology may have limited students' thinking. While we do not have evidence as to whether or not the students knew what correlate meant, we argue that using this terminology likely did not provide an opportunity to support students' reasoning.

Discussion

Our analysis of student pairs conducting an EDA using CODAP has provided evidence of how students make use of artifacts in CODAP to create instruments to answer meaningful questions of their own interest. We found that students who were able to transform the artifacts in CODAP to meaningful tools (i.e., going from unsystematic to systematic to expanding instrumentation) were able to pose and answer more robust questions that surfaced during EDA. All of the student pairs, except one, were able to move from using an unsystematic to a systematic approach to making sense of data. Two of the six pairs were even able to use an expanded approach that transformed features in CODAP into a more usable and powerful instrument that were used in a meaningful way in new situations. While this did not occur many times and for all student pairs, we hypothesize that this was likely due to the fact that this was students' first exposure to CODAP, as well as many students' first experience engaged in EDA. Nonetheless, this provides evidence that even students' initial experiences with using CODAP during EDA can support them in developing statistical reasoning as they make sense of data. An important implication for designing learning environments is that given an appropriate tool and well-designed task that uses real data, students can learn to use a tool while engaging in EDA. While teachers often acknowledge the affordances of using technology to support student learning, they sometimes argue they have insufficient time to do so. We suggest that these findings indicate that teachers do not need to teach students to use a tool first and then provide opportunities to engage in statistical thinking later.

Additionally, we found that students' interactions with teachers often impacted how they moved between *different* types of instrumentation. In some cases, students move from unsystematic to systematic approaches was preceded by an orchestration by the teacher. In fact, in all cases of students using expanded instrumentation, the approach was always preceded by an interaction with the teacher. We were not surprised that most orchestration types categorized as suggest a data move since this was students' first experience with CODAP. Nor were we surprised that merely suggesting a move and then the teacher making explicit their own conclusions about relationships between attributes limited students' opportunities to reason statistically. However, this work provides direct evidence of what we know anecdotally. Our findings indicate that at least one way a teacher can support students moving from unsystematic to systematic or systematic to expanding instrumentation is to explicitly encourage them to notice a trend or relationship. Further, this work shows that different orchestration types provided different learning opportunities for students to develop statistical thinking. Future work should examine this relationship between students' instrumentation and teachers' orchestration more closely.

In conclusion, we believe that providing opportunities for students to engage with well-designed tasks that use real, motivating data are fundamental aspects of designing learning environments that support students' statistical thinking. We also argue that providing opportunities for students to reason about data using dynamic statistical tools, like CODAP, is a fundamental component of learning environments that develop students statistical reasoning. Interactions with such technologies and teachers' orchestration impact learning opportunities for students.

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INTERPRETING WORKED EXAMPLES OF INTEGER SUBTRACTION

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Drawing on research around the utility of worked examples, we examine how 29 first- and 27 third-grade students made sense of integer subtraction worked examples and used those examples to solve similar problems. Students first chose which of three worked examples correctly represented an integer subtraction problem and used the example to solve a similar problem. Later, we presented only the correct worked example and had them solve another similar problem. Our results highlight how their initial ideas around which worked example was correct supported or constrained their later interpretation and use of the correct worked example. Students were attuned to the number of jumps shown in the examples; however, they sometimes misinterpreted the jumps' direction. Students' visual answers were correct more than their written answers, suggesting further attention to visuals could support students' reasoning.

Keywords: Number Concepts and Operations, Cognition, Elementary School Education

Students with strong schemas for a particular concept may be more resistant to changing these schemas, even with instruction; for example, upper elementary students who more frequently used a limited addition schema (i.e., A + B = C) had difficulty solving equivalence problems (i.e., A + B = ___ + D) even after being shown correct solutions to the problems (McNeil & Alibali, 2002). To help students revise their existing schemas, many studies described the use of worked examples as a support that can effectively promote middle-school students' increased conceptual understanding (Booth et al., 2013), especially for students with lower prior knowledge (Atkinson et al., 2000; Schwartz et al., 2016). Worked examples can engage students in productive struggle, drawing students' attention to relevant features in their current schemas or important underlying features when extending their schemas and highlight alternative ways of thinking about problems (Booth et al., 2015; Lange et al., 2014). Yet, students with lower prior knowledge might not know which features are relevant to pay attention to (Booth & Davenport, 2013; Crooks & Alibali, 2013). With integer operations, there are many features for students to pay attention to, and their use of them can vary greatly depending on their number schemas (e.g., Aqazade & Bofferding, 2019; Bofferding, 2019). In this study, we further explore how elementary students, who attend to different problem features based on their schemas, interpret integer worked examples and investigate how their prior knowledge schemas correspond to their application of worked examples.

Theoretical Framework

From a blended theory of conceptual change perspective (Scheiner, 2020), students' understanding of negative integers and operations changes through an interaction between their number schemas (e.g., a mental integer number line, Bofferding, 2014; see also Case, 1996, McNeil & Alibali, 2002) and pieces or features that comprise the schemas (e.g., order, value, symbols, operations, Bofferding, 2019; see also Booth & Davenport, 2013; Case, 1996; Crooks & Alibali, 2013). Students might solve 3 – 5 in many different ways, depending on their integer schemas and interpretation of the problem features. Some students might interpret the problem as 5 – 3 (Bishop et al., 2014; Bofferding, 2010) due to a positive integer schema that you cannot

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subtract a larger number from a smaller number (Karp et al., 2014) together with a flexible interpretation of the order of the features in the problem (i.e., a student could read the problem as three taken away from five). Students who understand that order matters in subtraction but have a strong whole number schema might pay attention to the features, start with three, and argue that the answer is zero or that you cannot take away more than three (Bishop et al., 2011, 2014; Bofferding & Wessman-Enzinger, 2017). Finally, students who have an integer schema might count back from three and answer negative two (Aqazade et al., 2016; Bishop et al., 2011).

Including negative numbers within the problems themselves may cause additional struggles for students. For example, when solving integer subtraction problems, such as -2 – 4, students need to distinguish the minus sign feature appended to the two (i.e., negative sign or unary meaning of the minus sign, Vlassis, 2004, 2008) from the minus sign feature between the two and four (i.e., subtraction sign or binary meaning of the minus sign, Vlassis, 2004, 2008). In fact, students who attend to subtraction signs and reason based on a whole number schema might ignore the negative sign or interpret it as an indication to subtract two (Bofferding, 2019). Other students might think the negative sign needs to be part of the answer and append it to the answer after solving 4 – 2, getting -2 (e.g., Bofferding, 2010; Bofferding, 2019). Students who know the order of negative integers, might still struggle with interpreting their value; therefore, they may vary in whether they subtract by getting numbers smaller in absolute value, counting toward zero and answering "2," or by getting numbers smaller in linear value and answering "-6" (Ball, 2013; Bofferding, 2019).

When analyzing worked examples, students with particular schemas might look for particular features that align with their schemas in order to make judgments about why worked examples are correct or not or to apply ideas from a worked example to a similar problem that they need to solve themselves. In this study, we add to previous literature by focusing in particular on the ways that elementary students interpret and use integer subtraction worked examples, highlighting what features (e.g., number of jumps, direction) students use and their reasoning when making use of the worked examples. Our research questions include:
How do first and third graders make use of integer subtraction worked examples?

1. How do they interpret and determine a correct worked example?
2. Which features are important to them when they try and solve a similar worked example?

Methods

Participants, Setting, and Data Sources

Twenty-nine first-grade and 27 third-grade students from a public elementary school in the midwestern United States with 9% English-Language-Learners and 46% economically disadvantaged participated in this study. As a part of a larger study, students completed two worked example tasks involving integer subtraction about one month apart. The first worked example task included three potential solution strategies for 3 – 5 associated with a number path model: one correct (B) and two incorrect: 3 – 5 = 0 (C) and 3 – 5 = 2 (A) (see Figure 1).

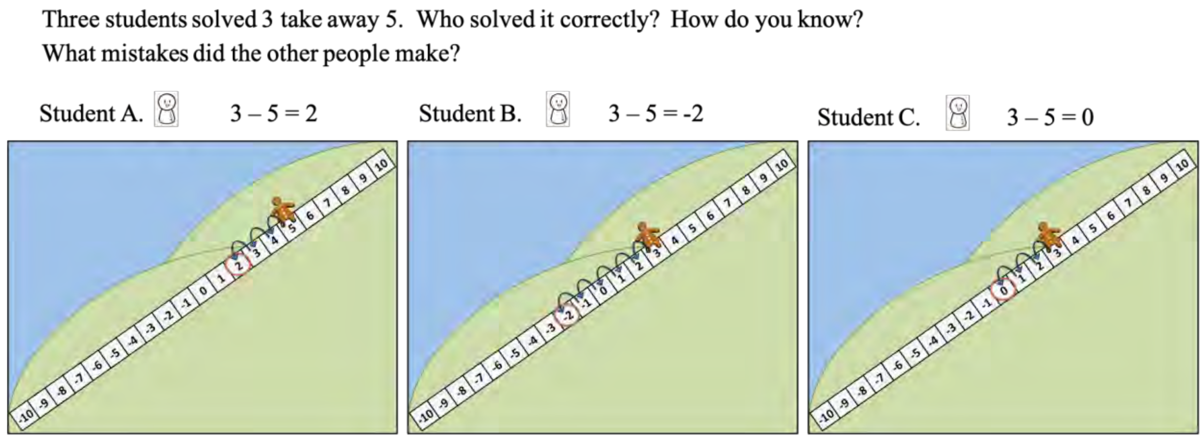


Figure 1: First Worked Example Task

Students were asked to choose the worked example that is correct, explain their reasoning, and describe the mistakes in the other two examples. Next, without receiving feedback on their response, we encouraged them to use that example to solve 1 – 4 and use an empty number path model to draw their solution (see Figure 2A). The second worked example task (see Figure 2B) showed students the correct solution for 3 – 5 = -2 illustrated on a number path. Next, students were asked to solve -2 – 4 and draw on an empty number path using this worked example.

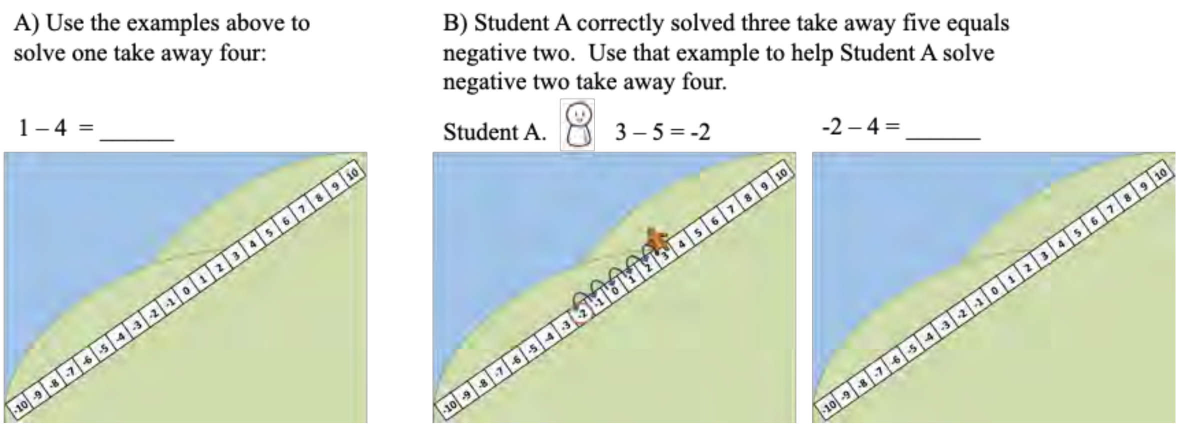


Figure 2: A: First Worked Example Task (cont.) and B: Second Worked Example Task

Data Analysis

On the first worked example task, we determined the number of students who chose A, B, or C as the correct worked example for 3 – 5. Next, we determined if they solved 1 – 4 by using a similar strategy to A from the worked example (i.e., reversed the order of the numbers and answered 3), B from the worked example (i.e., counted through zero and correctly answered -3), C from the worked example (i.e., stopped at zero and answered 0) or *other* (i.e., an answer not aligned with one of the three strategies presented in the first worked example or if their picture did not match their written answer). Then, we noted whether their strategies for both problems matched (e.g., did they answer zero for 1 – 4 if they had selected C for 3 – 5). Finally, we analyzed the explanation of their choice and identified elements or features they focused on, including number of jumps, starting number, ending number, or direction.

On the second worked example task, we calculated the number of students answering correctly and if their written answers matched with the number they landed on on the number path or the number they circled. To better understand their strategies, we analyzed their drawings based on the direction of jumps, starting number, number of jumps, ending number, and circled number. Lastly, we looked for any patterns between the first and second worked example tasks in students' solution strategies.

Findings

First Worked Example Task

The majority of students at both grade levels chose 3 – 5 = 2 (A) or 3 – 5 = -2 (B) as the correct worked example for solving 3 – 5 (see Table 1). When applying this example, overall, 17% of first graders and 44% of third graders correctly solved 1 – 4 as represented in their written answer; however, 38% of first and 48% of third graders showed the answer correctly through their drawing on the empty number path. Among these students, only 14% of first and 37% of third graders solved 1 – 4 correctly both on the written response and on the number path.

Table 1: Students' Choice of Correct Worked Example for 3 – 5 and Solutions to 1 – 4

		First Worked Example Task			
	First Graders (*n*=28) [a]	**1 – 4 → 3**	**1 – 4 = -3**	**1 – 4 → 0**	**1 – 4 → other**
(A)	**3 – 5 = 2** (*n*=9, 31%)	2 (22%)	4 (44%)	0 (0%)	3 (33%)
(B)	**3 – 5 = -2** (*n*=14, 48%)	0 (0%)	6 (43%)	0 (0%)	8 (57%)
I	**3 – 5 = 0** (*n*=5, 17%)	1 (20%)	1 (20%)	1 (20%)	2 (40%)
	Third Graders (*n*=27)	**1 – 4 → 3**	**1 – 4 = -3**	**1 – 4 → 0**	**1 – 4 → other**
(A)	**3 – 5 = 2** (*n*=7, 26%)	2 (29%)	1 (14%)	1 (14%)	3 (43%)
(B)	**3 – 5 = -2** (*n*=16, 59%)	1 (6%)	11 (69%)	0 (0%)	4 (25%)
I	**3 – 5 = 0** (*n*=4, 15%)	1 (25%)	1 (25%)	2 (50%)	0 (0%)
	Overall (*n*=55) [a]	**1 – 4 → 3**	**1 – 4 = -3**	**1 – 4 → 0**	**1 – 4 → other**
(A)	**3 – 5 = 2** (*n*=16, 29%)	4 (25%)	5 (31%)	1 (6%)	6 (38%)
(B)	**3 – 5 = -2** (*n*=30, 54%)	1 (3%)	17 (57%)	0 (0%)	12 (40%)
I	**3 – 5 = 0** (*n*=9, 16%)	2 (22%)	2 (22%)	3 (33%)	2 (22%)

Note. The table shows the number and percent of students who picked a particular worked example (i.e., A, B, or C) who also solved 1 – 4 in a particular way. For example, in the first row with the nine first graders who thought the worked example A (3 – 5 = 2) was correct, two of those students (or 22% of those selecting A), also solved 1 – 4 with an answer of 3. [a] One first grader did not choose a worked example for 3 – 5, so this student was not included in the data presented here.

Choosing A: 3 – 5 = 2 and Applying This Worked Example to Solve 1 – 4

Students' explanations for why they chose A as the correct worked example were often focused on the number of jumps or reinterpreting the subtrahend as being three (seeing the problem as 5 – 3), which was coupled with referencing the starting and ending number (or answer) in the problem. For example, Horse2$_{(3rd)}$ focused on the expected answer and the subtrahend as being three and said, "Took away three and they had two." Bat6$_{(1st)}$ chose to focus on the subtrahend as three and the starting number as five, "He's on five, and he only has to move three spaces." Likewise, when students then explained why B and C were not the correct worked examples for solving 3 – 5, they also pointed out the starting number, ending number, or the number of jumps. For example, Rabbit5$_{(3rd)}$ said, "He put a minus two" for B and she "did five minus three–one, two, three, so it's two." Robin3$_{(1st)}$ chose A because "he jumped three" and B was not correct because "he was supposed to jump three."

Out of 16 students who chose 3 – 5 = 2 (A), only four students' (two first and two third graders) responses to 1 – 4 were aligned with this choice. They made use of 3 – 5 = 2 the same way and applied it to their strategy for 1 – 4 to answer 3. Interestingly, five students (four first-grade and one third-grade) solved 1 – 4 correctly, which corresponds to 3 – 5 = -2 (B). One third grader's answer to 1 – 4 represented the incorrect 3 – 5 = 0 worked example as they answered 0. Even though Sheep6$_{(3rd)}$ wrote "3" and said, "One take away four," when drawing on the number path, her movements showed that she interpreted 1 – 4 as -1 + 4, perhaps because she misinterpreted the direction of the arrows on the examples: "Because they're subtracting one minus four, and I thought you started at the minus one and you go up to the three like those two [options B and C]." Finally, five students' solution strategies for 1 – 4 were not aligned with their choice of 3 – 5 = 2 and indicated either making an exact copy of 3 – 5 = 2, doing 1 + 4, putting the number of jumps as the answer (i.e., 4), or jumping the wrong number on the number path. We classified these types of responses as *other*.

Choosing B: 3 – 5 = -2 and Applying This Worked Example to Solve 1 – 4

Similar to students who incorrectly chose A as the correct worked example for 3 – 5, students with the correct choice of B also referred to the number of jumps, starting number, and ending number when explaining why A and C were incorrect worked examples and B was correct. For instance, Robin4$_{(1st)}$ counted the number of jumps to justify her choice, "One, two, three, four, five" and for C, said, "It didn't get far enough." Duck3$_{(1st)}$ explained why A was not the correct worked example, "They started on the five and landed on two." He, for B, counted from -2 to 3 and confirmed it was 5 and said, "I think it would be in the minuses." Finally, for C, he referred to the answer, "It's not on the zero, it's not on the minus."

Out of 30 students with the choice of 3 – 5 = -2 (B), 17 students' (six first and 11 third graders) responses to 1 – 4 were aligned with this choice. They correctly made use of the 3 – 5 = -2 worked example, wrote -3 for 1– 4, and correctly showed their solution on the number path. Nine students (six first and three third graders) made use of the 3 – 5 = -2 worked example to some extent when solving 1 – 4. Some of them started at an incorrect number (e.g., 0 or 3) but counted backward the correct number of jumps to get into the negative numbers. Some other students answered 1 – 4 correctly when using the empty number path but said the answer was three and wrote three on their paper. Only one third grader—Goat2$_{(5th)}$—despite choosing B for the correct worked example of 3 – 5, answered 3 for 1 – 4 seeing it as 4 – 1. Two first graders' responses to 1 – 4 did not align with their choice of B because they started at an incorrect number and jumped an incorrect amount. Finally, one third grader skipped this problem.

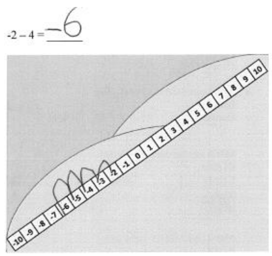
Choosing C: 3 – 5 = 0 and Applying This Worked Example to Solve 1 – 4

Interestingly, rather than primarily referring to the number of jumps, students who chose C for the correct worked example of 3 – 5 often focused on the starting or ending number. As an example, Goose9$_{(1st)}$ rejected A and B because their answer was not zero. Duck1$_{(1st)}$ used his fingers to take away five from three and, similar to Goose9$_{(3rd)}$, thought A and B were incorrect worked examples because “It only gets two” or “Three minus five is not two, it’s zero.”

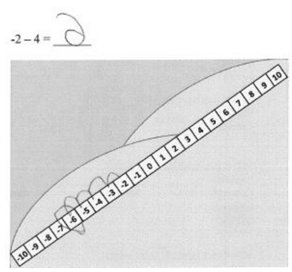
Out of nine students choosing 3 – 5 = 0, only two third graders and one first grader applied the same strategy when solving 1 – 4 and answered zero. One first and one third grader answered -3, corresponding to the correct 3 – 5 = -2 worked example. One first and one third grader solved 1 – 4 as 4 – 1 reflecting the incorrect 3 – 5 = 2 worked example. One first grader—Bat5—used her fingers to count and answered 0. However, when drawing on the number path, she correctly showed -3 as the answer for 1 – 4. Lastly, one first grader did not provide any answer for 1 – 4.

Second Worked Example Task

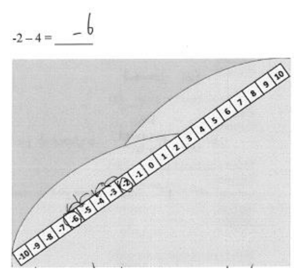
On the second worked example task, 33% of first and 54% of third graders correctly solved -2 – 4 on the written response (43% overall), but even more students solved it correctly using the empty number path (52% of first and 65% of third graders; 58% overall). In fact, 43% of students (33% first and 54% third graders) correctly answered on both the written response and the number path because if they wrote the correct answer, they also illustrated it correctly. Among these students, seven (four first and three third graders) had chosen 3 – 5 = 2 (A), 13 (four first and 11 third graders) had chosen 3 – 5 = -2 (B), and three (one first and two third graders) had chosen 3 – 5 = 0 (C) in the initial worked example task. These students’ drawings on the empty number path indicated that many correctly identified the important features of the 3 – 5 = -2 worked example including the starting number and number of jumps to use when solving -2 – 4 on an empty number path (see Figure 3 for examples). However, many did not show the ending number (or answer) by circling it or circled both the starting and ending numbers. Some of them did not show the directional movements on their jumps. Thus, from only drawings, it was not clear where the starting and ending numbers were and which direction they jumped. In fact, Rabbit3$_{(3rd)}$ only put a mark on -6 and explained, “I went to four [-4] and one, two [referring to the jumps],” and she ended at -6.



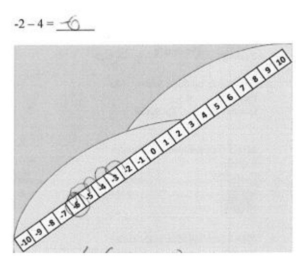
Duck2$_{(1st)}$ showed non-directional jumps and did not circle the answer.



Goose3$_{(1st)}$ showed non-directional jumps and circled the answer.



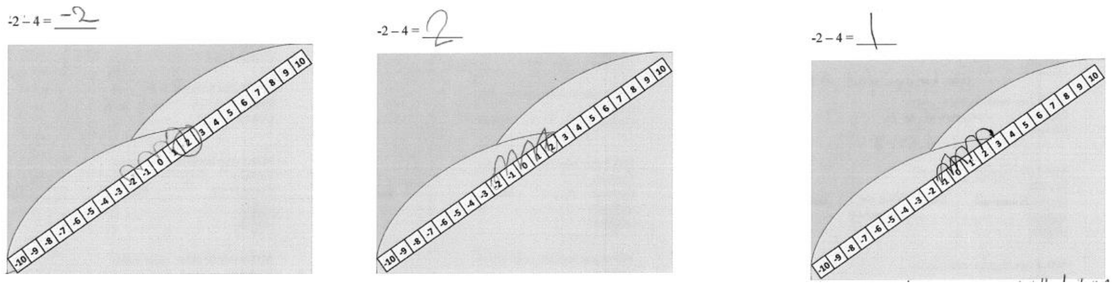
Horse4$_{(3rd)}$ showed directional jumps and circled the start and answer.



Horse8$_{(3rd)}$ showed directional jumps and circled the answer.

Figure 3: Examples of Students Solving -2 – 4 Using an Empty Number Path

Of those students who chose 3 – 5 = 2 (A) on the first worked example task and did not answer -2 – 4 correctly, a few answered two–solving it as 4 – 2—or negative two—solving the problem as 2 – 4 (see Horse9$_{(3rd)}$ in Figure 4). Their drawings on the empty number path demonstrated that they often started at an incorrect number (e.g., copying the exact 3 – 5 worked example and starting at 3) or jumped in an upward direction (see Figure 4 for more examples).



Horse9$_{(3rd)}$ started at 2 and jumped downward.

Goose4$_{(1st)}$ jumped in an upward direction.

Bat6$_{(1st)}$ started at 3 as in the example but jumped 6 (using the 2 and 4) by going down and up.

Figure 4: Examples of Students Solving -2 – 4 Incorrectly

One first grader—Robin3—correctly made use of the worked example when using the empty number path, saying, "Start up at negative two, and three, and four, and six [ending at -6]" but wrote down six as his answer. Among students who chose 3 – 5 = -2 (B) on the first worked example task, seven (four first and three third graders) also correctly made use of the second worked example task when using the empty number path but did not answer correctly on the written response. For example, Robin2$_{(1st)}$ said, "Take away four," drew four jumps between -2 and -6 on the number path, but said, "It's six" and wrote "6."

Other common answers of -2 – 4 among the students with the choice of B included -2, 2, and 0. Students' strategies for -2 – 4 as shown in their drawings on the empty number path often resulted in an incorrect answer because they started at 2 and jumped downward 4 to get to -2 or started at -2 and jumped 4 upward to get to 2. An interesting example was Goose6$_{(1st)}$ who started at -2 and made jumps to 4. She then counted the jumps (or distance) between -2 and 4; however, she did not take account of the direction and answered 6.

Some of the students who chose 3 – 5 = 0 I on the first worked example task and were incorrect on -2 – 4 responded 0 or 2. For example, Bat5$_{(1st)}$ said, "Of course I'll have to start at 2" but actually started at -2 on the number path and justified starting at -2 because, "It showed me (pointing to the -2 in the problem)." Then, she made four jumps upward, "One, two, three, four" and said, "It equals two." After she wrote 2, the interview asked her why she went up, and she referred back to the worked example, misinterpreting it by explaining, "Because this one, it says 3 – 5 equals negative two, and I saw that you had to go up instead of down." Goose9$_{(1st)}$ answered 0 on the written response because she used her fingers; she held up two fingers and then put them down when trying to take away four. To model this, on the empty number path, she correctly started at -2 but then jumped upward twice and stopped at 0.

Discussion and Implications

When given the option to choose which worked example correctly illustrated 3 – 5, students' inclination to choose 3 – 5 = 2 (A) indicates that about a third of the students had a strong prior schema for subtraction as subtracting a smaller number from a larger one (i.e., 5 – 3) (e.g., Bishop et al., 2014; Bofferding, 2011; Murray, 1985). Their whole number subtraction schema was strong enough that even when presented with the correct example (worked example B; 3 – 5 = -2), they did not determine it as a match. However, when then asked to solve 1 – 4, a few of these students—especially first graders—were able to answer correctly by starting at the correct initial number. These results suggest that introducing negative numbers as a result of subtraction (i.e., 3 – 5 = -2) could support students in developing a directional interpretation of subtraction and weaken (or eliminate) the schema that you can only subtract a smaller number from a larger one.

Students often focused on the number of jumps but, especially on the first worked example task, aligned their interpretation of jumps to how they viewed the problem (e.g., if they interpreted the problem as 5 – 3, they talked about it as having three jumps); sometimes they misinterpreted the direction of the jumps in the worked example, which was more prevalent with the second worked example task (-2 – 4). Students might have counted from -2 to -1, 0 and so on instead of -2 to -3, -4, and so on to align with interpretations of subtraction as getting smaller in absolute value (because it wouldn't make sense for them to go in a direction where the numbers were increasing in absolute value). Thus, students might need more experience interpreting and using number path and number line visuals, which could support their developing understanding of integer order and values and help those students who primarily relied on using their fingers and thought the answer to the first worked example task was zero.

In our previous work, we found that many students would solve integer subtraction problems by ignoring the negative signs, subtracting the number with smaller absolute value from the number with larger absolute value, and then append a negative sign to their answer (e.g., Aqazade et al., 2018); in this case, students would solve -2 – 4 as 4 – 2 = 2 and then make the answer -2. However, we did not see any students use this strategy, suggesting both their focus on the jumps and use of the worked example visuals helped them avoid this misinterpretation.

Overall, encouraging students to make sense of and use the integer worked examples provides opportunities for productive struggle and potential to resolve those challenges over time. Particularly, such encouragement in using the visual as presented in the worked example did help the students because they had higher performance on the visuals than when writing numerical answers. Part of the difference between the two formats is that students who were not familiar with negative numbers did not include the negative sign in their written answers, even if they landed at a negative number. Therefore, the tasks also revealed what elements or features students interpreted as important and provided insight into their number schemas. Further, our work adds to our understanding of the usefulness of worked examples (e.g., Booth et al., 2013; Booth & Davenport, 2013); by the second subtraction worked example task, the first graders' performance was closer to the third graders, so the worked examples seemed to help the novices begin to make sense of the problems.

Acknowledgments

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POSITIVE OUTCOMES OF A STUDENT'S STRUGGLE WITH NEGATIVES

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Operations with negative numbers are taught to students several years after solely dealing with positive numbers. However, only one unit of one year is devoted to learning and becoming proficient in operating with both negative and positive numbers. This particular study reports on a student who has been taught the concept of negative numbers but not formally introduced to strategies or rules for operations. During clinical interviews, the student was given open number sentences and asked to explain how she would solve for missing values. Her explanations reveal that working with adding and subtracting negative numbers was a form of productive struggle that had potential to build connections and illuminate mental inconsistencies. These type experiences could be beneficial for both students and teachers.

Keywords: Instructional Activities and Practices; Middle School Education; Number Concepts and Operations

Introduction

Working as an 8th grade math teacher, I noticed that my students' experiences with the algebra content was greatly shaped and influenced by their ability to accurately operate with integers, which is a concept addressed in a 7th grade standard (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). Many of my students did not feel confident and often misapplied integer rules; they would tell me that they are just "not good with negatives". Research into student learning of integers has consistently found that operations with negative numbers is a difficult concept for students (e.g., Bishop et al., 2014; Bofferding et al., 2018; Chiu, 2001; Prather & Alibali, 2008). So why is it that students spend most of their elementary years developing a robust understanding of operations with positive numbers, but then only a couple of months on operations with negative numbers? Negative numbers are introduced in the 6th grade curriculum (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010), but as one 6th grade teacher told me, "I make sure to not assign problems with negative number operations, because they don't learn those until 7th grade". As a response, I thought why not? Could students reason through such problems before formal instruction but after being introduced to negative numbers? How would they reason and struggle through it? And most importantly, what benefits could arise from these struggles?

Research conducted by Laura Bofferding and colleagues (Bofferding et al., 2018) pointed to the idea that students as young as first grade can invent their own notations for negative numbers and use it to work through adding and subtracting positive and negative numbers. Bishop et al. (2014) investigated how fourth graders, who had not been formally introduced to negative number in the classroom setting, used their knowledge of whole number arithmetic to solve problems with negative numbers. However, I found few studies that explored how students who *had* been formally introduced to the concept of negative numbers in the classroom, but not yet to integer operations could reason about arithmetic with this newly learned concept**.** This gap in research led me to work with 6th grade students to conduct an exploratory qualitative study to investigate the type of thinking that the students would demonstrate when asked to work with integers. The research question for my study was:

How do these 6th grade students, who perform at grade level, reason about the addition and subtraction of negative number before receiving formal instruction?

Conceptual Framework

Theoretical Framework

I situate my study under the theory of constructivism. Under this theory, learning is an active process of constructing a system of concepts, the smallest unit of which are referred to as schemas (Skemp, 1987; von Glasersfeld, 1995). This construction of knowledge occurs at the individual level; the learner organizes new information into already existing schemas, assimilating to build new knowledge, or reconstruct previous schemas in order to fit new information (Skemp, 1987). This reconstruction can happen as a result of cognitive dissonance, which is when one encounters a piece of information that does not fit into previous schema (von Glasersfeld, 1995). I view this cognitive dissonance as an example of productive struggle, where students engage in "some perplexity, confusion or doubt" (Dewey, 1933, p. 12).

In this study, my assumption is that each student has their own unique set of schemas and constructs for the concepts of numbers. I will use the phrase formal instruction to refer to formal instruction of integer addition and subtraction.

Student Reasoning

Reasoning is a fundamental aspect of mathematics and the main construct in my study. I define reasoning as the process of mental actions involving the use personal mental schemas to solve a problem (Piaget, 1970). I look to research that studied students without experience of formal, classroom instruction on integer arithmetic, particularly exploring their thinking and reasoning on tasks via interviews. Lamb et al. (2018) conducted an investigation of student reasoning with integers and integer subtraction and addition across multiple grades (2, 4, 7, and 11) using clinical interviews and a constructivist theoretical lens. They previously developed and refined a framework which categorized solutions and views of numbers as order-based, analogy-based, formal, computational, or emergent. After analyzing interviews of 160 students, researchers saw that the type of reasoning used varied across those students without formal instruction in grades 2 and 4 and students right after instruction in grade 7. Specifically, in students without formal instruction, the researchers saw that students applied knowledge of addition and subtraction of positive numbers to solve problems with all negatives. For example, so solve -9 + -1, student would state that since 9 + 1 is 10, -9 + -1 would be -10.

Another way that reasoning has been studied is through metaphors of addition and subtraction. Kilhamn (2018) studied three metaphors used in arithmetic reasoning: measurement, motion along a path, and object collection. She observed how these metaphors appeared in formal instruction of the concept of subtraction. The measurement metaphor was defined as students viewing numbers as lengths of segments. Motion along a path was viewed as interpreting numbers are locations and operations as movement along the locations. She categorized the idea of comparing sets of numbers and phrases such as smaller number taken from larger number as examples of object collection metaphor. These metaphors apply to arithmetic with positive numbers and therefore could be constructed by individuals based on their previous experiences.

Methodology

Setting and Participants

There were two participants in the study. However, in this brief research report, I will focus only on Julia (pseudonym) because she demonstrated reasoning that was not using algorithms or rules. Julia attended a public middle school for grades in Northeastern USA. I chose grade six because it is the grade where students are introduced to negative numbers but not yet shown strategies for operating with them.

Data Collection

Two clinical interviews (Ginsburg, 1997) were conducted with each participant. Since the main goal of this study was to explore students' reasoning, clinical interviews were appropriate because this methodology is used to "gain insight into many aspects of the children's thought" (Ginsburg, 1997, p. x). The interviews were conducted virtually through Zoom due to schools and public places being shut down during the Covid-19 pandemic.

In the first interview, I asked general open-ended questions (Ginsburg, 1997) about integers, such as "how would you describe negative numbers?" and then open number sentence (Bishop et al., 2018) questions such as $-8 + 12 = \square$. In the second interview, the number sentence questions had the missing number in different places, such as $-5 - \square = -14$. A total of 13 number sentence questions were asked in the two interviews. Using the virtual whiteboard feature on Zoom, I asked the participants to visually demonstrate their thinking. I used their written work as a secondary data source.

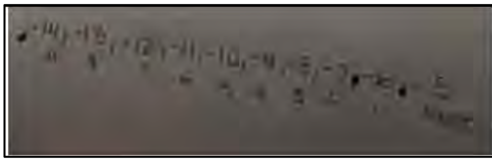
Data Analysis

After I transcribed the interviews, I conducted an initial round of open coding (Corbin & Strauss, 1990), coding for the indicators of definitions and descriptions of negative number, as well as my interpretation of the participant's explanation. My analytical framework was inferred by the Ways of Reasoning framework developed by Bishop and colleagues (Bishop et al., 2014) and Kilhamn (2018) and then evolved as I worked through the data. I looked for codes that were indicators of the concept of negative/positive numbers, such as statements about negatives to the right of zero or viewing numbers as lists. Statements about reasoning such as breaking into parts or jumping to zero were indicators of the types of reasoning. Lastly, strategies such as counting by 1s or moving on a number line were indicators of arithmetic metaphors. I grouped another set of themes based on my interpretation of how the participant thought of the addition/subtraction: increase/decrease sequentially or combining groups. During this process, I re-watched the recordings to ensure consistency between my notes, the transcript, and my memory. To ensure validity, I triangulated the data from transcripts with the data of participant's written work. Additionally, I presented two pieces of data to other research colleagues for feedback on my interpretations.

Results

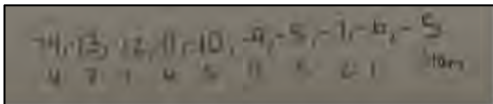
In this section, I report on my findings regarding the way that Julia approached the open number sentences. I chose to showcase Julia because she did not use any computational rules or procedures to solve the tasks but rather seemed to be using strategies from positive number operations.

Julia's main way of reasoning to determine the missing values in the number sentences included writing out numbers as a list and counting on. The lists were similar to a number line, but distinct in that the order was not always consistent; reading left to right, the order of the list was decreasing for two problems and increasing for the rest. However, when asked about how she visualized positive and negative numbers, Julia drew a number line. When using lists of numbers, Julia started with the first value given and then sequentially by 1s counted up to the other value in the number sentence in solving -5 – □ = -14 (see Fig 1). She then counted how many numbers it took to get to the final value and gave that as the answer, similar to the measurement metaphor (Kilhamn, 2018). The noteworthy finding was that Julia did not change her approach when the task was slightly changed to -5 + □ = -14 (see Fig 2).



R: So walk me through how you got 9.
J: Okay so I did a number line. I started at negative 5 on the right and I went all the way back to negative 14. I started at negative 5 and counted it to negative 14 and I got 9.

Figure 1: Julia's explanation and visual for the task -5 – □ = -14



R: Show me how you worked out this problem.
J: So this is my number line. I started at negative 5, kinda like the same as the previous number. I went up to negative 14 and got 9.

Figure 2: Julia's explanation and visual for the task -5 + □ = -14

My interpretation is that it seems Julia is not attending to the operation of addition versus subtraction, but instead focusing on the values, with the missing number representing the amount of numbers between the given values. Julia did not seem conflicted about her answers until I pointed out the fact that she answered 9 for both tasks, to which she changed her answer for -5 – □ = -14 to -9 because " maybe instead of adding 9, subtract negative 9 to get -14".

Discussion

The experience of the two tasks reported in the results can be viewed as Julia engaging in productive struggle as defined by Hiebert & Grouws (2007) – when students "expend effort to make sense of mathematics, to figure something out that is not immediately apparent" (p. 387). During this struggle, Julia's reasoning is illuminated so it could be interpreted for any hindrances or obstacles. After these tasks, Julia has the potential to make connections between the connection of adding negatives and subtracting positives. Despite not being formally introduced to strategies for operations with integers, Julia was able to apply her reasoning to solve problems with negative numbers. This finding is consistent with Bofferding et al. (2018) and Bishop et al. (2014) who saw that young learners can conceptualize negative numbers.

The main implication from the presented results is that giving a student the opportunity to reason about integer arithmetic prior to teaching him/her the commonly used strategies can provide access to student thinking that may both inform the teacher and motivate the student. The experience may illuminate any mental inconsistencies the student has; activities can be planned that use student thinking as a starting place, such as looking at similarities and difference between operations. Such experiences would allow for student engagement in ways that applying integer rules cannot support. For example, true/false number sentences can be used to develop and challenge students' conception of equality with both positive and negative numbers (see Carpenter et al, 2003 for more).

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AN EXPLORATION INTO CHILDREN'S THINKING ABOUT LEARNER-GENERATED INTEGER DRAWINGS

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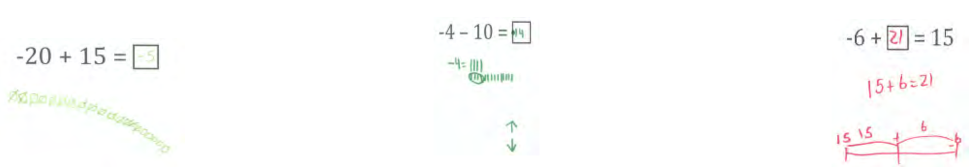
Learner-generated integer drawings are representations for integer addition and subtraction created by children. We present a study where six grade 5 participants examined and unpacked other children's learner-generated integer drawings for integer addition and subtraction. Themes that emerge included: (a) Participants' initial reasoning often did not align to the drawings; (b) and, without prompting the participants critiqued the drawings. Because participants' reasoning differed from the drawings presented and they critiqued the drawings, this offers potential for learner-generated drawings as a pedagogical tool.

Keywords: Cognition; Elementary school education; Mathematical representations; Number concepts and operations

Children's thinking about integers is beautiful. It is beautiful in the sense that children extend and enhance their whole number reasoning with negative integers (Bofferding, 2014; Bishop et al., 2014); they invent their mathematics with negative integers as young mathematicians. Children create unique drawings (i.e., learner-generated drawings; van Meter & Garner, 2005) when they solve integer addition and subtraction problems for the first time (Wessman-Enzinger, 2019a, 2019b). Integer learner-generated drawings (van Meter & Garner, 2005; Wessman-Enzinger, 2019b) are constructions that children create for solving integer addition and subtraction problems. The integer learner-generated drawings are a type of representation for integers that is created by the children, rather than teachers or researchers. An important pedagogical consideration is how children make sense of other children's learner-generated integer drawings. The following research question guided this work: *How do children make sense of other children's drawings for integer addition and subtraction?*

Theoretical Framing: Learner-generated Integer Drawings

Learner-generated integer drawings can exhibit children's thought processes as they work through solving unfamiliar problem types (Wessman-Enzinger, 2019a). Children create a variety of different types of drawings for integer addition and subtraction: single set of objects (e.g., Figure 1a), double set of objects (e.g. Figure 1b), number sequences, empty number lines (horizontal or vertical; see, e.g., Figure 1c), number lines (horizontal or vertical), number sentences (horizontal or vertical; see, e.g., Figure 1c), emphasis on plus, minus, and negative signs (see, e.g., Figure 4 below where a minus signed is changed to a plus sign).



(a) (b) (c)

Figure 1: (a) Single set of objects, (b) double set of objects, and (c) empty number line.

Learner-generated drawings for integers vary from typical integer instructional models (Wessman-Enzinger, 2019a, 2019b). Consider the empty number line in Figure 1I. The negative numbers are on the right-hand side and positive numbers on the left-hand side of the number line; this differs from traditional number line instructional models where negative numbers are on the left-hand side of the number line (e.g., Nurnberger-Haag, 2007, 2018; Saxe et al., 2013; Stephan & Akuyz, 2012).

As teachers and researchers, if we wish to support students' constructions and representations in classroom discourse, it is first necessary to consider the ways in which children invent, use, and make sense of their own ***and others' drawings***. This study, therefore, aimed at extending the prior work that has been done with learner-generated integer drawings (e.g., Wessman-Enzinger, 2019a, 2019b) by studying how children make sense of other children's drawings. It is critical to understand how children make sense, not only of models of integers, but other children's learner-generated integer drawings. Insight into how children make sense of these drawings will offer insight not only into research on children's thinking about integers, but also understanding of the potential these drawings have as instructional models or representations for integers.

Methods

Six grade 5 participants (Lucy, Maggie, Evan, Hudson, Megan, Estrella) volunteered to participate in three, structured task-based interviews (Goldin, 2000) in May and June 2020. We met virtually on Google Meet for the interviews for 45 to 60 minutes and some children could not participate in all three interviews. All children participated in the first interview (n = 6). One child did two of the three interviews (Evan); three children did all of the interviews (Hudson, Megan, Estrella)—resulting in a total of thirteen interviews.

For the interviews, we created tasks by using actual learner-generated integer drawings and we tried to vary the types of drawings we used (e.g., single set of objects, double set of objects, empty number line). Figure 2 shows a sample task from interview 1: *Students solved problems with negative numbers for the first time and you will see their solutions and drawings. Help us make sense of the student's thinking. Share if you agree or disagree with anything* (see, e.g., Figure 2).

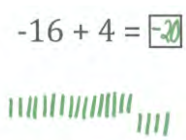


Figure 2: Sample task from interview 1 that uses a double set of objects.

Figure 3 shows a task from the second interview with the following directions: *Students worked on problems. Sometimes they were unsure. The students shared different solutions they thought might work. Sometimes they determined correct solutions; sometimes they did not. Sometimes none of their solutions were correct.* We asked the children to explain the drawing and if the student changed their mind correctly or incorrectly.

Figure 3: Sample task from interview 2 with an empty number line drawing highlighting a student changing their mind from 3 to -1.

The third interview consisted of showing the participants two drawings, where one was correct and one was incorrect. We provided the following directions: *We are going to show you students' solutions, drawings, and work again. The students solved open number sentences (e.g., 7 - __ = 9; ___ + 3 = -10). They found the number that goes in the box. We will play: This or That? One of the solutions is correct and one is incorrect. Help us!* We asked the children to select "this or that" as the correct or incorrect solution and describe the drawings.

For data analysis, we transcribed all of the interviews, took notes, and engaged in open coding (Corbin & Strauss, 2015; Teppo, 2015). As part of open code discussion, we drew on our previous discussions, the transcripts, and notes. As we identified themes, we created intermediate codes, where we started writing out the themes and revisiting the transcripts. We then coded all of our units of analysis (i.e., transcripts of student responses) and negotiated differences (coded 103 units of data related to making sense drawings). In this research brief, we will discuss two of the themes we found: reasoning and critique of drawings.

Results: Reasoning

There were three types of reasoning that our participants used when they were presented with the tasks: (1) using the drawing explicitly, (2) using reasoning related to the drawing, or (3) using unrelated reasoning.

When using the drawing explicitly, participants directly referenced what they saw in the drawing to aid their reasoning (8% of units of coded). When using reasoning related to the drawing, students did not explicitly state that they were referring to a specific part of the drawing, but the language they used aligned with part of the drawing in some way (39% of units coded). When using unrelated reasoning, students expressed their thinking, used examples, or shared analogies that did not align to anything seen in the drawing (53% of units coded).

When shown the open number sentence -6 + • = 15 with a number line drawing, our participant Hudson said "as you can see in the drawing" and reference the movement on the number line with an explicit reference. When shown the open number sentence, -4 – 10 = • and drawings with tallies in Figure 1b (above), our participant Evan began talking about the idea of numbers being leftover. Although he never explicitly referenced the drawing, his reasoning aligned with the drawing because the drawing also had the idea of leftovers, as referenced by the circled tally marks in the drawing that were leftover. For the category of using reasoning not related to the drawings, our participant Megan was shown the task below (see Figure 4) and started using an analogy of owing her mom money. Although a helpful analogy, her reasoning was not related to the drawing because there is nothing in the drawing that indicates debt or money.

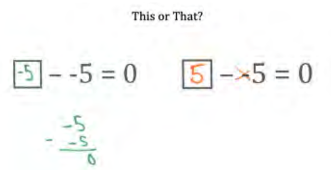


Figure 4: Vertical number sentence (left) and symbolic markings (right).

Results: Critique on other Students' Drawings

Another theme of examining others' drawings is that our participants critiqued the drawings they were shown. Critiquing other students' drawings occurred in 20% of the units of data. When our participants did not comment on the drawings and interacted with them without making additional commentary on the design of the drawing itself, we did not consider this a critique (80% of the units of data).

Evan, when examining the drawing in Figure 1c (see above) for -6 + • = 1 5, critiqued this number line drawing, with negative integers non-traditionally placed on the right-hand side, Evan stated, "I think it [reference to -6] should be switched with the 15," he suggested moving -6 to the left side of the number line and moving the 15 to the right-hand side of the number line, a more traditional approach to ordering numbers on a number line. He stated, "I think it just maybe makes it a little more confusing." When he referenced the negatives on the right-hand side of the number line as confusing, he offered a critique of the drawing.

Lucy, when examining the drawing in Figure 5 for -11 – -2 = •, bot h critiqued and affirmed the drawing presented to her. Lucy explicitly critiques the number sentence and offers a different interpretation; she also affirmed the vertical number line and offers how it provided her a new way of thinking. Lucy noticed that -11 – -2 in the problem presented does align with -11 + -2, which the student wrote:

> I would do different with that problem is so I would take away the negatives so that it would be less confusing and I would change it back so I would show, so that means like I would show that is also another way to solve negative 11 minus negative 2 instead of showing that negative 11 plus negative 2 gets you negative 13. …

Because we considered number sentences that were learner-generated to be drawings as well, we considered this to be a critique of the drawing. In a similar way, Lucy also noticed that the number line created by the student is something different than what she would create (e.g., "I actually wouldn't think of it like that…); but, she affirmed that it can be solved that way and provided her a different way of thinking about integers.

> So it looks like there is a number line going up and down which I actually wouldn't think of it like that I would think of it across but I like the way they thought of that, um it does make a little bit more sense than going crosswise.

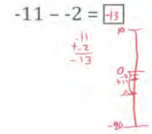


Figure 5: Problem presented to Lucy in the first interview.

Concluding Remarks

The themes described here highlight ways that children engage with drawings they did not produce. A noteworthy take-away is that children may not prefer using the drawings directly; however, their reasoning may be related to the drawings. Using learner-generated drawings may offer new insights or ways of thinking to a child that they had not previously been considered.

Critiquing others' reasoning is an important part of doing mathematics (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). In this study, children critiqued the learner-generated drawings without being prompted to do so. When students critique drawings, it invites the possibility for teachers to leverage that critique in a follow-up task wherein, students are asked to make their own drawings.

Acknowledgements

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EL MODELO DE LA RECTA NUMÉRICA COMO ESPACIO DE REPRESENTACIÓN HOMOGÉNEO

THE NUMBER LINE MODEL AS A HOMOGENEOUS REPRESENTATION SPACE

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En el presente trabajo se investiga el tránsito de los signos desde índices hasta símbolos en tareas de desigualdades numéricas e inecuaciones lineales, usando el modelo de la recta numérica. En un estudio de caso con una estudiante que tiene buen manejo de la numerabilidad, encontramos que existen complejidades en el uso de los signos y otra de tipo lógico en la interpretación de indicaciones simultáneas cuando el modelo es una representación homogénea.

Palabras clave: Conceptos numéricos y operaciones, cognición, educación secundaria, representaciones numéricas.

Antecedentes

El modelo de la recta numérica ha sido usado como recurso didáctico del que se han destacado distintas propiedades que van desde la medida hasta la operación. Este modelo se usa con o sin marcas, como una guía, Diezmann et al. (2006, 2010), o se abunda en el orden sin considerar la distancia, Teppo et al. (2013).

Aquí consideramos a la recta numérica como un espacio de representación homogéneo, Nemirovsky (2003) donde la posición de los signos es su única propiedad, para abordar tareas de desigualdades numéricas e inecuaciones lineales. En particular, analizamos el papel de las marcas y números enteros respecto al orden, posición relativa y ubicación espacial con carácter numérico. En este entorno los segmentos unitarios son la base del modelo tanto numérico como espacial.

Marco referencial

Desde el punto de vista de Radford, en la teoría de la objetivación el saber no puede ser algo de lo que podamos apropiarnos o lo que podamos poseer, sino que "es un proceso de elaboración activa de significados" (Radford 2006[a], p. 116).

La objetivación está ligada al uso de instrumentos, signos o artefactos que no son ayudas para el aprendizaje, sino medios cuya presencia y uso imprimen un sello distintivo sobre lo re-construido a través de una mediación semiótica, desarrollada en una praxis reflexiva (Radford, 2004, p. 13)

Para establecer las funciones de los distintos tipos de signos usados en las tareas planteadas vamos a considerar que los segmentos pueden ser tanto índices como símbolos matemáticos, Pierce (2005) dependiendo del uso que le dé el estudiante.

Finalmente, en este trabajo la recta numérica será un artefacto regulado por las propiedades de orden, posición relativa y ubicación espacial donde los números, las marcas e incluso los segmentos unitarios son parte de la estructura semiótica del modelo.

Metodología

El objetivo de este estudio es indagar el uso semiótico del modelo de la recta numérica para la interpretación de las desigualdades numéricas e inecuaciones lineales del tipo $x < b$ o $x > b$ con x y b enteros, a través de un enfoque cualitativo que produce datos descriptivos, apoyándose en un estudio de caso llevado a cabo en condiciones de cuarentena durante el año 2020, cuyos instrumentos fueron un cuestionario a través de GeoGebra y una entrevista – intervención vía Zoom que propició una práctica reflexiva en el caso presentado.

La participante es una estudiante de octavo grado con 14 años del Instituto Genaro Muñoz Hernández de Siguatepeque, Honduras, quien a lo largo de este estudio mostró tener un buen manejo de la estructura semiótica del modelo de la recta numérica, tanto de números enteros como de decimales con base en la numerabilidad previamente adquirida.

El cuestionario fue aplicado en dos partes, en la primera se plantearon desigualdades numéricas y en la segunda las inecuaciones, la entrevista se realizó en línea sobre las respuestas incorrectas o incompletas en la entrevista intervención, las cuales se centraron en 1. La localización de la zona solución de dos desigualdades numéricas y 2. La detección de la zona solución de dos inecuaciones simultáneas.

Resultados

En las respuestas sobre la localización de la zona solución, las marcas usadas originalmente por la estudiante eran inexactas, como en el siguiente ejemplo, donde se solicita que marque la zona de los números mayores que 5 y la zona de los números mayores que 7.

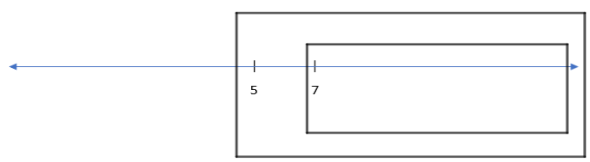

Imagen 1: Respuesta de la estudiante a la tarea

La respuesta que se dio por separado dio motivo al siguiente diálogo:

28. I. … ¿Algún entero cumplen las dos condiciones?
29. E: Mmm… (Silencio)
30. I: ¿Algún entero cumple que es mayor que 5 y es mayor que 7?
31. E: Sí
32. I: ¿Cuáles?
33. E: Todos los que siguen después del 5 y después del 7
34. I: Pero ¿6 cumple esa condición?, ¿Cumple las dos condiciones al mismo tiempo?
35. E: Ahhh … no.
36. I: Ah ok, entonces solo cumple una

Aquí confrontamos el carácter de índice de la solución usada por la estudiante y la no consideración de la zona con dos condiciones numéricas simultáneas, que pudo resolver correctamente por separado, debido a una interpretación lógica inadecuada. El diálogo continúa como sigue:

40. I: Para que me cumpla las dos condiciones ¿Qué número tiene que ser?
41. E: Tiene que ser un número que sea mayor que 7

La estudiante puede darse cuenta de sus errores, sin embargo, considerar las dos indicaciones al mismo tiempo representó un reto para su interpretación.

Esta problemática, aparecerá nuevamente en las inecuaciones, como vemos enseguida:

129.I: Ok aquí vuelvo a ver la misma zona, ¿Qué le pedían en esta tarea? En esta tarea se le pedía que representara en la siguiente recta numérica la solución. Esto es, marca la zona de los números si son enteros o no y cumplen las dos condiciones x > 5, pero también x > 7. ¿Cuál sería esa zona? ¿Es una zona o son dos zonas?
130. E: ¿Es UNA zona que cumple las dos condiciones?
131. I: Correcto
132. E: Ahhh … entonces sería la zona después del 7, a partir del número 7 hacia después
133. I: Ah perfecto. ¿La puede dibujar?
134. E: ahí está (La dibujó inmediatamente)

Ella no se había percatado de que la solución es una única zona, aunque ya se había discutido una situación parecida en el caso numérico (*ver línea 132*), discusión que transcurre en adelante desde 1. Un momento de duda ante la anticipación, 2. La verificación y 3. La posterior *objetivación,* todo ello producto de una práctica reflexiva desde el punto de vista de Radford (2006b).

En la siguiente imagen podemos ver que en esta ocasión si responde a las restricciones numéricas adecuadamente.

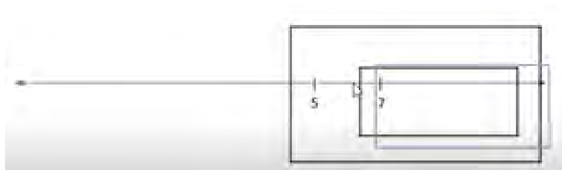

Imagen 2: Respuesta de la estudiante a la tarea en la entrevista

Por último, la tarea presentada a la estudiante estaba relacionada con el cumplimiento de dos condiciones simultáneas con inecuaciones, que incluía una combinación del signo de desigualdad y números positivos, donde pedía lo siguiente:

Da un valor entero para x que cumpla con las siguientes condiciones: x > 2, pero también x < 6

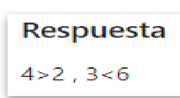
Respuesta
4>2 , 3<6

Imagen 3: Respuesta original de la estudiante a la tarea 19 de la actividad 2

Se llevó a cabo el siguiente diálogo para discutir la respuesta original dada a la tarea en el cuestionario.

155.I: Ahora ésta otra, un valor entero para x que cumpla con las siguientes condiciones x > 2, pero también x < 6
156.E: El número 1, ehh … no. ¿Un solo número para los dos verdad?
157.I: Correcto
158.E: El número 3

Hay un nuevo momento de duda en su interpretación (*línea 156)* aumentado con la complejidad de los dos tipos de signos de desigualdad usados, pese a haber resuelto con éxito el caso numérico con dos condiciones simultáneas. El siguiente diálogo se refiere a esta tarea:

167.I: Entonces cuando le preguntan acá, ¿Cuántos enteros cumplen las dos condiciones al mismo tiempo? ¿Cuál fue su respuesta?
168. E: Todos
169. I: Entonces ahora ¿Cuál es su respuesta?
170. E: Eh sería, mmm … todos los números que son menores que 6
171. I: Menores que 6 ¿Segura? Recuerde que la condición es que sea mayor que 2, pero también menor que 6 ¿Cuántos enteros hay que sean mayor que 2 y menor que 6?
172. E: Que sean mayor que dos y menor que seis. Ah no… Eh (silencio largo)
173.I: Dejémoslo ahí, ahora vamos a pasar a su representación. Ahí está representando los números mayores que 2 y los números menores que 6 ¿Sí? ¿Cuál sería la zona?

Luego de un largo momento de reflexión, el uso del artefacto medió su respuesta y finalmente logro una interpretación adecuada para los dos eventos .

179.I: Ok, ahora que me cumpla las dos condiciones al mismo tiempo
180.E: Mayores que dos y menores que 6, ¿Verdad? …
181.I: Marque la zona, ¿Cuántas zonas serian, una, dos o cuantas?
182.E: (repetía en voz baja varias veces: mayores que dos y menores que seis) Sólo sería la zona que está entre el número 2 y el número 6.
183.I: ¡Correcto, ahora márquela!
184.E: ¡Yuju!
185.I: Marque la zona
186.E: (Guarda silencio y la marca correctamente) …

La reflexión interna observada que tuvo el apoyo del artefacto (*línea 182*), permitió marcar la zona correcta

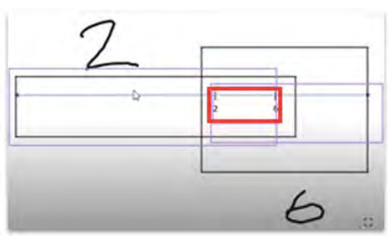

Imagen 4: Respuesta de la estudiante en la entrevista a la tarea 22

Luego del diálogo interno de la estudiante, que da cuenta de la práctica reflexiva mediada por el artefacto e interpreta la solución correctamente sobre el modelo de la recta numérica usado como un espacio de representación homogéneo.

Conclusiones

Este trabajo permitió observar el papel de los diferentes signos presentes al usar el modelo de la recta numérica para trabajar con tareas de desigualdades numéricas e inecuaciones lineales de la forma $x < b$ o $x > b$ con x y b enteros. Encontramos que los signos del modela presentan distinto tipo de complejidades, una de ellas se surge cuando la estudiante hizo uso de los segmentos como índices para establecer una dirección, lo que fue resuelto por el carácter de representación homogénea del modelo. El otro conflicto se representó cuando se deben atender dos indicaciones simultáneas y el uso de diverso de signos para marcar la zona solución, lo que fue resuelto mediante una práctica reflexiva que dio paso a una objetivación al verificar y estimar sus supuestos sobre la representación homogénea y los intervalos unitarios.

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THE NUMBER LINE MODEL AS A HOMOGENEOUS REPRESENTATION SPACE

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In this paper we investigate the change of signs from indexes to symbols in numerical inequalities and linear inequalities tasks, supporting by the model of the number line. In a case of study one student who has a good manage of numeracy, we found that there are complexities to use of signs and their logical organization when she has to interpret two simultaneous indications to the solution zone under the look of that this model is a homogeneous representation, complexities that was solved by a praxis reflective.

Keywords: Numerical concept and operations, cognition, secondary education, numerical representations

Background

The number line model has been used as a didactic resource of which different properties ranging from measurement to operation have been highlighted. This model is used with or without marks, as a guide, Diezmann et al. (2006, 2010), or no marks in order without considering distance, Teppo et al. (2013).

Here we consider the number line as a representation space homogeneous, Nemirovsky (2003) where the position of the signs is their only property, to address tasks of numerical and linear inequalities. We analyzed the role of integers and theirs marks with respect to order, relative position and spatial location with numerical character. In this setting unit segments are the basis support both numerical and spatial modeling.

Referential framework

From Radford's point of view, the objectification theory of knowledge cannot be something we can appropriate or possess, but "is a process of active meaning making" (Radford 2006a, p. 116).

Objectification is linked to the use of instruments, signs or artifacts that are not learning aids, but means whose presence and use imprint a distinctive stamp on the re-constructed through a semiotic mediation, developed in a praxis reflexive (Radford, 2004, p. 13).

In order to establish the functions of the different types of signs that are using in the proposed tasks we will consider that segments can be used both as indexes as mathematical symbols, Pierce (2005) depending on the use given to them by the student.

Finally, in this work the number line will be an artifact regulated by the properties of order, relative position and spatial location where numbers, marks and even unit segments are part of the semiotic structure of the model.

Methodology

The goal of this study is to investigate the semiotic use of the model of the number line for the interpretation of numerical and linear inequalities of the form: $x < b$ or $x > b$ with x and b integers when the model is a representation space homogeneous. We development a qualitative approach with data that supported by a case study carried out in quarantine conditions during the year 2020, whose instruments were a GeoGebra supported questionnaire and an interview – intervention by Zoom to promote a praxis reflective, Radford (2004) in the case presented.

The participant was a 14-year-old student of eighth grade from the Genaro Muñoz Hernández Institute of Siguatepeque, Honduras, who showed a good treatment of numerability and the semiotic structure of the number line model, both of whole numbers and decimals in a previously learning.

The questionnaire was proposed in two connecting sections, first part was about numerical inequalities and second she solves linear inequalities, the interview was conducted online where we based on her incorrect or incomplete answers to develop the intervention-interview, which focused on: 1. The location of the solution zone of two numerical inequalities and 2. The detection of the solution zone of two simultaneous linear inequalities.

Results

Originally, the marks to frame the solution zone were inaccurate, in the following dialogue between Researcher (I) and Student I, we were asking her to mark again the zone of numbers greater than 5 and the same time the zone of numbers greater than 7.

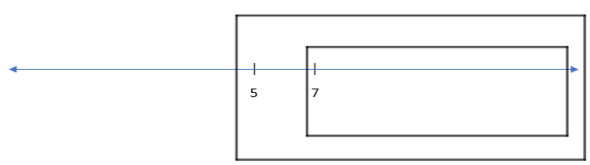

Figure 1: Student's response to the question

The solution was given by parts and then the following dialogue occurs (to see Figure 1):

28. I. ... Do any integers meet both conditions?
29. E: Mmm ... (Silence)
30. I: Do any integers meet that it is greater than 5 and at the same time is greater than 7?
31. E: Yes
32. I: Which ones?
33. E: All the ones that follow after 5 and after 7.
34. I: But does 6 meet that condition, does it meet both conditions at the same time?
35. E: Ahhh ... no.
36. I: Ah ok, then it only fulfills one

Here we confront her about the index character of the solution used by the student and the non-consideration of the zone with two simultaneous numerical conditions, due to an inadequate logical interpretation, which she was able to solve correctly by parts. The dialogue continues as follows:

40. I: In order to meet both conditions, what number does it have to be?
41. E: It has to be a number that is greater than 7.

The student can realize her mistakes, however, considering the two indications at the same time represented a challenge for her interpretation.

This problem will appear again in the linear inequations, as we see below:

129. I: Ok, here I see the same zone again, what we were asked in this task? In this task I asked you to mark the solution on this number line. That is, mark the numbers zone if they are integers or not and meet two conditions: x > 5, and also x > 7. What would that zone be? Is it one zone or two?
130. E: Is it ONE zone that meets both conditions?
131. I: Correct
132. E: Ahhh ... so it would be the zone after the 7, from number 7 onwards.
133. I: Ah perfect, can you draw it?
134. E: there it is (she drew it immediately)

She had not realized that the solution zone is a single area, although a similar situation had already been discussed in the numerical case (see line 132), the discussion goes from 1. She had a moment of doubt before the solution 2. Verification and 3. Subsequent objectification, like a product of a praxis reflective from the point of view of Radford (2006b).

In the following image we can see that on this occasion she answers to the numerical frame adequately.

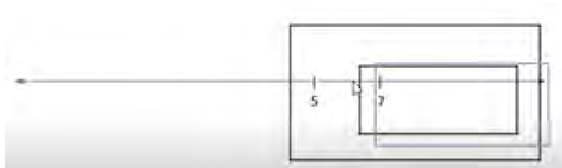

Figure 2: Student's response to the task in the interview.

Finally, the task presented to the student answer was successful to two simultaneous conditions with linear inequalities, which included a combination of the different inequality sign and positive numbers, where we asked the following:

"Give an integer x that satisfies the following conditions: $x > 2$, and $x < 6$"

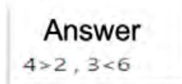

Figure 3: Student's original response to Task 19 of Activity 2

Then the following dialog was held to discuss the original answer given to the task in the questionnaire.

155. I: Now this one, an integer value for x that satisfies the following conditions $x > 2$, but also $x < 6$.
156. E: The number 1, ehh ... no. One number for both, right?
157. I: Right
158. E: The number 3

There is a doubt again in her interpretation (line 156) it increased with the complexity of the two types of signs of inequality used, despite have been successfully solved the numerical case with two simultaneous conditions. The following dialog refers to this task:

167. I: So, when asked here, how many integers meet the two conditions at the same time? What was your answer?
168. E: All of them
169. I: So now what is your answer?
170. E: Uh it would be, um... all numbers that are less than 6.
171. I: Less than 6, are you sure? Remember the condition is greater than 2, but also less than 6 How many integers are there that are greater than 2 and less than 6?
172. E: Greater than 2 and less than 6. Ah no... Eh (long silence)
173. I: Let's leave it there, now let's move on to its representation. There you are representing the numbers greater than 2 and the numbers less than 6, Yes? What would be the area?

After a long moment of reflection, the use of the artifact mediated her response and finally achieved an adequate interpretation for the two events:

179. I: Ok, now that it fulfills both conditions at the same time.
180. E: Greater than two and less than 6, right? ...
181. I: Mark the zone, how many zones would it be, one, two or how many?
182. E: (repeated in a low voice several times: greater than two and less than six). It would only be the zone between number 2 and number 6.
183. I: Correct, now mark it!
184. E: Yay!
185. I: Then you mark the zone
186. E: (Keeps silent and marks it correctly) ...

The internal reflection observed was supporting by the artifact (line 182), allowed to mark the right zone

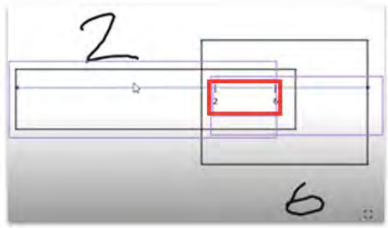

Figure 4: Student's response in the interview to the assignment 22

After the student's internal dialogue (line 182), which accounts for the praxis reflective mediated by the artifact and she interprets the solution correctly on the model of the number line used as a representation space homogeneous.

Conclusions

This work allowed us to observe the role of the different signs present when using the number line model to work with numerical and linear inequalities tasks of the form $x < b$ or $x > b$ with x and b integers. We found that the used of signs of the model present different kinds of complexities, one of them arising when the student made use of the segments as indices to establish a direction, which was solved by the homogeneous representation character of the model. The other conflict was rise when two simultaneous indications must be attended and also the use of different signs to mark the solution zone, which was solved by a praxis reflective that gave way to an objectification by verifying and estimating of her assumptions about the homogeneous representation and the unit intervals.

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IDENTIFYING AND FIXING DOUBLE COUNTING ERRORS IN MATHEMATICS AND PROGRAMMING

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We investigated how 28 first graders and 27 third graders, who analyzed worked examples as part of a programming intervention, debugged (identified and fixed bugs) and reasoned about double-counting errors in mathematics and programming tasks. Students completed the tasks on a pretest, a midtest (only programming tasks), and a posttest. Results showed that identifying double-counting errors positively correlated with fixing those errors in both programming and mathematics tasks and that students made more gains if they had analyzed worked examples during their programming, game-playing sessions prior to the test. The results suggest the importance of two-dimensional coordination in programming and mathematics debugging.

Keywords: Computational Thinking, Computing and Coding, Number Concepts and Operations, Elementary School Education

Computational Thinking (CT) includes cognitive skills, e.g., abstraction, problem-solving, or debugging (e.g., Wing, 2006, 2011), which align to key computer science standards (NGSS Lead States, 2013) and mathematical practices (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). Prior research in elementary education has shown a correlation between programing and mathematics scores (Grover et al., 2016; Lewis & Shah, 2012) and indicated that learning programming helped students extend mathematics content knowledge and develop problem-solving skills (Ahmed et al., 2011; Fessakis et al., 2013; Friend et al., 2018). At the same time, elementary students encountered difficulties in counting while debugging a program (Bofferding et al., 2020; Kocabas et al., 2019). We further explore the relation between debugging in programming and mathematics for early elementary students by focusing on this fundamental skill: counting.

Debugging and Counting “Bugs”

Debugging is difficult for students who have little programming experience (Fitzgerald et al., 2008; Murphy et al., 2008). Studies have reported that fixing errors in a program is harder than identifying them (Fitzgerald et al., 2008; Katz & Anderson, 1987; Lewis, 2012) and that fixing an error becomes easier if the error has already been identified, when students pay attention to relevant features (e.g., Lewis, 2012). On the other hand, having no or little programming experience might lead students to introduce new errors while trying to identify the existing error in a program (Gugerty & Olson, 1986; Nanja & Cook, 1987). Therefore, they are more likely to do extra, unneeded, modifications in a program (Ahmadzadeh et al., 2005; Nanja & Cook 1987).

Double counting, counting the same object or space twice, is a common difficulty for young students in programming (e.g., Kocabas et al., 2019) and mathematics (e.g., Fuson, 2012). Fuson

(2012) found that three- to five-year-olds made more double-counting errors when objects were disorganized than when they were displayed ordinally. Kocabas et al. (2019) reported that first and third graders double counted the spaces on a programming path where it switched directions. Similarly, Battista and colleagues (Battista, 1999, 2010; Battista et al., 1998) reported that second graders without row and column structures may double count where rows and columns overlap. When counting down, as for solving 14 – 6, children say 13 while putting up one finger to indicate that one less than 14 is 13 and gradually say "12, 11, 10, 9, 8" while sequentially raising five more fingers (Maclellan, 1995; Wright et al., 2006); however, some students may count the 14 as one taken away. Encouraging students to debug could help draw their attention to such counting errors. We combine a focus on debugging and counting in this study through the following research questions: (1) How do first and third graders make sense of double counting errors in programming versus mathematics debugging tasks? (a) To what extent does success in debugging double-counting errors correlate between programming and mathematics tasks? (b) How does students' success in debugging change after counting to make programs in a coding game? Does analyzing worked examples earlier versus later affect the changes? (c) What are possible explanations for students' debugging reasoning (different or similar) in programming and mathematics?

Methods and Analysis

For this study, we analyze data from 28 first graders and 27 third graders from a public elementary school in the Midwest. The students completed a pretest, three 20-minute sessions playing *Osmo™ Coding Awbie* in pairs, a midtest, participated in a 30-minute presentation on programming applications, three additional 20-minute sessions of game play, and a posttest. Before the sessions, students were randomly assigned to either the immediate-worked-examples (immediate) group or the delayed-worked-examples (delayed) group. During the first three sessions, students in the immediate group analyzed a set of programming worked examples (<10 minutes) and then played the game without interruption (>10 minutes), while students in the delayed group just played the game. After the midtest, during the second three sessions, the immediate and delayed groups switched their activities.

In this paper, we focused on one programming debugging item (see Figure 1, left panel: bug 1) included on the pretest, midtest, and posttest and one mathematics debugging item (see Figure 1, right panel) included on the pretest and posttest. In both cases, students watched a video of the counting bug occurring and were asked to find and fix the bug. We interviewed the students individually. We ran correlational analyses to determine if there was an association between students' identifying and fixing (debugging) programming and mathematics counting errors.

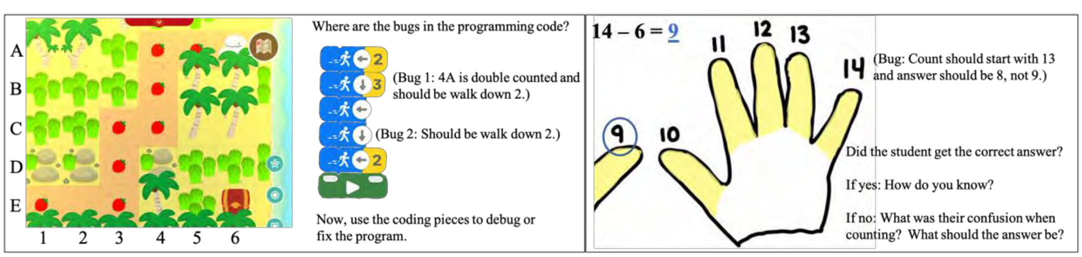

Figure 1: Program Debugging (left panel) and Mathematics Debugging (right panel) Items

Further, we used McNemar tests and Mann-Whitney U tests to determine whether there were significant differences between immediate and delayed groups from the pretest to posttest. For qualitative analysis, we then grouped students based on whether they (1) did not identify or fix, (2) identified but did not fix, (3) did not identify but did fix, or (4) identified and fixed the errors. To provide a clear picture of students' debugging performance, we identified qualitative descriptions of students' reasoning based on their interpretation of the given visuals and their use of additional strategies (e.g., creating their own code rather than modifying given code or counting on their own fingers to determine the answer). Within each group, across the mathematics and programming items, we looked for commonalities in their reasoning.

Findings

Students' identifying and fixing the mathematics bug were significantly correlated within the pretest (immediate group: r=.750, p<.001; delayed group: r=.727, p<.001) and within the posttest (immediate group: r=.806, p<.001; delayed group: r=.802, p<.001). Similarly, identifying and fixing the programming bug were significantly correlated within the pretest (immediate group: r=.650, p=.001; delayed group: r=.606, p=.002), midtest (immediate group: r=.793, p<.001; delayed group: r=.512, p=.009), and posttest (immediate group: r=.651, p=.001; delayed group: r=.592, p=.004). The only other significant correlation for the delayed group was fixing the pretest math counting bug with fixing the midtest programming bug (r=.421, p=.029). On the other hand, the immediate group had a significant correlation with identifying the pretest programming bug with identifying the posttest math bug (r=.430, p=.022) and identifying the math and programming bugs on the posttest (r=.426, p=.027).

Overall, based on a McNemar test of change, students in the immediate and delayed groups made significant gains in fixing the programming bug from pretest to posttest (χ^2= 5.06, p=.021 and χ^2= 8.64, p=.002 respectively), but did not make significant gains in fixing the mathematics bug (χ^2= 1.13, p=.289 and χ^2=.13, p=.727, respectively) (see Table 1). Further, based on Mann-Whitney U tests, the gains in fixing bugs between the two groups from pretest to posttest did not differ significantly on the programming debugging item, U=320.00, z=-.60, p=.550, or on the mathematics debugging item, U=416.00, z=1.46, p=.146. However, based on a McNemar test of change, students in the immediate group made significant gains in fixing the programming bug from pretest to midtest, χ^2 = 6.13, p=.008, unlike the delayed group, χ^2= 2.50, p=.109. Neither group made significant gains from midtest to posttest.

Table 1: Percent of Students Who Identified and Fixed the Math and Programming Bugs

	Mathematics				**Programming**					
	Identify		**Fix**		**Identify**			**Fix**		
Group	**Pre**	**Post**	**Pre**	**Post**	**Pre**	**Mid**	**Post**	**Pre**	**Mid**	**Post**
Immediate [a]	32%	54%	29%	43%	29%	42% [b]	54%	29%	54% [b]	64%
Delayed [c]	30%	28%	26% [d]	20% [d]	19%	22%	56% [d]	15%	37%	60% [d]

[a] n=28. [b] n=26 because two students missed the midtest. [c] n=27. [d] n=25 because two students moved before the posttest.

Programming and Mathematics Debugging Reasoning

For the program debugging item, students were more likely to fix the bug once they identified it. Interestingly, students who did not identify the bug sometimes inadvertently fixed the bug when rewriting the code. Yet, when rewriting the code, some of these students created new double-counting or directional bugs. For instance, two first graders succeeded in correcting “walk down 3” to “walk down 2” but changed the first correct code “walk left 2” to “walk left *3*” or “walk *right* 2” (see Figure 1). A few students identified the bug but did not know how to fix it. Students who did not identify or fix the bug often double counted the space A4; they had difficulty structuring their counting, i.e., separating the horizontal code (walk left) from the vertical code (walk down), and identifying that space A4 was counted in the first line of code.

For the mathematics debugging item, although approximately 25% more of the immediate group identified and fixed the bug on the posttest compared to pretest, students in the delayed group did not show similar improvement. Similar to programming, once students identified the bug, most of them fixed it. Students who did not identify the math bug but still fixed it either knew the answer should be eight or correctly took six fingers away to get eight; however, they did not have a problem with the picture showing the count starting at 14. Students who neither identified nor fixed the error agreed that “six fingers are taken away.” However, they did not realize that the first count incorrectly started with 14 instead of 13. On the other hand, about 10% of students did identify the bug but failed to fix it. These students indicated that the answer of nine was not correct (often by counting on their own fingers to check), but when they counted the fingers on the picture, they ended up agreeing with the counting strategy and did not fix it. The group of students who succeeded in identifying and fixing the bug often reasoned that “fourteen doesn’t count” and avoided double counting.

Discussions and Implications

Our study confirmed previous findings that once identifying bugs, students could fix them in programming (e.g., Fitzgerald et al., 2008) and mathematics contexts. For the delayed group, there was a correlation between fixing the pretest mathematics bug and the midtest programming bug, building on similar correlational findings by Lewis and Shah (2012). On the other hand, these items were not correlated for the immediate group, possibly because thinking critically about the worked examples from the beginning helped students even if they had not fixed the mathematics bug on the pretest. Moreover, identifying the bugs in the mathematics and programming items on the posttest were correlated for this group, once again suggesting some relation between mathematics and programming. Future studies could balance programming and mathematics debugging experiences to further investigate how they relate.

Overall, we found that students struggled with coordinating horizontal movements with vertical movements (lines 1 and 2 of the programming code in Figure 1) and with aligning pictorial representations with their own finger counting. In both situations, students showed a lack of global structuring of the information (Battista et al., 1998; Battista & Clements, 1996). In the programming item, the grid organization may not have alleviated students’ inclination to double count because they were asked to track the position and its result in their heads and may not have considered overlapping spaces. Likewise, for the mathematics item, they saw a static representation of the finger counting (with highlighting to show action), so they may have had difficulty tracking what the count corresponded to in relation to the picture. The fact that even some third graders, who had initially indicated the answer should be eight, ended up agreeing with the answer of nine because the picture looked right, highlights the need to help students analyze and reason about visuals (as was done with the programming worked examples).

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STUDENTS' REORGANIZATIONS OF VARIATIONAL, COVARIATIONAL, AND MULTIVARIATIONAL REASONING

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In this paper we examine sixth grade students' constructions and reorganizations of variational, covariational, and multivariational reasoning as they engaged in dynamic digital tasks exploring the science phenomenon of weather. We present case studies of two students from a larger whole-class design experiment to illustrate students' forms of reasoning and the type of design that supported those constructions and reorganizations. We argue that students constructed multivariational relationships by bridging, transforming, and reforming their reasoning and that the nature of the multivariational relationship being constructed affected this process.

Keywords: Algebra and Algebraic Thinking, Design Experiments, Integrated STEM/STEAM

Background

According to the National Research Council (NRC) and Mathematics Learning Study Committee (2001), students "must learn to think mathematically, and they must think mathematically to learn" (p. 1). As the NRC argues, mathematics has facilitated the advancement of science, technology, engineering, business, and government. Mathematics interacts with these disciplines in the form of expressing the variation of multiple quantities. For example, in science, weather forecasters study the variation in air temperatures and dew points to predict the chances of a rainy day. These phenomena usually involve complex relationships between multiple quantities that vary. Although people need to understand this complex variation in many facets of life, school often neglects the study of change in multiple quantities and focuses only on changes in one (variation) or two quantities (covariation). Only one source was found to examine multivariational reasoning, with a focus on undergraduate education (Jones, 2018).

In this paper, we discuss how our project that engaged students in a study of earth and environmental phenomena supported them in reasoning multivariationally. In previous iterations, we found that by engaging with our tasks, simulations, and questioning, students were not only coordinating the change in two quantities but they also reasoned about changes in multiple quantities (e.g., Basu et al., 2020; Panorkou & Germia, 2020a; 2020b; in press). These findings informed our subsequent iterations that aimed to engineer more opportunities to prompt students to study the variation in multiple quantities and reason multivariationally. This paper describes three of those opportunities and discusses how students' thinking progressed from variational, to covariational, and then to multivariational reasoning. Specifically, we explored: 1) How does students' reasoning progress from variation to covariation and then multivariation while engaging with our design? And 2) How does our design support this progression of reasoning?

We use a quantitative reasoning (Thompson, 1994) lens to examine and characterize students' thinking. A quantity is a measurable conceptual attribute that exists in the conception of a situation. Reasoning quantitatively involves constructing the quantities involved in a situation, recognizing which quantities change, and constructing relationships between the changes in pairs of quantities. Thompson and Carlson (2017) define variational reasoning in terms of envisioning "that the quantity's value varies within a setting" (p. 425) while covariational reasoning involves envisioning two quantities' values varying simultaneously.

Our goal was to examine the progression of students' reasoning from variation to covariation and then to multivariation. Because knowledge is dynamically constructed through constructive activity, we aimed to understand how students' meanings about varying quantities could be shaped and reorganized as students interact with our task design, simulations, and questioning. By *meaning*, we refer to "the space of implications that the current understanding mobilizes – actions or schemes that the current understanding implies, that the current understanding brings to mind with little effort" (Thompson et al., 2014, p. 12). By *reorganization* (Piaget, 2001) of students' meanings, we refer to humble inferences we make about their reflections and projections of particular meanings about the quantities and their relationships to a higher conceptual level where these initial meanings become part of a more coherent whole.

Methods

We followed a whole-class design experiment (DE) methodology (Cobb et al., 2003). Our Des were conjecture-driven, in that the research team constructed some initial conjectures about supporting students' quantitative reasoning and these conjectures evolved as the experiment unfolded. In this paper, we present the design of one task focusing on weather, which involves asking students to explore a dynamic simulation and the variation of its quantities.

We designed the Hot Air Balloon simulation to show the relationship between the size of the flame in a hot air balloon, the temperature of the air inside the balloon, the density of that air, and the balloon's altitude. We chose to model a hot air balloon to encourage students to reason about the properties of air masses, such as temperature and density, which can affect how air masses interact to form weather patterns. The student can change the temperature of the air inside the balloon using the "turn flame up" and "turn flame down" buttons. Increasing the size of the flame also increases the temperature of the air inside the balloon, which decreases the density of that air, which increases the balloon's altitude (Figure 1).

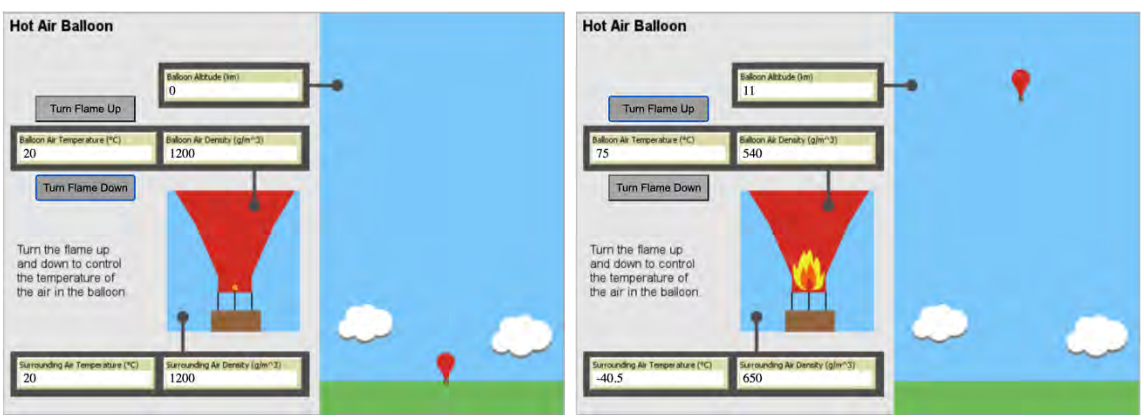

Figure 1: The Hot Air Balloon simulation

We collected data from a sixth-grade classroom from the Northeast of the US. The Des consisted of 15- to 50-minute sessions in which we interviewed the students during their virtual classes in Google Meet. In this paper, we focus on the retrospective analysis (Cobb et al., 2003) of one pair of students, Anne and Violet, to discuss their constructions and reorganizations of variational, covariational, and multivariational reasoning.

Findings

We organize our findings according to how Anne and Violet's constructions and reorganizations took place: by *bridging*, *transforming*, or *reforming*. We also present the type of questioning that might have supported these constructions and reorganizations.

Bridging

Anne and Violet first identified varying quantities as they explore the simulation and its controls. For example, when asked to describe what she noticed in the Hot Air Balloon simulation, Violet clicked to change the flame height and described how altitude, density, and temperature all changed. This showed that Violet constructed variational reasoning about these quantities during her initial explorations of the simulation. Our questioning then turned the students' attention to relationships between these quantities. For example, Anne described relationships between the flame and the balloon's altitude ("when I was turning the flame up, it [the balloon] would like go up") and the flame and the air density ("whenever you turn it [the flame] down, it goes, the density becomes higher"). These excerpts show that she was making connections between pairs of simultaneously changing quantities, thus reorganizing her initial variational reasoning into covariational relationships.

To encourage students to merge the relationships they had reasoned about, we then asked students about the relationships between more than two variables. For example, Anne reasoned, "for the temperature, when you turn it [the flame] down, it gets cooler. And then for the density, it decreases." In this statement, she expressed her reorganization of the covariational relationship she had previously identified into a multivariational envisioning of all three quantities changing at the same time, thereby *bridging* her multiple covariational relationships into a single multivariational relationship.

Transforming

In one case, we observed Violet expanding a single covariational relationship rather than bridging such relationships together in pairs, instead *transforming* one by including new quantities. Violet originally constructed a covariational relationship between her control of the flame and the resulting changes in the balloon's altitude. Then, when we asked her to describe the changes in the density of the air inside the balloon, she clicked to turn the flame up three times and observed, "What happens is that when I go higher [turn up the flame to lift the balloon], the density inside the balloon gets lower." Then, immediately following this, she clicked to turn the flame up three more times and added, "But the temperature goes higher." We interpret her statements to show that she had added two new quantities to her reasoning, thus *transforming* her single covariational relationship between the flame height and altitude by reorganizing it to construct a multivariational relationship in which changes in the flame height resulted in changes in both the density and the temperature, as well.

Reforming

We also observed students *reforming* their multivariational relationships into relationships with different structures after considering more covariational relationships they found during their explorations. The Hot Air Balloon simulation offers a nested multivariational relationship in which changes in one quantity (flame height) affect the next (air temperature), which affects the next (air density), and then the next (balloon altitude) in a nested sequence. Initially, both Violet and Anne constructed multivariational relationships in which changes in the flame height caused simultaneous changes in the simulation's other variables. However, in subsequent DE sessions, both Violet and Anne further considered other covariational relationships in the simulation and then used these to reorganize their multivariational reasoning.

For example, Anne reasoned that "whenever you turn it [the flame] down, it goes, the density becomes higher." Similarly, Violet argued that "the hotter the air inside the balloon is … the more its density decreases." Then, when we asked Violet to explain her reasoning in this statement, she added, "when you turn up the flame, it gets hotter, the density decreases, and it makes the balloon fly up higher." Violet's wording in this excerpt seems to indicate that she had reorganized her reasoning about the multivariational relationship to construct it as a chain of related dependencies, rather than describing a change in one variable causing simultaneous changes in three other variables as she had before. She had *reformed* her multivariational relationship to include her reasoning about the new covariational relationships.

Similarly, Anne first reasoned that "for the temperature, when you turn it [the flame] down, it gets cooler. And then for the density, it decreases," constructing a multivariational relationship in which a change in one variable caused changes in two others. Later, after she had constructed the covariational relationship between temperature and altitude, we again prompted Anne to reason about all of the quantities. She responded, "When I turn up the temperature, the density starts getting low and then altitude, it shows how like the balloon is going up." We consider this excerpt to show that Anne had reorganized her construction of the multivariational relationship into one in which changes in each of the quantities caused a change in the next in sequence, engaging in *reforming* similar to Violet.

Conclusions

Our analysis shows that the simulations provided opportunities for students to see, control, and reason about multiple changing quantities. As we questioned them about the relationships among higher numbers of these quantities, we observed that the students progressed along a trajectory of first constructing variational reasoning and then reorganizing this into covariation and then into multivariation. Specifically, questions about noticing and describing change such as "What is changing in this simulation?" encouraged students to identify variables and reason variationally about individual quantities. Questions about noticing and describing relationships such as "What is the relationship between depth and temperature?" or "What is the relationship between temperature, dew point, and cloud altitude?" then encouraged students to reorganize their thinking first into covariational and later multivariational relationships.

In this paper we have discussed how students engaged in bridging, transforming, and reforming of their reasoning in different multivariational situations. Specifically, students engaged in a *bridging* form of reorganization in which they first constructed two covariational relationships and then merged these into a single multivariational relationship. However, we also saw Violet engage in *transforming* her existing construction of a single covariational relationship into multivariation by reorganizing it to include the addition of new variables. Moreover, both Violet and Anne engaged in *reforming* their initial multivariational reasoning after considering more of the covariational pairs that make up the larger nested relationship in the simulation. This may indicate that the nature of nested relationships has some effect on students' progressions of multivariational reasoning. This shows that students go through different mental actions, and thus different constructions and reorganizations, based on the type of relationship they have to construct. We thus believe that more research is needed on characterizing students' constructions and reorganizations in different types of multivariational situations.

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STUDENTS' EARLY CONSTRUCTING AND INTERPRETING THEIR DATA DISPLAYS OF CATEGORICAL DATA

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Constructing and interpreting data displays are crucial for statistical learning as well as college and career readiness. The findings presented in this paper are part of a larger study of the development of statistical concepts and skills. Thirty students organized data, constructed their own graphs, and interpreted values and patterns in their graphs. Preliminary findings indicate that organizing data by groups may relate to understanding that data can be aggregated in the data display. In addition, choosing which type of data display to construct appears to impact interpretation of data shape or patterns, even in this simple case. Implications for educators and researchers are discussed.

Keywords: Data Analysis & Statistics, Elementary School Education, Mathematical Representations

Study Objectives and Background

The U.S. Common Core State Standards for Mathematics (CCSSM, Common Core State Standards Initiative [CCSSI], 2010) indicate that instruction in data display should begin at Kindergarten and continue through the upper grades, and the American Statistical Association's *Guidelines for Assessment and Instruction in Statistics Education (GAISE) Report* (Franklin et al., 2007) addresses the use of "graphical displays" of data (data displays or statistical graphs) in data analysis and interpretation of results of the statistical problem-solving process, promoting the statistical competencies students need for college and career readiness. Thus, students need to learn how to construct and interpret data displays. The findings presented here are part of a larger study of the development of statistical concepts and skills investigating the relationship between constructing and interpreting data displays. Our research question is: how do students construct and interpret statistical graphs for categorical data?

Framework

The CCSSM expects students in grades 3 to 5 to "represent and interpret data" (CCSSI, 2010), yet attending to both data and its context can be challenging for students. They may disregard data and attempt to make statistical inferences based on personal beliefs or, conversely, focus on numerical quantities without considering the data context (Groth, 2021). Unfortunately, instruction in data display at the elementary level tends to "facilitate a view of graphs as illustrations, rather than reasoning tools …" (Fielding-Wells, 2018, p. 1125). In this study, we investigated how students analyze and display raw data in an extended task and with a given purpose in mind. We previously published a hypothesized learning progression (LP) to represent qualitative shifts in constructing and interpreting data displays from 2nd grade to post-secondary education (Kim et al., 2020). This LP is used in the current study as an analysis framework to analyze student knowledge and understandings about constructing and analyzing data display.

Methods

This study reports on student work with *Summer Math Camp*, an extended task designed to elicit evidence of students' knowledge and understandings of constructing and interpreting statistical graphs for categorical data. We recruited students in grades 4 to 6 because the CCSSM expects students to construct and interpret bar graphs as early as 2nd grade (CCSSI, 2010); thus, we could expect that participants would have worked with this type of graph. Students were presented with a set of completed paper surveys from 45 fictional summer math camp participants indicating which of four activities were their favorite among Forecasting the Weather, Reading Math Stories, Building a LEGO® City, or Programming Robots. Students were asked to organize the categorical data, create a graph to show the results of the survey, and identify quantities, group differences, and patterns in the data. Students independently created a graph and provided written responses to questions. We then conducted semi-structured cognitive interviews with each student to better understand their thinking as they engaged with the task. Responses were audio-recorded, and the workspace was video-recorded (i.e., no student's faces were captured in the video).

A diverse sample of 30 of 31 students completed the task and interview (one student chose to stop participating). The sample included eleven 4th-grade, eleven 5th-grade, and eight 6th-grade students. Slightly more girls than boys participated, with 18 students identifying as female. One student identified as Native American or Alaska Native, two as Hispanic or Latino, six as Asian American, five as Black or African American, and 16 as White. Most students attended traditional public elementary or middle schools, with one student attending an independent school and two attending parochial schools. Reported school math achievement varied from grades of C or "meets expectations" through A+.

We used Nvivo 12 to code transcripts of the video and student work by using the LP descriptors in Kim and colleagues (2020). This allowed us to describe students' knowledge and understandings of constructing and interpreting data displays. We then produced a matrix of all descriptors, pre-graphing activities, and types of graphs by participant examined the display for patterns.

Results

Pre-Graphing Activities

We had not initially planned to attend to students' pre-graphing activities but were struck by the variation in student approaches to organizing the given set of data (i.e., surveys indicating a favorite math camp activity). Seventeen students began by sorting the surveys into four piles by preferred activity. Five other students hesitated or tried different ways of sorting the data before realizing that making four piles could support their construction of a display of categorical data. Four students used the graph itself as an organizer by going through the surveys one-by-one and increasing the height of one of the bars or the number of Xs for each new survey in the pile. Four remaining students used a tally chart as an organizer before constructing their graph. The type of organizing was not related to students' choice of appropriate graphs for the given data set, but only students who sorted surveys into piles or who used tally charts later produced graphs with cases aggregated into fives. It is possible that some pre-graphing organizing is more likely to lead to an aggregate view of data.

Choosing Appropriate Graphs for Categorical Data

We observed great variation in the types of graphs produced, including a few students who answered questions about the data without creating any graph at all. Sixteen students chose graphs appropriate for displaying categorical data (i.e., pictographs, pie graphs, and bar graphs), while others created inaccurate bar graphs (e.g., no spaces between the bars), histograms, tables, tally charts, idiosyncratic diagrams, or iconic drawings. Most students constructed their bar graphs by individual cases and used a one-unit scale on the y-axis, yet some students realized that the number of surveys each chosen camp activity was divisible by 5, and thus used a five-unit scale on the y-axis. In spite of their success in creating appropriate graphs for categorical data, however, most of these students could not explain why they chose that type of graph or explained constructing a bar graph because they were familiar with the graph type, such as Student 1, who responded: "Well, it was just the first thing, like I usually do this [points to the bar graph] in school."

Constructing and Interpreting Bar Graphs

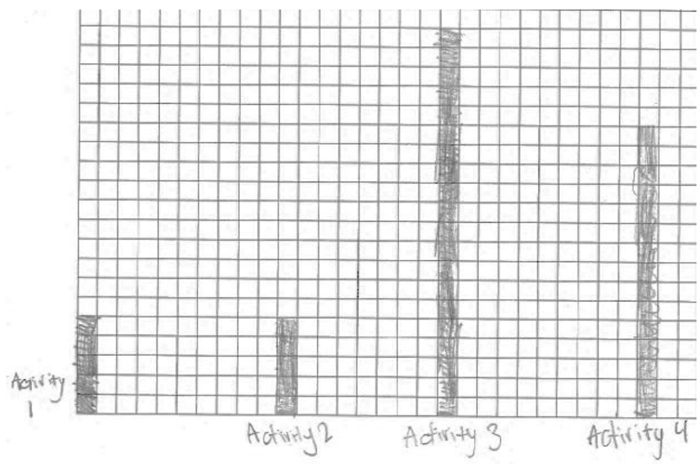

Figure 1: Bar Graph from Student 22

Almost all students who accurately constructed bar graphs were able to interpret the meaning of individual data values (e.g., how many students chose Activity 4 [Programming Robots] as their favorite activity) and describe differences in values between groups (e.g., how many more or fewer students chose Activities 1 and 3 [Forecasting the Weather and Building a LEGO® City] than Activities 2 and 4 [Reading Math Stories and Programming Robots]?). However, students who produced no graphs (e.g., iconic drawings) could also answer those three questions, probably because the data set was small enough to allow them to recall the relevant values.

Only students who accurately or inaccurately constructed bar graphs were able to write about or comment on the shape of or patterns in the data. In all cases, students used everyday language to describe these patterns, such as Student 13, who wrote "low, low, higher, medium" to describe the order of bars representing values of 5, 5, 20, and 15, respectively. Although students could preview all the questions we were going to ask them on their worksheet, most students constructed graphs with the bars in the order of activity number (see Figure 1) and did not seem to consider that because the data were categorical, the bars could be presented in any order to more effectively see the shape of or patterns in the data. Student 22, for example, described that she constructed the graph "So I started the activity one and then it's from lower because activity one and two they have five, and activity three and four have fifteen and twenty. So, I wrote them from lower to higher," but did not follow this logic to the end by presenting her bars in order of

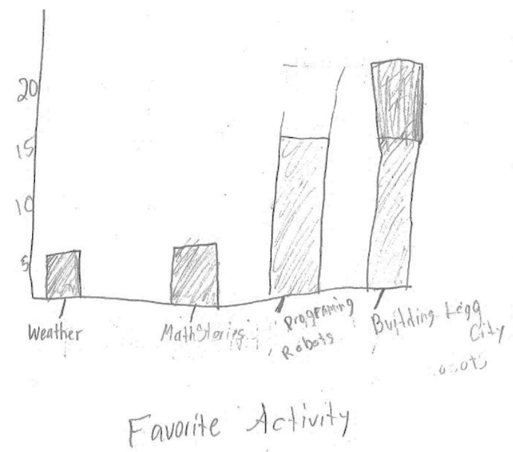

Figure 2: Bar Graph from Student 21

increasing value. Student 21 (see Figure 2) was the only student who recognized that she could change the order of the bars so that they formed a more consistent pattern; she switched the order of the bars for Programming Robots and Building a LEGO® City and explained that she intended to create a consistent order: "One pattern of my graph is ordering them by the amount of students who chose … who chose their favorite thing that they did."

Conclusion and Discussion

The findings of this study are limited in generalizability because the sample of students is not representative of U.S. students in grades 4 to 6. In addition, the findings apply to students' work with categorical data only, while the larger study will include these students' engagement with numerical data in the final results. In spite of the limitations, these preliminary findings suggest that it is important to investigate connections between pre-graphing, constructing, and interpreting data display as students begin to work with formal graphs (see Lehrer & Schauble, 2000). In the case of categorical data, organizing data by groups may relate to understanding that data can be aggregated in the data display, although the direction of this relationship cannot be determined from this study. Choosing which type of data display to construct appears to impact interpretation of data shape or patterns, even in this simple case. Few students, however, could explain their reasons for choosing to construct a specific data display, even when they constructed an appropriate and accurate one. Educators should consider the opportunities that students have to work with raw data and to choose the most appropriate data display for the intended purpose. Research can continue to investigate pre-graphing activities and the relationships between these activities and the construction and interpretation of data displays.

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EXHIBITING INTEGERS' CONFLICT AND RESOLUTION USING A MATHEMATICS STORYBOOK: THE CASE OF FOUR FIFTH GRADERS

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Students often struggle to make sense of integer concepts because they contradict their whole number understanding. In this paper, I unfold how four fifth graders interpret a story conflict and its resolution within different interactive versions of a mathematics storybook, which was designed to highlight the contradictory ideas between the absolute and linear values of integers in the context of temperature. Over three sessions of reading and retelling, students paid more attention to the differences between positive and negative numbers' order and value and more often referred to the mathematical conflict and its resolution. This study informs the effectiveness of introducing the concept of integers by presenting conflict and illuminates ways that students can be supported through the resolution process.

Keywords: Number Concepts and Operations, Cognition, Elementary School Education

Negative numbers are difficult to learn because of their abstract nature, the difficulty of relating them to quantitative interpretations, and their contradictions with whole number knowledge (Bofferding, 2014; Whitacre et al., 2011). Therefore, identifying ways to help students make sense of the conflict between numbers' absolute values ($|-5| > |-3|$) and linear values ($-5 < -3$) is essential. One way to expose students to a conflict and propose a resolution of it is through stories. However, a story's effectiveness in helping students make sense of a conflict and assimilate the story's resolution may depend on students' prior knowledge of the topic (in this case, integer order and value) as well as the supports provided to make sense of the story's content. In this paper, I investigated the role of three types of supports within a temperature-related mathematics storybook—*Temperature Turmoil*—on students' understanding of the conflict between negative and positive numbers' order and value and its resolution.

Theoretical Grounding and Literature Background

Based on Piaget's theory of cognitive development, children's learning process begins with noticing a conflict, internalizing it, and engaging in assimilation, accommodation, or equilibration (Piaget, 1952). Although exposing students to a conflicting situation may initiate the learning process by triggering students' cognitive conflict, doing so is not sufficient without also providing opportunities for resolving the conflicts (e.g., Limón, 2001). Research not only needs to detail students' cognitive conflicts but also should explore and facilitate their resolution process. Mathematics storybooks can convey conflicting, new mathematical ideas that build on children's prior knowledge and further provide opportunities to make meaning of such conflicts through the story's narrative and illustrations within a real-world context (Ginsburg et al., 2018; Moyer, 2000). When reading a mathematics storybook, children can construct their knowledge through reciprocal interaction between their social environment and their internal interpretation of the story. Some of these interactions can emerge during children's explorations (e.g., questions that they ask an adult or peers), some might be embedded in the story for the children to act upon (e.g., stories that have questions in them), and some can reflect a teacher's (or adult's) conversations with children. One way to promote children's interactions with a storybook without an adult or peer present is to embed interactive features within the storybook.

In this study, the narrative and illustrations in the mathematics storybook—*Temperature Turmoil*—introduce a mathematical conflict between positive and negative numbers' order and values and later demonstrate the mathematical resolution of the conflict. For instance, a comparison between two lands' temperatures—one with positive temperatures and one with negative temperatures—along with directed magnitude language unfolds in a conversation between two characters. Curt Cozy: "Your 20 [-20] is more cold than back in my land [land of positive temperatures only]." Ilana Icy: "Your 20 [+20] is more hot than back in my land [land of negative temperatures only]." The resolution of this conflict begins when one character says, "Each 20 is far from zero but in an opposite spot." Besides the story resolution, interactive features provided additional support to facilitate the resolution process. I examined:

(1) What are the effects of embedded interactive features within a mathematics storybook on students' ways of resolving the positive and negative numbers' order and value conflict?
(2) How did students interpret the story conflict and its resolution during retellings?

Methods

Participants, Study Design, and Data Analysis

Six fifth graders participated from a public elementary school in the Midwest, United States (45% were economically disadvantaged and 11% were English-Language-Learners).

Pretest and posttest. I interviewed students individually, examining their integer knowledge including order and value test items. Each test took approximately 40-minute.

Sessions 1-3: Reading and retelling. After the pretest, for three sessions, students listened to a version of *Temperature Turmoil* written by Laura Bofferding: *control*, *interactive question*, *interactive visual*, or *interactive mixed*. The interactive versions differed in terms of additional resolution support in the form of hotspots. In the **Control**, students listened to the electronic version of the book without any interactive support. In the **Interactive Question**, activating a hotspot resulted in a mathematical question + feedback after responding, which was designed to reinforce the directed magnitude language in conjunction with numbers. In the **Interactive Visual**, activating a visual hotspot resulted in an animation, a slider, or a combination thereof that centered on a thermometer representation, which emphasized the numbers' continuity. In the **Interactive Mixed**, hotspots contained both interactive visuals and questions. After each session, students retold the story and drew a thermometer to show the story resolution.

I present a multiple-case study (Yin, 2018) of four fifth graders—Harry $_{\text{(control)}}$, Lola$_{\text{(question)}}$, Claire$_{\text{(visual)}}$, and Chase$_{\text{(mixed)}}$. Based on Bofferding's (2014) integer mental model levels and scores on order and value test items, I classified the extent to which students exhibited conflicts between positive and negative numbers' order and value and identified a resolution: no-resolution (i.e., initial level and scoring at most 40% correct); partial-resolution (i.e., transition I level and scoring at most 60% correct, magnitude level and scoring at most 85% correct, or transition II level and scoring at least 85% correct); or complete-resolution (i.e., formal level and scoring 100% correct). I categorized students' descriptions of story conflict and its resolution during retellings (see Table 1). Further, I classified students' drawn thermometers as *no-resolution* (i.e., *one* thermometer of positive numbers); *beginning-resolution* (i.e., *two* thermometers: one of positive and one of negative numbers); *middle-resolution* (i.e., *one* thermometer with 0 and -0); or *complete-resolution* (i.e., *one* thermometer of positive and negative numbers).

Table 1: Classification of Students' Story Conflict and Resolution

		Categories		
Story conflict	**Ignore-conflict**	**Contextual-conflict**	**Partial-mathematical-conflict**	**Complete-mathematical-conflict**
Example	"I don't know"	"There are two lands: hot and cold lands and they don't like it"	"The people of two lands used different numbers"	"One land's temperature was above 0 and one land's temperature was below 0"
Story resolution	**No-resolution**	**Contextual-resolution**	**Partial-mathematical-resolution**	**Complete-mathematical-resolution**
Example	"They will get used to being cold and hot"	"They will switch lands"	"They will use a new thermometer"	"They will use positive and negative numbers to see the difference"

Findings

On the pretest, all four fifth graders were classified as partial-resolution with Harry$_{\text{(control)}}$, Lola$_{\text{(question)}}$, Claire$_{\text{(visual)}}$, and Chase$_{\text{(mixed)}}$ scoring 89%, 76%, 55%, and 99% respectively. Harry and Chase exhibited the transition II level and Lola, and Claire showed the magnitude level. Only Harry improved to the complete-resolution category by the posttest, and Lola scored higher (91%) and exhibited transition II level. The other two were classified the same as their pretest.

Sessions 1-3: Readings and Retellings

Session 1. Harry's$_{\text{(control)}}$ and Chase's$_{\text{(mixed)}}$ first retellings contained evidence of complete-mathematical-conflict and complete-mathematical-resolution. Further, except for Lola$_{\text{(question)}}$ drawing a thermometer representing no-resolution, all students' drawn thermometers represented the middle- or complete-resolution. Harry$_{\text{(control)}}$ described two lands' temperatures:

> So, their temperatures are different because this one is *going up from zero* and this one is *going down from zero*. This one, I think it's *negatives* and this one, is *positives*. So, they think that's the same temperature cause thirty-three positive and thirty-three negative.

On the other hand, Lola's$_{\text{(question)}}$ and Claire's$_{\text{(visual)}}$ first retellings referred to a contextual-conflict and contextual-resolution. For instance, Claire focused on the "hotness" and "coldness" of two lands, "Each other temperatures are different from what they have it. Cause there is hotter people and colder people." She described the resolution of the story as "going back home."

Session 2. Lola$_{\text{(question)}}$ was the only one to not refer to at least partial-mathematical-conflict in the second retelling. Lola explained the two lands' conflict as "it was getting too cold in Cozyland and it was getting too hot in Icyland" and resolution of "figuring out the problem like drawing out and seeing the difference," which implicitly reflected her way of thinking about two lands' opposite thermometers. Therefore, Lola referred to contextual-conflict and partial-mathematical-resolution of the story. Although Claire$_{\text{(visual)}}$ more explicitly described the differences of two lands' thermometers, "They drew the thermometer and saw that theirs was *opposite* from each other," she did not articulate a way to distinguish them and exhibited partial-mathematical-conflict and partial-mathematical-resolution. Chase$_{\text{(mixed)}}$ said, "They didn't know what to do and they couldn't figure out what's wrong cause they both had the *same temperature, but it wasn't*" and their conflict was resolved by using "*negatives* and *positives*." Therefore, Chase was classified as complete-mathematical-conflict and complete-mathematical-resolution. With the same classification as Chase, Harry$_{\text{(control)}}$ described the notation that distinguishes positive and negative numbers, "They *made a line* next to the negatives because there's a line [for numbers] below zero and it was colder. So, then you could tell it was different going down below [zero]." Only Chase and Harry represented complete-resolution in their drawn thermometers (Lola: Beginning-resolution and Claire: No-resolution).

Session 3. All students' drawn thermometers were categorized the same as session 2. All students referred to a partial- or complete-mathematical-conflict and partial- or complete-mathematical-resolution in the third retelling. For instance, Lola's$_{\text{(question)}}$ interpretation of the story conflict and its resolution showed the most growth:

> Why it was getting *more cold* in one land and *more hot* in the other and they got confused because they didn't know that hot is like that way [thermometer going up] and cold is like going [thermometer going down] …They drew a little chart out on the ground, and they saw their thermometers in their world. That's how they figured out that *there's difference*.

Even though compared to the previous session Lola provided a more detailed description of the mathematical conflict and its resolution, the absence of referring to positive and negative numbers put her in partial-mathematical-conflict and partial-mathematical-resolution category. Despite Claire's$_{\text{(visual)}}$ reference to the opposite thermometers, she did not articulate further how the oppositeness of two categories of numbers can be distinguished and was classified as partial-mathematical-conflict and partial-mathematical resolution. Harry$_{\text{(control)}}$ and Chase$_{\text{(question)}}$ both exhibited complete-mathematical-conflict and complete-mathematical-resolution. Harry said:

> So, they decided to make a different thermometer. Since that one [Icyland thermometer] is *going down* and that one [Cozyland thermometer] is *going up*, they decided to make a big thermometer that showing *positives* and *negatives*. So, for the Icyland, they're in the negatives and they're cold, so *they're not really the same thirty*; they're different and so in the Cozyland it's hot [and] up here in the positives.

Discussion and Implications

On the pretest, all students were categorized as partial-resolution but Harry$_{\text{(control)}}$ and Chase$_{\text{(mixed)}}$ started higher compared to Lola$_{\text{(question)}}$ and Claire$_{\text{(visual)}}$. On the posttest, only Harry exhibited the complete-resolution category, but Lola made the most growth. Even though it is difficult to make a strong argument about the role of interactive features in the development of the mathematical resolution, Lola seemed to benefit the most based on her scores and integer mental model level from pretest to posttest. Overall, the reading and retelling sessions gradually drew students' attention to the differences between absolute values versus linear values (Bofferding, 2019) and resulted in more references to the mathematical conflict and its resolution during their retellings. The results show that students needed a second reading to absorb the complexity of the story elements and attend more to the story's mathematical conflict and its resolution rather than contextual conflict and its resolution. The use of directed magnitude language and reference to the thermometers increased throughout the sessions, which suggests the promises of using the *Temperature Turmoil* book in highlighting the mathematical language and visual representation and in establishing opportunities for productive struggle. Perhaps, the mixed interactive version has a higher potential to draw students' attention to both the visual and language. The implication of this research provides insight into viable forms of literacy and mathematics integration.

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USE OF ZIG-ZAG TO REPRESENT MATHEMATICAL THINKING ABOUT ANGLE

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Angle is a prominent feature of shapes which make it important to provide students with opportunities to carefully explore the idea of angle beginning in the elementary grades. But developing the understanding of angle concept is complex (Tanguay & Venant, 2016; Alyami, 2020). The purpose of this study was to understand students' use of angle within the Tinker Lab, a non-formal making space. We examined the situated use of the informal term zig-zag to develop and communicate mathematical ideas about angle from an embodied perspective (Kirsh, 2013). With the research question – *How is the term 'zig-zag' used to convey ideas of angle in the context of a making activity?* The video data was collected from three groups, one group each in Grades 3-5, participating in a 25-minute activity. The activity consisted of two phases: (1) making a path by placing the tape on the floor from one side of the room to the other; and (2) programming Dash (a toy robot) on the path laid by the other group.

In answer to our research question, results showed that students regularly used 'zig-zag' to communicate in two different types of scenarios: (a) **about a single angle** – communicating about a specific angle between two adjacent sections of tapes. To form an angle, students negotiated about two parameters – the length of the tape (i.e., length of rays) and direction of connection between two sections of tape (i.e., the angle between the rays) (Refer Figure 1); and (b) **about a broader pattern which consists of multiple angles –** communication about the path without referring to a particular section of tape as a base arc for the angle or about a particular angle composed of rays (Refer Figure 2);. We found that students struggle to communicate about angles, but the use of a casual term 'zig-zag' allowed them to facilitate their group communication. Zig-zag' was found to be a more powerful – if less precise – way to communicate that does not require students to have the same perception of magnitude or even be talking about the same arc (Williams-Pierce et. Al., 2021).

Figure 1: A representation of 'zig-zag'

Figure 2: A broader zig-zag pattern

The findings of this study implies that students' perceptions, intuitions, and physical gestures compose their understanding and use of angles, these non-formal representations can be leveraged to better develop their understanding of the concept of angle. This study signifies students rely on the use of informal mathematical language and physical gestures to form and communicate mathematical reasoning around angle within the situated use of zigzag.

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INVESTIGATING THE IMPACT OF COVID-19 ON ELEMENTARY MATHEMATICS TEST SCORES

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The COVID-19 pandemic has ravaged onward over the last year and has greatly impacted student learning. An average student is predicted to fall behind approximately seven months academically; however, this learning gap predicts Latinx and Black students will fall behind by 9 and 10 months, respectively (Seiden, 2020). Moreover, the shift to online instruction impacted students' ability to learn as they encountered new stressors, anxiety, illness, and the pandemic's psychological effects (Middleton, 2020). Despite the unprecedented circumstances that students were precipitously thrust into, state testing and assessments continue. Assessments during the pandemic are likely to produce invalid results due to "test pollution," which refers to the systemic "increase or decrease in test scores unrelated to the content domain" (Middleton, 2020, p. 2). Considering the global pandemic, test pollution is prominent and worth exploring as it is uncertain whether state testing can identify the impact COVID is having on student learning.

NWEA produces the Measures of Academic Progress (MAP) assessment test. NWEA argues that MAP testing can also provide school districts with the ability to "identify trends for students, create flexible learning groups, and target professional development for teachers" (Belgard, 2017, p.1). This case study research aims to identify if current learning conditions have impacted MAP scores in a rural school setting due to the pandemic. The three research questions are: (a) Is there a statistically significant difference in 4th grade MAP math scores between students in the September 2019 cohort and September 2020 cohort? (b) Is there a statistically significant difference between 4th-grade math MAP scores based on gender within this rural school district between September 2019-September 2020? (c) Is there a statistically significant difference between this school districts' 4th grade math scores and national 4th grade math MAP scores for September 2019 -September 2020?

Data analyses suggest no statistically significant difference between the two cohorts, nor is there a statistically significant difference in 4th grade MAP math scores between male and female students. Data findings suggest no statistical significance between our case study cohorts' mean math scores and the comparison groups: the case study district, and the national norm data. While COVID has a wide-reaching impact on school activities, our findings indicate that the COVID-19 pandemic may not necessarily affect student learning outcomes as measured using MAP scores. Future studies that utilize qualitative methods, such as teacher and student interviews, should be conducted to problematize these findings.

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A LOCAL INSTRUCTION THEORY FOR EMERGENT GRAPHICAL SHAPE THINKING

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Keywords: Learning Trajectories and Progressions, Design Experiments, Algebra and Algebraic Thinking, Middle School Education

Emergent graphical shape thinking (or *emergent reasoning*) entails conceiving of a graph as a trace that represents a covariational relationship between two quantities' magnitudes (Moore, 2021; Moore & Thompson, 2015). Despite the importance of such thinking to graph construction and interpretation in mathematics and other subject areas (e.g., Glazer, 2011; Paoletti et al., 2020; Potgieter et al., 2008), researchers have indicated emergent graphical shape thinking is non-trivial, even for U.S. teachers (Thompson et al., 2017). Though such studies reveal challenges in expressing this way of thinking, some middle and high school students have demonstrated elements of emergent reasoning during teaching experiments designed to achieve other goals (i.e., Ellis et al., 2015; Johnson, 2015). These studies suggest that such reasoning is within reach for these populations.

Our study sought to develop a learning progression to address the following research question: "How can we support eighth grade students to develop emergent graphical shape thinking as part of their stable meanings for constructing and interpreting graphs?" Our report responds with a *local instruction theory* (hereafter, LIT), which is a generalized, reasoned, and adaptable learning path that can inform instruction toward a specific mathematical goal (Gravemeijer, 2004). We conducted a series of six small group teaching experiments in a diverse middle school in the northeastern United States, culminating in a full-class teaching experiment with eight eighth-grade students who had just completed a high school level Geometry course. For each teaching experiment, we obtained parental consent and student assent, openly video- and audio-recorded the students as they worked, and collected and digitized written work samples. We analyzed the data using ongoing and retrospective analyses, consistent with the teaching experiment methodology (Steffe & Thompson, 2000). During each iteration, we designed and revised the task sequence as well as our underlying LIT.

In this poster, we present our theoretically- and empirically-grounded LIT to support students in developing stable meanings for graphs that entail emergent graphical shape thinking. Specifically, our LIT posits repeated engagement with and a fundamental relationship between 1) quantitative and covariational reasoning (e.g., Thompson & Carlson, 2017), 2) reasoning within a coordinate system (e.g., Lee, 2016; Paoletti et al., 2020), and 3) emergent reasoning. We illustrate the interrelationship of these ways of thinking through examples from two focal students as they engaged in a task sequence designed in alignment with our LIT. We present implications that span both research and practice, with particular emphasis on designing instructional supports for supporting middle and high school students' graphical fluency.

Acknowledgments

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THE EFFECTS OF OPERAND POSITION AND SUPERFLUOUS BRACKETS ON STUDENT PERFORMANCE IN MATH PROBLEM-SOLVING

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Early middle school students have a strong tendency to adhere to the left-to-right principle when solving math problems, which may lead them to overlook the role of brackets within math expressions (Blando et al., 1989; Gunnarsson et al., 2016). However, perceptual features, such as the spacing between symbols, can direct students' attention to salient features of math expressions that support efficient problem-solving (Goldstone et al., 2017; Harrison et al., 2020). Similarly, superfluous brackets—brackets that do not change the meaning of notation, but can guide learners' attention to the correct procedure—can help students achieve higher success rates in solving problems (Hoch & Dreyfus, 2004; Marchini & Papadopoulos, 2011). Here, we examine how the presence of superfluous brackets and the position of higher-order operands (i.e., multiplication and division) within an expression affect student performance in an online assignment, as measured by assignment mastery speed and average response time.

We are conducting a 3 (Operand position: left, center, or right) X 2 (Superfluous brackets vs. no brackets) design study in an online tutoring system, ASSISTments (Heffernan & Heffernan, 2014). Approximately 280 sixth and seventh-grade students will complete an assessment of their baseline math knowledge, then be randomly assigned to one of six conditions: 1) brackets-left (e.g., (5*4)+2+3), 2) no brackets-left (e.g., 5*4+2+3), 3) brackets-center (e.g. 2+(5*4)+3), 4) no brackets-center (e.g., 2+5*4+3), 5) brackets-right (e.g., 2+3+(5*4), and 6) no brackets-right (e.g., 2+3+5*4). Students will complete problems in an ASSISTments' Skill Builder where the goal is to "master" the content by answering three questions correctly in a row (Kelly et al., 2015). We will conduct two ANCOVAs to examine how operand position and superfluous brackets affect student performance as measured by average response time (i.e. total time on Skill Builder divided total problems a student attempted) and mastery speed (i.e. the total count of problems a student attempted before reaching mastery).

Data collection is on-going; results will be included in our final submission and presented at the conference. This study will advance research on the roles of perceptual cues in math notation by shedding light on how the presence and position of superfluous brackets affects student performance. We aim to provide recommendations for the presentation of expressions in online learning platforms to support learning.

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WHY DO STUDENTS PICK THE WRONG ANSWER ON FRACTION MULTIPLE CHOICE WORD PROBLEMS?

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Solving word problems with understanding is challenging as it demands the comprehension of the quantitative meaning of two (or more) numbers, how these quantities are related to the result by a mathematical operation, and an anticipation of the meaning of the number attained after the procedural manipulation (Duzenli-Gokalp & Sharma 2010; Wyberg et al., 2012). Multiple choice questions force students to pick an answer, and researchers, teachers, and other stakeholders often assume that this choice meaningfully indicates what students understand. In this study, we examined the connections between how students interpret and solve a strategically designed multiple choice word problem, with an aim to explore why students picked the wrong answer. The goal of our larger effort is to understand how teachers and researchers make inferences about student knowledge based on students' multiple-choice answers, and to what extent such inferences are accurate.

Our work is guided by the ideas related to *operation sense*, defined as the ability to make sense of the quantitative meaning related to basic arithmetic operations (Baroody et al., 2006; Verschaffel et al., 2000). Operation sense involves understanding the underlying structure and properties that the operation possesses (Briars and Larkin, 1984), and the various forms and contexts in which the operation could exist (Carpenter, 1985). Verschaffel et al. (2000) discussed two ways students' operation sense is revealed in their problem solutions: how a word problem *situation* is transformed into a simplified model in the *translation process* and how *mathematical symbols and operations* are used to produce a result. We designed multiple choice problems to assess how students interpreted the problem (*situation)*, chose a specific arithmetic operation (*translation*), and performed computation (*mathematical operation*). As a paradigmatic example, we present results related to one problem that addresses a Grade 5 standard for fraction subtraction (CCSS.MATH.CONTENT.5.NF.A.1). We analyzed both the answer choices as well as qualitative data drawn from students' written work to examine the operation sense that supported their choices. The data is drawn from a sample of Grade 5 students ($N = 1465$) from a Mid-Western state and part of a larger study. The students' answer choices and written work was examined using thematic coding (Braun & Clarke, 2006). Within each response option, we coded for distinguishing features and themes in students' operation sense.

Around 35% of the students struggled in the translation process because they used addition or multiplication. Although most students (64%) translated the problem as a subtraction problem, only 47% chose correct answer. Notably, there were substantial differences among the students who selected the same multiple-choice option in terms of their mathematical operation. For example, some operated with the mixed fraction as a sum of a whole and a fraction whereas others converted to an improper fraction.

These results illustrate how inferences about students' operation sense from multiple choice word problems depend largely on the set of provided answer choices. This demonstrate that even with a carefully designed set of options such inferences may be more warranted for translation than they are for mathematical operation. This finding has implications for teaching and research by clarifying the affordances and limitations of multiple-choice assessments.

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YOUNG CHILDREN'S USE OF GESTURING DURING DURATIONAL REASONING

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Keywords: Measurement, Cognition, Early Childhood Education, Elementary School Education

Children attend to many different attributes when reflecting on the duration of past experiences, such as their perceived efforts or the accumulating activities of their experience (Smith, 2021). Their verbalizations of these attributes provide evidence of how they might quantify the duration of their experiences. For example, a 6-year-old explaining that vacuuming took a long time because it was hard work. Here, it seems that the child correlated her efforts (hard work) with the duration of her experience (long) and expressed this through her words.

I found that when describing the duration of their lived experiences, some 4- and 5-year-olds used iconic gestures (gestures that parallel the action or object being described; McNeill, 1992) in conjunction with their words. Such spontaneous gestures can provide an "observable…interpretable…index" of children's understandings (Goldin-Meadow, Wein & Chang, 1992). I conjecture, therefore, that these gestures might serve as further evidence of young children's conceptions of duration as a measurable quality of their world (Smith, 2021).

During one conversation, 4-year-old Cody (pseudonym) described the duration of flying on an airplane to a different state. As he described his experience, Cody first verbalized that it took a long time. When I asked how he knew it took a long time, Cody spread his arms as wide as he could (Figure 1). I inferred that Cody was reflecting on the far distance of his travel, which he embodied through his iconic gesture (McNeill,1992) of a wide arms-length. Here, it seemed that Cody was relating duration with a length-based measure (Earnest, 2019). I inquired about the length that he displayed by posing, "That long?" Cody then changed his gesture to a fast movement of an "airplane" (Figure 2) as he verbalized that "because the airplane is fast and then it took a very fast time." Here, again, Cody seemed to use his gesturing to support his conception that the speed of the airplane impacted the duration of his experience, a common conception among children (Piaget, 1969).

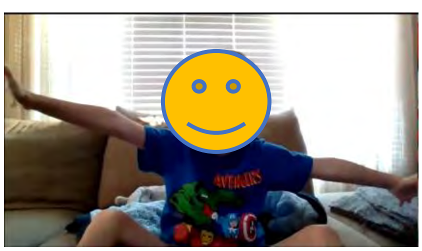

Figure 1: Cody Gesturing the Far Distance of his Travel

Figure 2: Cody Gesturing the Fast Movement of the Airplane

McNeill (1992) asserted that "Gestures and speech grow up together" (p. 295). When considering duration—an invisible, intangible quantity (Earnest, 2019; Long & Kamii, 2001)—how young children might utilize gestures in their descriptions of duration may enable researchers and teachers to better recognize what conceptions they have.

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INVESTIGATING CONNECTIONS BETWEEN COVARIATION AND UNITS COORDINATION IN MIDDLE SCHOOL STUDENTS

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This poster presents some preliminary results from study investigating connections between the middle school students' Units Coordination (UC) and Covariational Reasoning (CR). UC has been shown to be important in elementary and middle school mathematics (Hackenberg and Lee, 2012; Olive, 2001; Steffe, 1992) whereas CR has been shown to be important in high school and undergraduate mathematics (Carlson et al., 2002; Johnson, 2015). However, little is known about connections between a student's UC and CR. This study used a Piagetian perspective to frame students' mathematics in terms of their mental actions in line with theoretical groundings of UC and CR to answer: How do middle school students' units coordinating structures contribute to their ability to conceptualize how two quantities covary?

To answer this question, semi-structed clinical interviews were done over Zoom with 6 middle school students. Students were selected based on UC stage (Norton et al., 2015). This poster reports results some preliminary results from three of the six students: a 6th grader at Stage 2, a 7th grader at Stage 3, and an 8th grader at advanced Stage 2. The sample task reported on here consists of an animation of a triangle with green side decreasing discretely exponentially with the orange side growing continuously linearly. Students were asked a series of questions designed to elicit their covariational reasoning about how the green side and orange side changed in relation to one another. Student work on the task is shown in Figure 1.

(a) (b) I

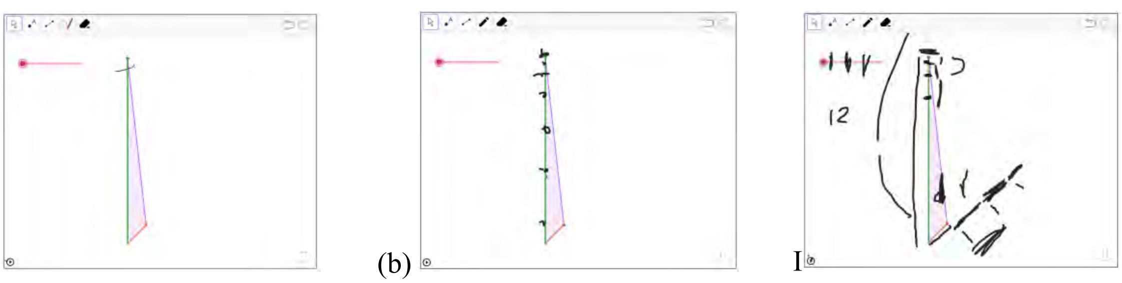

Figure 1: (a) 6th grade Stage 2; (b) 8th grade advanced Stage 2; (c) 7th grade Stage 3

Video data and student written work were analyzed using Thompson and Carlson's (2017) Levels of Covariational Reasoning framework. Based on some preliminary analysis, students at different UC stages reason differently in covariation tasks. For example, in Figure 1, we see that the Stage 3 student was able to attend to both quantities whereas both Stage 2 students only focused on one quantity. Both Stage 2 students were not able to form a relationship between the orange and green line beyond gross coordination of the direction of change; however, the advanced Stage 2 student was able to describe how the green line was changing for equal amounts of time passes. This indicates an ability to reason at a higher level than just gross coordination when one of the variables is time. The Stage 3 student was able to coordinate amounts of change in both quantities and seemed to create a multiplicative object of the two quantities changing together. In line with Hackenberg and Lee's (2015) and Boyce and colleague's (2020) findings, I've observed that students with high-level units coordinating structures (higher UC stage) are able to engage in more sophisticated mathematical reasoning.

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A TOOL FOR COMPARING FRACTIONS OR A TOOL FOR DISPLAYING CONCLUSIONS? STUDENTS' FRACTION NUMBER LINE USE

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The number line is a powerful visual model of quantity and is widely prevalent in mathematics curriculum. Empirical studies found that students' ability to use a fraction number line for comparing fractions predicted their later mathematical achievements, such as their algebra performance (Booth & Newton, 2012; Torbeyns et al., 2015). Given algebra is also a gateway to post-secondary mathematics and STEM; it is no exaggeration to say that mastery of the number line should be a critical goal of elementary mathematics education.

Researchers (e.g., Teppo & van den Heuvel-Panhuizen, 2014) have synthesized a wide range of pedagogical uses for the number line across several curricular topics in the elementary and secondary curriculum. Still, we know little about how students use the number line as a tool to solve mathematical problems. In this study, we examined fourth graders' use of fraction number lines to answers the following two research questions: RQ1. To what extent do fourth graders use number lines to solve fractional tasks? RQ2. How are fraction number lines used?

We surveyed 214 fourth graders from ten elementary classrooms in a Midwest state as part of a larger research project. These ten classes were in seven different elementary schools across five different counties. The survey included four open-ended tasks with one task addressing each of the following topics: fraction comparison, fraction equivalence, fraction additions, multiplication of fractions by a whole number. To answer RQ1, we counted all the student responses that include a number line for justification in each task and calculated the percentages. To answer RQ2, we examined the fourth graders' written work who used number lines on the fraction comparison task. We only examined the case of the fraction comparison task because only one student used a number line on any of the other tasks. We grouped the responses shared similar features to identify the themes of students' number line use.

We found 13 students used number lines for fraction comparison (6%, N=214), one did so on the fraction equivalence task (0.5%), one on the fraction addition task (0.5%), and none on the fraction multiplication task. These results revealed that very few students chose number lines to explain fraction concepts. By scrutinizing the 13 responses, we identified three fraction number line use themes: to determine the size of a fraction (one student), (2) to support benchmark reasoning (two students), and (3) to represent comparison conclusions (ten students).

Our findings revealed that only a limited number of students chose number lines to explain fraction concepts. What's more, most students used the number line to display number order among this limited number of students. Very few used the number line for displaying equivalence or additive or multiplicative relationships (Teppo & van den Heuvel-Panhuizen, 2014). We also noticed a connection between students' conceptual understanding of fractions and their conceptualization of number lines. In particular, six of the ten students used number lines to represent comparison conclusions with whole number biased misconceptions (Ni & Zhou, 2005; Lai & Wong, 2017; Vamvakoussi & Vosniadou, 2010). Students' low tendency to use the number line model is an alarm that emphasizes the need for more regular and meaningful instructional experiences with the fraction number line. Moreover, the result suggests that professional development and teacher education may be needed to expand teachers' knowledge for teaching number lines more effectively throughout the curriculum.

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Chapter 5:

Math Content — Later Years

Olanoff, D., Johnson, K., & Spitzer, S. (2021). *Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Philadelphia, PA.

ISOMORPHISM AND HOMOMORPHISM AS TYPES OF SAMENESS

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Isomorphism and homomorphism are topics central to abstract algebra, but research on mathematicians' views of these topics, especially with respect to sameness, remains limited. This study examines 197 mathematicians' views of how sameness could be helpful or harmful when studying isomorphism and homomorphism. Instructors saw benefits to connecting isomorphism and sameness but expressed reservations about homomorphism. Pedagogical considerations and the dual function-structure nature of isomorphism and homomorphism are also explored.

Keywords: Advanced Mathematical Thinking, Undergraduate Education

Students' understanding of isomorphism in abstract algebra has been studied for over twenty-five years (Dubinsky et al., 1994), but research explicitly on students' understanding of homomorphism has begun more recently (e.g., Melhuish, et al., 2020; Rupnow, 2021). In an effort to position students' understanding, we wanted to learn more about how mathematicians position these topics in relation to notions of sameness. Thus, in this paper we address the following research questions: (1) What connections do algebraists see between sameness and isomorphism? (2) What connections do algebraists see between sameness and homomorphism?

Literature Review and Conceptual Framework

Prior work on students' understanding of isomorphism has shown associations between isomorphism and sameness. Leron et al. (1995) described a course in which students were taught to focus on sameness with isomorphic groups. Subsequent literature has confirmed references to sameness in the context of isomorphism by other groups of students and professors (e.g., Rupnow, 2021; Weber & Alcock, 2004). However, large-scale research has not verified whether this emphasis on "sameness" is normative across mathematicians.

Furthermore, small-scale research on mathematicians has revealed other types of language commonly used to describe isomorphism and homomorphism. Weber and Alcock (2004) highlighted algebraists' references to relabelings. Hausberger (2017) observed use of "structure-preservation" to refer to the homomorphism property. Rupnow (2021) observed renamings, relabelings, and structure-preservation as well as references to operation-preservation, disembeddings, and use of equivalence classes to describe isomorphism and homomorphism. However, the prevalence of these types of language among algebraists has remained unknown.

Our theoretical lens is conceptual metaphors (e.g., Lakoff & Nú ez, 1997), in which a source domain is used to structure understanding of a target domain. For example, "An isomorphism is an operation-preserving map" is a conceptual metaphor that describes the target domain (isomorphism) in terms of a source domain (operation-preserving map) to provide a way of thinking about isomorphism. In this case, the metaphor encourages focus on the homomorphism property, which guarantees a similar type of behavior in both structures (e.g., groups). In this paper, we build on Rupnow's (2021) previous isomorphism and homomorphism metaphors.

Methods

Data were collected from a survey sent to every 4-year college/university math department in the United States. This survey addressed how algebraists think about sameness in general and in specific mathematical contexts. Participants were 197 mathematicians from 173 institutions who had taught at least one abstract algebra or category theory course in the last five years.

The four survey questions relevant to this paper, numbered below, queried participants' beliefs about sameness related to isomorphism and homomorphism, and were the first reference to isomorphism or homomorphism in the survey text itself. These questions followed questions on the nature of sameness in math and about how similar particular objects were.

1. How might sameness be helpful when thinking about isomorphism/isomorphic structures? (Q1)
2. How might sameness be harmful when thinking about isomorphism/isomorphic structures? (Q2)
3. How might sameness be helpful when thinking about homomorphism? (Q3)
4. How might sameness be harmful when thinking about homomorphism? (Q4)

Responses to the isomorphism questions were grouped together for coding as were responses to the two homomorphism questions. Each paired response could receive multiple codes. To ensure coding validity, we used investigator triangulation with two members analyzing the data. Each member would independently code the data using the agreed codes; we then discussed any coding discrepancies and came to consensus on the final codes. These discussions included any modifications for future coding, such as refined code definitions or new codes for consideration.

The data were analyzed in accordance with thematic analysis (Braun & Clarke, 2006). First, we used versus coding (Saldaña, 2016) to identify different beliefs about sameness based on the help vs. harm contrast. However, after coding, we determined that these codes did not effectively capture all nuances in the data. We then revised codes, using descriptive coding (Saldaña, 2016) to supplement our initial coding. These second-round codes permitted clearer connections to our conceptual framework by incorporating codes based on Rupnow's (2021) prior work.

Results

We present mathematicians' responses about the helpfulness and harmfulness of sameness to considering isomorphism and homomorphism. Code frequencies and percentages are presented in Table 1. Participants largely viewed sameness as conceptually relevant to isomorphism. However, pedagogical issues and the context-dependent nature of isomorphism were noted as potential difficulties if using "sameness" as a substitute for isomorphism. In contrast, participants viewed sameness as needing to be qualified or viewed sameness as irrelevant to homomorphism.

Helpful or Harmful

Based on the question format, where we asked participants about how sameness might be helpful or harmful for thinking about isomorphism and homomorphism, our default expectation was for respondents to address both helpful and harmful aspects of sameness. This was the case for isomorphism, where 72% of respondents were coded as helpful/harmful. For example:

> [Helpful:] Isomorphism is a kind of sameness, so certainly you have to have some sense of sameness to understand the idea behind isomorphism. [Harmful:] Maybe thinking that sameness = identical in every aspect? At some point you always have to move away from

intuition coming from English (and "sameness" is certainly not a mathematical concept) and rely only on mathematical definitions to make progress.

In this participant's view, isomorphisms are a type of sameness, but issues arise if one relies on a concept without a formal mathematical definition. Another participant had a different interpretation, focusing on the specific aspects that are and are not the same:

> [Helpful:] We all like to group things that are the same together, and it is useful to think that two very different looking objects (e.g., rings) that have the same algebraic properties should be put in the same group. We want to emphasize that algebraic objects should be studied based on their algebraic properties, not on the choice of names for their objects. [Harmful:] If a student starts to think isomorphic objects are the same as sets and mixes up elements of the objects, that could be harmful. If we use the wrong sort of sameness and think that the identity of a group must always be 0, for example, we could easily become very confused.

Table 1: Frequencies of Codes.

Category	Code	Isomorphism n(%)	Homomorphism n(%)
Help/Harm	Not harmful	18(9%)	5(3%)
	Helpful/harmful	142(72%)	72(37%)
	Not helpful	15(8%)	35(18%)
	Similar	0(0%)	38(19%)
	Not relevant	1(1%)	12(6%)
Pedagogical Considerations	Motivating instruction	15(8%)	7(4%)
	Leveraging intuition	33(17%)	8(4%)
	Misconceptions	25(13%)	20(10%)
	Imprecise language	29(15%)	14(7%)
Types of Sameness	Context-dependent	77(39%)	16(8%)
	Levels of sameness	11(6%)	6(3%)
	Generic identical	30(15%)	2(1%)
	Generic equal	20(10%)	6(3%)
	Isomorphism vs. homomorphism	1(1%)	76(39%)
Informal Sameness	Relabeling	10(5%)	0(0%)
	Matching	13(7%)	4(2%)
	Same behavior	48(24%)	36(18%)
	Same properties	7(4%)	7(4%)
	Structure preservation	9(5%)	12(6%)
	Operation preservation	5(3%)	20(10%)
	Disembedding	0(0%)	18(9%)
	Equivalence classes	0(0%)	3(2%)
Functions vs. Structures	Isomorphism vs. isomorphic	15(8%)	0(0%)
	Homomorphism vs. homomorphic	0(0%)	38(19%)
	Fundamental Isomorphism Theorem	0(0%)	12(6%)

This participant described isomorphisms as a way to classify objects into categories and viewed sameness as helpful for that grouping but emphasized that identification between objects was not required and could cause confusion for students (e.g., names of elements can differ).

In contrast, only 37% of respondents clearly highlighted both helpful and harmful aspects of homomorphism. For example:

> [Helpful:] It can be helpful, say to emphasize that it preserves some information, but not all. For instance, I like to say that homomorphisms from something complicated to something easier to work with or better understood are often the right approach (e.g., representations). Though one may lose info, by working with something easier you may still learn something new about the original. [Harmful:] Similar to the above, it should be emphasized that a lot of info can be lost, or that homomorphism is far from saying they are exactly the same, but is maybe a tool for extracting some information about sameness.

Note that the participant highlighted preservation of some aspects but a loss of some information, indicating utility but the need for care when discussing sameness with homomorphism.

Some participants only saw benefits to using sameness. For isomorphism, 9% of participants only expressed a helpful view of sameness: “Well, it’s the essential notion of isomorphism. In no way [harmful], but it is important that we understand sameness to mean sameness of underlying structure, not sameness of superficial characteristics, like labels.” Here we see the participant considered sameness to be the conceptual point of isomorphism, and thus did not consider sameness harmful. 3% of participants expressed an exclusively helpful view of sameness for homomorphism. For example: “[Helpful:] Same as isomorphism, except now we are only identifying a part of each of the two structures that behave the same algebraically. [Harmful:] Again, with carefully presented examples I don’t think there is harm per se.” Notice, even though the participant claimed sameness was helpful and not harmful for homomorphism, this sameness only referred to parts of structures instead of whole structures.

Although most isomorphism responses received a not harmful or helpful/harmful code, 19% did not. One participant was coded as not relevant: “When Isomorphism is being considered, isomorphism defines the sameness, and what makes the isomorphic objects “different” is to some extent obvious, but not really of interest. So considering sameness is neither a help nor a hindrance.” They saw the reverse connection of isomorphism giving some insight into sameness but did not consider this notion to be important for understanding isomorphism. Others saw sameness as relevant, but it was unclear whether they viewed sameness as helpful, harmful, or both: “[Helpful:] I like distinguish equality (for subsets of a given ambient object) and isomorphism. [Harmful:] The idea [of] flexible notions of equality or sameness is pretty subtle and counterintuitive.” While this participant typed distinct responses in the two boxes for the “helpful” and “harmful” responses, their response did not directly address how sameness might be helpful or harmful for understanding isomorphism, so it was not given any of those codes. Finally, 8% of respondents only expressed a harmful view: “Helpful? I don’t think it is. There’s nothing added to the concept of isomorphism by saying the word “same”. Well, homomorphisms also preserve something. Bijections also preserve something. So, talking about “same” is going to blur some distinctions.” This participant only saw a lack of clarity arising from sameness.

Participants expressed more skepticism to using sameness to discuss homomorphism, with 61% of responses *not* receiving a helpful-related code. 18% of participants considered sameness not helpful to describing homomorphism: “I think tying “sameness” to any homomorphism that is not an isomorphism is misleading at best. Not a fan.” Others were unwilling to use sameness but allowed similarity: “A homomorphism provides a notion of similarity.” 6% of participants considered sameness irrelevant to homomorphism: “Homomorphism is restricted version of the “sameness” defined by isomorphism. Usually when trying to show homomorphism exists, it is trying to show that a certain defined property holds and I do not see how sameness either helps or hinders.” Other participants acknowledged the relevance of sameness to homomorphism, but whether they viewed it positively or negatively was unclear. For example: "[Helpful:] Can there be a connection between these two structures even though they are different? We are locating a connection that is not as deep as isomorphism. [Harmful:] Homomorphic structures may not be isomorphic." Here, the participant described some connections between structures and compared the relationship to isomorphism, but it was unclear what they meant by this connection.

Pedagogical Considerations

In addition to the notions of helpfulness and harmfulness explicitly prompted by the questions, a number of respondents focused on pedagogical implications of using sameness. Some specifically highlighted how sameness could be useful for motivating isomorphism or for connecting to students' intuition: "Different levels of "sameness" and different informal definitions of "sameness" can be used to motivate the formal definition of isomorphism." The participant here explained how formalizing sameness could provoke a need for isomorphism. Leveraging intuition was also described: "It might be helpful for students to think of sameness in a familiar context (e.g., geometry or linear algebra) in order to appreciate the notion of isomorphism in algebra." Here, the participant described how students' intuition and prior experiences with sameness in math could be used to help them understand isomorphism.

However, some respondents highlighted pedagogical concerns like student misconceptions or imprecision. Misconceptions often addressed difficulties with names of elements or objects: "Students often think that if two sets have different looking objects (integers vs matrices, for example), then they can't be "the same." This makes it more difficult for them to understand the more meaningful examples in class." Here the participant observed students could struggle with identifying superficially different objects. Another common concern was that using sameness may lead to imprecision in exercises and proofs: "Two objects can be isomorphic as groups under their additions, but not as rings, when both addition and multiplication are involved. The idea of sameness must be carefully used especially with students since they tend to forget the context." Here, the respondent worried that using sameness haphazardly could lead students to confuse different types of isomorphism and to not attend to context.

Although some participants described ways sameness could be helpful for teaching homomorphism (motivating instruction and leveraging intuition) this happened less than with isomorphism. Some motivated a specific aspect of homomorphisms: "Help understanding the importance of study of kernels." Others described how sameness can aid intuition: "If [a homomorphism] is injective, you could talk about how the structure of the domain is the "same" as the structure of the range, and again this informal notion could make the concepts accessible for students." Again, this participant did not make a blanket statement about sameness in homomorphism but qualified it as useful for considering injective (one-to-one) homomorphisms.

Pedagogical concerns about using sameness for homomorphism were similar to isomorphism concerns. Respondents often described misconceptions about the strength of sameness in homomorphism. For example: "Again, the wrong sort of sameness, as in equality of elements of sets, could be problematic if the student, for example, thinks all identities are actually the element 0." Participants also described issues with being imprecise, including difficulties that could arise when students wrote proofs: "If students get too comfortable expressing things "are the same" without being formal, their proofs can very quickly become incorrect." Although this participant had previously noted utility in thinking about sameness with homomorphism as it connected to the isomorphism theorems, they acknowledged dangers in using loose definitions.

Types of Sameness

Other responses focused on the nature of sameness and specific types of sameness. Some detailed problems if using sameness for specific concepts by highlighting the context-dependent nature of sameness. For example: “The context and criteria for sameness need to be clear for isomorphism to be something that can be empirically verified as true.” This highlights the necessity of describing the context in which sameness is used but gives few details. Others were more specific, describing different levels of sameness: “There are different “strengths” of sameness: equals, equivalent, related to, almost/weakly equivalent, etc. There is not a one size fits all to sameness.” This provides a variety of types of sameness that might be placed on a continuum for strength comparison.

Similarly, the participants were specific about other concepts that could be confused with isomorphism, such as being equal or identical. Consider a confusion with equality example: “Isomorphic is not the same thing as “equals” as it does not imply a canonical identification. The word “same” can trip people up in that way.” Here, confusion between different mathematical understandings of sameness, isomorphism and equality, are specifically highlighted. Similar issues arose with identical: “In common, nonmathematical parlance, same means identical, so when students hear the word “same” they may think identical.” Note this participant’s identification with sameness and identical, a strong type of sameness.

Many respondents compared isomorphism and homomorphism (39%), with a focus on the strength difference. For example: “Same has too strong a connotation in most students minds and they may interpret this to mean isomorphism rather than homomorphism.” Implicitly, the participant seems to suggest that sameness implies a strong relationship, so students will identify the stronger concept (isomorphism) with sameness.

Informal Sameness

Participants used descriptive language of varied specificity to highlight sameness in isomorphism and homomorphism. Some participants highlighted shared defined properties or generally referred to same behavior. For instance, this participant was coded as same properties: “…Also, the facts that properties like cyclic and Abelian are preserved by isomorphisms.” Notice, the respondent referred to defined properties that are shared by isomorphic structures. Other participants wrote generally of shared behavior for isomorphism or homomorphism: “The idea of an isomorphism is that two different sets of objects can behave the same in certain scenarios.” And “With a homomorphism, the objects of the image will behave in “the same” way as the domain (or quotient based on the domain).” While highlighting the sameness of objects linked by a morphism, such responses did not provide details on the shared sameness.

Participants also used renaming/relabeling and matching language to describe isomorphism and, to a lesser extent, homomorphism. This participant described isomorphism in terms of a renaming of elements: “I like to emphasize to students that algebraists care about the algebraic structure and equations, and we don’t care nearly so much about what we choose to name the elements in these structures.” Note they highlighted the arbitrary nature of element names in keeping with Rupnow’s (2021) distinction between renaming/relabeling and matching. Another participant described isomorphism in terms of matching: “For finite groups where Cayley tables are not too time-consuming either to make or to understand, one beneficial way is to see that they can be arranged to have the same overall pattern.” Notice this respondent referred to rearranging Cayley tables to demonstrate a matching between appropriate elements in isomorphic objects.

Structure-preservation and operation-preservation were used to describe both isomorphism and homomorphism, but slightly more often for homomorphism. For example: “It gives a colloquial way of saying 'the algebra doesn't change' for particular structures. Things like order, dimension, and so on are preserved." Some explicitly connected structure-preservation to homomorphisms: "I teach homomorphisms as functions which preserve group structure. Homomorphic images are 'large scale structure' while subgroups are 'small scale structure' (at least in examples like symmetric groups and matrix groups)." We believe this participant means that homomorphisms reveal aspects of the domain group's structure by examining a simpler image. Most operation-preservation seemed focused on the homomorphism property: "A homomorphism preserves the operations of the algebraic structures. For example, it will take the identity element of one algebraic structure to the identity element of another algebraic structure." This explanation of operation-preservation foregrounds an identity connection, which highlights a specific type of shared structure.

Disembedding examples highlighted shared properties of the domain and codomain. This example highlighted how relevant shared structure could give insight into a group:

> Sometimes it is useful to think of a group as "sitting inside of" another group, even if in a literal sense the subgroup you are thinking of is not a subset of the bigger group. For example, one might think of some copies of the dihedral group D_4 sitting inside of the symmetry group of the cube…

Notice, although D_4 describes the symmetries of a square and a similar pattern of symmetries exist in the symmetry group of the cube, the underlying elements are not interchangeable, and we would not consider D_4 a subset of the symmetry group of the cube. Nevertheless, recognizing their shared structure could yield insight into the symmetry group of the cube.

Forming equivalence classes was used to discuss sameness of elements in homomorphisms:

> We often build new structures from old by a quotient structure which makes use of an equivalence relation. A homomorphism is one source of such an equivalence relation (but not the only example). I certainly believe that this is an immensely useful way to build structures. And the 'sameness' concept is at its root (in the quotient structure, elements are identified as 'the same' if they lie in the same equivalence class).

Observe that equivalence class language groups elements of a similar nature together into the same equivalence class, which highlights a similarity among these elements within the structure.

Functions vs. Structures

8% of respondents contrasted mapping (isomorphism) and structural (isomorphic) aspects of the concept of isomorphism. For example: "Two groups (for example) can be isomorphic, but the isomorphism may not be obvious….the groups $(C, +)$, and $(R, +)$, are isomorphic because they are isomorphic as Q-vector spaces, but it is fundamentally impossible to write down an explicit isomorphism!" Here the respondent emphasized that objects being isomorphic did not imply that an isomorphism specifying which elements act the same would be easy to find or define, despite such an identification being a likely criterion for considering objects the same.

More commonly (19%), responses detailed the difference between mapping (homomorphism) and structural (homomorphic) interpretations of homomorphism. For example:

> Students who are used to thinking about isomorphic = "the same" will want to think the same thing about homomorphism and will start taking about "G and H being homomorphic" without realizing that the concept is meaningless, and that when studying homomorphisms, we are typically more interested in the properties of the function itself rather than in what it tells us about the structures independently from the function.

Unlike isomorphism, where function and structural aspects are both commonly discussed, the participant here emphasizes that the mapping is the important part of homomorphism.

Some participants provided a way to interpret homomorphism structurally by involving isomorphism. Specifically, 6% of participants referenced the Fundamental Isomorphism Theorem to provide a way to connect sameness, isomorphism, and homomorphism. For example, one participant observed: "I guess the First Isomorphism Theorem should come to mind here. If you quotient out by the kernel, then you get the "same" group as the image, right?" Here we see how the concepts of homomorphism, isomorphism, and quotient groups are linked via theorem: the quotient and image are isomorphic, and the homomorphism defines the kernel.

Discussion and Conclusions

In this large-scale study, we confirmed some findings of prior small-scale studies. Specifically, relabeling/renaming, matching, structure-preservation, operation-preservation, and generic sameness metaphors like same behavior (Hausberger, 2017; Leron et al., 1995; Rupnow, 2021; Weber & Alcock, 2004) were all used by some mathematicians to describe isomorphism. Similarly, structure-preservation, operation-preservation, disembedding, and equivalence class metaphors (Hausberger, 2017; Rupnow, 2021) were used by respondents when describing homomorphism. However, none of these particular metaphors were used by more than a quarter of participants. Furthermore, though it appeared, only three mathematicians described homomorphism in terms of equivalence classes, although it was commonly used by one of Rupnow's (2021) algebra instructors. These differences may indicate that Rupnow's (2021) instructors used uncommon language for homomorphism or could suggest that examining language in instruction as well as out-of-class contexts is important to examine the breadth of language used for isomorphism and homomorphism. Future research should examine the prevalence of these metaphors in instruction for larger groups of mathematicians.

This study also shows a difference between mathematicians' perceptions of the relevance of sameness to isomorphism and homomorphism. This was demonstrated through limited resistance to the concept of sameness for isomorphism (81% of respondents coded with a partly helpful code), and resistance to "sameness" largely related to imprecision, not irrelevance. In contrast, a majority of respondents resisted or did not clearly relate sameness to homomorphism (39% of respondents coded with a helpful-based code), and the "sameness" in homomorphism related only to parts of structures, not whole objects. Differences were also emphasized through participants' portrayals of the function and structure aspects of these concepts (isomorphism vs. isomorphic and homomorphism vs. homomorphic) that highlighted whole object and partial object differences between isomorphism and homomorphism. While these results are not very surprising, they confirm that sameness is relevant to isomorphism and can be a conceptual base for making connections to other subjects as long as the reduction in precision is acknowledged.

Finally, the context-dependence of sameness was a clear theme in participants' responses. Distinguishing isomorphism from other, potentially "stronger" forms of sameness, like equality and being identical, as well as "weaker" forms like homomorphism relates to the importance of precision: what exactly or how much needs to be the same in a particular situation. Similarly, participants' concerns about misconceptions largely related to confusion about whether elements or groups need to look the same or what happens when intuition about sameness leads astray. However, content-dependence can also be viewed as a purpose for examining mathematical sameness. Considering how sameness appeared through equality in prior classes and relating that to isomorphism could create new connections for students and help them appreciate the subtleties of mathematical definitions, which we know are often problematic for students (Edwards & Ward, 2008). Future research should examine how many of these sameness connections are already made by students as well as examine how to help students make such connections, both to help future teachers appreciate how different notions of sameness have appeared in K-12 settings and to help math majors reexamine their prior learning.

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OPERATIONALIZING AUTHENTIC AND DISCIPLANARY ACTIVITY FOR THE UNDERGRADUATE CONTEXT

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Many educators and researchers advocate for student engagement in disciplinary activity. This is especially the case in advanced undergraduate courses taken by mathematics majors. In our respective design-based research projects, we found a need to better operationalize the activity of mathematicians in order to both plan for and document student engagement in disciplinary activity. In this report, we share our literature-based efforts to identify the tools and objects used by pure mathematicians in their work. We share the overarching framework we developed, Authentic Mathematical Proof Activities (AMPA), and illustrate the ways we have used this framework to analyze teacher-student activity using an activity theory lens. We conclude with reflections on how tensions between authenticity-to-the-discipline and authenticity-to-the-students shape the teacher-student activity system.

Keywords: Undergraduate Education, Advanced Mathematical Thinking, Design Experiments

What does it mean to document participatory learning in a proof-based classroom? Analysis of classroom activity at this level often focuses on cognitive analogs (such as documenting taken-as-if-shared practices) and analyzing argumentation through lenses such as Toulmin's argumentation scheme (e.g., Rasmussen & Stephan, 2008). While such analyses can provide important insights into student activity, we have found them insufficient for analyzing student activity in relation to our *authentic mathematical activity* design principles. That is, such analyses enable documentation of students' progressing related to content and arriving at mathematical argumentation goals but may oversimplify the nuances involved in the disciplinary mathematical activity we hope to engender.

In mathematics education, the term authentic mathematical activity or practice is often used in conjunction with two simultaneous, but sometimes competing goals: (1) Staying authentic to the disciplinary activity of mathematics and mathematicians; (2) Staying authentic to student communication, activity, and thinking. Lampert (1992) noted that authenticity needs to go back and forth between "being authentic (that is, meaningful and important) to the immediate participants and being authentic in its reflection of a wider mathematical culture" (p. 310). Herbst (2002) has referred to the tension between students having the opportunity to engage in authentic activity and need for proofs to progress in normative ways as a *double bind* when teaching proof. Dawkins et al. (2019) has elaborated on this in the undergraduate proof setting in which such a bind was felt between "supporting success for all students and authentic mathematical activity" (p. 331). As design-based researchers, we have observed a similar tension in our work, resulting in a need to better operationalize authentic activity at this level in order to plan for and then analyze such activity. In this report, we share our efforts using Activity Theory (Engeström, 2000) to better articulate the authentic activities from the discipline and how such activities may or may not be authentically observed in student activity.

Theoretical and Analytic Framing

Activity occurs within larger systems informed by cultural history and norms. Both research mathematicians and students operate within *activity systems* (Engeström, 2000) which account for individuals' goal-driven actions and the way a community works together when they share a common object. These systems consist of: the acting subject, objects (where the action is focused; the motive will be embedded within the object), tools (the means by which the subject acts in relation to the object), the community, the norms and rules of the community, and the division of labor between members of this community as they work towards a goal. We focus heavily on *tools* as the culturally-situated ways that a subject can transform an object toward a desired outcome. Further, we focus on *objective* as a means to capture the compound notion of both a focal object and the embedded motive consistent with Kaptelinin et al. (1995) and Engeström's (2000) treatment of objects.

Researchers in science education have pointed to the key role of tools and usage of tools towards disciplinary objectives in both engendering and analyzing students' disciplinary activity (Nolen et al., 2020). Classroom activity systems may differ substantially from mathematician activity systems in terms of the community, norms, and division of labor; however, tools and objectives can theoretically exist across systems. Instructors, members of both communities, often play a boundary crossing (Akkerman & Bakker, 2011) role connecting between the disparate settings. While work in other contexts has focused primarily on material tools, we argue that conceptual and procedural tools play a more substantial role in the activity of pure mathematicians due to the abstract nature of the discipline. This leads to the natural question:

- What tools towards what objectives do mathematicians use in their discipline that have the potential to be used by undergraduate students in proof-based contexts?

Authenticity to the Discipline: Tools and Objectives

In order to answer this question, we conducted a thorough literature review of both mathematics education research journals and journals that publish mathematician research activity. We created categories of tools and objectives found in student activity from research projects that shared our basic assumptions around desiring student engagement in authentic disciplinary activity (such as inquiry-oriented and anthropological theory of didactic studies). In alignment with concerns voiced in a recent issue of *ZDM* (Hanna & Larvor, 2020; Weber et al., 2020), we verified that such tools and objectives have been documented in empirical studies of mathematicians. We worked reflexively from the two literature bases to arrive at a three-dimensional framework focused on tools and objectives decomposed into motives and objects. Motives include *understanding*, *testing*, and *constructing* (cf. Selden & Selden, 2017) which exist in relation to mathematical objects: *proofs*, *concepts* (including definitions), and (propositional) *statements* (cf. Dawkins, 2015). For example, the mathematical activity of conjecturing would link to the objective: constructing a statement. In terms of tools, we identified nine categories described in Table 1.

Table 1: Mediating Tools in Mathematician Activity

Tool	Description
Analyzing/ Refining	A process of analyzing and/or refining a proof, statement, or definition via attention to the strength and consequence of assumptions.
Formalizing/ Symbolizing	A process of translating informal ideas into symbolic or formal rhetoric form.
Warranting	A process of inferring why a particular claim is true based on the provided premises.
Analogizing/ Transferring	A process of importing a proof, statement, or concept across domains and adapting to the new setting.
Examples	A specific, concrete instantiation of a mathematical statement, concept, or proof representing a class of objects.
Diagrams	A visual representation of a mathematical object (statement, concept, or proof) that captures structural features.
Logic	The rules of logic which allow for precisely quantified statements and deductive arguments.
Structures/ Frameworks	A top-level structure for a proof (or modular section of a proof) which is determined by statements to be proven.
Existent PSC Objects	Proofs, statements, and concepts (definitions) that are accepted as valid in the community.

Detailed case studies of mathematician's work (e.g, Fang & Chapman, 2020, Fernández-León, et al., 2020; Martín-Molina, et al., 2018) reflect several other themes in the ways in which mathematicians use tools towards objectives. Notably, their activity involves coordination of tool use (both in tandem and succession and the use of tools within and outside of the formal-rhetoric system) and transition of objects to tools for continued mathematical activity. Consider an example from Fernández-León et al.'s (2020) study of a mathematician's conjecturing and proving activity. The mathematician (and their colleagues) began with an *existent statement*: "all complete CAT(0) spaces satisfy the (Q4) condition" (p. 7). They then *analyzed/refined* the statement through the process of exploring *examples* to arrive at a new, stronger statement *formalized* as: "every CAT(0) spaces satisfy the (Q4) condition" (p. 12). This statement is *tested* with additional *examples* aided by a *diagram* which rejection of the statement, and a new *refined* statement was *constructed*: "any CAT(0) space with constant curvature satisfies the (Q4) condition" (p. 12). The mathematicians then *tested* this statement with a new *example* producing a proof of the Q4 condition being met (using *logic/framework*) and *analogizing* the proof process in this context. This proof then served as a generic example for *constructing the proof* of the statement (and thus a final *testing* of the statement). Such illustrations help to bolster the claim that authentic mathematical activity is nonlinear, or "zig-zagging" as argued by proponents of authentic activity (e.g., Lampert, 1992) in which a multitude of tools are used to meet objectives and prior objects become tools for continued mathematization.

Authenticity to the Student: Division of Labor, Norms, and Community

A set of mathematician tools and objectives provide a means to document some engagement in disciplinary activity; however, they need to be paired with exploration of other components of the activity system to reflect authenticity to students as well. A traditional undergraduate course tends to contain a division of labor in which students are responsible for taking notes and answering largely closed-form questions, while the instructor presents definitions, theorems, and formal proofs accompanied by verbal informal explanations (Artemeva & Fox, 2011; Paoletti et al., 2018; Weber, 2004). Thurston (1994) and others have questioned the authenticity of the focus on formal products rather than the informal, nonlinear processes involved in the creation of such products. Further, advocates for students to engage in more authentic activity have suggested images of instruction with differing norms and division of labor focusing on student activity driving the mathematical agenda (e.g., Laursen & Rasmussen, 2019).

From an observable standpoint, the division of labor can evidence whether activity is more or less authentic to students. For example, Herbst (2001) illustrated the division of labor in a geometry class in which the community was working collaboratively to produce a proof, but ultimately the teacher introduced the key idea. Thus, this activity may become less authentic to students in order to meet goals of staying authentic to disciplinary aims. Division of labor can provide insight into *agency* and *authority* (e.g., David & Tomaz, 2012; González & DeJarnette, 2012). We operationalize agency as the freedom to make decisions and create tools in the activity system such as who prompts the use of a tool or who creates a tool. Whereas authority reflects how mathematical tools and objects are determined to be valid. The more division of labor reflects students taking on these roles, the more authentic the activity is to them.

Setting and Analysis Process

The driving force behind this theoretical exploration was a need to better analyze the activity of undergraduate students participating in our design-based research studies. Our framework affords analyses of how a tool is introduced by who, and the degree to which student and teacher contributions shape the overall activity (working towards a particular objective). In order to illustrate the potential of our framework to document student activity and provide a means to make claims about authenticity, we share an episode from one of our projects. This episode stems from a larger project aimed at adapting instructional practices from the K-12 literature base (e.g., Stein et al., 2008) to an undergraduate proof-based setting (introduction to abstract algebra) in order to promote student engagement in more authentic proving activity. The project consists of cyclic task development and instructional supports through a series of task-based design experiments starting with small groups of students and then tested in a classroom setting. For the scope of this paper, we share an episode from the second cycle in which we engaged four undergraduate students in a series of task-based interviews. This episode stems from a task (Melhuish, et al., 2020) in which students compare proof approaches and use these proofs as a springboard to refine and test versions of the theorem establishing that the Abelian property is structural (that is, if two groups are isomorphic, and one is commutative, the other is as well).

Our analysis process was as follows: First, we focused on the degree to which the activity was authentic to the discipline. To do so, we coded all tools and associated objectives from the AMPA framework. We then considered how this tool use approximated the complexity of mathematicians (e.g., Are multiple tools being used in conjunction and in succession towards an objective? Do prior objects become tools for new objectives?). To analyze the degree to which the activity was authentic to the student, we analyzed the division of labor in goal-directed actions (e.g., Who introduces the tools? Who uses the tools? Who connects the tools to the objectives?). As a result of analyzing the two facets of authenticity, we further identified shifts in authenticity and provided robust descriptions of these changes to better understand the activity system. The analysis proceeded in several passes – first focusing on the tools and objectives, then focusing on the division of labor and shifts in authenticity. In each stage, at least two researchers analyzed the data with one researcher serving as a first reader and second researcher serving to challenge interpretations. Disagreements were resolved through discussion.

An Illustration of Analyzing Student-Teacher Activity

The following episode describes the student-teacher activity as they used disciplinary tools toward the objective: *construct* a *statement*. This episode occurred after students spent time *understanding* a theorem *statement* (including subdividing the theorem into a set of assumptions and conclusion) and then *understanding* two student *proof* approaches and comparing across them. During this activity the students identified several differences between the proofs including the use of *warrants*. One proof did not appear to use the fact that an isomorphism is 1-1 and onto to warrant any claims whereas the other approach did. The episode began with the teacher researcher prompting students to use the *analyzing/refining* tool in conjunction with the *existing* theorem and proofs to decide, "So, the big question is, did we actually need all of the assumptions in this statement?"

The teacher-researcher positioned the statement and proof as existent objects in which students used their understanding developed in the prior episodes to move forward. The students responded:

> StudentC: Yeah, we definitely, if the final proof is H is abelian, for sure G is abelian.
> StudentA: Because that's the property that we use, and we also used isomorph- [cross talk]
> StudentD: You would need everything for isomorphic, because you need to know that it is isomorphic.
> StudentC: I mean, couldn't we prove it with homomorphism?
> [continued cross talk]
> StudentC: Our proof worked [proof that did not use onto or 1-1]
> StudentD: All you need to know is that G and H are homomorphic.

The teacher-researcher asked the students to explicitly construct a revised statement based on the homomorphism-only suggestion. One of the students suggested replacing "isomorphic" with "homomorphic." The teacher-researcher slightly altered the statement and wrote the conjectured theorem on the board: *Suppose there exists a homomorphism from G to H. Then if G is Abelian, H is Abelian.*

At the beginning of this episode, we can see the students *warranting* by referencing the necessity of abelian in the proof and disagreeing over the necessity of 1-1 and onto warrants. After some cross talk, the students arrived at the antecedents needed to *construct* a new *statement*. The division of labor at this point included the teacher-researcher prompting a tool to use (*analyze/refine*) and the students using this tool in conjunction with their understanding of the statement and proof from their prior activity. Further, the students engaged in debate, reflecting authority in determining what is mathematically true. At the end of the episode, the teacher-researcher rephrased the student suggestion to align with convention reflecting the teacher-researcher using the *formalizing* tool. This was a place in which authenticity to the discipline and authenticity to student objects converged, with the teacher-researcher preserving some aspects of the student object while also acting to bring it closer to the mathematical community standards.

After the new statement was written, the teacher-researcher again prompted for a specific tool use, *testing* the *statement* with *examples,* or producing a counter*example*. This was a consistent role the teacher-researcher assumed. The students began trying to generate *examples* while explaining their strategies such as, "I'm trying to think of groups under specific operations that wouldn't map correctly to other groups. So, maybe a different operation, but I don't know exactly how a homomorphism looks in that sense," and "... so, since we lost one-to-one and onto, maybe think of some element in H that doesn't have a pre-image." We can see in these contributions that the students were linking the objective (*testing the statement*) with the tool (*examples*) drawing upon *existent* concepts such as the meaning of onto and relationship with the pre-image. The students began suggesting different groups, but also voiced their uncertainty about whether they were creating counterexamples. At this point, the division of labor shifted as the teacher-researcher began scaffolding the tool generation by asking questions about what would need to be true about G, *phi*, and H with students suggesting, "abelian," "homomorphism," and "non-abelian," respectively. While the students confidently answered abelian and homomorphism, their "non-abelian" response conveyed hesitancy, to which the teacher-researcher took on authority to endorse the correctness of non-abelian.

The teacher-researcher next asked for example groups to meet the abelian and non-abelian requirements. The students suggested a number of examples which the teacher-researcher would challenge with questions by asking if the examples met the requirements to be abelian or a group. For example, a student suggested the *example*, "Integers under subtraction, they don't have the abelian property." When the teacher-researcher asked if this was a group, the students disagreed. To resolve this disagreement, the teacher-researcher prompted for the use of an *existent definition*, "So, what properties are you checking right now to decide if it's a group or not?" The students could list the properties of a group, and the teacher-researcher began asking about them one-by-one. The students suggested "0" as the identity to which the teacher-researcher prompted for the definition of identity. Several students made suggestions including, "any other element yields that element," to which another student responded, "So, a minus zero would still be a." The teacher-researcher then asked, "What about zero minus a?" A student shared, "negative a" with two students voicing that was not a group structure. We can see that students used their *definition* for group and the various group properties. We can also see that the division of labor reflected the teacher using the *definition* of group to ask a series of questions to check the properties. As such, the students' agency was more limited. This was another instance in which the teacher-researcher provided scaffolding questions to implicitly challenge a student *tool* that was not conventionally accurate. In terms of authenticity to the students, this episode reflects a shared distribution of labor in which the teacher-researcher never explicitly stated the structure was not a group, but asked questions that implicitly alerted students that more needed to be explored. Through asking these questions, the teacher-researcher changed the *objective* from *testing* the revised *statement* to *testing* the implied *statement: The integers with subtraction is a group.* The students did appear to link the tool and the objective agreeing ultimately that the failure of the identity property (using the *definition*) meant that statement was untrue.

At this point, the teacher-researcher resolicited for a non-abelian group with students making some suggestions and the teacher-researcher taking up the suggestion of the dihedral group *example*. Unlike the first instance, the students engaged in using the *existent definition* to *test* the implicit *statement* that the dihedral group was non-abelian, and came to an agreement using the *example* elements r and s ("$rs \neq sr$"). The spontaneous use of the definition and example reflected a different distribution of labor with students taking on more agency. The students then suggested an abelian group ($\{-1,1\}$), but voiced confusion about creating the homomorphism map. At this point, the teacher-researcher interjected to introduce the *diagram* tool and drew a function diagram with the co-domain and domain group. The teacher-researcher further asked the leading question, "If we have a homomorphism, where do we know this identity has to go?"

with a student stating to the "identity." Another student asked, "[can] we just pick another one for the negative one to go to?" The teacher-researcher challenged, "do we need to pick a different one?" with a student returning to the assumptions in the statement to say, "it's not 1-1 or onto." Ultimately, the teacher-researcher introduced the map of sending all elements to the identity. While the counterexample was co-constructed by the teacher-researcher and students, there were a number of places in which the authenticity to the students was limited due to the teacher-researcher introducing the tool or providing guiding questions that resulted in students having less degrees of freedom.

The teacher-researcher then asked if this was in fact a counterexample and prompted for the students to explain. A student shared, "Because we have an abelian group that maps to non-abelian group, therefore H does not always have to be abelian." This student's contribution evidenced that they were seeing the *example* as a tool to meet the objective of *testing* the statement, which also provides some indication that limited authenticity to the students' contributions may still be authentic to their activity. This was further evidenced in the next episode when the teacher-researcher had students return to proofs and examples to determine whether 1-1 and/or onto was needed in the statement. The labor shifted from the teacher-researcher providing a specific tool for students to use to allowing students the agency to use whatever they wanted to *test the statement.* For the sake of space, we do not share a detailed analysis of this next portion, but we do note that students used the prior tool for further reasoning. They repurposed the counter*example* and *diagram* (Student C: If you just say one-to-one …) as a means for continued *analysis* and *testing* the onto assumption of the statement, noting that altering the map to make it 1-1 did not fix the issue (Student D: So, yeah it would still be wrong; Student A: Yeah, it contradicts.). The remainder of this task session involved the students and teacher-researcher using both the proofs and examples to arrive at a final statement.

Discussion

We selected the above episode because it provided nuance to authenticity and illustrated a time in our design experiment in which the teacher-researcher shifted the division of labor. We would conjecture that some researchers may read the exchange and feel it was inauthentic because the teacher-researcher engaged in much of the labor, including focusing student objectives and suggesting the type of tools for students to use. Further, at some points, the students themselves did not generate the tool without the teacher-researcher scaffolding. However, it is likely that other researchers would see this episode as illustrative of authentic activity because students engaged with tools of the discipline towards disciplinary objectives and students' contributions were an expected part of the labor throughout. These differing, yet viable, interpretations lend credence to the notion that authenticity is not a binary construct.

Some of this nuance may be attributed to differing types of authenticities and the tensions involved between maintaining authenticity to the discipline and authenticity to the students (Lampert, 1992; Ball, 1993, Dawkins et al., 2019). Weiss et al. (2012) further identified two distinct types of authenticity to the discipline which they deem authentic to practice and authentic to discipline. We conceptualize this divide as a practice and content distinction. For example, this distinction can be seen in Chazan and Ball's (1999) discussion of convention for "testing ideas, for establishing the validity of a proposition, for challenging an assertion" (p. 7) and "definitions, language, concepts, and assumptions" (p. 7), respectively. From our framing, practice is reflected in types of tools and motives, and content is reflected in types of objects. Although unproblematized in the literature, we would argue that student authenticity can similarly be subdivided into content and practice. Practice can be thought of as authentic to students if students have agency to create tools and use them to meet objectives related to knowledge generation. From a content perspective, we can also consider the authenticity in terms of how student-generated tools and objects are positioned in the activity. That is, are student contributions legitimized, and are objects (proofs, statements, and definitions) that students generate used for further mathematical activity?

Because of these differing aspects of authenticity, a student-teacher activity system is likely to be rife with tensions stemming from differences in traditional classroom activity systems and mathematician activity systems. For example, the division of labor in the teacher-student activity system likely necessitates teachers providing tasks and thus setting some of the objectives in the classroom setting. Further, the student-teacher activity system has competing goals related to apprenticing students in mathematician activity while also developing proficiency with conventual mathematical content. Engeström (2001) points to the role of "historically accumulating structural tensions within and between activity systems" (p. 137) as propagating system change – a notion we see reflected in our own work in which the teacher serves a unique role as a member of both the mathematician and undergraduate community, and thus serves as a boundary crosser for the respective activity systems. Teacher scaffolding serves both the role of "help[ing] learners use cultural tools" (Belland, 2016, p. 32) and managing the tensions involved in authentic activity (Williams & Baxter, 1996). We point to these tensions, and the role of the teacher, to emphasize that authenticity is not a binary construct, rather, activity can be authentic in different ways and to different degrees when analyzing an activity system.

The particular episode we selected illustrates a situation in which students were prompted to use certain tools for a certain objective, but had agency in the specifics involved while generating those tools, leading to the construction of a statement that would not be valid in the mathematician community. In terms of content, the student objects stayed centered (although formalized by the teacher-researcher). However, there was variation in how much agency students had in creating the examples to test the statement. In particular, the students seemed at an impasse around generating a counterexample, and so the division of labor shifted to the teacher-researcher. Although, there was a dip in authenticity for the students in trade for authenticity to discipline and practice, we would argue that this ebb in authenticity opened a space for students to engage more authentically in the next portion of the task as students both had increased agency in what tools to use and repurposed the co-constructed counterexample.

We developed the AMPA framework to provide a concrete way to analyze and evidence student engagement in authentic activity. The framework contains operationalizations of mathematician tools and objectives that had the potential for use in the undergraduate setting. As an analytic tool, we complemented the tool and objective analysis with considerations of complexity and division of labor. Student activity more closely approximates disciplinary activity when tools are used in more complex ways and prior objects become tools to meet new disciplinary objectives. This activity is likely to be authentic to student practice if students play a substantial role in the division of labor, have agency to generate tools towards an objective, and authority to evaluate the validity of objects. While our initial attention was to focus on elements of practice, we note that the tension in terms of student-generated and disciplinary content also played a larger role in the activity system. Future research could develop additional analytic tools to further parse the ways that the four types of authenticity (authentic to disciplinary content and practice / authentic to student practice and content) shape school activity systems in both research and classroom settings.

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ANALYTIC EQUATION SENSE: A CONCEPTUAL MODEL TO INVESTIGATE STUDENTS' ALGEBRAIC MANIPULATION

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In this paper, I propose a new construct named analytic equation sense *to conceptually model a desired way of reasoning that involves students' algebraic manipulations and use of equivalent expressions. Building from the analysis of two existing models in the field, I argue for the need for a new model and use empirical evidence to explain the new model.*

Keywords: Algebra and Algebraic Thinking, Cognition

Students' success in learning algebra has concerned educators for decades, and researchers have stressed algebra's importance to students' learning and growth extensively and repeatedly (Kaput, 2000; Usiskin, 1995; Wu, 2001). A central difficulty to students' algebra learning, as Behr et al. (1980) captured, is a sense of "extreme rigidity about written sentences," which includes "an insistence that statements be written in a particular form" and "a tendency to perform actions (e.g., add) rather than to reflect, make judgments, and infer meaning" (p. 16). Such a sense of rigidity in doing algebra appears in scenarios such as students interpreting an equal sign as "calculate the left side" (e.g., Knuth et al., 2006), students meeting difficulties in using the substitution method in solving equations (e.g., Jones, 2008), and students hesitating to transform an expression into its equivalent expressions when beneficial (e.g., Ying, 2020).

The field has conducted many studies regarding a sense of rigidity that appears in students' conception of the equal sign and students' use of the substitution method (e.g., Alibali et al., 2007; Baroody, 1983; Knuth et al., 2008; McNeil., 2006, Jones et al., 2012). Comparatively, less research has focused on the sense of rigidity in students' symbol manipulation and use of equivalent expressions for problem-solving (such as given $x+y=2$, $xy=2$, students should be able to evaluate x^2+y^2 without solving for x and y but realize $x^2+y^2= (x+y)^2-2xy$). On the one hand, such an area that is challenging to research as 1) the idea of using equivalent expressions or symbol manipulation is so general that converting $2+x=5$ to $x=5-2$ can also be argued as using equivalent expression; 2) a flexible use of symbol manipulation and equivalent expressions may be influenced by a complicated set of mathematical knowledge and is hard to list out clearly (Hoch, 2006); 3) it is doubtful whether some algebraic manipulations are just symbol playing which carry little educational value (Booth, 2018). However, on the other hand, studies have reported an important connection between students' flexibility in using algebraic manipulation and their success in mathematics (e.g., Novotona & Hoch, 2008; Vincent et al., 2017; Kieran, 2006) and how some delicate algebraic manipulations echo the essential aesthetic nature of mathematics and are accompanied by deep mathematical thinking (e.g., Arcavi, 1994; Dreyfus & Eisenberg, 1986).

The aforementioned difficulties and affordances of reasoning flexibly with algebraic equations collectively suggest the need to construct a conceptual framework in studying students' algebraic manipulation and the use of equivalent expressions. Accordingly, the paper reports a result from an ongoing research effort in constructing such a conceptual framework. Specifically, the paper begins by discussing the affordances and limitations of two existing constructs. Building from this analysis, the paper proposes a new construct named *analytic* *equation sense* with empirical evidence illustrating a) how the construct was conceptualized from analyzing students' algebraic manipulations and b) three factors that the construct captured as the core elements in supporting students' algebraic manipulation and use of equivalent expressions.

Existing Constructs in Studying Algebraic Manipulations and Use of Equivalent Expressions

One of the early studies that drew attention to algebraic manipulation and equivalent expressions was Arcavi's (1994) work on symbol sense. Arcavi established the construct of symbol sense as an analog to number sense in the context of algebra. The definition of symbol sense included all sense-making activities relevant to symbols, which includes students' algebraic manipulations and use of equivalent expressions but extends much further.

Specifically, the idea of symbol sense addresses algebraic manipulations that are complemented with what Arcavi calls *reading through symbols*. As Arcavi (2005) indicated, the detachment of meaning in symbol manipulation helps with efficiency, but reading through symbols adds a layer of meaning and connectedness to performed manipulations. One example Arcavi (1994) provided was the problem of finding the numerical property of the result n^3-n when n is an integer. Arcavi suggested the problem could be solved by converting the expression n^3-n to the expression n(n-1)(n+1) and realizing that the latter term was the product of three consecutive integers, which further implied that n^3-n can be divided by 6. Arcavi argued in such a solution, one had to both apply manipulations (convert n^3-n to n(n-1)(n+1)) and read through symbols (conceive n(n-1)(n+1) as representing three consecutive integers) to fully solve the problem. Consequently, Arcavi argued that algebra manipulations and reading through algebra symbols are complementary to each other.

In the case of using equivalent expressions, Arcavi suggested that equivalent expressions can be conceptualized with non-equivalent meanings. Using the same example above, Arcavi believed the expression n(n-1)(n+1) helps students to interpret the term as the sum of three consecutive integers, which is an observation that the original expression n^3-n may not afford directly. Similarly, many expressions, when transformed into different equivalent expressions, can generate a richer set of implications and meanings. Therefore, Arcavi suggested an important aspect of symbol sense is to treat the result of manipulations not only as results but also as "potential sources of new meaning" (p.28).

Collectively, Arcavi's idea of symbol sense stresses the importance of incorporating a search for meaning while performing algebraic manipulation and using equivalent expressions. In other words, educators should attend more to symbol manipulations that are accompanied with meanings.

Nevertheless, since symbol sense also contains many other aspects, researchers have adopted the term in a broad range of areas. For instance, symbol sense was also used in studying students' conception of the minus sign (Lamb et al., 2012), students' understanding of the quantitative relationship between different expressions (Pope & Sharma, 2001), students' calculus performance (Thompson et al., 2010), and students' function graphing skills (Kop et al., 2020). As a result, the versatile use of the construct symbol sense has the risk to obscure researchers' real interest when working with such a construct. As Pierice & Stacy (2004) categorized, the applicability of symbol sense contains almost everything involving symbols. Furthermore, as Bokhove & Drijvers (2010) stated, "observing symbol sense is not a straightforward affair," as students "exhibit both symbol sense behaviors and behavior lacking", and it was hard to decide whether students "are relying on standard algebraic procedures or are actually showing insight into the equation of expression" (p.48). In summary, it is questionable whether the idea of symbol sense is at an appropriate grain size for studying algebraic manipulations and the use of equivalent expressions, and the definition of symbol sense might be too general to provide sufficient details to pinpoint students' cognitive difficulties in algebraic manipulations.

Another widely adopted framework in the field was Hoch's (2003) idea of structure sense. Building on Arcavi's symbol sense, Hoch narrowed her scope of interest to manipulations that leveraged algebraic structures. Hoch & Dreyfus (2005) defined algebraic structures as combinations of external appearances (the way an expression is written) and internal orders of an expression (the potential implications of an expression). As an oversimplification, one might interpret Hoch's definition of algebraic structures as almost all possible information that one can derive from algebraic expressions, and the conception of structure sense is then a collective set of skills in leveraging the derived information to make manipulations and use equivalent expressions. In short, Hoch's idea of structure sense is a trimmed version of symbol sense that focuses on students' flexible algebraic manipulations.

Many researchers have used Hoch's idea of structure sense and studied relevant students' algebraic manipulations. Hoch and Dreyfus (2005) primarily investigated students' structure sense by asking students to solve problems that contain some "cancelable" parts on both sides of the equation, such as solving for x knowing $(x^3 + 2x) - x = 5 + (x^3 + 2x)$. During the study, Hoch found only 6.3% of the students recognized the cancellation without a bracket, 13.6% with one bracket, and 17.7% with two brackets. Hoch and Dreyfus (2006) found that structure sense increases students' accuracy in solving algebra problems, but even high performers lack structure sense. Jupri and Sispivati (2017) reported that experts (mathematics lecturers in college) would solve some challenging problems in a consistent way with Hoch's picture of structure sense. An interesting observation by the authors was that sometimes experts started the problem by following procedural solutions without exploiting algebraic structures, and then these experts came back to leverage structures when they met difficulties. Researchers have also used the idea of structure sense in a broader setting, including college algebra and basic arithmetic (e.g., Novotna et al., 2006; Novotna & Hoch, 2008; Meyer, 2017; Bishop, 2018), and the lack of structure sense among teachers and students was a common theme across many findings (e.g., Musgrave et al., 2015; Vincent et al., 2017).

The idea of structure sense has a much smaller grain size than the idea of symbol sense, and researchers have applied the term with more coherence in studying students' algebraic manipulations. However, the construct still suffers from salient constraints: namely, if one carefully reviews the mathematical tasks that researchers have used in studying structure sense, one may find a lack of clarity in the mathematical understanding that the idea structure sense tries to capture. Consider the following three questions as examples:

Q1: $\frac{1}{4} - \frac{x}{x-1} - x = 5 + (\frac{1}{4} - \frac{x}{x-1})$; Q2: (x-3)4-(x+3)4 ; Q3: 10012-9992;

All three questions are taken from Hoch and Dreyfus's (2005, 2006) research. Hoch and Dreyfus believed that students' abilities in solving these questions elegantly measured their structure sense. In the appearance, all four questions do measure students' abilities in performing certain algebraic manipulations, but it is doubtful whether the intellectual capacities required in each task are well-connected or consistent. For instance, Q1 requires students to be sensitive toward a potential cancellation on both sides of an equation, Q2 expects students to view a compound expression (x-3) as a single entity, and Q3 asks students to apply the property of (a^2-

b^2)=(a+b)(a-b) into a numerical expression. Cognitively, each task seems to demand a different set of mathematical knowledge. In relating those tasks to the broader field of mathematics education, one may find Q1, Q2, and Q3 all indeed align with different existing research topics. For instance, Q1 overlaps with Carpenter's (2005) idea of relational thinking, which models students' coordination between both sides of an equation (e.g., realizing cancellation). Q2 touches on the broader topic of transitioning between arithmetic and algebra, and many researchers have studied students' difficulties in forming an algebraic way of thinking and mastering algebraic rules (e.g., Carraher et al. 2006; Filloy & Rojano, 1989; Herscoviss & Linchevski, 1994; Kirshner, 2004). Q3 brings the theme of creativity in mathematics problem solving and number sense. As a result, the cognitive commonalities between these tasks and between the thinking required in these tasks are unclear. Consequently, without explicating students' thinking behind all those tasks, to group those tasks under the same quilting of structure sense might be counter-productive in helping teachers to locate students' real struggles with the learning of algebra.

The lack of cognitive explanation on the thinking behind the construct structure sense is most salient for the question that asks the student to prove $(x+y)^4=(x-y)^4+8xy(x^2+y^2)$ (Hoch & Dreyfus, 2006). In Hoch and Dreyfus's writing, this question should be solved with certain manipulation tricks. However, why students should not just expand the polynomial on both sides? As Jupri and Sispivati (2017) illustrated, experts also attempt problems by procedural solutions, and it is psychologically natural for students to take an approach that is less cognitively demanding. Therefore, we remain cautious in believing all manipulation problems share equal values. Moreover, to help differentiate between random symbol playing and desired manipulations, I believe a cognitive explanation to the thinking behind algebraic manipulations is needed.

In summary, both the constructs of symbol sense and structure sense have helped researchers studying students' symbol manipulations. However, both constructs lack specificity and cognitive explanatory power in a) identifying beneficial and preventive factors that are relevant to students' algebraic manipulation; b) explicating a way of reasoning that teachers and students can adopt in engaging algebra manipulations. Some studies also touch on such an area, such as Harel & Soto's (2017) work on structural reasoning, Hausberger's (2015) work on structuralist thinking, and Schoenfeld's (2014) work on problem-solving. Similarly, their works situate in different grain sizes and lack specialized cognitive analysis of the thinking behind desired algebraic manipulations. Still, all aforementioned works are indispensable, and they are the giants' shoulders the paper stands on.

Method and Methodology

The ongoing research project aims to design a conceptual framework in studying students' flexible and meaningful algebraic manipulation and use of equivalent expressions in problem-solving. The term conceptual framework follows Thompson's (2008) writing on conceptual analysis. Epistemologically, we share many premises with general constructivism (e.g., Glaserfeld, 1995) and believe that students construct their own mathematics. Accordingly, the building of a conceptual framework creates a hypothetical thinking model through observing and analyzing students' thinking so that such a framework becomes a viable way of assessing students' mathematical knowledge and provides a viable way of thinking that students and teachers can adopt in relevant tasks (Thompson, 2008, 2013).

Up to the date when the manuscript was written, we conducted four semi-structured clinical interviews (Barriball & While, 1994) separately with four pre-service high school teachers, and the length of each interview varied between two to three hours. We have only recruited pre-service teachers so far as 1) pre-service teachers' knowledge is an important factor that influences the general teaching quality (e.g., U.S Department of Education, 2000); 2) pre-service teachers share similar mathematical understandings with high school students on K-12 math content (Carlson, Oehrtman, & Engelke, 2010). All participants have completed several college-level math courses (e.g., multi-variable calculus) but not high-level analysis courses (e.g., complex analysis). In each session, we asked the participant to go through 6-8 sequenced algebra problem, and a talk-aloud approach was adopted. Such approach asks interviewers to encourage the participant to share their thinking at every step verbally, while staying cautious in not intervening participants' own thought process (Carlson & Bloom, 2005). Most of the problems were challenging algebraic questions with multiple solutions, and we do not expect nor push participants to solve all of them. Instead, we encourage each participant to try as much as possible, and view both their successful attempt and unsuccessful attempts as valuable data in indicating their thought process. We transcribed all recordings and used open coding (Khandkar, 2019) to find emergent themes that assisted the modeling of students' thought process.

Analytic Equation Sense along with empirical supports

Based on the empirical findings, the paper proposes a conceptual model named analytic equation sense (AES). We define AES as a positive cyclic reasoning process with three important aspects:

1. Equation aspect: Students should conceptualize an equation as generative to further equivalences.
2. Analytic aspect: Students should analytically navigate between different equivalences in a given problem beyond solely relying on visual clues.
3. Sense aspect: Students should reflect on the encountered problem to gain more knowledge about the potential affordances and limitations of different manipulations and equivalent forms. The reflection, in return, strengths students' awareness that an equation have multiple equivalent forms and helps students to develop stronger skills in navigating between various equivalent forms.

Equation Aspect

We chose the term equation as we found that students' conceptualization of an equation plays an important role in performing algebraic manipulation and using equivalent expressions. In specific, we build off Ying's (2020) research on differentiating between two different conceptions of the equation: Students with a type A conception conceive an equation as representing one equivalent relationship, and that students will be able to substitute quantities that are shown in the relationship. For instance, when given the equation $x^2-x+1=0$, students with type A conception can substitute the term x^2 with the term x-1 when needed. Students with type B conception will further conceive an equation as also representing a family of equivalent relationships, and that students will be able to transform the equation to generate substitutions for new quantities. For instance, when given the same equation $x^2-x+1=0$, students with type B conception can also generate a substitution for unappeared terms, such as $\frac{1}{x}$. The student may realize $x^2-x+1=0$ implies $x-1+\frac{1}{x}=0$ and then aware that $\frac{1}{x}$ can be substituted with the term 1-x when needed. We argue students' flexible algebraic manipulations require students to develop the type B conception.

We use students' work on the following task as an illustration: "Given $a^2-3a+1=0$, find the value of $3a^3-8a^2+a-1+\frac{3}{a^2+1}$". A challenge in solving this problem is tackling the term $3a^3$ and $\frac{3}{a^2+1}$ (using the fact $a^2-3a+1=0$, the term $\frac{3}{a^2+1}$ equals $\frac{1}{a}$). One way to substitute both terms is to transform the given equation $a^2-3a+1=0$ into $a^3-3a^2+a=0$ (multiply a on both sides) and $a-3+\frac{1}{a}=0$ (divide a on both sides) and use those two new equations for substitution. All participants displayed a sense of struggle with this problem, especially with tackling the term $3a^3$ and $\frac{3}{a^2+1}$. We believe many of the observed struggles related to their lack of type B conception, which is the conception that an equation represents a family of equivalent relationships, consider: In dealing with the term a^3, students did not attempt substitute a^3 directly (which can be accomplished through converting $a^2-3a+1=0$ to $a^3-3a^2+a=0$). Rather, students rewrote a^3 as $a(a^2)$ and substitute a^2. Such manipulation displays a sense of preference to operate only with the term that was shown in the given equation $a^2-3a+1=0$. Similarly, in dealing with the term $\frac{3}{a^2+1}$, all participants deduced that $a^2-3a+1=0$ implies $a^2+1=3a$ and rewrote $\frac{3}{a^2+1}$ $as\,\frac{3}{3a}$. However, when tackling the term $\frac{3}{3a}$ or $\frac{1}{a}$, all participants were puzzled and confused. When we asked participants whether they could infer anything about 1/a from the given equation $a^2-3a+1=0$, they suggested no. Since all participants performed substitution, we believe students have developed the type A conception of an equation. However, their inabilities to deal with the term 1/x and their preference to only operate with the term that was shown in the original equation indicated their potential lack of the type B conception.

After showing the solutions to the students and asking for their feedback, all of the participants expressed a sense of shock regarding the possibility of transforming the given equation to generate new equations. Their feedback reaffirmed our hypothesis that students may not conceptualize an equation as representing a family of equivalent relationships. In specific, one participant said, "I automatically think of modifying what's already there as opposed to changing the equation itself before we begin to solve, before we begin to work and solve actual problem." He also elaborated, "you are given these two equations, so the major response was to, ok, what can we do with these two, by themselves, to get the answer. Rather than what can we change about these two, you know like multiplying by a on both sides and dividing a on both sides before we begin actually go about solving." Similarly, another student stated, "I was thinking a lot of it like taking things like this (circling the original equation $a^2-3a+1=0$) as it was instead of moving terms around." In another problem, one participant also shared a similar sense of reluctance in transforming the given equation and stated that "these numbers are kind of sets, and usually I guess, these are usually presented in the way that is easiest to solve." Based on those responses, we infer that many students do not conceptualize an equation as a potential source to generate new equations, and such thinking thwarts students' flexibility in performing algebraic manipulations and their use of equivalent expressions.

In short, we use the term equation to highlight the need for students to understand that an equation can be transformed and leveraged in various equivalent forms, and educators should be aware of some unproductive beliefs, such as that equations "are usually presented in the way that is easiest to solve."

Analytic

We chose the term analytic as we found that an analytic way of reasoning plays an important role in doing algebraic manipulation and use of equivalent expressions. In specific, we followed Stylianou's (2006) research on differentiating between three different types of proof schemes, which are external (random guess), empirical (based on old memory or visual similarity), and analytic (with mathematical rationale). We believe a student displays an analytic way of reasoning in algebra manipulations if the student can provide a mathematical rationale or justification for the manipulation he or she wants to perform and performed. In contrast, we believe that a student does not display an analytic way of reasoning if he or she performs an operation solely based on random guesses or visual similarities. We argue that an analytic way of reasoning helps students with flexible algebraic manipulations. Consider the following two scenarios:

The first scenario is the case where the student adopted an analytic way of reasoning. The problem is "given a=2003, b=2007, c=1997, Evaluate $a^2+b^2+c^2$-ab-bc-ac". One way to solve the problem is realizing that the targeted expression equals $\frac{(a-b)^2+(a-c)^2+(b-c)^2}{2}$.

The student started the problem by writing down the expression $(a-b)^2$. Interestingly, he did not remember the exact formula but quickly calculated (a-b)(a-b) on paper to derive the expansion. He then wrote out the expansions for $(b-c)^2$ and $(c-a)^2$. And he said that he was going to try to use these three perfect squares expressions to get the answer. Finally, he realized that $(a-b)^2+(a-c)^2+(b-c)^2$ is $2(a^2+b^2+c^2$-ab-bc-ac) and solved the problem. When the interviewer asked about his thought process in deciding such an approach, he replied, "the way the question is framed, with the squares, and also the subtraction of ab, bc, and ac. That makes me think of this formula how a different of squares will get you… get you there…Also I am seeing, after I saw this that, it will be easier to get a square if I can subtract out some of the larger number from each other". Later, he also explained that he wrote out all three expressions because he believed all three perfects squares were needed to substitute the terms "ab," "bc" and "ac that were shown in the expression.

Such a process displayed a desirable analytic way of reasoning. The student started the problem by trying to establish associations between the expression that he needed to evaluate ($a^2+b^2+c^2$-ab-bc-ac) and the expression that he was acquainted with ($(a-b)^2$). After making such an association, he reaffirmed those associations' usefulness by realizing their potential in simplifying calculation (notice the difference between a,b, and c are relatively small). He further noticed that since the three middle terms were "ab," "bc," and "ac", if he wanted to rewrite the entire expression based on those perfect squares, he would also need $(b-c)^2$ and $(a-c)^2$. In such a thought process, his final success in finding the solution was accompanied by mathematical rationales, and those rationales guided and reaffirmed his choices of manipulation.

The second scenario is where the student adopted a non-analytic way of reasoning. When solving one problem, the student needed to evaluate $x^2-1+\frac{1}{x^2}$ from given equation $\frac{1}{x}+x=1$. One possible approach was to take squares on both sides of the equation $\frac{1}{x}+x=1$. Facing the problem, the student stated, "this expression (referring to $x^2-1+\frac{1}{x^2}$) was kind of similar to the one we were given (referring to $\frac{1}{x}+x=1$) , but I need to substitute something to replace the

x^2". After some thought, the student decided to substitute x^2 by x-1 (which derived from timing x on both sides of the equation $\frac{1}{x}+x=1$) and transformed $x^2-1+\frac{1}{x^2}$ to $(x-1)-1+\frac{1}{x-1}$, and then she was puzzled and stuck. We asked why she performed such a substitution, and she explained, "you wanna have similar terms on each of these, so just thinking about how a manipulation will help you give you something similar to whatever the expression is you are trying to find the value of."

From her response and writing, we infer the mathematical operation that she performed was largely motivated by pursuing visual similarities, and she might regard $\frac{1}{x}+x=1$ and $(x-1)-1+\frac{1}{x-1}$ as similar since those two expressions visually appear so. Nevertheless, visual similarities between equations do not always translate into the similarities between equations' mathematical meanings. In this case, the student's pursuing of visual similarities thwarted her chance to find the solution, since converting between $\frac{1}{x}+x=1$ and $(x-1)-1+\frac{1}{x-1}$ takes much more effort than converting between $\frac{1}{x}+x=1$ and $x^2-1+\frac{1}{x^2}$. Indeed, during our study, many students displayed non-analytic ways of reasoning and chose to perform some manipulations for reasons such as "this is what I did in the previous problem" or "I want to make this look like that". Frequently, those non-analytic ways of thinking lead students in an unproductive direction. More importantly, without analytic ways of reasoning, students frequently meet difficulties in evaluating whether a particular approach is worth continuing.

Based on the contrast between these two scenarios, we argue that students who are guided solely by visual features of expressions without analytically considering their mathematical meanings will have a more challenging time performing appropriate algebraic manipulations and use appropriate equivalent expressions. In summary, we chose the term analytic to highlight the necessity of helping students to generate a mathematical rationale regarding every manipulation that students made in solving algebra problems.

Sense

We inherit the word sense from Arcavi and Hoch's writings as we believe students' algebraic manipulation and use of equivalent expression is essentially a sense-making process in solving algebra-related problems. Since we do not believe one may develop his or her sense-making ability all in a sudden. We believe students' skills in algebraic manipulation and use of equivalent expression, as one's skill in sense-making, requires continuous effort in practicing and reflecting.

In the example provided above where the student decided to substitute x^2 by x-1 for the expression $x^2-1+\frac{1}{x^2}$, it is worth noticing that such substitution was derived from actively transforming the given condition $\frac{1}{x}+x=1$ to $x+x^2=x$. But that student, in the earlier problem which is "Given a^2-3a+1=0 and evaluate $3a^3-8a^2+a-1+\frac{3}{a^2+1}$", did not attempt to change the given equation. Similarly, after solving one problem which required taking the reciprocal, that same student actively started to try to take reciprocals in the next problem. Unfortunately, we cannot prove that she gained these insights by reflecting on the earlier problem. However, her performance does raise the possibility that one's intuitions and skills for algebraic manipulations can be gained through practice and reflecting on encountered problems. Those practices and reflections, in return, can strength students' awareness that an equation have multiple equivalent forms and helps students to develop stronger skills in navigating between various equivalent forms. Therefore, we use the term sense to indicate our belief that the development of algebraic manipulation skills is a constant learning process that requires continuous effort in practicing and reflecting.

Conclusion

In a nutshell, the conceptualization of AES represents a sincere effort to capture the potential sense-making process in which students can engage in algebraic manipulation and use of equivalent expressions. AES can be used both as a way of reasoning that students can adopt in solving algebra problems or as a research framework in understanding relevant students' difficulties. In a broader context, AES speaks directly to the general theme of rigidity that educators try to tackle, and the construct encourages students to engage in algebra problems flexibly, analytically, and creatively.

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THE PRODUCTIVITY OF TRANSFORMATIONAL REASONING: STUDENTS' WAYS OF UNDERSTANDING CONGRUENCE BASED ON LEARNING EXPERIENCES

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This study explores how students reason about congruence based on their high school learning experiences. We developed an online survey to characterize how students understand congruence and gather their recollections about high school geometry coursework. Three ways students understood congruence emerged from data analysis: pictorial, measurement-based, and transformational. Our results indicate that (1) transformational reasoning was the most productive among students' meanings but was exhibited by less than one-fifth of participants and (2) that diverse contexts encourage critical and productive thinking about congruence. In this paper, we discuss the implications of our findings to research and teaching of congruence.

Keywords: Transformational reasoning, Geometry, Congruence, Secondary education

Introduction

There are two main approaches to introducing congruence in school geometry. One approach focuses on polygons and defines the congruence between two polygons based on the equivalent measures of their corresponding parts (Bass et al., 2001; Boyd et al., 2004). The other approach focuses on congruence as a mapping from one figure to another using rigid motions (translations, rotations, and reflections) (Wu, 2013). This transformational approach might provide a way to promote coherence in students' understanding of congruence of all figures including non-polygons (Wu, 2013). Math educators thus proposed to structure school geometry curricula with definitions of congruence and similarity based on rigid transformations and dilations (e.g., Common Core State Standards for Mathematics, CCSSM) (National Governors Association Center for Best Practices, 2010). Though many studies document the importance and strengths of CCSSM (Harel, 2014; Gaddy et al., 2014; Wu, 2013), few document classroom teachers' instruction of congruence via transformational geometry and the meanings students make from this approach after its implementation (Hollebrands, 2003; Yanik, 2014). This paper focuses on the following research questions: *What meanings do students have for congruence based on their secondary school experiences, and are these meanings productive across various contexts?*

Theoretical Perspective

We adopted Piaget's genetic epistemology as the lens for analyzing data in this study. This perspective assumes that individuals idiosyncratically organize their experiences within mental schemes (Glasersfeld, 1995; Piaget, 1971; Piaget & Inhelder, 1969). The contents of those schemes provide a space of implications for an individual's reasoning (what is also called the student's *meaning (in the moment)*) (Thompson, Carlson, Byerley, & Hatfield, 2014). Patterns in how an individual reasons describe that person's *way of thinking* relative to those meanings (Harel, 2008; Thompson et al., 2014). For example, when a student is presented with two triangles and asked if they are congruent, the student's answer might include an explanation of the figures both being triangles and both having corresponding angles of equal measure (saying nothing about other features). This gives *some* insight into the student's scheme of triangle congruency. Looking at how that student answers other questions that we (as researchers) interpret to be about triangle congruency can provide additional support for hypotheses about the student's way of thinking. Thus, we focused on looking at trends in how students reason about a variety of tasks that we interpret to be about congruence in an effort to gain insights into students' schemes related to this idea.

Methodology

We distributed an online survey to thirty-three undergraduate students at a large public university in the United States. These students were enrolled in a college-level mathematics course but had not yet begun any college-level geometry course. The survey was divided into four parts. The first three consisted of tasks that measured participants' comprehension of congruence and required participants to explain their reasoning or solution process. In Part 1, students were asked to identify congruences in real-world images and create a definition of congruence to use throughout the survey to complete the tasks. In Part 2, they were asked to determine whether two geometric shapes were congruent. In Part 3, they were asked to construct congruent shapes digitally (given support on how to use the tools). At the end of these two parts, participants were given the opportunity to revise their definition of congruence if they felt it was warranted. The fourth part of the survey asked participants to reflect on their experiences in high school geometry courses.

Our goal in data analysis was to examine students' observable behavior (the answers they gave and the work they provided) to gain insights into their ways of thinking. We did not pre-determine categories for data analysis; rather, we searched for trends in students' solution processes, vocabulary, definitions, and explanations to create normative categories by which to classify their thinking. In this exploration, it became evident that students' schemes commonly involved one or more of three expectations about congruent figures: (1) a general "sameness" in two shapes' form or characteristics; (2) the equal measure of the shapes' corresponding parts (such as sides and angles); and (3) the existence of one or more transformations that maps one shape to the other. We refer to these three meanings for congruency relationships as pictorial (P), measurement-based (M), and transformational (T), respectively.

Results

In Part 1, the majority of responses (84.7%) reflected P, M, or P&M categories of meaning. Students typically found congruences between objects that resembled geometric shapes, such as rectangular windows. Commonly-noted congruences between non-geometric shapes occurred in objects that were close in proximity or formed patterns, such as adjacent slices of toast and equally-spaced candies on the roof of a gingerbread house. Students also considered whether non-mathematical, qualitative characteristics of objects were requirements of congruence. Some of these characteristics included color, perspective, and whether objects were machine- or hand-made (and therefore likely to contain accurate or inaccurate measurements, respectively).

In Part 2, responses were coded both by the meaning(s) they reflected and whether students were successful. Being "successful" entailed correctly identifying whether two geometric shapes were congruent, even if the reasoning was incorrect or incomplete. Table 1 shows the success rate for students' responses reflecting each category of meaning (bottom row) along with the success rate of the general population on each question (right column).

Students were most successful on Question 2, an image of two congruent circles. Some noted that one could imagine congruent radii or diameters (M) or map one circle to the other by one rigid motion (T). Others offered less specific reasoning, such as the circles being the same shape (P) or the same size (M), or even incorrect reasoning, such as "all circles have 360 degrees" (M).

Table 1: Success Rates per Question and per Category of Meaning

Meaning / Question	P	M	T	P & M	M & T	P & T	P, M, & T	Population Success Rate**
1 (equilateral and isosceles triangle; not congruent)	5/5	14/14	4/4	7/8	-	-	1/1	31/32 = 96.9%
2 (two circles; congruent)	2/2	7/7	7/7	8/8	-	-	3/3	27/27 = 100%
3 (two curved polygons; congruent)	¾	4/4	8/9	10/10	1/1	-	1/1	27/29 = 93.1%
4 (two similar triangles; not congruent)	¼	15/18	1/1	2/3	1/1	1/1	0/2	21/30 = 70%
5 (two squares; congruent)	¾	5/6	6/7	14/14	-	-	1/1	29/32 = 90.6%
6 (two line segments; congruent)	5/6*	11/12*	3/3	8/9*	-	0/1*	-	27/31 = 87.1%
Success Rate Per Interpreted Category of Meaning	19/25 = 76%	56/61 = 91.8%	29/31 = 93.5%	49/52 = 94.2%	2/2 = 100%	½ = 50%	7/8 = 87.5%	

*non-represented students indicated they were unsure (and therefore were not successful)
**some students responded with no or inconclusive reasoning, and were thus not included

Students were least successful on Question 4, an image of two similar triangles. Students who described a dilation and transformation(s) (T) or reasoned that the corresponding sides of the triangles did not appear equal in length (M) were successful. Unsuccessful students determined the triangles were congruent because their angles were congruent (M) or because the size was not relevant (only being the same "shape" was) (P).

The most commonly employed categories of meaning were P, M, T, and P&M. The most successful categories of meaning were M, T, P & M, and M & T, with success rates between 91% and 100%. Students were much more likely to reason based on a transformational meaning in questions that dealt with curved shapes and squares than the remaining questions concerning line segments or triangles. We note that Question 6—which presented an image of two congruent line segments—led some students to doubt whether congruent shapes had to be polygons.

In Part 3, responses were again coded both by the meaning(s) they reflected and whether students were successful. The criteria for success in Part 3 was reproducing precisely the original shape and providing an explanation of the solution process. Our analysis for Part 3 is based on the responses of two-thirds of participants who completed the activities (the remaining students did not submit completed responses, so their responses could not be analyzed). Table 2 relates each approach taken in the construction of congruent shapes to a category of meaning.

Table 2: The Understanding that Underlies Each Student Approach to Construction of Congruent Shapes

Meaning	Approach Taken to Construct a Congruent Figure
Pictorial (P)	Create a Shape of Same Type of Polygon
Measurement-based (M)	Create a Shape with Equivalent Corresponding Measurements (and therefore the same type of polygon)
Pictorial & Measurement-based (P & M)	Create a Shape of Same Type of Polygon and Approximately or Exactly Equal Size
Transformational (T)	Create a Transformation that Maps the Original Shape to a New Shape

Part 3 consisted of two tasks: constructing a congruent triangle and constructing a congruent circle. Participants had a 40.9% success rate on the first task and a 57.1% success rate on the second task. In both tasks, roughly half of the students chose a measurement-based approach, attempting to replicate the exact lengths of the triangle's sides on Task 1 and the exact length of a radius or diameter on Task 2. These students were more successful in Task 2 but averaged a 47.8% success rate on both tasks. Students who chose a pictorial approach (and did not consider size as a factor by any means) had a 0% success rate. Students who employed both a pictorial and measurement-based approach, who accounted for roughly one-quarter of the responses, had a 12.5% success rate. In contrast, students who employed a transformational approach accounted for roughly one-quarter of responses and had a 90% success rate. These students created a point, line, or vector about which to rotate, reflect, or translate the original shape, respectively.

Part 4: Classroom Experience

Part 4 of the survey gathered information on students' impressions of their geometry course (which was not coded by meanings reflected in responses or accuracy). Of the thirty participants who took high school geometry, 73.3% felt they understood the content at least at an average level and 90 % indicated the course was coherent or somewhat coherent. Almost all of the students recalled determining whether two shapes were congruent, and roughly two-thirds recalled learning about congruence between polygons and triangle congruence criteria, using rigid transformations to create patterns or copy shapes, and constructing congruent shapes using construction tools. On the other hand, only about one-third of the students recalled proving two shapes congruent with or without a two-column (or "t-table") proof. There was no significant correlation between the meanings we coded in Parts 1-3 and responses to these questions.

Discussion

Due both to the format of the study (which did not allow for clarifying questions) and many instances of inconsistent reasoning among individual participants, the most powerful data resulted from our analysis in trends rather than individuals' schemes. Our conclusions are therefore based on the frequency and accuracy for responses of each category of meaning.
The study's results indicate that reasoning based on all four of the most common categories of meaning (P, M, T, and P&M) are conducive to successfully identifying congruent shapes, while only a transformational meaning—which was exhibited by less than one-fifth of participants in their definitions—is consistently conducive to successfully constructing congruent shapes. However, our results confirm a common observation among educators and researchers: that

students rely more on empirical data rather than definitions and theorems to solve problems (Mizayaki et al., 2017; Herbst & Brach, 2006).

With inconsistent reasoning across tasks came another important insight: the power of varying contexts, specifically those with non-geometric or non-triangle shapes and real-world images. In particular, contexts in our survey with curved shapes and non-triangle polygons led more participants to consider the relationship between two shapes as a whole rather than their corresponding parts. This variation reveals that different types of tasks assimilate to different schemes for students. Exposure to diverse tasks (diverse meaning likely to assimilate to different schemes about congruence) may encourage students to reflect on and modify their schemes to reorganize them to be more robust, flexible, and interconnected. Even further, real-world contexts inspired questions from participants that would likely not come to mind in a purely geometric context.

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UNDERGRADUATE MATHEMATICS STUDENTS' SELF-REGULATION IN ROUTINE AND NON-ROUTINE PROOFS

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This study examined how undergraduate mathematics students engaged in self-regulatory activities while performing routine and non-routine proofs. We used Zimmerman's model of self-regulated learning (SRL) theory, which emphasized the sequential, cyclic nature of self-regulation feedback loops in learning and task performances to better understand the difficulties students faced with proof-construction. Two students participated in think-aloud interviews, solving both a routine and non-routine number theory proof. Using qualitative data analysis, we found that students engaged in fewer cycles of self-regulation during the routine proofs, while self-regulation in non-routine proofs involved a broader range of strategies and more negative self-reactions. Our findings can inform instructors about practices that better support proof-construction – particularly approaches that are more metacognitive and reflective.

Keywords: Metacognition, Reasoning and Proof, Undergraduate Education

Introduction

Proof-construction and comprehension are important skills for undergraduate mathematics students to develop as they transition to upper-division courses. However, they face substantial difficulties in developing these skills, including focusing their attention to format, rather than content (Stylianou et al., 2015; Harel & Sowder, 1998), and these difficulties contribute to students not continuing in upper-level mathematics (Selden & Selden, 2008). The cognitive demands of proof-construction and its associated processes, including proof-writing and proof-validation, are starkly different than the demands of prior courses, such as calculus or linear algebra, that require more procedural fluency (Mujib, 2015). As a result, introductory proof courses are significant mathematical junctions, because they signify a drastic shift in the problem-solving experience of students. This poses a problem for undergraduate STEM retention, particularly in mathematics, because proof-based courses serve as a gate-keeping course to the major (van den Hurk et al., 2018).

While research on proof-construction has grown in the last few decades, there is still limited work on exploring how mathematics students engage with proofs through a metacognitive lens (Papaleontiou-Louca, 2003). Finding its roots in social cognitive theory (Bandura, 1986), self-regulated learning (SRL) theory is one cyclic model that attempts to explain students' contextual metacognitive processes (Zimmerman, 1989; 2000). This study provides a better understanding of mathematics students' cycles of self-regulation to potentially equip instructors of proof-based courses with tools and strategies. We engaged students in routine and non-routine proofs (Kablan & Ugur, 2020) in order to examine their cycles of self-regulation in tasks with varying cognitive demands to contribute to the existing body of task-specific, metacognitive research in mathematics education. Our research question was: How did undergraduate mathematics students engage in self-regulatory activities while performing routine and non-routine proofs?

Framing

We used Zimmerman's (2000) cyclical phase model of SRL, which emphasized the sequential, cyclic nature of self-regulation feedback loops in learning and task performances (Cleary & Chen, 2009). These feedback loops, or cycles of self-regulation (Zimmerman, 2000; 2013), attend to students' cognitive processes and accompanying motivational beliefs and are situated in three self-regulatory phases: (1) *forethought*, (2) *performance*, and (3) *self-reflection* (Zimmerman, 2013). The *forethought* phase encompasses students' task-analysis and self-motivation beliefs about themselves and the task at hand. This includes identifying a strategy to pursue and is related to students' self-efficacy beliefs. In proof-construction, this could involve analyzing a proof, deciding how to approach the proof, and having feelings related to completing the proof (e.g., confident). In the *performance* phase, students execute cognitive processes associated with their chosen strategy and engage in self-observation and monitoring of their progress. In proof construction, this could involve self-instruction or self-guidance through tasks, while monitoring progress by comparing their work to prior proofs they had done. Finally, the *self-reflection* phase includes self-evaluations and judgements of performance, including associated reactions and affect. This could involve making a judgment of a strategy that was ineffective and perhaps feeling frustrated that their efforts were not successful. Each subsequent cycle of self-regulation is the result of evaluating one's performance, identifying one's errors, and deciding to re-engage in task-analysis, thus signaling the forethought phase of a new cycle of self-regulation. We posit that the key distinctions between routine and non-routine proofs (Kablan & Ugur, 2020) are also reflected in students' cycles of self-regulation.

Methods

Eleven undergraduate students in a Minority-Serving Institution in California participated in a larger research project about transfer students' experiences in mathematics courses. Purposeful sampling (Creswell, 2013) was used to select two of these students for this study because of their course enrollment; one self-identified as male, and the other self-identified as female. In Fall 2020, the participants enrolled in a transitory course that introduced students to rigor and proof in mathematics designed to prepare them for upper-division coursework. Number theory and set theory were used as a backdrop to introduce students to proof-writing strategies and conventions.

We conducted think-aloud, task-based interviews (Leighton, 2009) in the quarter after students enrolled in the transitory course. Students verbally reported their thoughts as they worked on a routine and non-routine number theory proof. Number theory was used as the content area for the proofs because students encountered it in the transitory course. The routine proof used in this study was (1) *Prove that if* n^2 *is odd, then* n *is odd*; and the non-routine proof was (2) *Is it possible to create a square, with whole number side lengths, that has an area equivalent to the sum of the areas of two other squares, each with an odd area and whole number side lengths?*

We qualitatively analyzed students' think-aloud by watching videos of the interviews, creating content logs, reviewing students' written work, and writing memos about the observable cycles of self-regulation in which students engaged. Within each cycle, a priori coding (Miles et al., 2020) was used to distinguish each SRL phase. We determined that changes in strategy or method of approaching the proofs indicated a new cycle; however, students self-regulated and engaged in other metacognitive processes beyond the scope of what they verbalized out loud.

Findings

Broadly, our findings center around the number of cycles of self-regulation (hereafter just referred to as cycles) in which students engaged and the differences between phases of the cycle in routine and non-routine proofs. We found that students engaged in fewer cycles of self-regulation during the routine proof in comparison to the non-routine proof. Jason (a pseudonym, used for all proper nouns), for example, engaged in three cycles during the routine proof, and seven cycles during the non-routine proof. Ava, engaged in two cycles during the routine proof, and five cycles during the non-routine proof. We linked this difference in number of cycles to students' experiences in early introduction to proof-writing, where they are often taught specific strategies (e.g., direct proof, contrapositive, contradiction, etc.) and conventions to logically connect a premise and a conclusion. Such rote approaches to proof-writing likely constrain the number of strategies students use, as was reflected in Jason's and Ava's engagement in fewer cycles in the routine proofs. We present findings specific to students' phases of self-regulation, particularly highlighting differences in *forethought*, *performances*, and *self-reactions*.

Phases of Self-Regulation in Routine Proofs

Students evoked prior understanding of formal proof-writing and exhibited implicit self-motivation beliefs during the *forethought* phases in the routine proof. Jason mentioned, "So you have to follow the premises first," exhibiting an understanding of premises and conclusions. Ava accessed her prior knowledge, noting, "I remember from the class the definitions," as she made sense of the odd numbers in the proof and wrote: "$n^2 = 2k + 1$." Both students' task-analysis heavily relied on prior knowledge of proof-writing strategies, definitions, and conventions, and this was likely attributed to their work with routine proofs in the transitory course. In terms of their self-beliefs, students were less vocal about self-motivation or perceptions about themselves and the task.

Both students easily performed procedures associated with their chosen strategies in their *performance* phases. We observed students self-instructing or students guiding themselves through the mechanics as they executed their task strategies. For example, Jason decided to begin with n to reach a conclusion about n^2. He wrote, "$n = 2k + 1$.", then he engaged in the mechanics of squaring n while describing the process of squaring the binomial $2k + 1$. The other primary component of the *performance* phase is self-observation, and we noticed students relied heavily on the phrasing of the proof, particularly using the premise and conclusion to determine if the logic of their work aligned with what they were being asked to prove. Interestingly, Ava also used her perception of formal proofs to monitor her progress. She continually asked, "Is this rigorous enough?," referencing prior expectations of proof-construction.

Results of students' monitoring of their progress were illuminated in their various *self-reflection* phases, particularly culminating in students' self-judgements and reactions to the effectiveness of their performances. For example, Jason's approach began with squaring n, and writing "$n^2 = (2k + 1)^2$." He then said, "Give me one second. [Paused to erase work.] I was thinking about it the opposite way." He monitored his work via the phrasing of the proof, particularly realizing his approach began with the original intended conclusion. His *self-reflection* phase involved judging his original efforts as ineffective. This then prompted a new cycle as Jason analyzed the task and pursued an alternative strategy. In terms of affect and emotion, both students were less verbal about this facet of the cycle during the routine proof.

Phases of Self-Regulation in Non-Routine Proofs

Students' *forethought* phases in the non-routine proof involved a wider range of strategies, a lack of clarity in goal setting or planning, and more instances of self-motivation, albeit with more negative motivations and lower self-efficacy. Both Jason and Ava began their non-routine proof task-analyses with geometric interpretations, but later pursued a different approach. Neither student attempted to approach the problem via formal proof until the very end, which we posited resulted in a lack of formal proof-writing as a problem-solving strategy. In the *forethought* phases, the students also verbalized more negative perceptions of themselves and the task than with the routine proof. For example, Jason shared, "Yeah, I don't know how well I'm gonna do with this one," exhibiting negative perceptions of his abilities and lower self-efficacy.

We found that students were less systematic about performing the procedures associated with their chosen strategy in the *performance* phases, likely influenced by a lack of confidence and clarity in their goals during the *forethought* phase. Contrary to the routine proofs, Ava and Jason sought out help from the interviewer and attempted to regulate with the non-routine problem. Both students elicited assistance in clarifying the question and made attempts to change the non-routine proof as a result of their perceptions of the level of difficulty. For example, Jason decided that since the non-routine proof did not have explicit constraints on the two squares being the same size, he would represent the two respective sides as "$a = 2k + 1$" and "$b = 2k + 1$" to "make it easier." Ava decided to disregard the whole number side-lengths portion of the proof to attend to the difficulties she faced. The students also monitored and tracked the effectiveness of their strategy based on the phrasing of the problem, similar to their monitoring in the routine proof. Re-reading the problem in each *performance* phase was a distinct behavior both students exhibited as a way of tracking their progress.

Students expressed more negative self-judgements and reactions in their *self-reflection* phases. As students evaluated their efforts, they repeatedly noted their ineffectiveness and shared that these efforts were not leading them to correctly answer the question. However, it was unclear whether or not their self-judgements were the result of comparing their work to the phrasing of the problem or if they had just exhausted a particular strategy. Lastly, with these self-judgements came more instances of negative self-reactions. In one particular *self-reflection* phase, Ava shared, "Oh no, I don't think I thought this through very well." Interestingly, while both students mentioned feeling unclear about what to do next, their *self-reflection* phases still elicited a subsequent cycle of self-regulation. Students still persisted in continuing to work on the non-routine proof until the end of the allotted time.

Discussion and Conclusion

We found key differences in students' cycles of self-regulation in routine and non-routine proofs. In routine proofs, students engaged in fewer cycles of self-regulation, used a limited number of strategies, and exhibited more procedural performance. On the other hand, students' self-regulation in non-routine proofs involved a broader range of strategies and interpretations, fewer systematic performances, and more negative self-reactions. Understanding students' self-regulation in proofs illuminates some of the challenges they face at this crucial junction. This research can inform metacognitive practices that better support students' proof-construction such as more reflective approaches to analyzing difficult tasks and exercising agency over their affect and motivations. Furthermore, situating SRL theory in this context allowed us to further distinguish routine and non-routine proofs through a metacognitive lens, and provided insight on how to support students to persevere through cognitively demanding proofs.

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FRACTION'S REPRESENTATION, FRACTIONS' ARITHMETIC AND WORD PROBLEMS SOLVING AS SKILLS TO SOLVE TASK INVOLVING FRACTIONS

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The students' fractions knowledge is considered a determining factor for success in their school years. Thus, the interest arises to evaluate the understanding high school students have about this concept. To achieve this goal, three skills are evaluated in this work: graphic representation, arithmetic and word problem solving; as well as the effect of the first two on the third. A test was applied on two different students' groups (198: first grade, 112: second grade) from public schools in Mexico City. After a quantitative analysis, the results show that students from the second-grade group had greater ability in word problem solving and representation, but not in the case of arithmetic, with which the first-grade students were more successful; in that group, a high association between representation and arithmetic with word problem solving is shown.

Keywords: Graphical fraction's representation, Fractions' arithmetic, Word problems solving

Problem Approach and Research Objectives

The specific study of fractions begins in primary school and gradually gives way to rational numbers, so those who begin high school education should have developed specific knowledge and skills to be competent when using fractions. However, previous research showed this is a difficult content to teach and to learn (Siegler & Pyke, 2013), because of that the idea about 15 years old students still have difficulties when solving fractions tasks persists.

Problem solving has been proposed for teaching and learning since the 1980 (Lambros, 2002). However, the school tradition uses exercise and practice-style problems to consolidate or to apply the knowledge students acquired (Sanz & Gómez, 2018), and not to reflect on the resolution process. The graphic representation of fractions is linked to the teaching-learning model in which the students participate, and the models used can be an obstacle to developing the understanding of fractions (Lamon, 1999). On arithmetic with fractions, Braithwaite, Pyke, and Siegler (2017) hypothesized that poor learning of fraction arithmetic procedures reflects poor conceptual understanding of them. Based on the above, the next objectives were proposed: 1) to evaluate the abilities of students between 15 and 17 years old (first and second grade) to solve tasks that involve solving word problems with fractions, their graphical representation, and their arithmetic. 2) to analyze the possible differences between first and second-grade high school students' skills.

Theoretical Framework

Nicolaou and Pitta-Pantazi (2015) determined that the fractions understanding in students between 11 and 12 years old is based on seven skills: a) fraction recognition; b) definitions and mathematical explanations; c) argumentations and justifications; d) relative magnitude; e) representation; f) Connections of fractions with decimals, percentages and division; and g) reflection during the solution of fractions problems. In this work, of these skills, the fractions representation and the problem solving were considered to evaluate the developed skills of students between 15 and 17 years old. In addition, the fractions arithmetic was incorporated. To describe the specificities with which the test was designed, the framework on the uses and aspects of fractions made by Valenzuela et al. (2017) was taken as a reference, which is derived from the ideas of Freudenthal (1983). In this framework, the uses of fraction as descriptor and comparator in everyday language are distinguished, and at a more abstract level the uses of fracture, comparer, measurer, operator, and number. Figure 1 details the particularities for the design of the instrument in this work.

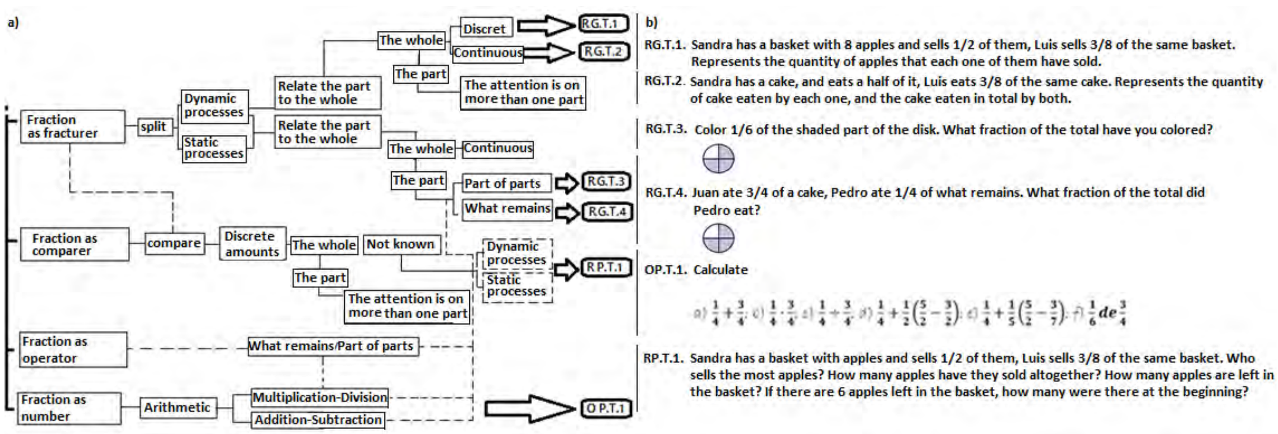

Figure 1: a) fractions uses that describe the design of the tasks and b) the instrument.

In Figure 1 (a) the dotted lines indicate the relationships between the particularities among each of the uses of the fraction involved in the design of the test. The codes enclosed in an oval, indicate the evaluated skills.

Methodology

Sample. The study was carried out on a sample of 310 students from five public schools in Mexico City. Two groups were distinguished: 198 students from the first grade (15 and 16 years old), and 112 students from the second grade (16 and 17 years old). All the data in the study sample are comparable since possible differences in the results of the test due to the geographical inequalities of the centers and the economics of the families, as well as those related to gender were rule out (test of difference of proportions each variable with p-values> 0.05).

Instrument. The test, see Figure 1 (b), is made up of six tasks supported by the Nicoloau and Pitta-Pantazi (2015) and Hart (1981) research. The four tasks that start with the RG code evaluate the graphic representation, in two of them it is requested to represent parts of a whole, but they differ due to their context -discrete (RG.T.1.) or continuous (RG.T.2.)-. The other two determine the skills related to the representation of parts of part, evaluating whether it is on the part (RG.T.3.) or on the complement to the unit -or what remains- (RG.T.4.). The fractions´ arithmetic is presented in OP.T.1. Finally, a word problem (RP.T.1) is presented in which the comparison of fractions, addition, subtraction, and arithmetic (assumed value determined) or algebraic resolution (generic value through unknown) is required.

Method. The test was applied without allowing the use of digital tools and lasted 60 minutes. To analyze the responses, the following were considered: 1) the descriptive analysis, which allows observing the trends of the two populations studied from the percentages obtained; and 2) an inferential analysis that determined the association between graphical representation and arithmetic with problem solving. In this case, Cramer's V coefficient was used as it allows knowing the degree of association between nominal variables.

Results

First, the general results on the three skills under study are presented, and then the results related to the association between the skills of graphical representation and fractions arithmetic with word problem solving are presented.

Graphic Representation tasks. In both grades, difficulties are highlighted in relation to the ability to represent situations related to "Part of parts" and "What remains". The deficit of success does not come from the representation, comes from the lack of interpretation since, of the percentage that was not successful, 78.3% (first grade) and 77.7% (second grade) drew the "part of parts" indicated in the sentence, and 58.3% (first) and 63.4% (second) drew the statement of "What remains". The percentage who was unsuccessful with representation in the discrete context (48.3% of the first grade and 52.7% of the second grade) used the area model instead of a discrete model in their representation. The area model was also used in the representation in the continuous context (88.3% of the first and 92.9% of second grade).

Arithmetic with fractions tasks. The results indicate that only 3.9% in the first grade and 1.8% in the second grade do all the operations correctly. The differences between school grades are notorious in all operations, being the first grade the most successful, and in the case of subtraction the difference is statistically significant.

Word Problem Solving task. Success in solving the word problem is 22.8% in the first grade and 24.1% in the second grade with no significant differences between both grades. In the resolution processes, was observed significant differences in the process to comparing the fractions, with higher success in first grade students. In addition, a greater success in arithmetic resolution than in algebraic processes was observed.

Association between graphical representation and arithmetic with word problem solving. It is highlighted that for first grade students it is obtained that the ability in graphical representation and arithmetic with fractions is highly associated with success in solving word problems. In contrast, in second grade, only the arithmetic ability in addition and subtraction of fractions is associated with word problem solving.

Table 4. Association of skills with Cramer's V fractions (p-value)

		1st Grade (15-16 years old)	2nd Grade (16-17 years old)
Graphical Representation vs success word problem solving	Discreet success	0.707(<0.0001) *	0.074(0.436)
	Continuous success	0.717(<0.0001) *	0.006(0.925)
	"Part of parts" success	0.710(<0.0001) *	0.087(0.358)
	Success what remains	0.733(<0.0001) *	0.181(0.055)
Arithmetic vs word success problem solving	Success Sum	0.712(<0.0001) *	0.208(0.028) *
	Success Subtraction	0.709(<0.0001) *	0.229(0.016) *
	Success Multiplication	0.715(<0.0001) *	0.099(0.297)
	Success Division	0.727(<0.0001) *	0.106(0.261)

	Success Part of Parts	0.717(<0.0001)*	0.068(0.773)
	Global arithmetic success	0.710(<0.0001)*	0.097(0.590)

Conclusions

The results show difficulties and the need to improve the understanding of high school students (15 and 17 years old) associated with the skills to solve word problems, represent graphically, and operate fractions with some particularities.

In relation to graphic representation, the need to incorporate in the instruction tasks that allow students to interpret and represent situations, in different contexts, related to "Part of parts" and "What remains" is highlighted. In addition, there is evidence of a preference to use the area model to make graphical representations, despite the task being in a discrete context.

On arithmetic with fractions, in correspondence with other investigations, there is evidence that the ability to solve addition and subtraction decreases as the age of the learners increases. In this regard, an instruction is suggested that associates the procedure with contexts and not only introduces algorithms to be memorized.

In word problem solving there is a tendency for the arithmetic process over the algebraic one. Thus, it is hypothesized that fractions are being widely taught in arithmetic contexts, but perhaps not in algebraic ones. Finally, to first grade students, graphing and arithmetic with fractions are highly associated with success in word problem solving, while for second graders it is arithmetic that becomes decisive for successful word problem solving.

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THE IMPACT OF COMPUTER SIMULATIONS ON LEARNERS' IDEAS ABOUT SAMPLING

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This report presents preliminary results from the initial survey and task within an ongoing design study investigating preservice elementary teachers' approaches to sampling. Nineteen preservice teachers enrolled in an elementary mathematics methods course completed an initial survey involving tasks related to sampling and inference, followed by a series of designed activities using computer simulations as a tool for inquiry of these concepts. Of interest were the preservice teachers' initial ideas, and how exposure to the simulations may have impacted their approaches. Initial analysis of the surveys suggests that participants' ideas about sampling showed similarities to research involving K-12 students. Their work during the first task suggests that computer simulations may help learners make more reasonable predictions within sampling contexts, along with providing a way to quantify the likelihood of those predictions.

Keywords: Data Analysis and Statistics, Probability, Instructional Activities and Practices, Technology

Topics related to sampling and sample size in the context of making conclusions or inferences are important for K-12 students and contribute to the development of statistical literacy (i.e., Pfannkuch et al., 2015; Shaughnessy & Ciancetta, 2002; Watson, 2006). Research indicates that students have strongly held misconceptions about sampling, sampling distributions, and sample size prior to any instruction which may be difficult to overcome even with instruction (Fischbein & Schnarch, 1997). Students may believe that "anything is possible" when sampling (Watson, 2006), have inaccurate models of sample space and variation (English & Watson, 2016; Noll & Shaughnessy, 2012), and struggle to differentiate between the distribution of a single sample and its sampling distribution (Saldanha & Thompson, 2014). Research also suggests that teachers' approaches to stochastic tasks may be similar to those of K-12 students (Leavy, 2010; Lovett & Lee, 2017).

Recent research in both K-12 and at the undergraduate level has also emphasized simulation within statistics and probability instruction (Garfield et al., 2015; Pfannkuch et al., 2015; Watson & Chance, 2012). These tools give students the opportunity to manipulate variables like sample size, observing the effects of these manipulations dynamically and making connections between various graphical representations (Pfannkuch & Budgett, 2016; van Dijke-Droogers et al., 2020). However, research suggests that students may not engage productively with these kinds of simulations without guidance (i.e. Lane & Peres, 2006; de Jong & Van Hoolingen, 1998). Students may change variables at random or change several variables at once, making it difficult to formulate and test hypotheses. In addition, more open investigations may not prompt students to confront their current ideas and common misconceptions. Instead, a more systematic/guided approach to inquiry with simulations is proposed.

In this research, preservice elementary teachers' approaches were investigated through implementation of a task which involved sampling from a bag of marbles. The following questions guided the inquiry: How do preservice teachers approach tasks related to sampling and inference, and what shifts occur in preservice teachers' thinking related to these concepts in the presence of designed computer simulations?

Theoretical/conceptual framework

This research is concerned with understanding how humans' approaches to probabilistic and stochastic tasks may be impacted as the result of engagement with a computer simulation which allows for dynamic sampling and the adjustment of variables like sample size. This aligns closely with the premises of sociocultural theory of learning developed by Vygotsky (1987) which emphasizes the important role that tools and interactions play in mediating learning; these tools include both physical and virtual manipulatives, along with language (Kazak et al., 2015a). Simulations in particular have the ability to allow students to confront the extent to which their initial probabilistic intuitions are accurate through implementing trials of a particular scenario (Kazak et al., 2015b). There has been extensive research on the ways that intuitions about chance concepts can lead to faulty conclusions, especially when contexts involve sample space and sample size (i.e. Fischbein, 2002; Fischbein & Schnarch, 1997; Kapon et al., 2015). These misconceptions can be deeply held. Lane and Peres (2006) suggest the importance of making hypotheses and predictions as a route to confronting initial conceptions in probability and statistics. As such, simulations can be a platform for testing predictions efficiently. They can provide an avenue to help students confront these ideas through the use of both physical manipulatives and technology, which can also provide dynamic visual representations to emphasize important probabilistic and stochastic concepts.

Methods

This study investigates participants' initial approaches to sampling and inference and how these approaches may be impacted by exposure to computer simulations using principles of design research (Cobb et al., 2003). These studies are interventionist, and often theorize sequences of tasks in order to examine the learning of a group or individual. Researchers engage in an iterative process of theorizing about the impact of an intervention, followed by refining/adjusting their hypotheses after implementation. Analysis is ongoing throughout a design study; researchers use what occurs to inform their next steps and refine/test their ideas, while also engaging in more in-depth retrospective analysis after the study is complete.

Nineteen preservice elementary teachers enrolled in a mathematics methods course during Spring 2021 at a large university in a Midwestern state completed an initial survey which involved tasks related to sampling, inference, and stochastic modeling. This survey was followed by three investigations involving computer simulations using the Common Online Data Analysis Platform (CODAP, www.codap.concord.org) related to these concepts. In addition, the participants completed focused reflections (either written or video via Flipgrid) after each task. Students submitted their work on both the survey and tasks virtually, including written documents and screen shots at various stages, along with their reflections.

This paper focuses on preliminary analysis of one item from the initial survey, along with the first designed task, both of which considered what would happen when 10 marbles are drawn without replacement from a bag containing 50 red, 30 blue, and 20 yellow marbles. Of interest were their responses for (1) Expected number of blues, (2) How likely they would consider a draw of 3 blue marbles (less than 50%, greater than 50%, or around 50%), and (3) What number of blue marbles they would find to be "surprising" in their draw. Theoretically, using the hypergeometric distribution, there is a 95% probability of drawing between 1 and 5 blue marbles, with a most likely outcome of obtaining exactly 3 blue (28%).

Analysis involved coding participants' responses before and after engaging with the simulation to the three questions highlighted above, looking for patterns in terms of both their initial responses/justifications and changes in response/justification after working with the simulation. In particular, researchers wondered to what extent patterns suggested by prior research with K-12 students in sampling contexts (i.e. Saldanha & Thompson, 2014; Noll & Shaughnessy, 2012; Watson, 2006) would be evident before and after the computer simulation. Moments when participants changed a prediction based on simulation results or connected their ideas to a theoretical model were also noted as potentially significant (Lane & Peres, 2006; English & Watson, 2016).

Preliminary Results

Survey results

On the survey, while a majority (74%) of respondents noted that they expected 3 blue marbles to be drawn out of 10, their ranges for what they would be "surprised" by varied and were at times at odds with their initial predictions. For example, one student predicted 3 blue marbles while also stating that she would be "surprised" by a draw of 3 blues later on. Over half (63%) provided ranges skewed above or below their expected outcome, often leading to fairly narrow intervals. This tendency to skew predictions either above or below the expected value and propose extremely wide or narrow intervals has been noted by researchers when working with K-12 students (Noll & Shaughnessy, 2012). In this case, the predictions tended to skew below the expected value (3); several students mentioned the large number of reds in the bag (50/100) as the reason why they would be surprised to get even 4 blues.

Ten students suggested that they believed the probability of obtaining 3 blues would be less than 50%, but their reasons for this belief varied. A common reasoning was that the probability would be close to 30% because 30/100 of the marbles in the bag were blue; other research has noted that students may struggle to differentiate between the distribution of an individual sample and the sampling distribution of all possible samples (Saldanha & Thompson, 2014). Only one student reasoned that obtaining 3 blues would have less than 50% likelihood because of how many other possibilities could occur (any number of blue other than 3). Four students suggested that the probability of obtaining 3 blues was actually greater than 50%; these students seemed to expect the theoretical breakdown of 3 blues, 5 reds, and 2 yellows would almost always occur, another characteristic of students' conceptions of sampling that was noted by Shaughnessy and Ciancetta (2002). The remaining students believed that the likelihood was around 50%, giving a variety of reasons. For example, one student suggested that she believed it was 50/50 because it was impossible to know whether you get the exact probability of anything, echoing the "anything is possible" notion of probability which has been found in research with K-12 students (Watson, 2006). Several responses mentioned not being sure why they made the choice they did or stated that they based their response on a feeling, emphasizing the role that intuition plays in student approaches to tasks involving chance (Fischbein, 2002).

Designed Task

A task with similar questions to the survey was implemented alongside a computer simulation designed using CODAP. The investigation allowed students to simulate draws of 10 marbles from a virtual bag of 100 marbles and observe how many of each color were selected, displaying their results with a dot plot representing how many of a certain color were obtained.

Students were prompted to run increasing numbers of trials (8, then 50, then 150) and observe how many blues were drawn.

Several student responses suggest that the computer simulation may have impacted their thinking about the likelihood of getting various combinations of marbles from a sample of 10, leading to more reasonable predictions for what may occur when compared to the theoretical probabilities. In general, their prediction intervals became wider and more symmetric. While on the initial survey it was common for respondents to provide intervals either ranging above or below their prediction of 3 blues, during the activity all responses were either symmetric (i.e. 1-5) or nearly symmetric (i.e. 1-4), and were often justified by what occurred during the simulation. One student who on the survey stated she would be surprised by any number of blues above 2 predicted a range of 0-6 blues within the activity. Another student used the simulation results to provide the empirical probabilities of each number of blues to justify why she would not be surprised to get anywhere from 1-5 blues. This student had considered 0-3 blues as reasonable on the initial survey, focusing on the large number of reds and yellows in the bag to justify her response. The simulation appeared to not only lead to more reasonable predictions, but also provided another way for the student to justify her conclusions.

The tendency to conflate the proportion of blues in the bag with the probability of drawing that number of blues was less evident in the activity responses (only one response indicated this reasoning), with most students justifying their thinking about the likelihood of drawing 3 blues using their simulation results. For example, one student obtained 3 blues in 17/50 trials (34%), and used this evidence (along with the large number of trials with 2 and 4 blues) to convince herself that her original prediction of more than 50% likelihood was probably incorrect. However, it is unclear whether the activity prompted students to truly confront the difference between the proportion of blues in the bag and the probability of drawing 3 blues. In addition, while researchers had hypothesized that some participants might consider how they could model/determine the likelihood of drawing 3 blues without a simulation after seeing the simulation results (for example using sample space), responses did not show evidence of this.

Discussion/Limitations/Next Steps

This paper provides initial analysis of one survey item and task within a larger study of preservice teachers' ideas related to sampling and inference, and how their approaches to these concepts may change in the presence of designed activities involving computer simulations. While these conclusions are certainly limited by the preliminary nature of this work, analysis suggests that preservice teachers showcase many of the same initial ideas of sampling in chance contexts as K-12 students. Most participants recognized the relationship between the proportion of blue marbles in the bag and what they would expect from a sample, but participants also predicted ranges which were skewed below the expected value and seemed to equate the distribution of a single sample with the sampling distribution. Within the designed task, the participants confronted some of these ideas after simulating the marble bag. Their prediction ranges became more reasonable, and the simulation also provided an opportunity for quantifying likelihood using the empirical results in a way that was not evident prior to exposure to the simulation. While the simulation did provide a way for participants to quantify the chance of drawing 3 blues, it did not appear to lead to insights on how they might determine the likelihood without the simulation. It was also unclear whether the simulation allowed participants to truly distinguish between the proportion of blues in the bag and the chance of drawing that proportion of blues. More research is needed to investigate these ideas further, and also to consider how exposure to designed computer simulations might impact approaches to sampling in other/future contexts. Analysis of the two additional simulation tasks in the larger study, along with future iterations of implementation of the Marble Bag Task, may provide insight on these questions.

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EXTRACTIVE AND INFERENTIAL DISCOURSES FOR EQUATION SOLVING

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We investigate the algebraic discourse of secondary mathematics teachers with respect to the topic of equation solving by analyzing five teachers' responses to open-ended items on a questionnaire that asks respondents to analyze hypothetical student work related to equation solving and explain related concepts. We use tools from commognitive analysis to describe features of teachers' explanations, and we use these survey responses as examples to illustrate a distinction in discourses about equation solving that has implications for students' learning of common procedures for finding solution sets of equations and systems.

Keywords: Algebra and Algebraic Thinking, Classroom Discourse, Reasoning and Proof

The Common Core State Standards suggest that students should come to view equation solving as a form of inquiry whose goal is to identify all solutions of an equation, and learn that steps in an equation-solving process represent successive deductions about a hypothesized solution (6.EE.5, A-REI.1, NGA & CCSSO, 2010). However, discourse about equation solving in algebra courses does not always capture this sense of discovery and deduction (Patterson & Farmer, 2018). In this report we investigate inservice teacher thinking about solving equations with respect to its treatment of mathematical objects, symbols, and routines and explore implications for classroom discourse and opportunities for students' algebraic reasoning.

Theoretical Framework

In describing discourse about equation solving, we mark a distinction between an *extractive discourse* and an *inferential discourse* for solving equations. In the extractive discourse, one describes an equation-solving routine as a sequence of actions on mathematical symbols. In the inferential discourse, one describes an equation-solving routine as using properties of numbers and relations to generate a sequence of endorsed narratives (or inferences) about a hypothesized solution to an equation or system. The routine as a whole produces a conditional: "If the original equation [or system] is true, then the value of the variable must be …" One defining distinction between extractive and inferential discourses is that extractive discourse contains more lexical markers of human agency in the equation-solving process, such as "I moved the $2x$ to the other side" or "you need to set both factors equal to zero." This is consistent with extractive discourse as primarily focused on actions on mediators ("moved", "set", "plugging") and is an example of *personalization* in mathematical discourse (Ben-Yehuda, Lavy, Linchevski, & Sfard, 2005).

Our focus on the concepts of extractive and inferential discourse is rooted in the work of Sfard (2016) who described *ritualized* and *explorative* participation in mathematical discourse. Ritualized discourse is defined as a "discourse-for-others" (Sfard, 2006) in which learners talk about mathematics according to the goals and motivations of others, without clearly identifying mathematical objects (such as numbers, functions, or solutions) as objects of the discourse. By contrast, in explorative discourse, participants strive to know more about mathematical objects and are not constrained to reasoning moves and routines set by others. Literature from the commognitive perspective indicates that explorative participation in mathematical discourse entails meaning for the objects of the discourse that may not be accessible to learners engaged in ritualized participation (e.g., Ben-Yehuda et al., 2005). We hypothesize that developing an inferential discourse for equation solving entails developing three mathematical meanings (Thompson, 2013): (1) to solve an equation is to find value(s) of the variable(s) that make the equation true; (2) we can assume that the variable(s) have value(s) that make the equation true, and each step in the process asserts that an equality is true provided that the previous equality is true; and (3) the converses of the conditional statements generated in this process may or may not be true, depending on whether the functions applied to both sides are invertible.

As part of a larger study (NSF Award #1908825), we use these ways of thinking about teacher mathematical discourse to investigate the question: *What language do middle and high school mathematics teachers use to describe and explain routines commonly used in algebra?*

Method

A 13-item survey on algebra concepts was administered to five teachers. This report focuses on responses to two items on procedures for solving equations (Table 1). The participants are teachers in an urban school district in the southern United States. Diann, Felicia, and Teodora are high school teachers, while Vanessa and Tanya are middle school teachers. All teachers were teaching at least one Algebra 1 class at the time they completed the survey. We analyzed each teacher's responses, noting how their uses of words and mediators, endorsements of narratives, and descriptions of routines aligned with extractive or inferential discourses for equation solving.

Table 1: Two Items in the Solving Equations Strand

Description / Questions Asked (Separate cells indicate separate pages)
Meaning of Solve: "What does it mean to *solve an equation*?"
[A correct solution of the equation $13 + 3x = 48 - 4x$ is provided.] "Thinking about this problem-solving process as a whole – without analyzing each individual step – why does this process produce a number ($x = 5$) that is a solution to the original equation given?"
Special Systems of Linear Equations [A correct solution of the inconsistent system of equations $\{15x + 3y = 33;\ 5x + y = 14\}$ using the substitution $y = 14 - 5x$ is provided. The hypothetical student obtains the equation "42 = 33" and writes, "This is never true, so the system has no solutions."] In the work above, is the equation $15x + 3(14 - 5x) = 33$ a true statement? Why or why not? The student then simplifies the equation $15x + 3(14 - 5x) = 33$ to obtain the equation $42 = 33$. Is this new equation a true statement? Why or why not? Does the reasoning shown here support the conclusion that the system has no solutions? If so, explain why. If not, explain what is wrong with the reasoning shown.
[A solution of the dependent system of equations $\{4x - 12y = 28;\ x - 3y = 7\}$ using the substitution $x = 3y + 7$ is provided. The hypothetical student obtains the equation "28 = 28" and writes, "This is true for all numbers x and y, so <u>all</u> ordered pairs (x, y) are solutions."] Does the reasoning shown here support the conclusion that all ordered pairs (x, y) are solutions to the system? If so, explain why. If not, explain what is wrong with the reasoning shown.

Results and Analysis

In our analysis we describe teachers' responses to the three items in Table 1, whose names we abbreviate *Meaning* and *Systems*. Our goal is not to characterize any one participant's discourse for equation solving as particularly extractive or inferential; we found that each teacher's responses contained elements of both, often within the same response.

Discourse About the Meaning of "Solve" and the Equation-Solving Process

Each teacher's response to the first page of *Meaning* suggested that solving an equation involves finding a value (or all values) of the variable that make the equation true. Responses differed with respect to the importance of finding all solutions of an equation: Felicia said, "Solving an equation is usually finding the value(s) of a variable that makes that equation true," while Teodora said, "Solving an equation means finding a value of x, that when substituted, will make the left side of the equation equal to the right." We do not consider this distinction as having any bearing on the extractive-inferential distinction. However, we also note that while Felicia describes solutions as "the value(s) of a variable that makes that equation true," suggesting independence from the actions of the solver, Teodora's description is suggestive of the process of substituting a value for the variable to verify that it is a solution. We therefore characterize Felicia's response to this question as closer to the inferential end of the spectrum because it describes the idea of solution in a manner independent of human action.

On the second page, Tanya's and Teodora's responses only verified that 5 is a solution of the original equation. By contrast, Diann and Felicia addressed the equation-solving process directly. Diann said, "By performing the inverse operations on both sides of the equation, you are reversing the operations on the $x = 5$ that ended with that result." This response focuses on performing appropriate actions on the "sides of the equation" based on the structure of each side; the word "reversing" may refer to the order in which these actions should be taken. This focus on actions on signifiers points toward extractive discourse. Felicia responded, "There is an assumption that both sides are equal and basically the whole process is manipulating things while keeping that equality until the x is isolated." This response also contains markers of extractive discourse ("manipulating things," identifying the step when the mediator x is "isolated" as a termination condition for the routine), but also stipulates the assumption that the two sides are equal and states that equality should be preserved at each step, which points toward inferential discourse. Both discourses have benefits to offer: the inferential discourse focuses on the equality of values of the expressions at each step, while the extractive discourse highlights strategic knowledge that would help a person reduce the equation to a simpler form.

Shedding Light on Special Cases: Discourse About Linear Systems

The item *Systems* asks respondents to address implications that occur in the process of solving a "special" system of two linear equations in two variables. The first question on the first page asks whether the equation $15x + 3(14 – 5x) = 33$, resulting from a correct substitution, is a true statement. The next question then asks whether the equation $42 = 33$, a correct simplification of the prior equation, is a true statement. From an inferential perspective, the equation $15x + 3(14 – 5x) = 33$ is true under the assumption that (x, y) is a solution, and the fact that this statement implies the false statement $42 = 33$, yielding a contradiction, shows that our original assumption (that there is a solution) must be false. We analyzed how teachers dealt with the apparent contradiction in saying that the equation $15x + 3(14 – 5x) = 33$ is true but $42 = 33$ is false.

Teodora considered the first question from a global perspective: "No, it is not a true statement. When solving, we are looking for a value that will make it true. The student finds that solution does not exist and therefore the statement will never be true." This points toward inferential discourse: because the truth of the equation $15x + 3(14 – 5x) = 33$ (along with the truth of the equation $y = 14 – 5x$, not mentioned in the response) would imply the existence of a solution, and we know that no solution exists, it is not possible for the equation to be true. On the other hand, Tanya, Diann, and Felicia all responded that this equation was true. Tanya said, "Yes, this used substitution. Substitution is one method for solving a system of equations." This response endorses the narrative $15x + 3(14 - 5x) = 33$ based on the use of a standard routine without reference to properties of numbers and relations, and thus we view it as an example of extractive discourse. Diann's and Felicia's responses were both suggestive of taking the truth of the original equation $5x + y = 14$ or the equivalent $y = 14 - 5x$ as a premise; Diann said, "From the above line if $y = 14 - 5x$ then y is equal to $14 - 5x$ so they can be replaced to represent the same value. It did not change the value of the equation since they were equal before the substitution." While this response contains some references to actions on symbols ("replaced"), it also grounds its argument in narratives about the equality of numbers based on the assumption that an original equation is true, which we expect to find in inferential discourse.

Asked about the truth of $42 = 33$, Diann responded, "The simplification was correct. After distributing the 3, the terms with x will cancel out to zero. The new equation $42 = 33$ is not a true statement because those two numbers are not equal or the same." Felicia said, "The new equation is not a true statement, $42 \neq 33$." We wondered how each teacher viewed the conclusion that the system has no solutions. Diann said simply, "The reasoning is correct, this system has not [*sic*] solution. There is no coordinate pair (x, y) that will make the equations true." Felicia said, "The reasoning is that there is no value of x that will give us a true statement, therefore no solution to the system." While we cannot be certain what Felicia meant by "true statement," the fact that she identified the statement $42 = 33$ as untrue suggests that she understands that no value of x will avoid this contradiction and concludes that no value of x (and y) can solve the system.

Discussion

We view extractive and inferential discourses as complementary ways of communicating about processes for solving equations. While the extractive discourse provides access to language and narratives that help solve equations mechanically and fluently, the inferential discourse offers a conceptual microscope under which learners can examine unexpected wrinkles in solution processes. Developing an inferential discourse for equation solving may unlock opportunities for productive struggle in students' learning of algebra, because this discourse allows learners to examine novel features of equation-solving processes based on foundational principles rather than uncritically memorizing routines for classes of problems. Investigating teachers' discourse about equation solving is an important first step in this work because their discourse can afford or constrain students' opportunities for conceptual thinking.

Our analysis is based on teachers' untimed responses to survey questions. Because teachers understood that they were explaining concepts for researchers and not for their own students, we cannot claim these survey responses as a model for explanations that teachers might give in the algebra classroom, where timing, assessment of students' needs, and curricular context might influence decisions about discourse. However, the variety of responses to the two items in this report is evidence of the diversity of explanations available to teachers when they encounter an opportunity for conceptual development. While teachers may not have access to all of these explanations depending on their knowledge and prior experience, we anticipate that through collaboration and professional development teachers may gain access to a greater range of discursive tools for helping students build conceptual understanding of algebraic procedures.

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INFORMAL ISOMORPHISM CONCEPTS IN ABSTRACT ALGEBRA TEXTS

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Isomorphism and homomorphism are important concepts in introductory abstract algebra courses (Melhuish, 2015). Notice that the term "isomorphism" can be taken two ways; it can refer to the property of being isomorphic (i.e., bearing a particular equivalence relation), or it can refer to the function—a particular kind of homomorphism—that witnesses this property. We use the term "isomorphism property" to refer to the former, and "isomorphism function" to refer to the latter. Observe that the concept of homomorphism does not have the property-function duality that isomorphism has—there is no meaningful sense in which two groups (rings, fields) are "homomorphic". While Melhuish (2015) describes the ways that textbooks informally describe the concept of isomorphism, she does not differentiate between the isomorphism property and the isomorphism function. Accordingly, we address the following question: how do textbooks informally describe the isomorphism function, the isomorphic property, and the homomorphism function?

We analyzed the same textbooks as Melhuish (2015) did, namely, those identified as the four most commonly-used introductory abstract algebra texts: Fraleigh (2003), Gallian (2009), Gilbert and Gilbert (2009), and Hungerford (2012). We examined each text's "isomorphism" and "homomorphism" sections to search for examples of informal descriptions of isomorphism. Additionally, we used optimal character recognition (OCR) to examine each place in which the texts included the strings "iso" and "homo". We also examined instances of what Melhuish (2015) calls an *Example Motivating a Definition (EMD).*

Every portion of text quoted by Melhuish (2015) to describe textbooks' usage of informal ideas of "isomorphism" is limited to the isomorphism property. In each text, the informal description appeals to the general notion of sameness. There are two senses in which textbooks have informal counterparts to the isomorphism function. The first uses informal language to describe attributes of the isomorphism function such as "preserve the operation". Since this language describes the homomorphism property, we identify it as an informal description of homomorphism functions. Such informal language occurs in the Gallian and Gilbert & Gilbert texts, whereas Fraleigh uses "structure relating map" to describe the homomorphism function. The second sense uses informal language to describe a function itself (rather than using the word "function"). We call this the *correspondence sense* of the isomorphism function. It occurs prominently in the Fraleigh and Hungerford texts in the context of EMDs involving tables, in the same block of text as the quotes highlighted by Melhuish (2015). The correspondence sense involves an alignment of elements ("relabeling") between structures in order to prove that they are isomorphic. Interestingly, both the Fraleigh and the Hungerford texts use this notion of correspondence as a way to demonstrate the informal counterpart of the isomorphism property (one must see the relabeling in order to see that the structures are isomorphic). Our future work will further examine how textbooks position the relationship between the isomorphism function and the isomorphism property.

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STRUCTURAL CONVENTIONS FOR EQUATIONS IN MIDDLE SCHOOL MATHEMATICS TEXTBOOKS

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An important topic in the teaching and learning of algebra is *expressions and equations* (NGA & CCSSO, 2010). While the literature has addressed student's difficulties in reasoning about the equal sign and making sense of equations (Kieran, 1981; Knuth et al., 2005; Matz, 1982; Prediger, 2010), less attention has been given to the way that equations are organized in solutions to show and justify the symbolic manipulations that lead to the answer. We refer to the norms associated with organizing related equations in a solution as structural conventions for equations. As students progress through middle school mathematics topics, equations play an increasingly central role in nearly all topics. However, little empirical research has been done to document the types of structural convections students must learn as they work with equations in middle school mathematics.

In this study, we examined the structural conventions for equations in two series of 7th and 8th grade mathematics textbooks: *Connected Math 3* Grades 7-8 (CMP3, Lappan et al., 2014) and *Eureka Math* Grades 7-8 (*Eureka Math*, 2015). According to a recent RAND report (Opfer et al., 2016), both of these textbooks are widely used and highly aligned with the Common Core State Standards for Mathematics (NGA & CCSSO, 2010). Equations were coded for equation type and structural conventions. Interrater reliability was 88% for CMP3 and 93% for Eureka Math. Over 90% of the data was coded by both authors, with the remaining 10% coded by the first author.

The two main types of structural conventions for equations in these textbooks were *lists of equations* (LOEs) and *strings of equalities* (SOEs). We defined LOEs as equations listed sequentially (either vertically or horizontally) without any text between them. LOEs were common in all of the sections we sampled from each textbook. We found that the equations in an LOE were often linked explicitly or implicitly by equation operations (Eos) that were performed on previous equations and reported in subsequent ones. We identified six different types of Eos, with reductions and deductions (Matz, 1982) being the most common. To successfully make sense of LOEs, students need to compare each equation with the equation that preceded it to identify the EO that links them. However, we found that some LOEs were written in a way that it was not possible to tell which Eos had been used to link equations. Other LOEs listed equations that were not related by Eos, and thus could be confusing to students.

We defined SOEs as lists of expressions linked by equal signs. SOEs were also common across the sections we sampled from the four textbooks. SOEs ranged from 3 to 8 expressions linked with equal signs. We noted that SOEs presented particular challenges for students, because they often involved different equation types and meanings for the equal sign in the same string of equalities.

Interestingly, neither curricula contained explicit discussions about structural conventions for norms, nor suggested instructional strategies for helping students learn structural conventions. Given the variety of equations types and equal sign meanings in LOEs and SOEs, an implication of our study is that students would benefit greatly from instruction that identifies structural conventions and explicitly models appropriate ways of reasoning about LOEs and SOEs.

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USING MULTIPLE STRATEGIES TO SOLVE ALGEBRA AND ARITHMETIC PROBLEMS

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Knowing multiple strategies and being able to apply them adaptively to various situations, which is also known as flexibility, links to higher academic achievement and can increase transfer across many problem-solving domains (Hiebert & Carpenter, 1992). This study explores high school students' use of standard and better-than-standard strategies when solving problems, with a particular focus on whether strategy differences exist between students' work on arithmetic and algebra problems.

A question of interest is whether students' strategy usage is influenced by the use of a particular task that has been commonly incorporated into prior research on mathematical flexibility (Star & Rittle-Johnson, 2008; Star & Seifert, 2006; Xu et al., 2017). In the present study, 450 high school students in the United States engage in the task of solving five arithmetic and algebra problems and then re-solving the same problems, after being instructed to use a different strategy. Our research questions are: (1) To what extent do students rely upon standard and better-than-standard strategies? (2) Are students' strategy repertoires influenced by whether the attempted problems are arithmetic or algebra problems? (3) Are strategy repertoires influenced by whether the problems are being solved for the first time or being resolved?

Results show that students relied much more upon the standard strategy on the first attempt and more upon the better-than-standard strategy on the second attempt. This raises the possibility that asking students to re-solve pushed them to consider strategies other than the standard and generate better strategies. We also see that students' strategy repertoires were influenced by whether the problems were arithmetic or algebraic, and by whether the problems were being solved for the first time or being resolved. In particular, there seemed to be higher preference to use the standard strategy on algebra problems than on arithmetic.

Table 1: Percentage of strategies used on attempted problems

	First Attempt			Second Attempt		
Strategies	All	Arithmetic	Algebra	All	Arithmetic	Algebra
Better	19%	28%	5%	32%	32%	32%
Standard	62%	44%	88%	23%	19%	29%

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MAKING SENSE OF NON-INTEGER EXPONENTS USING A NUMBER LINE MODEL

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One area of difficulty for students when reasoning about exponential expressions is correctly manipulating and making sense of exponents (Berezovski, 2004; Cangelosi, et al., 2013; Gol Tabaghi, 2007). Common curricular approaches develop the idea of an exponent as the number of times a number is multiplied by itself (Ellis et al., 2015). However, a central limitation of this “number of factors” meaning for exponents is an inability to make sense of non-integer exponents. While progress has been made in addressing this concern through the expansion of approaches to developing meaning for exponents (Thompson, 2008; Ellis et al., 2015; Kuper and Carlson, 2020), questions remain about how to engender scaling-continuous covariational reasoning (Ellis et al., 2020) to supports students in calculating non-integer exponents. While in scaling-continuous covariation students think about change as it happens over an interval of a fixed size, they can also continuously resize the intervals, a process called zooming. We argue that exponentially scaled number lines can support students in applying scaling-continuous covariational reasoning about non-integer exponents.

An exponentially scaled number line is a number line where same-sized segments of the line represent an increase by the same multiplicative factor. For example, if students were asked to model the growth of bacteria whose amount triples each hour, students might create equally spaced tick marks on a number line labeled 1, 3, 9, etc., on one side of the number line to represent the number of bacteria and 0, 1, 2, etc., on the other side to represent the elapsed time. With support, students could eventually come to realize any same-sized segments of the line represent an increase by the same multiplicative factor of the bacteria. They could also come to realize that on one side of the number line there are exponents, while on the other there are the corresponding powers of three. Students could then make sense of expressions such as $3^{1/2}$ by leveraging the idea that this represents the number of bacteria after half of an hour and will be represented on the number line by a segment that is half as long as the whole hour segment. They could then reason that over the two half-hour segments the number of bacteria grew by the same factor, which means they need the number that when multiplied by itself gives 3, namely $\sqrt{3}$.

We see this model as productive because we believe that it fosters scaling-continuous reasoning. An exponential number line is consistent with representing growth that is continuous and is also consistent with resizing chunks continuously. As students use the number line to explore values between their chunks they will need to reason simultaneously with the change in time and the change in the number of bacteria. Analogous to the linear reasoning behind positioning day 0.5 at the midpoint between of hour 0 and hour 1, the multiplicative reasoning of the number line directs students to find a multiplicative value for a half hour period growth such that two half hour growths results in a one hour growth. This process can then be repeated and the same reasoning applied for successively smaller segments of a growth. We believe this allows for both zooming in on the number line and a continuous image of the exponential function, vis-à-vis the number line, to emerge.

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THE MULTIPLICATIVE REASONING OF A HIGH SCHOOL SENIOR

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Multiplicative reasoning (MR) is essential for understanding elementary and secondary math concepts. MR can be measured by how a student conceptualizes and utilizes groups of numbers in multiplicative situations defined by how many groups of numbers they can coordinate simultaneously known as their level of unit coordination (Hackenberg, 2010; Steffe, 1992; Ulrich & Wilkins, 2017). Studies have found that students who have not fully developed MR as measured by their unit coordination struggle to understand improper fractions (Hackenberg, 2007) and linear equations (Zwanch, 2019). MR has been measured for students in the 1st through 4th grades (Askew et al., 2019; Kosko & Singh, 2018a; Mulligan & Mitchelmore, 1997; Smith & Smith, 2006; Steffe, 2002) and 5th through 6th grades (Brickwedde, 2011; Hackenberg, 2007; Kosko, 2019; Norton et al., 2015; Ulrich & Wilkins, 2017; Zwanch, 2019), however, the MR of high school students is currently understudied.

Our question for this research project is: What is the MR of a high school senior struggling with high school math? To answer this question, we focus on a high school senior (Sunil, pseudonym) who was selected based on their poor performance on a pre-assessment. We followed up with a written based MR assessment (Ulrich & Wilkins, 2017) that measured his unit coordination and then follow up with a diagnostic interview which gave evidence of not having fully developed MR.

On the written based MR assessment, Sunil struggled to accurately estimate the size of partitions. This suggests that he was not able to mentally iterate the smaller piece accurately across the whole to determine the correct size. In the follow up interview when asked how he determined his answer, Sunil mentioned he counted the small spaces he drew on the smaller bar to determine the partition size. After asking him why, he mentioned he didn't know. Upon guided questioning, he determined that the smaller piece "fits into" the larger piece 4 times. On another interview task, Sunil was asked if he earned \$360 a week, how many weeks would he have to work to earn \$7,000. After considerable work, he determined 20 weeks by skip counting with \$360 to 20 weeks. When further asked for a more accurate answer using division, he was not able to use division to determine a different approach.

Although this sample of his reasoning is not complete, this research further illuminates Sunil's reasoning on other situations and includes a high school junior's reasoning. The overarching goal of this research is to highlight determining struggling students' MR as a pathway to build their reasoning through targeted instruction.

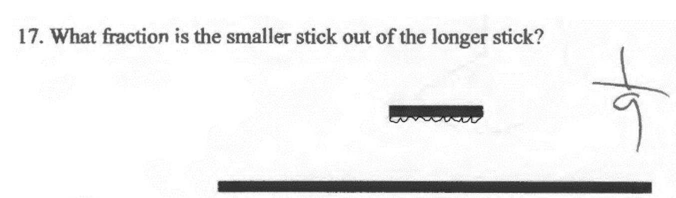

Figure 1: Written based evidence for not fully developing MR

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CONSISTENCY IN TERMINOLOGY AND ITS EFFECT ON EQUITY IN LEARNING MATHEMATICS: THE CASE OF ORIENTATION

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Keywords: Preservice Teacher Education, Geometry and Spatial Reasoning, Equity, Inclusion, and Diversity

One of the most important factors in teaching and learning mathematical concepts is connecting mathematics to our real-world experiences. As mathematics educators, we either start with a concept and then connect it to a phenomenon in the real world, or we model a real-world problem, then mathematize it to clarify the concepts and to deepen our students' understanding. Regardless of the instructional path, it is crucial to make these connections in consistent and coherent ways. Yet, in some instances, when it comes to academic mathematics vocabulary, we use terms and definitions in different ways in everyday language and in academic contexts. As a result, students who come from diverse linguistic and cultural backgrounds often find mathematics in their home language or culture disconnected from academic mathematics. One of the primary focuses of high school geometry includes geometric transformations (NCTM, 2018), and students spend a good amount of time exploring the concept of orientation. However, within the mathematics education research literature, the term orientation has two different definitions, and these definitions are not consistently used within mathematics education research and curriculum material.

1. Orientation as a shape's position with respect to the axis and the Cartesian System's origin, or any other points as the origin (Battista, 2007; Sinclair, Cirillo & De Villiers, 2017).
2. Orientation as the relative position of a shape's vertices to each other (Sinclair, et al., 2012).

While the first definition is the closest to the everyday use of the word and is most often discussed in mathematics education research, the second is the one that is often what we focus on in school mathematics, particularly as it connects to attributes of different transformations.

These inconsistent uses and definitions often cause confusion for students as their everyday experiences are not always coherent with academic experiences. In order to support each and every student, and particularly multilingual students, it becomes necessary to discuss strategies to leverage the role of language to support students in learning math while also engaging them in learning the language through math. Examples of strategies include the use of multiple representations and the use of dynamic software applications that can help students transfer mental images of concepts to visual interactive representations that can lead to more robust understandings and connections between language and mathematical concepts (Dick & Hollebrands, 2011). In general, to achieve equitable learning experiences for each and every student, as mathematics educators we need to leverage our students' linguistic and diverse cultural backgrounds as we support them in rich mathematical experiences.

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REPRESENTATIONS AS TOOLS OR TASKS IN FUNCTIONS OF TWO VARIABLES

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Keywords: Mathematical Representations, Advanced Mathematical Thinking

Access to mathematical objects occurs through their representations (Duval, 2017). Understanding students' understanding of mathematical objects is assisted by analysis of their interactions with representations of those objects. The concept of functions of two variables has grown in prominence and importance in today's technological world. Nevertheless, this concept is underdeveloped in school mathematics and is understudied in students who are prospective secondary mathematics teachers (PSMTs). This study examines PSMTs' understanding of functions of two variables by investigating how they deal with related one-variable functions.

Perspective

As Duval (2017) has pointed out, two inscriptions represent the same objects when there exists a one-to-one mapping of pieces of meaning between the features of the representations. Researchers have studied students' understanding of representations of functions of one variable (Even, 1998) but few studies (Martínez-Planell & Trigueros Gaisman, 2009, 2012; Weber & Thompson, 2014) have investigated of student understanding of two variables. The purpose of this research was to examine PSMTs' mapping of pieces of meaning (or features of the representation) within a representation and across representational registers.

Methods

PSMTs took part in task-based interviews in which they were given a symbolic rule for a function of two variables and asked to describe the corresponding graphical representation. The researchers analyzed video recordings of these interviews, using thematic analysis to identify patterns among the participants' work with representations. The researchers constructed descriptions of participants' actions and verbalizations, developed consensus on participants' use of representations, and, identifying patterns, characterized participants' use of representations.

Results

Our analysis of the data has led to features of our participants' work with representations.

Participants used representations as tasks lacking meaning or meaningfully as tools.

An instance of P6's work provides an example of using representations as tools. In her analysis of the graph of $g(x,y) = xy^3$, she compared the width of $g(x,y)$ for varying values of x, noting that as the value of x increased, the shape of $g(x,y)$ evolved from a downward sloping cubic function to a horizontal line to an upward sloping cubic function. This revealed her mental construction of the 3D graph. Another participant, P7, used representations as tasks, consistently following the procedures exactly as she had learned them and producing template conclusions.

Participants reacted differently after noticing conflicts among representations.

For example, when P1 isolated one feature of the graph of $g(x,y)$ (that inputs of 0 yield outputs of 0) that was not consistent with her reasoning about the graph slices being cubic. She resolved the dilemma by concluding that there must be a break in the graph. When P6 encountered the same aberration, she reasoned correctly that the cubic functions evolved into a flat line, which was followed by cubics of the opposite orientation.

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STUDENTS' CONCEPTION OF CONFIDENCE INTERVALS: STATISTICAL LITERACY REFLECTED ON THE COVID-19 VACCINE MEDIA REPORTS

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Due to the COVID-19 pandemic, the need for statistical thinking, ability to understand, interpret, and evaluate data from numerous media reports in decision-making has been increased. Such ability was defined as statistical literacy in the literature (e.g., Gal, 2002): "(a) ability to interpret and critically evaluate statistical information, data-related arguments, or stochastic phenomena, which they may encounter in diverse contexts, and when relevant (b) their ability to discuss or communicate their reactions to such statistical information" (pp. 2-3). Inarguably, statistical literacy entails conceptual understanding in statistics, which allows students to transfer knowledge to novel problems (e.g., Bude, Imbos, van de Wiel, & Berger, 2011), understand data (e.g., Garfield & Chance, 2000; Jones et al., 2011), interpret results (e.g., Gal & Garfield, 1997; Jones et al., 2011), and think critically (e.g., Garfield & Chance, 2000) (Crooks, Bartel, & Alibali, 2019). Learners' adaptation of their knowledge structures appeared to be often inaccurate in some contexts (Smith, diSessa, & Roschelle, 1994; Crooks, Bartel, & Alibali, 2019).

The purpose of this pilot study was to examine students' conception of Confidence Interval (CI) and identify student difficulties associated with Cis presented in media reports on the COVID-19 vaccines. Studies suggested that CI is one of the difficult concepts for students (e.g., Coulson, Healey, Fidler, & Cumming, 2010; Cumming, 2006, Henriques, 2016). The study's participants were fourteen graduate students in the STEM education program at a private research university in Southeastern State. The assessment consists of six questions, including open-ended items based on news media reports on the efficacy rates of COVID-19 vaccines presented with Cis, which the author created. The questions are based on the key concepts of CI: (a) understanding the definition of the term "confidence interval," (b) understanding the distinction between sample and population means and how they are related, (c) understanding the notion of confidence level (i.e., 90% vs. 95% CI), (d) understanding how various factors (e.g., sample size, sample variability) affect CI width, I understanding what can be inferred about future replications based on Cis, and (f) understanding how to interpret Cis accurately (Crooks, Bartel, & Alibali, 2019). Students were asked to complete the assessment and answer probing questions through individual interviews. Students' interpretations of Cis were coded by two independent raters using the conceptions and difficulties identified by the previous studies (Castro Sotos, Vanhoof, Van den Noortgate, & Onghena, 2007; Cumming & Maillardet, 2006; Fidler, 2006; Grant & Nathan, 2008; Greenland et al., 2016; Henriques, 2016). Findings highlight students' difficulties in interpreting confidence intervals displayed in graphic representations. The poster will present the assessment items and prevalent student conceptions probed by the items in the context of media reports on the COVID-19 Vaccines.

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RE-ENCOUNTERING RATE OF CHANGE IN DIFFERENTIAL EQUATIONS

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Keywords: Undergraduate education, Preservice teacher education, Problem based learning

Rate of change is a core idea that cuts across secondary school and tertiary mathematics education. Prospective secondary school mathematics teachers re-encounter rate of change in a variety of undergraduate mathematics and science courses. Hohensee (2011) introduced the construct of *backward transfer* to capture how prior knowledge changes as new knowledge is built upon it. Rasmussen and Keene (2019) identified rate of change in differential equations as a concept ripe for backward transfer. However, simply re-encountering an idea in more advanced courses may not necessarily lead to any substantive enrichment in one's understanding of the idea. We conjecture that intentional instructional efforts are needed to realize the transformative potential of re-encountering an idea for enriching conceptual understanding of that idea. In this poster we take up this conjecture by analyzing the work of eight prospective secondary school teachers in an inquiry-oriented course in differential equations. Data includes students' end of semester portfolios and video-recorded oral presentations of those portfolios. The portfolio included seven self-selected entries that showcased their mathematical work and a rationale statement explaining the personal significance of each entry. The portfolio also asked for an eighth entry focused explicitly on all of the ways in which they now think about rate of change, with examples of their mathematical work from the semester. We used thematic analysis (Braun & Clarke, 2006) to collaboratively interpret and code the rate of change entries and presentations. Results revealed that students significantly enriched their understandings of *dy/dt* as a rate of change, as compared to the meanings they reported on a pre-semester survey. These new understandings of dy/dt include: a dynamic slope, a tool to know solutions, an operator that acts on other functions, and a function in and of itself. The poster presentation will provide illustrative examples of each of these expanded ways of understanding rate of change. We also discuss implications for the teaching of differential equations and other mathematics courses in order to deepen future secondary school teachers' understandings of core mathematical ideas.

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Chapter 6:

Math Knowledge for Teaching

REPRESENTING PROPORTIONAL REASONING ALGEBRAICALLY TO PROBLEM SOLVE

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Proportional reasoning problems can be solved using algebraic reasoning. Therefore, making connections between proportional reasoning and algebraic thinking is important for solving problems. This study examined K-8 teachers' problem-solving strategies as they worked out a real world multi-step problem that involved proportional reasoning and algebraic thinking. The findings revealed that many teachers found this problem challenging. Particularly, some teachers had difficulty figuring out how to translate the variables into an algebraic equation. Some teachers who used variables as labels tended to engage in additive reasoning. They had difficulty representing the proportional problem context algebraically and solving the problem for the unknown quantity. Implications for further research are discussed.

Keywords: Teacher Knowledge, Proportional Reasoning, Algebraic Thinking

Proportional reasoning and algebraic thinking are often taught independently of each other. Therefore, when encountering a real-world problem that involves proportional reasoning and algebraic thinking, students who only developed procedural knowledge find such problems difficult to solve. This is because teachers tend to focus on aspects of problems that require only procedural knowledge, with a singular solution, strategy, and representation (Glassmeyer & Edwards, 2015). Teachers need to develop a deep understanding of the interrelationship between the conceptual and procedural knowledge to support their students to engage in problem solving and reasoning (Ma, 1999; Rittle-Johnson, Siegler & Alibali, 2001). Researchers suggest that many teachers struggle with understanding proportional reasoning (Riley, 2010; Cohen, Templin & Labato, 2010; Weiland, Orrill, Brown & Nagar, 2019). Particularly, this is the case with distinguishing proportional and non-proportional situations. Furthermore, teachers tend to focus on additive reasoning as opposed to proportional reasoning. There is very little research on teacher knowledge on proportional reasoning (Weiland, Orrill, Brown & Nagar, 2019).

This study investigated teacher's ability to translate a proportional relationship into an algebraic equation. More specially, pre and post test data on how teachers solved a multi-step problem involving proportional reasoning and algebraic thinking after participating in content based professional development was analyzed.

This study focused on answering the following research questions:

1) Did the professional development improve in-service teachers' overall conceptual understanding involving proportional reasoning and algebraic thinking?
2) Did the professional development improve in-service teachers' conceptual understanding in algebraic thinking?

3) Did the professional development improve in-service teachers' conceptual understanding in proportional reasoning?

Proportional Reasoning

A proportional situation is one that has "structural relationships among four quantities, (say *a*, *b*, *c*, *d*) where there is a covariance of quantities and an invariance of ratios, where a ratio is a comparison of two quantities" (Weiland et al, 2019. P. 233). The components involved in reasoning with proportions are unitizing, rational numbers, ratio sense, partitioning, quantities and change, and relative thinking (Lamon, 1999). Uniting is a cognitive process that assigns a unit of measurement to a specific quantity (Lamon, 1996). The ability to form and operate within complex unit structures allows for higher ordered and flexible thinking. An example of uniting would be referring to an hour as 1 unit of 60 minutes or 2 units of 30 minutes, or 6 of 10 minutes, or 12 of 5 minutes depending on the context. Partitioning refers to the ability to break down a unit into equal parts (Lamon, 1999).

When proportions are represented in fraction notation (ie. $\frac{a}{b}, \frac{c}{d}$, not *a*:*b*, *c*:*d*), the information is structurally represented where it can be manipulated algebraically in any given calculation process. It is important to remember that all of the four quantities (*a, b, c, d*) can each be equal to one (1) in a given situation, which is simply the multiplicative identity property as a proportion. This models a transfer and flexibility of thinking which is necessary to manipulate the proportion. Critical thinking is foundational to proportional reasoning, as it involves abstracting then possibly manipulating that information (depends on the situation).

Proportional reasoning situations can also be represented algebraically. Representing proportional reasoning situations algebraically involves a flexible understanding of the meaning variables such as representing a category, a known value, an unknown value, or a changing value (Moss & Lamberg, 2019) to represent a problem situation. Representing proportional reasoning problems algebraically involves the ability to model real world situations which is considered a main objective of algebra (Izsak, 2003; Kaput, 1999; Schoenfeld, 1992).

Algebraic Reasoning

Algebraic thinking involves engaging in reasoning and sense making (Kaput and Blanton, 2005, Swafford and Langrall, 2000). It is the ability to model quantitative situations by being able to represent relationships quantitatively (Driscoll, 2001). According to Driscoll (2001), algebraic thinking involves developing habits of mind to think about quantitative relationships such as the ability to organize information by discovering patterns, relationships, and rules. As these 'habits' are listed as a structure of steps, in essence, this too then is a procedural skill. When this is practiced in order to become a habit, it becomes a behaviorist model with the incentive of possibly developing a conceptual understanding at any given point in this habitual practice. However, algebraic reasoning is a practice with the distributive, commutative, associative, and identity properties of addition and multiplication, and its abstracted symbolic representation is used to denote the calculations which deliver the final analysis and result.

The cognitive process in algebraic thinking includes encoding information, then retrieving and manipulating it to produce a final representation—a function in the brain also known as working memory (Gluck et al., 2016). Reasoning algebraically with known, unknown, and changing values is an evident example of the working memory function in the brain, and multi-step reasoning problems require unitizing and managing the transitions as they are modeled, which are then abstractly denoted symbolically (or vice versa). Algebraic reasoning models the various stages which demonstrate one's depth of knowledge—a conceptual understanding of the mathematics in any given situation.

Method

Twenty-three K-8 teachers participated in four-week content based professional development in a western state and the data presented here is from a larger study. A pre and post test was administered at the beginning and end of the week-long institute. The week-long institute focused on developing teachers' understanding of fractions and proportional reasoning content and pedagogical knowledge. The following problem was analyzed in this study.

> *Jeff had one-fourth as much money as Peggy. Ed had twice as much money as Peggy, they counted their money and then gave $20 to one of their friends. If they now have a total of $84, how much money did they initially have. Write an equation for this problem and solve it.*

The data was coded based on strategies that teachers used in three categories, *evolving, emerging* and *effective* in proportional reasoning and algebraic thinking, as illustrated in Figure 1.

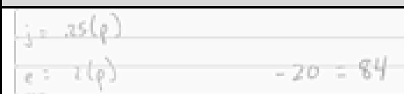
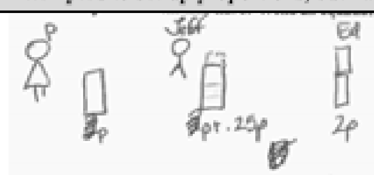
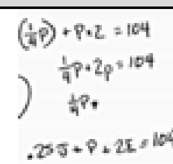
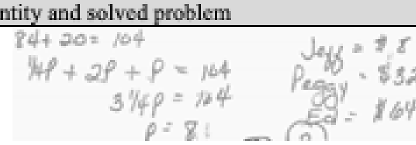
Score	Proportional Reasoning	Algebraic Thinking
1 *Evolving*	No Proportions, or no attempt	Variables used as labels, unable to model problem context problem and set up algebraic equation
		
2 *Emerging*	Attempted to set up proportions, but has errors	Variable used to represent an unknown quantity and modeled problem but did not solve original problem
		
3 *Effective*	Identified proper proportions correctly	Modeled problem using one variable as an unknown quantity and solved problem
		

Figure 1: Variables Rubric: Proportional Reasoning, Algebraic Thinking

In proportional reasoning, *effective* scores were able to identify the correct proportions. In this stage, participants were able to identify the unit (Peggy) creating a one-to-one relationship. They were also successful in demonstrating a correct ratio and partitioning understanding in forming the relationship between Jeff and Ed. For *emerging* scores, Peggy was correctly identified as the unit. However, for there was an incorrect partitioning or ratio relating Jeff or Ed to Peggy. For *evolving* scores, there was a lack of proportional understanding, in that there was

Olanoff, D., Johnson, K., & Spitzer, S. (2021). *Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Philadelphia, PA.

no attempt to utilize a proportional relationship between Peggy, Jeff, or Ed. These strategies use a guess and check, and there was no clear demonstration of unitizing, partitioning or ratio concepts.

In algebraic thinking, *effective* scores were able to create an equation using one variable and correctly solve for all three values using algebraic properties of equality. For *emerging scores,* an equation was created using one variable, but was either not correctly solved or contained incorrect or missing proportions for either Jeff or Ed. For *evolving* scores an equation was used but contained multiple variables representing Peggy, Jeff and Ed or variables for Peggy, Jeff and Ed were identified but no equation was developed.

In full effect, an *effective* score in algebraic thinking includes correct referent units when presenting the solution. The presentation of referent units exhibits a thorough analysis and conceptual understanding of a problem, and demonstrate a focus on the unit that was manipulated in the problem. In this case a dollar symbol ($) was used to denote the referent unit in the rubric example. The role of referent units is for tracking information and changes between the known and unknown values.

A McNemar test (McNemar, 1947) was used to determine whether there was difference in proportion of participants classified as *non-effective (evolving & emerging)* and *effective* in both proportional reasoning algebraic thinking between the pre and post tests.

Results

The data revealed there was growth between the pre and post test in relation to teachers' ability to engage in proportional reasoning and algebraic thinking. In a paired t-test there was a significant difference in overall scores between the Pre ($M = 1.35$) and Post ($M=2.07$) scores, $p <.001$. This shows there was growth between the pre and post test (see Figures 2 & 3). This shows that overall, the professional development did have a positive impact on teachers' overall skill in a problem involving proportional reasoning and algebraic thinking.

The McNemar test (McNemar, 1947) revealed the following: In algebraic thinking the results were statistically significant *($p=.016$)* meaning there was a significant difference in the proportion of participants found to be *effective.* In proportional reasoning the results were statistically insignificant (*$p=.063$)* meaning that there was no significant difference in the proportion of participants considered to be *effective* in proportional reasoning between pre & post tests.

In an analysis of the pre and post tests, most teachers were able to set up the proportional reasoning aspect of the problem. However, they struggled modeling the problem algebraically using variables to solve the problem. More teachers initially struggled setting up an algebraic equation and solving the problem. While this was not the initial focus of this study, possible gaps of the teachers understanding in proportional reasoning are discussed.

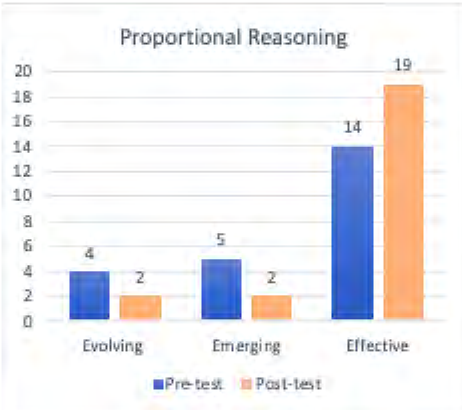


Figure 2: Proportional Reasoning Results

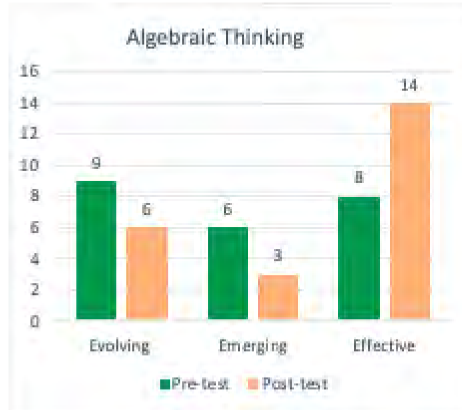


Figure 3: Algebraic Thinking Results

Discussion

The findings reveal that while there was improvement in the proportion of *effectiveness* in both proportional reasoning and algebraic thinking, the greatest gains and statistically significant results were in algebraic thinking. It was noted that some teachers initially struggled with proportional reasoning, and their ability to represent the multi-step proportional problem algebraically. These findings are consistent with other research findings that teachers struggle with conceptual understanding of proportional reasoning (Riley, 2010; Cohen, Templin & Labato, 2010; Weiland, Orrill, Brown & Nagar, 2019). The teachers that struggled with proportional reasoning were likely engaging in additive reasoning. Initially, many teachers had difficulty meaningfully modeling the proportional reasoning problem context using variables and equations. The ability to model and engage in algebraic reasoning is critical for understanding algebra (Izsak, 2003; Kaput, 1999; Schoenfeld, 1992). Algebraic reasoning involves being able to model the problem using expressions, equations, and variables. Specifically, it is helpful to distinguish between how variables are used such as labels or unknowns (Moss & Lamberg, 2019). Teachers who struggled with variables used them as labels to keep track of their thinking but were unable to set up an algebraic equation that involved proportional thinking. The post test results revealed that teachers became more proficient at solving a similar problem after they had engaged in professional development aimed at conceptual understanding of proportional reasoning and algebraic thinking.

More specifically, when setting up the initial proportions, teachers were able to effectively represent the proportions in one variable, which is foundational to the task to write an equation, and then solve it to answer the question. In further analysis, it was noted that all the participants in the *evolving* and *emerging categories* for algebraic thinking had a conceptual barrier in both unitizing and equality properties for proportional reasoning, whereby a one-to-one relationship allows for substitution in a problem, which then changes a multi-variable equation into a one-variable equation through properties of equality that is slated to be solved for first. Specifically, these participants were unable to identify Peggy as the unit and create a proportion or relationship between Jeff and Ed based on the unit (Peggy). In the *effective* category, it was noted that participants were successful in unitizing, creating a one-to-one relationship for Peggy, allowing for the multi-variable to be translated into a single-variable problem, using the correct proportions. See Figure 1.

This suggests that identifying Peggy as the unit algebraically may have been a barrier in setting up the algebraic equation in terms of one variable. An example of this gap in algebraic thinking can be found in the *evolving* score as in Figure 1. This shows there is an algebraic idea developing, with denoting the three people as three different variables are to be added. However, no further steps are taken to solve the equation (the final task) represented in three different variables.

In the *emerging score,* the teacher starts building an equation in one variable (p), there is a complete mid-stop and disconnect to a final presentation of an equation, which thereby demonstrates there is an incomplete understanding in relating the unit or one-to-one proportional representation that is necessary for the algebraic representation in the equation, ultimately limiting their ability to solve the problem. This is an example of when Driscoll's habits of mind used as a practice set of steps to foster algebraic thinking (2001) represents a classical conditioning model (a behavioral process) that may lead to habituation over time (response to a stimulus declines). However, either does not necessarily ascertain the development of the necessary conceptual understandings of the algebraic content and reasoning being presented.

Classical conditioning and the working memory in the brain (processing new and incoming information) are not directly related and have different neural underpinnings in the brain (Gluck et al., 2016). The latter has been found to predict the learning of underlying conceptual structures when connecting multiple pieces of information (Banas & Sanchez, 2012). Flexibility and transfer of thinking when connecting information are developed in the function of the working memory. The only flexibility in thinking shown *emerging* teachers' written response in levels, was in changing fractions into decimal representations. The process of identifying the unit in a problem that used a proportion and algebraic thinking was not a specific concept covered in the professional development. More research should be done to document the cognitive processes and relationship between the proportional reasoning unit and algebraic translation.

Given that this study was limited to a single math problem, more research is needed in teacher's knowledge and ability to translate a proportional reasoning problem into an algebraic equation to model and solve the problem. Furthermore, more research is needed to understand and develop best practices to support instruction and student thinking when encountering problems that involve both proportional reasoning and algebraic thinking. Additionally, in future professional developments, more attention needs to be spent in the cognitive process of unitizing in the proportional reasoning to determine whether it enhances teachers' ability to improve their algebraic thinking.

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USING PROPORTIONAL TASKS TO EXPLORE TEACHERS' ABILITY TO MAKE SENSE OF STUDENT THINKING

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In this paper, we extend our previous work on challenges teachers face when engaging with proportional reasoning contexts to investigate two contexts that included four problems for middle grades teachers to solve as well as eight student solutions. Analysis included coding for correct solving of the problem as well as making sense of and determining reasonableness of the associated student work. Results indicate that making sense of student work was not dependent on correctly solving the problem. Determining reasonableness of student work was more challenging for our 32 participants. The think aloud interview, we argue, mimics responding to student thinking in a live setting. Implications for teacher knowledge as well as professional development and teaching will be discussed.

Keywords: Mathematical Knowledge for Teaching, Professional Development, Rational Numbers & Proportional Reasoning, Teacher Knowledge

Purpose

Proportional reasoning is an important mathematical concept for succeeding in K-12 math. However, not only students, but teachers, struggle with proportions (e.g., Akar 2010; Harel and Behr 1995; Post et al. 1988; Riley 2010). Teachers are often challenged to reason conceptually about proportions. Likely teachers, like their students, have an over-reliance on algorithms, like cross-multiplication, that leads to correct answers while not attending to multiplicative structures (e.g., Berk et al. 2009; Lobato et al. 2011; Modestou and Gagatsis 2010; Siegler et al. 2010).

Using Lamon's (2007) description of proportional reasoning as, "supplying reasons in support of claims made about the structural relationships among four quantities, (say a, b, c, d) in a context simultaneously involving covariance of quantities and invariance of ratios or products" (p.637-638) suggests the importance of teachers to identify what stays constant and recognize what varies in proportional relationships. One key idea is recognizing that a proportion is a multiplicative comparison and not an additive one (Lamon, 2007). In addition, teachers need to understand representations that highlight various components of proportional relationships, such as ratio tables and double number lines (Lobato & Ellis, 2010). Teachers need to understand how these representations support reasoning about the proportional structures.

One noted area of struggle is correctly identifying proportional reasoning situations and the tendency for students and teachers alike to use proportional reasoning in non-proportional situations (e.g., De Bock et al, 2002; Izsák & Jacobson 2017; Modestou & Gagatsis 2007). For example, De Bock and colleagues (2002) investigated students' persistent use of proportional thinking in a task focused on an area relationship. Of the 40 high school student participants in the study, 32 could not determine a correct answer even after being prompted with five scaffolds designed to highlight the area relationship. In our own work, we used a similar task, the Santa Task (see Figure 1), with middle grades math teachers and found several teachers were misled by the task. By the end of the task and our three scaffolds, only 13 of the 32 teachers correctly applied an area interpretation. Thus, this topic is challenging for teachers as well.

Santa Task		
A painter painted a 56 cm high Santa on the door of a bakery. He needed 6ml of paint. Now he is making an enlarged version of the same painting on a supermarket window using the same paint. This copy should be 168 cm high. How much paint will Bart need to do this?		
Scaffold 1: Compare favorite answers of 18ml and 54 ml	Scaffold 2: A student drew rectangles around both images	Scaffold 3: A student used easier numbers. For smaller picture used 1 tube of red paint and figured out larger would use 9 tubes of red paint.

Figure 1: Santa Task and student work

Given teachers need to not only work math problems correctly, but also make sense of students' work, we were interested in the relationship between teachers' abilities to solve the task and to make sense of sample student work on that task. We were also interested in their ability to determine whether the students' work was reasonable. We previously shared results of the Santa task analysis (Brown & Orrill, 2019). Our hypothesis was if teachers cannot solve a problem correctly, they are less likely to make sense of student thinking and less likely to determine if the solution is reasonable. Our Santa data suggest many participants could make sense of student thinking with or without solving the problem correctly; however, determining reasonableness was much more challenging when these participants had not solved the problem themselves correctly. Given this finding, we wanted to expand our focus to find out whether these trends stayed consistent across other items. Thus, we expanded our analysis to include the Milkshake task (see Figures 2-4). In this paper, we provide our analysis of both the Santa Task and the Milkshake Task. For each problem included in the tasks, we considered: (a) whether teachers engaged with students' reasoning and (b) whether they could determine the reasonableness of a student's approach. Our intent was to explore whether there was a connection between teachers' demonstrated content knowledge and their ability to make sense of students' reasoning in terms of how the student worked the task and whether the student's approach was a reasonable one. In our work, we define reasonableness to include determining whether the approach was mathematically viable or identifying the usefulness of a representation.

Perspective

Teachers facilitate students' interactions with mathematics in ways that allow them to develop meaning. Kilpatrick, Swafford, and Findell (2001) argue teachers decide when to allow students to struggle, ask questions, and provide guidance. Teachers also facilitate classroom discussions around key mathematical ideas. To do this well, teachers must engage with students' mathematical reasoning. *Principles to Actions* (NCTM, 2014) suggested teachers "elicit and use evidence of student thinking" (p. 10), including being able to assess student understanding in order to make instructional decisions.

While many have defined knowledge of teachers to include making sense of students' understanding (e.g., Shulman, 1986; Ball, Thames, & Phelps, 2008), little research has been done connecting teachers' understandings of mathematics to their understandings of students' ideas about mathematics. Rowland and colleagues' (e.g., Rowland, 2013; Turner & Rowland, 2011) Knowledge Quartet framework explores the connection between teacher content knowledge and how that knowledge is visible in teaching practice. Rowland (2013) wrote of the differences between the quartet and the Mathematical Knowledge for Teaching framework (Ball et al., 2008), "In the Knowledge Quartet, however, the distinction between different *kinds* of mathematical knowledge is of lesser significance than the classification of the situations in which mathematical knowledge surfaces in teaching." (p. 22). The Knowledge Quartet includes four dimensions: Foundation, Transformation, Connection, and Contingency. Foundation is teacher knowledge learned through schooling and professional development. The Transformation dimension is around using knowledge to support student learning in instruction. Connection addresses the coherent planning and teaching of mathematics. The fourth dimension is Contingency, a teacher's response to events in the classroom. This study addresses Contingency by investigating how a teacher reacts to unplanned student ideas about a task and how, if at all, that reaction relates to the teacher's ability to solve the same task.

Methods

In this study, we analyzed the data from the same 32 middle school teachers as in our previous work. The participants ranged from one to 26 years of experience. They were a convenience sample of middle school teachers from four states. Eight participants identified as male.

Milkshake Problem 2

Katrina wanted to make 3 cups of the recipe. How much of each ingredient did she need?

Student C:

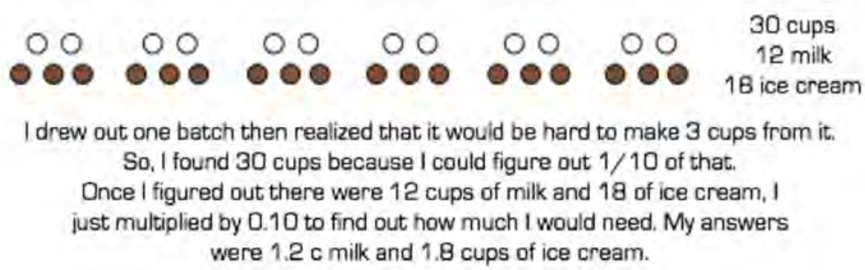


Figure 2: Milkshake Problem 2 and student work

Each participant completed a think-aloud interview that included the Santa Task and the Milkshake Task. The Santa Task was around the middle of the protocol and was inspired by De Bock et al's (2002) study. The Milkshake Task was inspired by the Orange Juice Task (National Research Council, 2001) and was at the end of the protocol. Teachers were asked to solve a mathematical problem and then respond to student work on the same task. Both tasks prompted the teachers to explain what the student was doing and whether it was reasonable. Figures 1-4 provide details about the two tasks. The Santa Task included one task with three different student solutions (we refer to these as Scaffolds 1-3). The Milkshake Task included three problems. Part 1 included two student solutions (Students A and B; Figure 3); Part 2 included one student solution (Student C; Figure 2); Part 3 included two student solutions (Team A and B; Figure 4). We considered in the verbatim transcript of each participant's interview (a) whether the participant's answer to the problem was correct; (b) whether the participant was able to make sense of each student approach; and (c) whether the participant identified the reasonableness of the approach. For participants who changed their initial solution, we analyzed only those responses that were given after the switch to correct reasoning. Transcripts were coded independently by each author and then discussed to reach 100% agreement.

Findings

Our intent was to determine whether there were relationships between participants' own mathematical thinking and their engagement with making sense of the students' thinking. Table 1 details the number of participants who solved each part correctly, were able to make sense of the student solutions, and were able to determine the reasonableness of the student solution process. The Santa problem was the more challenging problem for our participants to solve with only 13 solving it correctly. Of the three milkshake problems (Students A & B, Student C, and Teams A & B), the first was solved correctly by the majority of participants (97%). The second was solved correctly by 72% of participants, and the third was solved correctly by 88% of the participants.

Table 1: Coding for Santa and Milkshake Tasks

	Number of participants	Solved correctly	Able to Make Sense	Able to Determine Reasonableness
Santa Task				
Scaffold 1	32	12 (38%)	23 (72%)	12 (38%)
Scaffold 2	32	13 (41%)	26 (81%)	14 (44%)
Scaffold 3	32	13 (41%)	19 (59%)	10 (31%)
Milkshake Task				
Student A	32	31 (97%)	14 (44%)	16 (50%)
Student B	30*	29 (97%)	26 (87%)	27 (90%)
Student C	32**	23 (72%)	27 (84%)	23 (74%)
Team A	32	28 (88%)	27 (84%)	25 (78%)
Team B	30*	26 (87%)	17 (57%)	17 (57%)

*Data were missing for two participants for Student B and for two different participants for Team B.
**Only 31 people responded to the reasonableness question for Student C.

Overall, eight participants were able to correctly solve all four of the problems. Only three of those eight were able to both make sense of the student work and determine the reasonableness for all shared student work. Thus 9% of the participants (8 out of 32) were able to solve the problems correctly, make sense of and determine the reasonableness of all approaches.

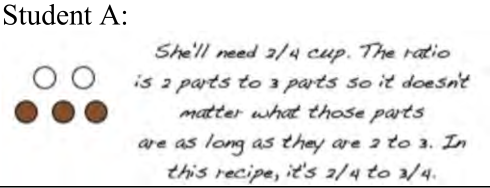
Another eight participants were able to solve one or two of the four problems correctly. Seven of these participants were able to make sense of student work on problems they did not solve correctly, but these same seven could not always make sense of the student work associated with the problem they did solve correctly. Thus, making sense of student work does not seem related to being able to solve the problems correctly. With respect to determining reasonableness, five of these eight participants had a harder time determining the reasonableness of the student work (the number of times they could determine reasonableness was less than the number of times they made sense of student work).

Looking only at the Milkshakes problems we can see that many participants solved the problems correctly and many were able to make sense of the student work and correctly determine reasonableness. However, we noticed that the responses for Student A and for Team B were markedly more difficult for these participants. Closer examination revealed that both of these student responses involved non-standard uses of common representations. For example, Figure 3 shows Student A using a common, discrete representation of the ratio two to three. In Student A's explanation, the reasoning applied to that representation used variable parts reasoning. That is, the student describes the idea that a ratio can be thought of as a fixed number of parts that can vary in size (see Beckmann & Izsák, 2015). Teachers who correctly reasoned about the student work often praised the approach as being "clever" (Charlotte), "wonderful" (Felicia) or "nice" (Greg). As Felicia so simply articulated, "I don't know why I didn't think about it, but basically, as long as she keeps the ratio to milk to ice cream as 2 to 3, it should work." For reasonableness, many teachers commented on the compatibility of the numbers. For example, Greg responded "This example was easier because you were given that she has three-quarter cups of ice cream. Therefore, each of those could equal one-fourth, and that was easy. If she was given two cups of ice cream, then you would have to figure out how to make the three circles equal to two, which would be a harder question for most students." These teachers not only solved this problem correctly but could make sense of Student A's work and reasoning. In correctly determining reasonableness they recognized the importance of the ratio remaining constant and often articulated when this strategy would be more challenging to use.

Milkshake Problem 1

Katrina wants to follow the milkshake recipe of 2 c milk and 3 c ice cream, but she only has ¾ c ice cream. How much milk will she need?

Student A:



Student B:

I used a ratio table to figure it out.

milk	2	1	1/3	1/6	3/6
Ice cream	3	1 1/2	1/2	1/4	3/4

Once I did that, it was easy to see that I'd need half a cu

Figure 3: Milkshake Problem 1 and student work

Team B, in Figure 4, used strips to demonstrate the ratios in the four recipes from Milkshakes Part 3. In this student work, we can easily see the ratios where each rectangle in the strip represents one cup. This representation is common, especially when students think more additively. The constant of proportionality is much harder to attend to in this representation. As seen in Table 1, only 17 out of 30 (57%) teachers could make sense of this work and determine whether it was reasonable. For example, Diana, in response to the question if this approach will always work, said no "Because I feel, I really feel like these students are simply counting their boxes and aren't taking into consideration the ratio." Participants who were able to determine reasonableness were able to articulate the potential pitfalls with this work and many suggested they would want to ask students a follow up question, such as when Ella said, "let's take the fractions five eighths and two thirds like those are one twenty-fourth apart. So if we were to draw those, I don't think students really could see which one is more chocolaty."

Milkshake Problem 3

Compare four milkshake recipes to determine which is the most chocolatey.

Team A: Team B:

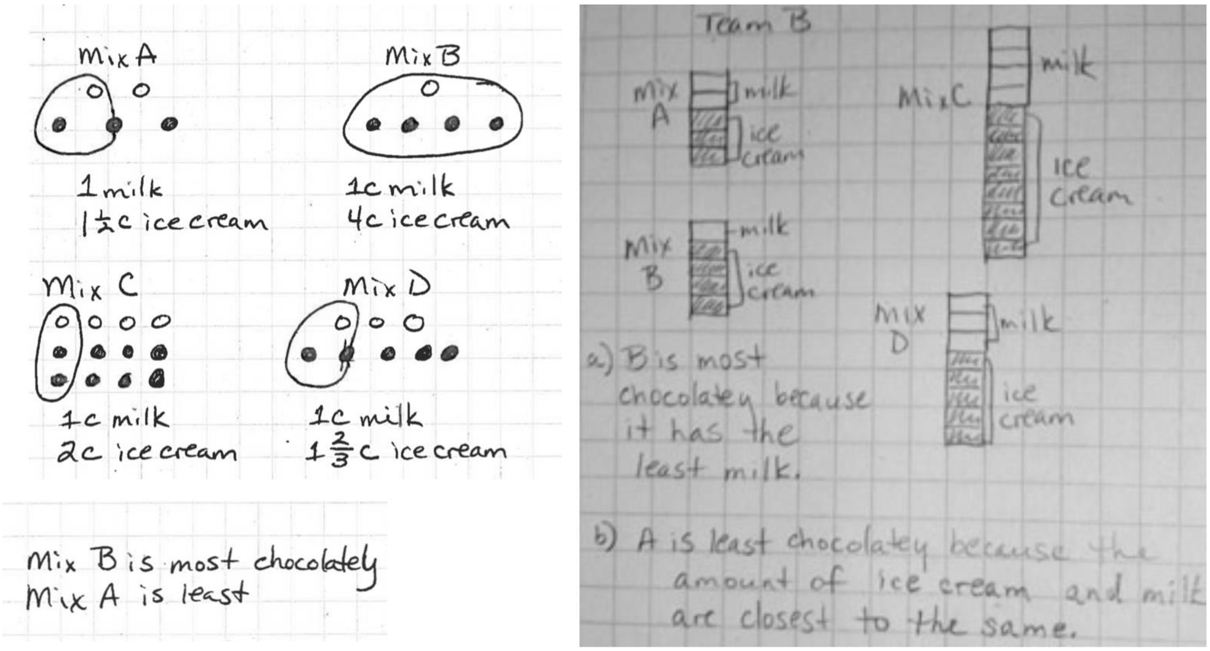


Figure 4: Milkshake Problem 3 and student work

Significance

These results contribute to the fields understanding of teacher knowledge, particularly using the Knowledge Quartet framework (Rowland, 2013) with practicing teachers (the Knowledge Quartet framework resulted from studies of preservice teachers). Our intent in this inquiry was to understand how teachers solve the problems and how they make sense of students' work, because those are both fundamental aspects of the work of teachers. We assert that evaluating seeing student work in an interview protocol is similar to seeing student work during an active lesson, only with the time constraints inherent in the classroom removed. Teachers need to make decisions about what a student is doing and whether it is a reasonable approach in order to respond in productive ways to students' work. The results from our participants indicate teachers, regardless of being able to solve a proportional reasoning problem themselves, can often make sense of what a student is doing to solve that same problem. Being able to determine the reasonableness of the approach appears to be more aligned with a teacher's ability to solve the problem correctly. Thus, Foundation and Contingency (2 of the four dimensions) may not result in the same teacher understandings used. The field should consider what teacher knowledge we are actually measuring.

In addition, these results suggest teachers are familiar with common representations, such as ratio tables. However, when common representations are used in unusual ways, such as our Student A and Team B examples, teachers have a harder time making sense of the work and determining whether it is reasonable. Thus, professional development providers and teacher educators should consider not only engaging teachers with these representations, but also engage them in considering unusual ways these representations could be used in productive and unproductive ways. This likely means, engaging teachers with the structures of the mathematics (e.g., attending to invariance and what remains constant) rather than using the representations.

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MATHEMATICAL KNOWLEDGE FOR TEACHING PROOF: COMPARING SECONDARY TEACHERS, PRE-SERVICE TEACHERS AND UNDERGRADUATE STEM MAJORS

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It has been suggested that integrating reasoning and proof in mathematics teaching requires a special type of teacher knowledge – Mathematical Knowledge for Teaching Proof (MKT-P). Yet, several important questions about the nature of MKT-P remain open, specifically, whether MKT-P is a type of knowledge specific to teachers, and whether MKT-P can be improved through intervention. We explored these questions by comparing performance on an MKT-P questionnaire of in-service secondary mathematics teachers, undergraduate STEM majors, and pre-service secondary mathematics teachers. The latter group completed the questionnaire twice- before and after participating in a capstone course, Mathematical Reasoning and Proving for Secondary Teachers*. Our data suggest that MKT-P is indeed a special kind of knowledge specific to teachers and it can be improved through interventions.*

Keywords: Mathematical Knowledge for Teaching, Reasoning and Proof, Preservice and In-service Secondary Teachers

In recent years, there have been welcomed shifts in the research on teaching and learning of argumentation and proof towards increased focus on classroom-based interventions for supporting students' engagement with reasoning and proving (Stylianides & Stylianides, 2017). These studies have shown that students' opportunities to participate in proof-related practices such as generalizing, conjecturing, posing and critiquing arguments, are dependent on teachers' ability to design learning environments that foster such engagement and on teachers' ability to advance students' learning of reasoning and proof (Bieda, 2010; Cirillo, 2011; Martin, McCrone, Bower & Dindyal, 2005; Stylianides, Bieda & Morselli, 2016).

Given the critical role of the teacher in facilitating student engagement with reasoning and proving (Nardi & Knuth, 2017) and following Ball, Thames and Phelps' (2008) and Shulman's (1986) notion of Mathematical Knowledge for Teaching (MKT), several researchers have introduced the notion of Mathematical Knowledge for Teaching Proof (MKT-P). The latter has been posited as a special type of mathematical knowledge teachers need in order to carry out the work of teaching mathematics with an emphasis on reasoning and proving (e.g., Buchbinder & McCrone, 2020; Lin, et al., 2011; Lesseig, 2016; Stylianides 2011).

Although this line of research is fast growing, several key questions about the nature of MKT-P remain open. Specifically, it is unclear whether MKT-P is a type of knowledge that is specific to teachers of mathematics, or whether it should be viewed as general knowledge of mathematical content. If MKT-P can be shown to be distinctive to the act of teaching reasoning and proving, another important question is whether it is possible to facilitate the development of MKT-P through targeted interventions. Both questions have critical importance for preparation and professional development of mathematics teachers, yet, as far we know, the literature on this topic has been scant. Our study aims to provide some initial answers to both questions.

The study reported herein is part of a larger, NSF-funded 3-year design-based-research project (Edelson, 2002), which investigated how content and pedagogical knowledge of prospective secondary teachers (PSTs) developed as a result of their participation in a uniquely designed capstone course *Mathematical Reasoning and Proving for Secondary Teachers* (Buchbinder & McCrone, 2020). The design of the course and the MKT-P assessment instrument grew out of our conceptualization of MKT-P, described below. As we explored the growth of PSTs' knowledge in the course (our original research objective), we became intrigued by the specificity of the nature of MKT-P, which led to this current investigation. We administered the same MKT-P questionnaire to 17 in-service secondary mathematics teachers, 22 undergraduate STEM majors and 9 PSTs. These PSTs participated in the capstone course in Fall 2019. We hypothesized that the in-service teachers' performance would be quantitatively and qualitatively different from the other two groups. We also hypothesized growth in the PSTs' MKT-P as measured on the pre- and post-test questionnaires.

The Course: Mathematical Reasoning and Proving for Secondary Teachers

Our prior interest in proof and reasoning (Buchbinder, 2010; McCrone and Martin, 2009) and the current work with preservice mathematics teachers has culminated in our design-based research project in which we designed a capstone course *Mathematical Reasoning and Proving for Secondary Teachers* and studied the development of PSTs' knowledge in it (Buchbinder & McCrone, 2020). The course comprised four modules focused on the following proof themes: (1) direct reasoning and argument evaluation, (2) conditional statements, (3) quantification and the role of examples in proving, and (4) indirect reasoning. These topics are known to be particularly difficult to learn and to teach (e.g., Antonini & Mariotti, 2006; Stylianides & Stylianides, 2018).

Each module includes activities aimed to crystalize, connect and apply the PSTs' knowledge of proof and reasoning across a range of secondary mathematics topics. The *crystalize* activities aimed to help PSTs refresh their memory of a particular proof theme. The *connect* activities provided opportunities to connect PSTs' mathematical knowledge with knowledge of students' proof related conceptions and misconceptions. PSTs were then required to *apply* their knowledge in actual secondary classrooms by developing lessons related to a specific proof theme and teaching those lessons to small groups of middle school and high school students. Collectively these activities aimed to enhance PSTs' MKT-P.

Mathematical Knowledge for Teaching Proof Framework

Our conceptualization of MKT-P draws on Schulman's original framework (1986), with the broad categories of subject matter and pedagogical knowledge. Within these categories, we distill those elements that have particular relevance to teaching of reasoning and proving. We also drew inspiration from the existing MKT-P literature (e.g., Corleis et al., 2008; Lin, et al., 2011; Lesseig, 2016; Stylianides 2011), but ultimately developed our own comprehensive MKT-P framework. The framework distinguishes between three interrelated facets: Knowledge of Logical Aspects of Proof (KLAP), Knowledge of Content and Students specific to proving (KCS-P) and Knowledge of Content and Teaching specific to proving (KCT-P). KLAP describes elements of subject matter knowledge specific to proof, such as knowledge of valid and invalid modes of reasoning, knowledge of logical forms of proof, such as direct proof or proof by contradiction, knowledge of a range of accepted definitions, theorems and their proofs, knowledge of logical connections and relations, such as converse, inverse, bi-conditional, etc. The pedagogical content knowledge specific to proof is represented by two types: KCS-P and KCT-P. KCS-P includes knowledge of students' proof related conceptions and misconceptions such as a tendency to rely on inductive reasoning when attempting to prove general statements, or view counterexamples as mere exceptions. KCT-P describes knowledge of pedagogical strategies for supporting student learning of reasoning and proving, such as designing and enacting proof-related tasks, questioning techniques and providing instructional feedback on students' arguments.

The three facets of MKT-P are interrelated. For example, designing proof-oriented tasks (KCT-P) must take into account students' conceptions (KCS-P); and assessing the validity of students' arguments (KCT-P) requires robust knowledge of logical aspects of proof (KLAP). Distinguishing between the knowledge facets was useful for designing both the MKT-P questionnaire and the capstone course targeting MKT-P development.

Methods

Participants and Data Collection

The participants in the study were nine PSTs who participated in the capstone course in Fall of 2019, 17 in-service secondary mathematics teachers and 22 undergraduate STEM majors. All three groups completed the same MKT-P questionnaire, described below. The PSTs were seniors who had successfully completed most of their mathematical coursework, including a *Mathematical Proof course*, and at least one methods course, but had no prior classroom teaching experience. The PSTs completed the MKT-P questionnaire twice, at the beginning and the end of the course.

The in-service teachers were recruited through in-person presentations at local schools and professional development workshops. Of the 17 participants, five teachers were from the same school; the rest were from different schools or districts. Their teaching experience ranged from two to 25 years ($\bar{x}$ = 12.18, *SD* = 8.00). The teachers completed the Qualtrics Research Suite online version of the questionnaire and received $35 honorarium.

The 22 undergraduate STEM majors were recruited through in-person presentations in three sections of a *Mathematical Proof* course at the same university in which the capstone course was given. The group comprised 11 computer science majors, 9 mathematics majors, 1 mathematics education major and 1 philosophy major. Twelve participants were sophomores, eight juniors and two seniors. The questionnaire was administered in a paper and pencil version during the final weeks of the *Mathematical Proof* course and students received extra credit for this.

MKT-P Questionnaire

We developed a 29-item MKT-P questionnaire, with some questions having a common stem. Ten items were in the area of KLAP, 11 in KCS-P and 8 in KCT-P. The items spanned four proof themes: (1) direct proof and argument evaluation, (2) conditional statements and logical equivalence, (3) quantification and role of examples in proving, and (4) indirect proof, matching the four proof themes of the capstone course (Buchbinder & McCrone, 2020). The mathematical content was middle- to high-school level algebra, geometry and functions.

The KLAP questions were multiple-choice items with a box for justification. The questions called for detecting correct assumptions for a proof by contradiction, determining logically equivalent statements, recognizing circular reasoning steps in given proofs, and identifying counterexamples. The KCS-P questions were grounded in pedagogical context (Baldinger & Lai, 2019), describing classroom situations where students presented arguments for or against a particular conjecture. The participants were to interpret the students' arguments, assess their correctness on a 4-point scale and describe any errors or potential misconceptions (if any) they notice. The KCT-P items had a similar setup as KCS-P items, but instead of numeric assessment, the participants were asked to provide feedback to the hypothetical student, highlighting strengths and weaknesses of their arguments (see Figure 1 for a sample KCT-P item).

Mr. Briggs asked his students to prove the following statement: ***The sum of any two rational numbers is a rational number.***

Molly's solution:

Suppose r and s are rational numbers. By definition of a rational number, let $r = \frac{a}{b}$ where a and b are integers, and $b \neq 0$. Similarly, s is rational so let $s = \frac{a}{b}$ where a and b are integers, and $b \neq 0$.

Then $r + s = \frac{a}{b} + \frac{a}{b} = \frac{2a}{b}$.

Let $p = 2a$. Then p is an integer because it is the product of two integers.

Hence $r + s = \frac{p}{b}$, where p and b are integers and $b \neq 0$.

Thus, $r + s$ is a rational number by definition of a rational number. ■

i) Identify errors (if any) in the student's argument. If none, write "no errors".

ii) Provide feedback to the student, highlighting strengths and weaknesses of their argument.

Figure 1: Sample KCT-P item

Large-scale validation of the instrument was beyond the scope of our original research. Thus, we used expert validation with three mathematicians and one mathematics education expert, and tested the instrument for two years. Cronbach alpha for the entire MKT-P questionnaire was 0.892, with 0.81 for KLAP, 0.71 for KCS-P and 0.76 for KCT-P.

Data Analysis

For the quantitative data analysis, each KLAP item was scored out of 3 points: 1 point for correct choice and 2 points for correct explanation, or 1 point for partially correct explanation. The KCS-P and KCT-P items were scored on a 0-4 point rubric, with 0 points given to a mathematically incorrect response and 4 points to a correct answer that showed deep engagement with the student's argument (exceeding expectations). The research team developed the scoring rubric jointly, by analyzing about 20% of the data. Next, two researchers scored the rest of the data individually and met regularly with the rest of the team to reconcile any discrepancies. The Kappa scores for inter-rater reliability were 0.78 for KCS-P and 0.8 for KCT-P.

For each group: Teachers, STEM majors and PSTs, we calculated the mean total scores for the overall MKT-P. Since the number of items in each subdomain: KLAP, KCS-P and KCT-P, was different, we calculated the mean average score per subdomain per group. Using JMP® Pro statistical software version 15.0.0 we performed one-way ANOVA to determine whether the three groups differed statistically. In addition, we used Welch's Test for the presence of non-constant variance and Tukey-Kramer's Honestly Significant Difference Test for multiple comparisons. Since PSTs' pre and post-course scores are dependent on each other, we performed two separate analyses, once comparing the performance of teachers, STEM majors and PSTs' pre-course scores, and once comparing teachers, STEM majors and PSTs' post-course scores. In the rest of the paper, we use PSTs-pre and PSTs-post to denote this distinction. We also used matched pairs t-tests to compare PSTs-pre to PSTs-post performance.

To capture qualitative differences among the groups we used open coding and thematic analysis (Miles, Huberman, & Saldana, 2018; Yin, 2011). In particular, we coded for the use of first-person language in providing feedback to hypothetical students, and for the types of negative and positive appraisals of student arguments given by the study participants.

Results

The overall MKT-P performance of the three groups is shown in Table 1. The maximum possible score on the test was 86, suggesting that teachers' mean total was about 60%, STEM majors scored around 50% and PSTs went from 41% on the pre- to 70% total score on the post. The one-way ANOVA and the Welch's Test showed that the three groups: teachers, STEM majors and PST-pre are statistically different from each other ($p = 0.0219$). The difference was due to teachers scoring significantly higher than PSTs-pre ($p = 0.0379$). The differences between STEM majors and PSTs-pre or between teachers and STEM majors were not statistically significant. However, it is notable that while the maximum total score in the STEM majors' group was 66, four teachers had a total score above 75, meaning that the lack of significance can be due to the high variability of performance in the teachers' group.

Table 1: Overall MKT-P performance of the three groups

Group	No of participants	Mean Total Score	SD
Teachers	17	51.4	21.11
STEM majors	22	42.5	13.12
PSTs **pre**	9	35.4	7.59
PSTs **post**	9	59.7	11.23

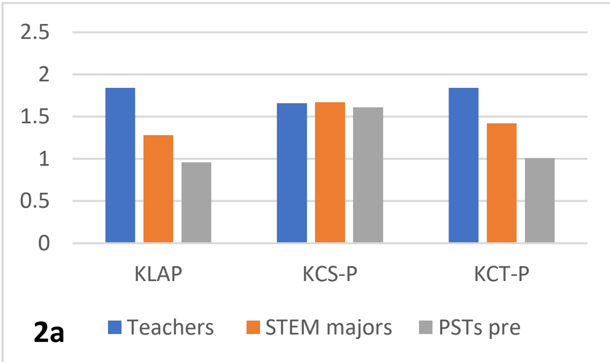
When comparing the mean scores of teachers, STEM majors and PSTs-post, the differences were still significant ($p = 0.0053$), but in this case, the difference was due to PSTs-post scoring higher than the other two groups. In particular, PSTs-post significantly outperformed STEM majors ($p = 0.0274$), but not the teachers ($p = 0.4319$).

The pairwise t-test comparing PSTs' pre and post-course performance revealed significant growth in overall MKT-P ($p < 0.0001$); the 95% confidence interval showing the average increase between 16 and 29 points. This outcome supports our assumption that MKT-P can be improved by targeted intervention, such as our capstone course, so much so that PST-post outperformed both the teachers and STEM majors.

Table 2 shows the results of the analysis broken down by MKT-P subdomains: KLAP, KCS-P and KCT-P. In this table, we calculated the mean scores for each domain, rather than total points, since the number of items (and points) in each domain was different.

Table 2: Performance of the groups by MKT-P subdomain

		KLAP		KCS-P		KCT-P	
Group	No	Mean Score	SD	Mean Score	SD	Mean Score	SD
Teachers	17	1.84	1.26	1.66	1.09	1.84	1.29
STEM majors	22	1.28	1.29	1.67	1.09	1.42	1.23
PSTs **pre**	9	0.96	1.17	1.61	1.01	1.01	1.12
PSTs **post**	9	2.15	1.15	2.04	1.03	1.97	1.22



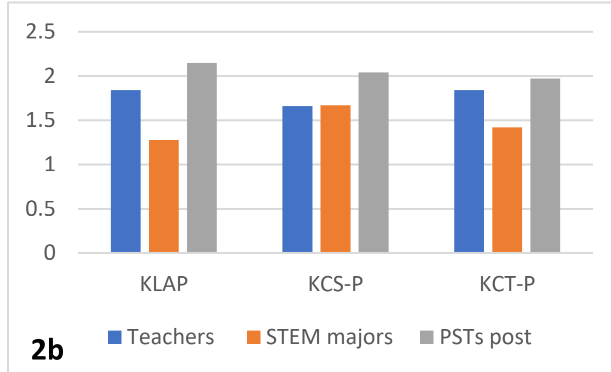


Figure 2 a & b: Performance of the groups by MKT-P subdomain

Figure 2 shows the same information as Table 1, but in a graphic format: Figure 2a (left) compares performance of teachers, STEM majors and PSTs-pre. Figure 2b (right) compares teachers, STEM majors and PSTs-post.

When comparing teachers, STEM majors and PSTs-pre, the analysis showed that there three groups differed significantly on KLAP ($p < 0.0001$) and KCT-P ($p < 0.0001$), but not on KCS-P ($p = 0.8843$). The KCS-P portion of the questionnaire intended to assess participants' ability to identify proof-related misconceptions. All three groups performed very similarly. We do not have an explanation for that, except that our KCS-P items probably measure mostly mathematical knowledge, despite their pedagogical framing. The PSTs performance on the KCS-P items improved significantly from pre to post ($p = 0.0013$), and was significantly higher than of teachers ($p = 0.0162$) and of STEM majors ($p = 0.0126$).

Considering the KLAP portion of the questionnaire, the teachers significantly outperformed both STEM majors ($p < 0.0001$) and PSTs-pre ($p < 0.0001$). This result is interesting since KLAP items measure pure mathematical knowledge. A closer analysis revealed that teachers were better than other groups at identifying logical forms such as converse and contrapositive and tended to use proper mathematical vocabulary. The PSTs' KLAP performance improved significantly on the post-questionnaire ($p < 0.0001$). The PSTs-post scored significantly higher than STEM majors ($p < 0.0001$) but not significantly higher than teachers ($p = 0.1343$).

A similar tendency was observed with respect to KCT-P portion of the test – items that called for identifying logical errors in student arguments and providing instructional feedback to the students. Not surprisingly, teachers significantly outperformed STEM majors ($p =0.0090$) and PSTs-pre ($p < 0.0001$). But when compared to PSTs-post, the PSTs closed the gap and scored very similar to the teachers, and significantly higher than STEM majors ($p = 0.0050$).

Qualitative Differences Between the Groups

The differences between the groups also had a qualitative nature, as revealed in the analysis of written feedback to hypothetical students' arguments on the KCT-P items. STEM majors tended to use third person language talking about the student work rather than addressing the student directly (contrary to the task requirements). STEM majors tended to compliment student work for brevity or clarity, focusing more on the presentation rather than on the content of the argument, e.g., "clear and appropriate assumptions, well ordered." Positive appraisals often merely reiterated the student's approach, e.g., "Anthony was smart in using variable *a* and *b* to help prove the conjecture." Despite praising the student, this comment shows neither analysis of nor engagement with the student's proof strategy.

In their critiques, STEM majors tended to point out that a student's argument did not constitute a mathematical proof but without clarifying the concern, e.g., "not proven enough," "isn't concrete enough to prove the statement." More substantive critiques referred to incorrect assumptions, e.g., "the student assumed their conclusion by saying the sum of two fractions is a fraction," and lack of generality, e.g., "there is no generality, it is only examples." Overall, STEM majors tended to focus feedback on the mathematical validity and form of a student's argument.

Alternatively, participants in the teacher group tended to speak directly to hypothetical students and focus their comments on students' conceptual understanding of the given problem. For example, "you show your strong understanding of what a rational number is and how to use variables to generalize a situation." Teachers' critiques of student arguments tended to focus less on the form and more on the mathematical validity of the arguments. Moreover, the critiques were often phrased as open questions, e.g., "can we look at this algebraically?" or "would your proof hold true if *r* and *s* were equal to different fractions?" or "is there a way to show this is true for all real numbers?" This rhetorical style of feedback shows teachers' concern for student understanding and engaging students in revisionary work.

The PSTs comments fell between the student-oriented feedback of the teachers and the mathematics-oriented feedback of the STEM majors. Some PSTs worded their feedback in the question format e.g., "How can you say that only numbers that satisfy Sam's conjecture are 2 and 0?" But the majority of PSTs used third person language and made mathematics-oriented comments, e.g., "Anthony made a valid argument by turning the numbers into a general expression." There were shifts towards more frequent use of first person language and question-posing feedback from pre- to post-questionnaire.

Discussion

The objectives of our study were to examine whether the Mathematical Knowledge for Teaching Proof, as measured by our MKT-P questionnaire, differs from pure knowledge of mathematical content. We hypothesized that if this knowledge is special to mathematics teachers, it would show as better performance on the MKT-P questionnaire when compared to STEM majors or PSTs. We also conjectured that it would be possible to facilitate MKT-P growth through a targeted intervention such as our capstone course *Mathematical Reasoning and Proving for Secondary Teachers* (Buchbinder & McCrone, 2020), which would be visible in improved PSTs' performance on the MKT-P questionnaire.

The data presented above supports both of our assumptions. The teachers outperformed STEM majors and PSTs-pre on the overall MKT-P, and on two MKT-P subdomains: knowledge of the logical aspects of proof (KLAP) and knowledge of content and students (KCT-P). The three groups performed similarly on the KCS-P portion of the questionnaire – items intended to assess knowledge of students' proof-related (mis)conceptions. This may be reflective of a limitation of our instrument, which did not discriminate between the different groups.

The fact that teachers outperformed PSTs-pre on almost every measure is not surprising; it is consistent with the general MKT literature (e.g., Phelps, Howell, & Liu, 2020). Our study adds to this literature by showing that the differences between prospective and practicing teachers appear also in MKT-P. The teachers also scored higher than STEM majors, whose knowledge of proof was fresh in their minds due to their enrollment in a proof course at the time of the study. This outcome may support our assumption that MKT-P is a special kind of knowledge, beyond mathematical content knowledge. Alternatively, this difference can be due to self-selection bias of the participants in the two groups of STEM majors and in-service teachers. Our study design does not allow distinguishing between these alternatives. Future studies should explore this issue.

Another support for our hypothesis about the special nature of MKT-P comes from the qualitative analysis of the feedback provided by the participants on sample student arguments. Particularly striking were the differences between teachers and STEM majors, while PSTs were somewhere between those two groups. The teachers' comments were characterized by a tendency to use first person language addressing the student directly, deeper engagement with a student's argument, attempts to gauge and advance student understanding through guiding questions and suggestions for revisions. On the contrary, the STEM majors' comments were characterized by the tendency to use third person language, focus on the form of the argument rather than its logical structure, critiquing student work for the lack of mathematical rigor but without explaining insufficiencies in student work. Thus, teachers' MKT-P is evident in their ability to provide feedback of higher potential for educative impact than STEM majors (Hattie, & Timperley, 2007).

Our second research question was whether MKT-P can be enhanced through intervention. Note that exploring *how* PSTs' MKT-P evolves throughout the capstone course, connecting the learning processes to the design features of the course and examining factors that promote or inhibit MKT-P development were the core objectives of our three-year long study. Presentation of these findings is beyond the scope of this paper. The significance of this paper is in comparing PSTs' pre- and post-course performance with other groups who may have similar characteristics to our PSTs. We do not see STEM majors or teachers as control groups in any sense. Comparing the MKT-P performance across all groups allows putting the observed changes in the PSTs' MKT-P into broader perspective, adding methodological strength to the simple pre-post design.

The data presented above show that STEM majors performed slightly better than PSTs-pre, although the differences were not statistically significant on any measure. Despite the fact that all PSTs had successfully passed the *Mathematical Proof* course in the second or third year of their program, prior to taking the capstone course, the proof-specific mathematical content was fresher in the minds of the STEM majors than of the PSTs. The course *Mathematical Reasoning and Proving for Secondary Teachers* provided the PSTs with opportunities to refresh and strengthen their proof-related content knowledge. More importantly, the course activities challenged the PSTs to connect this knowledge to teaching secondary mathematics by analyzing sample student arguments, providing feedback on hypothetical student work, planning proof-oriented tasks, enacting them in real classrooms and reflecting on their teaching. These types of activities help to bridge the gap between university-level mathematics preparation and the practice of teaching secondary mathematics (Grossman et al., 2009; Wasserman et al., 2018).

Our study concurs with that literature. After participating in the capstone course, the PSTs' MKT-P improved significantly both overall and in each subdomain. The PSTs closed the gap with in-service teachers on the overall MKT-P, KLAP and KCT-P, and scored significantly higher than the teachers did on KCS-P. The PSTs-post also performed significantly higher than STEM majors did on the overall MKT-P and on each of the MKT-P subdomains. Overall, these results support our hypothesis that MKT-P can be enhanced through intervention.

Our study is exploratory, small scaled and localized. In our data analysis, we utilized statistical techniques that are robust to small numbers of participants (see methods section). However, we make no claims to generality and the results should be interpreted as preliminary. Nevertheless, our study makes several notable contributions to the existing body of knowledge. We proposed an MKT-P framework and a questionnaire for assessing MKT-P at the secondary level, which spans four proof themes – key areas of difficulty with reasoning and proof, according to the research literature. The comparison of the MKT-P performance of in-service teachers, STEM majors and PSTs suggests that MKT-P, as measured by our instrument, is indeed a type of knowledge that is special to teachers, as opposed to other groups with presumably similar mathematical content knowledge. Finally, our study has shown that MKT-P can be enhanced by targeted intervention, such as our capstone course. This was evident in the significant improvement of PSTs' performance from pre- to post, and in comparison of PSTs-post scores with STEM majors and teachers. It would be important to replicate this study on a larger scale and with other, more diverse, populations.

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CHARACTERIZING PROSPECTIVE SECONDARY TEACHERS' FOUNDATION AND CONTINGENCY KNOWLEDGE FOR DEFINITIONS OF TRANSFORMATIONS

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One promising approach for connecting undergraduate content coursework to secondary teaching is using teacher-created representations of practice. Using these representations effectively requires seeing teachers' use of mathematical knowledge in and for teaching (MKT). We argue that Rowland's (2013) Knowledge Quartet for MKT, in particular, the dimensions of Foundation and Contingency, is a fruitful conceptual framework for this purpose. We showcase an analytic framework derived from Rowland's work and our analysis of 85 representations of practice. These representations all featured geometry. We illustrate examples of combinations of "high" and "low" Foundation and Contingency, and show results of coding juxtaposed with performance on an instrument previously validated to measure MKT. We describe the potential for generalizing this framework to other domains, such as algebra and mathematical modeling.

Keywords: Mathematical Knowledge for Teaching, Undergraduate Education, Preservice Teacher Education

Many secondary teachers find their undergraduate mathematical preparation disconnected from their teaching (Goulding et al., 2003; Ticknor, 2013; Wasserman et al., 2018; Zazkis & Leikin, 2010). The mathematicians who teach these teachers may want to connect content coursework to secondary teaching (e.g., Lai, 2019; Lischka et al., 2020; Ticknor, 2013). At the same time, mathematics faculty may lack the resources to say precisely what connections may be there, and how to give feedback to teachers regarding the connections (Lai, 2019).

In recent years, several groups have addressed this problem by developing (a) tasks for content courses where teachers create representations of practice, and (b) design principles for such tasks (e.g., lvarez et al., 2020; Wasserman et al., 2019). Mathematics faculty can now use these principles to create such tasks, but not necessarily to provide constructive feedback to teachers about their responses. To our knowledge, the field lacks frameworks for characterizing the mathematical knowledge in and for teaching (MKT) observable in representations of practice created by teachers, in ways that would support feedback. Such frameworks could position mathematics faculty to bridge mathematics and teaching more powerfully.

Our purpose is to characterize dimensions of MKT visible in teacher-created representations of practice, and to do so in a way that can potentially inform feedback to prospective teachers about their mathematical understanding and its use in teaching. In our work, representations of practice are snapshots of discourse used in responding to student contributions. Using such representations, created by prospective teachers, we asked: *What MKT is observable in teacher-created representations of practice?* We argue that Rowland's Knowledge Quartet framework for MKT is a productive analytic framework for analyzing representations of practice. We contribute a framework for observing "high" and "low" levels of knowledge in two dimensions of the Knowledge Quartet, namely, the Foundation and Contingency dimensions. Our corpus consists of teacher-created representations of practice featuring geometry from a transformation perspective. We conclude by considering how our work can generalize to other mathematical domains.

Conceptual Perspective

Mathematical Knowledge in and for Teaching (MKT)

Across the various literatures on MKT (e.g., Ball et al., 2008; Davis & Simmt, 2006; Heid et al., 2015; Rowland, 2013; Thompson & Thompson, 1994) and on mathematics learning (e.g., Daro et al., 2011; National Academies of Sciences, Engineering, and Medicine, 2018; Simon, 2006; Thompson, 2000), we have found ideas of Rowland and colleagues (2013, 2016), Simon (2006), and Thompson (2000) most generative for our work.

Among the four dimensions that compose Rowland's (2013) Knowledge Quartet framework for MKT, we focus on two: *Foundation* (knowledge and understanding of mathematical ideas, the nature of mathematics, as well as principles of mathematical pedagogy) and *Contingency* (the ability to respond to unanticipated events ranging from network outages to learners' alternative strategies). The remaining dimensions are *Transformation* (presenting ideas to learners) and *Connection* (cohering ideas over time). Foundation knowledge is observed through actions associated with the other dimensions, as well as in teacher talk outside of teaching (e.g., in a debrief of student teaching). Rowland, Thwaites, and Jared (2016) validated the use of this framework for identifying instances in student teaching at the secondary level where an observing teacher educator can infer the use of content or pedagogical content knowledge.

Rowland and colleagues used videos of teaching across multiple topics in multiple schools. We examined teacher-created representations of practice responding to a limited set of prompts. Hence, we found it useful to delimit and elaborate on Foundation and Contingency as follows. First, we delimited the Foundation dimension to knowledge of mathematics, because of our interest in content coursework. Second, the dependence of Foundation on *mathematical understanding* suggested that we be theoretically clear about a conception of mathematical understanding. We used Simon's (2006) characterization: mathematical understanding is the "learned anticipation of the logical necessity of a particular pattern or relationship" (p. 364). For instance, we consider understanding mathematical procedures to include relating that procedure to its underlying definitions or concepts, and to anticipate doing so when explaining procedures or troubleshooting a use of a procedure. Then, we delimited Contingency to the ability to integrate given student thinking into teacher talk. Finally, we used Thompson's (2000) notion of interacting reflectively to elaborate on integrating student thinking (cf. Ader & Carlson, 2018). When teachers interact reflectively, they interpret and leverage student thinking. When teachers interact unreflectively, they do not adopt the student's perspective.

Rowland and colleagues' work results in a framework for identifying instances where MKT may be used, but it does not result in a framework for characterizing levels of use such as would be useful for guiding an instructor to provide feedback to a teacher.

Teacher-Created Representations of Practice

In all assignments we analyzed, prospective secondary teachers created representations of practice based on a description of a teaching situation provided to them, where the teaching situation included various samples of student work. These representations of practice may be considered an approximation of practice in Grossman et al.'s (2009) framework, meaning that they are "opportunities for novices to engage in practices that are more or less proximal to the practice of a profession" (p. 2058).

Mathematical Context for Analysis

A transformation perspective is characterized by defining congruence and similarity via transformations (Usiskin & Coxford, 1972). The transformations critical to congruence and similarity are reflections, rotations, translations, and dilations. Across the two units in the materials used in this study, prospective teachers developed community definitions for reflection, rotation, translation, and dilation. Then, prospective teachers used these definitions to construct images of these transformations, as well as to determine whether two proposed figures can be connected by one of these transformations.

There were four prompts for creating representations of practice in the modules. Two prompts asked teachers to video-record themselves, and two asked teachers to write narratives. All prompts provided images of secondary student work and asked prospective teachers to respond in a way that would move students toward understanding connections between definitions and constructions of images of relevant transformations. All prompts provided the secondary level task that the sample secondary student work was responding to. Figure 1 shows images of student work from some of the prompts.

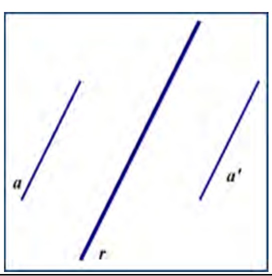
(Sample for prompt focused on rotation) As students are working on rotations of a flag, you observe two students with the following work completed.

Student 1:

Student 2:

(Sample for prompt focused on reflection) As students are working, you observe two students with the following work completed.

Student 1:



Student 2:

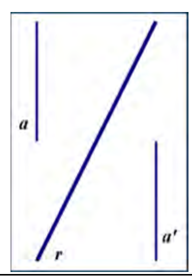


Figure 1: Images of secondary student work from two prompts

Data & Method

Overview

To develop a framework for characterizing Foundation and Contingency knowledge in teacher-created representations of practice, we analyzed teacher-created representations of practice in two rounds of coding. The first round aimed to characterize three levels of Foundation knowledge using all representations of practice from the first year of data collection. The second round used a purposive sample from three years of data collection, as detailed below. In this round, we sought to streamline coding for Foundation and develop coding levels of Contingency. We focused on streamlining because we sought a framework that could be ultimately usable by mathematics faculty who may not be education researchers, and that could potentially generalize across domains. In the second round, our process for coding and reconciling for two levels each of Foundation and Contingency knowledge ("High", "Low") took an average of 10 minutes per coder, per representation of practice.

Sampling

Data were drawn from the *Mathematics of Doing, Understanding, Learning, and Educating for Secondary School*s (*MODULES*[2]) project, which has developed curriculum materials for content courses for prospective teachers in four content strands (algebra, geometry, statistics, and modeling). We analyzed prospective teachers' responses to tasks in the geometry materials. Data included more than 300 teacher-created representations of practice from 93 prospective teachers in different regions of the US. The first round of coding used 54 teacher-created representations (2 representations x 27 teachers) from the first year of data collection. The second round of coding used a purposive sample of 31 teacher-created representations of practice (4 representations x 7 teachers + 3 representations x 1 teacher). The purposive sample was selected to document the range of potential MKT. Among the 93 teachers, 61 had completed pre- and post-semester forms of GAST, an instrument validated to measure knowledge for teaching geometry at the secondary level (Mohr-Schroeder et al., 2017). After assigning "high-GAST" and "low-GAST" thresholds for each item, we narrowed the pool to 20 teachers, consisting of the top 10 teachers ranked by proportion of "high-GAST"-"high-GAST" pre-post item scores, and the top 10 teachers ranked by "low-GAST"-"low-GAST" pre-post item scores. Only 8 of these teachers had submitted all 4 representations of practice assigned in the modules. During analysis, we realized that one assignment was scanned incompletely. This resulted in our sample of 31 teacher-created representations of practice.

Analysis

To develop a framework for characterizing foundation and contingency knowledge, we first considered Rowland's (2013) descriptions and Weston and Rowland's (n.d.) elaborations of Foundation and Contingency dimensions. We then reflected on how these considerations may apply to the specific teacher-created representations of practice analyzed and, at the same time, how they may apply to other domains. To do this, we involved researchers with expertise in a variety of mathematical domains – such as mathematical modeling, algebra, and geometry – in our discussion. In both rounds, we were blind to pre/post-test scores; no coders scored the pre/post-tests. We created lists of characteristics of representations of practice demonstrating "High" and "Low" Foundation and Contingency knowledge. We used these lists of characteristics to classify representations of practice first individually, then reconciling differences collaboratively, following a constant comparison method (Strauss & Corbin, 1998).

Results

Characterizing Levels of Knowledge in Teacher-Created Representations of Practice

Our main result is a framework for characterizing levels of Foundation and Contingency knowledge in teacher-created representations of practice. This framework is shown in Figure 2. Codes from the second round of analysis, using the framework, are shown in Figure 3, along with those teachers' post-test scores.

In our full presentation, we illustrate characterizations of all four combinations of Foundation/Contingency (High/High, Low/High, High/Low, Low/Low) with teacher-created representations of practice in response to multiple prompts and discuss contrasts in knowledge observed in video and written representations of practice.

	FOUNDATION	CONTINGENCY
HIGH	Recognizes the logical necessity connecting the definition of a transformation to ways of constructing a transformation image *Examples:* ● Explains a method of construction by marking points on a preimage and then "applying the definition to each of the marked points" to obtain the image ● Reasons that an attempted image is incorrect by showing that it does not satisfy the transformation definition	Frames questions or explanations about connection between construction and definition in terms of students' thinking *Examples:* ● Directs attention to student work to understand the idea that all properties of the definition must be followed to produce a correct construction. ● Engages students in selecting locations in sample student work, and reasoning whether the definition is satisfied
LOW	Explicitly or implicitly treats the definition of a transformation as separate from constructing images, and/or demonstrates lack of understanding of definition *Examples:* ● Describes a method for constructing, and never mentions any definition. ● Provides incorrect definition	Does not integrate student thinking into explanation of connection between construction and definition *Examples:* ● Evaluates student work as "right" or "wrong"; does not cite work otherwise ● Provides a correct explanation that does not reference student work

Figure 2. Framework for Characterizing Foundation and Contingency Knowledge

		FOUNDATION			
PSMT	post	📹	✏	📹	✏
GMM 205	12	H	H	H	H
GMM 202	11	H	H	H	L
GTA 218	10	L	L	L	L
GMM 302	9	L	H	L	L
GMM 308	9	H	L	H	L
GMM 201	9	L	L	L	L
GTA 206	6	L	H	L	--
GTA 208	3	L	L	L	L

		CONTINGENCY			
PSMT	post	📹	✏	📹	✏
GMM 205	12	L	H	L	H
GMM 202	11	H	H	H	L
GTA 218	10	L	L	L	L
GMM 302	9	L	H	L	H
GMM 308	9	L	H	H	H
GMM 201	9	L	L	L	L
GTA 206	6	L	L	L	--
GTA 208	3	L	L	L	L

PSMT = prospective secondary mathematics teacher, listed by pseudonym
post = score on post-test administration of GAST
• = video representation practice, ✏ = written representation of practice
• Representations of practice are listed in order they were assigned
H = High, L = Low

Figure 3. Characterizations of knowledge observed in purposive sample, with GAST scores

In this paper, we illustrate two combinations of Foundation and Contingency knowledge (High/High, Low/High), using responses from one prompt. In this prompt, teachers were asked to write a narrative describing how they would "elicit student thinking about these reflections, with specific use of the two example students work, and move the class toward understanding connections between methods of reflection and the definition of reflection." The sample student work was shown previously in Figure 1. A class definition for reflection is given in Figure 4.

A reflection across a line L is a transformation that, for every point P in the plane: • $P' = P$ if P is on L • L is the perpendicular bisector of segment PP' if P is not on L.
Note: These materials teach prospective teachers the convention that P' refers to the image of a preimage P under the transformation discussed.

Figure 4. Version of class definition of reflection

Illustration 1: High Foundation-High Contingency

In our framework, the quality of Foundation knowledge is characterized by linking constructions to the definition, and the quality of Contingency knowledge is characterized by integrating student work into the work of connecting constructions and definitions. GMM302 was a representative case to illustrate High Foundation - High Contingency knowledge. GMM302 began their representation of practice:

> "To start, I would draw the student responses and our definition of Reflection on the board. [...] Pointing to the first response, [I would ask,] if we were to draw a line between the points P and the corresponding P's, what can we tell about the line segments made by P and P'? As students respond, I draw and make the corresponding changes to the figure on the board." (see Figure 5a).

Then, after describing some potential responses from students, GMM302 prompted students to link construction and definition: "What is it we know about our line of reflection in regard to our definition of reflection?" GMM302 then marked the angles (see Figure 5b), asked students questions to review the two defining properties of perpendicular bisectors (bisecting, and with perpendicular angles), and posed the question, "Since our main problem here is the angles, how might we approach this in a way that results in right angles instead?" Finally, after drawing a correct reflection (see Figure 5c) but without evaluating it as such to the students, GMM302 asked, "Looking at our new figure, does this hold true to the definition of a Reflection?" GMM302 concluded, after describing potential responses, "Yes, it does hold true. So, we know [segment] a' is the reflection of [segment] a across the given line." We characterized this response as High/High because GMM302 created tight connections from incorrect and correct images to the definition, positioned students to engage with these links, and did so while centering student work.

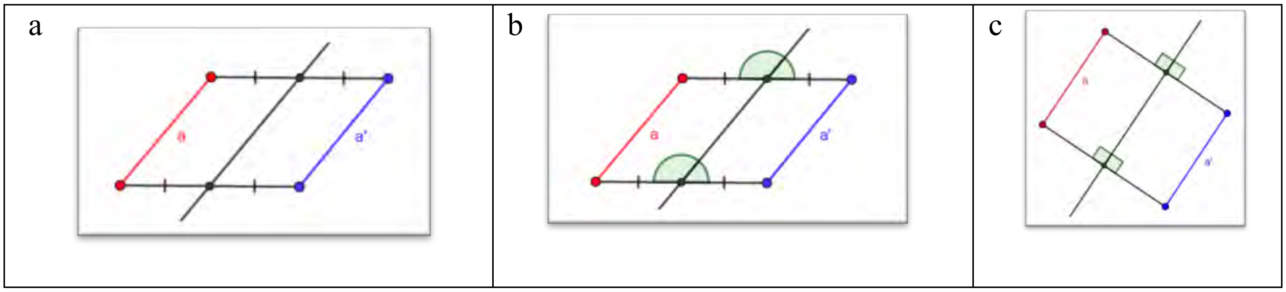


Figure 5. Board drawings proposed by GMM302 to link construction to definition

Illustration 2: Low Foundation-High Contingency

We chose participant GMM308 as a representative case to illustrate Low Foundation - High Contingency knowledge. GMM308 began their representation by analyzing the student work, and suggesting what may have been going on in the students' mind that led to these constructions.

> It looks as though they have drawn lines across the line of reflection from each point to the reflected point. I believe that they have thought that since it is reflected that the distance from the line of reflection is now opposite for each point (the point on top of the reflected image is the same distance as the point on the bottom of the pre-image and vice versa).

In this way, GMM308 exemplified the notion of interacting reflectively with student thinking (Thompson, 2000). GMM308 then described how the student work could be linked to the definition:

> I would use [Student 2's work] to discuss with students how this attending to some points of the definition, but not quite (sic). They have used the idea of the same distance from the line of reflection, but it was utilized incorrectly. I would use this to be able to discuss with students how this doesn't fully fit the definition of a reflection and how we can fix that. We would work as a class to improve the original reflection and make sure it fits all of the necessary components of the definition needed.

This representation of practice exemplified High Contingency knowledge because GMM308 identified how specific student work could be connected to the definition of reflection, especially the role of perpendicular bisectors. However, GMM308 did not articulate the reasoning about perpendicular bisectors precisely, and so we characterized the Foundation knowledge as Low. GMM308's analysis of Student 2's work was similar to their analysis of Student 1's work in that it did not articulate how precisely students might be able to determine whether an image and preimage could satisfy the definition of reflection.

Discussion

We analyzed teacher-created representations of practice in two rounds, resulting in a framework for observing Foundation and Contingency knowledge, characterizations of combinations of Foundation/Contingency levels, and the potential for comparing these characterizations to performance on an instrument previously validated to measure MKT. Previous research has identified dimensions of MKT (e.g., Rowland, 2013), conceptualized mathematical understanding (e.g., Simon, 2006), and unpacked teachers' actions to understand

and act on student thinking (e.g., Ader & Carlson, 2018; Thompson, 2000; Weston, 2013). These scholars grounded their work in videos of teaching and interviews. We synthesized previous research to contribute a framework for observing varying levels of different dimensions of MKT, applied to a pedagogy of teacher education, that of teacher-created representations of practice.

In evaluating the robustness of our framework, we consider the limitation of our data to four prompts for representations of practice in geometry, and the potential for our framework to generalize across domains. Our framework, as reported, is tailored to the use of definition to a particular concept of geometry, and derived from the analysis of a limited number of prompts. However, we see our framework as generalizable to mathematics more generally because of its underpinnings in the Knowledge Quartet (Rowland, 2013), mathematical understanding (Simon, 2006) and interacting reflectively (Thompson, 2000), all of which are intended to apply broadly to mathematics teaching and learning. Moreover, the centrality of definition to mathematics, as well as reasoning with definition or other assumptions (Kitcher, 1984), suggests the potential for adapting this framework to domains of mathematics with strongly structured logical systems, such as algebra. For instance, in place of linking definitions with construction methods, the framework could emphasize connecting definitions with common procedures or tests (e.g., ways to solve equations, vertical line test), and to what extent student thinking is centered in engaging with these procedures or tests. For domains such as mathematical modeling, which apply mathematics in phases of distinctive practices (e.g., Blum & Leiß, 2007), the framework could emphasize the rationale for each phase as well as anticipation of movement across phases, for instance, knowing that the proposal of a mathematical model can be followed by considering the real world or the results of the model, that that these phases can work together to refine one's model (e.g., Czocher, 2018).

We would be remiss to not issue caveats about the use of "levels" of knowledge. Most importantly, these characterizations, like other hierarchical characterizations in the literature (e.g., Ader & Carlson, 2018; Serbin et al., 2018), are not intended to be characterizations of teachers or their ultimate potential. Rather, we present these levels as descriptions of observable features of representations of practice that may be ultimately usable by teacher educators to guide formative feedback for prospective teachers. When the teacher educator is a mathematics faculty member, characterizing only an ideal may not be sufficient for helping that teacher educator articulate, for example, where a teacher might have involved student thinking more. A teacher educator could use the framework as a way to begin a dialogue with prospective teachers to support their growth. We believe that the risk of characterizing "levels" is outweighed by the potential benefit of supporting mathematics faculty members in seeing how to connect mathematics and teaching.

When we began this work, we had in mind conversations with mathematics faculty members as well as the research that indicates that mathematics faculty may want to connect mathematics and teaching, but do not know how. We also had in mind the mathematics faculty members that pilot our materials, which come in four domains: geometry, algebra, mathematical modeling, and statistics. We argue that attention to the dimensions of Foundation and Contingency are a fruitful framework for characterizing knowledge in teacher-created representations of practice. GMM308's representation of practice, and other examples of High/Low and Low/High combinations, illustrate that the dimensions of Foundation and Contingency can be viewed as separable, and therefore be distinct categories for feedback to teachers. Whereas Foundation knowledge might be a dimension that mathematics faculty find familiar, the Contingency dimension may be more foreign. We hypothesize that narrowing the scope of the unfamiliar to

Contingency, in the way that we have delimited it, may make it more accessible to mathematics faculty. If our hypothesis holds, then we will have a framework that can shape instruction and curriculum for mathematics content courses in many domains. Our future work involves testing the promise of this framework for building stronger connections between mathematics and teaching.

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TEACHERS' REPRESENTATIONAL AND CONTEXTUAL JUSTIFICATIONS FOR SELECTING PEDAGOGICAL REPRESENTATIONS

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There have been many efforts to measure pedagogical content knowledge with multiple-choice survey instruments, but little is known about how different types of items contribute. In this study, we examined interviews with 9 Grade 4 teachers to develop a deeper understanding of how teachers select pedagogical representations in the context of a survey assessment. Our analysis revealed two broad themes: representational justification (focused on how teachers interpreted features of the representation) and contextual justification (focused on how teachers considered their students and their own perspectives and experiences). These results indicated that content and pedagogical knowledge were highly intertwined in teachers' work on these tasks. However, the results also identify limitations for using this item type to measure teachers' pedagogical content knowledge in mathematics. Implications are discussed.

Keywords: Assessment, Elementary School Education, Mathematical Knowledge for Teaching, Rational Numbers.

In the time since Shulman (1986, 1987) first described pedagogical content knowledge (PCK) as an important part of the teacher knowledge base, the term has gain wide currency and accumulated a large body of scholarship. In mathematics education, perhaps even more than in other areas, concerted effort over the last two decades has been made to measure PCK and the related domain of mathematical knowledge for teaching (MKT) with survey instruments comprising multiple-choice items (Hill et al., 2005; Saderholm, et al., 2010).

Multiple choice teacher knowledge items are often written to measure specific categories within the domain of PCK or MKT, but the responses to a variety of different types of items all contribute to the same overall score on these instruments. Conceptualizations of teacher knowledge and psychometric results from several independent projects suggest that PCK and MKT are multidimensional constructs even though they have been measured primarily with unidimensional scales (Jacobson, 2017). Qualitative studies of teachers' written responses (Fauskanger, 2015) and teacher interviews (Lai & Jacobson, 2018) have revealed that teachers often draw on more than one category of knowledge when answering multiple-choice survey items. Even so, such items are still widely used because they provide an efficient means to assess teacher knowledge at scale.

MKT and PCK items often pair a specific mathematical topic with a pedagogical decision about that topic. Ball et al., (2008, p. 400) list 16 different "mathematical tasks of teaching" around which assessment items could be written such as, "evaluating mathematical explanations" and "modifying tasks to be easier or harder." Items on existing MKT and PCK instruments include a wide selection of these tasks of teaching, but little is known how well each type of task reflects teachers' knowledge. To investigate this issue, we designed a set of PCK items that share a single task of teaching: selecting pedagogical representations.

Theoretical Framework

In this study, we describe our efforts to use multiple choice survey items to efficiently assess teachers' knowledge of pedagogical representations, one of the two kinds of PCK described by Shulman (1986). Rather than framing our analysis in terms of PCK categories (and several different categorizations have been used in prior research), we ground our work with a foundational idea in one of Shulman's original articulations: PCK is an "amalgam of content and pedagogy", "the blending of content and pedagogy into an understanding of how particular topics, problems, or issues are organized, represented, and adapted to the diverse interests and abilities of learners" (1987, p. 8). This study is part of a larger project to design assessments of teachers' PCK and to link teachers' scores on these assessments to classroom teaching and student learning.

We focus on pedagogical representations because of their central importance in mathematics instruction. Mathematics is inaccessible by direct experience, hence, teaching and learning are necessarily mediated by representations (Bruner, 1966; Duval, 2006). Following Cai and Lester (2005), we make a distinction between solution representations used by problem solvers to make sense of a problem and communicate their solution to others and pedagogical representations which are "the representations teachers and students use in their classroom as carriers of knowledge and thinking tools" (p. 223). These two kinds of representations can overlap, but the critical difference is that teachers are uniquely responsible for curating the pedagogical representations available in their classrooms to ensure they are both mathematically accurate and comprehensible to learners (Cai & Lester, 2005; Leinhardt, 2001).

The dual nature of teachers' responsibility vis a vis pedagogical representation is aligned with the dual constituents of PCK understood as an amalgam of content and pedagogy. To select a mathematically accurate representation, teachers need to attend to the way features of the representation map to features of the problem situation and the underlying mathematical ideas. Content knowledge is implicated in this attention: for example, teachers must know to check that a representation which purports to show a fractional quantity is equally partitioned. Mathematical accuracy is necessary but insufficient. Teachers must apply their pedagogical knowledge to recognize which representations are unfamiliar or confusing for students and thus provide more hindrance than help. Teachers who coordinate their knowledge of content and pedagogy to select accurate, comprehensible pedagogical representations demonstrate PCK.

The research question which guided this study was, *To what extent do teachers' rationales for selecting a pedagogical representation reflect PCK versus other factors unrelated to PCK?*

Method

Item Design

We designed eight items (denoted Q1 – Q8) to assess Grade 4 teachers' proficiency in selecting pedagogical representations for fraction and decimal instruction. Each item consisted of a mathematics problem and two diagrams illustrating different pedagogical representations for the mathematics problem (see Figure 2). The mathematics problems were aligned with state standards for Grade 4 fraction and decimal topics. The pedagogical representations were based on interviews with teachers and teacher educators, a review of fraction and decimal literature in practitioner and research journals, and a review of student thinking and misconceptions with fraction and decimal topics. Teachers responded by indicating whether or not the representation in each diagram was useful for students who were learning to solve the problem (A, B, both, or neither).

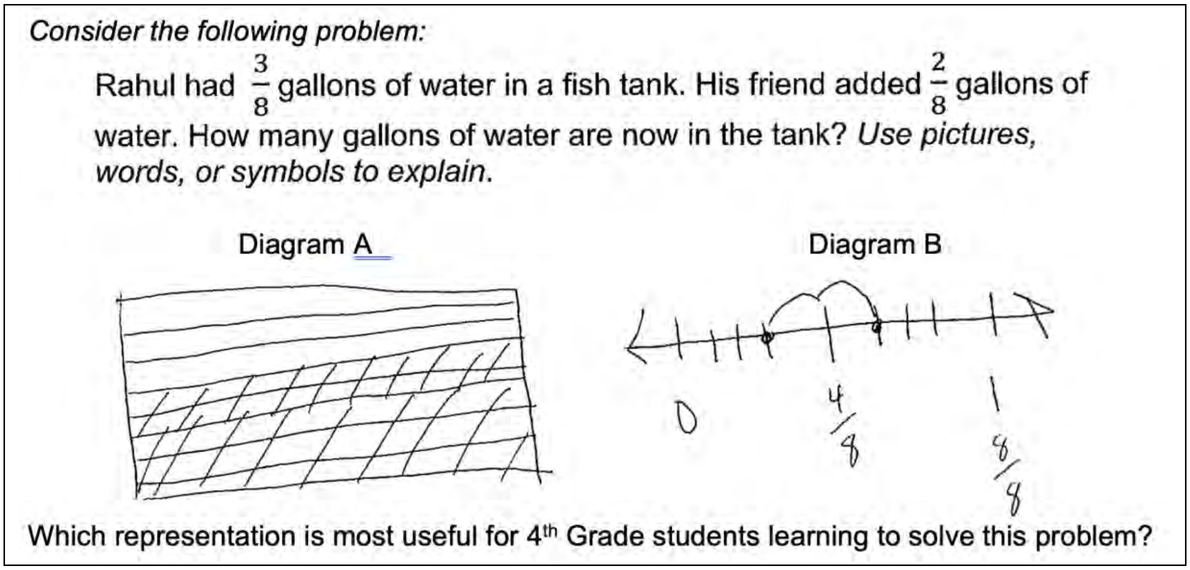

Figure 1. An example item for selecting pedagogical representations. Teachers could select A, B, both, or neither diagram.

The representations were designed with key features for teachers to notice that had either mathematical or pedagogical implications. All diagrams included representations of numbers in the problem, the solution, or both. The diagrams we intended teachers to endorse had representations that were both pedagogically warranted and mathematically accurate. Other diagrams that we intended teachers would not endorse included features that made them inaccurate or unclear.

We included one or more key features to make representations inaccurate. One inaccurate feature was using a representation that had a different sized whole for two fractions in a problem that were supposed to be commensurate. Another inaccurate feature had to do with the problem-representation alignment and focused on teachers' ability to distinguish between semantic equivalence of a problem and representation which can influence children's comprehension and the mathematical equivalence that an adult (but not a child) might use to solve a problem. For example, students would likely struggle to recognize a word problem describing 36 copies of 1/8 in a representation show 36 shared equally into eight groups, although an adult might use 36/6 as a way to compute 36 x 18. The third inaccurate feature was mathematical errors. For example, in one diagram we made a representation with a circle partitioned into non-congruent sections.

We also included key features to make diagrams unclear. (Although inaccurate representations are also unclear, here we describe features that are accurate but unclear.) One feature of unclear diagrams were representations that displayed the result instead of showing the process. For example, one unclear representation for a fraction comparison was an open number line with correctly labeled points but without any benchmarks or regular partitions by which order could be found. Another unclear feature was the alignment between the representation and common solution strategies students might use. For example, students often use a think addition strategy to solve change unknown subtraction problems. Teachers selecting a representation for a subtraction problem who were unaware of this strategy might inappropriately reject a representation that apparently showed addition. The third unclear feature was whether the representation illuminated or obscured a primary learning goal of the problem. Considering students who are learning how to add fractions with different denominators, a representation that already shows common denominators is less clear than a representation that also shows the original fractions.

Data Collection and Sample

Nine Grade 4 teachers (all White females), were selected from an economically and geographically diverse set of schools in a Midwestern State to complete a think-aloud item response interview. Teachers were in 7 different elementary schools across 5 different counties. Student eligibility for free or reduced lunch in these schools ranged from 8% to 85%, and enrollments of White students ranged from 64% to 98% across schools. Rates of students scoring at or above "proficient" levels on state mathematics tests ranged from 40% to 79% across the schools, with state school rankings ranging from the bottom 50% to the top 10%. This sample provided an opportunity to study the scope of teachers' thinking across a wide range of schooling contexts. Pseudonyms were used to report results.

Teachers answered the items and justified their representation selection. All interviews were video-recorded and transcribed. Because we limited interviews to 90 minutes and teachers varied in the time it took to discuss each item, not every teacher answered every item. Two teachers answered all 8 items, four answered 7 items, one answered 6 items, and two answered 5 items. In all, there were 60 item responses across all teachers. We used the selection (non/endorsement) and justification for each diagram (two diagrams per item per teacher) as the unit of analysis for this study. Thus, the size of the analytic sample was 120 diagram responses and the corresponding justifications.

Data Analysis

We first summarized teachers' responses and justifications with direct quotes (e.g., Vicky did not endorse Q1, Diagram 2 (a number line) because, "they [students] don't think about physical placement on the [number] line."). Then we conducted an inductive thematic analysis to analyze patterns in teachers' justifications, by reading the summaries and developing an initial codebook (Boyatzis, 1998; Rice & Ezzy, 1999). We then modified the codebook to remove ambiguities and overlapping codes. Two authors used a new codebook to iterate the code-reconcile-modify cycle until they arrived at a consensus on all codes. Once the codes were finalized, we reviewed each group separately considering to what extent the responses illustrated content knowledge, pedagogical knowledge, or an amalgam of the two.

Findings

Our analysis revealed two broad themes in teachers' response justifications: (1) some teachers based their justifications on the key features of the representation (as we had intended) and (2) some teachers based their justifications on contextual factors that teachers supplied from their own knowledge of students, the curriculum, and from their teaching experience. Among the 120 responses, responses were as likely to be justified based on features of the representation (*representational justification*; 56/120) as they were to be justified by non-diagrammatic concerns (*contextual justification*; 55/120). The themes were not strictly exclusive: 9 responses included both representational and contextual justifications. On some responses (18/120), teachers did not provide justification.

Representational Justification

Three subthemes emerged from the analysis of teachers' representational justifications. Teachers *interpreted key features*, *attended to some key features* of the diagram (while overlooking others), or *misinterpreted key features*.

Interpreted key features. In this subtheme, teachers noticed the key features of the representation and they endorsed (or did not endorse) a representation as pedagogically useful because of the mathematical or the pedagogical affordance of the representation in light of these key features (n = 41).

We considered teachers' interpretation reasonable if they noticed the key features and provided an explicit, coherent justification to support their non/endorsement of a specific diagram. For example, Sarah justified why Diagram 2 in Q1 was not useful by observing that:
Although the number line is a good measurement, you have no benchmark. The number line is very abstract for them [students] to say this is 3/8 without benchmarks such as one and a half. When denominators are different, the number line with benchmarks is still hard.
In this example, Sarah noticed that the number line did not have benchmarks, something we left out intentionally to make the representation unclear. (Note that this teacher also comments on student difficulty in her response, an example of contextual justification which is discussed in the next section.)

Similarly, Amy asserted that Diagram 2 in Q4 was not useful "because some of the kids are going to count the number of coins, not the value of coins". Another teacher, Vicky, said "Well, some of them might see so many pennies and say that [pennies] must be more than that [dimes]." We chose to use coins because values in this kind of representation are based on convention instead of an observable quantity like count, length, or area. This key feature makes the representation unclear for some students because it obscures a primary learning goal of the problem: the relationship between tenths and hundredths. These teachers closely attended to the key features of the representations and used these features to justify their non/endorsement of the representations.

Attended to some key features. In this subtheme, teachers paid attention to some key features of the representations while overlooking others (n = 12). For example, in Q1, Rose noticed that both diagrams (bar model and number line model) showed how large a fraction is compared to one whole: "We can see that 3/¾s getting closer to one whole on the number line..Ind of like a bar graph"; however, this teacher overlooked the absence of benchmarks on the number line. This situation also occurred when teachers endorsed mathematically inaccurate diagrams. For example, a set model representation in Q5 incorrectly added denominators, and both Brooke and Molly apparently overlooked this feature while asserting that this diagram would be useful for students.

Misinterpreted key features. This subtheme captured responses of teachers who noticed the key feature of the diagram but interpreted them incorrectly (n = 3). This code was prevalent for Q2, Q3, and Q5 because one of the options was designed to be mathematically inaccurate. For example, three teachers (Megan, Vicky, and Cathy) asserted the usefulness of a non-equipartitioned circle model (Q3, Diagram 1). Vicky said students "would be able to figure it out pretty easily and tell." Megan reasoned: "With the circle graph, that might give a little trouble…but I think they could see that this is a fourth out and this is a fourth out." Teachers noticed the representational feature but still considered the diagram to be useful for their students despite its inaccuracy.

Overall, responses in this category illustrated the potential for this kind of teacher knowledge task (selecting pedagogical representations) to assess teacher knowledge for using representations. Teachers who interpreted key features of the representations demonstrated pedagogical content knowledge whereas teachers of overlooked or misinterpreted key features did not.

Contextual Justification

The theme of contextual justification includes reasoning that influenced teachers' responses but had less to do with the features of the representations than with the teachers' own context and professional experience. Four subthemes emerged within the theme of contextual justifications: students' familiarity, students' competence, teachers' competence, and teachers' preference.

Students' familiarity. These non/endorsement responses were justified based on teachers' knowledge of students' familiarity with the representation (n = 30). Teachers used phrased like "used to", "have seen", or are "familiar with" while stating that their students' had a high degree or lack of familiarity with a representation. They used phrases like "never used" or "[they] haven'tcused those much" to indicate students' low degree of familiarity with a particular representation. In general, this category was not correlated with identifying inaccurate or unclear representations because key features were used to modify both frequently used (e.g., hundred grid) and less-frequently used representations (e.g., abacus) when we designed the items.

Students' Competence. Some teachers justified their diagram selection by anticipating students' competence (n = 13). Teachers justified their non-endorsement with statements, like, "They [students] don't think about physical placement on the line"," "number lines are very, very difficult for my fourth graders"," "even though Diagram 2 showed the whole, kids might not understand the pictures unless an explanation was given" and "the diagrams had too many things going on, which might be confusing to some students." sometimes, teachers selected a diagram over another by anticipating which one would be easier for their students (e.g., Laura chose Diagram 1 in Q7, stating "it might be easier to use [than Diagram 2]".)

Sometimes, teachers who were focused on students' competence seemed to neglect the pedagogical value of some representations that might be challenging to use, such as number lines. At other times, the anticipation of student struggle was associated with an unclear or inaccurate representation, and we could not determine whether these teachers may have had some tacit understanding of the problematic features of these representations. On the other hand, from our position as teacher educators, we found this stance towards selecting pedagogical representations concerning because it may lead to a classroom experience with only a very limited set of mathematical representations. Endorsements based on students' competence were weakly correlated with identifying inaccurate or unclear representations, and therefore can be taken as an indication of pedagogical content knowledge.

Teachers' competence. Teachers sometimes used their own anticipated competence with the representation to justify their selection (n = 7). In a typical example, Cathy explained why the abacus in Q6 would not be useful by stating "That one confuses me so I wouldn't even know how to explain it to my kids." Teachers also used phrases like "a little harder [for me] to understand" to explain why they thought the representation would not help their students.

Teachers' preference. A small number of teachers justified their selection based on their personal preference (n = 5). Teachers made statements such as "I like number line(s)" (Brooke, Q1) or "I am a fan of rectangle diagram(s)" (Sarah, Q1), usually without offering a rationale based on specific features of the representation. Only one teacher (Rose) justified their preferences based on the features of the diagram. Rose preferred Diagram 1 on Q2 because the zoomed-in portion of the number line representation allowed students to "see not only are tenths smaller than one but then each tenth is also broken down into smaller and smaller pieces." She continued "I haven't used this [number line with zoomed-in portion], but I like this." The teachers who made choices based on either their sense of competence or their preference often overlooked problematic key features, suggesting that both of the last two subthemes illustrate reasoning which was associated with a lack of pedagogical content knowledge.

Discussion

In this study, we examined interviews with nine Grade 4 teachers to understand how teachers select pedagogical representations in the context of a survey assessment. Our unit of analysis—a teacher's endorsement and justification of one diagram for one mathematical problem—provided a fine-grained tool to examine patterns in the variability across teachers and items. We present results that summarize 120 justification responses into two main themes with seven subthemes. These themes emerged from our analysis provide a comprehensive description of how teachers justified their decisions about the usefulness of specific pedagogical representations for particular mathematical problems.

Early in the analysis, we expected that responses involving representational justifications would provide opportunities for us to consider how content knowledge was used to select pedagogical representations, and that responses involving contextual justifications would help us understand how pedagogical knowledge was used. However, content and pedagogical knowledge were more intertwined in our data. As the example of dimes and pennies illustrates (Q4, Diagram 2), the largest subtheme—interpreted key features—involved some teachers who had made a sound, pedagogically-informed justifications for key features of representations which we had designed to be problematic. Similarly, contextual justifications sometimes provided information about teachers' content knowledge; the teacher who liked the zoomed-in number line representation did so because she saw how features of the representation mapped to important ideas about the content. These findings suggest that selecting pedagogical representations is an item type is well suited to engage teachers in reasoning that draws on pedagogical content knowledge.

This study offers researchers useful insights for developing and validating assessments of teachers' 'CK for selecting representations. For the most part, the interviews provided information about teachers' PCK that was aligned with what we would have inferred from their representation selection. However, the responses also illustrate the variety of reasoning that teachers used to justify their selection of pedagogical representations, including some justifications that were based on inaccurate mathematical thinking and inadequate pedagogical knowledge. An even larger number of responses were based on the teachers' personal experience and context, factors that are not related directly to the pedagogical content knowledge we aimed to assess.

The prevalence of justifications based on teachers' context and professional experience when selecting pedagogical representations is an important discovery because it highlights factors that are consequential to teachers' responses but are beyond the scope of survey item design. Every teacher sees the same text and inscriptions of an item, but these data reveal how the different experiences they have had with students, curriculum and their professional training provide a lens through which they interpret the task and their own response.

In practice, the best instructional decision when selecting a pedagogical representation will always depends on the context. A teacher with high knowledge may be able to use a new representation effectively, whereas a teacher with low knowledge may not be able to use a new representation without training. In such cases, it would be better for students to experience a familiar representation with mathematical accuracy than have the content misrepresented because of an unfamiliar representation. Even knowledgeable teachers' must balance the trade-off between the time and effort of investing in a representation that is unfamiliar to students and whatever pedagogical gains they anticipate once students have adopted the new representation. If the focus is on a new mathematical idea, then a familiar representation will help support student understanding, whereas an unfamiliar representation may impede students understanding by putting students in the position of needing to learn both the new mathematical idea and a new kind of literacy required to write and read with the new representation.

Can multiple choice survey items be designed that support valid inferences about teachers' PCK for selecting representations even though teachers' personal experience and context are equally salient as the key features of the representations in question? The responses that were justified by teacher's preference do not present a problem for this goal because they were generally aligned with low PCK. However, the remaining subthemes of contextual justification are all problematic to some degree. Responses based on the student competence subtheme are the least problematic, because these responses sometimes reflect PCK. On the other hand, these responses seem to come from a stance toward teaching and learning we find concerning because it might limit students' education experiences. Responses based on student familiarity—by far the most frequent contextual justification—are the most problematic, because these responses would reduce the accuracy of each item by adding noise to the signal. Noisy instruments are inefficient and must have more items to reach an acceptable level of reliability. If the noise could be mitigated through careful item design, it might be possible to build a trustworthy instrument to measure PCK for selecting representations. Addressing the important question of whether such a difficult task is possible is beyond the scope of this paper and will certainly require additional empirical work which we have begun to undertake.

The contribution of this study is to illuminate teachers' reasoning across a set of multiple choice teacher knowledge items of the same type: selecting pedagogical representations. Without a set of items with the same design focused on the same pedagogical decision and varying only in content (i.e., the mathematics problem, the representations, the key features), we could not have drawn generalizations about how teachers' reason about this type of item. In prior work, a PCK or MKT instrument might have had one or two items of this type among 20 to 30 items on an instrument, and interviews with teachers (e.g., Fauskanger, 2015) across a diverse set of items would not have supported the kind of discoveries we report about the range of teacher reasoning on a single type of question. We expect that there are important affordances and limitations of many other types of MKT and PCK items commonly in use that are not known because they have not been adequately investigated.

The data we have presented is drawn from a small qualitative study. The present study is focused on better understanding how teachers select pedagogical representations, and whether inferences about teacher knowledge can be drawn from multiple choice surveys about representation selection. The larger project engages in similar work for three additional kinds of pedagogical decisions. Further work is ongoing to examine whether these themes generalize to an independent sample of 40 preservice and inservice teachers.

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EXPLORING THE PEDAGOGICAL CONTENT KNOWLEDGE FOR TEACHING FRACTIONS BETWEEN TWO IRANIAN EXPERIENCED TEACHERS

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Teachers' knowledge of teaching fractions and their ability to teach such a challenging topic plays an essential role in children's learning fractions. While some scholars have investigated teachers' pedagogical content knowledge for teaching (PCK) fractions, there is less attention on how PCK for fractions is operationalized by teachers of different nationalities. This study reports on initial efforts to understand how PCK for fractions operates between two expert teachers from Iran. Findings from interviewing these teachers suggests that the pedagogical reasoning of both teachers in confronting PCK-Fractions scenarios is similar. The implications of the commonalities in teachers' PCK-Fractions reasoning are discussed in the paper.

Keywords: Teacher knowledge, Rational Numbers & Proportional Reasoning

Teaching involves a set of highly professionalized skills. To acquire such skills, a teacher needs, among other things, to have sufficient content knowledge as well as the ability to apply such knowledge to students. Shulman (1986) named this intersection of teachers' subject knowledge and pedagogical knowledge as Pedagogical Content Knowledge (PCK). The PCK is the skill that teachers use to transfer the subject matter knowledge to facilitate student learning. Built on Shulman's notion of PCK, researchers have developed frameworks for it in different areas such as science (Van Driel et al., 1997; Halim & Meerah, 2002), literacy (Love, 2009), and physical education (Ayvazo & Ward, 2011). In their learning mathematics for teaching (LMT) project, Ball, Hill and others applied Shulman's construct into mathematics and created mathematical knowledge of teaching (MKT) framework (Ball et al., 2008; Hill et al., 2008).

Several studies have investigated the nature of teachers' understanding of students' reasoning. For instance, Peng and Lou (2009) examined the nature of mathematical misconceptions from both students' and teachers' perspectives. They found that teachers usually were able to identify students' errors, but they had difficulty interpreting the rationale underlying those errors. Sorto et al. (2014) investigated teachers' difficulty explaining students' reasonings and found that teachers mostly used a procedural explanation for interpreting students thinking and analyzing learners' errors. Rather, teachers are aware of students' errors in solving problems, but they usually fail to reason about those errors conceptually. Prediger (2010) called this ability to understand students' thinking a diagnostic competence. He explained that: "This notion is used for conceptualizing a teacher's competence to analyze and understand student thinking and learning processes without immediately grading them." (Prediger, 2010, p. 76)

Teachers' conceptual understanding of students' mathematical reasonings influences their classroom instructions and thus students' learning (Cai et al., 2016). Such conceptual understanding intertwines within other aspects of norms, culture, experience, knowledge of subject, and the whole instructional situation that a teacher lives in it (Herbst and Chazan, 2011). In order to have a more in-depth insight into how teachers understand students' reasoning, it is important to see how those elements integrate with teachers' interpretations (e.g., if any at all, they use the norms or their culture to interpret students' reasoning).

Given the importance of instructional situations, specifically the role of instructional and cultural norms in how teachers interpret students' reasoning and the importance of being familiar with the teacher's culture, this study aims to understand how experienced Iranian teachers understand students' reasonings of fractions. Therefore, in this exploratory study, I am addressing the following research question: How do experienced Iranian teachers understand students' misconceptions of fractions?

Method

For this exploratory study, two expert teachers participated from Iran. Iran has a centralized educational system. All K-12 educational textbooks are produced and published by the Ministry of Education. Therefore, all teachers across the country use the same textbooks as their main source of teaching the content. However, teachers are able decide on any text as supplementary material. The two participants, Dorsa and Roya, (pseudonyms) had experience teaching at upper-grade elementary schools for more than ten years. Both had experience teaching at upper-grade elementary schools for more than ten years.

The cognitive interview was selected as a technique of interviewing for this study. Both teachers were asked to read each scenario, then interpret each scenario, and lastly explain their rationale for their response to each item. For the interview, I used tasks of teaching, in the form of test items, from a PCK measure for fractions that were tested for content validity and reliability measures (Zolfaghari et al., in review). Each interview includes twelve scenarios: seven multiple-choice items and five multiple-response items. The items for the interview were adopted from a PCK measure for fractions. Initially, twenty questions were developed for the PCK measure for a pilot study (Zolfaghari et al., 2019). Considering the Iranian primary school knowledge and based on my knowledge of teaching mathematics in Iran, I finally selected 12 questions of the PCK measure and translated in Persian.

Analysis and Findings

Constant comparative analysis (CCM) was used to analyze the data (Lincoln & Guba, 1985). CCM is one of the qualitative analysis techniques that rely upon the systematic coding of the interview data alongside analyzing those codes to create a theoretical framework based on emergent themes (Ridolfo & Schoua-Glusberg, 2011). Guidelines provided by Ridolfo and Schoua-Glusberg (2011) were applied to analyze the cognitive interview data for this study.

The analysis began with line-by-line coding of the transcripts for all four interviews, considering the question in mind: how did the respondents answer to each PCK scenario interact with their backgrounds and culture? The findings' codes were organized to identify the analytic themes as well as the thematic categories that make up each theme.

Open coding

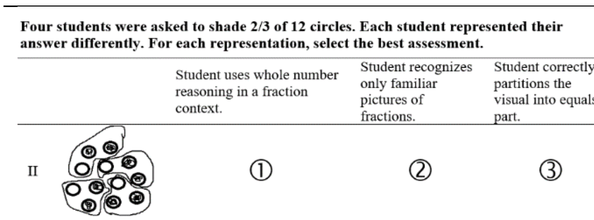
During the cognitive interview, I investigated the initial interpretation of teachers' responses to each PCK scenario. Respondents were probed on how they answer each 15 designed PCK scenarios and their explanation and rationale for their answers regarding their background (e.g., what item they think is the answer and why they select that item). While reviewing the transcripts, several key terms and notes were found. Interestingly, both teachers frequently use similar notes to answer each of the PCK scenarios. Those notes merge into several categories, including their reactions to atypical' strategies, their answers aligned with the intended answer, using self-experiences, citing books or curriculum, explicitly using correct/incorrect in their responses, explained their preferred strategies, showing interests in students' 'olutions. After gathering the above information from the transcripts, I used axial coding to refine those notes and clustered them to resolve any discrepancies among those categories and noticed the similarities (Ridolfo & Schoua-Glusberg, 2011).

Axial coding

Based on reviewing the initial codes, four categories emerged including: 1. Different attitudes toward atypical strategies, 2. Considering pieces of student's solution, 3. Methods of teaching fractions operations, 4. Referring to prior experience and textbooks. Following, I provide examples of how the teachers' answers were placed in those three categories.

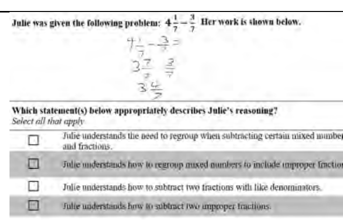
The first category emerged because in some scenarios, the students' strategies was so atypical and made it difficult for both experienced teachers to interpret them. Such difficulty in interpreting students' strategies leaded the teachers to make opposed decision on either to accept them as or reject the entire strategy. For instance, when Roya was confronted on a scenario in which a student divided 12 circles into 4 groups to get 2/3 of 12, she admitted that the students' answer was right, however, she didn't accept it as "truly" correct because the strategy was not as clear and typical. In contrast, the other teacher admired a student's strategy and called it "smart work" and accepted it as a truly correct answer (see table 1 for teachers' quote to a scenario).

Table 1: Category #1[Different attitude toward atypical strategies]

	Dorsa' Quote	Roya's Quote
 **Four students were asked to shade 2/3 of 12 circles. Each student represented their answer differently. For each representation, select the best assessment.** Student uses whole number reasoning in a fraction context. / Student recognizes only familiar pictures of fractions. / Student correctly partitions the visual into equals part. II ① ② ③	"Wait, what did he do? …. oh, he divided into three, then colored two of each group. It is smart work. Nice.I am going to say he did his work accurately"	"he got the correct answer which would fit to what question asked, but his strategy is not. We always (in classroom) tell our students that for such problems they need to first divide the circles into three sections, then all the colored circles should be in a group and uncolored in another part. His representation of partitioning has problem."

The second category included when teachers considered a piece of students' 'nswers and interpret the entire strategy based on it. For instance, in the example shown in Table 2: the student used the strategy of regrouping to subtract a mixed number from a fraction, however, she did it inaccurately and get $3\frac{1}{7}$ instead of $3\frac{8}{7}$ and get the incorrect answer as a result. In examining her work, Roya interpreted the student's regrouping as a negligible mistake and said, "b"cause the hardest part of these scenario is knowing [that] there is a need to do regrouping and I think the student know it [how to regroup] … the computation is the student's problem… she just forgets to add a 1 part to the whole". This consideration of a partially correct solution was not always in favor of students and sometimes teachers ignored the correct piece of students' 'olution and rejected the whole of it (See table 2).

Table 2: Category #2 [Considering piece of student's solution]

	Dorsa' Quote	Roya's Quote
 **Julie was given the following problem:** $4\frac{1}{7}-\frac{3}{7}$ **Her work is shown below.** **Which statement(s) below appropriately describes Julie's reasoning?** *Select all that apply* ☐ Julie understands the need to regroup when subtracting certain mixed numbe and fractions. ☐ Julie understands how to regroup mixed numbers to include improper fractio ☐ Julie understands how to subtract two fractions with like denominators. ☐ Julie understands how to subtract two improper fractions.	"I can't tell if she understands to subtract two fractions, because she couldn't get the correct answer"	"and I think the student knows it [how to regroup] … she just forgets to add one part to the whole"

The third category included the teacher's approach of doing fractions operations. In PCK scenarios related to fractions operations (e.g., addition, subtraction, etc.), teachers used the expected norms of doing fractions operation.

Table 3: Category #3 [Method of teaching fractions operations]

	Dorsa' Quote	Roya's Quote
Ms. Weasley asked Wesley to tell her which two of these fractions is the smallest? $\frac{6}{7}$ or $\frac{7}{8}$? Wesley answered: They are equal because there is one piece missing $\frac{6}{7}$ and one piece missing from $\frac{7}{8}$. **Based on Wesley's answer, what is the best assessment of his reasoning?** ① Wesley is not considering the relative size of the two wholes. ② Wesley is not considering the relative size of the fractional pieces. ③ Wesley is not considering the relative size of either the parts or the wholes. ④ None of the above.	"nor nominators, neither denominators are not similar to each other…" then Dorsa started to mentally calculate the common denominator."	"even, as the boy said, one pieces missing… [as we know the rule textbook] if two nominators are same, then the fraction with smaller denominator is bigger"

The last category contains moments in which both teachers refer students' reasonings to their experiences or the textbooks. For instance, in a scenario that a student mistakenly wrote fourth in a denominator to change four to the fraction ($4 - \frac{6}{8} \ldots \frac{4}{4} - \frac{6}{8}$). Dorsa sighed and stated: "this is a common mistake that you would see every year.

Selective coding

Based on evidence from the above two steps, the unifying link between all patterns and categories was identified. The findings show the commonalities between teacher's approaches toward students' reasonings. In fact, both teachers used the same approach while they confront a same scenario. Interestingly, using the same approach of examining scenarios was independent from teachers' final decisions toward students' understanding of the concept. For instance, in a scenario of subtracting an improper fraction (see table 2), both experienced teachers used the same approach of *considering part of student's reasoning* to analyze student's reasoning. However, their decision in whether student understand the regrouping or not was different. Such commonalities observed across all themes within PCK scenarios. Indeed, both teachers used the same norm of reasonings while they interact with a similar situation. The norm that is defined as part of the instructional situation (Herbst and Chazan, 2011).

Conclusion

This study provides information about how two expert Iranian teachers interpret PCK fractions scenarios. The findings suggest the similarities across both teachers' approaches in terms of interpreting students' reasonings. Prior studies suggested the process of understanding students' misconceptions, interpreting and engaging to those misconceptions as the steps of teachers' work with students' reasoning (Peng & Luo, 2009; Prediger, 2010). This study provides evidence of some commonalities with teachers' approach to interpreting students' reasonings. Teachers' PCK has different aspects that are bounded by the schools' environment, culture, and other elements; however, some elements of the pedagogy aspect of teaching are common across teachers. Such similarities can play an important impact on PCK constructions. Future research is needed to compare more experienced teachers with more diverse nationalities.

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TEACHERS' KNOWLEDGE RESOURCES FOR SOLVING PROPORTIONS

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In this report, we consider whether there are differences in the fine-grained understandings teachers use to reason about proportional situations. To examine these differences, we divided a group of 32 teachers into one of three groups based on their performance on the Proportional Reasoning LMT assessment. Then, we used a knowledge in pieces lens to analyze the teachers' performance on a series of proportional reasoning tasks. Based on that analysis, we were able to use Epistemic Network Analysis to determine which knowledge resources were most commonly connected for the groups of teachers. Implications for teacher development will be discussed.

Keywords: Teacher Knowledge; Rational Numbers & Proportional Reasoning, Cognition

Purpose of the Study

Proportional reasoning is a critical mathematical concept as evidenced by its status as a strand unto itself in the Common Core (National Governors Association & Council of Chief State School Officers, 2010). Despite this importance, students struggle with proportions (e.g., De Bock et al., 2002; Modestou & Gagatsis 2007) as do their teachers (e.g., Izsák & Jacobson 2017; Orrill & Brown, 2012). In this inquiry, we wanted to move beyond identifying whether teachers are able to solve proportions to understand what knowledge resources might be important for them to use in such solutions. To this end, we used the Learning Mathematics for Teaching Survey for Proportional Reasoning (LMT, 2007) as a proxy measure for teachers' proportional reasoning knowledge. We then divided the participants into three groups based on their LMT scores and examined each group's use of knowledge resources on a different collection of proportional reasoning items. We were interested in seeing whether there were differences in the knowledge resources used by teachers to solve proportion items based on their ability as measured by the LMT. Thus, our exploratory research question was: Are there different patterns of use in the proportional reasoning knowledge resources teachers used based on their LMT scores?

Theoretical Framework

This work relies on Knowledge in Pieces (KiP; diSessa, 2016; diSessa, Sherin, & Levin, 2006). KiP is a conceptual change theory that posits that knowledge exists as fine-grained understandings that work together *in situ* to allow people to make sense of their world. From this perspective, knowledge is comprised of pieces that are connected in a variety of different ways that allow them to be called upon for any given situation. Learning can occur through the development of new fine-grained understandings, refinements of existing understandings, or building connections between understandings, which allows that knowledge to be more readily accessed. KiP has a commonality with research on expertise in that both support the notion that expertise not only implies a person has more knowledge resources, but also that the ways in which those resources are connected differs from novices. Thus, attending to the connections between knowledge resources may be a useful way to understand teachers' knowledge.

In order to see the connections between knowledge resources, we rely on Epistemic Network Analysis (ENA; Shaffer et al., 2009). ENA is an analytical approach that focuses on the connections between knowledge resources. ENA presents a visual mapping of the knowledge resources that are used in connection with other resources along with information about the strength of that connection, as denoted by the thickness of the line, and the relative frequency of the use of each individual knowledge resource, as denoted by the size of the node.

Methods

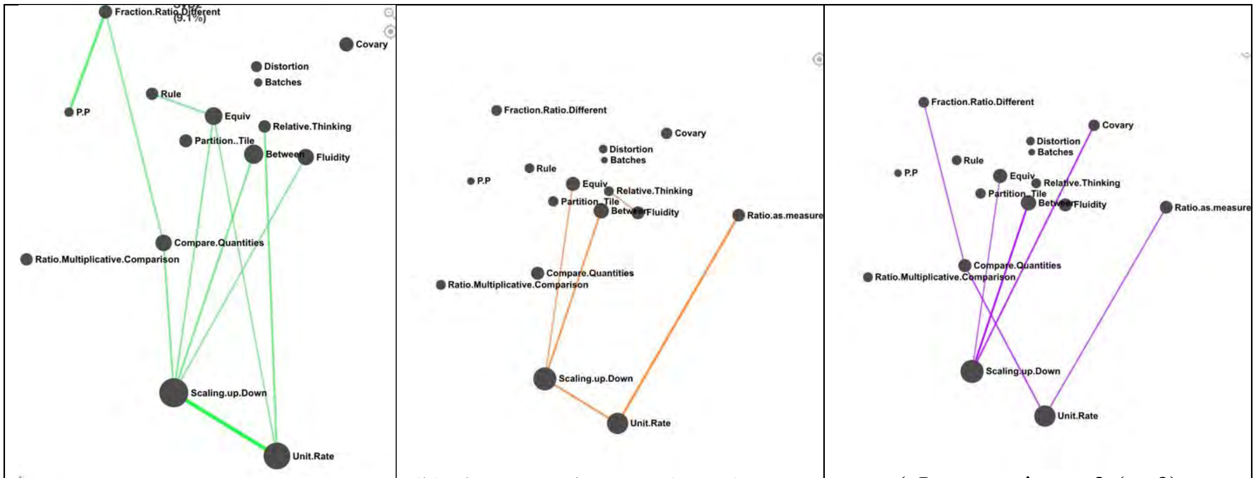
We consider a convenience sample of 32 middle grades mathematics teachers from four states. The participants ranged from one to 26 years of experience. Eight participants identified as male.

Participants completed two task-based interviews. One interview was completed using a LiveScribe pen, which recorded markings on the paper as well as participants' spoken words. The other interview was a 90-minute video-recorded clinical interview. Tasks asked participants to make sense of student work or other teacher's mathematical observations as well as to solve novel proportional situations. Both interviews were transcribed verbatim and analyzed by at least two members of the research team. Each utterance was coded for the presence of knowledge resources in participants' reasoning based on a set of 26 codes for teaching proportional reasoning (Weiland et. al, 2020). For this analysis, we focused on subsets of the codes to help us better understand how the teachers were reasoning.

Using ENA (https://www.epistemicnetwork.org/), equiload graphs were generated showing the ways in which participants used two or more knowledge resources together. The equiload graphs show each knowledge resource as a node. The lines connecting the nodes indicate that two resources were used in the same utterance at least two times. Thicker lines indicate more co-occurrences of resources. Using ENA, the mean of a participant's equiload is used to place that participant in the space generated by ENA. Participants are located within the field of codes based on the relative frequency of co-occurrences between codes. All of the equiload graphs in this paper represent the average equiload for the group of participants, rather than one single participant.

Findings

To understand how teachers reasoned about the proportional reasoning items, we first considered the teachers' use of the full set of knowledge resources we had identified for reasoning about proportional situations (see Weiland et. al, 2020 for more information). To determine whether there were differences in patterns among our participants, we grouped them into three groups: those who scored below 0 on the LMT (n=3), those who scored above 0 but less than 2 on the LMT (n=20), and those who scored above 2 (n=9). We then used the ENA tool to plot the group "average" interactions for each of these groups using only 16 codes that were related to proportional reasoning knowledge resources. These included structural codes, such as attending to the between measure space relationship, as well as codes more focused on solving proportions such as applying a rule. Based on that analysis, we were able to generate the equiloads shown in Figure 1.



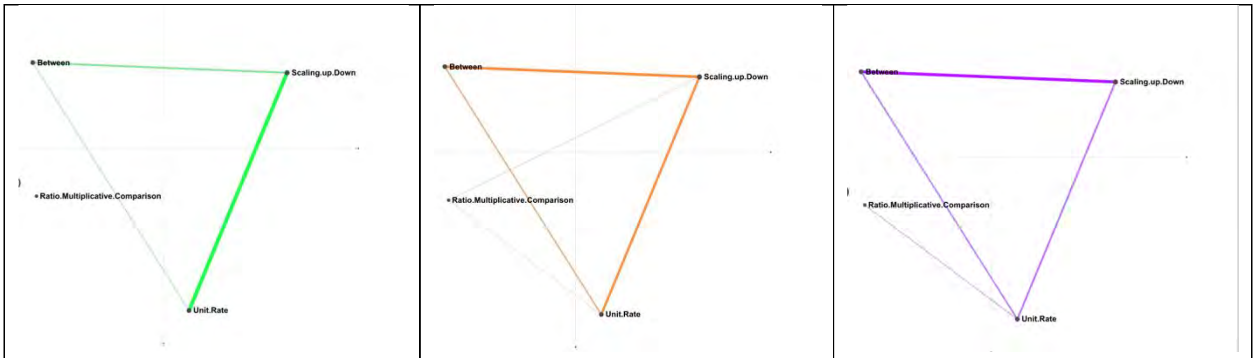
Those scoring < 0 (n=3) (b)Those scoring 0-2 (n=20) (cIose scoring > 2 (n=9)

Figure 1. Equiloads of 16 proportional reasoning knowledge resources for each group of participants.

These ENA maps show considerable differences in the knowledge resources that were used in combination by the different groups. The two lower-scoring groups, for example, often used scaling up and down (that is, applying a scalar to find an equivalent ratio) and using unit rate together to solve tasks. However, in the highest-scoring group, that connection was no longer a factor. Instead, the teachers in the highest group tended to group scaling ideas with equivalence, between measure space reasoning, and covariation. This suggests a strong attention to structural characteristics rather than different solution paths. A second difference we noted between these groups was that the lowest-scoring participants used more combinations of knowledge resources (as seen in connecting lines) as compared to both stronger-scoring groups. Our hypothesis is lower-scoring teachers have many of the same knowledge resources as higher-scoring teachers; however, they may have not developed as many connections between and among those knowledge resources. Therefore, they may not see the commonalities between tasks to cue the use of particular resources. That is, stronger teachers have more organized, and therefore more accessible, knowledge resources.

Because we noticed differences between the three groups of teachers and particularly noted the interaction between unit rate reasoning and scaling, we wondered how different the groups might look if we focused only on a small number of knowledge resources focused on key structures for proportions. To this end, we narrowed the codes for which we were analyzing to 4: unit rate, using multiplicative comparisons, scaling up and down, and reasoning about the between measure space relationship. We selected these four because they get at the heart of proportional relationships: the constant of proportionality, the invariant nature of proportions, and the multiplicative nature of proportions.

In this analysis (see Figure 2), there is an interesting progression in which the lowest-scoring participants again show a strong reliance on using unit rate and scaling up and down together. In the middle group, we saw our first signs of attention to the multiplicative relationship. That group also started to place more emphasis on the relationship of between measure space reasoning and scaling up and down, which suggests attending to both the constant of proportionality (the between measure space constant) and the maintaining of equivalence through using a scale factor. Finally, in the highest-scoring group, the co-occurrence of multiplicative reasoning and unit rate becomes stronger and the connection of between measure space and scaling up and down becomes more dominant (meaning these participants used both resources to solve tasks more frequently).



(a) Those scoring < 0 (n=3)	(b)Those scoring 0-2 (n=20)	(cIose scoring > 2 (n=9)

Figure 2. Equiloads of 4 key structural knowledge resources for each group of teachers

Implications

This study is important because it starts to unpack teacher content knowledge in ways that highlight differences among the ways in which teachers reason about proportions. This begins to uncover differences between teachers who have various levels of success with the LMT that could be addressed in professional development. For example, the findings in Figure 2 strongly suggest that more focus on reasoning about proportions in multiplicative ways would be a fruitful approach for all teachers in professional development. Similarly, this analysis demonstrates that all teachers have a strong set of knowledge resources as shown in Figure 1, but that they draw on them in different combinations. By understanding these combinations, we can create tasks that help teachers further develop their connections between knowledge resources in productive ways that highlight the structure of proportional relationships.

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THE PARAMETER IN TEXTBOOKS A DOCUMENTAL ANALYSIS

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Parameters are not explicitly mentioned in textbooks from elementary school to higher education to solve Systems of Linear Equations (SEL) with infinite solutions, these almost always present methods to solve square systems with unique and numerical solutions, although later the parameters appear and are addressed in the undergraduate level but are introduced in an ambiguous manner which makes it difficult to understand its nature, its potentiality and proper use.

Keywords: Algebra, Algebraic Thinking, Parameters, textbooks.

Background

Parameters were introduced by Viète (1540 - 1603) to distinguish different types of variables, while in Mathematics Education they have eventually been interpreted as: *"…variables which are replaced by constants where at least one other variable remains*" (Šed . At the same time, they have been considered as a special type of variable, as Drijvers (2001) points out where *"The parameter can be considered a metavariable: the a in y=ax+b, it can play the same roles as an 'ordinary' variable, such as a placeholder, an unknown or changing quantity, but it acts on a higher level than a variable" "p.2).* The parameter is considered here as a second order variable, because they can make vary what is already varying, as Ursini & Trigueros (2004) point out, where they assume the role of unknown or related variable, depending on the situation.

The use of parameters for their later professional use can initially be addressed with SEL that have infinite solutions, so we are going to investigate the treatment of parameters in the school taking as a source of information the institutionally recommended textbooks, in order to know the basis on which the parameter could be proposed as a control variable, which includes both the treatment as a variable and as a constant.

Theoretical Framework

From the textbook format point of view, O'Keeffe (2013) suggests that the analysis of a book should consider: *content, structure, expectations, and language*. Concerning the structure of the textbook it is established that it increases or decreases the comprehension of the textbook, which suggests that the succession and connections between the elements of the textbook should be carefully analyzed. Textbooks are the result of a certain way of thinking about the contents that are in fact considered important to teach in the classroom and how they should be teaching.

In this case, we will address the mathematical content in textbooks associated with the parameters and analyze aspects related to the dominant procedure and their presence in relation to infinite solutions from high school to undergraduate level.

Methodology

In this study we carried out a documentary research on the textbook contents in order to detect the way how SEL solutions are presented and how parameters are introduced in the case of infinite solutions. For this purpose, we have reviewed some of the syllabi of different levels of mathematics education in Mexico with respect to SEL, their solutions and the treatment of parameters in these materials, for which we have chosen some representative secondary, high school, and undergraduate institutions to analyze the textbooks recommended in such materials.

The syllabi and sample textbooks we reviewed were as follows:

- **Basic level**: Current syllabus recommended by the Secretaría de Educación Pública SEP (2017) in it is recommended: (1) Trigueros et al (2019), (2) Bosch & Meda (2018), (3) Riva Palacio & Santana;(2019), (4) Martínez & Contreras, (2019) and finally the text of (5) Block, García & Balbuena (2019).
- **High School:** The syllabus of the Dirección General del Bachillerato (DGB,2018) and the UNAM corresponding to the Escuela Nacional CCH. Recommends: (6) Miller, Heeren & Hornsby (2013).
- **Undergraduate level:** The Linear Algebra texts recommended by the Faculty of Sciences of the UNAM and by the Division of Basic Sciences and Engineering of the UAM recommend: (7) Espinosa et. al. (2004) and for science we review the books by (8) Hoffman (1971), (9) Curtis (1984) and (10) Rincón (2006).

In order to carry out the analysis of the contents, we chose those we considered representative, so we will now discuss the text productions at the secondary (1-5), high school (6) and undergraduate levels (7-10), analyzing 10 texts in total.

Results and data analysis

The review performed on the content of the textbooks suggests that there are certain constants among them, which refer to 1) The shape and size of the systems, 2) A dominant type of algebraic procedure, 3) The type of solutions associated to the SEL and finally, we will see any mention of the parameters to deal with infinite solutions in the books analyzed, aspects that we will summarize below.

- The middle school books (1-5) present 2x2 square systems; use of solution methods by equality, substitution; dominant method: elimination; unique and numerical solution; no mention of parameters.
- The high school book (6) uses square and rectangular SEL, although his solution of this case is not addressed; solution methods by equality, substitution, and elimination; dominant method: elimination; numerical and infinite solution; no mention of parameters.
- The undergraduate books (7-10) address square and rectangular SEL; the same methods and Gaussian elimination is added; the case in which there is no solution is incorporated; dominant method: Gaussian elimination; regarding the apparition of the parameter, two cases out of four mention a change of variable, one presents it as a variable and finally in the last one its nature is not clarified.

The analyzed books' contents show that the predilection of the authors of secondary and high school textbooks is directed to the treatment of SEL squares of order 2x2 and 3x3, while in the undergraduate level, rectangular SEL are incorporated, which will give rise to infinite solutions that are eventually solved.

In all the cases analyzed, the texts rely on the development of multiple procedures to determine that the solution is unique and numerical, while the infinite solutions are almost not addressed and it seems that these solutions cannot be found because this problem is discarded and the books do not promote interest in them, nor do they lay the foundations to calculate them through the figure of the parameters.

Concerning the introduction of the parameters in the SEL, we observe that they appear functionally to the tertiary level when the infinite solutions of the SEL are attended to.

We found that in some texts it is used as a special type of sign (Hoffman, 1971; Curtis, 1984), while in others it is assigned the role of a constant (Espinosa et. al., 2004), in both cases an explanation of these facts is omitted, which generates confusion.

In the following we present excerpts from the textbook by Espinosa et. al. (2004) that support the above:

La última matriz está en su forma escalonada reducida, ya no se puede reducir más, de donde obtenemos:

$$x + \frac{5}{8}z = \frac{3}{2}$$
$$y - \frac{9}{16}z = -\frac{1}{4}$$

despejando x, y

$$x = \frac{3}{2} - \frac{5}{8}z$$
$$y = -\frac{1}{4} + \frac{9}{16}z$$

luego x, y dependen de z, si $z = t$, $t \in \mathbb{R}$, tenemos

$$x = \frac{3}{2} - \frac{5}{8}t$$
$$y = -\frac{1}{4} + \frac{9}{16}t \; ; \quad t \in \mathbb{R}.$$
$$z = t$$

Es decir, el sistema de ecuaciones tiene una infinidad de soluciones ya que para cada valor de t habrá un valor para x, y, z.

En esta parte se dan ejemplos de sistemas de ecuaciones lineales donde se determinan valores de constantes para que el sistema tenga o no solución.

15) Obtener el valor de λ para que el sistema de ecuaciones:

$$2x + y = 6$$
$$x + \lambda y = 4$$

tenga:

a) Solución única.

b) Infinidad de soluciones.

c) Carencia de solución.

Figure 1: Parameter use in SEL (Espinosa et. al. 2004, P.17 & 25)

On the left side of Figure 1 we observe an SEL with more variables than equations, which justifies the use of the parameters for its solution, but in the text there is no explicit mention of it, your presence is justified through a sentence that says: *". "I can't be reduced anymore."* and then they search for a "*variable*" that is defined as usual and the parameter *t* is introduced, accompanied by another sentence that says: *". "Ien x and y depend to z , if z = t"* "hat strictly is not justified and then it do not seems to be any rules to know how and when it should be used. The phrases that accompany the activity seem more like a kind of story, of which we do not know when and why we must use it, despite the mention of dependence, which involves specific conditions for this to be carried out.

When we avoid an explanation, we may generate a misconception particularly when one variable is simply substituted for another, in this case $z = t$, leading to possible doubts about the procedure, e.g., why z should be equal to t and not x or y, this procedure seems to be part of an apparent substitution ritual that avoids presenting the parameters and their properties explicitly.

On the other hand, on the right side of Figure 1 we observe a particular situation of SEL treatment, in which the parameter is initially introduced as a constant named λ, but never is clarified. This type of practice promotes the idea, as in the previous case, that the type of letter indicates the difference between the variables as well as of its role and the proper operation.

Another use to the parameter is the fragment of Hoffman's'(1971) work in Fig. 2, where the parameter appears as a substitution of a variable (x_4) for a literal (c), as is mentioned in the phrase *"... "t is evident that if we assign any rational value c to x_4 we obtain a solution ..."* the first obstacle that we observe is to affirm that the substitution *"i" is obvious"*, as if this were the only possible way to do it, so again we have a substitution like a ritual in which implicit indications are not given, such as the fact of choose x_4 as the variable to solve due it appear at the right hand, therefore it should be used to make the substitution.

EXAMPLE 5. Suppose F is the field of rational numbers, and

$$A = \begin{bmatrix} 2 & -1 & 3 & 2 \\ 1 & 4 & 0 & -1 \\ 2 & 6 & -1 & 5 \end{bmatrix}.$$

$$\begin{bmatrix} 0 & 0 & 1 & -\frac{11}{3} \\ 1 & 0 & 0 & \frac{17}{3} \\ 0 & 1 & 0 & -\frac{5}{3} \end{bmatrix}$$

The row-equivalence of A with the final matrix in the above sequence tells us in particular that the solutions of

$$\begin{aligned} 2x_1 - x_2 + 3x_3 + 2x_4 &= 0 \\ x_1 + 4x_2 \qquad\quad - x_4 &= 0 \\ 2x_1 + 6x_2 - x_3 + 5x_4 &= 0 \end{aligned}$$

and

$$\begin{aligned} x_3 - \tfrac{11}{3}x_4 &= 0 \\ x_1 \qquad + \tfrac{17}{3}x_4 &= 0 \\ x_2 \qquad - \tfrac{5}{3}x_4 &= 0 \end{aligned}$$

are exactly the same. In the second system it is apparent that if we assign any rational value c to x_4 we obtain a solution $(-\frac{17}{3}c, \frac{5}{3}, \frac{11}{3}c, c)$, and also that every solution is of this form.

Figure 2: The parameter presence as a C-value (Hoffman, 1971, p.9)

In this case, the parametric solution is made explicit in the following lines: *". "Id also that every solution is of this form* $\left(\frac{-17c}{3}, \frac{5}{3}, \frac{11c}{3}, c\right)$" and despite the fact that every solution can be written with this expression, the statement is incomplete since the form of the solution set depends on the variable that we are working with and it is also ambiguous because the solution set is unique, but not the expression to determine it, besides the fact that the solution must be $\left(\frac{-17c}{3}, \frac{5c}{3}, \frac{11c}{3}, c\right)$.

Finally, let us analyze an excerpt from the book by Miller, Heeren & Hornsby (2013), in Figure 3.

The infinite solutions here are presented meant of the idea of linear dependence, this is made explicit in the sentence: "*T“e set of ordered pairs of a system of dependent equations is written as a set of ordered pairs that expressing x in terms of y...*" ”nd in the procedure where any of the variables involved in the problem is declared. An action that can confuse students when they do not know yet this type of resource that consists of clearing one of the variables *(x)* and then presenting the solution set by means of the expression $\frac{y-2}{4}$ in the first entry of the ordered pair associated to the solution set and the second by *y,* in this procedure the *x* disappears and then is found again in the next line without warning. Again, we have a change in the nomenclature without apparent justification and with the features of a substitution ritual as part of the procedure.

Final Discussion

In this inspection of the secondary school textbooks, we observed that the contents deal exclusively with SEL squares of 2 and 3 dimensions with emphasis on unique and numerical solutions from which we can infer that the language and the teaching expectation is directed to the exercise of the procedures to acquire algebraic language skills.

Regarding the treatment of parameters in university textbooks, we observe that they are used as constant or variable as convenient, which makes that their function be seen as informal by the student or that depends on the situation, this treatment prevents their nature.

The square dimension of the systems studied at this level is extended to rectangular ones, which requires the use of parameters to find the solution set, they are treated indirectly through phrases like: *"w“ can't’clear anymore"* ”r *"i“ is evident that if we assign any rational value c to* x_4 *we will obtain a solution ... "* “hat seem more like a warning to use them, rather than an explicit and normed procedure, which takes the form of a ritual in which variables are substituted when the conditions are same like commented it and in the same way they are presented in the texts. Therefore, we find that the language associated to the parameters is ambiguous and these

Este enunciado verdadero, es decir, $0 = 0$, indica que la solución de una ecuación también es solución de la otra, de modo que el conjunto solución es un conjunto infinito de pares ordenados. Las dos ecuaciones son dependientes.

El conjunto solución de un sistema de ecuaciones dependientes se escribe como un conjunto de pares ordenados expresando x en términos de y, como se muestra a continuación. Se elige cualquier ecuación y se despeja x. Nosotros seleccionamos arbitrariamente la ecuación (3).

$$-4x + y = 2 \qquad (3)$$

$$x = \frac{2-y}{-4}$$

$$x = \frac{y-2}{4} \qquad \text{Se multiplica por } \tfrac{-1}{-1}.$$

El conjunto solución se expresa como

$$\left\{\left(\frac{y-2}{4}, y\right)\right\}.$$

Los pares ordenados del conjunto solución se pueden obtener seleccionando valores de y y calculando los valores correspondientes de x. Por ejemplo,

$$\text{si } y = -2, \text{ entonces } x = \frac{-2-2}{4} = -1$$

y la solución es el par ordenado $(-1, -2)$. ■■■

Figure 3: SEL with infinite solutions (Miller, Heeren & Hornsby, 2013, p.380).

are only highlighted as auxiliary bridges that we allow to write the solutions in very particular cases, they leave aside the role of this type of variables play in the solution, which causes that they are not considered as part of the procedure, but as a story for the general expression, but not as a mathematical resource.

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THEORETICAL FRAMING FOR PRESERVICE TEACHERS' VIRTUAL CUISENAIRE ROD USE WHEN EXPLORING FRACTION TASKS

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Given the most recent need for virtual teaching, many mathematics teacher educators have taught pre-service teachers with virtual manipulatives, which often stood in for physical manipulatives. This brief considers the theoretical framing for pre-service teachers' actions with physical Cuisenaire rods before considering benefits and limitations virtual Cuisenaire rod features afford when making such a transition to virtual Cuisenaire rods. In particular, we seek to consider the theoretical question: what is gained and lost when trying to replicate preservice teachers' 'se of physical Cuisenaire rods with virtual Cuisenaire rods?

Keywords: Preservice Teacher Education, Mathematical Knowledge for Teaching, Rational Numbers, Mathematical Representations

In response to the COVID-19 pandemic, many mathematics teacher educators are trying to promote productive struggle while transitioning from teaching with physical manipulatives to teaching with virtual manipulatives. If we are trying to prepare teachers to teach with physical manipulatives, to what extent can we replicate that use with use of virtual manipulatives? Motivated by Herbst and Chazan's (2017) call for subject-specific theories of mathematics teaching, we focus on the specific mathematical goal of conceptualizing fractions as measures—that is, of understanding fractions as the result of coordinating mental actions of partitioning, disembedding, unitizing, iterating, and splitting (Wilkins & Norton, 2018)—and on one specific manipulative, Cuisenaire rods. Cuisenaire rods are constructed of wood or plastic, of size 1cm by n cm, for n=1 to 10. Each rod length is represented by color (i.e., orange rods = 10 cm in length, red rods = 2 cm in length). They have been used for decades to teach arithmetic (Aurich, 1963; Egan, 1990) and fractions (Robinson, 1978; Wallace, 1974). We examine Affordance theory (Gibson, 1979) and Units Coordination theory (Steffe & Olive, 2010) to theorize how features of virtual Cuisenaire rods support teaching fractions as measures.

Theoretical Framing

To frame this discussion, we integrate an ecological affordance theory for virtual manipulatives (Gibson, 1979) with a constructivist theory of fractions learning (Steffe & Olive, 2010). Moyer-Packenham and Westenskow (2013) constructed an interrelated framework with five different types of affordances with virtual manipulatives that explain learning effects. *Focused constraint* (1) is the degree to which features focus and constrain students' attending to particular mathematical aspects of the manipulatives. *Creative variation* (2) allows students opportunities to experiment with mathematical ideas in creative and novel ways. *Simultaneous linking* (3) promotes students to link actions, dynamic/static pictures, and/or symbols simultaneously. *Efficient precision* (4) describes the degree of fidelity of particular dynamic objects and mathematical properties of manipulatives. *Motivation* (5) involves affective qualities students express in manipulatives use. We wondered how these affordances may benefit or limit preservice teachers' (PSTs) construction of fractions as measures.

Steffe and Olive (2010) describe a theory for children's construction of fractions as measures via a reorganization of mental operations for constructing whole number concepts. Units are constructed through children's external activity before becoming internalized (imagined actions) and then interiorized (able to anticipate relationships between levels of units). *Partitioning* involves projecting a "composite unit into a continuous whole to create equally sized parts within the whole" (Wilkins & Norton, 2018, p. 2). By *disembedding*, students are able to take "parts out of a whole as separate units while maintaining their relationship with the whole" (p. 2). Students evidencing *iterating* repeat a "unit of length or area to produce a connected whole" (p. 2). By coordinating three levels of units (i.e., unit fraction, composite fraction, referent whole) students are able to conceptualize a fraction larger than the whole.

Fraction Development with Physical Manipulatives

We first consider features of physical Cuisenaire rods as a PST solves the fractions task, "if the purple rod is ⅔, what rod represents ½?". The student first locates the purple rod and places it on the desk. Next, the student places her fingers on the rod in a "cutting" motion (see Figure 1a). The student explains that she is partitioning the rod to determine the unit fraction (⅓). After determining the red rod is equal to ⅓, the student then iterates the red rod three times to construct the referent whole (see Figure 1b). Finally, the student determines a green rod iterated twice represents two halves. One of these two halves is then placed alongside the purple rod to illustrate the relationship between ⅔ and ½ (see Figure 1c).

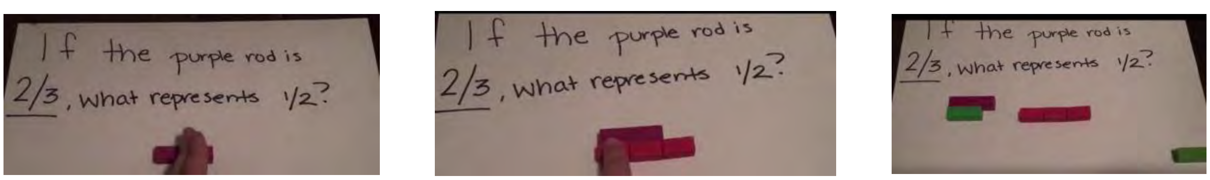


Figure 1a (left), b (middle), c (right): A Preservice Teacher's Activity with Cuisenaire Rods.

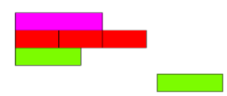
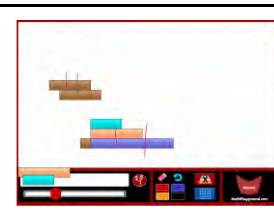
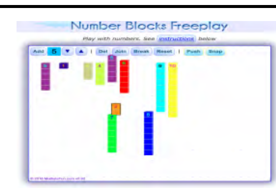
When PSTs engage with physical Cuisenaire rods, partitioning is constrained because individual blocks cannot be broken into smaller parts. We theorize that this constraint would encourage iterating, which is critical for understanding a fraction as a measure. The proportional relationship between blocks affords students a "guess and check" strategy; PSTs who do not immediately choose the correct rod are able to choose rods a bit smaller or larger in length due to the accuracy of their solution, allowing multiple access points.

Fraction Development with Virtual Manipulatives

When considering PSTs' potential engagement with virtual Cuisenaire rods when solving fraction tasks, we sample five virtual Cuisenaire rods. In this examination, we compare features of each virtual tool and determine possible affordances these features may promote in PSTs (see Table 1). When reviewing the tools, it seems clear that there were features that were similar between virtual Cuisenaire rods (e.g., grid features, ability to overlap or not) and features that varied (e.g., pen tool, organization of rods by length). To examine these features relative to potential affordances, we consider each feature individually.

Grid and proportional relationships. When virtual Cuisenaire rods have a grid and rods are organized in proportional order, it removes the necessity to partition and iterate because the unit and referent whole length are measured with the grid or illustrated with the relationships. These features afford PSTs simultaneously linking between measured lengths and the rods. Conversely, we argue these features might also afford PSTs' efficient precision when solving fraction tasks. Thus, PSTs may not actively partition and iterate with virtual tools with these features but provide accurate solutions to fraction tasks.

Table 1: Comparison of Virtual Cuisenaire Rods

Cuisenaire Rod and Website	Cuisenaire Rod Features	Cuisenaire Rod Image
Maths Bots https://mathsbot.com/manipulatives/rods	Grid (on/off), rotate blocks, Rods overlap, Add text box, Add rods with 10+ units, Blocks easily draggable	
Cuisenaire Environment https://nrich.maths.org/4348	Grid (on/off), Labels (on/off), Rotation, Rods overlap, Arrows to move blocks, Rods can be show as organized from shorter to longer, representing proportional length relationships (on/off), Blocks easily draggable	
Modeling Fractions with Cuisenaire Rods https://opb.pbslearningmedia.org/resource/rttt12.math.cuisenaire/modeling-fractions-with-cuisenaire-rods/#.YEZVpy2cbBI	Grid always on, Labels always off, Only horizontal, Rods cannot overlap, Rods organized - representing proportional length relationships, Blocks easily draggable	
Math Bars https://www.mathplayground.com/mathbars.html	Grid (on/off), Labels (on/off), Rotation, Rods can overlap, Slider forms sizes - representing proportional length relationships, Pen tool, Blocks easily draggable	
Number Blocks https://www.mathsisfun.com/numbers/number-block-freeplay.html	Grid (on/off), Labels (on), Only vertical, Rods cannot overlap, Rods are organized from shorter to long, Blocks easily draggable	

Labels. When reviewing labels, we considered how PSTs might be afforded with simultaneous linking between whole numbers (cm length) and proportional relationships between rods. For example, a fraction measurement task might state that the “red rod equals ½”; this might present the PST with conflicting values between fractions and whole numbers if the red rod is labeled “2” in the virtual manipulative. Again, these simultaneous links between whole numbers and rods remove the necessity to partition and iterate. These links may inhibit PSTs from considering rods with this feature as representing fractions and flexible units.

Rotation of rods. Some virtual Cuisenaire rods had the option to rotate rods from horizontal to vertical and vice versa and some do not allow rotation. Often measurement models for fractions use horizontal orientations. By allowing rotation of the rods, we believe PSTs are afforded creative variation. This variation may allow PSTs to change the orientation of the models to one that most often aligns with their understanding of measurement. It is unclear how this may provoke or prevent iteration and partitioning.

Overlap of rods. Many virtual rod tools allow for these rods to overlap. For example, a rod 3 cm in length might overlap another rod 2 cm in length, resulting in a rod 4 cm in length. Again, we believe this feature affords PSTs creative variation. In this case, creative variation has the possibility of representing inaccurate lengths than the physical rods may represent. Additionally, this feature may discourage iteration because the PSTs’ actions do not represent full copies of units. This feature may also afford PSTs’ focused constraint because they need to continually attend to the length each rod represents to prevent rods from overlapping. Thus, this feature may have benefits for supporting knowledge for teaching a fraction as a measure.

Pen tool. Only one of the virtual manipulatives reviewed had a built-in pen tool. We believe this feature affords PSTs’ simultaneous linking, creative variation, and efficient precision. These three affordances would promote sequential partitioning and iterating activities because they would have to construct segmenting marks, simultaneously link these marks to their actions, and use these links to attend to the mathematical precision of a potential solution.

Blocks easily draggable. All virtual manipulatives had features, which allowed individuals to drag blocks across the screen with traditional mouse-driven devices and multi-touch devices. We determined this feature afforded PSTs’ creative variation and simultaneous linking because they were given creative opportunities to explore mathematical ideas while also linking their actions to the placement of the rods. This affordance has the potential to mimic iteration in similar ways that physical manipulatives afford PSTs.

Conclusions

In conclusion, our theoretical framework and examination of virtual manipulative features suggest that the Math Bars virtual Cuisenaire rod tool has features which allow PSTs’ the ability to turn on and off grids, turn on and off labels, rotate rods, overlap rods, rely on proportional ordering of rods, and draw with a pen tool. Many of these features are helpful because mathematics teacher educators can specify which features afford PSTs to partition and iterate. Conversely, the proportional relationships feature also limits these actions. The Number Blocks virtual Cuisenaire rod tool does not have this proportional relationship feature but uses labels which also provide simultaneous links between whole number symbols and rods. The Cuisenaire Environment and Maths Bot tools do not provide a pen tool, but features such as a grid, labels, and proportional relationships can be turned off. Moreover, this tool features overlapping rods if users do not attend to the precision of the rod’s length. Thus, these tools may benefit PSTs’ active engagement when constructing fractions as measures. Given the limited space this paper provides, we wonder which affordances are often beneficial and/or limiting when PSTs evidence and construct fractions. Additionally, larger questions center on how frameworks may change when blending theoretical frames.

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MATHEMATICAL KNOWLEDGE FOR TEACHING FORMATIVE ASSESSMENT: RECOMMENDATIONS FOR MATHEMATICS TEACHER EDUCATORS FROM A META-AGGREGATION

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*Qualitative research explores the complexity of teaching and learning, but there are barriers to using qualitative findings to inform practice/policy. We see a critical need to integrate finding*s *across qualitative studies to develop synthesized, actionable recommendations for mathematics teacher educators. Using a meta-aggregation of 11 qualitative studies, we investigated how teacher educators' mathematical knowledge for teaching can inform their support of formative assessment practices in secondary mathematics classrooms. Synthesized findings led to nine lines of action that we provide as recommendations for mathematics teacher educators.*

Keywords: Assessment, High School Education, Research Methods

Mathematics teacher educators play a key role in connecting theory into practice when it comes to using formative assessments (FAs) to inform instructions in secondary mathematics classrooms and supporting teachers' development of effective FA practices. In this study, we used meta-aggregation, a qualitative research synthesis method, to understand how mathematics teacher educators' (MTEs) pedagogical content knowledge (PCK) informs their work with mathematics teachers on FA practices in secondary classrooms. From our results, we generated \nine actionable recommendations for MTEs who work closely with teachers to connect research to classroom FA practices. Our research questions are:

- What is needed in terms of MTEs' PCK to support teachers' collection and use of FA data to move student thinking forward in secondary mathematics?
- What recommendations can we make for secondary MTEs to connect teachers' 'CK to their development of FA practices?

Theoretical Framework

As a part of classroom assessment, FA is a crucial component of instructional decision-making and developing students' mathematical thinking (Black & Wiliam, 2009). FA is a complex set of "informed actions" (Andrade & Cizek, 2010; Black & Wiliam) concerned with identifying and understanding what and how students are thinking (Black & Wiliam; Looney, 2005; McManus, 2008). Collecting FA data often permeates the various interactions that teachers have with learners as they discuss, listen to, and observe mathematical thinking in the classroom.

In order to identify and assess students' mathematical understandings, it is essential for secondary mathematics teachers to have extensive knowledge of mathematics content, how it connects to student thinking and learning across the broader mathematics curriculum, and how it informs pedagogical practice (Gotwals et al., 2015). In other words, effective FA practices are connected to teachers' PCK. MTEs can support teachers development of PCK as it relates to their FA practices; but knowing how to facilitate this development also requires MTEs to develop and reflect on their own teaching knowledge (Borko et al., 2013; Carney et al., 2019). In this paper, we report research-based recommendations for MTEs who are supporting teachers' FA practices to move students' thinking forward.

Method

Qualitative Synthesis as a Methodology in Mathematics Education

Qualitative research studies play a critical role in education as they push the field "beyond process–product research toward considering the complexities and nuances significant for understanding factors that influence outcomes" (Thunder & Berry, 2016, p. 334). The wide range of qualitative methods and the nature of the findings, however, make it difficult to integrate findings across studies. Recently, meta-aggregation has gained attention as an effective methodology for bridging research findings to practice (Aromataris & Munn, 2020). It is a specific type of qualitative synthesis intended to treat findings from individual studies as *data* for analysis to derive a new theory or interpretation from the original findings. The aim of meta-aggregation is specifically to inform lines of action that are directly and immediately applicable to decision-making practices (Lockwood et al., 2015). This method is well suited to our objective of linking practice and research and generating recommendations for how MTEs can best prepare teachers to use FAs to move students' thinking forward.

Data Collection

In a meta-aggregation, the research question drives the search for qualitative studies to include in the synthesis. Before searching for existing studies, we set criteria for consideration in the search. Our data collection steps are described as follows.

Locating and sampling studies. The target population of studies for our meta-aggregation were qualitative studies that explored FA practices of secondary mathematics teachers. To be included, we set criteria that a study should: (a) be empirical, qualitative research; (b) target United States secondary mathematics students, teachers, and classrooms; (c) investigate in-service teacher's FA use; and (d) meet the criteria for reporting quality through a critical appraisal. We searched using multiple databases (e.g., Google Scholar, JSTOR, PsychINFO). From abstracts, we identified 47 potential studies for inclusion, but excluded 36 after screening for not meeting inclusion criteria.

Appraising reporting quality of identified studies. We used a critical appraisal form with 23 appraising questions to evaluate the sufficiency of reported information of each of the remaining potential study and to enhance the credibility of our results (Lockwood et al., 2015). All 11 identified studies met the criteria for reporting quality and were included for the meta-aggregation (these are indicated with an * in references), comprising three journal articles and eight dissertations. These *primary studies* provided the data for our analysis which includes verbatim extracts from the studies and authors' interpretation of results (Lockwood et al., 2015).

Qualitative Synthesis

After extraction, there are three steps for analysis. For clarity, we use *claims* below to refer to the data we pulled from the primary studies, and we use *findings* to refer to the results of our own data analysis of these claims.

Step 1: Rating extracted claims. We considered authors' claims to be unequivocal (no room for debate), credible (open to challenge) or unsupported (when no evidence backed authors' statements). Our four-person research team worked to find coding consistency and discussed any challenges encountered. Through this process, we identified 656 claims as credible or unequivocal, and 79 as unsupported across 11 studies. Coding was conducted using NVivo.

Step 2: Categorizing and synthesizing codes. We used descriptive coding (Saldaña, 2013), assigning a word or short phrase from each claim to describe the key idea, and then sub-coding (Miles & Huberman, 1994) to reduce the number of descriptive codes and organize the codes for further analysis. We next analyzed portions of data with an extended thematic statement (Saldaña) to identify conceptual commonality (Hannes & Lockwood, 2011; Major & Savin-Baden, 2010). Finally, we used provisional coding (Saldaña) to look within each of 23 themes to see if we could identify connections among our data and PCK framework. We wrote synthesis statements for each theme individually, and reviewed and finalized as a whole research team.

Step 3: Generating lines of actionable recommendations. The final step of the analysis was to translate the synthesis statements to actionable recommendations. We first independently developed a recommendation for each synthesis statement. Then, the developed recommendations were compared until we reached full consensus among team members.

Results

The results below detail our meta-aggregation findings and actionable recommendations for MTEs to (a) support secondary mathematics teachers' development of specific FA practices that they can implement as part of their daily instruction and (b) develop, enact, and reflect on *their own* teaching knowledge to effectively facilitate PD around FA.

Synthesis of MTE's Knowledge of Teachers

Mathematics teachers at all levels of experience benefit from ongoing, differentiated professional development. It is important for MTEs to understand secondary teachers' prior teaching experiences (as a teacher and a learner) with FA as well as their current FA strategies. Unpacking these things could be established through teacher interviews, small group discussions, or classroom observations. Understanding teachers' dispositions and levels of self-efficacy in their mathematical content knowledge and PCK is also critical, as a lack of confidence in themselves as mathematics teachers and a fear of not understanding students' mathematical thinking will hinder effective use of FA for learning. Developing an environment that positions teachers as autonomous partners in learning is shown to improve teachers' sense of self-efficacy with FA. Resulting lines of action are shown in Figure 1.

Unpack Prior and Current Experiences. Recognize that teachers have prior experiences engaging their students in FA. Use teachers' experiences to build common vocabulary, norms, and practices.	**Unpack Affective Factors.** Work with teachers to identify levels of confidence in themselves as mathematics teachers and their dispositions, beliefs and values related to FA practices in the secondary classroom.	**Support Individual Practices, Preferences and Needs.** Differentiate PD and instructional support to meet mathematics teachers' 'iverse needs related to the implementation of effective FA strategies.

Figure 1. Recommendations for MTEs: Focus on knowledge of teacher-learners.

Synthesis of MTE's Knowledge of Content and Curriculum

Targeted instructional support/professional development can lead to positive changes in teachers' understanding of short- and long-cycle assessment use, regardless of their years of experience. Teachers use FA for a variety of purposes including: (a) assessing student's proficiency of mathematics content; (b) evaluating whether students "got it" or not and how instruction should be adjusted; (c) understanding student thinking and reasoning; and (d) assessing students' engagement and attitude. Teachers used different types of FAs in terms of its nature, occurrence, format, and strategy. Thus, when teachers are learning about FA, it is important for them to develop their own definition of FA, to understand the purposes that FA can serve in a mathematics classroom, and to identify the factors that may influence FA practices in their classrooms. When teachers know the broad goals of FA for specific mathematical topics, they are better prepared to alter instructional plans in response to FA data to move students' thinking forward. Figure 2 shows three lines of action for MTEs to consider in relation to curriculum when facilitating experiences that support secondary mathematics teachers' FA practices.

Understand the Purpose for FA	**Understand Conditions for FA.**	**Use Learning Objectives and Trajectories**
Ask teachers to unpack multiple purposes of FA as they engage in FA practices. Build a collective understanding that all can relate to, internalize, and incorporate into practice.	Understand the complex thinking that accompanies FA, and support teachers' understanding of the factors that influence their choice of FA tools and strategies.	Work with teachers to look across grade levels, follow the curricular trajectory of mathematical topics, and explore trajectories in students' thinking on related learning targets.

Figure 2. Recommendations for MTEs: Focus on knowledge of PD content and curricula.

Synthesis of MTE's Knowledge of Teaching

Teachers benefit from observation and written feedback on their FA practices from instructional leaders or peers who have the knowledge and ability to provide differentiated support for teachers' diverse needs regarding FA practices. With the implementation of FA, teachers learn to attend to diverse student thinking, different learning approaches, and their understanding of specific content area, difficulties, and challenges students face in learning mathematics. Therefore, teacher's self-reflection, self-evaluation, and self-regulation of their own learning and teaching effectiveness are critical to identify their strengths/weaknesses around supporting students and building confidence in their teaching. Thus, ongoing internal and external supports for teachers' learning and use of FA practices is beneficial.

Develop Teachers' 'elf-Efficacy with FA	**Recognize Teachers' Roles as Learners**	**Provide Ongoing Instructional Feedback and Support.**
Develop authentic collaborative mathematics learning spaces where the MTE is a facilitator, not authority, in the development of FA practices to build teachers' 'elf-efficacy with FA.	Examine how decisions guide and are guided by FA. Collaboratively analyze how FA informs which type of adjustments could be most productive for a given situation. This can support teachers' applications of past learning to a new group of students in a similar setting.	Create and sustain a system for ongoing instructional support with coaches and peer teachers. Engage teachers in observing and analyzing their own and others' 'A practices and providing focused reflection or feedback on target goals.

Figure 3. Recommendations for MTEs: Focus on knowledge of teaching

Discussion and Conclusions

Effective use of FA in secondary mathematics classrooms involves a complex set of practices. When considering teachers' varying experiences and beliefs related to FA, supporting teachers to engage their students in FA is equally complex. Our meta-aggregation highlights the importance for MTEs to reflect on multiple components of their own PCK when working with secondary mathematics teachers to offer customized, ongoing and differentiated support for all levels of secondary mathematics teachers regardless of their experiences with FA practices. Our findings and nine lines of action recommendations motivate a call for using needs-driven PD

(Marra et al., 2011) for supporting FA practices. Teachers who receive targeted instructional support demonstrate improvement in their knowledge and use of FA core practices. We recommend that help secondary teachers practice FA as a normal, integrated component of instruction instead of as a stand-alone construct. Our synthesis also illustrates the potential of qualitative synthesis methodology for generating empirical-based practical knowledge directly applicable for future PDs and classrooms.

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ARTICULATING THE UNARTICULATED: PROSPECTIVE SECONDARY MATHEMATICS TEACHERS' NARRATIVES OF PROOF WITHOUT WORDS

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This paper reports a preliminary analysis of prospective secondary mathematics teachers' diagrammatic reasoning and understanding of visual proofs. The data comprises responses to a task in which participants were asked to articulate how a given claim was proven through diagrams and to compose a dialogue between two students and a teacher about the proofs. The findings show that the participants' approaches to articulate "the unspoken" of the diagrams greatly differed; their pedagogical preferences to teach the same mathematical content were also distinct. The unspoken and unarticulated, from prospective teachers' perspective, were different from what was anticipated.

Keywords: Visual proofs, Diagrammatic reasoning, Prospective secondary mathematics teacher, Teacher knowledge

If linguistic representation is sequential and linear, then diagrammatic representation is planar (Larkin & Simon, 1987). In both geometric and algebraic proofs, figures and diagrams are often created and used not only in presenting a problem, but also in illuminating a solution (e.g., Alsina & Nelsen, 2010; Brown, 1999; Nelsen, 1993, 2000). In proof construction, in particular, visual means are much more than an aid to understanding; they can be resources for discovery and justification, and even for proving (Arzarello, Micheletti, Olivero, & Robutti, 1998; Giaquinto, 2007). Researchers in mathematics education have noticed content and structural distances between informal argument and acceptable proof (Garuti et al., 1998, Pedemonte, 2001, 2007). While acknowledging the formal and informal approaches to proof as different ways of thinking, recent studies suggest an optimal approach that simultaneously uses logic and visualization (Nardi, 2014; Zazkis, Weber, & Mejia-Ramos, 2016). This study uses the suggested approach to investigate prospective secondary mathematics teachers' understanding of visual proofs. Particularly, the paper focuses on prospective secondary mathematics teachers' narratives of visual proofs that the sum of an integer and its reciprocal is at least two.

Theoretical Framing

Visual Proofs

The concept of visual proof, as Davis (1993) stated, is an ancient one. Azzouni (2013) emphasized the use of geometric content presupposed in visual proofs as it allows an experience in 'seeing' the proofs in question and seeing that the result is true. From a philosophical perspective, visual representations can be seen as adjuncts to proofs, as an integral part of proof, or as proofs (Hanna & Sidoli, 2007). A *proof without words* (PWWs) is such an example of the latter case. These proofs use pictures or other visual means to communicate a mathematical idea, statement, equation, or theorem (Casselman, 2000; Gierdien, 2007). More importantly, a *proof without words* offers insights into how one might begin to go about proving it true (Nelsen, 2000). With an emphasis on the provision of visual clues to the viewer, many *proof without words* in Brown (1999) and Nelson's (1993) work exhibit an interesting conservative element in mathematics and grant opportunities to promote and develop visual mathematical thought

(Azzouni, 2013; Gierdien, 2007; Stupel, Sigler, & Jahangiri, 2019). Recent studies on *proof without words* suggest that the relationships embedded in a good diagram represent real mathematics awaiting recognition and verbalization (e.g., Katz, Stupel, & Segal, 2016; Stupel, Sigler, & Jahangiri, 2019).

Diagrammatical Reasoning

On the one hand, what makes visual representation and diagrammatic reasoning effective is the directness of its interpretation – the capacity a user has to read off key features of the target structure from the appearance of the diagram (Stenning & Lemon, 2001). On the other hand, diagrammatic reasoning is also considered deceptive, particularly when it causes one to treat an apparent relationship as being valid in general. Due to its particularity, a diagram has been considered a heuristic device (Radford, 2008), a useful instrument for the discovery, formulation, and intuitive comprehension of a proof. Even though diagrammatic reasoning seems insufficient to fulfill a justificative role in proof, Kidron and Dreyfus (2014) suggest that mathematical justification takes into account the learner's point of departure with its intuitive thinking, visual intuitions, and verbal descriptions, rather than starting from formal mathematics.

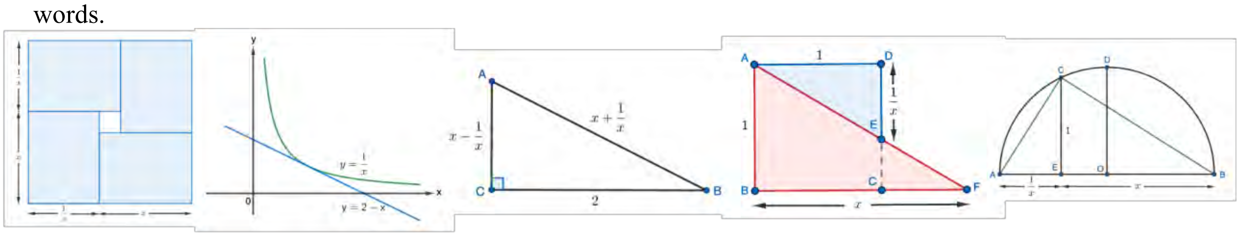
In the interest of using graphical arguments to support proof construction, Nardi (2014) found that some teachers used graph-based argumentation as part of the learning trajectory towards proof construction. Zazkis, Weber, and Mejia-Ramos (2016) further suggested that writing a proof based on a graphical argument could engage prospective teachers in elaborating and syntactifying that leads to a verbal-symbolic proof.

Method

Participants were 22 prospective secondary mathematics teachers enrolled in a mathematical content course as a part of their undergraduate teacher education programe. The goal of the course was to examine secondary mathematics from an advanced standpoint, to broaden the understanding of key topics by drawing connections among various topics and representations. The means towards this goal is intensive problem-solving experience, followed by reflection.

The Task

1. **Articulating.** The five diagrams below all illustrate "the sum of a positive number and its reciprocal is at least 2. Please choose TWO diagrams out of five and articulate how the two diagrams prove the claim in your own words.



2. **Script-writing**. Based on your articulation of the two diagrams you have chosen, write an imaginary dialogue about them between you (the teacher) and two students. Please start the dialogue with the prompt below.

Frank: I tried to add a positive number and its reciprocal. Look what I found… I sense that this must lead to something groundbreaking.

$$3+\frac{1}{3}=3\frac{1}{3},\ 2+\frac{1}{2}=2\frac{1}{2},\ 1+\frac{1}{1}=2,\ \frac{2}{3}+\frac{3}{2}=2\frac{1}{6},\ \frac{4}{5}+\frac{5}{4}=2\frac{1}{20},\ \dots\ \frac{100}{101}+\frac{101}{100}=2\frac{1}{10100}$$

Francis: How do you know?

Teacher: …

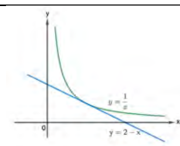
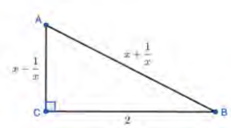
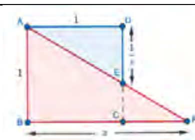
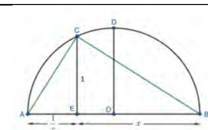
Figure 1. The Task

The participants were given five visual proofs of $x + \frac{1}{x} \geq 2$ for $x > 0$, selected from Nelsen (1993). The task consists of two parts: 1) select two out of five diagrams and articulate how they prove the claim; 2) write an imaginary dialogue between two students and a teacher based on the prompt (see Figure 1). The first part of the task was designed for prospective teachers to draw on their mathematical content knowledge; the second part, their pedagogical considerations regarding visual proofs. The imaginary dialogue is also referred to as a "scripting task" to explore and strengthen teacher knowledge while considering instructional situations (e.g., Zazkis & Kontorovich, 2016; Zazkis & Herbst, 2018). In analyzing participants' articulations and scripts, three themes emerged regarding "the unarticulated": 1) the extra bit, 2) the minimal value, and 3) the algebraic approach. This brief report focuses on the first two themes for discussion.

Preliminary Findings

The majority of the participants selected two out of five visual proofs to articulate, while a handful of participants attempted to explain all five proofs. The frequency of distribution of selected proofs can be found in Table 1. More than two-thirds of the participants selected proof 1 and 2; about half of the participants selected proof 3; one-third, proof 4; and 5 participants chose proof 5. In this paper, our preliminary analysis focuses on the first two proofs: the "square" proof and the "function" proof.

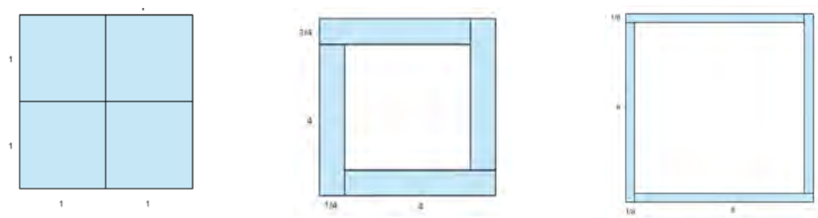
Table 1: Frequency of distribution of selected proofs

	Proof 1	Proof 2	Proof 3	Proof 4	Proof 5
Diagram					
Total	15	14	10	7	5

The Unarticulated – The Extra Bit in the "Square" Proof

When asked to articulate the "square" proof, the participants attended to the dynamic that is evident yet implicit in the diagrams. For example, Dave's approach to explaining the "square" proof focused on a discussion between the two student-characters around "the extra bit":

Frank: Okay, how do we know that the "1/x + x" by "1/x + x" square is bigger than the 2-by-2 square?
Francis: Well, because we can compare it to the original.
Frank: Hmm… what happens if we keep taking bigger and bigger x values?
Francis: Nice sketch… the inner square gets bigger too. But the area of the blue-shaped rectangles is still 4.
Frank: Oh, so that means $(x + 1/x)^2 \geq 4$…
Francis: ..Iich means $x+1/x \geq 2$!



The discussion took an empirical approach to test out the area changes of the inner square when the value of x changes. To reach the result that "$1/x + x$" by "$1/x + x$" is greater than 4, three examples were drawn to illustrate when x increases from 1 to 4 to 8, the side-length of the inner square consequently increases from 0 to 15/4 to 63/8, which leads to the changes of the area of the outer square from 4 to 4+15/4 to 4+63/8 to support the result $(x + 1/x)^2 \geq 4$. In a similar light, Judy's script focused on losing the extra bit to explain $(x + 1/x)^2 \geq 4$.

Teacher: The area of this square must be greater than 4 because we also have this little square in the middle to consider.
Francis: So then, the sum of $x + 1/x$ must be greater than 2.
Teacher: Exactly! Now notice that when $x = 1$, the area of the square is exactly 4.
Frank: Yes because the side length will be 2 and we will lose that middle center square.
Teacher: So then, $x + 1/x$ can also equal 2.

One distinction between Dave's and Judy's scripts lies in the fact that different characters participated, generated, and led a discussion. In Dave's script, the teacher-character was absent, while in Judy's, the teacher-character led the discussion. This indicates different pedagogical preferences of prospective secondary mathematics teachers when discussing the same mathematical content.

The Unarticulated: The Minimal Value of the "Function" Proof

Regarding the minimal value of the "function" proof, the majority of the participants focused on the instance when $y = \frac{1}{x}$ and $y = 2 - x$ intersect. For example, in Brian's script, the teacher character pointed out that the minimal value could be obtained when the graphs of the two functions touch. Carol noted that $y = \frac{1}{x}$ lies above $y = 2 - x$ for $0 < x < 1$ and $x > 1$, indicating that $x = 1$ was a "special" point. "Lying above" was interpreted differently by Mike: "We want to consider the 'area' where $1/x$ is larger than $2 - x$, which from the graph is always true except for $x = 1$." By 'area' he referred to the space below the curves even though the area below $1/x$ is unclosed.

In contrast, Niaj approached the minimal value of $x + \frac{1}{x}$ by differentiating $f(x) = x + \frac{1}{x}$ and identifying the zeros of the differentiated function:

Now let $f(x) = x + \frac{1}{x}$. If we differentiate $f(x)$ then,

$$f'(x) = 1 - \frac{1}{x^2} = 0 \Rightarrow x^2 - 1 = 0 \Rightarrow (x+1)(x-1) \Rightarrow\ x = 1 \; or \; x = -1$$

Since x is a positive number, when $x = 1$ we will have a minimum value: $f(1) = 1 + \frac{1}{1} = 2$. Thus, the sum of a positive number and its reciprocal is at least 2.

Discussion

This study explores prospective secondary mathematics teachers' understanding of visual proofs and their pedagogical considerations. All visual proofs, in one way or another, contain the unspoken or unarticulated elements that can be verbalized and articulated. The task consisting of five visual proofs was designed for prospective secondary mathematics teachers to create their narratives of proof without words. The preliminary analysis of the data indicates that the participants' approaches to articulate "the unarticulated" of the diagram greatly differed and that their pedagogical preferences to teach the same mathematical content were also distinct. While some participants read and interpreted diagrams in compelling ways, others focused on non-geometric approaches, shying away from diagrammatic reasoning. This speaks to Davis's (1993) observation that despite the importance and usefulness of the visual proofs, it was overshadowed by the rise of formal logic. Further investigation and a more in-depth analysis would shed more light on how prospective secondary mathematics teachers approach, interpret, and articulate visual proofs.

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TRANSFORMING AN ELEMENTARY TEACHER'S CONCEPTUAL UNDERSTANDING OF MATHEMATICS THROUGH COACHED PRODUCTIVE STRUGGLE AND REFLECTION

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Providing professional coaching opportunities to explore mathematical content and pedagogy can assist teachers in improving their conceptual understanding of mathematics (Association of Mathematics Teacher Educators, 2017; National Council of Teachers of Mathematics, 2000, 2014). With COVID-19 prompting a transition to remote learning, some teachers lacked access to professional coaching and shifted their pedagogical practices to rote learning. Recognizing the implications of this paradigm shift has encouraged teachers to rethink their own conceptual understanding of mathematics and how they might improve their teaching using meaningful tasks, manipulatives, problem-solving techniques, and child-friendly vocabulary (Appova & Taylor, 2020). This study reports on an elementary teacher's effort to improve their conceptual understanding of division through professional coaching used to elicit productive struggle and reflection when solving standard-aligned tasks. The following research question was examined: How does intellectual preparation through professional coaching encourage teachers to reflect on their conceptual understanding of mathematics and rethink their pedagogy?

This study used a participatory action research approach where one of the researchers acted as a professional coach to facilitate three intellectual preparation sessions with a 5th grade co-teacher in the Northeast. Intellectual preparation was defined as one-on-one coaching pre-instruction focused on solving standard-aligned tasks to improve conceptual understanding and identifying best practices for teaching the targeted concept. The coaching included conversations around content, pedagogy, and student learning, which evident in other studies improves teacher reflection and practice (Russell et al., 2020). Given that the researchers relied on the teacher's view of intellectual preparation, a social constructivist theory grounded this study (Creswell & Poth, 2018). Data collection included session observations regarding the teacher's observed conceptual understanding and the teacher's written reflections after each session. Data was analyzed using in vivo and descriptive coding techniques to recognize themes (Saldana, 2016).

We report on three themes that provide insight on how the teacher transformed their conceptual understanding to improve their pedagogy through coached productive struggle and reflection. The first theme addressed the teacher's realization of the importance of using appropriate mathematical terminology and child-friendly language to guide conceptual understanding. This is echoed in his statement: "I now see importance in thinking about the language that is necessary for teaching a specific concept conceptually." The second theme emphasized a new understanding of how models (e.g., tape diagrams) can be used to aid in visualizing abstract ideas. The last theme highlighted the teacher's efforts to use multiple problem-solving methods to make connections. After solving the standard-aligned task, the teacher was able to articulate the connection between a model and an algorithm. This research signifies how teachers can transform their conceptual understanding of targeted concepts with the aid of productive struggle during professional coaching to improve one's practice.

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PRESERVICE TEACHERS' MATHEMATICAL KNOWLEDGE FOR TEACHING: FOCUS ON LESSON PLANNING AND REFLECTION

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Keywords: Mathematical knowledge for teaching, Preservice teacher education, Instructional activities and practices, Lesson planning

Mathematics preservice teachers (PMTs) often take mathematics content courses in the Department of Mathematics and education courses within the College of Education, which limits PMTs' opportunities to learn how to apply content knowledge in their teaching (Burton et al., 2008; Wasserman et al., 2019). Because Mathematical Knowledge for Teaching (MKT; Ball et al., 2008) offers a framework with six MKT domains to explore how content knowledge is associated with mathematics teaching (Thomas et al., 2017), PMTs would benefit from attention to MKT-related dispositions and language. In this study, I provided opportunities for secondary PMTs, enrolled in a mathematics methods course, to engage in the following activities: planning, implementing, reflecting on, and discussing lessons; and reflecting on and discussing the six MKT domains. Utilizing a collective case study approach (Yin, 2017) and content analysis methods (Schreier, 2012), I investigated the following research questions: (a) Which MKT domains were evidenced in PMTs' lesson plans? (b) How did PMTs describe MKT domains in their lesson plan reflections? Data for the study were PMTs' lesson plans and reflections.

Initial findings indicated that three MKT domains, *Common Content Knowledge (CCK)*, *Knowledge of Content and Teaching (KCT)*, and *Knowledge of Content and Curriculum (KCC),* were evidenced in PMTs' lesson plans. For example, *KCT* was evidenced through PMTs' selections of instructional strategies while *KCC* was evidenced when PMTs built their lessons on students' prior knowledge. However, these domains were not evidenced in many lesson plans because some PMTs included generic teacher and student actions without connecting those actions with the content of their lessons. For example, one PMT, in her lesson plan, offered that she will "check students' understanding of the material and reflect on it." Here, the instructional strategy involved the PMT's plan for addressing students' understandings without anticipating what those understandings could be and how they plan to address those. PMTs most often highlighted *KCT* in the second data set (lesson plan reflections), describing why they included specific instructional strategies in their lesson plans. *Horizon Content Knowledge* was the least discussed domain in PMTs' reflections, which was not evidenced in their lesson plans. When PMTs were prompted to reflect on how their content knowledge contributed to lesson planning, they often mentioned that knowledge helped them to choose several mathematical strategies without specifying them. Further, PMTs acknowledged that they could not anticipate students' unconventional strategies during lesson planning, indicating that PMTs were able to reflect on *Knowledge of Content and Students* even though it was not evidenced in their lesson plans. Overall, even though some domains were evidenced in PMTs' lesson planning, PMTs tended to pay less attention to mathematics content while planning and reflecting on lessons. Thus, PMTs would benefit from content-specific instructional activities that require them to explicitly utilize their content knowledge in several aspects of teaching because such activities potentially assist them in exploring and utilizing the content knowledge for rich mathematics teaching.

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DESIGNING ASSESSMENT ITEMS FOR MEASURING PCK FOR PROPORTIONAL REASONING

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Conceptual Framework

Measures of teacher mathematical knowledge are notoriously difficult to develop (e.g., Orrill et al., 2015). This is in part because of the multidimensional nature of teacher knowledge. As part of two separate projects being undertaken by this research team, we have attempted to write assessments of teacher pedagogical content knowledge (PCK) in the area of proportional reasoning. Building from Shulman's (1986) conceptualization of teacher knowledge as comprised of content knowledge, pedagogical knowledge, and PCK, we have attempted to write items that capture only PCK. To this end, we endeavored to write items that measure PCK to teach proportions separate from the knowledge needed to solve proportions. The specific topics focused on analysis of student work, assessment of student understanding, planning for teaching, and issues of implementation (e.g., Smith & Stein, 2018).

The purpose of this poster is to report on findings from our development efforts. In prior papers, we have reflected on some of the challenges in writing items to measure teachers' specialized content knowledge (e.g., Orrill et al, 2015). In this paper, we reflect on our analysis of think-aloud interviews to identify what we have learned about the development of PCK items for proportional reasoning.

Methods

Data were collected on two assessments, one tied to an online course and the other being developed for broader use. As part of the item validation process, five in-service middle school teachers were interviewed for the first assessment and 11 were interviewed for the second. Teachers' responses to each item were analyzed to determine whether the item was measuring the intended knowledge as well as whether the item was interpreted by teachers as intended.

Findings

In this poster, we will report on some of our main findings related to the development of PCK items. These include teachers' reactions to the items, elements that obscure the measurement of PCK, and other observations about the interaction of CK and PCK.

Acknowledgments

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JAPANESE TEACHER INSTRUCTIONAL CIRCLES

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Keywords: Professional Development, Mathematical Knowledge of Teaching, Teacher Knowledge

Introduction

Interest in Japanese mathematics teaching has been motivated through their high achievement scores in international mathematics assessments (e.g., TIMSS). Lesson study is perhaps the most influential idea from the study of Japanese mathematics education. This mode of professional development has been widely studied and implemented in adapted versions in various countries (Fujii, 2013). Lesson study implementation outside of Japan has had mixed results (Doig, Groves, & Fujii, 2011). Miyakawa & Winslow (2019) argue that Japanese lesson study cannot be fully understood without understanding the entire "infrastructure" of work and development of a Japanese teacher. One part of this "infrastructure" that can support teachers' ability to engage in lesson study is *teacher instructional circles*.

Teacher instructional circles are widely unknown in the field of mathematics education. Therefore, the goal of this study is to provide an in-depth look at the purpose of these teacher instructional circles in the instructional system of Japan. Specifically, I was guided by the research question: *How do teacher instructional circles support Japanese teachers in the development of high-quality instructional plans during lesson study?*

Methods

Data Collection

To answer my research question, I used an ethnographic approach to collecting data. I participated in three different teacher instructional circles in three different areas in Japan. I collected data through participant observations, interview with members of the instructional circles, collection of artifacts, and self-journaling as my understanding of this professional development opportunity changed. I triangulated my data through member checks to make sure that the things I observed were accurate.

Data Analysis

Data analysis was done through an open coding approach. I began with the idea about the purposes of teacher instructional circles developed from my own experience, then added, tested, and refined categories as they emerged from the data analysis of observations and interview data.

Findings

I found that there are two types of teacher instructional circles, namely *study meeting (benkyoukai)* and *research meeting (kenkyuukai)*. The purposes and goals of these meetings vary however, the main focus is to improve teachers' understanding of student thinking, and to deepen their understanding of the mathematics they are teaching. Teacher instructional circles are adaptable to different teaching cultures, thus lending a possible support to countries attempts to implement lesson study. Even if lesson study is not the overall goal, teacher instructional circles provide an interesting alternate to current professional development programs.

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PSYCHOMETRIC ANALYSIS OF 2019 KNOWLEDGE FOR TEACHING EARLY ELEMENTARY MATHEMATICS (K-TEEM)

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Keywords: Assessment, Mathematical Knowledge for Teaching, Research Methods

Introduction

Hill and Shih (2009) presented a compelling argument for improvements in quantitative research methods in mathematics education. The quality and usefulness of quantitative research rests on high-quality measurement practices (APA, AERA, & NCME, 2014). In this poster, we present a method for examining the structural validity (Flake et al., 2017) of the 2019 Knowledge for Teaching Early Elementary Mathematics (K-TEEM), a web-based assessment of mathematical knowledge for teaching (MKT) at the early elementary level (Ball et al., 2008; Schoen et al., 2017; 2019; 2021).

Methods

The 2019 K-TEEM test serves as a pretest measure for a randomized controlled trial of a teacher professional development program based on Cognitively Guided Instruction. Teachers of grades K–2 completed the web-based assessment in spring 2019.

Analysis

Using methods based on both classical test theory and item-response theory, we went through the following process: missingness in data, dimensionality analysis, model fit and selection, item and test analysis, person-ability estimates, reliability estimates, and equating posttest to pretest scores.

Results

The sample size was 645. Item-level analysis suggested that 31 out of 32 items were adequate. Reliability estimates indicated that the test was useful for group-level analyses and was reasonably well aligned with the MKT of the population of interest.

Discussion

The process used for analysis and scoring of the K-TEEM can present a model for researchers in mathematics education to use as they increase the methodological rigor of their measurement practices.

Acknowledgments

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Chapter 7:

Math Processes

SUPPORTING MIDDLE-SCHOOL STUDENTS' DEVELOPMENT OF EMERGENT GRAPHICAL SHAPE THINKING

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Emergent graphical shape thinking (Moore & Thompson, 2015) is a way of reasoning that is critical across numerous STEM fields. However, evidence indicates that the underlying component ideas for emergent thinking are underdeveloped in school mathematics education (e.g., Thompson & Carlson, 2017), and few studies directly report on students' development of this way of thinking. We present the results of a teaching experiment conducted with eighth-grade students to support stable meanings for emergent graphical shape thinking. We focus on the in-the-moment meanings expressed by a pair of students as they engaged in a sequence of tasks that we conjecture could support stable meanings for constructing and interpreting graphs.

Keywords: Algebra and Algebraic Thinking, Middle School Education, Learning Trajectories and Progressions

Across STEM fields, constructing and interpreting graphs is a crucial skill (e.g., Glazer, 2011; Potgieter et al., 2008). For instance, in a study looking at the use of graphical representations across numerous science textbooks and practitioner journals, Paoletti et al. (2020) determined that, at least implicitly, an individual must engage in *emergent graphical shape thinking* (hereafter emergent thinking) to interpret most graphs in these sources. Moore and Thompson (2015) defined emergent thinking as conceiving a graph simultaneously in terms of "what is made (a trace) and how it is made (covariation)" (2015, p. 785). Specifically, with a conception of a point as a multiplicative object, a student can conceive of a graph in terms of an emergent, progressive trace generated by the point's movement and dictated by the covarying quantities' magnitudes represented on the axes. The resulting graph represents the tracking of the two quantities' simultaneous covariation. Although there is some evidence that students in grades 6-12 can engage in emergent thinking in-the-moment (e.g., Ellis et al., 2015; Johnson, 2015), other research suggests that pre-service (e.g., Moore & Thompson, 2015; Moore et al., 2019) and in-service (e.g., Thompson et al., 2017) mathematics teachers in the United States often do not reason emergently in tasks designed to elicit such reasoning. Therefore, there is a need to examine how to productively support students in developing emergent thinking.

In this report, we address the research questions: *How do two eighth-grade students develop meanings for graphs that entail emergent thinking?* To investigate this question, we conducted a teaching experiment (Steffe & Thompson, 2000). In this report, we examine the work of two eighth-grade students as they completed the *Faucet Task* (Paoletti, 2019). Prior to this, we define components of emergent thinking to help readers understand how the task could support students' developing meanings for graphs. We then describe the in-the-moment meanings (Thompson, 2016) the two students developed as they engaged in the task. Finally, we share the results of a task developed by Thompson et al. (2017) that the students completed after the instructional sequence to determine whether such meanings may have become part of the students' stable meanings for constructing and interpreting graphs.

Components of Emergent Thinking

Covariational Reasoning and Multiplicative Objects

Several researchers (see Thompson & Carlson, 2017, for a review) have explored ways in which students' covariational reasoning can support them in developing productive meanings for various mathematical ideas. Researchers have contended covariational reasoning is developmental (Carlson et al., 2002; Saldanha & Thompson, 1998). Initially, a student is likely to coordinate two quantities by thinking "of one, then the other, then the first, then the second, and so on" (Saldanha & Thompson, 1998, p. 299) until the student has developed an operative image of covariation that entails a relationship between quantities that results from imagining both quantities being tracked for some duration. Saldanha and Thompson (1998) elaborated:

> [Covariational reasoning] entails coupling the two quantities, so that, in one's understanding, a multiplicative object is formed of the two. As a multiplicative object, one tracks either quantity's value with the immediate, explicit, and persistent realization that, at every moment, the other quantity also has a value. (p. 299)

Saldanha and Thompson's use of *multiplicative object* stems from Piaget's notion of 'and' as a multiplicative operator (the Cartesian product). Thompson et al. (2017) noted, "A person forms a multiplicative object from two quantities when she mentally unites their attributes to make a new attribute that is, simultaneously, one and the other" (p. 98). Hence, covariational reasoning entails understanding the simultaneity of two quantities' values in relation to each other.

Reasoning in a Coordinate System

To represent and coordinate two conceived quantities, students can construct a coordinate system (Lee, 2016; Lee et al., 2020). In the Cartesian coordinate system, once a student has conceived that quantities' magnitudes can be represented via line segments, the student can consider changes in the lengths of these segments, oriented orthogonally on horizontal and vertical axes, as the situational quantities covary. With such a coordinate system in mind, a student can then conceive of a point as a multiplicative object (Lee, 2016; Lee et al., 2020; Thompson, 2011) that simultaneously represents the two covarying quantities via the two segments' magnitudes. Such a meaning is a prerequisite for reasoning about (or imagining) a graph as representing an emergent trace of a point representing covarying quantities.

Setting and Methods

The middle school where the study took place serves a diverse student population (over 75% students of color) in the northeastern United States. We conducted the teaching experiment in an accelerated eighth grade math class with eight students who had completed high school level Algebra I and Geometry courses. The experiment occurred over five days in June after administration of the Geometry end-of-course assessment. The first author, who was not the students' normal teacher, served as the classroom teacher-researcher (TR).

All portions of the teaching experiment were video- and audio-recorded. The two focus students for this study, Kendis (female, African American) and Camila (female, Hispanic), were a pre-established group in the class. During the instruction, Kendis and Camila used a Chromebook computer to view and manipulate interactive applets and recorded their work on paper worksheets and a dry-erase board. To analyze this data, we watched the videos to identify occurrences providing insights into each student's in-the-moment meanings for constructing, interpreting, or representing quantities and relationships between quantities (Thompson, 2008).

Additionally, we collected data from the *uv*-Task described in Thompson et al. (2017) one day after the instruction concluded. In the *uv*-Task, a coordinate system is shown, and bolded segments representing quantities *v* and *u* (on the horizontal and vertical axes, respectively) vary as the animation plays. (The animated task can be seen at http://bit.ly/CovaryMagnitudes.) Consistent with Thompson et al.'s (2017) methods, we gave participants a paper with a set of axes and the initial segments representing *v* and *u* shown, and the animation was played six times. The students were asked to sketch a graph that depicted the value of *u* relative to the value of *v* (see Figure 1 (left) for an accurate graph). Using the rubric from Thompson et al. (2017) shown in Figure 1 (right), we independently coded the student responses on the shape of the sketched graph (92% interrater reliability). Although the *uv*-Task was not explicitly designed to measure emergent thinking, we contend imagining the graph as the trace of the (imagined) point corresponding to the endpoints of the two segments as they covary is required to produce a more accurate graph shape; we infer scoring a 2 or higher is likely indicative of a person engaging in emergent thinking.

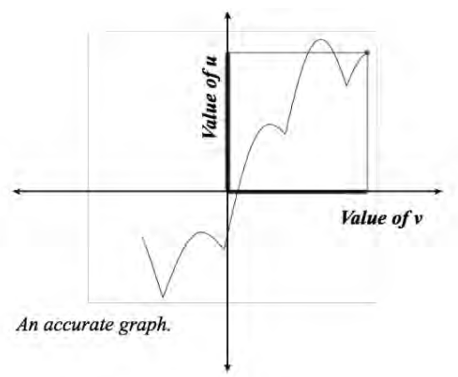

Table 4. Scoring rubric for Dimension B, shape of teachers' sketched graphs (SSG).

Level B4a:	The graph has four local minima in ascending order and three local maxima in ascending order*
Level B4b:	The graph begins decreasing, is generally increasing, has at least 2 local extrema, and has these points of inflection: – 6 if graph starts concave up and ends concave up – 7 if graph starts concave up (down) and ends concave down (up) – 8 if graph starts concave down and ends concave down
Level B3a:	The graph has 6 or 8 local extrema with minima in ascending order and maxima in ascending order.
Level B3b:	Same as B4b except that the graph has one too few or one too many points of inflection given the way the graph starts and ends.
Level B2:	The graph is generally increasing and has 2-5 or 9-12 local extrema, ignoring ascending order.
Level B1:	The graph has no more than 1 local minimum and is otherwise monotonically increasing.
Level B0:	The graph does not fit any of the above levels.

* By ascending order, we mean that from left to right each local minimum's *y*-coordinate was greater than the previous one, and likewise for each local maximum's *y*-coordinate.

Figure 1: (left) The accurate graph and (right) the scoring rubric for shape of the sketched graph on the *uv*-Task (Thompson et al., 2017).

Kendis and Camila's Development of Emergent Thinking

In the sections that follow, we present evidence that Kendis and Camila developed in-the-moment meanings for graphs that entailed emergent thinking. We first present evidence of their construction of component meanings to highlight how this thinking developed.

Constructing Quantities and Reasoning Covariationally

Critical to thinking emergently is conceiving of two covarying quantities. To help students construct quantities situationally, the TR presented the class with a GeoGebra applet (https://www.geogebra.org/m/rdxkrwek) intended to represent a faucet with hot and cold knobs (Figure 2). The TR directed students to use sliders to represent turning each knob on or off; changing the sliders changes the representations of amount of water (width of the rectangle below the faucet) and temperature (color of the rectangle). We intended for students to reason about the changing amount of water and temperature as two quantities to coordinate.

Figure 2: Screenshots of part one of the *Faucet Task* (turning cold water on).

The TR asked, “What are some of the things that this applet is trying to represent?” Camila stated, “the more to the right you dragged [the slider], the wider it [the rectangle below the faucet] got.” Kendis added that the width of the rectangle represents “how much water comes out of the faucet.” When the TR drew students’ attention to the changing colors of the rectangle, Kendis volunteered that the color represented “temperature,” and she explained that turning each knob outwards (cold on and hot off) would result in lowering the water temperature. Kendis’s responses demonstrated that she conceptualized two situational quantities.

Next, to support the students in coordinating two covarying quantities, the TR asked students to make predictions for what would happen to the amount and temperature of the water in four scenarios, assuming that both the hot and cold knobs start halfway on. Each scenario consisted of turning one knob either all the way on or off. By making predictions for changes in both amount and temperature of water in each scenario, we provided students opportunities to coordinate simultaneous changes in two quantities and thereby understand the simultaneity of the two quantities changing as a multiplicative object. As evidence of such reasoning, when asked what would happen if the hot knob were turned all the way off, Kendis responded, “the water is going to get colder, and it’ll be less [water].” Kendis’s response explicitly described changes in the magnitudes of both quantities, indicating her meaning that the quantities simultaneously covary.

Constructing and Using a Coordinate System

The next prompts were designed to develop two ideas related to constructing and using a coordinate system: using line segments to represent quantities and understanding a point in a coordinate system as a multiplicative object. These components support emergent thinking.

Using line segments to represent quantities. In the next prompt, students accessed a revised applet that included (a) a vertical (graduated, but unlabeled) thermometer (colored red) to represent the water temperature and (b) a horizontal pink line segment that corresponded to the width of the rectangle that represented the water stream (Figure 3, left). The positioning of the segments as vertical and horizontal was designed to foreshadow the creation of a coordinate system using segments to represent quantities’ values on the vertical and horizontal axes.

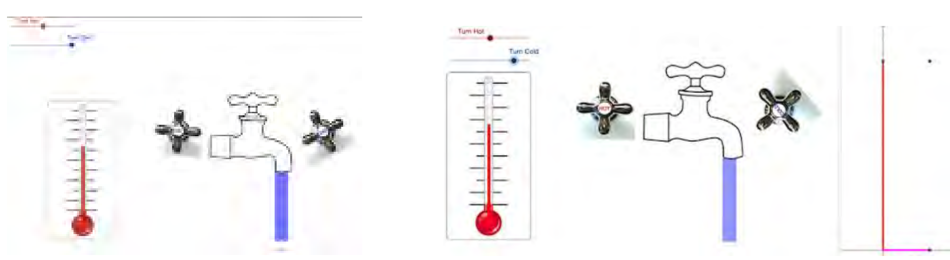

Figure 3: Screenshots of (left) part two and (right) part three of the *Faucet Task*.

The TR then asked students to describe how the segments varied for the same four knob-turning scenarios as before. Our goal was to provide opportunities to connect the lengths of the segments to the previously established quantities. To exemplify the productivity of such opportunities, consider a dialogue about turning the cold knob off between Camila and the TR:

Camila: Um, you turn the cold to the left, and then the temperature will increase, and the red line will get longer because of that. And the pink line will be shorter. [*TR asks Camila to repeat.*] The red line is going to get longer.
TR: It's going to get longer? Why?
Camila: Because you're eliminating the cold water, so the hot is left, and the hot water increases the temperature.
TR: [*TR restates what Camila said.*]…[A]nd then the pink segment's going to go?
Camila: It's going to get shorter.
TR: It's going to get shorter because there's going to be?
Camila: Less water overall.

In this dialogue, Camila connects changes in the quantities in the situation (temperature and amount of water) to changes in the lengths of the corresponding line segments, indicating that she understood the segments as representing the situational quantities. Further, we note Camila readily transferred this reasoning when presented with the segments on the coordinate system.

Understanding a point in a coordinate system as a multiplicative object. Shortly after the previous exchange, the TR showed students a new applet. This applet included a coordinate system with the pink segment (representing amount of water) positioned along the horizontal axis, the red segment (representing temperature) positioned along the vertical axis, and a point with position corresponding to the endpoints of both segments (Figure 3, right). The TR directed students to describe the motion of the point as they explored the applet to provide an opportunity to conceive of relationships between the point's movement and variations in both segments.

While working as a pair, Camila and Kendis had the following conversation with the TR:

TR: So how is this point moving around the screen?
Camila: In accordance with the…
Kendis: [*moves fists horizontally back and forth*]
TR: In accordance with what?
Kendis: The, the temperature… [crosstalk]
Camila: [crosstalk] Temperature.
Kendis: …and the, and how much water was coming out.
TR: With both?
Kendis: [*nods and gestures a vertical line with hand*] It stays in line with both of them.

We interpreted Kendis's reference to "both of them" as the segments representing amount of water and temperature. We inferred that Kendis's horizontal gesture was intended to show that the top endpoint of the temperature segment and the point on the coordinate system formed a horizontal line (and similar for the vertical gesture and the amount of water segment). We infer Kendis understood that the point's movement was dictated by the two quantities' magnitudes represented by segments; the point served as a multiplicative object in the coordinate system.

Using the same applet, the TR told students to investigate a point's movement in several scenarios. Responding to a scenario starting with both knobs turned halfway on, and asked to predict how the quantities will change when they turn hot all the way on, both Kendis and Camila related segment lengths to the situational quantities. Kendis stated:

> Yeah, this is going to go up [*traces finger along the vertical axis from the origin upward beyond the length of the red segment]*… more temperature…. [*traces finger along the horizontal axis from the origin to the right beyond the length of the pink segment*] It's going to move to the right and up.

In response to Kendis's reasoning, Camila used the Chromebook to do a Google Image search for a compass and produced a drawing (reproduced in Figure 4a). We infer that Camila interpreted the described action ("move to the right and up") as occurring simultaneously, and the diagonal line segment represented her understanding of the point's movement.

Indicative of not yet explicitly connecting the coordinate point to the situational multiplicative object she had constructed earlier, Kendis initially disagreed with Camila's representation, stating:

> [I]t's going this way [*traces right along the horizontal axis, as in (1) in Figure 4b*] and, look, it's going to stay in a line with [the red segment], so it's just going to move over and up [*traces from the point to the right a short distance (2) and then up (3) in Figure 4b*].

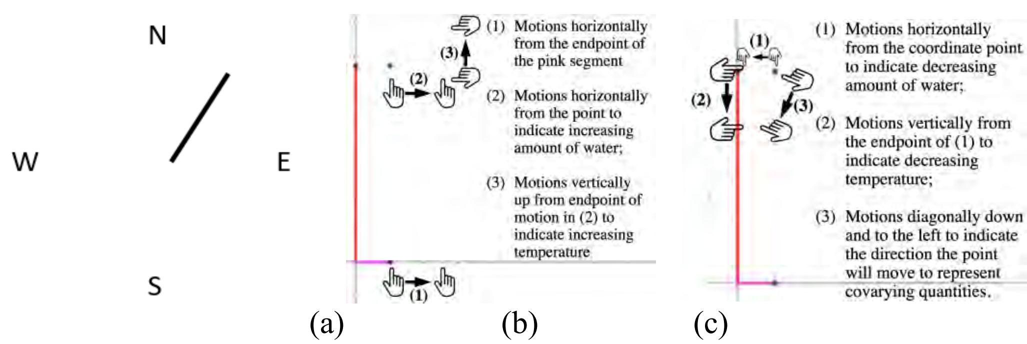

Figure 4: (a) Recreation of Camila's drawing. (b, c) Recreations of Kendis' hand motions.

Although both Camila and Kendis understood where the point would end up relative to its starting position, they conceptualized the point's movement differently. Consistent with the developmental nature of covariational reasoning (Saldanha & Thompson, 1998), Kendis initially conceived of the changes in the underlying segments as sequential (the point would move to the right, then up) as opposed to the simultaneous movement Camila had described.

As the pair continued to discuss the scenarios, evidence emerged that Kendis also began to explicitly connect the motion of the point with the simultaneously covarying situational quantities. For instance, when predicting the point's movement when the two knobs start halfway on and the hot knob is turned off, Kendis described "[the red segment]'s gonna go down, and then [*points to the horizontal axis*] it's less water also so it's gonna go diagonal [*making a diagonal cutting motion with her hand*]." Immediately after this, Kendis silently engaged in a series of movements. She first motioned horizontally to the left from the point as if indicating a decreasing amount of water (indicated by (1) in Figure 4c), then motioned down as if indicating a decreasing temperature of water ((2) in Figure 4c). Critically, and differing from her earlier activity, after these two motions, Kendis lastly motioned diagonally down-and-to-the-left ((3) in Figure 4c) to indicate that the point would move in such a way to reflect the simultaneous variations of the two segment and quantities magnitudes. We took this as evidence of Kendis's formation of a multiplicative object.

Reasoning Emergently and Interpreting Graphs in Multiple Ways

For the final activity in the *Faucet Task*, the TR provided students with several graphs that we told them resulted from turning the knobs. The TR asked students to determine what position each knob was in initially and what action(s) occurred. It is important to note that the graphs were undirected (i.e., no starting or ending point was identified). Thus, each graph had at least two possible interpretations. Through this activity, we intended to provide students the opportunity to reason emergently by interpreting (at least) one possible trace of the graph.

As the pair discussed the actions that would produce a graph (Figure 5 (left)) that was moving "down and to the right," Camila reasoned that the action was turning the cold knob on. She stated, "It's going down in temperature and to the right, so it means you're increasing water, and it's going down, so it means you have to be adding cold water." Camila's reasoning moved between imagining the tracing of a point on the graph, the underlying quantities and how they covary, and the action in the situation. We infer she was reasoning emergently.

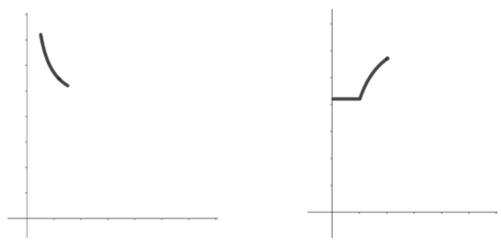

Figure 5: Two trace graphs the TR asked students to interpret.

Kendis and Camila did not independently consider that more than one action could produce the same graph. However, during the class discussion of Figure 5 (left), another group described an interpretation of the graph as turning the cold knob off (reading the graph from right to left). Once the discussion revealed that reading the graph as a trace from right to left could be produced by a different action that would result in the same final graph, Camila was able to apply this idea to describe two different possible productions of the graph in Figure 5 (right):

Camila: First step is to turn the cold on, then turn the hot one on.
TR: [T]hey're both starting completely off, turning cold on then turning hot on…. [S]o in terms of the two quantities, how did you know that was [*trails off*]?
Camila: Well, it continued to go to the right, so it means [the amount of water]'s increasing in quantity, and then, after the second transition, it's going up in temperature, which means you're going to be adding hot water. So, the first one we started off as cold adding it, and then we had to add more of hotter temperature.
TR: … Could there be another way this plays out?
Camila: Hot water off.
TR: Hot water, so you start with both of them on, turn hot water off get to here…
Camila: And then the cold is at halfway and then you could also turn it off.

We take Camila's independent description of two different action sequences that would produce the graph as strong evidence that she was engaging in in-the-moment emergent thinking.

uv-Task Results

As shown above, the work on the *Faucet Task* provided evidence of Kendis and Camila developing in-the-moment emergent thinking. We hypothesized that repeated experiences with such thinking in different contexts would allow emergent thinking to become part of the students' stable meanings for constructing or interpreting graphs. The remaining sessions of the teaching experiment provided students with seven additional opportunities to construct and five additional opportunities to interpret graphs in different tasks and contexts. We use the results from the *uv*-Task (Table 1) to provide some evidence that such opportunities were productive for both Kendis and Camila, as well as their classmates, in developing stable meanings (Thompson, 2016) for constructing and interpreting graphs that entail emergent shape thinking.

Table 1 presents the results of US secondary mathematics teachers as reported in Thompson et al. (2017) and our participants' results on the *uv*-Task. We interpreted a score of 0/IDK ("I don't know") on this task as no evidence of employing covariational reasoning, a score of 1 as evidence of employing gross covariational reasoning (Thompson & Carlson, 2017), and a score of 2 or greater as evidence of employing some level of emergent reasoning. Kendis and Camila each received a score of 2 for the shape of their sketched graphs (see Figure 6). These scores, which exceeded the performance of over 70% of US mathematics teachers in the Thompson et al. (2017) study, indicated to us that Kendis and Camila may have developed emergent thinking as a component of their stable meanings for constructing or interpreting graphs, as evidenced by their ability to apply such reasoning in an unfamiliar, decontextualized situation.

Table 1: Scores on the Shape of Sketched Graph Rubric for the *uv*-Task

	0/IDK	*1*	*2*	*3*	*4*
US teachers (n = 121)	65 (53.7%)	22 (18.2%)	11 (9.1%)	14 (11.6%)	9 (7.4%)
8th graders (n = 8)	2 (25.0%)	0 (0.0%)	4 (50.0%)	1 (12.5%)	1 (12.5%)

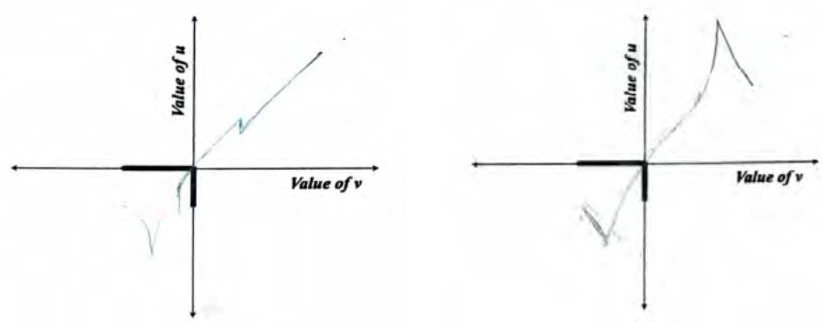

Figure 6: (left) Camila's graph and (right) Kendis's graph in response to the *uv*-Task.

Conclusion

Addressing our research question, we described two students' activity as they engaged in an instructional sequence that emphasized aspects of covariational reasoning (Thompson & Carlson, 2017) and reasoning within a coordinate system (Lee, 2016; Lee et al., 2020) to support them in developing emergent thinking. We highlight that despite individual differences in students' in-the-moment meanings during instruction, each student demonstrated evidence of stable meanings that entailed emergent thinking by the end of the study; each student conceived graphs as "what is made (a trace) and how it is made (covariation)" (Moore & Thompson, 2015, p. 785). We add to the literature by providing a proof-of-concept that at least some middle school students can develop emergent thinking as part of their stable meanings for graphs.

Although our quantitative results present students' performance on a task relative to a sample of U.S. mathematics teachers (Thompson et al., 2017), we do not intend for comparisons to be drawn between these populations. Rather, we intend to provide a frame of reference that conveys the non-trivial nature of constructing meanings that entail emergent thinking. Our study demonstrates that a purposeful learning progression can develop eighth-grade students' stable meanings for graphs via emergent reasoning; we conjecture other populations (e.g., teachers) could develop comparable meanings if provided similar opportunities.

We acknowledge that the small sample size and the use of an accelerated math class limit the generalizability of our findings. Given the importance of emergent thinking as a way of interpreting graphs across STEM fields (Paoletti et al., 2020), it is critical to continue to investigate ways to develop such meanings throughout school mathematics education.

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HARMONY AND DISSONANCE: AN ENACTIVIST ANALYSIS OF THE STRUGGLE FOR SENSE MAKING IN PROBLEM SOLVING

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This work seeks to understand the emergent nature of mathematical activity mediated by learners' engagement with multiple artifacts. We explored the problem solving of two learners as they aimed to make sense of fraction division by coordinating meanings across two artifacts, one being a physical manipulative and the other a written expression of the standard algorithm. In addressing the question, "How do learners make sense of and coordinate meanings across multiple representations of mathematical ideas?" we took an enactivist perspective and used tools of semiotics to analyze the ways they navigated the dissonance that arose as they sought to achieve harmony in meanings across multiple representations of ideas. Our findings reveal the value of such tool-mediated engagement as well as the complexity of problem solving more broadly. Implications for learning mathematics with multiple artifacts are discussed.

Keywords: Problem Solving, Mathematical Representations, Learning Theory, Technology

Hiebert and Grouws (2007) synthesized evidence from a number of studies to argue that the conceptual learning of mathematics is associated with teachers' and students' "explicit attention to the development of mathematical connections among ideas, facts, and procedures" (p. 391). Much research has been done regarding the ways in which teachers can support students' engagement with multiple representations. What is less well understood is the process by which multiple representations of a concept can be leveraged and connected in order to contribute to learners' meanings of the referent of those representations.

Findings from an enactivist analysis of strategy development in mental mathematics contexts suggest that the nature of the processes at play are dynamic, emergent, and contingent on "an ongoing loop" (Proulx, 2013, p. 319) of interactions between the problem and the solver(s). Since sense making results from problem solving, and since problem solving is dynamic, emergent, and contingent (Proulx, 2013), it follows that sense making should be, as well. Moreover, sense making is inextricably linked to the material and symbolic tools that mediate its learning (Artigue, 2002; Verillon & Rabardel, 1995). Following this line of inquiry, we consider what an enactivist analysis might reveal about the processes at play in mathematical meaning making as it develops through the complex interplay of signs and meanings (Maffia and Maracci, 2019) associated with learners' engagement with multiple representations. Thus, this work seeks to address the following question: *"How do learners make sense of and coordinate meanings across multiple representations of mathematical ideas?"* We do so through an analysis of the mathematical activity of two learners as they aim to make sense of fraction division mediated by two representations: the flip-and-multiply algorithm for fraction division and a physical manipulative designed for learners' engagement with fraction concepts.

Theoretical Framework

This study is grounded in the enactivist theory of cognition, which asserts that: "1) perception consists in perceptually guided action, and 2) cognitive structures emerge from the recurrent sensorimotor patterns that enable action to be perceptually guided" (Varela, Rosch, &

Thompson, 1992, pp. 172-173). Thus, cognition, or active knowing, is not some "outward manifestation of some inner workings" (Davis, 1995, p. 4), but rather a dynamically co-emergent phenomena that arises and is brought forth (Maturana & Varela, 1987) through one's goal-directed, "embodied (enacted) understandings" (Davis, 1995, p. 4). In Davis's (1995) adaptation of Maturana and Varela's (1987) words, "*Knowing* is *doing* is *being*" (p. 7).

By viewing knowing in the interactivity of learners, the enactivist perspective offers an alternative to a view of knowledge as the static accumulation of facts and ideas that one may select in response to a problem at hand. Instead, "to know is to respond adequately; it is a situated doing that emerges through the interaction of the organism (e.g., a student, a researcher) and [their] environment" (Maheux & Proulx, 2015, p. 212). But fit is more than that. We use *harmony* to emphasize that fit is an internally "felt dimension of experience" (Petitmengin, 2017, p. 144) that drives problem solving. This drive toward a harmony of goals and actions is theoretically linked to the concepts of structural coupling and structural determinism.

Structural coupling is the process associated with the Darwinian concept of co-evolution, whereby an organism and its environment co-adapt through recursive and repeated inter-actions (Maturana & Varela, 1987). As they do so, the organism and environment experience mutual structural changes so that the fit between them is dynamic. Moreover, this fit is contingent upon unique histories of recurrent interactions and structural changes (Maturana 1988, as cited in Reid & Mgombelo, 2015, p. 175) that are determined by the organism's own structure, a phenomenon referred to as *structural determinism* (Maturana & Varela, 1987). Proulx's (2013) analysis of students' emergent problem-solving activity is committed to this concept as it assumes that a problem solver's strategies are determined by the solver's own way of making sense.

We take this enactivist perspective on mathematical activity as knowing-in-action to investigate the emergent problem solving of two learners as they aim to understand fraction division by finding harmony in meanings across what for them are recurring and competing interpretations in the various elements of two artifacts: 1) the flip-and-multiply algorithm for fraction division, and 2) a manipulative that one of them designed for engagement with fraction concepts. Maffia and Maracci's (2019) concept of *semiotic interference* is used to analyze these dynamic, emergent, and contingent (Proulx, 2013) interactions with the two artifacts. This concept, framed within the Theory of Semiotic Mediation (Bartolini Bussi & Mariotti 2008) relies on Peirce's (1998) triad of sign relations to analyze how meanings emerge from the translation of personalized signs into new signs and eventually into generalized mathematical signs.

According to Peirce, a *sign* is a triadic relationship among a *representamen* (the perceivable part of a sign), an *object* (what the sign stands for), and an *interpretant*, which Presmeg (2006) describes as follows: the "interpretant involves *meaning making*: it is the result of trying to make sense of the relationship… [between] the object and the representamen" (p. 170, emphasis added). Thus, semiotic interference becomes useful for analyzing the process of meaning making across multiple artifacts whenever "the interpretant of a sign whose object belongs to the context of [one] artifact is translated by a student in a new sign whose object belongs to the context of another artifact" (p. 3-58). That is, as the two learners aim to "make meaning" by negotiating their interpretations of signs across the orange and the algorithm, each of the artifacts affords them with differing semiotic potentials (Bartolini Bussi & Mariotti, 2008) for the emergence of a relationship between the personal use of the artifact and mathematical meanings associated with the artifact and its use. Semiotic interference provides a window into their chaining of signs (Presmeg, 2006; Bartolini Bussi & Mariotti, 2008) as they negotiate these interpretations in order to converge upon a meaning for fraction division. In this sense, meaning making is understood as emergent phenomena arising from this “complex interplay of signs” (Maffia & Maracci, 2019, p. 3-57). We thus frame the activity of problem solving from an enactivist perspective and leverage tools of semiotics to depict the evolution of meaning making to better understand how learners make meaning through the coordination of multiple representations of mathematical ideas.

As a critical point of clarification, “representation” in the Peircean sense is a thing perceived by a learner, and that is the meaning we will be using throughout the remainder of this paper. What the field of mathematics education terms a “representation” (e.g., tables, graphs, symbolic expressions) is what we will refer to as an “artifact.”

Methodology

This project is part of a larger study that aims to test and refine the hypothesis that a pedagogically genuine, open-ended, and iterative design experience centered on the Making (Halverson & Sheridan, 2014) of a physical manipulative for mathematics learning would be formative for the development of practicing and prospective mathematics teachers’ (PMTs’) inquiry-oriented pedagogy. Data collection for this study took place across several semesters of a graduate-level mathematics course for PMTs at a mid-sized university in the northeastern United States. For the project reported here, we took a revelatory case study approach (Yin, 2014) in order to determine what an enactivist perspective might reveal about the phenomena involved in the problem-solving activity of “Dolly” and “Lyle” (both pseudonyms).

Dolly was a participant in this larger study; she is a participant-researcher on this project. She calls the tool she designed a “fraction orange” (Figure 1, left), and in designing it, she aimed to create a tool with affordances for the exploration of fraction concepts. The orange is a sphere partitioned into two hemispheres; one hemisphere is further partitioned into fourths, eighths, and sixteenths of the whole; the other into sixths and eighteenths.

Figure 1: The Orange and the Algorithm

The manipulative Dolly created and the thirteen-minute problem-solving interview she conducted with Lyle are artifacts of her participation in the larger study. They also constitute the data for this case study. Three researchers on this project, including Dolly, enacted interpretations of data both individually and in collaborative dialogue. Dolly’s role as both participant and researcher offers validation by permitting a strengthening of the interrelationship between a research context and its participants.

We undertook the analysis by transcribing the recorded video and analyzing the “verbal utterances through line-by-line analysis of the transcripts; stud[ying] body language and intonation by viewing video tapes...; and in511andeng] mathematical forms and objects from the participants’ actions, utterances and notations” (Simmt, 2000, p. 154). Specifically, we focused our analysis on the particular interactions where Dolly and Lyle aimed to coordinate meanings for fraction division in the manipulative and in an algorithm that presumably substantiates those meanings (Malafouris, 2013). As we take our learners’ activity to be driven by an evolutionary imperative to maintain harmony through their problem solving, we used the enactivist concepts of structural coupling and structural determinism to analyze these inter-actions. And in order to analyze their emergent and recursive processes of meaning making across multiple representations, we employed Peirce's (1998) triad of sign relations and Maffia and Maracci's (2019) concept of semiotic interference to refine the analysis.

Results

Given the duration and non-linearity of Dolly and Lyle's problem solving, space constraints only permit us to share selected excerpts uniquely revealed by enactivist and semiotic lenses that elucidate critical moments in their emergent mathematical activity. As a note for the reader, Dolly and Lyle only make use of the hemisphere of the Fraction Orange that is partitioned into fourths, eighths, and sixteenths. In our analysis of their activity, unless otherwise indicated, all fraction pieces are named as Dolly and Lyle do, that is, as if that hemisphere of the orange is the whole.

Embarking on a path of problem solving

We set the stage for the presentation of these findings at the beginning of Dolly's interview with Lyle. Dolly poses the problem, ½ ÷ ¼, on paper alongside her fraction orange. Lyle chooses the pen and paper, performs the flip-and-multiply algorithm: $\frac{1}{2} \div \frac{1}{4} = \frac{1}{2} \times \frac{4}{1} = \frac{4}{2} = \frac{2}{1}$, and declares his answer to be 2. We interpret this application of the standard algorithm as a structurally determined action informed by a lived history of structural coupling with traditional school mathematics, where a *knowing* of fraction division as the execution of an algorithm and the answer it yields was deemed *good enough* to "survive." It constituted what Lyle needed to do to achieve harmony within his mathematics learning environment.

Next, Dolly directs Lyle's attention to the orange and asks, "Can you show me with this?" With two artifacts affording them differing semiotic potentials, both Dolly and Lyle set off to navigate a complicated interplay of signs literally at (their) hand. As we will observe, they experience semiotic interference (i.e., meaning making through the enchaining of these signs) as they pursue a non-linear path of problem-solving activity punctuated by moments of what we refer to as either *harmony*[1], a pleasing fit, or *dissonance*, a displeasing conflict or lack of fit. The cognitive/affective underpinnings of these terms is intentional, because cognition from an enactivist perspective is synonymous with *effective* action.

First dissonance

This exchange captures the first moment of dissonance as Lyle responds to the task Dolly posed to him and as the two learners realize that their understandings of fraction division do not harmonize across the two artifacts.

> Lyle: A half divided by a quarter… <*removes what he considers to be a half piece*> a half divided by a quarter <*points to the fourth pieces inside of the half*> is four.
> Dolly: <pointing to the algorithm and the answer on the page> But that's not what you got.
> Lyle: Uh oh. <*Lyle pulls out the fourth pieces from the half pieces and looks back and forth between the paper and the orange. His gaze then shifts more rapidly between the two artifacts, and the timbre of concern in his voice grows as he continues*.> Uh oh. A half divided by a quarter. Why doesn't that work?

In analyzing this excerpt, we first point out that we are able to observe Lyle's embodied knowings of mathematics precisely because those actions *are* his knowings. They are not inferences of *a priori* knowledge possessed internally; they are only "discovered in action" (Malafouris, 2013, p. 174). In our observations of his interactions with the orange – selecting, removing, gesturing, and communicating about pieces – we can see that the tool mediates new affordances for Lyle's actions. In this first moment, these new affordances evoke an emergent sense of dissonance, which is evident in Lyle's puzzled utterances and frantic glances – somatic markers (Damasio, 1996, as cited in Brown & Coles, 2011) of his negative affective response to seemingly conflicting interpretations of the same mathematical idea. We take these actions to indicate that his knowing of fraction division as expressed through the algorithm is discordant with his knowing of fractions and division as he perceives them in the fraction orange. This experience of semiotic interference between the two representamens (the orange and the algorithm) catalyzes an embodied drive to find harmonized meaning between them, an essential motivation for their problem solving.

The messiness of multiple representations

This next exchange features an extended moment of semiotic interference that is a particularly complicated one for Lyle and that we suggest speaks more broadly to the complexity that is characteristic of meaning making through the connections of multiple representations (Lesh et al., 1987; Hiebert & Grouws, 2007). Dolly and Lyle, motivated by a desire of sense making, strive for harmony in meanings between the orange and the algorithm as they evaluate the expression, ½ ÷ ¼.

Dolly: Here's our half. <*She picks up the half piece and confidently places it next to the algorithm on paper. Lyle points to the piece and looks back to the paper*.> And how many quarters go into a half?

Lyle: <*Looking at the orange*> Two. <*shifting his attention to the paper*> Four. <*shifting his attention back to the orange, and then again back to the paper*> Is that half of a quarter, though? It's half <*pointing to the ½ on the paper in the expression, "½ ÷ ¼"*> of a quarter. <*pointing to the ¼ on the paper*> It's not half of a whole thing. <*As he says, "whole thing," he circles the "4" of the* $\frac{4}{1}$ *in the flip-and-multiply part of the equation on his paper.*>

Dolly: It's a quarter of a half, right? <*Lyle looks at the orange, back at the paper, and back at the orange*>

Lyle: <*with uncertainty*> Yeah?

Dolly: How many quarters of a half are there? <*pauses and laughs*> Why is this so hard?

Through Dolly and Lyle's varied interpretations of both fractions and fraction division in relation to the orange and the algorithm, we observe expressions of semiotic interference. Through their words and gestures, we see Dolly begin by enacting her knowing (interpretant) of "a half" (object) in the orange (representamen) and physically placing the piece on the paper, as if to propose a common meaning between the two by creating a physical bridge between the piece of the orange and the symbolic form of the fraction on paper. She interprets the posed problem, ½ ÷ ¼, as "How many quarters go into a half?" – an interpretation that is for Dolly both meaningful and actionable. Lyle, referencing the orange and evoking his own meanings of both one quarter and one half, determines that two quarter pieces fit into a half piece and (correctly) answers, "2." Immediately thereafter, however, he shifts his attention to the algorithm on the page, and possibly seeing $\frac{4}{1}$, he changes his answer to "4." Doing so provokes dissonance in the pair's meaning-making process, since the outcomes of what Lyle had enacted with the orange did not match what he had enacted with the algorithm. We conjecture that this shift from "2" to "4" was provoked by Lyle's prior knowing of fraction division as the execution of an algorithm, and as a result, he seems to privilege the algorithm over the orange as an anchor of certainty against which his own reasoning is measured.

Next, Lyle aims to resolve the dissonance he experienced as he produced two different solutions to the posed problem. Turning back to the dividend (½) and quotient (¼) in the problem, he seems set on finding a harmonious interpretation of the "whole thing" (object) across both artifacts and wonders yet again just what ½ ÷ ¼ means.

In our interpretation, Lyle's actions are directed at finding harmony across three instances of dissonance: 1) His expression, "Is that half *of* a quarter, though?" [emphasis added] corresponds to a (mis)interpretation of fraction division as one fractional part *of* another; 2) Lyle's ongoing endeavor to identify the whole in his interpretations of fractions – including the utterance, "It's not half of a whole thing" as he repeatedly circles the "4" on the paper – is an indication that he has yet to settle on what that whole is; and 3) His contemplative circling of the "4" could indicate that the number is a perceived point of both importance and confusion resulting from the actions of the flip-and-multiply algorithm. Dolly's utterance, "Why is this so hard?" is an expression of the messiness of engagement with multiple representations and what it feels like for her and Lyle to find themselves amidst spirals of semiotic interference across different *artifacts* (the orange, the algorithm), their wonderings about *objects* (e.g., What is a whole? What is division? What is $4/1$?), and the *relationships* between artifacts and objects across signs (e.g., What is the whole across these different representations?, What does ½ ÷ ¼, mean, and how does it relate to an enactment of "How many quarters go into a half?" with the orange?).

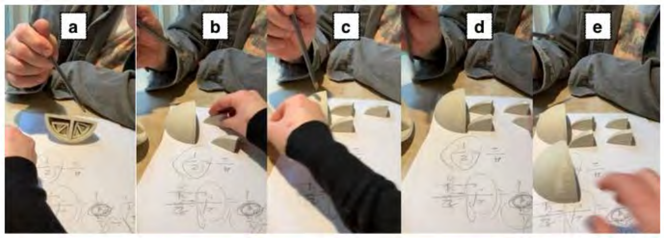
A crowning achievement

In this next excerpt, we present what appears to be a crowning achievement for Dolly and Lyle in their search for harmony in meanings for fraction division mediated by two artifacts. By enchaining signs across pieces of the orange and elements of the algorithm, more specifically by translating interpretations of parts of the orange to interpretations of quantities in the algorithm (i.e., $4/2$ and $2/1$), they have just made sense of those quantities. Next, they engaged in similar sense making in order to find interpretations for the ½ and ¼ in the posed problem, ½ ÷ ¼.

> *Dolly: <referring to the expression, ½ ÷ ¼>* We wanna take a half of one and divide it by a quarter of one, right?
> *Lyle:* Yes.
> *Dolly:* Take a half of one and divide – oh, that's what it is!
> *Lyle:* It's 2.
> *Dolly:* We wanna take this <points to the half piece of the orange> and see how many of those <*now pointing to quarter piece*> fit in there <*points to the half piece again. Then, with confidence:*> And that's why our answer is 2.
> *Lyle:* Yes.
> *Dolly:* There's still two halves in a whole, 'cuz this <*the expression, ½ ÷ ¼*> is in regards to a whole. <*rephrasing*> This is in regards to 1. So a half of 1 divided by a quarter of 1 is 2, because 2 quarters fit into 1 half. Or <*returning to the expression,* $\frac{4}{2}=\frac{2}{1}$> 4 quarters fit into 2 halves.

Lyle: Yeah.

In this excerpt, we observe the meaning Dolly makes of the expression, ½ ÷ ¼, by enchaining interpretations of ½ and ¼ in light of the measurement meaning of division she and Lyle enacted earlier, as well as the meanings they enacted for $\frac{4}{2}$ and $\frac{2}{1}$ in the algorithm. Next, Lyle re-enacts the interpretation for himself.



Figures 2a - –e: Lyle re-enacts Dolly's understanding of "$4/2 = 2/1$."

Lyle: <*pointing to ½ on the page:*> So this is half of a whole <now pointing to ¼ on the page.> and this is a quarter of a whole. <*Next, he turns his attention to the orange (Figure 2a) and points to the half piece resting on the paper. He mutters quietly as if he's reassuring himself:*> Half of a whole. <*Next, he takes his pencil and points to each quarter piece in a sweeping motion of the pencil across each piece*:> Quarter of a whole <*Then, pointing to the two quarter pieces, he continues:*> is 2. <*Thus, he appears to be establishing that the number of quarter pieces he's identified – 2 – is the answer to the posed problem, ½ ÷ ¼*>.
Dolly: <*pointing to the 2 quarter pieces*> Yeah, 'cause there's two quarters of a whole.
Lyle: Yeah, that makes sense.
Dolly: 'Cause there's two of these <*She pulls out the quarter pieces and sets them next to the half piece (Figure 2b)*.> for every one of these <*she says as she touches the half piece*>.
Lyle: <*with a sigh, perhaps of relief*> Yes.
Dolly: Or there's four of these. <*She takes the quarter pieces out of the other half piece.* >
Lyle: <*points to the half piece and extends Dolly's thinking (Figure 2c*)>: For two of those.
Dolly: <*revoicing Lyle*> For two of those. <*As she speaks, she aligns all of the quarter pieces as well as the second half piece on the page (Figures 2d and 2e*).>

As if to establish his own meanings for fraction division and its coherence in representations across artifacts as Dolly has just done, Lyle uses the pencil in his hand to re-enact a physical bridge between the elements of the problem (½ ÷ ¼) and the pieces of the orange. He utters "half of a whole" as he points to the ½ on paper, and "quarter of a whole" as he points to the ¼. Then he repeats these phrases on the other side of the bridge he's establishing: "half of a whole" as he points to the half piece, and "quarter of a whole" as he points to the quarter piece. We interpret this activity as a matching of his interpretation of half of a whole and quarter of a whole in the symbolic representations (½ and ¼, respectively) to the representations he's identified in the orange (the half piece and the quarter piece, respectively). These embodied epistemic actions seem to reify the harmony that has finally emerged from recursive interactions that culminate in an enchaining of signs signifying the sense he and Dolly have made. This reification can be viewed as a newly coupled structure of fraction division for Dolly and Lyle, one that offers a stark contrast to the structurally determined response to fraction division that they enacted at the outset of their problem-solving activity. That is, rather than performing a rote algorithmic process as fraction division, they actually come to do (be/know) fraction division and enchain multiple mathematical signs in order to do so.

Concluding Discussion

This work set out to address the question, *"How do learners make sense of and coordinate meanings across multiple representations of mathematical ideas?"* We did so by analyzing Dolly and Lyle's sense making of fraction division through the complex interplay of signs and meanings that emerged from their engagement with multiple representations. In particular, we analyzed problem-solving interactions that were driven by an imperative to *make sense* of the complicated ideas of fraction division mediated by both an algorithm and a "Fraction Orange" manipulative. The course of their moment-to-moment activity beckoned us to leverage an enactivist framework for its stance on interactions *as* knowing, and for its appreciation of the doing of mathematics as a recursive, nonlinear, unfolding, embodied activity influenced by a system's lived history and its ongoing strive for *fit*.

In analyzing the iterative cycles of harmony and dissonance experienced by Dolly and Lyle, the analytic concepts of structural coupling, structural determinism, semiotic interference, and fit enabled us to discern valuable insights into learners' activity as they navigated multiple representations of mathematical ideas. In particular, structural coupling and determinism enabled a particular focus on the co-constitution that takes place between the individual and their environment through dialectic interactions that result in action-as-knowing. Dolly and Lyle's structural couplings with traditional school mathematics became apparent to us as they navigated felt experiences of harmony and dissonance throughout their drive for fit. For quite a while, they struggled to establish and maintain coherence in meanings across representamens (artifacts, symbols), objects (mathematical ideas), and interpretants (their own meanings of relationships between artifacts and ideas) at hand. Eventually, their dissonance gave way as they established harmony by enchaining meanings across signs through interactions with multiple representations of the complex network of mathematical ideas involved in fraction division. Ultimately, this harmony made way for deep (and felt) ways of doing/knowing mathematics.

The implications of this finding for practice are in recommendations for pedagogical and material resources that enable, support, and honor this sort of loosely structured problem-solving activity to occur in mathematics classrooms. On this point, we wish to re-emphasize that it was this activity that was fundamental to Dolly and Lyle's learning and not their assimilation of a path constructed by others. As Proulx (2013) reminds us, students' paths of problem solving emerge in interactions with the environment and are contingent on their particular mathematical structures and interactions. "Average" paths and tools presumed viable for sense making simply cannot be determined *a priori*. Rather, tools should be provided that are responsive to students' creative and agentive efforts at sense making as they lay down their own path while walking (Varela, 1987). And it is only in such walking that learners can define and refine their own authoring of mathematical ideas and meanings, and find confidence as a mathematical doer with membership in a classroom community.

Note

[1] We use the word *harmony* in a sense similar to Mariotti and Montone's (2020) concept of *synergy*, to denote "the emergence of a phenomenon of semiotic interference [that] fosters the

evolution of signs in an effective semiotic chain," which is an indication of a "deepening and weaving [of] the semiotic web" of mathematical meaning (p. 113).

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THEO'S REINVENTION OF THE LOGIC OF CONDITIONAL STATEMENTS' PROOFS ROOTED IN SET-BASED REASONING

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This report documents how one undergraduate student used set-based reasoning to reinvent logical principles related to conditional statements and their proofs. This learning occurred in a teaching experiment intended to foster abstraction of these logical relationships by comparing the predicate and inference structures among various proofs (in number theory and geometry). We document the progression of Theo's emergent set-based model from a model-of the truth of statements to a model-for logical relationships. This constitutes some of the first evidence for how such logical concepts can be abstracted in this way and provides evidence for the viability of the learning progression that guided the instructional design.

Keywords: mathematical processes, reasoning and proof, advanced mathematical thinking

Teaching logic for the purpose of supporting students' apprenticeship into mathematical proving imposes fundamental challenges regarding how the content-general relationships of logic can be operationalized within students' reasoning about particular mathematical concepts. Scholars affirm that this requires that logic be understood in both its syntactic and semantic aspects (Barrier, Durand-Guerrier, & Blossier, 2009; Durand-Guerrier, Boero, Douek, Epp, & Tanguay, 2012). In other words, students must be able to reason about the form of statements and arguments as well as the way they refer to mathematical objects. Previous studies find that logic taught syntactically often does not foster understandings that are useful in context (e.g., Hawthorne & Rasmussen, 2014), and textbooks downplay the referential aspects of logic (Dawkins, Zazkis, & Cook, 2020). How then are students to abstract logical relationships that generalize across contexts and yet interface with their meanings for particular concepts? How do such logical understandings become functional for comprehending mathematical proofs?

In our ongoing investigations of these questions (Dawkins, 2017, 2019; Dawkins & Roh, 2020), we have found that set-based reasoning can provide a unifying structure by which students abstract key logical relationships. Set-based reasoning is propitious for student thinking and it provides a clear means by which students can interpret statements about very different topics as being in some sense the same (Hub & Dawkins, 2018). We claim the two questions above can be answered by guiding students to formulate logical understandings by comparing interpretations and generalizing their reasoning about mathematical texts in particular contexts.

In this report, we share a case study that illustrates one student's pathway to reinventing some basic logical principles of conditional statements: proof by universal generalization, converse independence, and contrapositive equivalence. As we shall explore, Theo's construction of these new logical relationships depended on his coordination of two ways of thinking: properties defining sets of objects and proofs showing implications between properties as relating such sets. Our teaching experiment methodology (Steffe & Thompson, 2000) allows us to provide a detailed account of Theo's learning process, rooted in his meanings and activity (Piaget & Garcia, 1991; Thompson, 2013). This account of a student abstracting logical relationships is a novel contribution to the literature. We analyze the episode using the emergent models framework to document the emergence of a new mathematical reality (Gravemeijer, 1999), namely that of content-general logical structure rooted in set relationships. We highlight Theo's learning pathway to demonstrate the viability of the learning progression, which closely matched what we intended in the instructional design.

Conceptual Analysis of the Logic of Conditionals

In our prior teaching experiments (Dawkins, 2017; Hub & Dawkins, 2018), we guided students to reinvent logical principles by comparing their interpretations of mathematical statements of the same logical form. In this experiment, we extended this task sequence by asking students to read theorems paired with 2-4 proofs each and to determine whether each proof proved its associated theorem. We encouraged students to associate to each property the set of objects that makes it true (*reasoning about predicates*, Dawkins, 2017). This allowed them to formulate generalizable truth-conditions for the statements and generalizable interpretations of the proofs. In this section, we shall present a conceptual analysis (Thompson, 2008) of these set-based understandings to clarify what we intended students to learn.

Each theorem was a universally quantified conditional: "For any [$x \in S$], if [$P(x)$], then [$Q(x)$]." We use brackets since the statements/proofs that students saw always had particular objects and properties in these slots (e.g., "For every integer x, if x is a multiple of 6, then x is a multiple of 3" and "For all quadrilaterals $\blacksquare ABCD$, if $\blacksquare ABCD$ is a rhombus, then it is a parallelogram"). Each proof was either a direct proof, a proof/disproof of the converse, a proof by contraposition, or a proof/disproof of the inverse (see Table 1). No proofs contained errors. All the proofs (as opposed to disproofs) used universal generalization. The principle of universal generalization (Copi, 1954) states that a proof regarding an arbitrary particular justifies the claim for the whole set of such objects. Choosing such an arbitrary particular is conventionally expressed using the imperative "let" and assigning a property to an object. The argument that follows must depend only on that property and thereby the argument will carry to all objects with the property (Alcock & Simpson, 2002). Such proofs that "property P implies property Q" justify a subset relationship: the set of objects with P is a subset of the set of objects with Q. This is the truth-condition for such conditional statements, which we refer to as the *subset meaning* (Hub & Dawkins, 2018). Such statements are false when there is an object with property P and not property Q. To connect the subset meaning to proofs, students must relate chains of inference to the underlying sets of objects. Proofs establish implication relationships among properties, which are tantamount to containment relationships between the sets of objects with the properties. Counterexamples show lack of set containment and property implication.

Notice that the direct proof of a conditional and the proof of its converse (if both possible) deal with the same two sets of objects. They prove two facts about those sets: the Q's contain the P's and vis-versa. In this case the two sets are equal, meaning the exact same objects have the two properties. Since not every implication involves two equal sets, these two proofs are taken as independent (the converse proof does not prove the original theorem). However, the contrapositive proof is understood to prove the theorem as the contrapositive statement is equivalent to the original theorem (arguments for this will appear in the results section). Since we expected students to abstract these structures from reading statements and proofs, we purposefully maintained a parallel structure to the proofs. The disproofs are somewhat oddly stated, but we intended for them to match the same first-line/last-line structure to help students associate each proof to the statement it is normatively understood to prove or disprove.

Table 1: Forms of proof presented for comparison

Direct proof	Converse proof	Converse disproof	Contrapositive proof	Inverse disproof
Original theorem	"For any [$x \in S$], if [$Q(x)$], then [$P(x)$]."	"For any [$x \in S$], if [$Q(x)$], then [P(x)]."	"For any [$x \in S$], if not [$Q(x)$], then not [P(x)]."	"For any [$x \in S$], if not [$P(x)$], then not [$Q(x)$]."
Proof: Let x have property P. … Thus, x has property Q.	Proof: Let x have property Q. … Thus, x has property P.	Proof: Let x have property Q. x could be a. a does not have property P.	Proof: Let x not have property Q. … Thus, x does not have property P.	Proof: Let x not have property P. x could be a. a does have property Q.

Guided Reinvention and Emergent Models

Our instructional sequence was inspired by the Realistic Mathematics Education design heuristics of guided reinvention and emergent models (Freudenthal, 1991; Gravemeijer, 1999). Guided reinvention entails providing students with experientially real situations they can easily imagine and from which they might elaborate key mathematical ideas. The emergent models heuristic describes how students may first develop a *model-of* the situation. They then elaborate the model by applying it to new situations until the model becomes a new body of understanding apart from the situation(s) it interpreted. The model then becomes a *model-for* reasoning about new problems and concepts. The model's elaboration for mathematical exploration constitutes the establishment of a *new mathematical reality* for the student. The key distinction between model-of and model-for is the extent to which the structure of the model reflects the original situation or alternatively comes to take on its own internal meaning for the student.

To apply these tools to teaching logic to undergraduates, we first wondered what kind of experientially real activity would lead students to perceive questions about logical structure. Logic generalizes across language and proofs, which led us to engage students in comparative reading of statements and proofs of parallel form. By focusing them on set structure, students can develop a *model-of* how each statement refers to sets of objects (reasoning about predicates) and what it means for conditional statements to be true and false (in terms of set relations). By considering how this set structure repeats across various statements and proof texts, students may extend their *model-for* reasoning about content-general logical relationships.

Context independence is a key aspect of how we study students' models. Students often draw the contrapositive inference in a particular context. For instance, they may infer that since all multiples of 6 are multiples of 3, a number that is not a multiple of 3 cannot be a multiple of 6. While this relates to a logical principle, it is not a *logical understanding* for that student if they only apply it locally. We call an understanding *logical* to the extent it generalizes across contexts. Only content-general understandings will support students in reasoning about the logical relationship between *any* conditional statement and its contrapositive statement/proof.

Methods

As part of a grant project developing constructivist models of students' learning of logical principles through guided reinvention, we conducted 8-12 session teaching experiments with pairs of undergraduate students recruited from Calculus 3 classes at two large public universities in the United States. The site for this study's data is a Hispanic Serving Institutions (HSIs). Students volunteered to participate and completed a screening survey to verify that they did not already know the target concepts to be taught (see Roh & Lee, 2018). The experiment featured in this paper was conducted remotely once per week over Zoom using OneNote as a shared space for reading and writing. The two participants chose the pseudonyms Theorem (which we abbreviate as "Theo" for clarity) and Phil. The lead author served as the teacher/researcher and the other three authors acted as outside observers (Steffe & Thompson, 2000). Each session lasted between 60 and 90 minutes and participants were compensated monetarily for their time.

This experiment consisted of an intake interview with a pre-test, nine instructional sessions, and an exit interview with a post-test. During the exit interview, we asked students to choose how they wanted to be identified in terms of their ethnic and gender identities and how those identities were significant for their mathematics learning at university. Theo identified himself as a white, non-Hispanic male. At the time of the study, he was in his first year of university as a finance and mathematics double major. He described himself as "passionate" about mathematics. Phil, an engineering technology major, identified himself as a Hispanic male. The two students worked productively and respectfully together, though they generally operated in parallel rather than interactively. We focus on Theo in this report because of the clear evidence of his progression toward our learning goals. Our models of other students' learning progressions will appear in other reports. Theo constitutes a clear existence proof for our intended learning path.

Consistent with teaching experiment methodology (Steffe & Thompson, 2000), the research team continuously made conjectures about the two students' understandings and tested those conjectures through questioning and iterative task design. The research team met once or twice between sessions to analyze and plan for subsequent sessions. All sessions were recorded on at least two or three screens: the interviewer screen that moved between pages in OneNote and two screens dedicated to capturing Theo and Phil's pages respectively. All main study sessions were transcribed. Our retrospective analysis drew upon field notes, transcripts, and compiled video.

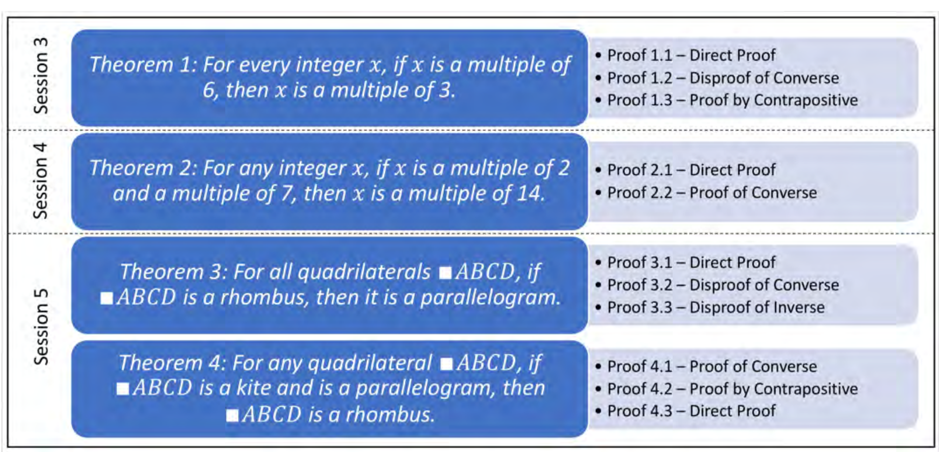

Figure 1: Sequence of proof reading tasks.

Teaching Progression

In the first two instructional sessions of the experiment, Theo and Phil read sequences of universally quantified conditional statements and considered the relationships between the sets of objects that made the if-part true and objects that made the then-part true (hereafter the "if-set" and "then-set"). We intended them to formulate the *subset meaning* (see Conceptual Analysis) for such statements and the conditions for a counterexample. In the next three sessions, they read theorems and proofs as shown in Figure 1. Theorems 1-4 were chosen to intentionally vary the relationships between the underlying sets (proper subset in 1 and 3; set equality in 2 and 4) and to vary the mathematical context (number theory in 1 and 2; geometry in 3 and 4). In the sixth session, which is the last featured in this report due to space limitations, Theo reviewed all of the theorems and proofs and his decisions about which proved the associated theorems. We call this the Comparison Task. We sought for him to systematize the relationship between the logical form of the proof and whether it proved the given theorem (evidence of a model-for logical reasoning that generalizes across context).

Results

Developing Set-Based Meanings (Days 1-2)

In the intake interview, Theo read a direct proof, inverse proof, converse proof, and contrapositive proof of the claim "For any integer x, if x is not a multiple of 3, then $x^2 - 1$ is a multiple of 3." He affirmed the direct proof proved the theorem and denied that the other three did. His rejection of the converse proof was based on whether the middle section of the argument worked, not based on its reverse order from the theorem. Productively, he showed early evidence of associating an equation such as $x = 3k + 1$ to a set of values (*reasoning with predicates*).

On the first day of instruction, once Theo and Phil had assigned truth values to all the given conditionals, the interviewer asked the students to explain the relationship between the if-sets and then-sets. Theo initially drew a diagram showing the then-set as a circle nested within the circle for the if-set. He then thought about a specific statement (Theorem 1) and revised his answer to say, "Because if you think of it, one to 100, that'd'be more multiples of 3. So that's the larger set in these multiples of 6. That's a subset." Over those first two days, the pair came to confirm this interpretation of the set relationship for true conditional statements. They also agreed that a conditional was false whenever an element of the if-set was outside the then-set.

Theo generally represented complement sets using separate circles rather than the inside and outside of a given circle. In reasoning about Theorem 1 and its contrapositive statement, Theo drew three circles that corresponded to the equations $x = 3k$, $y = 3k - 1$, and $z = 3k - 2$. This reflects a common tendency to replace negative categories with a positive description (Dawkins, 2017). He then drew a smaller circle inside the x circle to represent the multiples of 6. This pattern of representing a partition by separate circles persisted throughout the experiment.

Reading Number Theory Proofs (Days 3-4)

During the third and fourth teaching sessions, Theo eventually adopted normative answers as to whether each proof proved the associated theorem based on his set-based reasoning developed in the first two days. At the beginning of Day 3, the interviewer asked Theo to summarize what he had learned the previous two days. He reported:

> It's true if the statement, if it exists inside then or is the same size as then… If the condition exists outside of the parameters of the then statement. Like if it goes beyond the bubbles or diagrams that we created, if it extends beyond it then that's when it's not true.

We note two things about this explanation. First, Theo acknowledged that either the if-set may be contained in the then-set or they may be equal. Second, he referred to the parts of the statement as having physical extension in space and being contained by one another. This constituted his initial model-of interpreting statements by which he determined whether conditional statements were true or false. Each part of the statement corresponded to a group of objects and those objects could be imagined as taking up a region enclosed by a closed curve.

Theo affirmed that Proof 1.1 (direct) proved Theorem 1. He did so focusing on the steps within the proof, not the order from first to last line. He denied Proof 1.2 (converse), saying:

> I don't agree with this theorem [sic] because we're trying to say that if it's a multiple of six then it's a multiple of three, not if it's a multiple of three then it's a multiple of six. It kind of goes into what we were saying last week, if the condition falls outside of the realm of all possibilities and the then statement, then it doesn't hold up, it's not true.

In the first part of the quote, Theo restated Theorem 1 as what "we're trying to say" and contrasted it with what Proof 1.2 is addressing, which he articulated as the converse conditional. He thus attended to the order of the theorem and the first/last-lines of the proof in order to distinguished the meaning of the theorem from what the proof accomplished and to show conflict between the two. He then elaborated what the proof (which presents the counterexample 15) proved: that the if-condition for the converse "falls outside the realm" of the then-statement. He thus shifted back into the language of sets of objects as spatial regions.

Both Theo and Phil agreed that Proof 1.3 (a proof by contraposition) proved Theorem 1. They had read the theorem and contrapositive statement on Day 1 and then noted then that the contrapositive should be true based on the fact that all multiples of 6 are multiples of 3. It is worth noting that Proof 1.3 contains 19 lines and explores how a number having a remainder of 1 or 2 when divided by 3 means it has a remainder of 1, 2, 4, or 5 when divided by 6. The interviewer invited Theo to draw a diagram for how he understood the proof to ascertain how he was making sense of the complex case structure.

> Int: Okay. Can y'all try to use the diagrams that we were using the last two times we met? We have this kind of meaning for what the theorem says in terms of the group of multiples of 3 covering the group of multiples of 6. Can y'all try to explain to me how is it that Proof 1.3 proves it using that idea?
>
> Theo: I think you got to look at, it would be the pattern of all the non-multiples of three and you could be like, 1, 2, 4, 5, 7, 8. And you have that subset of numbers, and then you have the other subset that's three and obviously they're not in each other. However, the multiples of six does not exist inside the non-multiples of three. It only lives inside the multiples of three… It's talking about the subspace when x is not a multiple of three [see Figure 2], which is going to be this whole range of numbers on the left side. And basically, it proves that there exists no of this smaller subset that's on the right side, the blue circle that exists in the non-multiples of three, not even like a cross over even.

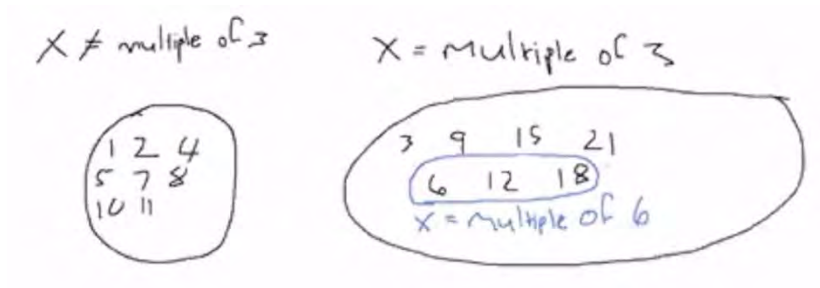

Figure 2: Theo's diagram for Proof 1.3

In this argument, Theo justified Proof 1.3 using what Hub and Dawkins (2018) called the *empty intersection meaning*, namely that "if x is not a multiple of 3, then x is not a multiple of 6" is true because there is no overlap between non-multiples of 3 and multiples of 6. While this is distinct from the subset meaning developed on the first two days, it supported Theo in perceiving symmetry between Proof 1.1 and Proof 1.3. He explained, "[Proof] 1.3 shows that the [multiple of 6] circle doesn't exist in the circle of non-multiples of three, while proof 1.1 would show it exists in the circle with multiples of three." By this point he was comfortable treating the complement of multiples of 3 as a category. However, his notation and reasoning in some sense expressed that x was *not in the set* of multiples, rather than saying it *had the property* of being a non-multiple. His empty intersection meaning similarly negated the "element of" relation, not the property of being a multiple of 6. We have found this preference common, meaning students often avoid treating a negative property as constituting a predicate (Dawkins, 2017). Theo's justification is similar to the arguments Yopp's students produced for how contrapositive proofs eliminate counterexamples (Yopp, 2017).

During Day 4, Theo affirmed Proof 2.1 (direct) and denied Proof 2.2 (converse). He did so using an analogy to Theorem 1 and Proof 1.2 (converse), desiring consistency. He explained:

> In this case, the if and the then are the same set. But, if you switch them around in a set where they're not the same, then it doesn't necessarily work out that way. In this example, it works out, but switching the if and then doesn't necessarily mean it will work out every time.

This argument marks a key development in Theo's thinking because his model-of the set structure allows him to make an analogy between Theorem 1 and Theorem 2 that determines how the proofs do or do not support the theorems. In Dawkins and Roh (under review) we discuss a prior study participant who similarly recognized the analogy, but she denied that it held force. That participant perceived that the difference between subset situations and equal set situations meant the proof relationships for Theorem 1 do not apply to Theorem 2. It is unclear why Theo took a different interpretation. Still, it shows how his set-theoretic model had become a model-for reasoning about more abstract relationships between theorems and proofs. However, we learned in the next sessions that his model still carried some contextual dependence.

Reading Geometry Proofs (Day 5)

Recall that our operating definition for a student's understanding as being logical is that it generalizes across semantic content. Theo's use of his set-theoretic model showed that to some extent he was attending to logical structure on the number theory tasks. In contrast with his prior reasoning, on Day 5 Theo affirmed that a Proof 4.1 (converse) proved the theorem and he denied that Proof 4.2 (contrapositive) did so. Initially Theo and Phil judged that Proof 4.2 was irrelevant to Theorem 4. Though Phil later developed an indirect argument for why Proof 4.2 supported the theorem, neither judged that it proved. Theo also had trouble applying his subset meaning to this theorem because he represented the if-set (kite and parallelogram) using two overlapping circles to show what is shared between the two properties. He then identified that the conclusion (rhombi) existed in the overlap, which led him to imagine the then-set as nested within the if-set. We hypothesize that the three-set structure and the more complex nature of negating the hypothesis kept Theo from structuring these theorem/proof pairs as he had on previous tasks.

Comparison Task (Day 6)

On Day 6, we presented Theo with all of the theorems and proofs from the previous three sessions along with his decision about whether each proof proved the theorem or not. We asked him to look for patterns among the proofs, his decisions, and how the proofs did or did not prove the associated theorem. He began by grouping all the direct proofs and affirming they all proved their associated theorem: "You start by saying like, okay, we have this if, it meets such criteria and then like, it continues on to conclude, okay, this criteria can be put inside this larger space." He initially placed Proof 4.1 (converse) in this group, and then removed it since it started with "the then." He then grouped the first three (dis)proofs of converse, omitting Proof 4.1. Later he decided to move Proof 4.1 into that group. When asked about his claim that Proof 4.1 proved the theorem, he decided to change his decision "to be consistent." He explained the pattern among the proofs that proved the theorem: "We're going to have to start by looking at the smallest subspace, I'm saying for the other yeses (sic), excluding 1.3, because that's doing it from the other way. But proving the converse doesn't necessarily prove the theorem."

Theo grouped all the rest of the proofs (inverse and contrapositive) together as not direct or converse. We explain this in terms of his set-based model, which operated with the structure of if-sets, then-sets, and everything else. Since his model did not include the complement of the if-set, he did not distinguish the order of inverse and contrapositive. Theo's groupings demonstrate how the structure of the proofs reflected the structure of the sets, not the statements per se.

The interviewer asked Theo to explain again his argument for why Proof 1.3 proved, which he did in terms of his empty intersection meaning. The interviewer then asked him to apply the same argument to Proof 4.2. Upon considering he decided Proof 4.2 proved, explaining:

> Because I think when we do it like the same way in 1.3, we're saying, okay, it has this property that it's a non-rhombus. And if it's not a rhombus, it either exists in the space that, like we don't have to look at it like is not a kite or is not a parallelogram… So it's basically saying that the non-rhombi don't have properties of both… Which is basically saying that the space where they have both of the properties is rhombi.

By adapting his empty-intersection argument, Theo began to construct contrapositive equivalence as a logical concept rooted in his set-based model-for reasoning about proofs.

Discussion and Conclusions

We proffer this account of Theo's learning as an account of how logical understandings can emerge from set-based reasoning about the structure of conditional statements and their proofs. We argue that Theo's ability to see necessity in logical relationships (e.g., converse proofs cannot prove for consistency) and to generalize logical arguments (e.g., applying his empty intersection argument from Proof 1.3 to Proof 4.2) as evidence that his set-theoretic model constituted a new mathematical reality for *reasoning about logic* (Dawkins & Cook, 2017).

To further illustrate what was involved in Theo's learning, we highlight some shifts in Theo's ways of talking about the statements and categories in the statements. First, he became comfortable talking about negative categories such as non-rhombus. Second, he shifted rather fluidly between using a) set language interpreted as spatial regions such as "smallest subspace," b) property language such as "meets such criteria," and c) syntactic/temporal order language of "if," "then," and "start." In this way, Theo coordinated quantification, property relations, and statement syntax to give meaning to these complex proof texts. What is more, these understandings allowed him to perceive theorems/proofs about number theory categories and geometric categories as the same, since they all shared set-theoretic structure. We conjecture that developing negative categories and exploring how properties stand for whole classes of objects are essential parts of his construction of a logic of conditional statements and proofs.

We began with questions about how students' understanding of logical relationships can interact with their content-specific reasoning. We claim that Theo's learning progression provides an actionable answer to this question. Specifically, logical concepts can be reinvented in context via the emergence of set-based models for the truth and falsehood conditions and the structure of mathematical proofs. Ongoing work is seeking to understand other students' pathway to these abstractions to create generalizable learning sequences for undergraduate students' introduction to mathematical proving.

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"WHAT DO YOU BELIEVE IS TRUE?" A ROUTINE FOR PROVING THEOREMS IN SECONDARY GEOMETRY

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We report findings from an investigation of one teacher's instruction as he guided students through the proofs of 21 theorems in a Grade 8 Honors Geometry course. We describe a routine involving four distinct phases, including Setting up the Proof and Concluding the Proof. Results from an end-of-course proof test are also presented to attest to the effectiveness of the teacher's approach. By engaging with descriptions of the theorem-proving routine, one can learn about different strategies that may support students to learn to prove theorems, such as asking students to put forth claims in the form of conjectures or other statements that they believe are true and seeking justifications for these claims as well as sanctioning a theorem once proven.

Keywords: Reasoning and Proof, Geometry and Spatial Reasoning

Purpose of the Study

In 1994, Alan Schoenfeld noted: "Proof is not a thing separable from mathematics, as it appears to be in our curricula; it is an essential component of doing, communicating, and recording mathematics" (p. 76). Yet, despite the fact that reasoning and proving are considered central to the discipline of mathematics, and geometry is typically the place in the school mathematics curriculum where proof is taught, the teaching of proof in school geometry has been considered to be a failure in almost all countries (Balacheff, 1988). Acknowledging this failure, Battista (2007) posed the question: "How can proof skills best be developed in students?" (p. 888). To address this question, in this mixed-methods study, we focus on the development of proof skills with respect to the "instructional situation" (Herbst, Nachlieli, & Chazan, 2011) of proving theorems. Proving theorems is an activity that differs from doing proofs of "configurations" (Herbst & Miyakawa, 2008, p. 470) whereby students are typically provided with "Given" information, a "Prove" statement, and a figure to go along with the proof. Research conducted by Otten and colleagues (Otten, Gilbertson, Males, & Clark, 2014; Otten, Males, & Gilbertson, 2014) suggested that U.S. textbooks primarily engage students in proving configurations rather than theorems. This is a problematic situation if one agrees with Schoenfeld's (1994) argument that proof is an essential component of doing mathematics.

Because we agree with Reid and Knipping (2010) who suggested that recommended changes to how teachers teach proof must be based on detailed understandings of how teachers currently teach proof, we designed a study that involved spending significant time in teachers' classrooms. After determining that the students of one of the teachers in the study, who we call Shane, were outperforming other teachers' students in seemingly similar courses on an end-of-course proof test, we observed 22 of Shane's lessons during the 2018-2019 academic year. For this paper, we posed the following research question: *How did a teacher whose students were overall* *"successful" on an end-of-course proof test teach his students to prove theorems in geometry?* We operationalize what is meant by "successful" in subsequent sections of this paper.

Theoretical Framework

In order to frame the purpose of the study, justify the study methodology, and focus and guide the reporting and discussion of the results (Cai et al., 2019), we review three areas of literature. We first describe past results from the end-of-course assessment used in this study. Next, we describe research on "doing proofs" in U.S. geometry classrooms to support the need for the study. Last, we highlight research-based competencies for proving to frame the findings.

Students' Past Performance on an End-of-Course Proof Test

In Senk's (1985) paper titled, *"How Well Do Students Write Geometry Proofs?"* Senk described some of her research instruments and summarized some key findings from her (1983) dissertation. Senk's (1985) research question was: To what extent do secondary school geometry students in the United States write proofs similar to the theorems or exercises in commonly used geometry texts? Her results were part of the larger Cognitive Development and Achievement in Secondary School Geometry (CDASSG) Project. To answer her research question, Senk administered three forms of the CDASSG end-of-course proof test. Each form contained six items. The first item required students to fill-in-the-blanks of a two-column proof. The second item required a translation from a verbal statement to an appropriate "figure," "given," and "to prove." The last four items required students to write full proofs (Senk, 1985). Senk administered the CDASSG assessments to 1520 students in 74 classes from 11 schools in five states in 1981. Each item was scored on a four-point scale, and students were considered "successful" if they scored at least 3 out of 4 possible points. Students scored a 3 if their proof steps followed logically from previous ones but contained minor errors. Overall, Senk (1985) concluded that only about 30% of students in the full-year geometry courses that covered proof reached a 75% mastery of proof (i.e., were "successful" on the test overall). Senk also concluded that proofs of textbook theorems were difficult for many students. For example, only 32% of students were successful in proving the theorem that the diagonals of a rectangle are congruent, and 34% of students scored a 0 on this proof. A common error was citing the theorem in the proof (i.e., using circular reasoning). Across the three forms, an average of approximately 13% of students were successful on all six tasks with only 3% of students receiving perfect scores on all six.

"Doing Proofs" in U.S. School Geometry

Building on the past work of Lampert (1993) and Schoenfeld (1986, 1988, 1989) who documented the role that proof has traditionally played in classroom teaching and learning, Herbst and colleagues examined both students' and teachers' perspectives on what "doing proofs" is like in American high school geometry classrooms. Herbst and Brach (2006) reported findings from 29 interviews with 16 students in five categories: Statements, Initial Conditions, Concepts, Targets of the Work of Proving, and The Work of Proving. Several findings from Herbst and Brach's study are relevant to this study, especially students' claims that:

- It is customary that the "given" and prove" will be specified in the problem statement,
- Students are rarely asked to prove theorems, and
- The first thing in proving is to mark the givens on the diagram.

Herbst's and colleagues' (2009) study of teachers' views outlined a set of 25 norms for the instructional situation of "doing proofs," including the following norms about the division of labor - the teacher or textbook is responsible for:

- Establishing the givens in terms of properties of a figure represented in a diagram, and
- Providing a diagram that fairly represents the objects to be used in the proof.

The 25th norm was that every single statement or reason is produced in a handful of seconds. Overall, this research demonstrates that when "doing proofs" it is the teacher, not the students, who seems to carry much of the cognitive load.

Developing Competencies for Proving

Cirillo and colleagues' research has focused on understanding the conditions in which teachers currently teach proof in geometry with the ultimate goal of improving the teaching and learning of proof (see, e.g., Cirillo & Hummer, 2019). After observing that the classroom teachers with whom she worked were unsure about how to teach proof and were particularly unclear about how to *begin* teaching proof, Cirillo et al. (2017) developed a pedagogical framework for teaching proof based on the research literature and classroom observations. The pedagogical framework decomposes proof so that understanding of the larger goal (i.e., doing proofs) can be built up sequentially by teaching particular sub-goals of proof over time. The sub-goals of proof included in the framework that are particularly relevant to this study include: Knowing Geometric Concepts, Conjecturing, Working with Diagrams, Drawing Conclusions, Understanding Theorems, and Understanding the Nature of Proof. Particularly relevant competencies, which are nested within the sub-goals, include: Being able to turn a conjecture into a testable conditional statement; knowing how to read a diagram and understanding what can and cannot be assumed from a diagram; using axioms, postulates, definitions, and theorems to draw valid conclusions from some "Given" information; and being able to identify the hypothesis and conclusion of a conditional statement and then writing particular "Given and "Prove" statements, typically making use of a generic figure (see the full framework in Cirillo & May, 2021). Many of these competencies were also observed in Cirillo and Hummer's (2021) smartpen interview study in the work of students who were "successful" in completing proofs during the clinical interviews. For example, the following competencies were observed in the work of students who were successful with the proofs - students: productively attended to the "Given" information; used the diagram as a resource; identified warrants as postulates, axioms, definitions, or theorems; and attended to important details in their proofs.

Methods

In this paper we share results from a sub-study of a larger study focused on improving the teaching and learning of proof in secondary geometry. The larger project, *Proof in Secondary Classrooms* (Cirillo, 2015-2020), is a mixed methods study that took place in the mid-Atlantic region of the United States. Here, we focus on a subset of participants from the larger study who did not receive the study treatment (i.e., they were in the control group).

Context and Data

Across the three years of the sub-study, with the help of the research project staff, six teachers who taught a total of 464 Grade 8 Honors Geometry students administered Senk's (1983) CDASSG assessment at the end of the school year. It is important to note that prior to adopting the CDASSG for our study, through an alignment analysis, we concluded that the

CDASSG was, in fact, well aligned with current standards and textbooks being used in the study classrooms. The assessments were scored, and results were analyzed each summer. Beginning in Year 1 of the test administration, we noticed that, in comparison to the other sections of Grade 8 Honors students, one teacher's students consistently scored higher on the CDASSG assessment. More specifically, we noticed that in Year 1, the students (n=43) of the teacher, who we call Shane, earned a mean score of 19.05 out of 24 possible points on the six-item proof assessment (i.e., 79%); whereas the Grade 8 Honors Geometry students (n=129) in other teachers' classes earned a mean score of 8.5 out of 24 (i.e., 35%). Upon noticing this, we became interested in observing Shane's teaching, and we asked to observe his proof-focused lessons. Consequently, we conducted 22 classroom observations in one section of Shane's Grade 8 Honors Geometry course during the 2018-2019 school year. We requested that Shane invite us in when he first introduced proof up until and including lessons focused on quadrilateral proofs.

Qualitative Data Analysis

Phase 1: Identifying a reduced data set. The research team initially watched and developed timelines of the 22 classroom observation videos. These timelines identified the portions of the class that were dedicated to various classroom activities such as whole-class work, seatwork, and going over homework; within each activity, researchers included descriptions of the content covered. Of the 22 observations, we identified 11 observations where theorems were proved in the whole-class setting. Within these 11 observations, a total of 21 theorems were proved, comprising of approximately 6 hours and 23 minutes of video data. Transana Multiuser 3.32d (Woods, 2020) was used to transcribe and create a collection of video clips of each theorem-proof (i.e., proofs of actual theorems rather than "configurations" (Herbst & Miyakawa, 2008) including the Pre-Proof activities). The video clip collection was then further analyzed.

Phase 2: Identifying themes. The research team watched all 21 video clips of the whole-class theorem-proofs, as well as any related activities conducted prior to the theorem-proof (i.e., pre-proof activities) and looked for patterns within these data. We identified three distinct activities that occurred during the teaching of theorem-proofs: Setting up the Proof, Making and Justifying Claims, and Concluding the Proof.

Phase 3: Coding the themes. We developed codes to further analyze the three activities. Codebooks for each activity were developed using constant comparative analysis (Boeije, 2002). The codebooks were continuously revised and improved as each activity was coded in teams of two. Each pair of researchers independently coded at least 3 of the 21 theorem-proofs for their specific activity. After achieving above 80% interrater agreement (i.e., 90% for Setting Up the Proof, 92% for Making and Justifying Claims, and 92% for Concluding the Proof) and reconciling differences, coders worked independently to code the remaining data.

Quantitative Data Analysis

To provide further information about Shane's students' performance on the end-of-course CDASSG proof assessment in comparison to other similar students' performance on the same assessment, we analyzed results from two particular items of the CDASSG. More specifically, we focused on results from Senk's CDASSG Items 4 and 5. These two items were selected because across all three forms of the CDASSG, the items were similar in nature and explicitly required students to write full proofs. In particular, each form of the test included a proof of a theorem for one of the two items (e.g., the measures of the angles of a triangle sum to 180° or the diagonals of a rectangle are congruent), and the second item was a configuration proof involving triangle congruence. Following Senk, we report the percentage of students who were "Successful" and "Not Successful" on these items, where "Successful" means that students scored at least 3 out of 4 points on the item. Results across three years of the study are shared for Shane's Grade 8 Honors Geometry students and for all other Grade 8 Honors Geometry students in the study who were also in the control group (i.e., did not receive the project treatment).

Results

We share findings from five related analyses. We begin by exploring the four parts of Shane's theorem-proving routine: Pre-Proof: Making and Justifying Claims; Setting up the Proof; During-the-Proof: Making and Justifying Claims; and Concluding the Proof (see Figure 1). We then share additional quantitative data from the Grade 8 Honors Geometry student assessment results. This last finding is included to provide evidence of the effectiveness of Shane's approach to teaching proof. We begin by describing the two Making and Justifying Claims activities since they are closely related to one another.

Figure 1: Shane's Routine for Proving Theorems

Making and Justifying Claims in Pre-Proof and During-the-Proof Activities

We considered student-generated claims and justifications that were made both during "Pre-Proof" activities, which preceded Setting up the Proof, as well as "During-the-Proof," which followed Setting up the Proof. We only considered claims and justifications that were made by the students, rather than the teacher. Claims that were truly generated by the students without teacher support were made 44% of the time, and claims that were generated by the students as a result of question-and-answer exchanges between Shane and the students occurred 56% of the time. Throughout all 21 whole-class discussions of the theorem-proofs, Shane used the word "believe" 130 times, asking questions such as: "What do you *believe* is true?," "Do you *believe* it's always true?," and "Do you have a reason for why you *believe* that?"

Pre-Proof Claims and Justifications. Pre-Proof activities included exploring definitions to better understand the geometric objects involved in the proof (e.g., developing or stating definitions of isosceles triangles or parallelograms) and making claims that were sometimes unsupported and considered to be conjectures or were valid conclusions that could be drawn from a proof assumption. Across the 21 theorems, we identified 28 claims made during the Pre-Proof activities. Three of these claims were related to establishing a definition of the geometric object centrally involved in the proof. Fifteen of the claims were conjectures that would ultimately be considered "worth proving;" that is, the students conjectured the proof idea through a discovery process led by Shane prior to the Setting-Up-the-Proof activity that followed. Two of the 28 claims were generated through a combination of some assumption that could be made about a diagram and a postulate (e.g., AB + BC = AC by the Line Segment Addition Postulate). The remaining eight claims were conclusions that could be drawn from the premise of the proof. For example, if Shane presented some quadrilateral ABCD that was assumed to be a parallelogram (i.e., eventually the proof hypothesis or "Given" statement), then students would state a valid claim that the two pairs of opposite sides of the quadrilateral were parallel. The justification for this claim would be the definition of parallelogram. By engaging students in a routine that involved Pre-Proof activities focused on claims and, where applicable, justifications, Shane provided students with opportunities to explore or "experience" mathematical objects (Schoenfeld, 1986) and develop conjectures prior to working on proofs about those objects.

During-the-Proof Claims and Justifications. Four codes were developed for Claims and Justifications made During the Proof. Across the 21 theorems, we identified 60 student-generated claims made during the proof. The first code, which was related to stating the proof assumption and justifying it by "Given" only occurred once. We hypothesize that this aspect of proving needed to be carried out only once so that students would understand this proof requirement. The next two activity codes were similar to activities that occurred during the Pre-Proof. There were 11 instances of students stating claims that were conclusions drawn directly from the "Given" statement. The justification for such claims was typically the definition of the mathematical object that was the subject of the theorem (e.g., definition of parallelogram), but, at times, it was also appropriate to use a theorem about the mathematical object to justify a claim made directly from the "Given" statement. Also, similar to an activity described above, there were six claims generated through combinations of a postulate and an appropriate assumption that could be made about the diagram. The majority of student-generated claims (n=41) were related to the statements and reasons that followed once the initial conclusions were drawn from the hypothesis and any valid, relevant assumptions made about the diagram were identified.

Setting up the Proof

Setting up the Proof involved a range of activities including: working with the theorem as a conditional statement, developing the "Given" and "Prove" statements, and developing or working with a diagram for the proof. During the Setting-up-the-Proof activities, Shane attended to different aspects of setting up the proof, working on different competencies across the 21 theorems, over time. For example, for 8 of the 21 theorems, rather than providing students with the conditional statements of the theorems to be proved, Shane drew from the conjectures students developed during the Pre-Proof activities. Since these conjectures were often written using "everyday language," such as "Opposite sides of a parallelogram are congruent," when Setting up the Proof, Shane led discussions that supported his students to identify the assumptions (or hypotheses) in the conjecture (e.g., [If] a quadrilateral is a parallelogram) as well as the conclusions of the conjecture (e.g., [then] the opposite sides are congruent). For 11 of 21 theorems, students were not provided with the "Given" and "Prove" statements but rather had to participate in developing them during the whole-class discussions. For 6 of 11 of these theorems, students also played a role in generating the particular figures that would be used in the proof. Six of the 21 theorems proved during the observations were converses of other theorems that the class had also proved. Thus, it is unsurprising that discussions about the truth values of the converses of six of the theorems occurred. Last, for 12 of 21 of the theorems, a figure was provided for the students, but it was not marked. For example, for parallelogram proofs, Shane had pre-populated parallelograms labeled ABCD on his advanced organizer, but for each of the theorems, the diagrams still needed to be marked to reflect what students knew to be true from what they determined to be the "Given" information. Across the 21 theorems, by modifying what information was provided and what Shane left blank for the students to develop, Shane provided students with opportunities to develop different competencies needed to set up the proofs.

Concluding the Proof

Across the 21 theorem-proofs, Shane's facilitation of Concluding the Proof activities included three noteworthy features. For 9 of the 16 theorems that did not have "names" such as The Midpoint Theorem or the Base Angles Theorem, Shane concluded the proofs by developing a shorthand version of the theorem that students could use moving forward. For example, Shane suggested that students could write "⊥ lines → ≅ Adjacent ∠s" rather than writing out the full text of the theorem: "If the two lines are perpendicular, then they form congruent adjacent angles." Referring to the shorthand notation, Shane stated, "Your options are either to write something like this, or you may just write the whole thing." Second, for 12 of the 21 theorems, Shane restated or rephrased the theorem after the class proved it, typically in a way that seemed intended to foster an understanding of what the class had just proved. For example, after proving the converse of the Isosceles Triangle or Base Angles Theorem, Shane stated: "So if you do have a pair of angles that are congruent in the triangle, it does imply that the sides opposite them are congruent, which implies it is an isosceles triangle." Last, upon completing 9 of the 21 theorem-proofs, Shane explained to students or reminded them that once a theorem was proven, it could be used in future proofs. For example, after writing shorthand notation for the third of four parallelogram theorems that they would prove that day, Shane asked his students, "So now we have how many properties of parallelograms we can use?" After establishing that they had three theorems plus the definition of parallelogram, Shane asked students to prove the fourth theorem of the day and reminded them: "Remember now, we, you can use any properties that we have already proved. So now you can use everything except for the one you're trying to prove." In doing so, Shane established that once a theorem was proved it could be used to prove other theorems; he also reminded the students not to engage in circular reasoning.

Proof Assessment Results

As explained in the Methods section, we calculated results for two of the full-proof items from Senk's (1983) CDASSG assessment that were administered in this study. For two groups of students - Shane's Grade 8 Honors students and Grade 8 Honors students who had teachers other than Shane (i.e., "non-Shane") - we calculated the numbers and percentages of students who were "Successful" (i.e., scored at least a 3 out of 4 points) on *both* Items 4 and 5, on *either* Item 4 or 5 but not both, and on *neither* Item 4 nor Item 5. As can be seen in Table 1, there were large differences between the results of the two groups of students. Acknowledging that the student populations for the two studies differed, as another point of comparison, in Senk's (1983) study, approximately 43% of students were successful on Item 4 and approximately 37% of students were successful on Item 5. Percentages of success for the same items in our study were 77% and 84% for Shane's students, respectively, and 30% and 25% for non-Shane students, respectively. This is noteworthy given that the students in this sub-study were all Grade 8 Honors students, whereas the students in Senk's study included a population of Honors and non-Honors students.

Table 1: Student Assessment Results for Shane's Students Compared to Other (Non-Shane) Grade 8 Honors Students

	Number (%) of Students *Successful on both 4 & 5*	**Number (%) of Students *Successful on either 4 or 5, but not both***	**Number (%) of Students *Successful on neither 4 nor 5***
Shane's Students (n=128)	83 (64.8%)	40 (31.3%)	5 (3.9%)
Non-Shane's Students (n=336)	53 (15.8%)	78 (23.2%)	205 (61.0%)

Discussion and Conclusions

As noted by Herbst and Miyakawa (2008), while all theorems have proofs, in U.S. geometry classrooms, not every theorem is proved. The study reported here is significant in that it describes a routine for proving theorems - an activity that is apparently lacking in many U.S. classrooms. Furthermore, our assessment results provide evidence that the strategies employed by Shane seemed to be reasonably effective given that nearly two-thirds of Shane's students were successful on the two full-proof tasks analyzed for this study. It is interesting to note that the percentage of Shane's students who *were successful* on both proof items analyzed (about 65%) is very close to the percentage of students from non-Shane classrooms who *were not successful* on either proof task (61%). One limitation of this study, however, is that due to space constraints, we did not report more sophisticated statistical analyses controlling for various factors, and we did not determine statistical significance when taking these factors into account.

In contrast to reports by Cirillo and colleagues (2017), who noted that proof is often taught in a show-and-tell manner, we saw evidence that, in Shane's classroom, students were expected to make claims and provide justifications for their claims. This was evident in the way that Shane continuously asked students questions about what they *believed* to be true during the 21 theorem-proof episodes. Summing together codes from the Pre-Proof and During-Proof activities, it is also noteworthy that, within the data set of 21 theorem-proofs, we identified a total of 19 instances of students drawing conclusions directly from the hypothesis of the theorem. This is important because drawing valid conclusions from the proof assumptions has been identified as an important competency in proving, particularly for *beginning* a chain of reasoning, a skill in which many students struggle (Senk, 1985; Cirillo and Hummer, 2019, 2021). Also, there were 8 instances, in total, of students generating claims through a combination of a postulate and an assumption about the diagram. Cirillo and Hummer (2019) pointed out that making valid assumptions about diagrams is an under-recognized, but important proof competency.

Across the three features Shane incorporated during the Concluding the Proof activity, one important take-away is that these activities often seemed to accomplish what Herbst and Miyakawa (2008) identified as "sanctioning" the theorem, which involves explicitly declaring it as having that label. Shane sanctioned theorems by restating them, establishing shorthand notation for writing them, and acknowledging that they could now be used in future proofs.

Herbst and colleagues (2006, 2009) provided evidence which suggests that teachers heavily control the work of proving in American classrooms. Although, as evidenced by the data, through his question-and-answer exchanges to support students' development of claims, and through the ways Shane scaffolded the Setting up the Proof activities by alternating which competencies students had opportunities to practice while proving any one theorem, Shane did seem to provide students with more opportunities to authentically engage in proving theorems than research suggests is typical. To start, in contrast to the claim made by Herbst and Brach (2006), that students were not expected to prove theorems, Shane *did* expect his students, not only to prove theorems, but to heavily participate in proving them. For numerous theorems, Shane also expected students to participate in sketching their own diagrams and in determining the "Given" and "Prove" statements from the conjecture or the conditional statement. Thus, in contrast to the teachers from Herbst and colleagues' (2009) study, Shane did, in fact, seem to expect his students to carry a good deal of the cognitive load. To be clear, we are not suggesting that we disagree with the norms put forth by Herbst and colleagues. Rather, we mention these norms to demonstrate that Shane's approach seems to be unusual, and, given his students' test results, is worthy of examination. One question that this study raises is related to how effective Shane's teaching approach would be with a non-Honors student population. In other words, would Shane's approach work well for heterogeneous groups of more "typical" students?

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TEACHER FEEDBACK AND AUTHORITY DURING INSTANCES OF STUDENTS' PROVING

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Teacher feedback during the process of students' proving is important to consider because proving is often challenging for students, requiring feedback and support. Additionally, feedback has implications for authority and agency, which are constantly being negotiated. We examined the authority dynamics evidenced in a teacher's feedback actions while students proved geometry claims. By analyzing audio and video recordings of classroom proving discussions, we found that various teacher feedback types position either the teacher or students as authorities with regard to providing and validating mathematical ideas. We provide suggestions for research and practice with respect to teacher feedback and authority in proof instruction.

Keywords: Proving, Authority, Classroom Discourse, Instructional Activities and Practices

Introduction

Mathematics proving has long been considered a central component of secondary students' mathematics (Schoenfeld, 1994; Stylianides et. al., 2017) because, among other reasons, it is useful as a process through which students make sense of and communicate mathematical knowledge (de Villiers, 1999; NCTM, 2000). Despite the significance of proving in mathematics learning, research evidence shows that secondary school students struggle with constructing and understanding proofs (Chazan, 1993; Harel & Sowder, 2007; Shongwe, 2020). This tension highlights the need to examine how teachers can support their students in learning this important yet difficult concept. Scholars have studied some teacher practices like the careful enactment of proof tasks (Bieda, 2010) and teacher moves (Martin et.al., 2005) to support students' learning of proving. The present study builds on the work of these researchers by examining how teacher feedback practices can support students' as they learn to construct a proof.

Teacher feedback is defined as the information provided by a teacher to students about their performance or understanding (Dempsey, et al., 1993; Hattie & Timperley, 2007) and is known to support student learning in a variety of ways. Feedback provides information that learners can use to confirm, reject, or modify prior knowledge (Fyfe et al., 2015), increases student motivation, and acts as a guide for what students should do to make progress towards the learning goals (Hattie & Timperley, 2007). Hattie and Timperely, however, caution that the effectiveness of feedback depends on the type of feedback and the way it is given, adding that feedback by itself may not have the power to initiate student action because students can accept, modify, or reject teachers' feedback.

Beyond academic performance, another way to examine the power of feedback is by focusing on how it promotes or hinders student authority in classroom learning. Powerful classrooms are ones where students provide the ideas that drive classroom discourse (Engle & Conant, 2002). Such classrooms are characterized by students having opportunities to talk elaboratively (Soter et al., 2008) and actively participate in the class by taking up positions of authority (Engle, 2012; Otten et al., 2017). Students' authority and active participation in classroom proving are important because they allow for student ownership of the proving process with students learning to make and justify conjectures (Otten et al., 2020) without relying on the teacher to always tell them what to do. In other words, feedback can be a way for a teacher to support a student and spur them forward or focus their attention without explicitly directing their actions. For these reasons, the present study considers authority in the context of students' proving, with the object of study being the teacher's feedback actions.

Theoretical Framing

Our view of proofs is guided by the work of de Villiers (1999), who viewed proofs as a tool to accomplish several purposes such as verification, explanation, communication, and intellectual challenge. These purposes are not solely for individuals but function within a classroom community in which teachers have a dual duty to represent the discipline of mathematics as a more knowledgeable other while also honoring students as learners of mathematics and attending to their needs (Stylianides, 2007). We connect these ideas to teacher feedback which can be a tangible act through which a teacher responds to students' learning needs while also representing mathematics in an authentic way.

Proving in a classroom community inculcates a discourse, during which patterns of interaction and authority dynamics are continuously established and negotiated (Otten et al., 2017). For example, teachers and students may negotiate what definition(s) and format(s) to use during classroom proving and there may be a negotiation over whether a proof is sufficiently complete or clear. During classroom negotiations, the person or object in authority is the one who takes the lead while others follow (Pace & Hemmings, 2007). Using this definition, we view the person in authority as the one who leads the proving discourse by deciding what ideas are foregrounded and validated.

Teacher's feedback may either maintain the authority of teachers or position students in authority of leading the proving discourse. We investigated the authority dynamics manifested during teacher feedback by examining both the focus and purposes of the feedback. In terms of focus, Hattie and Timperley (2007) outline that teacher feedback may focus on task correctness, students' processing of information and/or students' motivation. Dempsey and colleagues (1993) on the other hand delineate various feedback types that play different purposes (e.g., providing correct answers, probing students to think towards the correct answers, among others) during classroom learning. To this end, we ask two key questions:

RQ1) What feedback types does a teacher provide students during classroom proving?

RQ2) How is authority manifested in the feedback types used by this teacher?

Method

Setting and Participants

Data for this study came from a larger teaching experiment that explored a non-traditional way of introducing proof to secondary students. The teaching experiment involved students using tasks and strategies that attended to the generality and purpose of proofs rather than direct instruction on the specific techniques for constructing arguments (Conner, 2018). The third author was the instructor in this teaching experiment that involved ten students (7 females; 3 males) enrolled in an accelerated 9th grade mathematics course at a rural, public school in the Midwest United States. Prior to the study, these students had not received any high school Geometry or formal proof instruction, although reasoning and justification had been a part of earlier mathematics courses. The teaching experiment consisted of 14 sessions (ap539anderb28–38 minutes each), with students primarily working on tasks in three small groups. This study focuses on Sessions 11-13, when students were engaged in conjecturing and proving claims about specific classes of polygons and whether or not they are all similar (e.g., “all squares are similar”). The instructor used non-traditional instructional practices because she hoped to share authority of proving with students by actively engaging them in both the discourse and practice of reasoning-and-proving and she intended for students to see the purpose of deductive reasoning instead of relying on teacher authority to direct them to use deductive rather than empirical reasoning when constructing proofs.

Data Sources and Analysis

As part of the larger study, all classroom discourse was audio and video recorded with recorders at each of the three groups as well as focused on the full class setting. Students’ written work was also collected. Selected sections of the classroom discourse (i.e., when there were interactions between the teacher and students) were transcribed and coded using MAXQDA software. We coded for oral teacher feedback given during both whole-class and small-group discussions. For this report we are not including instances of student-to-student feedback, although this did occur.

Data analysis comprised three phases of coding. Phase one involved flagging any instance of teacher-to-student discourse that contained (or functioned as) feedback to the students about their performance, understanding, or directing them toward or away from a solution path. We tried to be overly inclusive in this flagging of teacher feedback. In phase two we coded for feedback types based on the works of Dempsey and colleagues (1993) and Hattie and Timperley (2007):

- *Knowledge of results feedback* - –informs learner whether their strategy or answer is correct or incorrect (e.g., “that is a good guess, but it is not correct”).
- *Knowledge of correct response feedback* - –informs the learner what the correct strategy or answer is (e.g., “the angles of an equilateral triangle are all the same size”).
- *Elaborated feedback on correct response* - –gives explanations for why the student’s response is correct and/or directs students to relevant materials or information that could strengthen the response (e.g., We can label the angles of a square with a representation of 90 degrees because the angles are always going to be 90 degrees).
- *Elaborated feedback on incorrect response* - gives explanations for why the student’s' response is incorrect and/or directs them to relevant materials or information that could lead to a correct response (e.g., “what you provided is a conditional statement, but it’s not a conditional statement for squares being similar”).
- *Questioning feedback* - –poses questions to respond to students’ strategy or answer with the questions functioning as an indicator of correctness or incorrectness (e.g., “[Student X] said that isosceles triangles are similar. What do you all think?”).
- *Revoicing feedback* – Uses a restatement of students’ strategy or answer to respond to students with the restatement functioning as an indicator of correctness or incorrectness *(*e.g., “[Student Y] just gave us a counterexample. He said …”).

The *questioning* and *revoicing* feedback codes were not in the prior literature but emerged from our analysis because these teacher moves, in some instances, functioned as a form of feedback to students. Note, however, that not all teacher questions nor all instances of teacher revoicing were necessarily feedback. For example, to start Session 11, the teacher asked students to read the task description then revoiced what the students read. In this context, this question and revoicing did not indicate to the student any information about their performance and was thus not coded.

Finally, in phase three, we coded for mathematical authority dynamics using our definition of authority. We examined the authority at play for each instance of coded feedback by asking the following questions: Who provided the mathematical ideas that were the focus of the discourse? Who critiqued or validated the correctness of the mathematical ideas? And who confirmed or rejected the completeness and correctness of proofs? We considered these questions with respect to the teacher and to students collectively (not individual students). We then considered patterns across instances in terms of the authority for each feedback type. Multiple authors analyzed the data and met regularly to clarify and reconcile coding differences.

Findings

The classroom proving discourse was marked by various feedback types, with questioning feedback being the most common. Table 1 shows the number of instances of each feedback type and the predominant authority figure for each feedback type.

Table 1: Authority Figure(s) and Number of Instances for Various Feedback Types

Feedback Type	Number of Instances	Predominant Authority Figure
Knowledge of results	9	Teacher
Knowledge of correct response	12	Teacher
Elaborated feedback for correct response	9	Student
Elaborated feedback for incorrect response	5	Teacher
Questioning feedback	24	Student
Revoicing feedback	13	Student

Authority dynamics in the feedback were two-fold. There was authority in terms of who provided mathematics ideas that propelled the classroom proving, and authority in terms of who validated the correctness of the proofs. Our findings show that *knowledge of results*, *knowledge of correct response*, and *elaborated feedback for incorrect responses* positioned the teacher as authority whereas *questioning feedback*, *revoicing feedback*, and *elaborated feedback for correct responses* positioned students as authority.

Feedback Types that Position the Teacher as Authority

Knowledge of results feedback positioned the teacher as the mathematical authority with regard to deciding whether students' responses or their steps within the argument they were formulating were mathematically correct. Moreover, through *knowledge of correct response* feedback, the teacher positioned herself in authority of both providing what counted as true responses and the ideas that moved the proving process forward. For example, consider the excerpt below from a whole-class discussion at the start of Session 12 when the teacher was introducing conditional statements.

T: To start off, does anyone remember anything about conditional statements? We talked about them when we were doing the diagrams? No? Okay.

S: They were true sometimes but not all the time, right? They were not for sure things.

T: That would be a good guess, but no. That is not what it is (*chuckles*). A conditional statement is just a statement that is written in a certain form. It is written in the form 'if something, then something. (*Writes the statement if _ then _ on the white board.*)

Here, a student attempted to give the definition of a conditional statement, and the teacher gave *knowledge of results* feedback, "that is not what it is." In this case, the teacher positioned herself in authority of determining that the students' response was incorrect. The teacher then went on to provide *knowledge of correct response* feedback on what the correct response was by saying, "A conditional statement is just a statement that is written in a certain form." Again, the teacher claimed authority in giving the correct response and also providing the definition which would be used later in proving activities.

Later in this class as the students were attempting to use the earlier given definition of conditional statements to prove that 'all squares are similar,' the teacher gave *knowledge of results, knowledge of correct response*, and *elaborated feedback for incorrect response*, again positioning herself as the mathematical authority. See the discussion below:

T: Can anyone take a stab at writing our statement 'all squares are similar' into a conditional if-then kind of format? (*Looking at student S1 specifically*) You want to take a stab?
S1: ..If all the angles on a square are 90 degrees, then they are all the same.
T: So that is a conditional statement, an if-then, but it is not a conditional statement for squares being similar. So, what could we--, what did we assume to be true when we were proving our statement about the squares being similar? What did we start off with?
S2: That they were similar?
T: That is what we are trying to prove. So, we did not start off with that.
S3: That they have four 90 degrees angles?
T: So that is actually a property of squares. So, we just started with, "if we have squares."

When Student 1 gave a guess of how to write a conditional statement for all squares being similar, the teacher used both *knowledge of correct response feedback* and *elaborated feedback for incorrect response* to tell the students why their response was incorrect ("that is a conditional statement ..It it is not..Ir squares..."). Here the teacher positioned herself in authority, determining what a correct conditional statement for all squares being similar would be. The teacher then rephrased the question by asking, "What did we start off with?" Again, when two students gave the responses "they were similar" and "they have four-90 degrees angles," the teacher gave *elaborated feedback*, wherein she did not explicitly tell the student that their responses were incorrect but rather elaborated the incorrectness by explaining that the responses were what needed to be proved or just "a property of squares." Finally, the teacher gave *knowledge of correct response feedback* by stating that "we just started with, 'if we have squares.'" In this way, the teacher assumed authority in validating the correctness of what the class "start[ed] off with" and in providing the correct response.

Feedback Types that Position the Students as Authority

Feedback that was in the form of *questioning*, *revoicing*, and *elaboration of correct responses* tended to position the students in authority of providing correct responses and ideas that moved the classroom proving discussion forward. For instance, see the discourse below when students were attempting to come up with conjectures for similar polygons at the beginning of Session 11:

S1: Is a circle a polygon?

T: Huuh. That was the question over there too. (*Asking the whole class*) Okay. Quick question. Is a circle a polygon?
S2: Yes.
S3: No.
S4: It has no defined sides.

In this case, the teacher avoided taking authority by giving the students the correct answer, and instead used *questioning* and *revoicing feedback* simultaneously by posing the same question to the whole class, thus positioning all students in authority of offering thoughts and potentially deciding as a group what they thought the correct answer to be. Indeed, a whole-class discussion continued until all students seemed to agree that circles are not polygons.

A similar authority dynamic happened in another instance during a small-group discussion in Session 11 when students were attempting to formulate a conjecture about specific triangles that are similar. One student asked the teacher, "All equilateral triangles are similar because they have 90 degrees--, no, which one has a 90-degrees angle?" The teacher responded using both *questioning* and *revoicing feedback* by re-asking the question to the other students in that group, "which one has 90 degrees?" again positioning students in authority of deciding together what could count as correct responses.

The prior examples involved shared authority as students clarified the scope of what they should consider in their proving process. The teacher also used *elaboration feedback on correct responses* in addition to *questioning* and *revoicing feedback* to position the students in authority of providing ideas that would propel the proving discourse forward. For example, see the excerpt below from Session 13 when the whole class was discussing how to label two squares before proving that they are similar.

T: Now, you all had a couple of different ways of correctly labelling this diagram, so we are going to talk about it. But first of all, let's talk about the angles (*the teacher has two unlabeled squares on the board*). What do we know, or how do we label the angles of our square?
S1: Put a box.
T: Put a box. And what does the box represent or tell us?
S2: 90 degrees.
T: 90 degrees. Now why can we label it 90 degrees instead of using like a letter?
S3: It's always going to be 90 degrees.
T: It is always going to be 90 degrees, because we are talking about squares.

In this example, the teacher started by giving a general *elaboration feedback* on students' previous correct work of labeling diagrams and invited them to share what they did. During the discussion, the teacher used both *revoicing feedback* to foreground what the students said and did so in a way that seemed to indicate the students were correct, and the teacher used *questioning feedback* (e.g., what "the box" means in angles and why we label squares using a "box") to solicit ideas from the students. This way, although the teacher was guiding the discussion, she centered students' responses, thus sharing with them the mathematical authority of giving ideas that propelled the classroom proving discourse.

Discussion and Conclusion

This study aimed to determine the teacher feedback types given to students during classroom proving and to examine the authority dynamics of the various feedback types. The teacher gave a variety of feedback types that positioned either the teacher or the students as mathematical authorities. The feedback types that tended to position the teacher in authority were those where the teacher gave verdict on the correctness of students' responses and where the teacher provided the students with the correct responses that moved the proving discourse forward. The feedback types that positioned the students in authority were mostly through questions that probed students to think deeper or questions that invited students to respond to each other's ideas and hence develop a shared notion of how the classroom proving process might continue.

It is our view that the variety of teacher feedback maintained a balance of authority between the teacher and students. This form of balance is worthwhile to consider because although it may be viewed as ideal for students to have more agency, giving entire authority to students is not always feasible, especially when they are learning formal proving for the first time (Otten et al., 2020). There are times when it is rational for the teacher to take up the role of the more knowledgeable other by providing corrective guidance to students. For example, there are incorrect or unproductive ways to turn the conjecture "all squares are similar" into a conditional statement and these could have negative implications for the proof the students were about to construct.

Our study contributes to the ongoing efforts to support teachers in sharing authority with students as a way of enhancing students' meaningful learning of math and active participation (Engle, 2012; Otten et al., 2020). Sharing authority in areas like proving may seem challenging due to the inherently complex nature of teaching and the difficulties students experience with learning proofs (Chazan, 1993; Harel & Sowder, 2007; Shongwe, 2020). We nevertheless encourage teachers to enact feedback practices that invite students to share in the authority of mathematical ideas to encourage students to become more adept at conjecturing, making arguments, and critiquing the arguments of others. Elsewhere, we encourage teachers to share authority gradually and strategically with students (see Otten et al., 2020) in the spirit of maintaining the dual role of teaching in a classroom community (Stylianides, 2007). Our findings in this study show that questioning feedback could be another strategy for sharing proving authority with students.

Finally, using feedback as a lens for viewing questioning is another way to think about the literature on questioning. Questioning has been documented as a teacher practice that is essential in promoting active student participation (Black et al., 2003). When used as teacher feedback, questions can invite students to take up authority of their own learning and that of their peers. Through questions, teachers can assess student thinking but also provide implicit feedback guiding them towards the learning goal. Good questions spur students to confirm or modify their prior thinking, detect errors and correct them without the teacher explicitly telling them the correct answer. Questions also serve to invite students to respond to and critique each other's arguments, thus promoting rich classroom proving discussions. For example, when the teacher re-asked a student's question on whether circles are polygons back to the class, the students held a discussion until they agreed that circles are not polygons. Future research might document the differential outcomes between a teacher providing directive feedback and questioning feedback, with questioning feedback not only having the possibility of promoting shared authority but also being aligned with the kinds of discourse that we hope to be common in the proving process.

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UNDERSTANDING JOINT EXPLORATION: THE EPISTEMIC POSITIONING IN COLLABORATIVE ACTIVITY IN A SECONDARY MATHEMATICS CLASSROOM

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This study examines how joint exploration is established and maintained among students and the teacher in secondary mathematics classrooms. We use the theoretical perspective of positioning to conceptualize joint exploration as involving the negotiation and coordination among participants to position students with epistemic agency and authority. Using a constant comparative method, we use classroom video data of two episodes containing joint exploration and closely analyze the shifts in epistemic positioning within them. We find that shifts in epistemic positioning, especially when students position one another with epistemic authority, help to support continued joint exploration. We also find that the teacher can play an important role in decentering themselves as the epistemic authority.

Keywords: Classroom Discourse, Problem Solving, Instructional Activities and Practices

Mathematics education reform has long called for students to collaboratively engage in the broad array of mathematical practices and reasoning used within the discipline (NGA & CCSSO, 2010; NCTM, 1989). To be authentically engaged in the discipline of mathematics, students should have opportunities to exercise epistemic agency and authority, which focus on their role in taking on the work of knowledge building. Epistemic agency goes beyond the idea of conceptual agency in mathematics, related to developing solution strategies and meaning of concepts (Cobb et al., 2009), by recognizing the roles students play in making decisions about the process by which ideas are constructed (Damsa et al., 2010; Stroupe, 2014). Specifically, this vision involves students making decisions as a part of mathematical inquiry or exploration, such as which mathematical questions and problems are worth pursuing or which approaches to take in investigating them. Within the context of a classroom community, these decisions are often made as joint negotiations between teacher and students (Krist, 2020). As many mathematics classrooms provide little opportunity for students to exercise joint epistemic agency and authority, it is crucial to better understand how teachers and students interact in ways that position students as active participants, particularly during mathematical explorations.

In this paper, we examine *joint exploration* in secondary mathematics classrooms when the teacher is present by analyzing the epistemic positioning of participants. It aims to answer the following research questions: *How are episodes of joint exploration established and maintained in a secondary mathematics classroom? What social and/or epistemic roles do teachers and students play, and how do these roles shift throughout the episodes?*

Exploration as a Form of Mathematical Activity

Exploration, or investigation, is an important aspect of knowledge generation across a wide range of disciplines, with disciplinary distinctions made in the methods for how this exploration is taken up. Exploration is centered around mathematical questions, ideas, or problems that are not sufficiently known and thus represent an intellectual need to be fulfilled (Harel, 2001). As such, exploration involves jointly orienting toward inquiry as a fundamental component of the work (Keifert & Stevens, 2019). Furthermore, processes by which questions, ideas, or problems are framed, clarified, and made investigable are an integral part of mathematical exploration. For example, research has examined students' problem posing in mathematics (Cai et al., 2015; Silver 1994) and the activities involved in problematizing in science learning (Suárez, 2020; Phillips et al., 2018). In addition to activity orienting toward an intellectual need, exploration also involves the investigation and potential closure around whether that intellectual need is fulfilled (Keifert & Stevens, 2019). These activities could include stating what is known and not known, offering suggestions of next steps, monitoring and reporting on the status of the activity to the group, and confirming a solution, among others. Thus, mathematical exploration provides a context for students to actively engage in many types of productive disciplinary work (Engle et al., 2002) for the purpose of fulfilling an intellectual need.

We focus on *joint* exploration, highlighting the instances in which students and the teacher participate collaboratively. This focus helps uncover the interactional and political dynamics involved in establishing and maintaining opportunities for exploration, given that it is not normative for mathematics learning to incorporate it. Thus, we claim that joint exploration likely necessitates re-negotiating the roles and participatory structures of traditional classroom contexts that distribute the social and intellectual authority and agency to students, rather than residing solely with the teacher (Ko & Krist, 2019). To investigate these complex interactions related to epistemic agency and authority, we apply the theoretical lens of positioning.

Positioning Theory as a Theoretical Lens

Positioning theory considers both social and intellectual roles and authority in analyzing interaction. It highlights the interactional nature of activity, which is afforded and constrained by normative possibilities of the authority and responsibilities associated with different roles (Davies & Harré, 1990). From this theoretical lens, participants in activity take up roles or positions, which afford them specific ways of acting and recognition among participants. These positions are flexible and can shift over time. Shifts tend to indicate important moments of activity because they typically involve participants negotiating and coordinating roles.

This theory has been recently used in analysis of discourse in mathematics classrooms in ways that help highlight the role of identity and power in micro-interactions among students and the teacher (Herbel-Eisenmann et al., 2017). For example, research on epistemic positioning (i.e., positions related to knowledge and its creation) in mathematics classrooms has identified two main positions students and the teacher commonly take up during mathematical activity: 1) a *knower* who provides mathematical information and 2) an *actor* who performs an action doing mathematics (DeJarnette & González, 2015; Lo & Ruef, 2020). Within these positions, participants can take on primary or secondary roles depending on whether they provide or request the activity to be completed. In a traditional classroom structure, the teacher is often in a position of authority as a primary knower, viewed by the students as the conduit of disciplinary knowledge. This traditional relationship between teacher and students can be conceived of as an inherent asymmetry in authority over knowledge, as well as bearing the responsibility for controlling the conversation (Mercer & Dawes, 2008).

In particular, we hypothesize that the presence of the teacher could both support and constrain students to be actively involved in joint exploration. The teacher could help facilitate collaborative work, model many disciplinary practices, and position students with epistemic authority to take on intellectual roles in exploration. However, the teacher may also be positioned as a knowledgeable disciplinary expert and authority, such that students could rely on the teacher to fulfill any intellectual need rather than take up exploration. Furthermore, we hypothesize that when a teacher seeks to create opportunities for students to participate in exploration, the inherent asymmetry of the teacher-student dynamic would need to be challenged by positioning students in roles that are associated with intellectual and social authority.

Methods

For this study we analyzed two comparative episodes drawn from a large classroom video dataset (Dyer, 2016). Below we describe the data, episode selection, and analysis of the episodes.

Data and Episode Selection

We use classroom video data from one focal teacher, Mrs. Perry, from a class with grade 10 and 11 students. Mrs. Perry was selected because her lessons contained a large amount of time with students working in small groups and previous research has documented that Mrs. Perry's teaching practice is responsive to student thinking (Dyer & Sherin, 2016). We believed that both of these factors would make it more likely for joint exploration to occur with the teacher present.

Data from Mrs. Perry's classroom included 10 videotaped 100-minute lessons, filmed approximately every week for the final three months of school, which corresponded to about a quarter of her lessons during that time period because her school used block scheduling. Video was collected from three different angles and separate audio was captured for each group of students. We selected two lessons, one from each of the first two months, from different units of instruction on 1) trigonometric functions, and 2) exponential and logarithmic functions.

We selected two episodes of joint exploration, one from each lesson. These episodes were identified by watching video of the two selected lessons and identifying potential instances of joint exploration as a sensitizing concept (Blumer, 1954) in which the teacher was physically present. We defined instances of joint exploration as the collaborative activity of investigating disciplinary questions and ideas among students and/or teachers through interaction. Thus, our analytic criteria specified that instances must have (a) at least two participants contribute substantively, intellectually or socially, to the group's sensemaking activity, and (b) participants seek to construct new knowledge related to a content idea or question to fulfill an intellectual need. We selected two contrasting cases based on the different types of participation from the teacher in each of these episodes as we hypothesized the teachers' participation would have a large influence on how episodes of joint exploration are established and maintained.

Episode Analysis

We used a constant comparative method (Strauss & Corbin, 1990) that involved constructing rich descriptions of the of the epistemic positioning for each of the episodes. We bounded episodes by starting with the teacher approaching the student(s) and ending with the teacher leaving the students to interact with another student group. We created transcripts of each episode, which we used in tandem with the video and audio records in all subsequent analysis. Two authors independently wrote descriptive accounts of the positioning with the aim of highlighting salient aspects of the epistemic positioning that occurred. These descriptions considered: Who is seeking knowledge? Who (or what) is being sought for knowledge? What is an individual's level of certainty about the knowledge? Who confirms the knowledge as certain?

We used the descriptive accounts of positioning to identify portions of the episodes in which the positioning of a participant in interactions shifted. We considered a shift in epistemic positioning to be: (1) an individual changing from expressing certainty to uncertainty (or vice versa) about a piece of information; (2) an individual being newly sought as a source of information or (3) a change in who confirmed information as valid.

Results

We present two contrasting episodes from the same classroom that we identified as instances of joint exploration and report shifts in epistemic positioning that we identified in each episode.

Episode I: Solve a Cosine Equation using a Graph

The first episode involves a group of three students, Ellie, Nick and Theo, and the teacher, Ms. Perry, jointly exploring a task that she provided to the class. This task asked students to solve the equation $-15=20\cos(30x)$ for the portion of the function shown in the provided graph (Figure 1, left), where x is measured in degrees. Using the inverse cosine function to solve the equation yields one solution (*x »* 4.62). Students could then use the graph and the period of the cosine function (12) to find the remaining three solutions (*x »* –4.62, 7.38, 16.62) by locating the x-values of the points of intersection of the function and the line $y = -15$ (Figure 1, right).

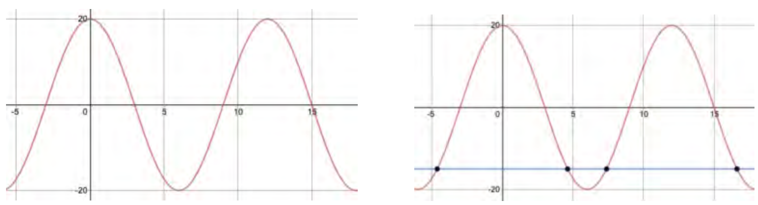

Figure 1: Solve a Cosine Equation Graph Provided (left) and its Solutions (right)

This episode involves joint exploration, as Ellie, Nick, Theo and Mrs. Perry (teacher) work collaboratively to find the remaining solutions. Each participant contributes by posing questions about the task, offering solution strategies, or directing the next steps of the activity. These questions, strategies, and directions are taken up by others to advance the search for remaining solutions. We present three main shifts in epistemic positioning that served to establish and maintain joint exploration below.

Ellie positions Nick as having epistemic authority. This exchange begins the episode of joint exploration. The first shift occurs when Ellie looks to Mrs. Perry to confirm her solution, who is standing up to leave. As Mrs. Perry leaves momentarily, Ellie then turns to Nick to confirm her answer of 4.61.

Ellie: Yes, I don't know if I'm right, is this right? (*moves paper towards Mrs. Perry who stands up to briefly leave group, turns to Nick*) What did you get Nick?

Nick: I got 4.62, 16.62

Ellie: Yesss (*exclaims, holds both fist in the air*). Wait, I just got 4.61. (*Mrs. Perry returns*)

Nick: You're supposed to find all this, just one more. All you have to do is add the period…

Ellie: But how do you make that into like an equation so I can solve it?

Nick: (*Stands up, leaning over Ellie's work across from him*) You don't necessarily, … put it into the equation. What you can do… knowing that this is going to be repeating over and over again … so say you want here… you would add this to 12.

Ellie seeks confirmation for her answer on the task and first looks to Mrs. Perry. When she leaves, Ellie instead turns to Nick to confirm her answer, indicating a shift in Nick's epistemic position. Ellie cheers as Nick gives an equivalent answer to hers, but then questions him when he lists a second answer. Ellie continues to ask Nick questions and he continues to fulfill his role as a source of information, offering an explanation of how to find the other solution.

Mrs. Perry shares epistemic authority. The second shift involves Mrs. Perry sharing epistemic authority with the students in the group by re-positioning herself as one who gives directives without giving mathematical information. By re-positioning herself, she shares her epistemic authority with the students and helps sustain the joint exploration in this exchange. When Mrs. Perry re-enters the conversation, she continues to generally redirect Nick, Theo, and Ellie to the graph, rather than providing a clear next step.

Mrs. Perry: Well, where's the one, what x does that say? (*pointing to Nick's work*)
Nick: 4.61, I got 4.62 because I rounded.
Mrs. Perry: Can you like find that, mark that on the x-axis where that is?
Nick: That would be about, let's assume, here (*writing on paper*)
…
Mrs. Perry: Maybe you should mark that on there, like you were about to, because maybe that will help you to think about, okay, how can I?
Nick: Because I know you can add to the period to get this
Mrs. Perry: Okay so that will get you that one, so you could at least get that one by adding the period and we just have to figure out how to get the other two.

Mrs. Perry's moves to re-position herself to the students include clarifying which x value the students are referring to and directing them to mark that value on the x-axis. Mrs. Perry does affirm Nick's proposed solution of adding the period to the first solution to find a second solution. Then, she summarizes what is left to find, the remaining two solutions, rather than providing more specific epistemic guidance. These moves position the students to continue the exploration independent of Mrs. Perry as the epistemic authority.

Theo and Nick position each other as having epistemic authority. The third shift occurs while Mrs. Perry is still with the group when Theo enters the conversation and begins to work with Nick to find the other solutions. Nick and Theo shift to position each other as sources of information. This shift also maintains the joint exploration. As Nick poses the question of how to find the remaining solutions, in addition to 4.62 and 16.62, Theo offers ideas that Nick takes up.

Nick: How would you solve for that one, though?
Theo: You would add 12
Nick: When you're adding 12, you're just going through an entire period, 6 you're going through half a period
Theo: Yes, which wouldn't work because it's a cosine..I, could you add the three? Possibly?
Nick: A fourth of the period?
Theo: Yeah.
Nick: Let's find out! (*typing into calculator*) No, that's not what I wanted..Ipe!... There we go, that's another one.

When Nick asks, "how would you solve for that one, though?" Theo responds with an answer, although it is restating the strategy Nick was employing, adding one period (12) to the first solution. This continues the joint exploration between Nick and Theo as they search for a way to find the remaining two solutions. Theo proposes a strategy of adding 3, which Nick reframes as "a fourth of the period" and proceeds to test out using a calculator. Nick reports back as he continues to use the calculator and eventually seems to have success. Nick and Theo, together take up the epistemic authority shared with them to find the remaining solutions. Theo's offer of a possible strategy and Nick's uptake of this strategy maintain the exploration of the episode, which ends when Nick finds another solution.

Episode II: "Is there a Natural Logarithm that is equivalent to *e*?"

This episode involves a student, James, posing a problem that he created. James calls for the teacher's'attention while she is at his group and asks whether there is a number whose natural logarithm is equivalent to e (it is $e^e \approx 15.15$, whose natural logarithm is e), which does not appear to be directly from the homework they discussed immediately before.

We consider this episode to be an instance of joint exploration among Mrs. Perry, James and a third student, Sergey, seated beside James. Each participant contributes intellectually (with information) or socially (with directives) to answer the question posed by James. Further, the participants are all not certain of their proposed answers right away. We present three main shifts in epistemic positioning that served to establish and maintain joint exploration below.

Mrs. Perry's initial release of epistemic authority. Mrs. Perry's initial reaction to James' question involves a change in epistemic positioning that serves to establish joint exploration. This shift involved a release of epistemic authority by Mrs. Perry.

James: Is there, I've just been messing around a little bit. Is there a natural log that is equivalent to *e*? (*looking at calculator, then looks up at Mrs. Perry, resting head on hand*) Like 15 point something?

Mrs. Perry: (*slowly*) Is there a natural log that is equivalent to *e*? (*pauses, steps back,*) You mean like, you take the natural log of something and you get *e*? (*looking at James, who nods*) Is that what you mean? (*pauses, puts hand up to mouth and then brings it down*) Ahhh so... (*leans slightly back briefly writing in the air*)

The interaction begins with James positioning Mrs. Perry as an epistemic authority, asking her this question and looking to her for an answer. Notably, James does have an accurate estimate to answer his question when he poses it. Yet, he is not certain and looking to Mrs. Perry as a source of epistemic authority to confirm his hypothesis of the existence of a natural logarithm that is equivalent to *e*. This is followed by Mrs. Perry restating James' question, pausing, and stepping back to lean backward, away from James and Sergey. Mrs. Perry's words and movement seem to suggest that the answer is unknown to Mrs. Perry, in contrast to being positioned with epistemic authority. By pausing to consider the question, Mrs. Perry creates an opportunity for James and Sergey to take up the intellectual authority within the conversation.

James' re-positioning of Sergey and himself as epistemic authorities. As Mrs. Perry shifts away from being an epistemic authority, James and Sergey position each other as epistemic authorities. In response to James' question, Sergey joins the conversation and affirms the existence of such a value, continuing the exploration.

Sergey: I mean, yeah, cuz it'd be a power.
James: of *e*, right? (Mrs. Perry: yeah)
Sergey: Would it just be 1? (*looking at James*)
James: It would be *e* to the power of *e* (*looking at Mrs. Perry*)…
Sergey: So the log, log base *e*
James: Oh, log of *e* to the *e* (*pauses*) is *e*? (*looking at Sergey, laughs*) Wait a second, is that right? Lemme check. (*picks up calculator*)
Mrs. Perry: Well wait, write it down, write it down. I can't think right. I have to see it.

bending down to table, leaning over student work) So,
James: I'm wondering, so I think we just figured out that (*writing*) log base *e*..It would be ln of *e* to the *e*. Okay. So, what's wait. Does that, does that work?
Mrs. Perry: That seems right. Log (*looking at work*)
James: Let's try that so (*typing into calculator*)
Mrs. Perry: Waaait no (*long pause, bends head all the way forward*)

Sergey supports James in exploring his question. He first affirms that such a value would exist, explaining that "it would be a power [of *e*]". Sergey initially offers 1 as a solution, which James quickly disregards. Sergey is undeterred, however, in contributing information. Sergey reminds James that he is interested in the natural logarithm (base *e*), rather than the common logarithm, and reorients James to this in several instances, which James acknowledges and takes up. Notably, James shifts his gaze from Mrs. Perry to Sergey, indicating a shift from positioning Mrs. Perry as an authority to Sergey as supporting his exploration. James begins to talk through his hypothesis that "log of *e* to the *e* is *e*" and uses the calculator to confirm his proposed solution, re-positioning himself as an epistemic authority. Rather than rely on Mrs. Perry for confirmation, James uses a calculator to confirm his solution.

James' and Sergey's epistemic authority begins to equalize. With Mrs. Perry still present, James and Sergey continue to take up the epistemic authority and their authority relative to each other appears to equalize. This can be seen as they reach an answer they are both satisfied with.

James: *e* to the *e* to the 1. Yeah, fifteen point, yeah, there it is. And then log of that is *e*
Sergey: No (James: Just kidding) natural log of that (*looking at James' work*)
James: (*mumbles*) ln. Yeah, my bad.
Sergey: Yeah
James: Yeah

At first, Sergey joins the conversation with lower epistemic authority than James: he offers 1 as a solution, which James disregards. Here, James acknowledges Sergey's contribution about it being the natural log (or log base e) rather than log and readily accepts by saying "yeah, my bad.". Their mutual affirmations of "yeah" at the end of this episode indicate that they are confident in their solution and acknowledge each other's confirmation as well, indicating a similar status of epistemic authority. At the conclusion of the exploration, Sergey and James arrive at a solution they were both satisfied with, without any affirmation from Mrs. Perry.

Comparison of Two Episodes

In both episodes, exploration is initiated by a student asking, or attempting to ask Mrs. Perry a question and request information. In episode one, this is initiated when Ellie asks Mrs. Perry if her work is correct and in episode two, when James poses his question to Mrs. Perry. Mrs. Perry's responses in both, either briefly leaving, or restating the question, are a shift away from her being positioned as the epistemic authority and coincide with the start of joint exploration.

In both episodes, joint exploration is maintained by two shifts in positioning: (1) Mrs. Perry, the teacher, redistributes authority by re-positioning the students as capable of seeking the answer, and (2) the students (Nick and Theo in Episode I and James and Sergey in Episode II) position themselves and each other as epistemic authorities in the situation, as they both exchanged ideas among each other, and took up the ideas offered.

The extent to which Mrs. Perry releases epistemic authority to the group, however, differs between episodes. In episode one, Mrs. Perry supports the students in working through the assigned task by directing Nick to mark his solution at a particular place on the graph and giving the status of the group's work. In doing so, Mrs. Perry shares some of the epistemic authority with the students yet still seems in command of the material as she guides the students toward the next step, implying she holds the solutions to the task. However, in the second episode, Mrs. Perry appears to release her epistemic authority almost entirely after James asks a question to which she does not immediately know the answer. The moment when Mrs. Perry says, "wait no" as she leans her head down, effectively bowing out of the conversation, appears to be when she fully releases her epistemic authority, as James and Sergey are not deterred by her refutation. Rather, they continue toward their solution in spite of her lack of direction or affirmation.

In both episodes, joint exploration ends when a student or students become the ones to determine whether knowledge was correct or incorrect, without confirmation from the teacher. This is particularly notable that students are able to resolve their own uncertainties in the presence of the teacher, yet without relying on the teacher as the source or confirmation of knowledge as valid. Additionally, as students worked to resolve their own uncertainty, students in both episodes utilized calculators as tools to verify their proposed solutions. While the student had to determine what to input into the calculator, and how to interpret the output, the calculator appeared to positioned by students as an authority for validating of knowledge.

Discussion and Conclusion

These episodes are encouraging as they suggest that joint exploration can happen in secondary mathematics classrooms with the teacher present. In fact, our findings suggest that the teacher's actions may be helpful to establish and maintain joint exploration. In both episodes, joint exploration was initiated by students while the teacher was present, with the teacher acting in ways that implicitly decenter herself as the epistemic authority and primary knower. It is not clear that this was the teacher's intended purpose in leaving the group briefly or expressing hesitation in response to James' question. However, it appears to have that result as Ellie redirects her question to Nick. Neither involves explicit or obvious positioning, such as the teacher directing the group investigate a question or asking how the group might figure it out. This suggests that the teacher's role in fostering joint exploration can be subtle. Future research could explore the possibility of whether teacher moves that more obviously and explicitly decenter herself as the authority can also be present when joint exploration is established.

The findings also show the teacher positioning students in more obvious and explicit ways while maintaining joint exploration. For example, the teacher takes stock of what solutions they know, frames the group's work as needing to figure out the other solutions, and restates a student's question with more clarity. These actions do not position students to provide information they already know about the problem in order to answer or solve a question, which would reflect the "knower" position in the literature (Gonz lez & DeJarnette, 2015; Lo & Ruef, 2020). Instead, these findings suggest that an additional position of "explorer" or "investigator" that better characterize the positions subtly implied by the teacher, which are taken up by students during the episodes of joint exploration. Future research could investigate whether coding schemes for positions of participants in groupwork (e.g., González & DeJarnette, 2015; Lo & Ruef, 2020) can be extended to distinguish between knowers, actors, and explorers.

We note that both of these episodes seem to be rich instances of productive disciplinary engagement (Engle & Conant, 2002), including mathematical sensemaking, reasoning, and a variety of mathematical practices. Future research could conduct an analysis of students' engagement with mathematical content and practices, and its relationship to positioning, in episodes of joint exploration. As previous research has done in relation to social positioning in collaborative groupwork in mathematics (e.g., Langer-Osuna, 2016), this line of future research could examine how different mathematical engagement may be afforded to different group members, potentially due to how they are positioned by other members in the group.

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PERSISTENCE OF PLAYING SCHOOL: EXAMINING AN IMMERSIVE 90-DAY SEMESTER-PROGRAM FOR SHAPING STUDENTS' MATHEMATICAL PRACTICES

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In this paper, we report on a study in which we investigated the outcomes of an immersive, 90-day, island-based semester-program that utilizes a place-based curriculum. Using interviews we investigated students' 'endencies to engage in sense-making (drawing on realistic considerations) in the context of story problems. Our findings suggest that such programs may not be enough to support students in unlearning the norms regarding the suspension of sense-making associated with doing story problems in school.

Keywords: Informal Education; Problem Solving; Problem-Based Learning

In this paper, we investigate the outcomes of an immersive, 90-day, island-based semester-program that utilizes a place-based curriculum. Program faculty leveraged the local culture and geography to help students connect their disciplinary learning and the broader world. The program's mathematics classes are focused on equipping students with the fundamentals of statistical and mathematical analysis needed to work problems about various sustainability and research projects of local significance.

Theoretical Framework

While place-based efforts in mathematics education are still emerging (Showalter, 2013), the foundations of such work are not new. Research on *ethnomathematics* (D'Ambrosio, 1985) demonstrates the role of context in student learning—sensitizing scholars to the differences between school mathematics and street mathematics (e.g., Carraher, Carraher, & Schliemann, 1985). Similarly, research on *funds of knowledge* (e.g., Civil, 2007) demonstrates how school mathematics tends to privilege institutionalized forms of knowledge over those drawn from students' lived experiences.

The *suspension of sensemaking* literature (e.g., Silver, 1993; Schoenfeld, 1991) has demonstrated ways that the story problem genre in particular, is woefully inadequate for eliciting students' realistic considerations—demonstrating that students often do not proffer realistic considerations when confronted with story problems. In that work, researchers illustrate children's tendencies to answer a question like, *A captain owns 26 sheep and 10 goats. How old is the captain?*, with nonsensical solutions like 36 (obtained by adding 26 and 10, see Baruk, 1985).

In an effort to resolve this, scholars have investigated the potential of curricular and instructional interventions (e.g., *Realistic Mathematics Education* (RME)—Van den Heuvel-Panhuizen & Drijvers, 2020) for shifting the story problem genre offered in schools—creating more realistic, interesting, and context-driven problems (Gerofsky, 1996). These efforts have demonstrated the potential for educational interventions for improving students' proclivities for mathematical sense- making (Verschaffel & De Corte, 1997).

Semester programs provide a fertile environment in which to test the value of a more fully-operational model of place-based curricula on student sense-making. Thus, this paper seeks to answer the question: What evidence of mathematical sensemaking can be observed in students' responses to story problems during their participation in such a program?

Methods

To investigate this question, we engaged 17 of the 51 students enrolled in the program in a 30-minute interview. Teachers recommended students of varying skill levels for interviews, and parent consent was obtained before proceeding. Students who participated received a $15 gift-card. In these interviews, we asked students to solve three story problems (see Table 1), drawn from the suspension of sense-making literature.

Table 1: Story problems administered in second half of interview

Problem	Rope Problem	Runner Problem	Bruce & Alice Problem
Problem Text	A man wants to have a rope long enough to stretch between two poles 12 meters apart, but only has pieces of rope 1.5 meters long. How many of these would he need to tie together to stretch between the poles?	John's best time to run 100 meters is 17 seconds. How long will it take him to run 1 kilometer?	Bruce and Alice go to the same school. Bruce lives at a distance of 17 km from the school and Alice lives at 8 km. How far do Bruce and Alice live from each other?
Expected Answer	8 pieces	170 seconds	9 km or 25 km
Realistic Answer	More than 8 pieces	More than 170 seconds	Between 9 and 25 km

Three criteria guided the selection of problems: The problem (1) required mathematics we could be reasonably assured that all participants would have prior experience with, (2) resembled the kinds of problems that could appear in high school curricula, and (3) had prior results reported for similarly-aged students, without intervention—useful for informing our expectations regarding the proportion of students likely to demonstrate realistic reactions (see Table 2).

Table 2: Percentage of students demonstrating realistic reactions to the given problems across four studies

	Rope Problem	Runner Problem	Bruce & Alice Problem
Study 1 (n=100)	12%	6%	N/A
Study 2 (n=67)	6%	3%	N/A
Study 3(n=75)	0%	3%	3%
Study 4 (n=45)	2%	7%	2%

Study 1: Greer (1993)—reported on 100 students between the ages of 13 and 14 years old from Northern Ireland; Study 2: Reusser & Stebler (1995)—reported on 67 students between the ages of 10 and 12 from Switzerland; Study 3: Verschaffel et al (1994)—reported on 75 students between the ages of 10 and 11 years old from Belgium Study 4: Yoshida et al (1997)—reported on 45 students in 5th grade from Japan.

We coded students' answers to the story problems (using their written and verbal response) using four of the five categories outlined by Verschaffel et al. (1994): Expected Answer (EA), Technical Error (TE), Realistic Answer (RA), No Answer (NA). The fifth category outlined by Vershaffel et al., Other Answer (OA), did not emerge in our analysis of the data. We also examined the transcript/video containing explanations students provided for indications of sense-making (hesitations, criticizing the problem, qualifiers) and augmented the five categories with a "+" if any such indications were found, and with a "-" if not. For example, in the Runner Problem, an EA- was used to code student responses that simply multiplied 17 by 100 to get 1,700 seconds; while EA+ was used if such responses were accompanied by reasoning like the following:

> But then again, you probably can't run that kilometer because if it's his best time to run 100 meters then you probably can't maintain that time for a full kilometer. But I feel like the number is 170, so I'm going to go with 170.

Results

Table 3 provides an overview of students' reactions to the three problems with the final row providing a summary of all the Realistic Reactions (RR)—combining categories with a "+". The percentage of students with realistic reactions is somewhat underwhelming for the first two problems, given that similar percentages have been reported in other studies of similarly-aged students without intervention (see study 1 from Table 2 conducted with 100 13 to 14 year olds).

Table 3: Number (percentage) of students' (n=17) according to various reactions to the problems

Answer Category	Rope	Runner	Bruce & Alice
EA+	0 (0%)	1 (6%)	8 (47%)
EA-	13 (76%)	13 (76%)	7 (41%)
TE+	0 (0%)	0 (0%)	1 (6%)
TE-	1 (6%)	1 (6%)	0 (0%)
NA+	0 (0%)	0 (0%)	0 (0%)
NA-	0 (0%)	1 (6%)	0 (0%)
RA+	3 (18%)	1 (6%)	1 (6%)
RA-	0 (0%)	0 (0%)	0 (0%)
RR	3 (18%)	2 (12%)	9 (59%)

We are less certain how to make sense of the larger percentage of RR responses in the final problem. One possibility is students' exposure to the triangle inequality in high school geometry gives them routine ways for thinking about the problem. Another is the back-to-back administration of problematic items may have sensitized students to the need to pay attention to context. Yet, we find it perplexing that such a large percentage of students engaged in these kinds of mathematical experiences gave EA+ type responses—unwilling to assert a realistic answer; choosing instead to say things like, "I'm just going to say it's nine kilometers," even after showing some signs of sense- making by asking questions like, "Do they live on the same side of the school?" or "Do they live like on—is this school between them or is this school like, are they both—Like would Bruce walk by Alice's house if he were walking to school?"

Discussion & Conclusions

These results are challenging to interpret. While one interpretation casts doubt on the impact of mathematical activities like those afforded at this semester program on students' propensities to make sense of problem situations, another casts challenges this genres' viability for gauging students' propensity to make sense. This echoes a concern expressed by others (e.g., Gerofsky, 1996, 2010). We suggest a third interpretation: this genre of items may not work well for gauging students' propensity to make sense *in this kind of supplementary program.* This interpretation draws on two premises: (1) the radically different organization of students' mathematical activity in such a program could be seen by students as something wholly different from the kinds of things valued and expected in school mathematics, and (2) the representation of the context in these problems evokes something closer to the norms of school and in this way may be at odds with students' experiences in the program. Premise one suggests that students may view the kinds of mathematical practices developed and used in such a program as not being applicable to the mathematical work expected of them back in their sending school. Premise two suggests to students that the story problem presented to students in the context of this interview "looks" like school, rather than like the work they have been doing in the program, and they respond accordingly.

Authentic problem posing and inquiry often leaves its users with more questions than answers. This is no less true in research as it is in mathematics classrooms. The ease with which students in this semester program reverted to the kinds of persistent, well-rehearsed routines of playing school when primed with a school-like story problem surprised us, which suggests the norms of school may be quite challenging for students to unlearn. We are interested in investigating ways to support students in developing the kinds of awareness and agency that would enable them to both question and challenge norms of schooling that may be unproductive for their learning. Such work would have important implications for supporting students to leverage the knowledge and experiences they gain in supplementary, out-of-school programs.

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UNDERSTANDING GENERALIZATION THROUGH THE LENS OF MATHEMATICAL WORK SPACES

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This study applied the theory of Mathematical Work Space (WMS) to examine learners' mathematical work in generalization activities. Data analysis suggested that the theory of MWS provided a useful tool to examine the quality of learners' mathematical work in the process of generalizing and to identify the obstacles encountered in this process.

Keywords: Advanced Mathematical Thinking; Learning Theory; Problem Solving

Generalization, both as a process and a product, plays an important role in mathematics teaching and learning. It has been argued that making, representing, justifying, and reasoning with generalizations are crucial components of mathematical thinking and should be at the heart of mathematics activity in school (Kaput, 2008). To help learners become more proficient at constructing, justifying, and reasoning with mathematical generalizations, it is important to understand the nature of mathematical work they engage in these processes. Rivera (2013) argued that learners' generalizing activities are sophisticated and influenced by inferential processes, kinds and sources of generalization, types of structures, ways of attending to structures, and modes of representing generalizations. The emphasis on the nature of inferential process and mathematical structures is implied in many categorizations of forms of generalization (e.g., Mason, Burton, and Stacey, 2010). There is also empirical evidence that semiotic representations mobilized in the process of generalizing mediate what is generalized and how generalization is produced, expressed, and validated (Wilkie, 2016). In addition, the study of mathematics learning in dynamic environments has shown that the use of mathematical tools mediates mathematical thinking processes. Therefore, the study of learners' mathematical work in generalization activities has to be framed in a way that simultaneously considers the mobilization of representations, the use of mathematical tools, and the use of chains of inference. Building on Duval's cognitive model of geometric reasoning (Duval,1998) and the theory of instrumental genesis (Artigue, 2002), the theory of Mathematical Work Spaces (MWS; Kuzniak & Rauscher, 2011; Kuzniak, Tanguay, & Elia, 2016) provides such a theoretical lens for examining the mathematical work of individuals when they engage in generalization activities.

Elements in the Theory of Mathematical Work Spaces

Building on the idea that mathematics work depends on the specific mathematical content under consideration and the individual's cognitive activity, the theory of MWS considers both the epistemological and cognitive aspects of mathematical work. The epistemological aspect concerns about the nature of mathematical knowledge and its historical development while the cognitive aspect is about how individuals as cognitive subjects acquire, develop and make use of mathematical knowledge in their practice of the discipline. The theory of MWS argues that mathematical meaning is constructed as a process of bridging the epistemological and cognitive planes through genetic development. The theory describes semiotic genesis, and instrumental genesis, and discursive genesis as the three dimensions of genetic development, which highlights the roles of representation, mathematical tool, and system of reasoning in mathematical work.

The theory of MWS uses three vertical planes to describe the interrelationships among the three geneses. The *Sem-Dis* plane describes the connection between the semiotic genesis and the discursive genesis of proof, which is critical for developing mathematical work that goes beyond the simple iconic perception of a sign. Within the Sem-Dis plane, the semiotic dimension is given priority when visualization and perception dominate mathematical work. In contrast, the focus is on the discursive dimension when a formal proof is deductively carried out with the heuristic help of visual representations. The *Ins-Dis* plane conjoins the instrumental genesis and the discursive genesis. It arises typically when instrument-supported exploration or experimentation is performed regarding one or several well-defined mathematical statements. The instrumental dimension dominates when conclusions are empirically drawn from data given by instruments while supported by inductive reasoning. On the other hand, mathematical work can strongly rely on the discursive dimension but gradually be constructed from steps that are supported by instrumented experimentations and exemplifications. The *Sem-Ins* plane connects the semiotic genesis and the instrumental genesis, which enables the production and transformations of representations for the purpose of shaping the conceptualization and understanding of a particular notion.

Although originally developed to analyze learners' mathematical work in the context of geometry, the theory has recently been used to study teaching and learning in other mathematical content, such as function (Miranda et al., 2016) and mathematical analysis (Delgadillo &Vivier, 2016) and mathematical thinking processes, such as problem solving (Santos-Trigo, et al., 2016), arithmetic and algebraic thinking (Hitt et al., 2016), and proof (Richard et al., 2016). Given its emphasis on the role of representation, tool, and system of reasoning in mathematical work, it is reasonable to ask the following question: To what extent can learners' mathematical work in generalization activities be analyzed and characterized by the theory of Mathematical Work Spaces? This study aimed to answer this question.

Methodology

The data for this study was collected from a series of task-based interviews that were a part of a larger research project aimed to investigate preservice secondary mathematics teachers as learners and teachers of mathematical generalizations. The task-based interview was chosen to obtain knowledge about individual preservice teacher's processes to generalize mathematical ideas and the mathematical knowledge resulting from those processes. Figure 1 is a task used in the interviews. The task included a GeoGebra file that allows its user to change the dimensions of the rectangle by sliding m and n. The participants were eight junior undergraduate preservice secondary mathematics teachers. Four are men and four are women. They were selected based on voluntary participation. Each participant spent about 45-60 minutes solving the above task. Each participant's interactions with the task were screen-recorded

Interior Crossings

In the rectangular grids below, the diagonal touches the interiors of some of the squares in the grid. For example, in the 5×2 grid, the diagonal intersects the interiors of 6 squares. In the 4×6 grid, the diagonal crosses through the interiors of 8 squares. In general, in an $n \times m$ rectangular grid of squares, a diagonal would pass through the interiors of how many squares in the grid?

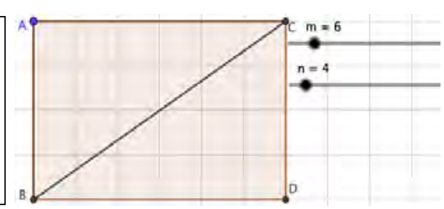


Figure 1. One task used in the interview

All the recordings were transcribed, resulting in annotated transcripts including what was said by the interviewer and a participant, a description of actions taken with the GeoGebra, and screenshots of work. Generalization attempts in each interview were then identified. These generalization attempts served as anchor points to understand what came before a generalization was articulated and what came after it. The mathematical work in these generalizing attempts was analyzed, focusing on its semiotic genesis, instrumental genesis, discursive genesis, and the transitions among the three geneses. A mathematical activity was coded as semiotic genesis when a participant created, manipulated, interpreted, or connected mathematical representations, as instrumental genesis when a participant relied on his/her use of GeoGebra to conduct experimentations, make observations, or confirm conjectures, as discursive genesis when a participant justified a generalization based on property and structure other than perception and empirical data. One activity might be supported by multiple geneses. The obstacles to making and justifying generalizations were also analyzed from the perspective of the three geneses.

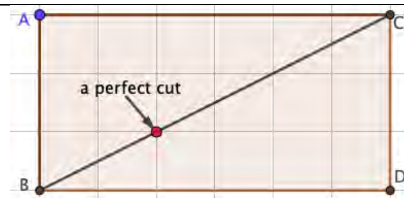
Results

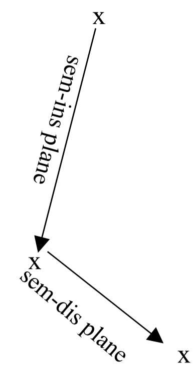
Data analysis has shown that the theory of MWS provided a useful tool to examine the quality of learners' mathematical work in the process of generalizing and to identify the mental blocks encountered in the process of making and justifying generalizations.

The process of generalizing can be triggered by various sources (e.g., visual perception, numerical pattern, analogy, experimentation with technology, and inherent structure) and result in different forms of generalization (e.g., empirical and structural generalizations). The quality of mathematical work produced in the process of generalizing is determined by the extent to which the work can support the construction, refinement, and justification of a generalization. This study has shown that mathematical work that led the participants to construct and justify a generalization often mobilized all three forms of geneses. This suggests that productive generalizing practices often involve purposeful creation and transformation of semiotic representations, skillful use of mathematical tools, and formation of a chain of reasoning. Although the use of dynamic software might trigger an individual to first conduct experimentations with technology or to manipulate dynamically-linked representations, it is important to note that a generalization attempt could start from any one of the three geneses and then move back and forth among the three geneses.

Speaker	What is said and done with GeoGebra	Sem	Ins	Dis
Jordan	When it's a rectangle it has to be at least m, or whichever one is bigger.			x
Intvr.	Okay.		ins-dis plane	
Jordan	(Sliding m and n to create a 6×3 rectangle) so this case is exactly 6, but I think that's probably because it's intersecting perfectly.	x		
Intvr.	What do you mean by intersecting perfectly?		sem-dis plane	

Jordan	Okay, like right in here (pointing to a "perfect cut" in the 6×3 rectangle). When I stretch it out (sliding m and n to create a 7×3 rectangle) it doesn't have those nice cuts. So, the diagonal is going to cut through more squares. Like here it starts with 7 because it has to go across, but then it has to make steps down. So it has to go all the way across and it has to go down two intersections. So would it be like m plus n minus 1? … 1,2,3,4,5,6,7,8,9. So that's nine. So I think in the case where it has no perfect intersections I think it will cut through m plus n minus 1 where m is the greater side.
Intvr.	Why is it m+n-1?
Jordan	So, I'm just thinking in this case it obviously has to go through at least m to get to the other side of the rectangle. But it also has to jump down two. Yeah, it has to jump down two, n-1.





The above short excerpt demonstrates the mobilization of all three forms of geneses and the associated vertical planes in the process of constructing and justifying a structural generalization. In this excerpt Jordan first reasoned that the number of interior crossings is at least the maximum of m and n because the diagonal has to first go across the rows or columns. The GeoGebra-generated diagram in the case of the 6×3 rectangle drew Jordan's attention to the "perfect cut" in the diagram. Jordan used the existence of "perfect cut" to reason why the number of interior crossings was exactly 6 in this case, although he had not yet established a numerical relationship between the number of "perfect cuts" and the number of interior crossings. By sliding m and n, Jordan then created a contrasting case (a 6×3 rectangle) where no "perfect cut" existed and observed that the diagonal had to pass through 7 columns and make 2 steps down. Based on this observation, Jordan made a generalization that when there is no "perfect cut" the number of interior crossings would be $m + (n - 1)$, where m is the number of steps to go across and $(n - 1)$ is the number of steps to go down. More importantly, his justification of the formula was based on a structure in the diagram rather than empirical results. Later, Jordan made use of his idea of "perfect cut" and further generalized that for any rectangular grid the number of interior crossings would be $m + ((n - 1) -$ the number of "perfect intersections"). The to-and-fro movements among the three geneses contributed to Jordan's productive work to generalize.

Obstacles to generalize can arise at different stages of the process of generalizing. These obstacles might include the difficulties to, for example, observe a useful numerical pattern or property, express a generalization with mathematical language, and formally justify a generalization. The study has shown that these obstacles to generalize can be identified and characterized through the lens of the three geneses. Certain blockages arose when one genesis and its associated vertical planes were absent in the process of generalizing. For instance, the absence of a particular way of seeing and manipulating mathematical signs (e.g., numbers, diagrams, and algebraic symbols) might lead to a blockage in a learner's process of generalizing; the lack of a particular technique for using a mathematical tool might hinder the construction of a specific generalization; and the inability to reason with and justify a mathematical idea might result in generalizations that are purely based on numerical patterns.

Discussion and Conclusion

In the past a few decades researchers have categorized generalizations based on different criteria. For instance, based on the status of cognitive schema in a generalization, Harel and Tall

(1991) differentiated between expansive generalization and reconstructive generalization.

fler's (1991) distinction between empirical generalization and theoretical (i.e., operative) generalization was based on the types of abstraction involved in generalization. The differentiation between empirical and structural generalization (Mason et al., 2010) is based on a learner's attentiveness to mathematical structures. The essential role of the three geneses in generalization activities suggests that learners' generalizations might be categorized based on the genesis that dominates the mathematical work in the process of generalizing. High-quality mathematical work in generalization activities requires the mobilization of all three geneses. It often involves purposeful creation and transformation of semiotic representations, skillful use of mathematical tools, and formation of a chain of reasoning.

Obstacles to generalize arise when one or more genesis are absent in generalization activities. In order to support learners to overcome a mental block in the process of generalizing it is important to first analyze the specific type of genesis that is missing and then identify pedagogical actions that are likely to mobilize the genetic development. Expanding the notion of instrumental orchestration (Drijvers et al., 2010), we might use semiotic orchestration and discursive orchestration to characterize teacher's pedagogical actions for guiding students' semiotic genesis and discursive genesis, respectively. Examples of these actions of orchestration include analyzing possible roles of semiotic representations, artifacts, and systems of reasoning, and their arrangements in a task environment, and deciding the schemes and techniques to be developed and established by the students in each genesis to be successful in problem solving.

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RETROSPECTIVE STUDY OF TEACHERS' EXPERIENCES THAT CONTRIBUTE TO THEIR DEVELOPMENT AS MODELERS AND TEACHERS OF MODELING

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Mathematical modeling is new to many teachers, especially in the United States. To complement studies of the effectiveness of professional learning programs, we use retrospective methods to elicit modeling teachers' perceptions of experiences that contributed to their capacities to understand mathematical modeling and to facilitate students' mathematical modeling experiences. Empirical evidence suggests a trajectory along which teacher understanding of mathematical modeling and the teaching of it might develop. Results inform the design of teacher professional learning activities as well as advance inquiry in mathematical modeling teaching and learning.

Keywords: Modeling, Mathematical Knowledge for Teaching

Mathematical modeling (MM) has been an essential element of many careers, the base of many technological advances, and a part of university mathematics curriculum for decades. Not surprisingly, mathematics education practitioners and researchers identify MM as important in K-12 education (e.g., Blum & Ferri, 2009) and elaborate its potential for school and college students (e.g., Garfunkel & Montgomery, 2019). MM also is an object of increased interest world-wide (e.g., Hankeln, 2020). Yet, many K-12 teachers historically have not integrated MM in their instruction (e.g., Schmidt, 2011), and MM has received little attention in teacher preparation programs (e.g., Lingefjärd, 2007). A growing body of literature (e.g., Anhalt & Cortez, 2016; Chapman, 2007; Fulton et al., 2019) offers insights into how teachers learn to model mathematically and to facilitate their students' MM activity. However, design research and studies of the effects of professional learning sessions take substantial time. In this study, we use theoretical and methodological lenses to listen to teachers' voices about what experiences matter and thus both complement existing work and yield immediate insights to inform professional learning design. Using retrospective methods framed by an adult learning theory, we ask: *What are secondary mathematical modeling teachers' perceptions of experiences and events that contributed to their capacities to understand mathematical modeling and to facilitate students' mathematical modeling experiences?*

Background

Literature in mathematical modeling (MM) includes theoretical discussions of what MM is (e.g., Kaiser & Sriraman, 2006) and offers numerous definitions of MM (Cirillo et al., 2016). We take the view that MM "is a process that uses mathematics to represent, analyze, make predictions or otherwise provide insight into real-world phenomena" (Garfunkel & Montgomery, 2019, p. 6). Developing insights into complex real-world phenomena requires knowledge of the phenomena and mathematical knowledge to model the phenomena and gain insights. Throughout this process, students (and teachers) encounter cognitive obstacles (Galbraith & Stillman, 2006).

In addition to deep knowledge of MM, teachers need knowledge about how to successfully facilitate their students' modeling (e.g., Blum & Ferri, 2009; Cai et al., 2014). Despite decades of studies of teacher knowledge and professional development in traditional areas of school mathematics (e.g., algebra, rational number, function, proof) (e.g., Jacobs & Spangler, 2017; Strutchens et al., 2017; Sztajn et al., 2017), a question that has not been answered adequately is what experiences help teachers of MM to develop such knowledge.

A shift from familiar problem solving to MM requires "a new set of teaching and learning skills" (Herget & Torres-Skoumal, 2007, p. 385). Beyond encountering obstacles associated with learning MM, teachers face such perennial challenges as limited teaching time (e.g., Blum & Niss, 1991). Unproductive teacher beliefs about mathematics and modeling (e.g., Gould, 2013; Zbiek, 2016) further challenge teachers' work. Recent studies suggest that completing a short MM module or a teacher preparation course can offer teachers opportunities to experience the MM process and deepen foundational understandings of the process (Anhalt et al., 2018; Cetinkaya et al., 2016; Jung & Newton, 2018) and to develop foundational ideas for teaching MM (Cetinkaya et al., 2016). However, these same researchers acknowledge a need for significantly more knowledge about how to develop experiences that facilitate the development of effective teachers of MM (e.g., Cetinkaya et al., 2016). Our theoretical perspective elaborates teacher knowledge and our methods underpin an alternative way to identify learning experiences.

Theoretical Perspective

Transformative learning theory (Mezirow, 1985, 1991, 2000) provides our theoretical grounding. An adult learning theory based in constructivism, transformative theory involves *meaning schemes* made up of an individual's specific knowledge or beliefs and *meaning perspective* as interconnected webs of assumptions and expectations among clusters of meaning schemes. Transformative learning occurs through *transforming meaning schemes* or *transforming meaning perspectives*. New, revised, or transformed meaning schemes typically arise through an individual's reflection on the content and process of problem solving (Mezirow, 1991). Meaning perspective transformation occurs through an accrual of transformed meaning schemes or in response to a triggered disorienting dilemma that precipitates critical reflection on presuppositions when current problem-solving processes do not provide resolution to the dilemma at hand (Merriam & Caffarella, 1999). Critical reflection involves reflecting on the premises of problem solving, that is, questioning the importance of or the validity and utility of the problem-solving content and process, often while engaging in rational discourse with others (Cranton, 2006; Mezirow, 1985). Conditions conducive to teachers' perspective transformations include dissatisfaction with current practices, occurrence of disorienting dilemmas, critical examination of beliefs, support and freedom to pursue alternatives, support and opportunity to engage in rational discourse, readiness for change, and openness to alternative perspectives (Cranton, 2006; Cuddapah, 2005; Merriam, 2004). Understanding teachers' perceptions of their experiences means exploring their recollections with respect to dilemma triggers and conditions.

Methods

To understand the nature of the conditions around the experiences that teachers perceived as contributing to their evolving understanding of MM and how to facilitate MM with students, we use a phenomenological methodology (Moustakas, 1994; Vagle, 2018). Because participants must have experienced the phenomenon (Moustakas, 1994) of learning to do and teach MM, we recruited teachers who met criteria for experience in doing and teaching MM. They (a) have experience in facilitating MM, (b) are committed to teaching MM, (c) have participated in professional development on MM at the national level, and (d) have served as leaders in MM teaching (e.g., led MM professional development, produced MM curriculum materials). The purposeful sample consisted of five teachers from across the United States.

We employed established retrospective techniques that minimize recall effects (Eisenhower et al., 19911). The teachers completed and submitted event history calendars (EHCs) (Morselli et al., 2019) and critical incident descriptions (CIs) (Eisenhower et al., 1991). EHCs aided teachers in accurately and completely reconstructing past events related to teaching and learning MM; CIs helped them to highlight significant events related to their professional development. Teachers also provided resumés. A first semi-structured interview elicited individuals' feelings about and experiences with the phenomenon (learning and teaching MM) under study (Seidman, 2006). It was conducted via Zoom with each teacher to reconstruct finer details of experiences detailed in EHCs and CIs (Seidman, 2006). A second Zoom interview was held to capture recollections of experiences not mentioned previously and for member checking of our evolving interpretations.

Our analysis of teachers' documents and interviews followed systematic procedures recommended for analysis in phenomenological studies (Vagle, 2018) and used by Peters (2014). Both interviews were videorecorded, transcribed, and annotated prior to analysis. Each researcher began by viewing the first interview videos followed by a line-by-line reading of the transcript while highlighting text and making margin notes. We created chronological listings of experiences that teachers identified as contributing to their capacities to understand MM and to facilitate students' MM experiences, as well as teachers' perceptions of characteristics that helped or hindered their development (Cuddapah, 2005). We sought evidence of transformative learning as well as evidence that transformative learning did not occur. For each teacher, after examining and discussing each researcher's notes and observations for the first interview, we crafted questions for the teacher's second interview. We then followed the same process to analyze the second interviews. We used constant comparison (Glaser & Strauss, 1967) in subsequent readings and discussions to identify emerging themes in the data to help us understand the teachers' experiences in learning of MM and its teaching and how the teachers described the experiences as shaping their current understanding of MM and the teaching of it.

Findings

As the five teachers recounted experiences that they perceived as helpful to their capacity to understand or facilitate MM, they reported similar kinds of experiences (e.g., professional conferences, conversations with colleagues, modeling problems that were of particular interest to them, reading coaching guides for MM competitions). Our findings are not the teachers' or our assessment of their experiences and events but rather an articulation of themes regarding what they experienced as triggers and described as valuable characteristics of their experiences.

Dilemma triggers common to all of the teachers included MM problems that interested them and supportive spaces in which to work on MM problems and to consider teaching MM. Triggers often challenged teachers' sense of what mathematics is, how one does mathematics, and what teaching is. Triggers often arose in engagement with others, in-person or through listening or reading. Importantly, triggers could be found in MM experiences and in experiences that were not explicitly about MM. For example, opportunities to use multiple representations and explore student thinking were noted as triggers for important early experiences.

The triggers connected to one or more conditions for learning. Karen twice participated in a MM competition as a high school student and engaged with teammates in MM problem solving. However, it was her experience in a college MM course that was a trigger to open her to doing MM. The college course presented a problem that interested her (what makes the "best" cookie) and provided a safe space to explore the problem and offer ideas about how to pursue it. She wrote in her EHC, "my world view of what math is was in constant disequilibrium and expanding each day but I had a strong basis to support me." In engaging with science and technology teachers during a summer workshop that required them to share lessons, Dwayne discovered that science teachers did wonderful demonstrations to inspire scientific ideas and formulas but did not pursue the mathematics behind the formulas. This trigger was about "bringing mathematics to life" and consistent with the "conversations" that he has with his students about variables and assumptions about real world phenomena that were at the heart of models in math problems. Teresa was interested in making math more "fun" for her students and readily looked for new problems and activities. Following presentations by other teachers at a professional conference, she examined her beliefs about students and became open to pursuing alternative lesson approaches as she realized she could trust her students' abilities to work with less-structured problems. Viv's dissatisfaction with her own practice as she realized students were offering only weak references to measurement error when they were asked to critique their models and encouragement offered by an expert MM teacher to engage in conversation led her to seek a lunch-time conversation with the expert teacher to explore specifically how to facilitate testing and revising models.

Each trigger was wrapped in opportunity for critical reflection and rational discourse, with readiness for change as a disposition. Opportunities fell short of being transformative when these features were absent. For example, a mandate from Phil's school system to implement problem-based learning (PBL) activities could have encouraged him to engage in MM. It fell short as he was informed by an administrator that his attempt at PBL fell short, with no opportunity for rational discourse. In contrast, when he had the opportunity to engage in conversation with others at a national MM workshop about a food waste problem that resonated with his social justice lens, Phil engaged in MM and shared that he found "a new way of thinking mathematically."

Our data analysis suggests a tentative pattern in experiences contributing to teachers' growth as modelers and teachers as they grow in five knowledge areas: MM, students, social nature of mathematics, curriculum, and contexts. Prior to learning what MM is, teachers have experiences that draw their attention to the usefulness of multiple linked representations and alternative strategies in solving mathematics problems and teachers encourage students to use alternative means to solve problems situated in real-world contexts. Greater engagement in MM and the teaching of MM occurs as teachers present problem statements that are open to different interpretations and prompt a variety of assumptions and variables. Teachers initially engage students in discussions that link aspects of the context with identified mathematical ideas, in the spirit of mathematizing to connect mathematical properties and parameters to situation conditions and assumptions (see Zbiek & Conner, 2006). Teachers become increasingly trusting of students and open to different interpretations of the problem and eventually to different questions and thus to different mathematical approaches for a single real world context. The last and most difficult part—perhaps due to the confines of time, grading, and school curriculum expectations— is the testing and revision of models in the spirit of settling on a model that, though imperfect, is appropriate to answer the real-world question that drove the MM activity.

Concluding Thoughts

This exploratory study reveals triggers and conditions of experiences that teachers perceive as helpful to their MM doing and teaching capacity. Our analysis also yields a potential trajectory along which teachers develop understandings of MM doing and teaching. The results offer insights into how teachers can be better supported in learning and doing MM.

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A CASE OF A STUDENT USING DIAGRAMS WHILE READING A PROOF

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Reading proofs is a significant part of mathematicians' and undergraduate mathematics students' work and is often viewed as challenging for students. Viewing the act of reading proof as a form of problem-solving, we investigated the different ways that a student analyzed and refined his diagram as he made sense of a proof. We found that (a) the student analyzed his diagrams in different and important ways and (b) that his diagrams could be viewed as a series of refinements in which the diagram evolved into a more descriptive representation of the formal proof. We argue that his diagram usage supported his proof comprehension.

Keywords: Reasoning and Proof, Mathematical Representations, Problem Solving, Advanced Mathematical Thinking

Reading proofs is a significant part of mathematicians' practice (Weber & Mejia-Ramos, 2011) and is a key aspect of mathematics students' university studies. However, the literature suggests that the practice is challenging for students (e.g., Selden & Selden, 2003; Weber, 2010). An important part of comprehending proofs is making sense of the mathematical objects introduced and the role they play in the logical argument. Research has shown that students may not understand the complex ways that these objects are used in proof (Lew & Meija-Ramos, 2019) potentially impeding their ability to fully comprehend the proof. We conjectured that supporting students to use diagrams would help with proof comprehension for two reasons. First, it is a productive tool for mathematicians' 'eading activity (Mejia-Ramos et al., 2012), and second, creating such diagrams necessitates attention to mathematical objects. In this paper, we report on a case of a student using diagrams in several distinct ways, answering: In what ways can a student analyze and refine their diagrams as they make sense of a proof?

Theoretical Perspective

Zazkis et al. (1996) introduced the Visualization/Analysis (V/A) model that assumes visualization and analysis work together in problem-solving. The model describes the thinking as beginning with an act of visualization, V_1 (e.g., constructing a diagram) and then is followed by an act of analysis, A_1 (e.g., comparing what is illustrated in the diagram to the given problem). A_1 then leads to a second act of visualization, V_2. As this cyclic process continues, the problem solver gains more understanding of the problem, and ideally, ends with solving the problem. Stylianou (2002) elaborated the model by describing categories of analysis that they observed mathematicians engaging in as they solved problems: (a) inferring additional consequences, (b) mathematical elaboration, (c) imposing a new goal, and (d) monitoring statements. We note that there are important differences in the problem-solving that the V/A model emerged from and the problem-solving that our students are engaged in. For instance, Zazkis et al.'s (1996) model emerged from students making sense of listing the elements of a particular dihedral group and calculating the product of two such elements. We view our students as engaging in problem-solving in the sense that they are investigating the prover's logical argument for a given claim (answering questions like: Why is a particular line true? How does the proof-structure organize a logical argument for the validity of the claim?). We take an individual view on this socially embedded activity in order to make sense of the thinking that the diagrams afforded a student.

Methods

This study is part of a larger project that aims to create inquiry-oriented curriculum materials for Introduction to Proofs courses. Our data comes from a design experiment (Cobb & Gravemeijer, 2008) with a pair of students, Piper and Neal. The primary goal of the experiment was to test and refine proof comprehension tasks to later incorporate into the course curriculum. Two teacher-researchers met with Piper and Neal on Zoom for multiple teaching sessions, each lasting 1.5 hours. During the sessions, the participants and teacher-researchers worked on a collaborative online whiteboard (i.e., Google Jamboards). Each session was recorded, capturing both the students' gestures and their markings on the whiteboard in real-time. A content log was created after each session, including the full transcription and pictures of tasks and students' work. Between each session, the teacher-researchers and another researcher on the project team discussed what happened in the previous teaching session and refined the upcoming session plan.

Statement:
Let $s \in \boldsymbol{R}$ be an upper bound for a sequence $\{x_n\}$. Suppose for every $\varepsilon > 0$, there exists a $k \in N$ such that $s - \varepsilon < x_k$. Then s is the least upper bound of $\{x_n\}$.

Formal version of the proof:
(1) Let s be an upper bound such that for all $\varepsilon > 0$, there exists $k \in N$ satisfying $s - \varepsilon < x_k$.
(2) By way of contradiction assume t is an upper bound of $\{x_n\}$ such that $t < s$.
(3) Let $\varepsilon^* = s - t$.
(4) Then, there exists $k^* \in N$such that $s - \varepsilon^* < x_{k^*}$.
(5) But, this means that $x_{k^*} > s - \varepsilon^* = s - (s - t) = t$ which is a contradiction.
(6) Therefore, s is the least upper bound of $\{x_n\}$.

Figure 1: Statement and proof that students were asked to read.

Data for this study comes from the second session when we noticed that the students leveraged diagrams in seemingly meaningful ways to make sense of the statement and proof given in Figure 1. In this report, we focus on investigating Neal's diagram usage as he read the proof. To begin this analysis, we first re-read the content logs and re-watched the corresponding video, searching for and making sense of acts of visualizations and acts of analysis. Analyzing these acts was an iterative process in which the two authors together identified visualizations, described Neal's analysis, compared and contrasted these descriptions to Stylianou's (2002) categories, and refined working definitions for the type of analysis we observed. We cycled through the data multiple times until our refined definitions captured the agreed-upon understanding of Neal's analyses. See Table 1.

Table 1: Definitions for Acts of Analysis

Act of analysis	Definition	Observable Evidence
Inferring	When the student searched the diagram to extract information about the proof.	Evidenced by the student's attention to a diagram followed by communicating (words or written) new information.

Elaborating	When the student connected how information from the diagram is used in the proof.	Evidenced by the student communicating (words or written) the utility of the information in the proof.
Monitoring	When the student compared the diagram to their current understanding of the proof or the proof itself.	Evidenced by the student referencing both lines in the proof and (parts of) their diagram.
Imposing goal	When the student identified information about the proof to add to their diagram.	Evidenced by the student altering their diagram.

Results

Prior to the following episode, the students interpreted the statement and created a diagram as they explained their thinking. After the students read the proof privately, the teacher-researcher prompted them to explain their thinking. The students expressed that they understood that the proof was by contradiction but were still unpacking the details. In response, the teacher-researcher suggested that they “relate this proof to the pictures that you drew before”. Neal then offered the first visual (see Figure 1-V_1), which was essentially the picture he created when making sense of the statement.

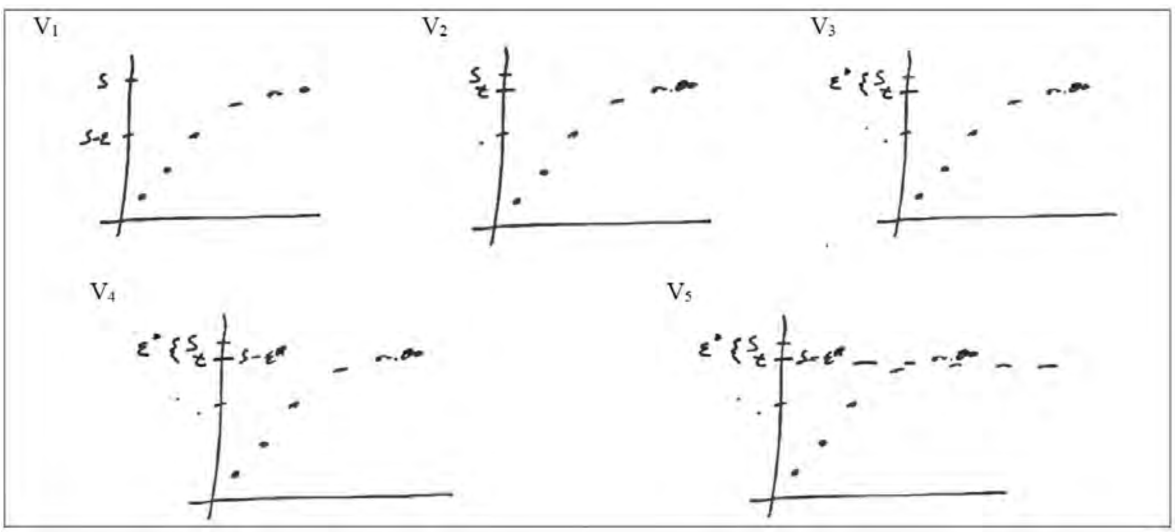

Figure 2. Neal’s Diagrams

Analyzed V_1 by inferring, elaborating, and imposing goal (A_1)

After Neal offered V_1, he said, “So I think I see that, like, I get the conclusion I get that the idea here is that x_k one of these guys [pointing to later terms in the drawn sequence] is greater than t which is our like fake upper bound.” Neal extracted from the diagram that there were sequence terms greater than $s - \varepsilon$ (*inferring*). Then, he explained the later terms were going to be used to show that there was some sequence term x_k greater than a “fake upper bound” t (*elaborating*). Neal identified that a “fake upper bound” t could replace the $s - \varepsilon$ (*imposing* *goal*) creating V_2 (see Figure 2-V_2). V_2 included a key mathematical object of the proof, an upper bound of $\{x_n\}$ that was less than s.

Analyzed V_2 by monitoring and imposing goal (A_2)

Neal then turned to the proof, explaining "…And we're gonna let $\varepsilon^* = s - t$ [referring to line 3]. Which should be this little area [pointing to the distance between s and t in V_2]." Neal compared the third line of the proof to his diagram (*monitoring*) and then identified that he could add ε^* to their diagram (*imposing goal*) creating V_3 (see Figure 2-V_3). V_3 depicted another key mathematical object, ε^*, a particular distance from s.

Analyzed V_3 by monitoring and imposing goal (A_3)

Neal read the first three lines of the proof, pausing after each line to point to the information in his diagram (*monitoring*). Then he read Line 4 multiple times, pointing to ε^* in V_3 and then indicated, "Well, this would be $s - \varepsilon^*$[pointing to t in V_3]" (*imposing goal*) suggesting that ε^* in V_3 did not explicate that $t = s - \varepsilon^*$. He then created V_4 by adding $s - \varepsilon^*$ to his diagram (see Figure 2-V_4). V_4 explicated the relationship between $\varepsilon^*, s,$ and t; if ε^* was the distance between s and t, then $t = s - \varepsilon^*$.

Analyzed V_4 by elaborating and imposing goal (A_4)

Next, Neal attempted to connect how $s - \varepsilon^*$ in his diagram was used in the proof (*elaborating*) by explaining: "And that means that x_k…" then paused for several seconds and added a horizontal line at $t = s - \varepsilon^*$ (*imposing goal*) to create V_5 (see Figure 2-V_5). The presence of the horizontal line in V_5 highlights a relationship between the sequence terms and t.

Analyzed V_5 by elaborating and monitoring (A_5)

Neal tried again to communicate the utility of $s - \varepsilon^*$in his diagram (*elaborating*). To do so, he also compared the diagram to his current understanding of the proof (*monitoring*), explaining:

> So I think the idea here, $s - \varepsilon < x_k$, so that's'right, any ε we choose. [...] Since ε is just greater than zero, we can kind of choose any ε for $s - \varepsilon$. We say t being our fake upper bound and is less than s. The difference there, we can call ε^*. That's'valid for $s - \varepsilon$, we know that there's'some $x_k > s - \varepsilon^*$, which means $x_k > t$ and that contradicts what we've said. Okay, I'm'there.

Here, Neal explained how $s - \varepsilon^*$in V_5 is used in the proof (*elaborating)*: First, he pointed to the assumption in Line 1 that 'for all $\varepsilon > 0$, there exists $k \in N$ satisfying $s - \varepsilon < x_k$' (*monitoring*). Then, he noted how their fake upper bound ($t = s - \varepsilon^*$) gave them an instance that satisfied the necessary conditions saying, "The difference there, we can call ε^*. That's'valid for $s - \varepsilon$." And then explained how $t = s - \varepsilon^*$ lead to the contradiction (*elaborating*). At this point, Neal seemed satisfied with his understanding of the proof ("Okay, I'm there.")

Discussion and Conclusion

In the above episode, Neal built a diagram in multiple steps in which each visualization (V_n) was followed by some analysis of the visualization (A_n). We described the different ways that he analyzed his diagrams as: inferring, elaborating, monitoring, and/or imposing goal. By engaging in this analysis, Neal's initial diagram evolved into a more descriptive representation of the formal proof in that each iteration of the diagram included more information than the previous. We argue that Neal's analysis and following diagram refinements supported his proof comprehension in the sense that it supported his awareness of the mathematical objects and the nuanced ways they were used in the prover's logical argument. Each act of visualization was motivated by Neal gaining awareness of mathematical objects and the potential role they played in the proof and how his diagram could better capture the objects (*imposing goal*). He gained this awareness by comparing mathematical objects introduced in the proof with the objects in his diagram (*monitoring*) and/or making connections to how these mathematical objects were used in the logical argument (*elaborating*). In this way, Neal's diagram usage gives us important insights for how we might support students in using diagrams to comprehend proof.

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PRESERVICE TEACHERS' LEARNING ABOUT ELEMENTARY STUDENTS' MATHEMATICAL REASONING

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We report on a study with 23 preservice teachers (PSTs) preparing to teach grades 1-8 who were engaged in analyzing a series of student-generated arguments for evidence of student mathematical reasoning (MR). We examined PSTs' assessment of student MR prior to and after instruction designed to support PSTs' understanding of how expert-like reasoning might look like in elementary mathematics classrooms. Prior to the intervention, PSTs interpreted and assessed students' MR looking for evidence of isolated reasoning actions (e.g., adapting, exemplifying, representing). After the intervention, rather than assessing student reasoning in terms of the presence or absence of specific reasoning actions, PSTs analyzed the identified reasoning actions on the continuum from less to more expert-like. We discuss the intervention and the specific nature of PSTs' assessment practices before and after class activities.

Keywords: Reasoning and Proof, Assessment, Teacher Education – Preservice

Introduction

Mathematical reasoning (MR), interpreted broadly as "the process of drawing conclusions on the basis of evidence or stated assumptions" (NCTM, 2009, p. 4), is fundamental to doing mathematics. MR involves "developing and communicating arguments" (Loong et al., 2017, p. 6) and takes many forms, from informal explanations and justifications to formal deductions and inductive observations. MR is often described in terms of reasoning actions such as analyzing, exemplifying, generalizing, conjecturing, inferring, adapting, investigating why, explaining, representing, evaluating, or justifying (e.g., Australian Curriculum Assessment and Reporting Authority, 2015; Clarke et al., 2012; Jeannotte & Kieran, 2017; Lannin et al., 2011).

Students show a wide range of mathematical expertise within any mathematics classroom and demonstrate varying levels of readiness for mathematical content. It is likely to expect then that students also demonstrate different levels of reasoning abilities in a typical elementary mathematics classroom. Preservice teachers (PSTs) preparing to teach elementary school mathematics need experiences that can help them make sense of different ways elementary students might reason. To promote and support students' reasoning skills, PSTs need the capacity to attend, understand, analyze, interpret, and assess students' MR. They also need to understand how disciplined expert-like mathematical reasoning can look like in the elementary mathematics classroom. In this paper, with a focus on PSTs preparing to teach elementary grades mathematics, we examine the following research question: *To what extent instructional intervention focused on MR facilitates PSTs' assessment of students' MR?*

Conceptual Framework: Assessing Students' Mathematical Reasoning

We drew on descriptions of students' reasoning skills delineated by the NRICH team at the University of Cambridge (NRICH, 2014). The NRICH team illustrated five levels of mathematical sophistication in elementary students' reasoning skills and described student's reasoning skills on a continuum from less to more expert-like. We used their descriptions to design the Student Reasoning Assessment Tool (SRAT) (see Table 1) to support PSTs' thinking about reasoning actions as PSTs analyze student-generated arguments. Our goal was to bring PSTs' attention to different reasoning actions, specifically justifying and generalizing, and bring their attention to a wide variety of student reasoning skills from novice to expert-like.

Table 1: Student Reasoning Assessment Tool (SRAT)

Levels	Descriptions of elementary students' reasoning levels
L0	The student tells what he or she did
L1	The student attempts to provide some reasoning (not necessarily relevant, complete, or valid) for what he or she did
L2	The student provides a chain of reasoning, which is incomplete, insufficient, or invalid, to support the assertation
L3	The student provides a chain of acceptable valid reasoning in support of the assertion; the argument is at best partial
L4	The student provides an exhaustive acceptable chain of valid reasoning in support of the assertion; the argument can be accepted as proof

Methods

Participants and Study Context

Participants were 23 PSTs enrolled in a semester-long mathematics content course for elementary education majors, *Algebra and Geometry for Teachers*. The course supported PSTs in developing conceptual understanding of mathematical ideas essential to the grades 1-8 mathematics curriculum. Throughout the semester, PSTs were engaged in analyzing, interpreting, and assessing students' MR about fundamental mathematics concepts in grades 1-8 mathematics. Figure 1 gives an overview of the class intervention.

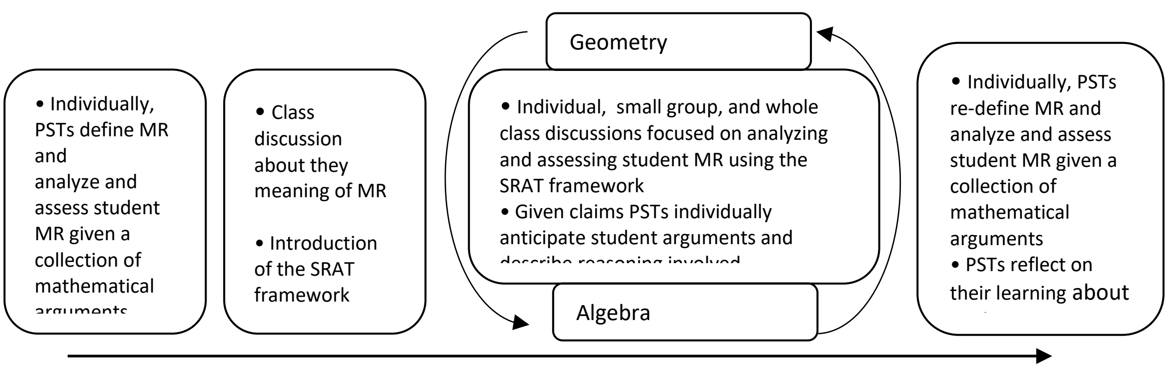


Figure 1: An Overview of the Instructional Sequence

Data and Data Analysis

We analyzed PSTs' written analyses of student arguments for evidence of MR (n = 69) and PSTs' reflections on their learning about MR (n = 23). We coded PSTs' responses for the specific reasoning actions that PSTs recognized in student arguments. The five codes and their descriptions below were derived from the existing literature on students' and teachers' MR (e.g., Clarke et al., 2012; Jeannotte & Kieran, 2017).

- Adapting: Recognizing what mathematical facts or properties students used to develop their arguments

- Justifying: Recognizing whether students validated their statements/claims
- Exemplifying: Recognizing how students used examples to reason about a given claim
- Generalizing: Recognizing whether students reasoned beyond particular cases
- Representing: Recognizing modes of representations students used to express their reasoning

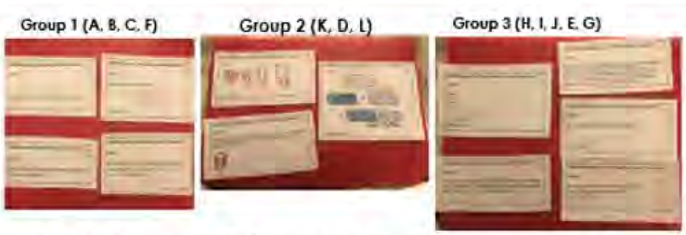
We then looked for patterns within and across PSTs' responses before and after being introduced to the SRAT and examined changes in the nature of PSTs' assessment practices. Finally, we used open coding (Hatch, 2002) to analyze PSTs' reflections and identify the impact of class activities on PSTs' learning about student MR.

Results

The analysis revealed vast differences in PSTs assessment of students' MR with and without the SRAT. Figure 2 includes a summary of PSTs' assessment practices prior to and after the intervention. In Figure 3, we illustrate the discrete emphasis in PSTs' assessment practices prior to class activities, with an excerpt from one of the participants (PST 17). The discrete approach of the assessment of student MR is evident in the provided explanations.

Assessment Foci	
Initial Assessment without SRAT	**Assessment with SRAT**
Assessing MR from less to expert-like (Developmental perspective) (4PSTs, 17%)	Assessing MR from less to expert-like (Developmental perspective) (23 PSTs, 100%) • Explicit attention to generalizing and justifying and delineating between the evidence of no justification, invalid or incomplete justification, valid but incomplete justification, valid and full justification, evidence of no generalizing, an attempt to provide generalization, and establishing generalization. • Delineating the quality of student justifications and generalizations focusing on how students used representations in their justifications, what specific mathematical ideas, properties, or definitions provided the basis for student justifications, and whether examples students used were generic or specific.
Discrete emphasis on the presence or absence of specific reasoning actions (19 PSTs, 83%) • Exemplifying: Paying attention to the use of examples (17 PSTs, 74%) • Representing: Paying attention to the use of mathematical symbols (15 PSTs, 65%) or pictorial representations (13 PSTs, 57%) • Adapting: Paying attention to whether to use similar mathematical ideas, properties, or definitions (8 PSTs, 35%)	

Figure 2: PSTs' Assessment of MR without and with SRAT



Group 1: In all of these examples, the students showed us that adding even numbers together always equals an even number by showing different even numbers added together.

Group 2: In this group, all the students made illustrations to demonstrate why adding even numbers is always even. All of these illustrations show some type of grouping.
Group 3: [In] all of these examples [students] …generated equations to support the argument that adding even numbers always equals an even number. For these equations, you should be able to plug in any even number, and it should work in the sense that the answer will also be an even number.

Figure 3: Example of PSTs' Assessment Practices Before Using the SRAT (PST 17)

At the end of the semester, the same PST recognized changes in the mathematical sophistication of student arguments and differentiated among ways in which students A, B, C, and F used examples to support their claims (yellow codes) (See Figure 4 below). PST 17 recognized that Student F reasoned about the sum of any two-digit numbers more generally by exploring sets of possible one's digits in the sum. PST 17 also noted that Student F's reasoning was not exhaustive to provide a proof-like argument generated by Students E and I.

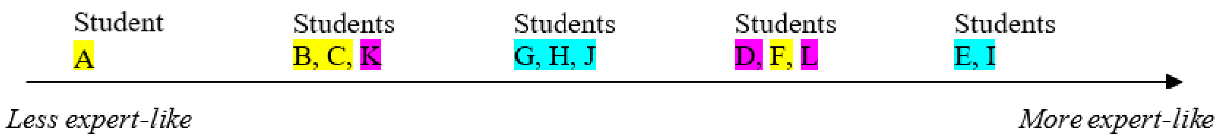


Figure 4: PST 17's Assessment Practices with the SRAT

When reflecting on their learning about MR from the class activities, PSTs commented on gaining awareness about different reasoning actions (11 PSTs, 48%), developing sensitivity for assessing progression in MR rather than taking a "discrete" approach (e.g., correct or incorrect) in their assessment of student MR (10 PSTs, 43%), seeing the benefits of SRAT in assessing their own MR (7 PSTs, 30%), and recognizing the need for instructional focus on MR in the mathematics classroom and their future work with students (5 PSTs, 22%).

Summary and Conclusion

Our work contributes to the research on MR by exploring how engaging PSTs in analyzing student-generated arguments for evidence of MR impacts PSTs' thinking about MR and their assessment practices focused on students' MR. Our results show that exposing PSTs to a broad range of students' mathematical arguments and reasoning increases PSTs' awareness of different reasoning actions. By providing PSTs with the framework for thinking about student MR developmentally (SRAT) we contributed to PSTs' understanding of MR on a continuum from less to more expert-like. At the beginning of the semester, almost all of our PSTs interpreted student reasoning in a discrete way by focusing on the presence or absence of isolated reasoning actions, particularly exemplifying, representing, or adapting actions. Using the SRAT as a guide for analyzing student reasoning, PSTs shifted their assessment practices to focus on justifying and generalizing actions and have begun considering the evidence of student MR on a continuum from less to more expert-like. In addition, our results also revealed that the SRAT and class activities positively affected PSTs' confidence in assessing students' MR, helped them reflect on their own reasoning skills, or develop a vision of their future practice with a focus on MR.

Our study provides important insights for mathematics teacher educators about supporting PSTs' learning about elementary students' MR. PSTs who learn to recognize and assess student reasoning actions along a progression from less to more expert-like can be more effective in helping their students become more sophisticated mathematical thinkers. Loong and colleagues (2013) advocated that teachers need a strong understanding of reasoning actions to be effective in promoting MR in their mathematics classrooms. In our study, the SRAT framework provided a scaffold for PSTs' learning about and assessment of student MR. This framework needs to be tested in future research with more diverse groups of PSTs and contexts.

Acknowledgments

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THE EMERGENCE OF STUDENTS' COLLECTIVE PROVING ACTIVITY

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While working with peers is seen as valuable for mathematical learning, this practice is understudied with undergraduate students in proof-based courses. I investigated how collective activity emerged among three students working on a proving task. Results show that as part of their collective activity, the group had to collectively create and coordinate multiple aspects of proof. I argue that this coordination was a critical component to their ability to move forward in their collective proving activity.

Keywords: Advanced Mathematical Thinking, Classroom Discourse, Reasoning and Proof

The claim that mathematical learning can be supported by working with peers has been repeatedly backed by literature (e.g., Francisco, 2013; Stahl, 2006). In the case of proof-based courses, research has also pointed to the benefits of engaging in social interactions during proving activity (Balacheff, 1991). However, literature specifically focused on students' collective activity in proof-based courses is slim (Ottinger, 2019). By studying how students work together to engage in proving activities, we can gain insights into how to support their learning in the classroom. The purpose of this study is to provide a detailed analysis of students' collective activity when working in groups on proving tasks. Specifically, it aims to answer the question: How does collective proving activity emerge among students working with peers in an Introduction-to-Proof course?

Theoretical Perspective

This study draws on the work of Martin and colleagues' (Martin et al., 2006; Martin & Towers, 2015) collective mathematical understanding framework. The framework is based on improvisational theories and focuses on the dynamical process of in-the-moment collective mathematical activity. Taking a sociocultural perspective, the emergent mathematical activity is how Martin and colleagues view collective mathematical understanding as developing. Martin and Tower (2015) identified three key types of collective mathematical activity that groups engage in as part of developing this understanding. These are: *Collective Image Making (CIM)*, *Collective Image Having (CIH)*, and *Collective Property Noticing (CPN).* The term image is used to represent any ideas that the learners might have about the specific topic at hand. These are considered to be primarily mental representations and it is the sense-making that happens surrounding these representations that constitutes the image (Martin & Towers, 2009). Students' progression through these stages is not meant to be linear but instead is comprised of a forward and backwards progress. When working at an outer stage, a group might encounter a problem that signals to the limits of their current image. When their image is challenged, students may *fold back* to an inner stage taking with them their newfound knowledge from the outer stage (Martin, 2008). I view students' proving activity as a form of problem-solving (Weber, 2005) and for this study, I use the stages of this framework, along with the concept of folding back, to frame the students' problem-solving process that makes up their collective proving activity.

Methods

This study was part of a larger, ongoing project with the goal of developing a modular inquiry-oriented Introduction-to-Proof curriculum and instructor support materials. Data for this study comes from a whole-class implementation of these curriculum materials in a university Introduction-to-Proof course. Content for the course was split between a group theory and real analysis context throughout the term. This course was offered remotely with synchronous meetings using video conferencing technology (i.e., Zoom). In line with inquiry-oriented instruction, the curriculum is designed to regularly engage students in small group work and as such, was rich in opportunities to study students' collective proving activity. Zoom breakout rooms were used to structure small group work where students used collaborative technology (e.g., Google Docs) to engaged in collective activity. Every time students worked in small groups I entered a breakout room with one group acting as an observer. Occasionally students directed questions to me, at those times I would take on an instructor role. Screen recordings were used to capture activity on Zoom and any collaborative technology that was being used in real time.

I used Powell et al.'s (2003) methodology for capturing students' mathematical reasoning to structure my analysis. First, I identified *critical episodes* that would undergo further analysis. These were episodes that featured groups working particularly well together, for example, by discussing and debating ideas often. For each critical episode, I developed multimodal transcripts (Hoffman, 2019) to capture all student interactions as accurately as possible. This study focuses on one critical episode in which three students; Justin, Abigail, and Alison worked together to collectively write a proof of what the class called the Sudoku Property of group Cayley tables. The Sudoku Property referred to the fact that for each row and column of a group's Cayley table, every group element appears exactly once (i.e., it exists and is unique). The results presented here focuses on the students' work in regards to the uniqueness part of the proof. I analyzed this episode by re-watching the video data and reading transcripts to identify the students' activity in relation to each stage of Martin and Tower's (2015) framework. This was an iterative process in which I first went back and forth between the data and framework to identify and develop descriptions of each stage of the framework. Through this process I found evidence of CIH, CIM, and folding back. With these descriptions (see Table 1), I coded segments of the transcript and interpreted these segments according to the surrounding context. I used these coded segments and interpretations to create an illustrative narrative of the students' emerging collective activity.

Table 1: Descriptions of Each Stage of Collective Activity

Collective Activity Stage	Description
Collective Image Making	When students offer up and/or discuss (but not yet take up) ideas for how to approach different parts of the proof.
Collective Image Having	When students agree on a direction and work on writing out a proof or give instructions to a peer for what to write in a way that is not challenged.
Folding Back	When an intervention is introduced that makes students' question and reexamine their approach/current train of thought.

Preliminary Results

After the instructor introduced the Sudoku Property and outlined two goals of the proof, the instructor sent the students into breakout rooms to work on the proof. The three students in this episode started working on the first part, proving uniqueness, with Alison offering a strategy:

Alison: Okay. So, we're like, kind of doing a proof by contradiction.
Abigail: (reading from the worksheet) Part One, show that each element appears at most once in each row. Suppose symmetry B showed up twice.
Alison: So, if symmetry B showed up twice, and we would have like Q being another symmetry. And I don't know, W being another symmetry, which means Q and W would have to equal the same thing. (As she speaks, she writes AQ=B and AW=B on the Google Doc.) They would have to be equivalent. Is that what we're getting at?

Abigail and Justin then agreed with this approach. Here the group began making an image of how to approach and structure the proof (*CIM*). Alison proposed a contradiction argument that assumed two symmetries Q and W composed with A would both result in the same symmetry B. With the lack of rebuttals regarding using a contradiction argument, the students at this point collectively had an image of their proof that centered on how it should be structured (i.e., by contradiction) (*CIH*).

While attempting to forward collectively to carry out their image of the contradiction proof, Justin introduced a question motivating the students to *fold back* to *CIM*.

Justin: What would the explanation part of it be. How does this prove that it's at most once?
Abigail: Well, well put- Because? Because, um, if? Well, because you wouldn't. So, on the top here (referring to the Cayley table), you only have like, each one shows up one time. So, if Q and W are identical, which they would be in this case, then-
Alison: there would be redundancy?
Abigail: Well, yeah, they would- that just wouldn't happen. Like they wouldn't show up more than once.
Alison: We just need another way to say all that.
Justin: Yeah, like I'm saying, I think this is right, is just with proofs we have to make sure to get that across. I just want to try to make sure that we can do cause I think it makes sense, is that we're supposed to- […], when you multiply the two different symmetries, that they're supposed to be unique symmetries? Right? I don't remember if that's something we explicitly said. So, I don't know if I can reference that. So, I'm trying to figure it out.

In order to complete the proof, Justin suggests that they need an "explanation part" which would outline why their argument showed that there is at most one solution. The group's inability to answer this question forced them to *fold back* to a *CIM* stage. At this stage, the students' CIM was focused on a different aspect of the proof, they were working to develop an understanding of *why* their proof worked (why they could assume distinct symmetries) rather than reexamining their understanding of how to approach the proof (i.e., by contradiction).

The episode continued with the members discussing the properties of Cayley tables, hoping these properties could provide insights to why their proof worked (*CIM*). Alison intermediately attempted to move their collective image making forward by connecting it to their task of explicitly writing the explanation that Justin originally suggested they needed. For example, consider the following exchange:

Alison: We have to find a way to say that there's a finite amount of symmetries. There are a finite amount of symmetries, right? There's just a bunch of different actions you can take to get each one?
Justin: A finite amount of non-redundant symmetries, I think.
Alison: Then let's say something like that, just to get that out of the way. Know what I mean?
Justin: Does it help us in any way?

The group continued to struggle to move forward to a *CIH* stage of their proof given that they had yet fully come to an understanding of why their approach works. The students eventually put this conversation on pause after the instructor added a comment on their Google Doc suggesting they justify each step in the proof they already had written down.

After addressing the instructor's comments, Abigail and Alison suggested moving on to the second part of the proof (i.e., proving existence). This indicates that they were satisfied with their proof and assumed that the group had completed a collective image of their proof. Justin on the other hand, did not see their proof as complete, stating "We didn't finish the first one is the only thing too." Justin suggested needing a statement for why their argument works to which Alison agreed stating "Yeah, like a conclusion." Justin's statement initiated the group to *fold back* to a *CIM* stage briefly to develop an image of what their final proof should include. With Alison's confirmation, and a lack of rebuttal from Abigail, Justin offered to complete that task (*CIH*).

Almost immediately upon entering the *CIH* stage, Justin encountered a problem which acts as a catalysis for the group to *fold back* again when he asks the "Do we want to agree on that there's supposed to be only unique- Each of the ones we're multiplying by have to be unique symmetries, meaning that Q and W can't be the same?". This comment brought the group back to the problem of why their argument worked (why they can assume distinct elements). The group again failed to move forward with their *CIH* due to not forming an understanding of one aspect of their proof. At this point, they decided to reach out to me and acting as a teacher, I informed them that they could assume the elements in their table were unique from each other. With this assurance, the students quickly checked in with one another and seemed to all agree on why their proof worked, marking a moment when they collectively had an image of their proof (*CIH*). At this point the students were able to successfully complete the problem.

Discussion and Conclusion

As part of their problem-solving process, the students in the above episode worked together to create an image of their uniqueness proof by developing a collective understanding of two related ideas: 1) what the proof should look like and how it should be structured (i.e., how to write the formal proof) and 2) the concepts underlying the proof (i.e., why their argument was valid). Moreover, even though the group was able to progress to the *CIH* stage of the proof by using the second idea, their inability to develop a collective understanding of why their argument worked (the first idea), often caused them to *fold back* to the *CIM* stage. It was not until the students developed a collective understanding of why their argument worked in relation to how they structured the proof, that they were able to successfully move forward with a collective image and complete the task. Thus, the preliminary results presented here suggest that to make progress in their collective proving activity, students need to 1) make explicit why and how they can prove their claim and 2) be able to coordinate those two aspects of their proof. The students in this episode were able to support one another in making progress to develop a rich image of the proof that they might not have gotten on their own. It is possible that without this activity the students might have instead struggled to work together. Further analysis will explore if a similar phenomenon occurs with other groups and with different proving activities.

Acknowledgments

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CONJECTURE OF THEOREM STATEMENTS AND THEIR PROOFS BY ANALOGY: THE CASE OF ANDREW

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Construction of proofs by analogy with previously constructed proofs is a ubiquitous strategy in abstract algebra due to the existence of several structural similarities between algebraic structures. Examples found in textbooks suggest that such proofs by analogy are straightforward. However, it is unclear how students make sense of these proofs by analogy. This preliminary report explores one student's process of producing conjectures for theorem statements and their respective proofs in ring theory by analogy with what they knew from group theory. In particular, I analyze the student's analogical reasoning with the use of the ARM framework (Hicks, 2020) to make inferences about their analogical proof activity. Results indicate that Andrew reasoned more productively when spontaneously reasoning by analogy, while he lacked confidence in analogous statements and proofs that were prompted by the interviewer.

Keywords: Advanced mathematical thinking, reasoning and proof, undergraduate education

Proof construction and comprehension is essential to undergraduate mathematics (Mejia-Ramos, Fuller, Weber, Rhoads & Samkoff, 2012). One proof approach is to generate a proof by analogy with a previously proven statement. Examples of proof by analogy are ubiquitous in abstract algebra due to the existence of several commonalities between the structures of group and ring theory by way of their historic development (Hausberger, 2018). For instance, consider the following quote from Gallian (2010): "The next three theorems parallel results we had for groups. The proofs are nearly identical to their group theory counterparts and are left as exercises" (p. 283). An underlying assumption is made in this quote: because the relevant analogous proofs are found in group theory, they are meant to be straightforward and do not require a proof written in the book. However, of the three theorems being referred to in this quote, one is the first homomorphism theorem for rings, a theorem that is hardly considered trivial in an introductory course in abstract algebra.

It is unclear to what extent students appreciate the apparent simplicity in suggesting that a theorem about a new context is obvious by analogy with a previously known theorem. Matters may become especially complicated when considering the potential lack of coordination between what students, their teacher, and the discipline may accept as an appropriate establishment of new knowledge (Solomon, 2006). In order to better understand students' analogical proof construction and comprehension, this preliminary paper seeks to answer the overarching question: *How do students leverage theorems and proofs in group theory to conjecture theorems and construct proofs by analogy in ring theory?*

Theoretical Framing

I utilize the Analogical Reasoning in Mathematics (ARM) framework (Hicks, 2020) as a foundation for describing analogical reasoning in this paper. In particular, ARM describes analogical reasoning as a process of comparing similarity and difference between two domains: collections of knowledge (of varying size and scope) of mathematical content. For example, one could reason about the domain of group alone, or about the domain of group theory taken as a whole. Analogies are formed by mapping content from a source domain to a target domain. The source is typically the domain that is known to the individual, while the target is the domain to be understood. Analogical reasoning is operationalized as: (1) mapping and non-mapping activity involving the source and target domains; (2) attention to similarity and difference between the source and target; and (3) the foregrounding of a domain during reasoning. In addition, ARM also outlines several specific analogical activities characterized with the three dimensions described above and suggests a categorization for the different types of mathematical content to be attended to during analogical reasoning. I outline these activities in the methods.

Consistent with ARM, I interpret student mathematical activity through the lens of the Actor Oriented (AO) perspective (Lobato, 2012). Several existing frameworks make assumptions of what an appropriate analogy entails (e.g., subrings in ring theory are the appropriate analogy for subgroups in group theory, and anything else is false). The adoption of the AO perspective in this study allows for an examination of student analogical reasoning that may or may not adhere to the established or accepted analogies one might expect.

Methods

The data in this study was collected as part of a larger study investigating how students might reason by analogy in the context of abstract algebra between structures in group theory and ring theory. In the larger study, five 60-90-minute-long interviews were conducted with each of four students who had previously taken a course in group theory: three undergraduates mathematics majors, and one graduate student in mathematics education. In this preliminary study, I explore the activity of the graduate student, Andrew (a pseudonym), as he conjectured about three theorems in ring theory by analogy with the following theorems in group theory:

A) The subgroup test.
B) Suppose phi is a group homomorphism from a group G to a group H. Then the image of a subgroup of G is a subgroup of H.
C) Given a group G, the set of cosets {gH | g is an element of G} is a group under the operation aH * bH = abH if and only if H is a normal subgroup of G.

The given task differed based on the focal statement. For Theorem A, Andrew was only asked to develop a structure in ring theory analogous to subgroups and attention to the subgroup test itself was a spontaneous development by Andrew. During this task, Andrew was given full freedom to generate an analogous theorem and proof with no intervention from myself. For Theorems B and C, Andrew was first provided the theorem itself and asked to generate an analogous theorem in ring theory. After formulating his conjecture, he was then given a proof of the theorem in group theory, asked to analyze the proof in group theory, and then consider how to prove his conjectured statement in ring theory.

Transcripts were produced for each of Andrew's interviews and the sections in which Andrew developed ring theoretic analogies with the three above statements were analyzed in this study. In particular, the relevant sections of transcript were segmented by attending to shifts in analogical activity and mathematical focus. To each segment, the Analogical Reasoning in Mathematics (ARM) framework was used to assign codes describing the analogical activity as well as the mathematical content central to the analogical activity. These codes can be found in Table 1. I present the results of this process in the next section.

Table 1. Analogical Activity Codes

Code	*Description*
Recalling	Recalling or remembering content about one domain, usually the source.
Distinguishing	Identifying differences between the source and target domain.
Associating	Identifying similarities between the source and target domain.
Exporting	Mapping (almost) exact content from the source to the target; often associated with assuming that domains are completely similar with respect to some content.
Importing	Purposefully selecting content from the source to map to the target domain; discriminately forming similarities rather than assuming content is similar.
Extending	Viewing one structure as being grounded within another and establishing the new structure by "decorating" the old.
Adapting	Making changes to content to account for differences found between domains.
Elaborating	Expanding on what is known about a domain, usually the target.

Results

Andrew's Conjecturing of Theorems by Analogy

Andrew's approach to conjecturing theorem statements by analogy varied depending on the theorem. For Theorem A, Andrew spontaneously recalled the subgroup test and considered what a subring test would entail on his own. In particular, Andrew had previously distinguished the structure and structural properties of group and ring (by noting that rings have two operations, and that multiplicative inverses need not exist) and leveraged these differences to make an adaptation to create the subring test: unlike subgroups, subrings required the property that the set was closed under the multiplicative operation.

While Andrew spontaneously produced the analogous theorem statement of Theorem A, he was provided the theorem statements for Theorems B and C and then asked to produce the analogical statement. In contrast to Theorem A, Andrew made conjectures for analogies to Theorems B and C in ring theory that closely resembled their counterparts in group theory. This is strongly evidenced by the following quote where Andrew conjectured a theorem statement in ring theory analogous to Theorem B above:

> I'm gonna let R and S be rings and let phi be a ring homomorphism. I'm literally just replacing the words. I mean, this is my best guess then phi of R is a subring… Right, yeah. Replace group with ring. Replace homomorphism with ring homomorphism and replace subgroup with subring.

Thus, Andrew's approach to generating an analogous statement for Theorem B was to replace the names of structures in group theory with their relevant analogues in ring theory. This same strategy was employed for producing an analogous statement for Theorem C, although Andrew briefly alluded to what might be different: "Yeah, right now I'm just replacing group with ring, and normal subgroup with normal subring. I am worried about dealing with the two operations, though." (I note here that Andrew was unaware of the definition of ideal when engaging with this task. Thus, the appearance of "normal subring" does not indicate that he was ignoring ideals as a suitable replacement for normal subgroups.) Unlike the subring test, Andrew did not consider the need to attend to any differences when formulating these theorems.

Andrew's Consideration of Proofs by Analogy

In addition to making conjectures about analogical theorems in ring theory for Theorems B and C, Andrew was also given proofs of the group-theoretic statements to analyze before considering the proof of his conjectured statement for rings. Andrew directly leveraged these provided proofs to organize his thinking of a potential proof for his analogically conjectured theorems. Consider his thinking about proving his conjectured analogue to Theorem B: "So really I think you can take this proof and just add a last little paragraph proving that phi of R is closed under multiplication. That would prove this theorem." Thus, Andrew considered the proof of his analogous theorem in ring theory as being an extension of the proof provided to him. While Andrew considered what might change, he claimed that there were only menial differences or changes to be made (e.g., "But I think you can scrap the first line..It's one of those things where it's not necessary, but it's not useless.") and doubled down on his belief of the analogous proof being an extension of the given proof:

> What part would I change? Maybe I would take the, the second paragraph and say next we check that phi of G is closed under each group operation and do each proof there. But yeah, I mean, it's really just an extension of this proof. This could be like a lemma for it.

Despite spontaneously establishing the subring test by analogy with the subgroup test, Andrew never presented a full proof of his conjecture; instead, creating the test by analogy appeared to be sufficient. Similarly, Andrew never went on to construct full proofs for the analogues to Theorems B and C either and once again relied on the analogy. However, Andrew was less confident in the analogues to Theorems B and C and their potential proofs:

> Well rings can do weird stuff… So there's a chance that I'm not considering something. Yeah, I feel like this is right, but I wouldn't be that surprised if… this actually isn't right.

Thus, Andrew maintained an awareness that his analogically conjectured statements and his strategy for constructing their proofs through analogy were perhaps insufficient or flawed.

Discussion

From these examples, it would appear that Andrew was more willing to create a meaningful (to him) analogous statement for the theorem that he spontaneously generated: he recalled the subgroup test without prompting, and the subring test he produced was a result of an adaptation to account for differences he had previously identified himself. In contrast, Andrew simply exported the theorem statements he was given in group theory in order to generate his conjecture for theorems in ring theory. Thus, it may be more productive to provide students an opportunity to spontaneously reason by analogy when possible.

In addition, Andrew appeared to be more confident in the validity of the subring test which he spontaneously produced. By contrast, when Andrew exported the statements of the theorems that were provided to him to formulate analogues to Theorems B and C in ring theory, he was not quite as confident in the viability of the analogous statements or their proofs. These preliminary results indicate that there is perhaps a disconnect between the apparent simplicity of formulating analogous proofs suggested in mathematics textbooks, and what students themselves view as being simple and straightforward analogies.

Further research can greatly refine these preliminary results on how students might productively conjecture theorem statements and construct proofs by analogy by observing the analogical proof activity of several students on a variety of tasks. Specifically, further research is needed to determine the balance between: (a) when and how it is productive for students to attempt spontaneous production of theorems and proofs by analogy, and (b) when and how students can productively leverage pre-existing statements and proofs to reason by analogy.

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MIDDLE SCHOOLERS' USE OF REPRESENTATIONS IN PROBLEM-SOLVING AROUND SLOPE

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In this study, I examine how the construction and use of representations might relate to the learning process in a problem-solving task involving slope. I report on one task-based interview with an eighth-grade student who had not yet engaged in formal study of slope to explore this relationship. Specifically, the construction of a graph coincided with a shift in the student's problem-solving approach, which I operationalize as the student's use of a representation. I discuss the implications of these moments in generating opportunities to learn.

Keywords: Algebra and Algebraic Thinking, Cognition, Mathematical Representations, Middle School Education

Representational activity is an important part of making sense of mathematical ideas and solving problems, particularly in the context of *slope*, or constant rate of change, in linear relationships. As learners work to conceptualize slope, meaningful construction and interpretation of graphs and tables, in particular, have been viewed as intertwined with robust conceptual understanding (Adu-Gyamfi & Bossé, 2014; Ellis et al., 2018; Peck, 2020; Zaslavsky et al., 2002). This study seeks to explore how students' representation construction might be leveraged in problem-solving involving slope. In this paper, I will address the following research question: *How does an eighth-grade student use her own constructed representation(s) to problem-solve about slope in a linear relationship prior to significant instruction on the topic?*

In the sections that follow, I will first present a conceptual framework for my constructs surrounding representation. Then, I present one case from a larger task-based interview study ($N = 2$) with the goal of developing a preliminary conjecture about how representations may be involved in the initial discovery of slope concepts. I conclude with a discussion of implications for opportunities to learn about slope through representations in problem-based environments.

Representations and Mathematical Thought

This work follows Stylianou (2011) in defining a *representation* as "a configuration that stands for something else" (p. 266). While internal representations are inaccessible to a researcher, this study focuses on learners' external representational activity, which includes expressions in both visual and discursive mediums (Goldin & Kaput, 1996). In this report, I seek to explore how a learner's own external representational activity may come to influence their internal representational schemes, as identified through shifts in external expressions.

Representations can lend themselves to different cognitive purposes for learners. While structural conceptions act as static mental objects (such as the perspective of slope as a measured quantity), operational conceptions are more dynamic (such as the process of calculation of slope, Sfard, 1991). Stylianou (2011) applied this theoretical distinction to a study of representations in mathematical problem-solving with both experts and middle-school students. In this account, learner-generated representations functioned statically when employed to understand a problem, record key information in a problem, or present an idea. Representations functioned dynamically when applied to explore a problem, evaluate an idea, or negotiate an understanding. In this report, I operationalize student *use of a representation* if they constructed a representation of the problem or strategy and interacted with it dynamically, per Stylianou's (2011) characterization.

Methods

Setting and participants

To examine external representational activity during a problem-solving task involving slope, I conducted a semi-structured, task-based interview (Maher & Sigley, 2020; Mejía-Ramos & Weber, 2020) with two students enrolled in a general eighth-grade mathematics course. The following report features only the case of one student, Marie (a pseudonym), as her interview involved the most marked strategic shift in *representational use*.

Marie was identified by her teacher for this project as a student who met but did not exceed grade-level performance standards (as measured through standardized test scores, course grades, or both) and was known or inferred to be comfortable with sharing her mathematical ideas out loud through Zoom video. Marie attended a public school in the northeastern United States that was approximately 70% White, 14% Hispanic or Latino, 6% multiracial, 5% Black or African American, and 5% Asian American. Marie, age 13, identified as female and White. She had not yet engaged in in-depth study of linear relationships at this point in the year. However, she and her classmates had been briefly introduced to the concept of slope as a geometric ratio for the purposes of reasoning with similar triangles.

Data collection and analysis

Marie was interviewed for 60 minutes via a recorded Zoom session to complete a problem-solving task involving slope. I recorded audio, a video focused on the student and their scratch work, and a screen capture of each student's live interaction with the Desmos-created task. The task was designed to encourage students' construction of tables and graphs in their problem-solving, as these representations were known to have prior meaning for the students in their instructional context. I introduced the task as one central question with several related prompts:

> Two friends go to the candy shop. They each pay for their baskets and collect their candy. The first friend buys 3 pounds of candy and spends a total of \$30.50 for the candy and the basket. The second friend buys 7 pounds of candy and spends a total of \$68.50 for the candy and the basket. How much are the friends paying for each pound of candy that they buy?

In each interview, I first asked students to solve the problem using any strategy to identify the price per pound (or the slope of this linear relationship). I then requested that the student construct three representations to solve the problem in the following order: 1) a visualization of their initial strategy, 2) a table, and 3) a graph. In cases 2 and 3, I also asked the student to explain if the representation had shifted their initial thinking about the task.

Explanations of thinking during problem-solving in conjunction with student reflection around that thinking aided in my identification of *representational use*. Representational use was determined if 1) the student explained how a constructed representation helped them to think about the problem differently, or 2) novel features of the mathematical strategies emerged compared to previous solution strategies. To support the determination of representational use, I developed a coding structure to first identify the mathematical strategies that occurred during discussion. Given the novelty of conceptions for slope to these students, I developed the coding scheme emergently yet also informed by related literature (i.e., Lobato et al., 2003; Peck, 2020) and prior experience. Ultimately, the codes of *price per pound as division* and *price per pound as* *difference* became salient strategies in Marie's interview. These codes are defined in the results that follow.

Results

Starting strategies: Marie representing the price per pound as division

Marie's initial strategies to solve the problem were classified as views of the *price per pound as division*. She described her preliminary ideas as follows:

> So the first thought, I was like, hm, maybe the basket's 50 cents, because they both have 50 cents. But, I think we might have to do some division to figure out, like, how much each pound is, so [trails off].

Though she described the potential role of the cost of the basket, Marie began to solve the problem by dividing the amount of money that the first friend spent by the number of pounds that were purchased. She repeated this strategy with both pairs of values, reaching a price per pound of $10.16 for the first friend and then $9.78 for the second. When I asked Marie why she thought she might be getting different results for these divisions, she responded, "Maybe they got different kinds of candy or, I'm not sure." Marie did not reach a conclusion for the price per pound, but she constructed a visual to support how she envisioned her general strategy of distributing one friend's total (in her example, $14, which was not a value from the problem) across the number of pounds (in her example, 3 pounds, represented by circles, Figure 1).

Figure 1: Marie's constructed representation of price per pound as division.

The theme of *price per pound as division* continued when Marie represented her thinking in a table (Figure 2). Still without a satisfactory solution from her initial approach, Marie resorted to dividing other values in the table (such as 7 and 3) to "see if there's, like, a relationship with like any of the numbers." Marie continued to focus on divisions (though now different divisions) as she constructed a table, but she again did not reach a value she identified as the price per pound.

friend 1	friend 2
3	7
30.50	68.50

Figure 2: A recreation of Marie's table representation.

Representational use: Marie representing the price per pound as difference

Marie's reasoning about *price per pound as division* began to change to *price per pound as difference* when prompted to construct a graph. Therefore, I describe Marie's construction of a graph as a case of *representational use*.

While constructing a graph, Marie drew a point to represent each friend. She connected each point horizontally and vertically to the x- and y-axes. She then explained her new strategy as a need to calculate the difference between costs: "So I'm going to try and count, um, like, this amount [creates Figure 3] um, where the green line is, like, from, from where the 68 is to where the, from where the top line is to the bottom line."

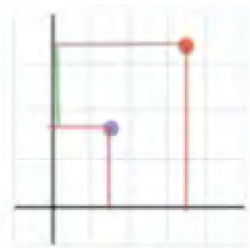

Figure 3: Marie's revised graph that connects points to axes; includes green segment to measure distance.

I asked Marie about her choice to construct this green line in the context of the solving the problem around the cost per pound:

> Hm. Um, I think it could, like, tell, like, um, the difference between, um, 30.50 and 68.50. Uh, I just had a thought. Maybe I could subtract those two numbers, so [types on calculator] 68.50 minus 30.50, yeah 38, hm. I wonder if that has to do with anything. So I just thought like, hold on, so I'm just going to write down the word 38 [writes "38" and circles it on paper], the number 38 in case I need it, so I remember it. I think I'm going to try like 7 divided by 38 [types on calculator] or no, 38 divided by 7.

Though the graph was the final representation with which Marie worked, reasoning involving the *price per pound as difference* emerged for the first time in her activity. However, despite this new idea, Marie still did not reach a solution for the price per pound by the end of work on the problem. In fact, when Marie was given a choice at the very end of the activity to try any final strategy to attempt a solution, Marie returned to her initial calculations of price per pound through division of each friend's total cost by the number of pounds purchased, still trying to reason through why these had given her different amounts. In reflection at the end of the activity, however, Marie described that the graph had shifted her conceptions during the activity:

> [The graph] kind of helped me to like, see the different, like, I don't know what the word is like the, the like, farness away of each dot, of each um, like number kind of, it, like, showed me like the difference.

Though Marie did not integrate her new reasoning around the *price per pound as difference* together with her previous strategies around *price per pound as division*, Marie described that the graph supported her to visualize a different element of the problem ("farness away"). This shift was unique to her work with a graph, thereby classified as *representational use*.

Discussion and Directions for Future Research

This research provides insight into the process of *representational use* in problem-solving with slope prior to significant formal instruction on the topic. Marie's initial reasoning had focused on a *price per pound as division* through her own representations and a table. However, Marie used her graph and uniquely adopted a new strategy of *price per pound as difference*. Though a graph became a dynamic representation (Stylianou, 2011) for Marie in this case, this trajectory is not assumed to be universal. Indeed, different representations led to different shifts for each of the two students in the interview study. I conjecture that moments of students' representational use could present potential opportunities to learn when accompanied by instruction and interaction outside of the task-based interview setting. Future research could explore how representational use is connected to conceptual development for slope over time.

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FROM NUMBER LINES TO GRAPHS: A MIDDLE SCHOOL STUDENT'S RE-ORGANIZATION OF THE SPACE

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We report on developmental shifts of a middle school student's (Ella) graphing activity as we implement an instructional sequence that emphasizes quantitative and covariational reasoning. Our results suggest that representing quantities' magnitudes as varying length of directed bars on empty number lines supported Ella re-organizing the space consistent with a Cartesian plane.

Keywords: Algebra and Algebraic Thinking, Cognition, Representations and Visualization

Constructing and interpreting graphs represents a "critical moment" in middle school mathematics for its opportunity to foster powerful learning (Leinhardt et al., 1990). Students, however, experience a number of challenges (e.g., conceiving graphs as picture of situation, event phenomena, literal motion of an object) in interpreting graphs (see Johnson et al., 2020 for a summary of these challenges). One potentially promising way to support students to develop productive meanings for graphs is to emphasize the role of seeing a graph as an emergent trace of how two quantities' measures vary simultaneously (Moore & Thompson, 2015). Although numerous researchers have investigated students' ability to interpret and construct graphs by plotting points and scaling axes, using slope and y-intercept, incorporating embodiment-based learning opportunities, and connecting with the other forms of multiple representations of functions, far fewer researchers have focused investigating how students construct graphs as emergent traces of quantities' covariation. Thus, we investigate the following questions: What ways of thinking do middle school students engage in graphing activities intended to emphasize quantitative reasoning? How can modeling with a quantitative reasoning approach support students' ability to develop productive and powerful ways of graphing?

Theoretical Framework: Quantitative Reasoning

This study focuses on middle school students' graphing activities involved in reasoning with relationships between quantities in real-world situations. We use *quantity* to refer to a conceptual entity an individual construct as a measurable attribute of an object (Thompson, 2011). In this study, we demonstrate ways in which students make sense of quantitative relationships in dynamic events and in graphs by reasoning with quantities' *magnitudes* (i.e., the quantitative size of an object's measurable attribute) independent of numerical values.

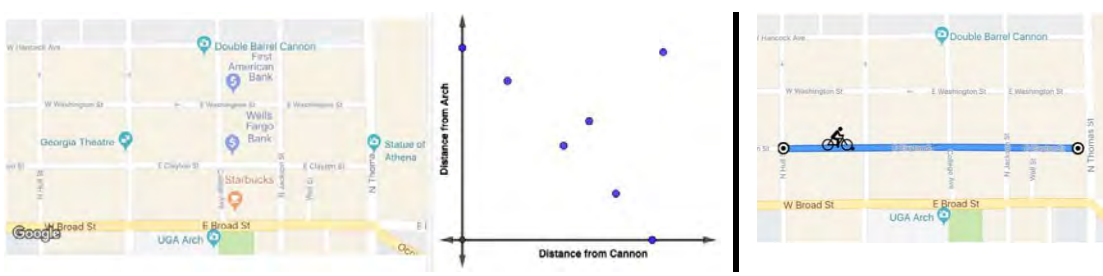
In the context of graphing, a relationship between two quantities is often represented in a coordinate system. Lee (2017) pointed out that researchers and educators have often taken coordinate systems for granted in students' graphing activity. Until recently, researchers did not question the importance of constructing a coordinate system because most researchers did not view it as a mental structure that students needed to construct (Lee, Moore, & Tasova, 2019). Furthermore, the idea of representing quantities' values or magnitudes on number lines is often taken for granted in students' construction of coordinate systems (and, in turn, in students' graphing activities), which is problematic because the construction of a plane requires conceiving of two number lines and using them to create a two-dimensional space (Lee, Hardison, & Paoletti, 2018). Thus, in this study, we investigated the nature and extend of students' abilities to represent varying quantities' magnitudes on number lines, and whether/how those abilities influence their construction of coordinate systems and graphs.

Methods

This study is situated within a larger study that examined four seventh-grade students' graphing activities in a teaching experiment (Steffe & Thompson, 2000) that occurred at a public middle school in the southeast United States. This study focuses on Ella's meanings for graphs and her developmental shift of those meanings over the teaching experiment.

Ella participated in 6 teaching sessions each of which last for approximately one-hour. Data sources included video and transcripts of each session that captured her exact words, gestures, and drawings. We conducted a conceptual analysis in order to understand her verbal explanations and actions and develop viable models of her mathematics (Steffe & Thompson, 2000). Our analysis relied on generative and axial methods (Corbin & Strauss, 2008), and it was guided by an attempt to develop working models of Ella's thinking.

Before conducting the teaching experiment, we developed an initial sequence of tasks each of which was designed with a dynamic geometry software and displayed on a tablet device (see https://www.geogebra.org/m/w9n4hn7r for digital versions of the tasks). *Downtown Athens Task* (DAT) includes a map with seven locations highlighted and labeled (see Figure 1a). We also present a Cartesian plane whose horizontal axis is labeled as Distance from Cannon (DfC) and vertical axis is labeled as Distance from Arch (DfA). Seven points are plotted without labelling in the coordinate plane to represent the seven locations' DfA and DfC (see Figure 1a, right). We asked students what each of these points on the plane might represent with an intention to observe their spontaneous responses and to explore students' meanings of points.



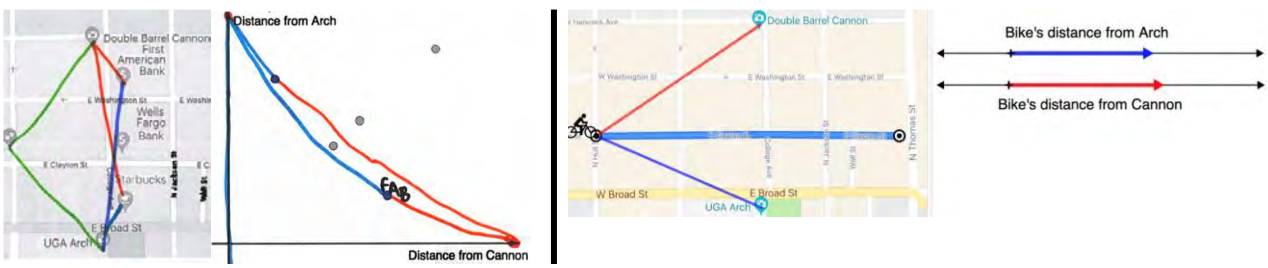
(a) (b)

Figure 1. (a) Downtown Athens Task (b) Downtown Athens Bike Task

In *Downtown Athens Bike Task* (DABT), we present the students with the same map of Downtown Athens highlighting a straight road (i.e., Clayton St.). We asked students to graph the relationship between the bike's DfA and DfC as the bike moves at a constant speed back and forth along the road. We also designed numerous tasks where students engage with quantities' magnitudes represented by varying length of directed bars placed on empty number lines (also called *magnitude lines*, see Figure 2b). The length of directed bars on the magnitude lines vary according to the bike's movement in the map. We conjectured that this representation might help students when they move to the two-dimensional space to represent two quantities by a single point in a coordinate plane. Note that we call the line "empty number line" in order to emphasize magnitude reasoning as opposed to numerical or value reasoning.

Results

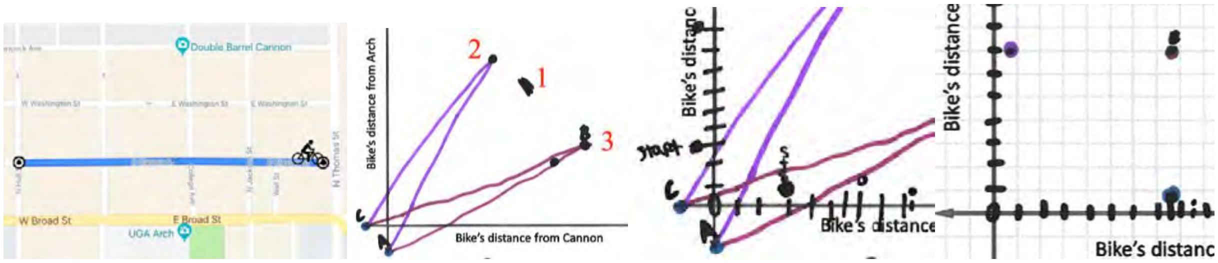
Initial meanings of the points and the organization of the space. We illustrate Ella's initial meanings by using her activity during DAT. Ella assimilated points in the plane as a location/object, however, her meanings were based in focusing on object's quantitative properties. After conceiving Arch and Cannon physically located on each axis as implied by the labels (see orange dots on each axis in Figure 2a), Ella made sense of the rest of the space by coordinating the radial distances between "places" on the plane and Arch and Cannon on each axis. For example, Ella labeled a point as "FAB" on the plane (see Figure 2a) to indicate First American Bank, and she conceived the point as FAB based on the orange and blue line segments that she drew on the plane. She stated, "the orange is shorter, and the blue is longer… [*referring to the orange and blue line segments on the map*] over here, like the same thing." Ella perceived FAB is closer to Cannon and farther from Arch in the map as well as in the plane. Therefore, we infer that Ella's meanings of the points included determining quantitative features of an object in the situation (i.e., its DfA and DfC as indicated by segments) and subsequently preserving these quantitative properties via the location of a point in the plane.



(a) (b)

Figure 2. (a) Ella's bipolar coordinate system, (b) DABT with magnitude lines

Representing a quantity's magnitude on an empty number line. In order to aid Emma in developing particular meanings for representing quantities in Cartesian plane, we engaged her in a dynamic tool that represented quantities' magnitudes as directed bars of varying length (see Figure 2b, right). We first wanted to get insights to how Ella could conceive this representation. While moving the bike to the right from its position seen in Figure 2b (red segment in the map and the corresponding red bar on the line were hidden at the moment), we drew Ella's attention to the fact that the right end side of blue bar on the magnitude line was moving to the left (indicating the bike's DfA was decreasing from our perspective). Ella determined that the bike's DfA is decreasing while moving the bike to the right in the map. She explained "it [*pointing to the blue bar*] is gonna get smaller because distance is smaller on the number line too." Moreover, Ella labeled the starting point as "zero." From this activity, we infer that Ella conceived the length of the blue bar on the magnitude line as a representation of the bike's DfA.



(a) (b) (c) (d)

Figure 3. (a) Map showing the bike's position when questioned, (b) Ella's graph, (c) marks and dots on axes, and (d) points in the plane.

Graphing bike's DfA and DfC. After Ella engaged with the magnitude line activity, we asked her to sketch a graph to represent the bike's DfC and DfA using a given piece of paper with two orthogonal axes. Ella re-organized the space different than her earlier actions in the teaching experiment (see Figure 3b vs. Figure 2a). For example, Ella conceived Cannon at very left side of the horizontal axis (labeled C) because "farther it is here [*sweeping her finger to the right from left over the horizontal axis*] means that farther it is from Cannon." This may show that Ella's re-organization of the space was an implication of her engagement with the magnitude line activity. Ella still assimilated the dot she drew in the plane as the bike (labeled B, #3 in Figure 3b) whose location was determined by coordinating the radial distances between the bike's DfA and DfC. Note that Ella wanted to change the location of the dots (see her earlier attempts in Figure 3b with the numbers showing the order in which she drew) "because it [the dot labeled as B] is like farther away from Cannon than it is Arch."

Ella's shift. Note that Ella plotted only one point on the plane (see Figure 3b), although the prompt was to graph the relationship as the bike traveled. We asked her whether her graph (i.e., the dot she plotted) illustrated the relationship between the bike's DfA and DfC as we animated the bike—the length of the bars on the magnitude lines also varied accordingly—in the tablet screen. She said no. Ella claimed, "I probably could have put a number line right here [*referring to the axes of the plane*]" to show how the bike's DfC and DfA changed as it moved. To illustrate this, she plotted tick marks on each axis in conjunction with tick marks plotted on the magnitude lines. She added dots near certain (and somewhat arbitrary) tick marks on each axis (see black dots in Figure 3c) to represent certain states of bike's DfA and DfC as the bike changed its location. During this activity, Ella did not focus on her purple line segments or the points that she drew earlier in the plane (see Figure 3b). She only worked on the axis to represent each quantity, and she did not plot points in the plane to represent them simultaneously. So, we repeated the same task with grid paper to see if she could join those quantities in the plane. By describing "this is what I did earlier" referring to her latest activity, Ella began plotting a dot on each axis to show the bike's DfA and DfC (Figure 3d). Then, she plotted a point in the plane "where those two [*tracing the pen in the air from the dots on each axis to the dot in the plane horizontally and vertically, respectively*] would meet up if they have like a little line." When asked to explain what that point represented to her, Ella said, "that is where the bike is." We infer that Ella seemed to establish a way to represent two quantities in her newly organized space as a single point; although she seemed to conceive the point that she plotted in the plane as the physical location of the bike.

Discussion

In this study, we illustrated different ways a student's graphing activity involved representing quantitative relationships. These examples illustrate alternative meanings of a coordinate system and coordinate points. Ella initially assimilated the points on the plane in relation to the physical objects that appear in the situation, and her meanings for points were based in quantitative properties (i.e., magnitudes from a fixed point). Ella conceived the length of the bar on the magnitude line as a proxy for the quantity that she conceived in the situation (i.e., the bike's DfA). In doing so, she conceived a constrain regarding how to represent the variation of a quantity on a magnitude line (e.g., only left and right on a horizontal line). Thus, she organized the space accordingly in later activities when considering two-dimensional space (see her shifts along Figure 2a, Figure 3b, and Figure 3d). Our results illustrate that explicit attention to quantities in the situation and mapping those quantities' magnitudes onto the empty number lines supported Ella's re-organization of the space consistent with a Cartesian plane.

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FEEDBACK PRACTICE FOR AN ABSTRACT ALGEBRA PROFESSOR AND STUDENTS' SUCCESSFUL PROOF REVISIONS

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This paper reports a case study of the feedback practice of one professor (Dr. X) in an abstract algebra course and the utilization of the feedback by four students from the class on three proof assignments. During interviews, Dr. X provided the rationale for each piece of feedback and described her general feedback practices. The students provided their interpretations of the feedback during interviews and were given a chance to revise the proofs to gauge whether they learned from the feedback. Dr. X wrote comments on her students' proofs to make them think, to correct notational or logical errors, or to address misinterpretations. Dr. X provided feedback even when no points were deducted. Students were generally able to successfully revise their proofs by addressing Dr. X's feedback.

Keywords: Advanced Mathematical Thinking; Classroom Discourse; Reasoning and Proof

Proof proficiency is an integral component of many upper-level mathematics courses. Many faculty members report spending considerable time and effort providing feedback to students on their proofs (Moore, 2016) and try to be thoughtful about their feedback practices (Weber, 2004; Lew, Fukawa-Connelly, & Mejía-Ramos, 2016; Miller, Infante, & Weber, 2018). However, this is under-researched at the collegiate level (cf., Speer, Smith, & Horvath, 2010).

Moore (2016) found that mathematics professors tend to focus on logical validity, clarity of writing, fluency of writing, and conceptual understanding when grading proofs. However, professors are rather subjective when grading proofs, often awarding vastly different scores for the same proof (Miller et al., 2018; Moore, 2016). Furthermore, professors report giving different scores based on students' past performances or the professor's interpretation of the error (Lew & Mejia-Ramos, 2019; Miller et al., 2018; Moore, 2016).

Additional research indicates that students are often unable to fully understand the feedback they are given (Byrne, Hanusch, Moore, & Fukawa-Connelly, 2018; Lew et al., 2016). This is not unique to mathematics; undergraduate students in many disciplines are often unable to fully interpret the comments instructors leave on their work (cf., Higgins, Hartley, & Skelton, 2001; Norton & Norton, 2001; Vardi, 2009). Byrne et al. (2018) found that students can fully revise direct edits on proofs without understanding the rationale for the feedback.

This project seeks to understand a professor's feedback practices in abstract algebra, and the students' reactions to the feedback. Our research questions are: (a) What are the professor's feedback practices on students' abstract algebra proofs when viewed through the commognitive framework? and (b) Can students revise their proofs successfully after receiving this feedback?

Theoretical Framework

Sfard's (2008) commognitive framework offers the tools to discuss detailed aspects of teaching and learning (Nardi, Ryve, Stadler, & Viirman, 2014) by adopting a participationist perspective towards human learning. Learning mathematics is the process of becoming a member of a mathematical discourse, which is distinguished by its *word use*, *visual mediators* (i.e., diagrams), *narratives*, and *routines* (Sfard, 2008). Narratives include definitions, proofs, and theorems. *Routines* are repetitive patterns (Sfard, 2008), which are governed by *object-level rules* and *meta-level rules* (or simply *metarules*). Object-level rules are the narratives about regularities in the behavior of objects of the discourse (Sfard, 2008), whereas *metarules* are patterns in the activity of the participants when trying to produce and substantiate object-level narratives (Sfard, 2008). In this study, *a proof* is a narrative that is produced by following certain routines and endorsed by those with authority. *Proving* is embedded within a commognitive social context (mathematical discourse) that puts emphasis on the activity.

Object-level learning is detected by the expansion of the discourse, such as extending the vocabulary, constructing new routines, and producing new endorsed narratives (Sfard, 2008). *Meta-level learning* is characterized by changing one's metarule of the discourse (Sfard, 2008), including knowing the *how* and *when* of a routine in a discourse (Ioannou, 2018).

For analyzing the students' proof revisions, we employed an analysis framework inspired by Conrad and Goldstein's (1999) strategy. We define a successful revision to be one that integrates the feedback in a way that improved the quality of the proof, even if some errors remained.

Methodology

Participants

The participants in this study were the course instructor (Dr. X) and four students from an undergraduate abstract algebra course at a large R1 university in the United States. Dr. X was an Assistant Professor and had a research specialty in Commutative Algebra and Lie Algebras. The students, recruited voluntarily, were three seniors, and one first-year graduate student. All four students had previously taken proof classes, so they were familiar with the general process of writing proofs.

Materials

Three problems assigned by Dr. X were intentionally chosen from three main topics (groups, subgroups, and isomorphisms) among the abstract algebra course assignments:

1. Let L be the set of positive real numbers. Define a binary operation $\heartsuit$ on L by $a \heartsuit b = a^{\log b}$. Determine if $(L, \heartsuit)$ is a group.
2. Let G be a group and let H be a subgroup of G. Let x be a fixed element of G. Define $xHx^{-1} = \{xhx^{-1} \mid h \in H\}$. Show that xHx^{-1} is a subgroup of G.
3. Show that $U_{26}/\langle 5 \rangle \simeq Z_3$.

Dr. X graded and provided feedback for each proof, as was her normal routine. We collected graded proof productions from each student (without Dr. X knowing who participated in the study). All collected proofs were rewritten for the purpose of confidentiality since they were shown to Dr. X during the meeting with her.

Procedures

We conducted two types of interview: interview with each student and interview with Dr. X. One of the researchers met with each student three times throughout the semester. During the interviews, students were asked to provide their interpretations of the feedback and answer general questions about feedback. Moreover, if Dr. X deducted points, the students were asked to revise the proof, explain what they changed, and identify how they used Dr. X's feedback. The revisions were not graded by the instructor, but they were evaluated by two of the researchers to judge whether the revisions were successful.

Two of the researchers met with Dr. X. During the interview, we asked Dr. X to offer her rationale for each piece of feedback on all 12 student proofs, along with insight into her more general process for providing feedback and her proof feedback practices. Note that all the students' 'roofs were rewritten by one of the researchers to hide students' identities, but the structure of the original proofs and feedback was maintained. All interviews were analyzed using an inductive approach, with the codes and themes derived from reading the interview transcriptions (Braun & Clarke, 2006).

Results and Discussion

Professor's Feedback Practices

Comments that prompt students to think. Dr. X wanted students to think when reading her comments. She highlighted this by probing students' understanding, bringing their attention to material from lecture, explicitly asking them to think about a part of their proof again, and encouraging the students to revise their proofs and discuss their revisions with her outside of class. For example, on Problem 2, Student A claimed that as a group in its own right, the subgroup H "must contain the identity element e" Dr. X left feedback indicating that Student A should elaborate and think this through, explaining in the interview, "I said all subgroups contain the identity element of G and particularly $\mathrm{e_H}$ equals $\mathrm{e_G}$, and I wanted them to think why that's'true." Another example of this type of prompting was found in the feedback on Student D's proof of Problem 1. Dr. X wrote "think again" to direct the student to think about Student D's assertion that every element of L had an inverse. In addition to the comment "think again", Dr. X also provided a hint on Student D's proof to focus the student on a counterexample.

In terms of general feedback practice, Dr. X stated, "I want them to see if their thought-process is properly transferred into the paper." Furthermore, she expressed that, at times, she wants students to "think" about the feedback and the problem, revise their proofs, and seek more feedback, sometimes asking students to turn the problem in again. Commognitively, such feedback prompts students to build a mathematical discourse with themselves. Dr. X intentionally avoided indicating what was wrong and how to fix it, leading the students to identify and correct what was wrong, helping the students improve their understanding of each topic and expand their own discourse.

Comments related to object-level learning. Dr. X provided much feedback on notation and quantifiers. This feedback was often provided in the form of direct edits, such as: crossing out multiple instances of H and writing xHx^{-1}; adding logical quantifiers; and other direct phrases conveying additional mathematical norms. With such feedback, Dr. X did not normally deduct points, stating, "if the ideas [are] there, if they don't use some time to write the right notation, I don't take points off, but then I say, 'look, your notation here should have been this.'" We did, however, find one instance where such an error received a substantial deduction.

Several times, Dr. X edited the students' proofs to add quantifiers. For example, on Student D's proof that $(L, \heartsuit)$ satisfied the associative law, Student D wrote: "Thus $a\heartsuit(b\heartsuit c) = (a\heartsuit b)\heartsuit c$, and so $(L, \heartsuit)$ has the associativity property," and Dr. X provided the direct edit "$\forall$“$, b, c \in L$" right after "$a\heartsuit(b\heartsuit c) = (a\heartsuit b)\heartsuit c$." Dr. X viewed this as a clarity issue and did not deduct points, but she did convey mathematical norms to the students through her feedback. In all, we found that Dr. X provided most of her feedback on object-level learning (logical reasoning, notation, and quantifiers), but this emphasis was often not accompanied by point deductions.

Comments related to meta-level learning. We found that there were two places where Dr. X would always deduct points: (1) for not showing non-emptiness when applying the subgroup test, and (2) for not ensuring that the subgroup is normal when working on quotient groups. In such cases, Dr. X explicitly noted what was missing. For example, she wrote to Student B that "you should also check that $xHx^{-1} \neq \emptyset$. $xHx^{-1} \neq \emptyset$ because $\mathrm{e_G} \in H$ and hence $x\mathrm{e_G}x^{-1} \in xHx^{-1}$." Viewing these commognitively, both of these errors are routines governed by metarules: the metarules of performing the subgroup test, and the metarules of quotient group construction. Ioannou (2018) suggested that "not addressing non-emptiness indicates that students are probably not aware of its importance due to problematic metalevel learning in the context of group theory" (p. 130). In a similar manner, not showing the normality of the subgroup could be an indication of deficient meta-level learning. The importance of awareness of routines and meta-level learning was punctuated by Dr. X's consistent deduction of points.

Revision Analysis

The four students successfully revised six out of eight revision attempts. Student B's unsuccessful proof of Problem 1 did not address all the relevant feedback items, neglecting to include the prerequisite sub-proof of 10 as the identity element of L. The student said that she would have included all four group axioms, including the missing sub-proof, if the revised proof was to be produced for Dr. X and not the study. However, by calling such portions "redundant," the implication may be that the student did not view the missing portion as logically necessary, but rather as satisfying the desires of a professor to see students' knowledge of group axioms on display. This highlights how feedback can fail to have the intended effect on a subsequent proof revision when a student filters the feedback and weighs its importance. In contrast, Student C received similar feedback and chose to include proofs of all necessary axioms. From a commognitive perspective, the metarules established for the classroom did not fully promote aspects of meta-level learning necessary to internalize which portions of the proof are logically necessary.

The other unsuccessful revision resulted from problematic meta-level learning of the metarules for normal subgroups, on Problem 3. Specifically, Students C and D needed to defend the *normality* of the subgroup $\langle 5 \rangle \subset U_{26}$. On this matter, Dr. X provided an essentially the same comment to both students. Student C provided incorrect justification to show the subgroup is *normal*, while Student D produced a correct proof. This could be due to students' preferences of the type of feedback and students' preexisting knowledge. A small nudge in the right direction may be just what one student needs to bridge the gap for internal discourse and complete a revision successfully, but that same nudge may be more frustrating than helpful for another student. This suggests that the notion of *effective feedback* might be very contextual and depends on many aspects involved in the feedback process. On Problem 2, Dr. X used a more direct approach with explicit corrections, which resulted in successful revisions from all students.

Conclusion

Dr. X wrote comments on students' proofs to prompt them to think, to indicate notational errors, or to illustrate misconceptions and logical errors. Point deductions were not the only reason for feedback. In fact, some of Dr. X's most pointed comments on significant inaccuracies (such as logical errors) came with little-to-no point reductions. Furthermore, Dr. X always deducted points and provided feedback relating to meta-level learning when students did not fully address the non-emptiness condition for a subgroup or did not establish that the subgroup is normal before articulating why a quotient group is isomorphic to $\mathbb{Z}_3$.

Having students revise proofs is a way to gauge whether students have properly interpreted the feedback, as seen in this study. However, understanding the error and professor's feedback did not imply that students could successfully revise their proofs. This result complements the earlier work by Byrne et al. (2018), which found that producing a successful revision does not imply that the student has fully understood the professor's feedback.

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THE INTERFACE OF QUANTIFICATION AND COVARIATIONAL REASONING IN REAL WORLD SCENARIOS

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Quantitative and covariational reasoning (QCR) are foundational to productive conceptualizations of functions, and especially to properties belonging to first and second derivatives. Through the lens of QCR, we explore how derivatives and rates of change might influence mathematical model construction. Drawing on sessions from an exploratory teaching experiment with an undergraduate STEM major, we illustrate reasoning with non-normative conceptions of derivatives that is consistent and robust and conjecture how this reasoning impacts model construction.

Keywords: modeling, calculus, mathematical representations, undergraduate mathematics

Many STEM disciplines rely on mathematical models to convey meaning. The kinds of models studied in advanced mathematics and STEM coursework regularly feature first and second derivatives, sometimes in relation to a single independent variable and sometimes in relations among themselves. Thus, relational properties among these quantities like dependence, directionality, and coordination-of-change take on additional importance when considering students' meanings for the models that recruit these quantities. It is yet to be understood how the manner of quantification of specific quantities may constrain covariational reasoning and subsequently the kinds of scenario-based conditions modelers may carry forward into their models. The purpose of this paper is to examine the interface of quantification and covariational reasoning about the first and second derivative.

Theoretical Perspective and Empirical Background

Researchers have elaborated theoretical constructs such as quantitative reasoning and covariational reasoning (QCR) for explaining, predicting, and extending students' thinking in the presence of variation (Thompson & Carlson, 2017) while at the same time, studies of mathematical modeling processes have been incorporating methods capable of accounting for modelers' QCR (e.g., Czocher & Hardison, 2021). By *mathematical model*, we mean a conceptual system accessible through the modeler's mathematical expression of locally meaningful representational systems for real-world phenomena. Coordinating two varying quantities and attending to relationships among them is *covariational reasoning* (Carlson et al., 2002). It presents as patterns of reasoning that compares quantities, combine them through operations, trace their changes, rates of changes, and intensities of changes (Johnson, 2015). According to Carlson et al. (2002), covariational reasoning passes through five levels of development based in the individual's imagery of the dynamics and relative to a task scenario. Each level corresponds to increasingly sophisticated mental actions while retaining the nature of mental actions associated with all lower levels: MA1 – *dependence* of one variable on another, MA2 – *direction* of change of one variable with changes in the other, MA3 – *amount* of change of one variable with changes in the other, MA4 – *average* rate-of-change of one variable with uniform increments of the other, MA5 – *instantaneous* rate-of-change of one variable with continuous changes in the other (Carlson et al., 2002). Jones (2016) studied students' conceptions of second derivative and concavity, elaborating on the covariational reasoning levels. He argued that, if one considers rate-of-change in a quantity as the dependent variable, then reasoning about concavity can be recast as mental actions MA4_1 (dependence of rate of change on independent variable), MA4_2 (direction of change in the rate of change with respect to the independent variable), and MA4_3 (amount of change in the rate of change with changes in independent variable). Conceptualizing mental actions 1, 2, and 3 applied to variation of rate-of-change along with mental actions 4 and 5 applied to variation in the base variable foreshadows ways of thinking reported in by Jones (2019), where participants conflated the magnitude of rate of change with its directionality.

Taken together, the literature points toward students' quantification as an explanatory mechanism for their modeling activities and especially toward the quantitative and covariational relationships students formulate as a basis for the graphical or symbolic expressions they create to communicate those relationships. Our methodology, described below, is borne out from these considerations as we seek to understand how an individual conceives of relations among time, a quantity, and a rate-of-change of that quantity and the models occasioned by those relations. We address the question: How does quantification of the first and second derivative influence covariational reasoning and what might be its collective impact on model construction?

Methods

Data comes from two task-based interview sessions drawn from exploratory teaching experiment with an undergraduate STEM major, Azure, focused on uncovering how to leverage and extend students' quantitative reasoning for the purpose of creating and expressing mathematical models of real-world scenarios. The sessions treat the Ice Melt Task, which presents a set of scenarios where ice is placed in contrasting environments. The participants are asked to distinguish among magnitude and sign of volume and rate of change of volume. Follow up questioning occasioned consideration of pairwise covariational comparisons of time, quantity, and rate of change of quantity, and to communicate properties of those conceived relationships through graphs and symbolic representations. Data analysis first sought instances of Azure's reasoning consistent with Jones's (2016, 2019) conceptual descriptions of concavity and where he made comparisons to physics concepts like velocity. We then catalogued the situationally-relevant quantities Azure imputed to the scenario, applying Czocher & Hardison's (2021) quantification criteria. Finally, we analyzed the instances identified in the first pass by examining the mental actions (MA1-MA5; MA4_1-MA4_3) and levels of covariation (CL1-CL5) the relevant pairs of quantities (identified in the second pass) permitted (Carlson et al., 2002; Jones, 2016).

Results

In total, Azure imputed 5 quantities with situational references relevant to the research question: Volume (amount of ice), Ambient Temperature (temperature of environment surrounding ice), Rate of Change of Volume (absolute change in volume between two distinct times; ice), Rate of Change (of Rate of Change) of Volume (rate at which ice melting), and Time (the indefinite continued progress of existence and events). Additionally, Azure quantified *slope* without a situational referent that nevertheless had a meaningful referent in his graphing activity. We discuss his non-normative situational referents for the rates of change below.

Azure described volume of ice in different scenarios in interesting ways, for example an iceberg in the ocean has a small/positive volume while ice in a cup has large/positive volume. He demonstrated coordination of V, volume of water, and time t through mental action (MA) 1 and covariational reasoning level (CL) 1. For example, he drew the graph in Figure 1, which along with his explanations suggested that as t changes, so must V. He was aware that V would decrease as time increased. In the next session, he stated that ice volume would decrease as time changed. In the tasks, he argued for a negative directionality between changes in V across two points in time, evident in his graphs in Figure 1. Thus, he evidenced MA2 for volume and time.

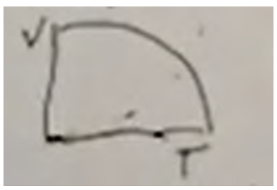

Figure 1: V-t graphs for Ice Melt, first session (left) and second session (right)

However, Azure did not extend his coordination of the direction of V with t in an anticipated way. Azure asserted that dV/dt would be positive or zero for the scenario-based conditions of the Ice Melt Task, for example, an iceberg in the ocean was said to have a small/positive dV/dt, and though he established that the direction of change of V with respect to t depended on ambient temperature, he did not evidence thinking that the rate-of-change of V with respect to t would change sign dependent upon ambient temperature. We interpret he coordinated dV/dt and t through MA1, MA3, MA4 and CL1, CL4. Azure did successfully and consistently coordinate direction of change of the magnitude of dV/dt, and so it is unclear whether 'credit' for MA2 should be given, according to the covariational reasoning framework. Specifically, he sketched Figure 2 (left) to represent an ice cube dropped into a hot cup of coffee. His figure shows dV/dt above the t-axis, with a positive sign, but he stated that the rate-of-change of volume was decreasing. He explained, "the slope of volume versus time graph is the magnitude of the dV/dt", suggesting that he associated slope with the directional coordination of V and t (MA2) but that he associated magnitude of dV/dt, which is always positive, with the rate that V changes with respect to t (also MA2, but for magnitude). Further complicating Azure's covariational reasoning, and our interpretations of it, was his quantification of rate-of-change of volume with respect to time. He defined rate-of-change of volume with respect to time as "the comparison [of volume] between two different points in time." Thus, he conceived rate-of-change as a displacement, an always-positive quantity. This would offer some confirmatory support to him when checking his own reasoning about the sign of dV/dt, or at the least, would not be a source of cognitive conflict in his reasoning with dV/dt.

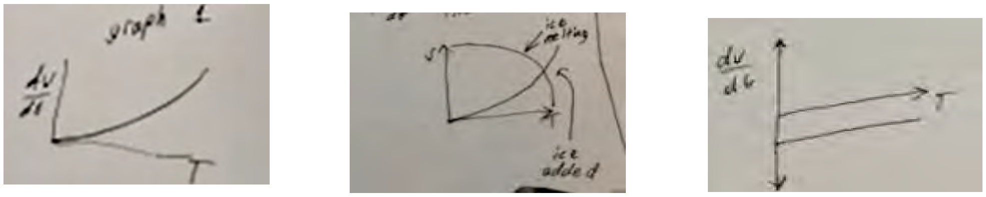

Figure 2: dV/dt-t graph for Ice Melt first session (left), V-t graphs for ice melting and ice added (middle), dV/dt-t graph for Ice Melt second session (right).

Azure evidenced imagery of amount-of-change of volume and rate-of-change of volume changing with respect to time. He stated that the rate-of-change of volume would be positive and increasing with respect to time. Azure drew two curves on the same V-t axes (Figure 2, middle) and stated, “either one of these, depending on how the problem is worded” would be correct, which suggests, similar to Jones (2019) argument, that the different quantifications of derivative can clash. Because Azure conceived of rate-of-change as always positive, the same dV/dt-t graph could represent either increase or decrease in volume. He explained that rate-of-change being positive either means that the volume is increasing (so ice is being added) or volume is decreasing (ice is melting). He noted there was no way to tell from the graph which scenario was modeled; he would need more information about the ambient temperature of the room and if water were available to re-freeze.

Azure supplied evidence that he could coordinate the amount of change of rate-of-change with change-in-time. In one instance, he argued “if it [dV/dt] is horizontal, regardless of if it’s above or below the x-axis [t-axis], it [volume] is changing. But if the V-t graph is anything but a straight line, if it’s one of these lines (indicating Figure 2 middle), the steeper this curve gets, the more dV/dt-t graph is a straight line up or down.” Working from Figure 1 (right) and Figure 2 (right), he appealed to a quantification of steepness of the graph (here the “situational referent” is figurative material in the graph). He referred to the steepness property of the V-t graph as an indicator of how closely dV/dt-t graph should resemble a vertical line. These latter instances are indicative of Jones’ extended MA4_1 and MA4_3. Azure attended to multiple attributes of volume, so it was difficult to clearly attribute MA4 and MA5 to his reasoning. He conceived negative rate-of-change of volume as equivalent to positive rate-of-change of volume when absolute “change between the initial state and the final state” are the same.

Discussion and Conclusions

Azure adeptly coordinated both change and change-in-change with time and was able to coordinate change and change-in-time with one another, by appealing to graphical properties, real-world reasoning, and without evidencing MA4 and MA5 for the V-t covariation. This observation supports Jones (2019) arguments. Azure’s conception of derivative was associated with multiple attributes of a situational referent. Sometimes dV/dt meant absolute change in volume across two times and at others meant changing intensity of that change. His conception of instantaneous rate-of-change corresponded to imagery of steepness of slope of the V-t graph, for an arbitrary time. Despite his non-normative conceptions and meanings for symbolic notation, his reasoning was consistent and correct when thinking through relations between change, rate-of-change, changes-in-change for volume and time, especially when illustrated graphically. However, his conceptions would be counterproductive for deriving models represented with arithmetic operations. Because he did not distinguish between freezing/melting graphically, he would not be able to use the graphs or his covariational reasoning to validate symbolic models and may come to inadequate conclusions about the validity of his models. We hypothesize that developing an (adequate to Azure) symbolic relationship as a model under these conditions would be challenging because of conditions he implicitly or explicitly imposed, like asymptotic behavior of the derivative directly caused by non-directionality covariation between the quantity of interest and time.

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TEACHERS' REMOTE INSTRUCTION PRACTICES THAT ENGAGE MULTILINGUAL LEARNERS

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This study investigated high school Math I teachers' methods for cultivating learning spaces in remote environments, and how teachers enriched language opportunities for multilingual learners (MLs) to engage in cognitively demanding work. Eight teacher interviews, from the 2020-2021 academic school year, were analyzed using two complementary theoretical ideas: key principles of reform-based instruction for MLs and mathematical language routines. We found that when teachers co-constructed math lesson with students building on their funds of knowledge while embedding digital platform(s) the mathematics language routines were complimented, and the learning environment fostered more active engagement. This research attended to the students' and teachers' productive struggle of during the COVID-19 pandemic and how mathematics language routines and technology supported their work.

Keywords: Diversity, Equity, High School Education, Inclusion, Professional Development, and Remote Instruction

National Public Radio recently reported that some parents expressed concern regarding their children's socialization and communication skills caused by a lack of peer interaction and learning during the COVID-19 pandemic (Kamenetz & Uzunlar, 2021). Many teachers have shared their struggles of having to quickly adapt and apply practical remote instruction techniques to increase student engagement during COVID-19. Before the pandemic, teachers already had difficulty engaging multilingual learners (MLs) in rich mathematical work (Iddings, 2005; Planas & Gorgorió, 2004). For these reasons, it is imperative to apply best instructional practices within distance learning classrooms, particularly for MLs. Our research question was: How did high school mathematics teachers cultivate safe learning spaces to encourage rich academic discourse for MLs to engage in cognitively demanding work in a remote environment?

Theoretical Framework

This study was organized around two complementary theoretical frameworks: key principles of reform-based instruction for MLs and mathematical language routines. Both frameworks helped teachers think about ways to engage MLs with content in meaningful ways. The five key principles to engage MLs in mathematical work reinforced and overlapped with one another. The first principle, *build on and use MLs' funds of knowledge and resources* (Moll et al., 1992; Moschkovich, 2002), has teachers identify, celebrate, and use the knowledge and skills students, their families and communities bring to the classroom. In the second principle, *provide multilingual learners with cognitively demanding work* (Stanford Graduate School of Education, 2013), teachers are expected to focus on the Common Core State Standards – Mathematics

(NGA Center, CCSSO, 2010) and provide MLs with the opportunity to engage in the same kinds of activities and assignments often reserved for students who are not MLs (Iddings, 2005; Planas & Gorgorió, 2004). The third principle provides MLs *opportunities for rich language production* (Khisty & Chval, 2002). Teachers provide comprehensible input through listening and reading, as well as opportunities for comprehensive output through speaking and writing. The fourth principle is to identify *disciplinary language supports for MLs* (Aguirre & Bunch, 2012). Teachers attend to those aspects of language that might prove challenging for all students, including MLs, and provide adequate scaffolding for students to both interpret and produce language (Aguirre & Bunch, 2012). The fifth principle is to *create a safe classroom* that allows for intellectual risk-taking (Luria et al., 2017), where everyone is part of a community of learners and where everyone values collaboration (Choike, 2000). A safe classroom is a place where students are free to learn regardless of their race/ethnicity, social class, or linguistic background (Hernandez et al., 2013).

These principles provide the foundation for mathematical language routines (Zwiers, et al. 2017), structures that support students' productive engagement with content, and supply them with tools they can regularly return to when completing mathematical tasks (Kelemanik et al., 2016). Routines provide opportunities for students to gain access to challenging mathematical content and build important mathematical thinking habits to engage in cognitively demanding work. Teachers can use routines specifically for MLs to amplify, assess, and develop their mathematical thinking and language simultaneously (Zwiers et al., 2017).

Methods

This study was situated in a California school district where teachers participated in a two-year professional learning program organized around four cycles of studio days (Von Esch & Kavanagh, 2018). Teachers developed, implemented, and studied lessons that focused on one ML principle and one mathematical language routine during each studio day cycle. The first two cycles were conducted in person: the second two, through Zoom.

Studio Days Enactment of a ML Principle and Mathematical Language Routine

The studio day cycle of interest here paired the principle *cognitively demanding work* with the mathematical routine Co-Craft Questions (Zwiers et al., 2017). Over Zoom, teachers participated in a series of three professional development meetings focused on *cognitively demanding work* and Co-Craft Questions. The language routine Co-Craft Questions is meant to engage students in communicating their reasoning by asking them to co-construct mathematical questions about a given context. Teachers first learned this routine in a pre-studio day Zoom session. Then, four of the teachers enacted a lesson using Co-Craft Questions over Zoom at their school during the studio day. One great advantage to using Zoom was that teachers for the first time observed one another implementing their lessons across school sites. During the final professional development day of this cycle, teachers examined student work, discussed their implementation of Co-Craft Questions, and shared challenges and successes.

Participants

Eleven teachers participated in our larger study, but we examined four focal teachers for this paper. The four focal teachers were interviewed before and after the Co-Craft Questions studio day cycle. Three of the four focal teachers were mathematics teachers: Mr. Ming, Mr. Huerta, and Ms. Parker. The fourth focal teacher, Ms. Lacrosse, was a special education co-teacher.

Data Collection and Analysis

We conducted two separate semi-structured interviews (Glesne, 2011) over Zoom with each of the four teacher participants to understand how they engaged their students, especially their MLs, in cognitively demanding work while engaging in mathematics. The pre-interview was conducted prior to the pre-studio day and the post-interview, following the post-studio day. Each interview was approximately one-hour long, was video recorded, and was transcribed. The interviews attempted to understand how well teachers provided MLs with rich language opportunities and access to mathematical content and reasoning. The interviews were divided into five parts: (1) teaching MLs in general; (2) experiencing mathematical language routines; (3) participating in studio days; (4) adjusting to remoting learning; and (5) attending to additional questions and comments.

Two researchers conducted several rounds of coding in NVivo. In the first round of coding, they independently open coded the transcripts. In the second round of coding, to categorize the data (Maxwell, 2013), they identified additional themes related to the following: (1) the five ML principles; (2) the Co-Craft Questions routine; and (3) the role of technology in remote instruction. Next, they used NVivo to produce comparative matrices (Yin, 2016).

Findings

We found that during the COVID-19 pandemic, teachers noted that their students participated more in mathematical discourse when they implemented mathematical language routines that were complemented by digital platforms. Teachers also shared successful conversations with their students in communal spaces in which both teacher and students were given opportunities to co-construct the learning environments together, intentionally intergrading funds of knowledge. Initially, each teacher indicated that their students were less likely to respond to language opportunities because of instructional changes associated with COVID-19. In response to this decrease in student participation, we identified two key themes in teachers' instructional practices. First, teachers used technology in the unfamiliar remote environment to implement mathematics language routines. Second, teachers recognized the importance of cultivating safe classroom communities, where students felt safe to ask mathematical questions and to discuss their responses. By using routines and technology, such as Co-Craft Questions with Pear Deck, teachers were able to elicit and display questions from their students. This allowed the class to review questions instantaneously, as a class collective, while making students feel comfortable to share their ideas privately, as well.

Using Technology to Implement Mathematics Language Routines in a Remote Environment

Teachers identified several ways they implemented mathematical language routines while simultaneously and strategically using technological platforms in remote learning environments. This allowed teachers to elicit sense-making and thoughtful responses from students, which they had found challenging in the new remote learning platform. Pear Deck, a Chrome add-on, allowed teachers to add interactive questions and formative assessments to their lessons. Mr. Huerta used Pear Deck in conjunction with Zoom and noted,

> For the first time this semester, it felt like…I get some insight on more than just quick math facts. That kids can respond in the [Zoom] chat or out loud, I get to see an actual constructed response from them [in Pear Deck] and having that space to like actually read it and then compare work with others. (01/12/21, 7:42-8:05)

Pear Deck made it seamless for Mr. Huerta to incorporate the language routine Co-Craft Questions into his remote instruction, "It could just be done at the beginning of every problem, and it could be…students typing. If it's on Pear Deck, it could be three kids sharing and people taking notes and conversing about it" (01/12/21, 15:51-16:06).
Routines and strategic use of technological platforms created a space for students to access mathematical texts and develop an understanding of rich mathematical language. Ms. Lacrosse explained that she also used Pear Deck with the language routine Three Reads to help students understand the language in the problems presented in class:

> We rip [the problem] apart, and we put [the problem] back together again in our Pear Deck slides. So, if we're doing something that is a word problem, we've really kind of broken it down to three or four sentences just so we can get it on the Pear Deck and start to then analyze it a little bit more. (10/12/20, 16:00)

Pear Deck provided user-friendly formatting to structure mathematical language routines and promote safe spaces for students' questions and broader student engagement.

Cultivating Safe Remote Classroom Communities

All four teacher participants valued safe classroom spaces where MLs could be supported with *disciplinary language supports* and engage in *opportunities for rich language production*. Mr. Ming explained, "I think you're able to develop a relationship with those students and to create a very safe and comfortable environment for them to go ahead and learn, and they feel comfortable making mistakes. I think that's first and foremost" (1/14/21, 23:24).

COVID-19 reshaped teacher perceptions of safe classroom spaces. For example, Ms. Parker shared that because some students had inequitable access to resources, such as the number of devices in their home, she limited the number of internet browser tabs students needed to open to participate in her class. Additionally, Mr. Huerta expressed that students in his classroom shared complaints of changing formats in their other classes, so he strived to provide consistency in the format of his class. Teachers perceived that their choices in presentation and platform correlated to students' responses and level of comfort expressed. Mr. Huerta highlighted this more clearly, explaining, "It's fascinating to see how many people will type on a Pear Deck, but won't type in a private chat on Zoom" (1/12/21, 36:42).

Discussion and Conclusion

We found that teachers strategically implemented mathematical language routines and technology to aid them in the construction of a safe and productive math learning environment in the midst of the COVID-19 pandemic. Teachers productively struggled through the pandemic with their students. By using mathematics language routines and technology platforms, teachers worked to increase the frequency of student responses, the depth of their mathematical responses, and the likelihood of persisting in *cognitively demanding work*. Complementary platforms to mathematical language routines, such as Pear Deck, provided safe spaces for students to articulate their questions and answers with peers. While strategic use of technology and mathematics language routines did not guarantee student engagement, our teacher interviews provided a deeper understanding of how teachers might effectively support some of our most vulnerable students in remote instruction.

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EMERGENT SELF-REGULATION STRATEGIES OF AN UNDERGRADUATE FRESHMAN IN CALCULUS I: THE CASE OF ISAAC

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Keywords: Calculus, Affect, Emotion, Self-Regulation, and Undergraduate Education

Self-regulation learning strategies are defined as those “actions directed at acquiring information or skills that involve agency, purpose, and instrumentality self-perceptions by a learner” by Zimmerman and Pons (1986, pg. 615). Self-regulation can then be thought of as those skills or strategies used by students in an academic setting while attempting to synthesize and process new information, as well as how to maintain agency in one’s own learning. During the transition to undergraduate mathematics courses from high school mathematics courses such as pre-calculus or first semester calculus, freshman must navigate obstacles in their mathematics learning that may relate to the difficulty or pace of a college level mathematics course.

Using self-regulation strategies frameworks established by Wolters (1998) and Zimmerman and Pons (1986) as well as noting the transitional difficulties described by Gueudet (2008) and Sonnert et al. (2020) this poster aims to describe how first-year undergraduates use self-regulation strategies to adapt to challenges encountered in undergraduate mathematics courses.

This case study focuses on three interviews with a first-year undergraduate student taking Calculus I, Isaac. Isaac was one of three undergraduates who participated in interviews. He was enrolled at a large, urban research university in the Southwestern United States during Fall 2020. He and the two other interviewed undergraduates were selected based on their responses to a questionnaire that twenty-one undergraduates completed. The questionnaires focused on undergraduates’ educational background, demographic background, mathematical identity, use of self-regulation strategies, and expectations of college coursework. Invitations were sent based upon whether the students were first-year students and had attended high school within the last year and indicated strong performance in their past high school courses. Interviews focused on the use of self-regulation strategies during weekly in-class group lab activities. An audiovisual recording of the interviews was made. This recording was transcribed word-for-word by the researcher. The subjects were given a pseudonym to protect their identity.

After facing difficulty in an early midterm Isaac resolved to adapt his learning strategies to focus on some conceptual difficulties he recognized, such as problems involving graphs, as well as his general study habits in preparing for exams. After his first midterm Isaac resolved to spend more time on problems outside of class as well as speak to peers and instructors more when faced with difficulty. He more readily employed strategies as outlined by Zimmerman and Pons (1986) involving seeking assistance and self-evaluation by continuing lab activities past what was assigned and his attitude of persistence while facing obstacles promoted continued study aiding him in his success in a calculus course. This process produced positive results in his next exam scores so he continued using these strategies throughout Calculus I, confident that he would earn a high mark in the class. Other students may experience a similar cause for strategy development which may be generalized with further analysis. In the ongoing data collection and analysis, it will be explored how students’ self-regulation strategies interact with their mathematics identity and performance through general study habits and content-specific strategies, as well as how to encourage this development of useful self-regulation strategies for undergraduates that experience obstacles in their learning of mathematics.

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A COMPLICATED RELATIONSHIP: EXAMINING THE EFFECT OF STRATEGY USE ON ACCURACY WHEN SOLVING MULTIPLE WAYS

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Strategy appropriateness lies at the core of flexibility (Star, 2005). Some problems can be completed using a so-called standard algorithm (e.g., Star & Seifert, 2006); other possible strategies may be better than the standard algorithm, where better may mean that the strategy is more elegant and/or better matched to the structural features of the problem. But what is the relationship between strategy appropriateness and strategy accuracy? Here we ask: (1) Are students more accurate when using standard approaches or better-than-standard approaches? (2) Is this relationship between accuracy and strategy appropriateness influenced by whether a problem is being solved for the first time or being re-solved?

A convenience sample of 450 high school students from 19 math classes participated in this study. Participants completed a two-part assessment where they were asked to solve five problems, each in two different ways. Responses were coded both for accuracy and type of strategy: standard, better-than-standard, and worse-than-standard. We fit a multiple regression model using the ordinary least squares regression technique and an interaction term to determine if the effect of strategy on accuracy depended on whether the student was resolving.

Our results indicate that while the standard approach was more related to accuracy as compared to the better-than-standard approach, this relationship differed by assessment part. For students' first strategy for solving each problem, the standard approach was related to a higher rate of accuracy compared to the better-than-standard approach $(t(3{,}044) = -6.728, p < .001)$. For students' second strategy, we found no significant difference in accuracy between the standard and better-than-standard approaches $(F_{1,3044} = 2.9, p = .0894)$. We also found a significant and positive interaction effect; the average effect of strategy on accuracy depended on assessment part $(t(3{,}044) = 5.901, p < .001)$. Finally, we find that the standard approach is more successful in Part 1 of the assessment compared to part 2 $(F_{1,3044} = 21.8, p < .001)$, while the above-standard approach is more successful in Part 2 of the assessment compared to Part 1 $(F_{1,3044} = 14.3, p < .001)$.

We interpret these results as suggesting that the standard approach is related to greater success in problem solving only when a student was solving a problem for the first time. However, when students are prompted to go beyond this strategy, they may be limited in the approaches they can apply successfully. It could be the case that the standard approach is more successful in part 1 because of students' greater familiarity with the standard algorithm as well as greater confidence employing this strategy. This combination of familiarity and confidence may not be the case for students who choose to employ better-than-standard approaches the first time they solve problems. This difference in order of approaches and the associated likelihood of obtaining accurate responses is a further area of study we recommend.

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Chapter 8:

Professional Development & Inservice Teachers

UNDERSTANDING MATHEMATICS TEACHERS' COLLABORATIVE SENSEMAKING IN THE CONTEXT OF TEACHERS' LEARNING ECOLOGIES

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Over the last two decades, researchers have portrayed quality professional development for mathematics teachers as collaborative and situated in teachers' instructional realities. However, empirical findings also point out various impediments to transforming teacher conversations into consequential learning. These findings illuminate the need to acknowledge additional resources that teachers bring to professional interactions and the need for ever more nuanced theories of teacher learning to inform teacher educators' work. Inspired by ecological models of learning, in this conceptual paper I work towards understanding teachers' collaborative sensemaking as part of broader teacher learning ecologies. I distinguish and name possible scopes and contexts for the study of teacher learning in conversations about instruction, and then identify directions for future research towards stronger connections between immediate and broader contexts.

Keywords: Learning Theory, Professional Development

Over the last two decades, many researchers have explored and documented ways to support teachers towards teaching rich mathematical content with extended student engagement, and more recently, with additional layers of responsiveness to the multiple cognitive, social, cultural and political dimensions of student learning. The resulting studies portray quality professional development (PD) as collaborative and situated in teachers' instructional context (Ball & Cohen, 1999; Borko et al., 2008; Horn, 2005; Horn & Garner, 2022; Kazemi & Hubbard, 2008). Here, I refer to these designs as Collaborative Sensemaking as Professional Development (CSPD). The situated nature of CSPD breaks away from prescription-based pedagogies that teachers often experience in typical top-down PD workshops (Kazemi & Hubbard, 2008), and its collaborative nature potentially counters the isolation that teachers often experience in schools (Lortie, 1975; Little, 1990). Other significant affordances of CSPD include providing teachers with opportunities to develop their adaptive expertise (Horn & Garner, 2022; Lefstein & Snell, 2013); supporting teachers in reconciling different perspectives on teaching (Ehrenfeld et al., 2020; van Es, 2012); and working towards more productive norms of participation in conversations about instruction (Horn & Little, 2010).

However, empirical findings also point out material, social, cultural, emotional, and cognitive impediments to transforming teacher conversations into consequential learning (Borko et al., 2008; Horn et al., 2017; Vedder-Weiss et al., 2018). For example, Horn & Kane (2015) provided evidence that limited engagement with rich conceptual resources in teacher workgroups results in limited learning opportunities, and vice versa. These findings illuminate the need to acknowledge additional resources that teachers bring to professional interactions and provide teachers with structures to reconcile these resources with local contexts. More generally, tensions between the potential and impediments for learning in CSPD underscore the need for ever more nuanced theories of teacher learning to inform teacher educators' work (Clarke & Hollingsworth, 2002; Horn, 2005; Horn & Garner, 2022; Opfer & Pedder, 2011).

Although researchers of teacher learning in conversations typically adopt sociocultural, situated and situative perspectives (Greeno, 2006; Lave & Wenger, 1991; Vygotsky, 1980)—all of which underscore the importance of context in interaction and learning—it is not always clear what contexts warrant careful attention, and which are overlooked. That is a theoretical and analytical gap central to this conceptual paper. As a review of teacher professional conversations by Lefstein et al. (2020) suggests, studies of teacher collaboration most often consider the immediate interactional context of learning, sometimes consider the institutional context of school and seldom acknowledge broader contexts, such as the multiple experiences teachers have in different settings external to school, or broader macro-level social structures. While some studies account for some of these aspects of teacher learning, there is nothing in the framing of sociocultural, situated, and situative perspectives that guides researchers towards being explicit and mindful of the contexts they account for and which they ignore.

Inspired by researchers of learning and development that take ecological perspectives (Bronfenbrenner, 1979; Erickson, 2004; Nasir et al., 2020), my overall goal is to understand CSPD environments in ways that account for the broader contexts of teacher learning ecologies, with a focus on the interactive impacts of multiple experiences in different settings, and social structures within which teachers work. Consequently, I ask *How can ecological models of learning inform research on mathematics teacher learning in CSPD settings?*

First, I discuss ecological perspectives on learning, with a focus on Bronfenbrenner's (1979) framework for studying the ecology of human development. Then, to distinguish and name possible scopes and contexts within teacher learning ecologies, I build on Bronfenbrenner's work and adapt it to the specific case of teacher learning. Finally, I move beyond distinguishing and naming contexts towards studying them as interrelated. In the final section I discuss how additional ecological models of learning (Cobb et al., 2018; Engeström, 2001; Erickson, 2004; Gutiérrez & Jurow, 2016; Horn et al., 2013; Hutchins, 1995; Nasir et al., 2020) can inform future research towards stronger connections between the immediate and broader contexts of teacher learning ecologies.

An Ecological Perspective on Learning

For the last four years, I have been part of a PD effort to support instructional growth among secondary mathematics teacher teams (Project SIGMa; Horn & Garner, 2022). Using the conceptualization of PD I introduced earlier, SIGMa would be considered CSPD, since it builds on teacher community and dialogue to respond to teachers' perceived instructional puzzles. Within this project, we learned about the teachers' personal and professional histories, their relationships with colleagues and students, and their approaches towards math teaching. Through these relationships, it became clear that the teachers' ongoing learning in the intervention was strongly related to other activities they participated in, such as workshops with our research partners, a professional development organization, ongoing conversations with colleagues, and their experiences in previous schools. This insight might seem obvious, but it stands in sharp contrast to the ways math teachers' professional learning in PD is typically examined within single activities or programs, with little to no attention to external settings and broader contexts.

Bronfenbrenner's Ecology of Human Development

Bronfenbrenner's (1979) framework provides a generalizable (Shelton, 2019) starting point for studying how teachers and their environments interact in professional development processes. Studying development "in context" could mean many things. Bronfenbrenner's call was not to study development "in context" in some general sense of development "in the real world" or "not in the laboratory" but rather to think about development in the specific context that is an ecological system, as he defines it (Bronfenbrenner, 1979; see also about his work in

Christensen, 2016; Shelton, 2019; Xia et al., 2020). It is hard to overstate the importance of this distinction for the study of teacher learning. As I mentioned, while researchers of teacher conversations often underscore the importance of context, they mainly consider the immediate local social situation and seldom acknowledge broader contexts. Considering broader contexts and their interactive impacts is essential for integrating issues such as power dynamics, class realities, and racial tensions more seriously into analysis of teacher conversation, and consequently, into teacher collaborative sensemaking.

Beyond the Microsystem

Bronfenbrenner's framework includes a collection of four nested structures of environment: microsystem, mesosystem, exosystem, and macrosystem. *Microsystems* represent the immediate settings in which the developing people engage in activities, relationships, and roles; all of which are directly influenced from participation in other settings. *The mesosystem*, rather than a layer that surrounds the microsystem, represents the relationships between two or more settings. For example, Bronfenbrenner discussed the case of mothers from two-parent families with part time jobs. Their partners might act as if they are full time mothers, and employers as if they are full time employees. The mothers themselves might experience the resulting frustration as parents, on the job, and more generally as human beings. (Bronfenbrenner, 1979, p. 212). In sum, participation in more than one setting has developmental consequences that are overlooked when we only attend to the immediate interactional context. *The exosystem* consists of settings in which the focal people of interest are not actively involved, but others who interact with them are. For example, for a child, if we consider the relations between home and school as a mesosystem, then the parents' workplace or social lives could be considered part of their exosystem, even if the child is not physically attending these settings. *The macrosystem* relates to the larger social and cultural structures within which development is taking place, including values, practices, resources, and the different types of identities they invite or discourage.

Interactive Impact of Multiple Contexts

According to reviews of literature by Tudge et al. (2009, 2016) many researchers wrongly see Bronfenbrenner's framework as a theory about the influence of context on development and use it to ask questions about the direct linear effects of individual factors (i.e., a "reductionist" or "mechanist" paradigm). In contrast, Bronfenbrenner's four nested structures of the environment are meant as a framework for exploring how different factors act synergistically towards multiplicative and non-linear outcomes. To adapt this perspective onto the realms of teacher learning, we would need to explore both the internal properties of CSPD settings themselves and how they gain their local meanings from their positioning in broader teacher learning ecologies.

Operationalizing Mathematics Teachers' Learning Ecologies

From a teacher perspective, teacher learning happens across time and settings, through a complex web of learning experiences. In contrast, a typical linear pathway perspective for teacher learning in PD assumes (often implicitly) that teachers attend PD where they develop their knowledge and beliefs, which, in turn, change their teaching practices and eventually improve students' learning (see Figure 1; Clarke & Hollingsworth, 2002). It too often focuses on the direct effects of PD interventions, where learning itself is an indicator of the effectiveness of specific curricula, programs, or core-features of PD activity (Goldsmith et al., 2014). This tendency within the field of math education reflects more general trends in research of teacher learning towards listing certain features of activity as optimal for teacher learning (e.g., Desimone, 2009). A binary perspective on features of PD as absent or present is problematic because it overlooks their specifications and interactions, which are highly consequential for teachers' learning (Opfer & Pedder, 2011). In response to such trends, many point out the lack of well developed theories that take into account more complex and nonlinear approaches towards professional development (Clarke & Hollingsworth, 2002; Horn, 2005; Opfer & Pedder, 2011).

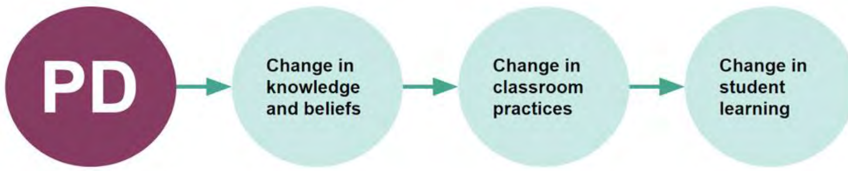

Figure 1: Linear Pathway Perspective (adapted from Clarke & Hollingsworth, 2002)

Inspired by Bronfenbrenner (1979) I suggest a framework for understanding teacher CSPD in the broader context of teacher learning ecologies. *The microsystems* included in this framework are the classroom, the settings of teacher collaborative sensemaking, informal teacher conversations, and other teacher learning settings such as PD workshops, conferences, organizations, and experiences in previous schools. *The school mesosystem* represents connections between classroom experiences, informal teacher conversations, and the focal collaborative and contextual PD setting. *The exosystem* represents connections to settings of teacher learning attended by one or more teachers or facilitators in the PD, but not necessarily by all (such as other PD workshops, conferences, organizations, and experiences in previous schools). *The macrosystem* represents the larger social and cultural structures within which the school operates, such as the school's neighborhood, or larger racial, ethnic, and civil structures. Figure 2 represents the suggested scope of an ecological perspective on teacher learning. Figure 3 represents possible contexts in the study of teacher learning in CSPD settings. Distinguishing and naming possible scopes and contexts for the study of teacher learning in CSPD settings can support researchers in being clearer about contexts they foreground and background in their design and analysis and for what reasons; in considering new aspects of learning that might be salient to their study; and in understanding the resources teachers bring to CSPD settings. The following vignette illustrates the framework as analytical lens for teacher conversations.

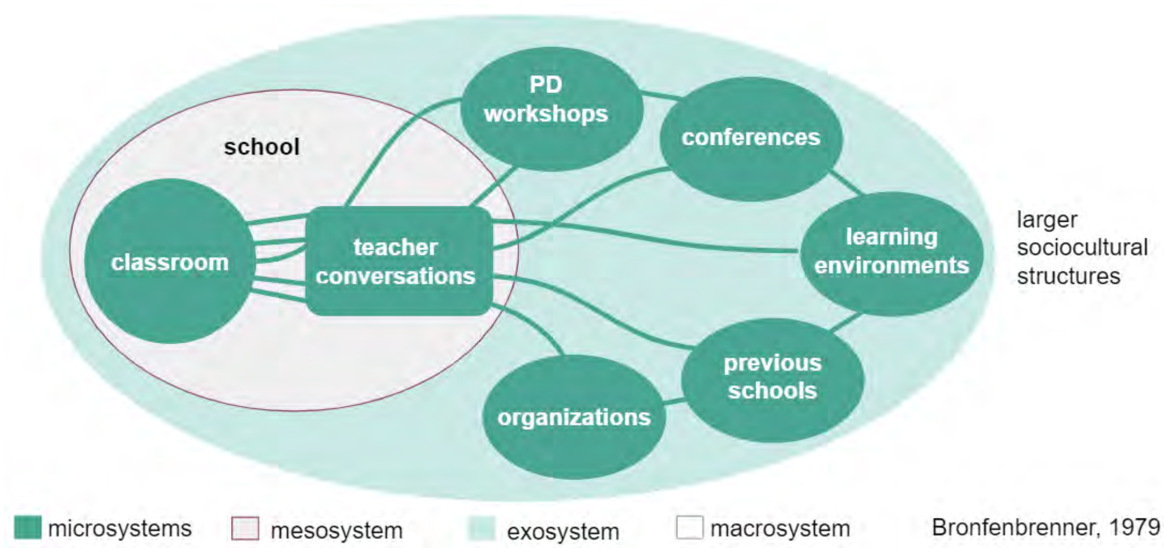

Figure 2: Suggested Scope of an Ecological Perspective on Teacher Learning in CSPD

Figure 3: Possible Contexts for the Study of Teacher Learning in CSPD

Vignette: Ezio and Veronica Discuss Grouping Strategies

To illustrate the analytical framework, consider the following vignette from Project SIGMa where two teachers discussed grouping strategies (for a careful analysis of this example see Ehrenfeld et al., 2020). In this conversation, the two teachers, Ezio and Veronica (pseudonyms), discussed Ezio's experiences in two different PD workshops: school-based Kagan training, where he learned about purposeful grouping (high and low achieving students in each group), and Park City Math Institute (PCMI), where he learned to group students randomly. Sharing his experience from the workshops, Ezio recalled how he first "did not agree" with PCMI random grouping. On the contrary, Kagan's purposeful grouping initially "made sense" to him. However, as the conversation progressed, Ezio and Veronica continued to make sense of these methods in light of their concern about tracking in their school. They discussed how purposeful grouping amplifies the consequences of tracking, in the shape of labeling kids as "dumb" or "awesome," while random grouping disrupts it. In addition, an interview with Ezio and Veronica revealed that they see tracking in their school as related to gentrification processes in their school's neighborhood. Their sense was that the principle was under pressure from newer and more affluent parents to increase tracking.

An ecological view of Ezio and Veronica's conversation highlights the following: First, teachers' opportunities to make sense of grouping strategies in the CSPD (microsystem), as an iterative process that involves experiments in the classroom (mesosystem) and experiences in two external PD workshops (exosystem). Second, it shows teacher agency with regards to both institutional structures of tracking (mesosystem) and broader social structures of gentrification (macrosystem) as structures to disrupt rather than to amplify. Third, it reveals other dimensions of the learning process, such as a trusting collegial relationship that affords ongoing inquiry into practice (microsystem). In sum, an ecological view of this example reveals learning in CSPD in the context of their broader learning ecologies (see Figure 4).

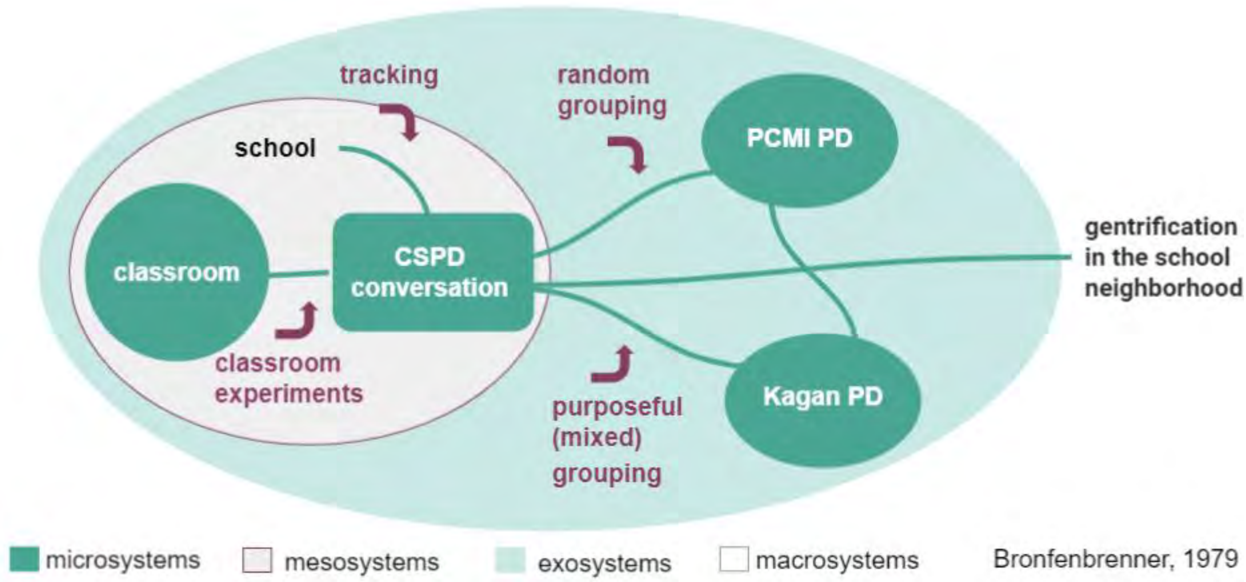

Figure 4: Suggested Scope of an Ecological Perspective on Ezio's CSPD Episode

My point is neither to diminish the value of research on the effects of specific activities (or certain features of activities) on teacher learning, nor to claim that every study of teacher learning must include all possible aspects of teachers' learning ecologies. Rather, I claim that (1) attempts to look at subsystems must be understood as partial (Opfer & Pedder, 2011) and (2) employing more complex perspectives on teacher learning would extend our ability to explain it and consequently to support teachers (Clarke & Hollingsworth, 2002; Horn, 2005).

Importantly, teachers do not experience scales and contexts as separated. In this sense, distinguishing and naming contexts should only be considered as a first step towards studying them as interrelated. In the next section I discuss how ecological models of learning can inform future research towards stronger connections between the immediate and broader contexts of teacher learning ecologies.

Future Research Directions: Connecting Immediate and Broader Contexts of Teacher Learning Ecologies

The claim that we need stronger theoretical connections between the immediate and larger contexts of teacher learning reflects more general calls to see interaction and learning both through a social microscope and a social telescope (Erickson, 2004). I will support this claim by looking at three examples of such calls. The first example comes from the recently published *Handbook of the Cultural Foundations of Learning*, wherein Nasir et al. (2020) conceptualize learning as "occurring along culturally organized learning pathways—sequences of consequential participations and transitions in learning activities that move (or do not move) one towards greater social recognition as competent in particular learning domains and situations" (p. 195). Nasir et al. made the overall claim that focusing only on local learning interactions limits our understanding of the cultural, relational, affective, and contextual nature of learning and their intersections with systems of power.

A second example comes from the work of Fred Erickson. In his book *Talk and Social*

Theory (Erickson, 2004), he likewise pointed out the need to better theorize connections between the immediate contexts of interaction and larger ones. Erickson discussed problems in contemporary social theories with regards to such connections, his main argument being that social theorists such as Foucault, Bourdieu, Gramsci and Fairclough are mostly showing “top-down” influences and are seldom attending to “bottom-up” or “inside-outside” ones. That is, they explain well how social order and structures of power are being reproduced by processes of socialization, normalization, hegemony, and the formation of discourses, but they fail to see the ways persistence and change of structures happens altogether by people in their everyday lives.

Along these lines, a third example is from the work of Gutiérrez and Jurow (2016). Conceptualizing social design experiments, Gutiérrez and Jurow extend traditions of design-based research and call for paying specific attention to ways participants reorganize their systems of activity to disrupt structural and systemic injustices. This emphasis is of particular interest to the world of mathematics education which is rife with normalized injustices and inequities (Chen & Horn, 2020; Louie, 2017). By building on their studies in the contexts of student leadership and food justice movement, Gutiérrez and Jurow (2016) describe different forms of learning, among them developing an understanding of oneself and other with relation to history and systems of power, increasing the capacity to use new conceptual tools, and giving rise to new forms of knowledge that develop across multiple contexts.

Tying the three examples back to the topic of mathematics teacher learning, I discuss how these and other ecological models for learning (Bronfenbrenner, 1979; Cobb et al., 2018; Engeström, 2001; Erickson, 2004; Gutiérrez & Jurow, 2016; Horn et al., 2013; Hutchins, 1995; Nasir et al., 2020) can inform research of teacher learning ecologies. Specifically, I identify three forms of investigations of learning that ecological models highlight (see Figure 5), and then “translate” them into three future directions for research of teacher learning.

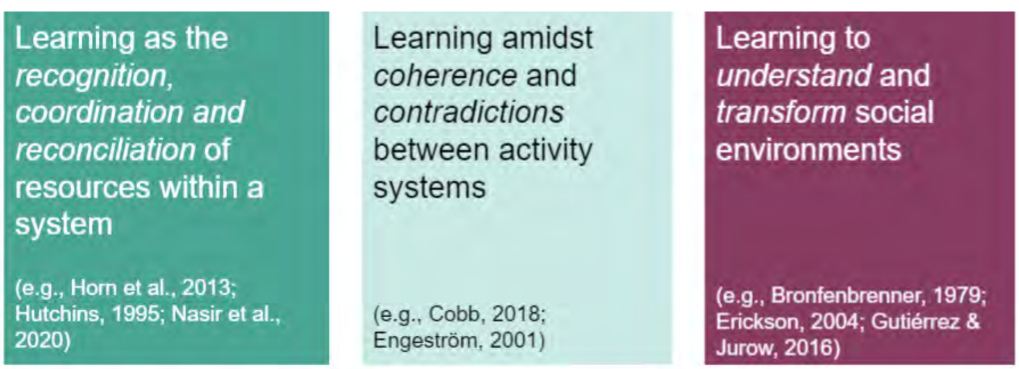

Figure 5: Ecological Forms of Learning Investigations

How do Teachers Reconcile Their Local Contexts With Circulating Resources?

Within the context of activity systems, a main form of learning is the recognition, coordination and reconciliation of resources (Horn et al., 2013; Hutchins, 1995; Nasir et al., 2020). Typically in research on math teachers’ learning, resource-centered approaches reflect the linear pathway perspectives (Figure 1), where instructional resources (e.g. practices, curricula, frameworks) are examined in the context of the PD interventions in which they are introduced (Sztajn et al., 2017). An ecological perspective on CSPD suggests the utility of a different resource-centered approach. In CSPD settings, as Ezio’s example illustrated, teachers themselves often draw on educational ideas that circulate across settings for their own sensemaking in their local contexts (Horn, 2005; Stengel, 1991). Indeed, I suggest that a critical aspect in the study of teacher collaborative learning is the articulation of mechanisms by which teachers reconcile local contexts with a range of circulating conceptual resources (Ehrenfeld et al., 2020).

What is the Role of Coherence and Contradictions in Teacher Learning?

In this section, I do not suggest that future research needs to decide whether coherence or contradiction are "better" for learning. Rather, I follow Opfer & Pedder (2011) and suggest that we should focus on causal explanations so that we understand under what conditions, why, and how teachers learn from coherence and contradictions between resources at hand. To do so, I compare and contrast Cobb et al.'s (2018) systematic perspective on teaching improvement efforts with Enge (2001) expansive learning. On the one hand, Cobb et al.'s (2018) systematic perspective on teaching improvement efforts specifically emphasizes coherence as conducive for instructional improvement towards ambitious and equitable math teaching. Others, such as Engeström (2001), describe contradictions as a force for learning. Engeström (2001) focuses on learning as constructing new practices that emerge from contradictions and hybridization (also Ehrenfeld & Heyd-Metzuyanim, 2019; Ward et al., 2011). We may hypothesize that coherence is more productive for normative and well-defined learning goals; whereas contradictions may be more productive for non-normative and disruptive trajectories.

How do Teachers Learn to Restructure Their Local Environment?

Bronfenbrenner's (1979) goal was to theorize the way people develop within and across changing settings "in both the immediate and more remote environment" (p. 11). He conceptualized human development as follows:

> Human development is the process through which the growing person acquires a more extended differentiated, and valid conception of the ecological environment, and becomes motivated and able to engage in activities that reveal the properties of, sustain, or restructure that environment at levels of similar or greater complexity in form and content. (Bronfenbrenner, 1979, p. 27, emphasis added)

In light of this definition, and in line with the argument that we need to better understand intersections between different contexts of teacher learning, I suggest that we need to investigate both separately and with relation to each other the processes by which teachers learn to reveal, sustain, and restructure their local environments, in particular, with relation to history and the disruption of harmful power relations (Gutiérrez & Jurow, 2016).

Discussion

The overall goal of this conceptual investigation was to work towards understanding teacher CSPD environments in ways that account for the larger contexts of teacher learning ecologies. First, inspired by Bronfenbrenner's work, I suggested an ecological perspective for teacher learning and analytic framework for considering different contexts and scales in the study of teacher conversations. Second, inspired by a larger set of ecological models for learning, I suggested three future research directions for studying these contexts and scales as interrelated. Ultimately, such perspective would open new spaces for thinking about, seeing, and designing for ecological teacher learning in SCPD settings.

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TRACING TAKE-UP ACROSS PRACTICE-BASED PROFESSIONAL DEVELOPMENT AND COLLABORATIVE LESSON DESIGN

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This study explored how two professional development approaches to reforming math instruction with different mechanisms for fostering change might have valuable synergies when used in tandem to support ***take-up****, i.e., teachers' acceptance, adoption, and incorporation of ideas into practice. This investigation of Practice-Based Professional Development and Collaborative Lesson Design found that take-up was a recursive process that occurred across both PD types as teachers iteratively moved between building and deploying knowledge. Both overarching and practice-specific struggles occurred during enactment, triggering shifts back to knowledge building. Struggles associated with learning to facilitate productive struggle included making sense of student thinking, identifying and providing appropriate scaffolds without lowering the cognitive demand, and helping students move from intuitive to mathematical arguments.*

Keywords: Professional development, Teacher knowledge, Instructional vision

Introduction

Decades of research suggest that aligning math instruction with how children learn math involves prioritizing student sense-making and instructional activities that require mathematical reasoning and productive struggle (Stigler & Hiebert, 1999; Boaler, 2016). Research-based reform of math instruction therefore involves changing the way teachers teach, shifting from an "I do, we do, you do" model to responsive engagement with students and their ideas, as well as changing the types of learning activities that are used in classrooms, shifting from repetitive practice and closed questions to rich, worthwhile math tasks. The challenge lies in finding an approach to professional development (PD) that addresses changes to the "how" of teaching (teachers' instructional practices) as well as to the "what" of teaching (the lesson plans and instructional activities teachers use in their classrooms).

Objectives

This study explored how two PD approaches to reforming math instruction with different mechanisms for fostering change might have valuable synergies when used in tandem to support the translation of a reform-oriented vision of math instruction into practice. While ample research has focused on opportunities for learning that occur within communities of practice like those present in these PDs, I have focused specifically on individual teacher *take-up*, i.e., teachers' acceptance, adoption, and incorporation of ideas into practice, in an effort to address a gap in existing research spotlighted by Lefstein et al. (2020). This investigation of Practice-Based Professional Development (PBPD) and Collaborative Lesson Design (CLD) was aimed at helping to conceptualize and identify instances of *take-up* and was guided by the following research questions:

1. How do individual teachers demonstrate take-up of ideas?
2. What connections are there between individual teacher take-up of responsive teaching practices in PBPD and take-up that occurs in CLD?

Theoretical Framework

Teaching is a complex art that involves not only what the teacher is doing but also what the students are asked to do-— interweaving of instruction and curriculum. Professional development (PD) that addresses changing one without changing the other can create "problems of enactment," i.e., teachers who want to teach in a new way but lack either the curriculum resources or the teaching skills to enact this new vision (Kennedy, 1999). Ineffective PD drains precious resources of time and money while fostering little change in classrooms, so endeavoring to better understand how take-up of ideas from PD occurs is a worthwhile avenue of investigation.

Collaborative Lesson Design (CLD) focuses on changing teaching by improving the planning process and lesson plans teachers use to enact lessons. In this professional development model, researchers and teachers work together within a community of practice and within a local context to co-create and continually revise lesson plans based on reform priorities (Hiebert & Morris, 2012). It is assumed that the lesson design cycle, which consists of planning, enactment, reflection, and revision, is a high-leverage opportunity for teacher learning and that the use of CLD could surface core teaching practices and give teachers the opportunity to become skilled in these practices through induction and refinement in their own classroom context. The challenges that arise when using the CLD model, however, include the possibility that without any training in a new set of pedagogical skills, teachers may face the "problem of enactment" described by Kennedy (1999), i.e., vision change without the necessary skills to enact the new vision, making teachers unable to execute the lesson plans as the creators intend.

Practice-Based Professional Development (PBPD), by contrast, focuses on changing teaching through pedagogical training in enacting core teaching practices, i.e., specific instructional skills including launching problems and facilitating discussions (Grossman, 2018). This model assumes that pedagogies of enactment, including representation, decomposition, and approximation, (Grossman et al., 2009) are high-leverage opportunities for teacher learning and that practices such as discussion facilitation are applicable in any classroom setting. Further, PBPD also assumes that learning core practices provides an opportunity for teachers to rethink their lesson design for their particular context. Challenges arise, however, in the transfer of pedagogical skills to specific educational contexts, and a parallel "problem of enactment" may occur if vision change occurs without the necessary resources to enact it (e.g., if a teacher, equipped with facilitation skills for high quality math tasks finds herself working with a curriculum devoid of those tasks). In the absence of a supportive community of practice and reform-oriented teaching materials, teachers may struggle to put their teaching practices to use as practice-based educators intend.

Research on professional learning communities and generative discourse has proliferated over the past two decades (Lefstein et al., 2020). Often, the focus of this research is on opportunities for learning and there is an implicit leap of faith involved in connecting what occurs in these communities with individual learning and particularly with what occurs in individual teachers' classrooms. Situative theory, which attends to "how various settings for teachers' learning give rise to different kinds of knowing" (Putnam & Borko, 2000, p. 6), provided a theoretical foundation for this case study research. I examined evidence of take-up in PBPD and CLD settings in order to ascertain whether and how these opportunities for learning impacted individual teachers' classroom practices.

Context

The Responsive Math Teaching (RMT) Project a research-practice partnership between university researchers and 13 schools within a large under-resourced urban school district, engages K-8 teachers and instructional coaches in three years of professional development focused on utilizing worthwhile math tasks as a vehicle for responsive teaching and for fostering student productive struggle (Responsive Math Teaching Project, 2021a). Since this is a departure from traditional teaching practices, participants spend Year 1 experiencing responsive math teaching as *learners* in monthly Math Circle PDs before moving on to focusing on how to teach responsively in Year 2. Prior to the pandemic, Year 2 professional development primarily utilized practice-based approaches supplemented with individual coaching to help participants shift their math instruction to align with the RMT instructional model (Responsive Math Teaching Project, 2021b), which emphasizes reform priorities that include student sense-making, use of low floor/high ceiling tasks, and teachers acting as facilitators of both productive struggle and rich, responsive discussions. In response to the move to virtual instruction and requests from participants for curriculum support to supplement PD focused on responsive teaching, the RMT Project began incorporating CLD in the fall of 2020. Although the RMT Instructional Model includes seven components, this study focused on four: 1) Launching a Task, 2) Facilitating Productive Struggle, 3) Making Student Thinking Visible, and 4) Connecting to a Mathematical Goal. These are the four practices that were represented, decomposed, and approximated most often during RMT PBPD and the four components of lesson planning emphasized most consistently during the CLD sessions involving planning, reflection, and revision of lessons.

Methods

Participants

RMT professional development offered to Year 2 participants consisted of six 5-week cycles that included one practice-based professional development (PBPD) session and two collaborative, cross-school, grade-specific lesson design (CLD) sessions: a planning session followed by a reflection/revision session. I utilized a comparative case study approach, purposefully selecting 14 participants who attended PD sessions most consistently. These participants represented classroom teachers and math leads (grades 1-8) from 10 different schools. All participants taught primarily in a virtual environment with some hybrid instruction integrated at the end of the year. Some participants were recommended for RMT PD by their principals and others were simply volunteers. In this paper, I focus on one case from the study, chosen because it is both illustrative of the overall study findings and because the focal participant was the "best case" (Patton, 1987) in the sense that she was strictly a classroom teacher and not a math coach, was not at a school that received supplemental coaching from RMT researchers or RMT-trained school personnel, and attended all PBPD and CLD sessions.

Data Collection

Data collected and reviewed included videotapes, audio transcripts, and chat transcripts of PBPD and CLD sessions; observational field notes; participant journals; participant responses to feedback forms for each cycle; and participants' artifacts of practice in the form of video and audio recordings and student work samples. For each of the 14 study participants, data from all of the aforementioned sources were compiled chronologically on a spreadsheet, wherein color coding was used to differentiate between PBPD, CLD Planning, and CLD Reflection session data. Direct quotes from comments made in PD sessions and from journal and chat entries were captured verbatim and parenthetical descriptions were added to contextualize each quote, including journal prompts, facilitator questions that prompted the comment or summaries of the preceding discussion. During the data collection and compilation process, analytic memos were written for each PD session summarizing observations about both individual and group take-up.

Data Analysis

A pilot study focusing on Cycle 1 data for five teachers was conducted in fall 2020 to create and test data analysis tools. I used the practice grain size and terminology established by the Core Practice Consortium (Grossman, 2018, pp. 186-189) to develop of a list of *aspects*, component parts of larger practices, and *approaches*, actions taken by teachers when enacting an aspect of a practice, using both emergent approaches and approaches included in RMT coaching materials. This list was reviewed and further refined with input from three RMT research team members. Although the full list is too extensive to include here, the aspects and approaches for Facilitating Productive Struggle (FPS) are shown below in Figure 1.

Practice	*Aspects*	*Approaches*
Facilitating productive struggle (FPS)	Supporting learner thinking without lowering the cognitive demand	• Relaunching the task with students who can't get started • Using models, diagrams, or acting out to help a student get unstuck • Using questioning and/or annotation to help a student make sense of their own thinking • Determining how much support/scaffolding is just enough • Pointing out an approach that has helped another student or group get started • Providing "just in time" tools or supplies • Coaching mathematical participation by suggesting a "what would happen if" scenario • Coaching mathematical participation by asking a student to convince others
	Providing opportunities for collaboration	• Providing opportunities for students to work in pairs or small groups • Scaffolding collaboration by orienting students toward each other • Strategically pairing students with similar or complementary strategies
	Monitoring student work	• Monitoring group work for progress and group dynamics • Keeping track of strategies being used • Looking for / capitalizing on opportunities to assign competence • Finding ways to observe student work in progress in a virtual setting

Figure 1: Aspects and Approaches for Facilitating Productive Struggle

Inductive data analysis during the pilot study also resulted in the identification of 9 emergent *take-up manifestations*, i.e., ways in which participants demonstrated take-up, shown in Figure 2 below. Kazemi & Hubbard (2008) drew on Cook & Brown's (1999) earlier work to distinguish between "*knowledge* that is possessed and *knowing* that is deployed in action" (p. 429), a distinction I used to sequence the manifestations in order of the level of action they entailed, moving from knowledge building to knowledge deployment during enactment and finally to sustained integration into classroom practice.

Noticing	Expressing awareness of a practice aspect. May occur with or without identifying the pedagogical reasoning behind the practice aspect.
Agreeing	Affirming another's comment about a practice aspect.
Asking	Asking a question or expressing confusion about a practice aspect.
Suggesting	Recommending a way to incorporate or improve upon a practice aspect. May occur with or without advocacy.
Prioritizing	Expressing a belief that a practice aspect is important.
Raising a concern	Noting a lag between one's vision of a practice aspect and one's ability to enact it.
Enacting	Executing a practice aspect. May be evident in a teacher's description of a lesson or in lesson video or audiotape artifacts.
Critiquing	Making critical comments about one's own execution of a practice aspect or giving critical feedback to others, including suggestions for improvement.
Sustaining	Integrating a practice aspect into regular classroom instruction beyond the task-based lessons enacted as part of the CLD professional development.

Figure 2: Manifestations of Take-Up (shown from early to late stage by gray coloration)

Working chronologically, data for each participant was coded inductively on two levels (Miles, Huberman, & Saldana, 1994): first for practice aspect using the list of approaches in Figure 1 and then for manifestations of take-up. For example, a participant journal comment might have been coded for FPS aspect "Supporting learner thinking" based on the presence of the approach "Using a model to help a student get unstuck" and for take-up manifestation "noticing." This coding made it possible to trace the development of each practice aspect chronologically over the course of the year in order to identify *threads*, i.e., progressions from low level to higher level take-up. Using a mapping process borrowed from expansive learning research (Bal, Afacan, & Cakir, 2018) to visually display these chronological threads also surfaced the presence of *struggles*, i.e., recurrent dilemmas that hampered take-up progress across one or more cycles.

Summary memos were written for each participant for each of the four focal practices. These memos were compared to exit interviews for triangulation purposes. Data was also validated via member reflection sessions in order to engage participants as collaborative partners and to ensure that their perspectives were accurately represented (Creswell & Poth, 2018). Dialogic engagement with strategically selected thought partners was used on 4 occasions to refine study design and to perform validity checks on data analysis processes and findings (Ravitch & Carl, 2016).

Results

While a number of additional findings emerged from this data analysis, here I will focus on two: 1) Take-up is a recursive, iterative process during which teachers cycle between knowledge building and knowledge deployment in action; 2) Two different types of struggles emerged that triggered a shift from knowing in action back into the realm of knowledge building: *overarching struggles* and *aspect-specific struggles.* Four types of overarching struggles spanning multiple practices were observed: *vision preceding skills, skill development with incomplete vision buy-in, belief that a practice cannot be enacted with particular content or with a particular group,* and *difficulties enacting practices virtually.* In addition to overarching struggles, other struggles emerged that were unique to specific practice aspects. Here, I focus on struggles that emerged from the FPS aspect "Support learner thinking without lowering the cognitive demand." These struggles included *difficulty making sense of student thinking different from one's own solution strategy, difficulty identifying and providing appropriate scaffolds in real time without lowering the cognitive demand,* and *difficulty helping students move from intuitive to mathematical arguments.* To illustrate these findings, I will focus on Melanie, a fifth and sixth grade math teacher whose case is representative of the larger group.

Demonstrations of take-up over time

Tracing take-up threads across PD sessions enabled me to construct narrative accounts of how take-up occurs, often progressing from low level take-up evident in noticing and agreeing remarks to higher level take-up evident in enactment and critiquing over the course of a single PD cycle or across multiple PD cycles as shown in Figure 3 below.

Tracing Take-Up for Melanie	
PBPD	Take-up of FPS aspect *support learner thinking without lowering the cognitive demand* first occurs when Melanie **agrees** with a comment about the difficulty of responding to a student strategy that you don't understand when the student hasn't articulated it well.
CLD	This practice aspect resurfaces in the next CLD planning session, when Melanie **prioritizes** finding ways to support students who can't find an entry point to start the task, noting, "*This is going to be the bulk for me. I'll have 20% of my class who will be able to navigate this task, who will be interested or intrigued. But then I do fear for the children who are just like, 'I don't know what to do.'*" Melanie continues to pursue this topic by **asking** the group how to best provide support. When Melanie returns to the CLD reflection session, she recounts **enacting** this practice aspect with a student in her classroom and **critiques** her enactment because she feels that she funneled the student to her own solution strategy instead of trying to follow the student's thinking.
PBPD	Melanie continues to focus on this FPS aspect, which surfaces again in a subsequent PBPD as **noticing** that FPS involves providing stepping stones that are enough but not too much of a stretch for the student. During the same PBPD, Melanie analyzes the RMT Instructional Model and **notices** that to support learners appropriately, "*you need enough scaffolds..You need to know your learners and have scaffolds so that the struggle is productive. It's not just straight struggle.*" Melanie's **noticing** continues as she analyzes a video of a teacher executing FPS and is able to pick out effective support moves that the

	teacher used. She also **suggests** another scaffold that the teacher in the video could have used.
CLD	In the next CLD planning session, Melanie **suggests** a scaffold for the task and also **prioritizes** "*helping a student get unstuck without doing it for them*," noting that the scaffold should only be used if a student were struggling and should not be given to the whole class just in case. After **enacting** the planned lesson, Melanie **critiques** her enactment specifically in terms of how much support she provided to the students and **raises a concern** about the balance she is striking in her class between supporting students and doing the work for them. She notes, *I don't feel like my kids take risks anymore. When I say work on it..they just sit there and wait knowing that I'm going to pop over to my whiteboard and sort of draw something, and you know, help them out, and I actually think that they're right. So, I'm coming to this conclusion: I'm taking on too much of the load.* When Melanie shares a video clip of her lesson, she **asks** whether she had responded to a student question by giving too much assistance. As the group discusses her video, Melanie **agrees** with an alternative talk move suggested by the group. Moving beyond agreement, Melanie **prioritizes** supporting students by making an FPS "talk moves wall" behind her computer screen with post-it reminders of questions to ask that she can refer to during her instruction. Melanie later completes a feedback form on which she describes using questions from the RMT FPS framework to support students in explaining their thinking during her regular, daily instruction, a **sustained** effort to integrate this practice aspect into her teaching.

Figure 3: Example of a Narrative Constructed from Take-Up Tracing

Melanie's narrative above highlights a common overarching struggle I have termed *vision preceding skills.* Here, her competence enacting FPS lagged behind her vision of what FPS should look like, prompting iterative returns to knowledge building. Melanie's narrative also exemplifies *difficulty making sense of student thinking different from one's own solution strategy* which surfaced as funneling a student towards Melanie's own solution path rather than helping her make progress on her own. Also evident in Melanie's narrative was *difficulty identifying and providing appropriate scaffolds in real time without lowering the cognitive demand.* Struggle points in Melanie's narrative and the resulting shifts into forms of take-up associated with knowledge building are visually displayed in Figure 4 below.

As evident in Figure 4, struggles often surfaced during enactment and reflection, prompting a renewed effort to build knowledge in order to refine skills. Narratives such as Melanie's examined across multiple cycles made clear that take-up is not simply a linear progression from low level to high level but rather an iterative process across both PD contexts in which noticing, asking questions, and suggesting remain essential in fostering enacting and critiquing and ultimately in the honing of teaching practices.

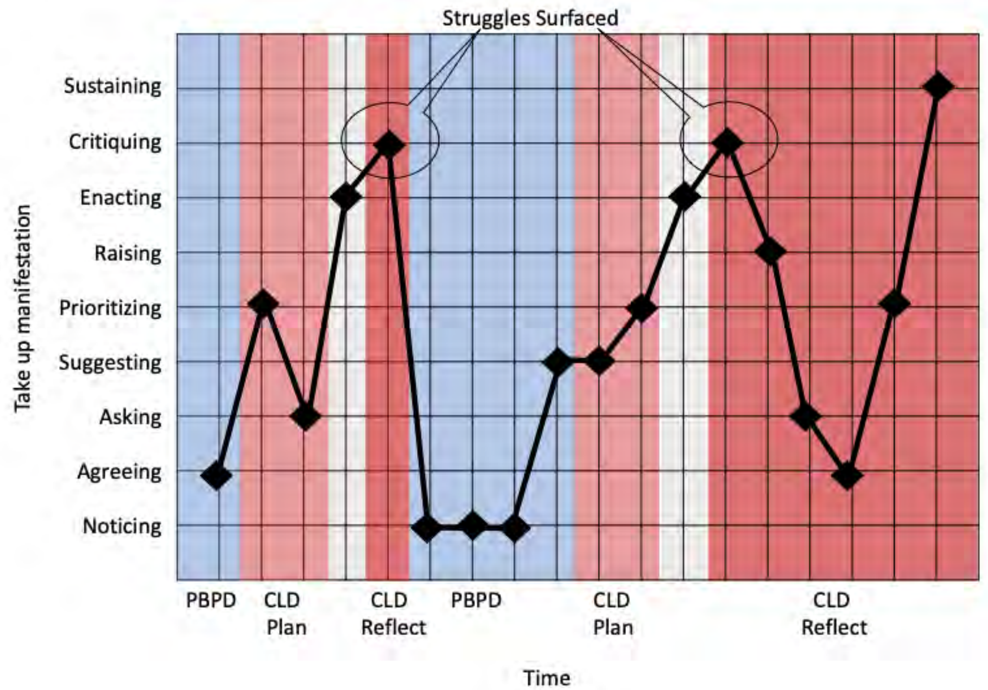

Figure 4. Melanie's Take-Up of FPS Aspect "Supporting Learner Thinking"

Discussion

Both take-up threads and their mapping made evident synergies between CLD and PBPD by highlighting instances when one form of PD provided opportunities for increased take-up of practice aspects originally taken up in the other. In early cycles, PBPD sessions focused on representation and decomposition of practices and most often fostered knowledge-building forms of take-up, including noticing, agreeing, and asking. Early on, suggesting, prioritizing, enacting and critiquing were primarily evident in CLD. CLD provided an early and consistent impetus to move beyond knowledge building and into knowledge deployment—beyond learning into experimentation, as 8th Grade teacher Leann noted, "*If we didn't need to do it for this [CLD Reflection Session], I might've not pushed myself to get it in.*" As the year progressed, however, and PBPD incorporated rehearsal and reflection on video artifacts, the types of take-up became more varied across both forms of PD. As learners focused their attention on specific practice aspects and specific struggles that emerged when enacting the practices, both PBPD and CLD sessions showed an uptick in the number of take-up manifestations across participants.

In my effort to focus on individual take-up, by no means did I intend to downplay the critical role played by the communities of practice formed within and across both types of PD. Discourse in each setting was not only generative but also fostered collective take-up in ways that were beyond the focus of this study. Instead, my intention was to shed some light on how participation in these communities impacted individual teachers' classroom instruction, a path less trodden in the field (Lefstein et al., 2020). Understanding how group and individual take-up intersect, with an eye towards classroom impact, remains an area in need of further investigation.

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EXAMINING THE USE OF VIDEO ANNOTATIONS IN DEBRIEFING CONVERSATIONS DURING VIDEO-ASSISTED COACHING CYCLES

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This study examined how mathematics coaches leverage written annotations to support professional discourse with teachers about important classroom events during synchronous debriefing conversations. Coaches and teachers created the annotations while asynchronously watching video of an implemented lesson as part of online video-assisted coaching cycles. More specifically, this project examined the extent to which a coach and teacher discussed the annotations during a debrief conversation in a coaching cycle. We present a rationale for needing new knowledge about the relationships between video annotations and professional discourse as well as the potential implications of such knowledge.

Keywords: Inservice Teacher Professional Development, Research Methods, Coaching

Coaching cycles have become a popular professional development activity to support teachers to plan for, implement, and reflect on ambitious instructional practices (Gibbons & Cobb, 2016). A coaching cycle typically consists of three parts including a coach and teacher collaboratively: (a) planning a lesson around specific learning outcomes for students and the use of instructional practices necessary to support student learning, (b) implementing the lesson and instructional strategies, and (c) reflecting on the success of the lesson using evidence of student learning and the teacher's use of new instructional strategies during a debrief conversation (Bengo, 2016; West & Staub, 2003).

Professional developers use video during coaching cycles for two primary reasons. First, video recording the lesson implementation, when paired with synchronous planning and debriefing conversations using distance technologies, allow coaching cycles to occur in a fully online space (Carson, Callard, Gillespie, Choppin, & Amador, 2019; Matsumura, Correnti, Walsh, Bickel, & Zook-Howell, 2019). Second, viewing video of one's own teaching has been shown to effectively support teachers to identify areas of improvement by providing a durable image of what occurred (Borko, Jacobs, Eiteljorg, & Pittman, 2008; Harlin, 2014; Rosaen, Lundeberg, Cooper, Fritzen, & Terpstra, 2008). Although using video during online coaching cycles has potential benefits, few researchers have examined how the content of what teachers and coaches notice during the viewing of lesson videos impacts debriefing conversations. Because the decisions made by a coach regarding how to facilitate coaching cycle conversations have been shown to significantly impact the learning opportunities of the teacher (Costa & Garmston, 2016; Heineke, 2013), this study examined the question, how do coaches and teachers discuss the annotations from lesson videos during debrief conversations within coaching cycles?

Theoretical Framing

Teacher noticing has become a common construct within research on mathematics teachers (Sherin, Jacobs, & Philipp, 2011; Star & Strickland, 2008). Teacher noticing describes the ability to sift through the many events taking place simultaneously within a classroom to identify important moments worthy of attention (Walkoe, 2015). The ability to notice what is important during complex classroom situations is a key characteristic of expert teachers (Berliner, 2001). In their framework, van Es and Sherin (2002) expanded the idea of professional noticing beyond simply identifying salient moments into three aspects: (a) identifying what is important during a teaching event, (b) reasoning about the event, and (c) making connections between this specific event and larger principles of teaching and learning. Productive teacher noticing also involves the ability to attend to and interpret student thinking so teachers can make decisions to respond in ways that positively impact student learning (Jacobs, Lamb, & Phillip, 2010; Miller, 2011).

A teacher or coach using an annotation to make their thinking public about a moment in a lesson video corresponds to the act of professional noticing (Amador, Carson, Gillespie, & Choppin, 2019). Furthermore, a teacher and coach have many choices throughout the annotation process, both about the events to mark as noteworthy and about how they communicate their thinking about these events of interest (Mason, 2011). Sherin (2007) adopted the term *professional vision* to emphasize the role of *selective attention* as a key subprocess of mathematics teachers' professional vision. Selective attention describes how a teacher focuses their attention given the many things happening within a single moment. In this study, we examined the interaction between the annotations (i.e. the highlighted moments) and what the teacher and coach discussed during the debrief conversation (e.g., their selective attention).

Related Literature

Several researchers have examined different aspects of the use of video within the specific context of online coaching cycles. Matsumura et al. (2019) found the use of video in online coaching cycles supported teachers to use new instructional practices, leading to improved student participation during class discussions. Gregory et al. (2017) argued that teachers involved in video-based online coaching cycles had improved student achievement, peer interactions, and a reduction in racial disparities. Both researchers made claims about the impact of video-based online coaching cycles, but neither articulated the ways in which viewing video supported professional discourse between a coach and teacher within the debrief conversations. Amador et al. (2019) examined the differences in the annotations created by coaches and teachers while watching lesson video within video-assisted online coaching cycles. They found coaches were more likely than teachers to focus on students and make connections within their annotations but did not explore the ways the coach and teachers made use of the annotations during the debrief discussion.

We explored the ways teachers and coaches took up the recorded noticing (i.e. the annotations) during a debrief conversation. Specifically, the study focused on how teachers and coaches used the annotations to conduct a debriefing conversation.

Methods

This study occurred within the coaching activity of a larger, fully online, professional development project created for middle school mathematics teachers working in rural areas (Choppin, Amador, & Callard, 2015; Choppin, Amador, Callard, Carson, & Gillespie, 2020). The project consisted of three parts: an online course, online teaching labs, and video-assisted online coaching cycles designed to improve teacher practices for implementing high cognitive demand tasks and facilitating mathematical discourse (Smith & Stein, 2011). Using a cohort model, 21 teachers from grades 5-8 participated in the project as part of two cohorts, each lasting two years. In the coaching cycle portion of the project, teachers were partnered with coaches using a content-focused coaching model (West & Staub, 2003).

Participants

This study focused on seven coaches and their assigned teachers who engaged in video-assisted coaching cycles in the professional development project. Each coach was assigned one or two cohort teachers, resulting in nine coach/teacher pairings. Data were collected from the debriefing conversations of the coach/teacher pairs in addition to the annotations created by the coach and teacher when watching the lesson video prior to the discussion.

Context: Video-assisted Coaching Cycles

The goal of each video-assisted coaching cycle was to support participating teachers to successfully implement productive discourse practices (e.g., Smith & Stein, 2011) discussed during the online course and teaching labs. Each coaching cycle followed the same structure and utilized both synchronous and asynchronous activities (see Figure 1). First, the coach and teacher participated in a planning discussion using video conferencing technology, Zoom, focused on a lesson proposed by the teacher. Guided by the content-focused coaching model, participants collaboratively analyzed the mathematical lesson goals, the tasks used in the lesson, the anticipated student strategies, and the instructional practice goals for the teacher (West & Staub, 2003). Following the planning meeting, the teacher video- and audio- recorded the teaching lesson using Swivl Technology (automated video camera and recording). After the lesson was taught, the coach and teacher asynchronously watched and annotated the lesson video. Annotations were written comments about the contents of the lesson video. The coaching cycle concluded with the coach and teacher engaging in a forty to sixty minute debrief discussion that utilized the annotations to reflect on the lesson.

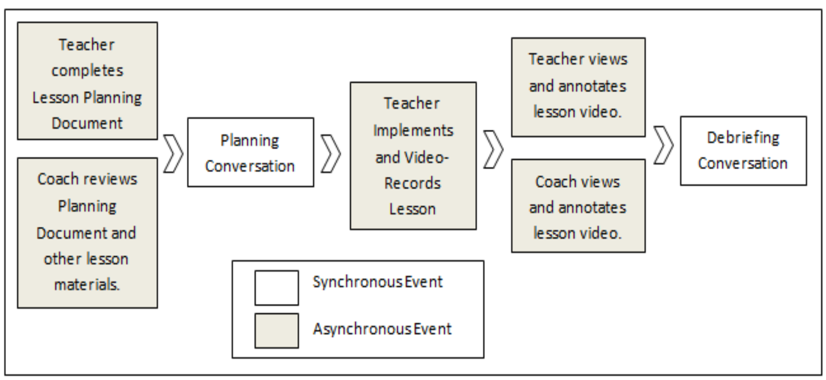

Figure 1*:* Video-assisted coaching cycle process

Data Collected

We analyzed the video annotations created by the coaches and teachers and the corresponding debriefing conversations from the third coaching cycle for nine coach-teacher pairs. The third coaching cycle was selected because it allowed the teachers and coaches time to become accustomed to each other and the video-assisted coaching process (Matsumura, Bickel, Zook-Howell, Correnti, & Walsh, 2016). Using the third coaching cycle data resulted in the analysis of video annotations from nine video-recorded lessons and the nine corresponding debrief conversations. All nine debrief conversations were video-recorded using Panopto screen-capture and then transcribed. Transcripts were parsed into stanzas which including the coach's discursive move and/or the teacher's discursive move about a particular topic (Sa , 2013). A single video annotation served as the unit of analysis.

Data Analysis

The data analysis process began with the researcher entering annotation data into a spreadsheet. This data included: the annotation text, the author of the annotation, the number of the annotation in the full set, and the timestamp connecting the annotation to a specific moment in the lesson video. The researcher then watched the video of the debrief conversations from the third coaching cycles and read the transcriptions of the conversations.

To identify instances when coaches and teachers discussed the annotations during debrief conversations, two binary variables were created to code the presence of a written annotation within an instance of verbal discussion. Both variables were assigned a code of yes or no for each annotation. The first variable, *indicated connection to annotation*, described instances when the coach or teacher clearly indicated that their verbal statement connected to a written annotation. The second variable, *verbatim use of annotation,* described instances when the verbal statement of the coach or teacher matched the written language in the annotation verbatim. If the coach or teacher explicitly indicated their verbal statement connected to an annotation or if a verbal statement matched a written annotation verbatim, there was reliable evidence that an annotation had been taken up in conversation. In instances when indication was coded no and it was debatable if a significant portion of an annotation matched a spoken statement verbatim, the researcher used the video of conversation to consider the context. In these ambiguous instances, if the annotation was present on the coach's screen at the time of the spoken statement, such instances were coded as verbatim. If the annotation was not on the coach's screen at the time of the spoken statement, such instances were coded as not verbatim.

An annotation was considered to have been discussed in the conversation if either variable was coded yes, since the presence of either variable provided a reliable indication that the written annotation influenced the discussion. If both indication and verbatim were coded no, the annotation was considered not to have been discussed in the conversation (see Figure 2). A single annotation could have been discussed multiple times throughout a conversation. Therefore, each time an annotation was brought into the conversation, the annotation was coded using the two variables and labeled as a new instance of annotation discussion.

Annotation considered present in the debriefing conversation. (green)

Annotation considered not present in the debriefing conversation. (red)

		Verbatim Does the spoken statement of the coach/teacher match language from the annotation verbatim?	
		Yes (Verbatim)	**No (Not Verbatim)**
Indication Does the coach/teacher clearly indicate (via spoken language or through technology features such as sharing a screen) that their verbal statement connects to an annotation?	**Yes (Indication)**	*Verbatim with Clear Indication*	*Not Verbatim with Clear Indication*
	No (No Indication)	*Verbatim with No Indication*	*Not Verbatim and No Indication*

Figure 2: Coding scheme for determining the presence of an annotation during debriefing conversations.

To illustrate the coding process with these two variables, an example is provided. Coach Alvarez created an annotation, "And what did you learn about students' understanding? How did this inform your lesson?" During the debriefing conversation, Alvarez said,

> I wondered then, again at 7:35, just what you thought about what you learned about students understanding, from the warmup, and then how that informed your lesson. Were there takeaways that you had from the warmup that made you think differently about your lesson?

Because Alvarez explicitly mentioned the timestamp of the annotation, she provided a clear indication her question was connected to the annotation. Therefore, indication was coded as yes. Alvarez also included the phrases "learned about student understanding" and "how that informed your lesson" in her verbal questions. Therefore, verbatim was also coded as yes.

If an annotation was considered to be discussed within an instance of the conversation, four additional codes were applied to each instance of annotation discussion to gain further insight into the research question. First, we coded for who initiated the conversation about the annotation, the coach or teacher. Second, we coded for who created the annotation. Third, we recorded the stanza number from the transcript in which the instance of annotation discussion began. Fourth, we recorded the stanza number from the transcript in which the discussion of the annotation ended. Coding the starting and ending stanzas for an instance of annotation discussion allowed us to analyze the length of discussion about an annotation and to determine if a single annotation was discussed multiple times throughout the conversation.

As an example of this coding process, coach Lowery created the annotation, "What do other people think about what he just said about using the difference of 5? His point highlights the relationship and bears repeating by another voice (preferably a peer before the teacher)." In stanza 12 of the debrief transcript, coach Lowery initiated conversation about this annotation. The discussion about the annotation continued until the end of stanza 13 when the conversation moved to a topic not contained in the annotation. In stanza 16, teacher Fernandez initiated additional conversation about this annotation which continued until the end of stanza 17. Therefore, the researcher recorded two instances of annotation discussion for this coach-created annotation; one initiated by the coach with a starting stanza of 12 and an ending stanza of 13 and the second initiated by the teacher with a starting stanza of 16 and an ending stanza of 17.

Findings

In total, we analyzed 308 annotations the nine coach/teacher pairs created during nine debriefing conversations. Of this total, coaches created 158 annotations and teachers created 150 annotations. In analyzing the extent coaches and teachers talked about the annotations, the process revealed 96 of the 308 annotations were taken up, resulting in an average of 10.7 annotations discussed per conversation. Because some annotations were discussed more than once during a conversation, 110 instances of annotation discussion were identified resulting in an average of 12.2 instances of annotation discussion per conversation (see Table 1). However there existed variability between the coach/teacher pairs with respect to their verbal uptake of written annotations. For example, coach Braithewhite and teacher Summers had only three instances of annotation discussion about three separate annotations despite collectively creating 50 annotations prior to the conversation. Conversely, coach Bishop and teacher Parsons had 23 instances of annotation discussion about 21 separate annotations after creating 23 total annotations prior to the discussion. This suggests differences in how these coach/teacher pairs interpreted the role of the annotations during video-assisted coaching cycles. This finding also highlights a range for the number of annotations that can be discussed within a single debrief conversation.

Table 1: Annotation Discussion Counts by Coach/Teacher Pair

Coach/Teacher	Annotations Created	Annotations Discussed	Instances of Annotation Discussion
Alvarez/Graham Marks	59	11	13
Bishop/Parsons	23	21	23
Bishop/Wise	14	9	12
Braithewhite/Summers	50	3	3
Hale/Swanson	47	9	11
Lowery/Fernandez	25	11	15
Riess/Larson Waters	27	10	10
Riess/Sandoval	23	5	7
Whilton/Morrison	40	15	16
Average	34.2	10.7	12.2

To further examine the extent coaches and teachers talked about the annotations during debriefing conversations, the percentage of transcript stanzas containing instances of annotation discussion were calculated. The number of stanzas containing an instance of annotation discussion was divided by the total number of stanzas in the conversation. For example, the Bishop/Wise conversation transcript contained 40 stanzas. Instances of annotation discussion occurred during stanzas eight and nine and in stanzas 18 through 29. Therefore, 14 of the 40 total stanzas (35.0%) contained instances of annotation discussion. When this analysis was done for all 364 stanzas within the nine debrief conversations, 41.4% of stanzas contained an instance of annotation discussion. This finding indicated annotations were taken up in debrief conversations but were not the sole focus on conversation since less than half of the stanzas contained instances of annotation discussion. Similar variability also existed when comparing the percentage of stanzas containing instances of annotation discussion between different coach/teacher pairs. For example, in the debrief conversation transcript between coach Whilton and teacher Morrison, 62.5% of the stanzas contained instances of annotation discussion. However, for coach Hale and teacher Swanson, only 20% of the conversation stanzas were found to have instances of annotation discussion.

Table 2: Instances of Annotation Discussion within Conversational Stanzas

Coach/Teacher	Total Number of Stanzas in Conversation	Number of Stanzas Containing an Instance of Annotation Discussion	Percentage of Stanzas Containing an Instance of Annotation Discussion
Alvarez/Graham Marks	37	23	62.2%
Bishop/Parsons	67	30	44.8%
Bishop/Wise	40	14	35.0%
Braithewhite/Summers	25	4	16.0%
Hale/Swanson	40	8	20.0%
Lowery/Fernandez	41	20	48.8%
Riess/Larson Waters	43	14	32.6%
Riess/Sandoval	31	12	38.7%

Whilton/Morrison	40	25	62.5%
Average	40.6	16.7	41.1%

Analyses also explored whether coaches or teachers were more likely to initiate conversation about the annotations. Coaches initiated conversation about the annotation more frequently than teachers. Of the 110 instances of annotation discussion, 91 (82.7%) were initiated by the coach and only 19 of the instances (17.3%) were initiated by the teacher. This finding was consistent across coach/teacher pairs as the coach initiated more than 70% of instances of annotation discussion for seven of the nine pairs.

Annotations coaches created were discussed more frequently than those teachers created despite the fact that roughly half of the annotations for the nine coach/teacher pairs were teacher created. Of the 110 instances of annotation discussion, 74 (67.3%) focused on coach-created annotations compared to only 36 of the instances (32.7%) focused on teacher-created annotations. This trend was found within instances of annotation discussion initiated by both teachers and coaches. Of the 19 instances in which teachers initiated discussion about annotations, 13 of these instances (68.4%) focused on coach-created annotations. Of the 91 instances when coaches initiated discussion about annotations, 61 of these instances (67.0%) focused on coach-created annotations. When combining the findings about initiating annotation discussion and the creator of the annotations, coaches initiating conversation about coach-created annotations was the most common occurrence with 61 of the 110 (55.5%) instances of annotation discussion meeting these criteria. The least common occurrence was teachers initiating conversation about a teacher-created annotation. This occurred in only six of the 110 (5.5%) instances of annotation discussion (see Table 3 for additional information).

Table 3: Comparison of Annotation Discussion Initiation and Annotation Authorship

	Instances of Discussion about Coach-Created Annotations	Instances of Discussion about Teacher-Created Annotations	Total
Instances of Coach-Initiated Annotation Discussion	61 (55.5%)	30 (27.3%)	91 (82.7%)
Instances of Teacher-Initiated Annotation Discussion	13 (11.8%)	6 (5.5%)	19 (17.3%)
Total	74 (67.3%)	36 (32.7%)	110 (100%)

These findings suggest coaches were more likely to initiate conversation about the annotations and tended to discuss annotations they created. These findings highlight that coaches exerted significant influence regarding the selection of annotations to discuss and focused on annotations containing their own ideas about the lesson.

Discussion

Findings from this study contribute to existing literature on coaching in three ways. First, prior studies have claimed that the use of video within coaching cycles held the potential to improve teaching practices (e.g., Gregory et al., 2017; Matsumura et al., 2019). However, prior studies did not provide any description about how the coaches and teachers took up the lesson video and annotations during conversations, leaving the actions of both coaches and teachers within the activities unknown. For professional development providers wishing to successfully implement video-assisted coaching, the findings from this study partially fill this crucial gap by highlighting a range for the number of annotations discussed within a single debrief conversation. This knowledge may support both teachers and coaches in selecting a limited number of focal annotations from a larger set when preparing for a debrief discussion. Additionally, findings about the percentage of stanzas containing annotation discussion also may support coaches and teachers to prepare for reflective discussions, which often are constrained by a limited amount of time. For example, this study revealed that even in extreme cases, less than two-thirds of the stanzas contained annotations discussion and eight of the nine coach/teacher pairs discussed 15 or fewer annotations within a single conversation. Thus, when given a fixed amount of time for a debrief conversation (commonly constrained by school logistics such as the length of a preparation period), these findings may support a coach and teacher to set realistic goals regarding how much of their conversation could be dedicated to annotation discussion.

Second, these findings contribute the existing literature about variability of coaching actions within coaching cycles. Prior studies have shown the actions of coaches when engaging teachers in conversation can vary significantly (e.g., Heineke, 2013; Sailors & Price, 2015). This study extends these claims about variability of coaching actions to the ways in which coaches take up annotations during debriefing conversations. The large range found in both the *number of annotations* discussed in the debriefing conversations and the *number of transcript stanzas* containing instances of annotation discussion suggest significant variability in the ways the coach leveraged the annotations to catalyze discussion. This variability may be due to different interpretations about the role of the annotations to support teacher learning. For example coach Braithewhite and teacher Summers created 50 annotations prior to the debriefing conversation. Yet, only three of these annotations were discussed with 16% of the transcript stanzas containing instances of annotations discussion. Conversely, coach Bishop and teacher Parsons created 23 annotations prior to the debrief discussion and discussed 21 of these annotations. In this case, 44.8% of the stanzas contained instances of annotations discussion. These differences suggest coach Bishop and coach Braithewhite may have held different views about how to use the annotations to initiate productive reflective opportunities for teachers. Such differences may have significant impact on teachers because diversity in the actions of coaches has been shown to influence learning opportunities of teachers (Heineke, 2013; Sailors & Price, 2015). Although these findings cannot be directly used to make claims about teacher learning, they do suggest teachers such as Parsons and Summers had different learning experiences when engaging in reflective discussion about video annotations.

Third, the finding that coaches initiated more annotation discussions than teachers and the tendency of coaches to initiate conversation about their own annotations connects to claims made by Mosley Wetzel and colleagues (2017) regarding implications of power within coaching conversations. They argued a coach holding a formal position of power is often perceived as being more accomplished and knowledgeable than the teacher. Therefore, the actions of the coach and their position of power may have implications for a teacher's learning experience. Akin to coaches positioning themselves as the authority through the use of *directive* discourse moves versus positioning the teacher as the authority through the use of a *reflective* discourse moves (e.g. Ippolito, 2010), coaches tendency to initiate conversation about their own annotations and the infrequency of teachers initiating conversations about their own annotations raises new questions about the power dynamics within video-assisted coaching cycles.

Specifically, if the goal of written annotations is to support teachers to verbally reflect on their practice, should coaches strive to position teachers to more frequently initiate conversation about their own annotations? Or, is it more beneficial for teachers to have coaches initiating conversation about coach-created annotations? Future research should examine these questions regarding how the differences in the ways coaches use annotations during debrief conversations impact teachers' learning experiences.

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PROFILES OF TEACHERS' EXPERTISE IN PROFESSIONAL NOTICING OF CHILDREN'S MATHEMATICAL THINKING

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Noticing children's mathematical thinking is foundational to teaching that is responsive to children's thinking. To better understand the range of noticing expertise for teachers engaged in multiyear professional development, we assessed the noticing of 72 upper elementary school teachers using three instructional scenarios involving fraction problem solving. Through a latent class analysis, we identified three subgroups of teachers that reflected different profiles of noticing expertise. Consideration was given to the noticing component skills of attending to children's strategy details, interpreting children's understandings, and deciding how to respond on the basis of children's understandings. We share theoretical and practical implications for not only the three profiles but also our choice to explore separately two versions of deciding how to respond (deciding on follow-up questions and deciding on next problems).

Keywords: Teacher Noticing, Professional Development, Elementary School Education

Our work is aligned with a vision of teaching that is responsive to children's mathematical thinking. In this vision, teachers attend to and pursue the substance of children's ideas and important mathematical connections within those ideas (Richards & Robertson, 2016). This type of responsive teaching builds on research on children's mathematical thinking and connects to numerous policy recommendations, but has proven challenging to achieve (Cai, 2017; National Council of Teachers of Mathematics, 2014; National Research Council, 2001).

Efforts to support teachers in achieving this vision have included attention to core practices of teaching (Grossman, 2018; Grossman et al, 2009; McDonald et al., 2013). We focus on one of the core practices—teacher noticing—that has been researched extensively in mathematics education (for compilations, see Schack et al., 2017; Sherin et al., 2011). Although multiple conceptions exist, *teacher noticing* fundamentally refers to how teachers focus their attention and make sense of what children say and do so that teachers' instructional responses are productive.

We chose to focus on teacher noticing of children's mathematical thinking, with an awareness that this type of noticing is only one of many that teachers must use to be successful. Examples of noticing research with different foci include curricular noticing (Amador et al., 2017), racial noticing (Shah & Coles, 2020), and noticing of participation and status (Kalinec-Craig, 2017; Wager, 2014). We view these different types of noticing as potentially mutually supportive in that using one focus as a starting point can provide entry into other types of noticing. In this study, we foreground noticing children's mathematical thinking as foundational for teaching that is responsive to children's thinking—one can only be responsive to what one has noticed. Further, research has shown that teachers usually do not gain this expertise solely from teaching experience (Copur-Gencturk & Rodrigues, 2021), but it can be learned (see, e.g., Casey & Amidon, 2020; Lee, 2019; Roth McDuffie et al., 2014; Schack et al. 2013; Simpson & Haltiwanger, 2017; van Es & Sherin, 2008).

Our conception of teacher noticing comes from our earlier work on professional noticing of children's mathematical thinking in which we identified three component skills: (a) attending to children's strategy details, (b) interpreting children's understandings reflected in their strategies, and (c) deciding how to respond on the basis of children's understandings (Jacobs et al., 2010). This final skill—deciding how to respond—refers to teachers' intended responses because teacher noticing is invisible, happening prior to teachers' observable responses. The three component skills are conceptually and temporally linked, and in the midst of instruction, they often occur almost simultaneously. They are not ends in themselves, but collectively are foundational for making productive instructional responses that build on children's thinking.

In this study, we extended our earlier work by identifying profiles of noticing expertise that include consideration of teachers' expertise with each of the component skills. By better understanding how teachers in multiyear professional development (PD) take up and engage in the complex practice of teacher noticing, we should be better able to support them in developing this expertise. Thus, we investigated the following research question: *What meaningful profiles of teachers' expertise in professional noticing of children's mathematical thinking exist among teachers engaged in multiyear PD?*

Methods

The data were drawn from a larger PD design study in which the goals included building a model of teaching that is responsive to children's mathematical thinking (Empson & Jacobs, 2021, these proceedings). In this paper, we focus on one instructional practice in the model—professional noticing of children's mathematical thinking—and use teachers' responses to a noticing assessment to identify profiles of expertise across the component skills.

Participants

We assessed the noticing expertise of 72 upper elementary school teachers—68 classroom teachers (grades 3–5) and 4 teaching specialists (instructional facilitators, resource teachers, etc.)—who had voluntarily enrolled in our PD. The teachers (64 females and 8 males) were generally experienced, with their teaching experience ranging from 2 to 36 years ($M = 11.8$).

To develop our noticing profiles, we purposefully studied teachers who were at different points in our 3-year PD and worked in a variety of contexts. Specifically, data were collected during one school year when teachers were at the end of their first ($N = 22$), second ($N = 26$), or third ($N = 24$) year of PD. Teachers worked in 3 districts in a state in the southern United States. The districts had varied instructional histories in that all administrations had endorsed teaching that was responsive to children's thinking, but for different amounts of time. Further, teachers were drawn from 36 schools that reflected a range of student demographics. Across the schools, students who qualified for free or reduced-cost lunch ranged from 10%–98% ($M = 59.7\%$) and students classified as Limited English Proficiency ranged from 2%–85% ($M = 33.3\%$). Student race and ethnicity classifications also varied. White students ranged from 6%–85% ($M = 49.6\%$), Hispanic students ranged from 4%–81% ($M = 34.8\%$), Black students ranged from 0%–20% ($M = 4.3\%$), Hawaiian and Pacific Islanders students ranged from 0–31% ($M = 5.4\%$), and students with race and ethnicity classifications of "other" ranged from 0%–14% ($M = 6.0\%$).

Professional Development

Our PD consisted of more than 150 hours of face-to-face workshops offered over 3 years, and the overall goal was to help teachers develop expertise in teaching that is responsive to children's mathematical thinking, with special emphasis on the teaching and learning of fractions (Jacobs, Empson, Pynes, et al., 2019). Key resources included research-based frameworks of children's mathematical thinking (Carpenter et al., 2015; Empson & Levi, 2011) and research-based frameworks of instructional practices, such as noticing children's mathematical thinking (Jacobs et al., 2010) and questioning to support and extend children's mathematical thinking (Jacobs & Ambrose, 2008; Jacobs & Empson, 2016).

Noticing Assessment

We captured teachers' noticing expertise using a written assessment that was structured around three instructional scenarios in which teachers had opportunities to notice children's thinking linked to fraction story problems. The scenarios were conveyed via authentic, strategically selected artifacts of practice—a classroom video, a set of children's written work, and a video of a teacher's conversation with one child. We chose the three scenarios because we wanted to capture teachers' noticing expertise throughout the multiple facets of their work.

For each instructional scenario, teachers responded, in writing, to prompts linked to the component skills of noticing children's mathematical thinking (see Table 1). Note that we included two prompts for the final skill of deciding how to respond. We chose to keep separate the prompts (and scores) for deciding on follow-up questions and deciding on next problems because the two categories of deciding how to respond are conceptually distinct, and we wanted to better understand their relationship to teachers' overall noticing expertise.

Table 1: Writing Prompts for the Noticing Assessment

<table>
<tr><th colspan="2">Noticing Component Skills</th><th>Sample Writing Prompts</th></tr>
<tr><td colspan="2">Attending to children's strategy details</td><td>Please describe in detail what you think each child did in response to this problem.</td></tr>
<tr><td colspan="2">Interpreting children's understandings</td><td>Please explain what you learned about these children's understandings.</td></tr>
<tr><td rowspan="2">Deciding how to respond on the basis of children's understandings</td><td>Deciding on follow-up questions</td><td>Imagine that you are the teacher of these children and you want to have a one-on-one conversation with one of them. Which child would you choose? Describe some ways you might respond to their work on this problem, and explain why you chose those responses.</td></tr>
<tr><td>Deciding on next problems</td><td>Imagine that you are the teacher of these children. What problem or problems might you pose next? What is your rationale?</td></tr>
</table>

Scoring

Each teacher received 12 noticing scores—4 scores for the noticing component skills within each of the 3 instructional scenarios. Drawing on our past research (Jacobs et al., 2010), scoring was done holistically on a 0–2 scale indicating the extent to which we had evidence for teachers' engagement with children's mathematical thinking: lack of evidence (0), limited evidence (1), or robust evidence (2). We double-coded all data (in a blinded format) and interrater reliability for all 12 noticing scores was 80% or higher. Discrepancies were resolved through discussion.

For the attending-to-children's-strategy-details score, we looked for inclusion of mathematically significant details such as how children used drawings to represent and partition quantities, how they combined fraction amounts, or how they described amounts using fraction names or notation. For the interpreting-children's-understandings score, we did not seek a single best interpretation but instead looked for an emphasis on what children understood (versus did not understand) and reasoning that was consistent with and grounded in the children's strategy details. For the deciding-on-follow-up-questions score, we did not seek a single best set of follow-up questions but instead looked to see if the questions and rationales were reasonable, meaning that they were consistent with the children's strategies and understandings. We also looked to see if the questions centered children's thinking not only by asking about details *inside* their existing strategies (Jacobs, Empson, Jessup, & Baker, 2019) but also by leaving room for children's ways of thinking (versus funneling children toward a particular strategy or answer [Wood, 1998]). For the deciding-on-next-problems score, we did not seek best next problems but instead looked for problems that were consistent with teachers' rationales. We further looked to see if the rationales linked to children's understandings and left room for children's thinking.

Findings

Our goal was to identify profiles of noticing expertise across the noticing component skills. We began by determining that the internal consistency for the noticing assessment was adequate, as indicated by Cronbach's alpha of .77. We then conducted a latent class analysis to empirically identify subgroups of the 72 teachers displaying similar patterns of responses across their 12 scores—4 scores for the noticing component skills within each of the 3 instructional scenarios. We considered the response patterns for these subgroups as profiles of teachers' expertise in professional noticing of children's mathematical thinking. Our goal was not to "label" teachers but instead to better understand variation in teachers' expertise in this practice.

We considered a 3, 4, and 5-profile solution, and we chose the 3-profile solution based on (a) the lowest Bayesian Information Criteria goodness-of-fit statistic (Schwarz, 1978), (b) conceptually interpretable profile patterns, and (c) sufficient sample sizes for comparison among profiles. We then assigned each teacher to the profile for which they had the highest probability based on their response pattern across the noticing assessment. The 3-profile solution generated ordered profiles that we labelled *Accomplished Noticing* ($N = 14$), *Mixed Noticing* ($N = 33$), and *Emerging Noticing* ($N = 25$). The profile means of teachers' overall noticing scores (computed as a mean of their 12 scores) reflected this ordering: 1.42, 0.98, and 0.61, respectively. Our assessment design allowed us to further characterize the expertise associated with each profile in terms of the noticing component skills, and we were especially interested in whether mean scores were above or below a score of 1—the midpoint in our 0–2 scale that indicated limited evidence of engagement with children's mathematical thinking (see Figure 1).

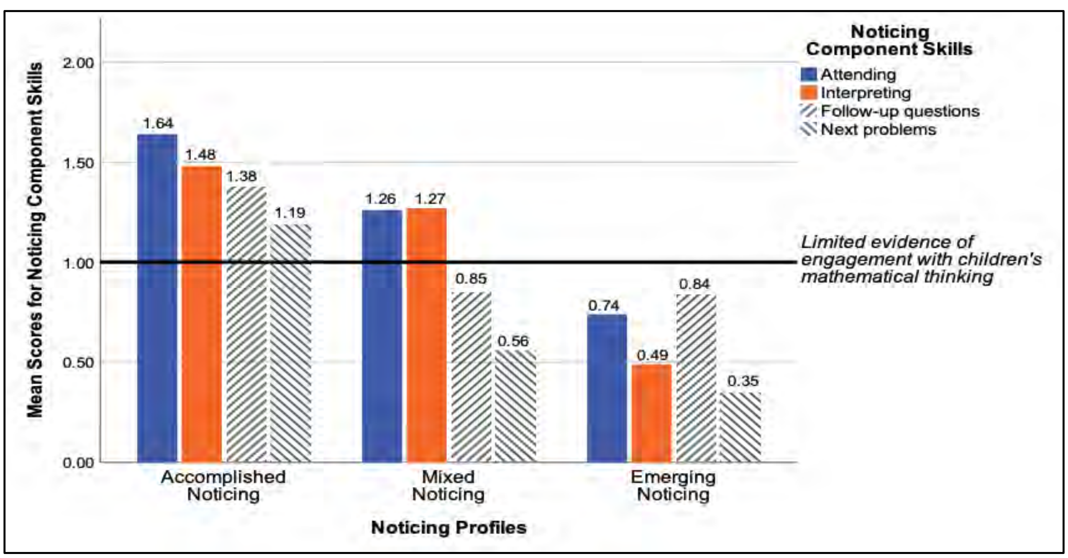


Figure 1: Mean Scores for Noticing Component Skills by Noticing Profile

The *Accomplished Noticing* profile was characterized by consistently strong expertise, with all mean scores above 1. For these teachers, expertise in attending to children's strategy details was their strongest skill and deciding on next problems was the skill for which they showed the most room for growth. The *Mixed Noticing* profile was characterized by a split performance, with mean scores above 1 for attending to children's strategy details and interpreting children's understandings, and mean scores below 1 for the two deciding how-to-respond skills. Thus, these teachers had developed substantial expertise in making sense of children's strategies, but they were still learning what to do with that information in terms of an instructional response. Finally, the *Emerging Noticing* profile was characterized by consistently weak expertise, with all mean scores below 1. However, their pattern of means scores showed that they were beginning to notice the details of children's thinking and pose follow-up questions about those details, but that they particularly needed support in making sense of what those details meant in terms of children's understandings and how to craft problems that built on those understandings.

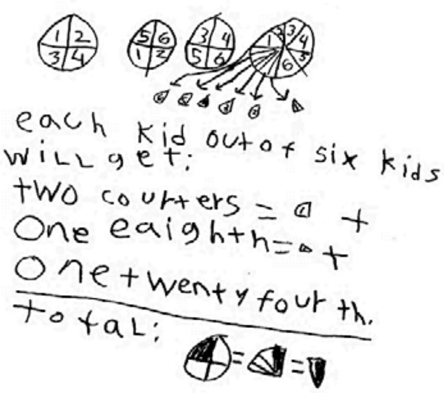
We also noted two major patterns across the profiles. First, the mean score for deciding on follow-up questions was higher than the mean score for deciding on next problems for all three profiles, reinforcing the importance of our separate consideration of these two categories. A related finding was that the mean score for deciding on next problems was the lowest score for all three profiles, which is consistent with earlier findings documenting this skill's challenging nature (Jacobs et al., 2010, 2011). Second, the mean score for attending to children's strategy details was one of the top scores for all profiles, suggesting the foundational role that details play in teachers' ability to make sense of and build on children's thinking (Jacobs & Spangler, 2017).

Figure 2 illustrates this second pattern for a teacher with an *Accomplished Noticing* profile. We share samples of her responses linked to Joy's written work for the pancake problem. Joy had a correct solution with a non-traditional final answer—she used words and pictures of fraction pieces (rather than fraction symbols) and did not combine her amounts into a single total. All sample responses were scored as robust evidence of engagement with children's mathematical thinking. For attending to children's strategy details, the teacher richly described Joy's strategy, highlighting details such as multiple partitions (4ths, 8ths, and 24ths) and why Joy might have made those partitions. For interpreting children's understandings, she focused on what Joy *did* understand, drawing on Joy's strategy details of (a) repeated halving, which is a common strategy for young children (Empson & Levi, 2011), and (b) correctly naming a fractional amount (1/3 of 1/8) which is challenging for many children. For deciding on follow-up questions, she made extensive use of strategy details, asking Joy how she partitioned, how she named the 1/24th-size pieces, and whether she had a sense of the amount each child would receive. She consistently centered Joy's thinking, and even her last question that moved beyond Joy's strategy to explore other possible partitions left room for Joy's thinking (versus funneling it toward the teacher's thinking). For deciding on next problems, she posed the same problem with new numbers that built on Joy's initial strategy that involved fourths (and repeated halving). Her rationale drew on this strategy detail to anticipate Joy's new strategy, and problem numbers were chosen strategically—9 pieces (for 9 sharers) cannot be reached by partitioning into fourths thereby making visible whether Joy could partition differently, such as by the number of sharers.

Across this teacher's responses for the component skills, strategy details were visible and integral to her reasoning. In contrast, for responses with scores of limited evidence or lack of evidence, we saw progressively less attention to strategy details in the initial strategy descriptions and throughout the other component skills. At times, teachers even focused on changing the child's strategy to one that they preferred.

<table>
<tr>
<td colspan="2">Joy's Strategy for the Pancake Problem

The teacher has 4 pancakes to share equally among 6 children. How much pancake does each child get?


</td>
</tr>
<tr>
<td>*Attending to Joy's strategy details*</td>
<td>Joy drew her 4 pancakes and cut them into 1/¼ I believe she did that because she is comfortable with 1/¼ When she reached her last pancake, she realized 1/¼wouldn't get each person a pancake piece. I think she then divided it into 1/8s. Again, I think 1/¼and 1/8s are comfortable for her. After she numbered 6, she realized she had 2 pieces left so she divided the last two 1/8s into 6 pieces. I believe she counted the pieces as if thirds were in each 1/8 to come up with twenty-fourths.</td>
</tr>
<tr>
<td>*Interpreting Joy's understandings*</td>
<td>**…** Joy has a good grasp of repeated halving (or fourth-ing). She continued to use 1/¼and 1/8s until she was able to solve the problem. She was however able to identify what [1/3] of 1/8 was. That impressed me and would be a question I'd pose to her.</td>
</tr>
<tr>
<td>*Deciding on follow-up questions for Joy*</td>
<td>• Can you tell me what you did? (To understand the thinking behind the work)
• Why did you split the first 3 pancakes into 4 pieces? (To understand the rationale, to see if she saw the relationship with the people)
• Tell me about the last pancake. (I want to see what she was thinking when she split this pancake)
• You wrote here 1/24. Can you show me 1/24 in the picture? How do you know that is 1/24? (What thinking was behind this decision to split the pieces? What understanding does she have about it?)
• Do you know how much the kids will get altogether? (Can she add her pieces?) Is it more than 1/½r less? More than 1 or less?
• Is there another way to split the pancakes? (Does she see the connection now?)</td>
</tr>
<tr>
<td>*Deciding on next problems for Joy*</td>
<td>*The teacher has 5 pancakes to share equally among 9 children. How much pancake does each child get?*

… I was curious to see if Joy would start with 1/¼and divide the pancakes into smaller pieces to solve the problem.</td>
</tr>
</table>

Figure 2: Sample Responses Linked to Joy's Strategy (*Accomplished Noticing* Profile)

Discussion

We began this study with the assumption that all participating teachers had strengths as teachers. They chose to engage in our PD to enhance their teaching by learning about children's mathematical thinking and its pivotal role in instruction—learning about noticing children's mathematical thinking was a piece of that learning. By assessing teachers' noticing expertise and empirically identifying three profiles of expertise, we hoped to better understand how teachers were taking up and engaging in this practice so that we could better support their development. We purposefully chose to assess teachers with varying amounts of PD because we know that teachers learn, and implement what they learn, at different rates, and we wanted to capture as much variety as possible. Our findings replicated our earlier work (Jacobs et al., 2010) in new grade levels (upper elementary grades versus primary grades) and with new mathematical content (problem solving with fractions versus problem solving with whole numbers). We also extended this work in two main ways: (a) elaboration of the deciding-how-to-respond component skill, and (b) identification of profiles of noticing expertise.

Elaboration of the Deciding-How-to-Respond Component Skill

In our earlier work, we introduced the inclusion of the deciding-how-to respond component skill in teachers' noticing of children's mathematical thinking, and we explored either decisions about follow-up questions or decisions about next problems, but not both together (Jacobs et al., 2010, 2011). In this study, we asked teachers to make both decisions for each instructional scenario so that we could compare teachers' engagement with children's thinking in the two categories of deciding how to respond with the same set of children's strategies. We found that teachers consistently showed more expertise when deciding on follow-up questions than when deciding on next problems, and this relationship held for each of the three profiles. This finding may reflect how teachers often have little experience deciding on next problems that build on children's understandings, and they may even wonder if they have the freedom to craft their own problems (or adjust existing problems) given the widespread, systemic use of resources such as pacing guides and mandated textbook materials.

In short, we would encourage the theoretical bifurcation of deciding how to respond because teachers engaged differently with each category, and both are important to teachers' work. We would also suggest including opportunities to practice both categories in PD, with an awareness that deciding on follow-up questions may initially be more accessible. Gaining expertise in posing these follow-up questions has other benefits as well because these questions can serve as leverage points for teachers' learning. Follow-up questions not only provide teachers with information about a specific child's thinking, but over time they also help teachers increase their understanding of children's mathematical thinking in general (Franke et al., 1998, 2001).

Identification of Noticing Profiles

Each profile had strengths and room to grow, and thus they provide snapshots of developing expertise. Theoretically, the profiles extend our earlier work in which we characterized expertise in each component skill but did not provide a conceptualization for how the skills might work together differently for individual teachers (Jacobs et al., 2010). Our profiles provide this conceptualization and suggest that teachers in different profiles may need different types of support (see also, Munson, 2020). We provide some initial suggestions for customization.

Teachers with an *Accomplished Noticing* profile demonstrated strong expertise across component skills, but still with room to grow. Focusing on challenging examples—complex or ambiguous strategies—could provide these teachers with opportunities to refine their expertise (Jacobs, Empson, Pynes, et al., 2019). Teachers with a *Mixed Noticing* profile demonstrated some expertise, with more expertise in attending to children's strategy details and interpreting children's understandings than with the two deciding-how-to-respond skills. Focusing on typical and straightforward strategies could provide these teachers with opportunities to easily make sense of children's strategy details and related understandings so that they could concentrate on how to build on these understandings with follow-up questions and next problems. Teachers with an *Emerging Noticing* profile had substantial room to grow in all component skills, but their mean scores for attending to children's strategy details and deciding on follow-up questions were relatively higher. Focusing on typical and straightforward strategies could provide these teachers with opportunities to solidify their ability to recognize strategy details and generate follow-up questions. Further, providing access to research on children's mathematical thinking could help them begin to interpret children's understandings reflected in strategy details, learn how those understandings are likely to develop, and consider next problems to support this development.

In addition to providing insights for PD, our profiles provide a starting point for conversations about teachers' developmental trajectories with respect to noticing expertise. However, caution is warranted given that our data are not longitudinal. Assuming that teachers are moving toward an *Accomplished Noticing* profile, the question is whether the other two profiles represent two separate paths or a single, connected path. Specifically, one possibility is that, as teachers learn about children's mathematical thinking, some may develop skills consistent with an *Emerging Noticing* profile and others with a *Mixed Noticing* profile, and then each group follows a different path toward an *Accomplished Noticing* profile. Another possibility is that, as teachers learn, they move from an *Emerging Noticing* profile to a *Mixed Noticing* profile and finally to an *Accomplished Noticing* profile in a single, connected path. We have some evidence to suggest that this second possibility may be more apt.

We looked at the relationship between the number of years of PD that teachers completed and their noticing profile. Teachers who had completed 1, 2, and 3 years of PD were found in all three profiles, but the distribution varied as one might expect with a single, connected path for development—there were more teachers who had 3 years of PD with an *Accomplished Noticing* profile and more teachers with only 1 year of PD with an *Emerging Noticing* profile. In fact, the membership of the two profiles were essentially mirror images of each other. The *Accomplished Noticing* profile had 7%, 36%, and 57% of teachers who had completed 1, 2, or 3 years of PD respectively, whereas the *Emerging Noticing* profile had 56%, 36%, and 8% of teachers who had completed 1, 2, or 3 years of PD respectively. The *Mixed Noticing* profile was in-between, with a more even distribution. These findings support earlier findings that teachers usually do not gain expertise in noticing children's mathematical thinking from teaching experience alone, but it can be developed with sustained time and support (Jacobs & Spangler, 2017).

Final Thoughts

We provided an initial exploration into profiles of teachers' expertise in professional noticing of children's mathematical thinking. The profiles we identified differed in terms of the overall expertise demonstrated and in the constellations of strengths and needed areas of growth related to the noticing component skills. Not only do these profiles help us better understand the construct of professional noticing of children's mathematical thinking, but they also form a basis for customizing PD to support growth in teachers' noticing expertise. Overall, the profiles increased our appreciation for the complexity of noticing expertise and raised our awareness that teachers may display inconsistent expertise across the component skills as they are learning.

Acknowledgments

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A VALIDATION ARGUMENT FOR THE PRIORITIES FOR MATHEMATICS INSTRUCTION (PMI) SURVEY

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Mathematics education needs measures that can be used to research and/or evaluate the impact of professional development for constructs that are broadly relevant to the field. To address this need we developed the Priorities for Mathematics Instruction (PMI) survey consisting of two scales focused on the constructs of Explicit Attention to Concepts (EAC) and Student Opportunities to Struggle (SOS) – which have been linked to increased student understanding and achievement. We identified the most critical assumptions that underlie the proposed interpretation and use of the scale scores and then examined the related validity evidence. We found the evidence for each assumption supports the proposed interpretation and use of the scale scores.

Keywords: Instructional Activities and Practices, Teacher Beliefs, Measurement

Teacher beliefs are important predictors of classroom practice (Stipek, Givvin, Salmon, & MacGyvers, 2001). The field of mathematics education needs measures of teacher beliefs that are broadly applicable and useful across multiple research studies (e.g., for comparisons), and linked to student learning outcomes of value to the field (e.g., student achievement). In many cases, this has led to development of surveys to assess the degree to which teachers hold beliefs aligned with preferred approaches to mathematics instruction. However, teachers' beliefs are just one aspect of a complex system affecting teachers' instructional practices (Leatham, 2006), and though survey scores may be associated with implementation, the competing priorities of teachers instructional practice have important effects on classroom practice. There is a need for a survey about mathematics instruction that describes teachers' beliefs while foregrounding the competing priorities teachers must consider when making instructional decisions.

Our interest in developing a survey stems for our involvement in multiple K-12 teacher professional development (PD) projects with a goal to influence teachers' beliefs about particular instructional strategies. We value our collaborations with teachers and the competing priorities they weigh while making instructional decisions (e.g., limited time vs. a desire for building both conceptual and procedural fluency). Therefore, we wanted a survey that does not devalue the knowledge teachers have about their contexts, and that gives us the ability to understand and use a broader perspective to support use of effective instructional practices. In particular, our survey is aimed to be applicable and useful for examining the impact of PD on teachers' beliefs and implementation across our PD projects, and with scales that recognize teachers' priorities without explicitly privileging particular instructional strategies.

Perspectives

Our instrument development work is framed through two perspectives. We first describe the theoretical framework for effective mathematics instruction from which our survey scales are based. We then draw from modern validity theory, explaining our choice to use an argument-based approaches to validation.

Theoretical Framework for EAC and SOS

Our perspective on effective mathematics instruction centers on Explicit Attention to Concepts (EAC) and Student Opportunity to Struggle (SOS), which come from Heibert and Grouws' (2007) synthesis of literature regarding classroom practices connected to increases in student conceptual understanding and mathematics achievement. EAC refers to instructional practices involving public noting of connections among mathematical facts, procedures, and ideas, while SOS occurs when students expend effort to make sense of mathematics or figure something out that is not immediately apparent. Recently, Stein, Correnti, Moore, Russell, and Kelly (2017) investigated the relationship between EAC, SOS, and student achievement across a large group of teachers. They found students in classrooms with high EAC and SOS performed higher on mathematics achievement assessments of both conceptual understanding and skills efficiency. Based on the extensive literature base, the connections to student achievement, and the likelihood for broad applicability, we used the constructs of EAC and SOS as the starting place to develop our survey scales.

Our goal was to identify and situate the EAC and SOS constructs in contrast to common competing priorities for instructional focus. Studies of traditional mathematics instruction highlight beliefs among teachers that emphasize ways in which beliefs about learning and context factors relate to teachers' choices to prioritize mastery of procedural skills (Philipp, 2007) and identify a need to 'funnel' tasks to reduce cognitive demand (Peterson, Fennema, Carpenter, & Loef, 1989). We label this set of priorities as Single Methodological Focus (SMF) and Highly Scaffolded Content (HSC), respectively, and situate them as contrasting priorities to EAC and SOS.

Argument-Based Validation

Modern validity theory has been articulating and promoting the idea of instrument validation through the lens of argumentation for many years (Cronbach, 1988, Kane, 1992, Messick, 1995), culminating in recommendations for argument-based validation in *The Standards for Educational and Psychological Testing* (AREA, APA, NCME, 1999, 2014). While a variety of approaches to argument-based validation have been articulated, there is not one generally accepted approach (Carney, Crawford, Siebert, Osguthorpe, Thiede, 2019). Therefore, we use the recommendations from *The Standards* (AREA, APA, NCME, 1999, 2014) and Kane (1992, 2001, 2016) to guide our work.

Validity involves the degree to which the score interpretation for proposed uses is supported by theory and evidence[1], and validation involves constructing and evaluating arguments related to the score interpretation for proposed uses (AERA, APA, NCME, 2014). Therefore, the articulation of the score interpretation for proposed uses must be the first step in validation (Kane, 2001, 2016). The argument is further developed by articulating the assumptions that underlie the score interpretation and use (AERA, APA, NCME, 2014). Once the assumptions have been articulated, it is incumbent upon the instrument developers to gather evidence to investigate the most critical or suspect assumptions first (Kane, 2001, 2016).

The goals of this paper are to (a) articulate the score interpretations for proposed uses for two survey scales we have developed, (b) articulate the most critical or suspect assumptions that underlie the score interpretations for proposed uses, and (c) examine evidence in relation to those assumptions. We see this work as an initial step in the iterative cycle of instrument development and validation, with the goal of others using the scales and continuing to gather evidence in support of, or to refute, the assumptions that underlie the score interpretation for proposed uses.

Methods

Instrumentation

Structure of the scales. As mentioned in the section Theoretical Framework for EAC and SOS, we wanted to structure our survey scales to recognize the most likely competing priorities for these constructs. We set up these competing priorities along a continuum for each construct. One continuum contrasts Explicit Attention to Concepts (EAC) with Single Methodological Focus (SMF), which prioritizes a compartmentalized approach to mathematics instruction that focuses on teaching one important mathematical idea and/or procedure at a time, often in an attempt to reduce student confusion between different approaches to solving problems. This approach is often manifested in classroom practice by asking students to correctly apply a particular procedure to a set of problems. A second continuum contrasts Student Opportunity to Struggle (SOS) with Highly Scaffolded Content (HSC), which prioritizes a gradual increase in complexity of mathematics, with scaffolding for students to move from relatively easy to more challenging ideas and procedures. This approach is often manifested in classroom practice by teachers breaking down students' work into progressively more challenging tasks, with the teacher providing explanations as needed, so students can gradually build fluency.

Using the two continuums as underlying constructs, the Priorities for Mathematics Instruction (PMI) survey has two scales focused on teachers' prioritization of beliefs - *PMI: SMF-EAC beliefs* and *PMI: HSC-SOS beliefs*. Each survey item starts with a common stem and presents instructional practices representative of the two ends of the target continuum. Respondents select one of six positions to describe the relative priority they place on the competing statements. See Figure 1 for the directions at the start of the survey and an example item highlighting the continuum:

Survey Directions: Each item asks you to choose between two statements. Both have value, and you may believe both are important to your teaching. Nonetheless, we ask you to choose one over the other. That is, please position the slider to indicate which statement you believe has greater priority in terms of your perspective on teaching mathematics. The further you move the slider to one side or the other, the greater priority you give that statement.

Survey Item: Students are more likely to succeed in math when they can:

Apply a particular method to solving similar math problems	•—•—•—•—•—•	Make connections among math topics
SMF	*-2.5 2.5*	*EAC*

**The information in italics is provided for reference and is not part of the item.*

Figure 1. Example of directions and an item for the PMI: SMF-EAC beliefs scale.

Interpretation and Use. The *PMI: SMF-EAC beliefs* and *PMI: HSC-SOS beliefs* scale scores (calculated as an average of the responses within the scale) can be interpreted in the following way. A score above 0 indicates beliefs more closely aligned with EAC or SOS practices, respectively. The closer the score gets to 2.5, the more closely the beliefs align with EAC or SOS. A score below 0 indicates the beliefs more closely align with SMF or HSC practices, respectively. The closer the score gets to -2.5, the more closely the teacher's beliefs align with SMF or HSC. A score near 0 indicates the teacher tries to balance the competing beliefs in their instructional priorities. The *PMI: SMF-EAC beliefs* and *PMI: HSC-SOS beliefs* scale scores can be broadly used by professional developers to examine beliefs relative to these constructs, inform professional development activities, and evaluate the effectiveness of PD activities in regards to their impact on teachers' beliefs related to EAC and SOS.

Critical Assumptions. Once the interpretation and use are clearly stated for an instrument, it is incumbent upon the developer to investigate the underlying assumptions (AERA, APA, NCME, 2014). The initial focus should be on the assumptions that are the most critical to demonstrate or the most likely to fail (i.e., are most suspect) (Kane, 2001). We have identified the following assumptions as particularly critical in our initial investigation of the interpretation and use of the *PMI: SMF-EAC beliefs* and *PMI: HSC-SOS beliefs* scale scores. For all instruments, there is an assumption that the operationalization aligns with the construct(s) theorized structure (assumption 1). For instruments such as the PMI survey where use is proposed (a) across a variety of professional development projects, the assumption is that the construct is broadly relevant to a mathematics education audience (assumption 2), and (b) related to measuring growth, the assumption is the instrument is sensitive enough to detect growth in an individual or group (assumption 3). Lastly, for instruments such as the PMI survey where social desirability of the response is a potential unintended factor, the assumption is social desirability is not impacting the scores (assumption 4).

Instrument Administration

Data Collection. The survey was administered to teachers participating in programs offered by a single K-12 math PD center in the Pacific Northwest. The programs are diverse in format, content, and duration, ranging from content-focused workshops to multi-year collaborative projects. There are clear differences in approach across the three PD groups [Blinded for Review]: Program 1, Program 2, and Other. Program 1 is a state-mandated 3-credit course in which K-12 educators build mathematical knowledge for teaching with a special emphasis on increased awareness of EAC and SOS, Program 2 is a federally-funded teacher-researcher alliance of Grades 6-8 teachers with an emphasis on adapting EAC and SOS strategies for their classroom practice, and the Other programs incorporate EAC and SOS ideas in their design, but not as the primary emphasis. Surveys were administered online via email invitation just before participating in the PD (pre, $N = 645$) and again (depending on program timing) 2 to 8 months later (post, $n=321$). Data collection spanned July 2019 to February 2021, with paired post/pre-response rates differing by PD group (Program 1: 48/107 (45%), Program 2: 78/106 (74%), Other: 195/432 (45%)).

Analysis

Statistical analyses of the survey response data was conducted in the statistical software package *R* (R Core Team, 2020), following recommendations for scale development by Jackson, Gillaspy, and Purc-Stephenson (2009). This included inspection of item response distributions, estimation of the bivariate correlational structure, and confirmatory factor analysis (CFA) using the *lavaan* software package (Rosseel, 2012). Missingness assumptions were evaluated under Little and Rubin's recommendations (1989), with iterative multiple imputation (van Buuren & Groothuis-Oudshoorn, 2011) used to augment incomplete responses (6.7%) without introducing bias into the fitted factor model. Evaluation and reporting of CFA model fit and parameter estimates followed guidelines by Cabrera-Nguyen (2010), with emphasis on indications of construct validity given the space restrictions of this report. Potential differences in pre-post PMI beliefs across subsamples were assessed using standard inferential statistical procedures (e.g., descriptive summaries, plots, ANOVA).

Results

Operationalization Aligns with Theory (Assumption 1)

The internal structure of the pre-responses were analyzed via a two factor CFA model using maximum likelihood estimation, with the eight EAC items loaded onto a latent "eac" factor, and the seven SOS items loaded onto a latent "sos" factor. The two factors were standardized (mean 0, standard deviation 1) and assumed to be correlated. The estimated model converged in 19 iterations with 31 free parameters, with indicators suggesting good fit between the theoretical model and the observed structure (model $X^2(89) = 304$, null $X^2(105) = 3228$, AIC $= 30140$, BIC $= 30279$, RMSEA $= .06$, CFI $= .93$, TFI = .92, SRMR $= .04$) with no areas of local strain and statistically significant factor loadings (all z $> 10, p < .0001$). Similarly, the model exhibited strong convergent and discriminant validity with standardized factor loadings strictly between 0.4 and 0.8 (see Figure 2). Follow-up principle component analysis identified no indications of cross-loadings (suggesting strong convergent validity), and the correlation between eac and sos beliefs (0.71) was below 0.80, suggesting strong discriminant validity. The evidence of model fit provides support for the unique operationalization of SOS to HSC and EAC to SMF as a continuum.

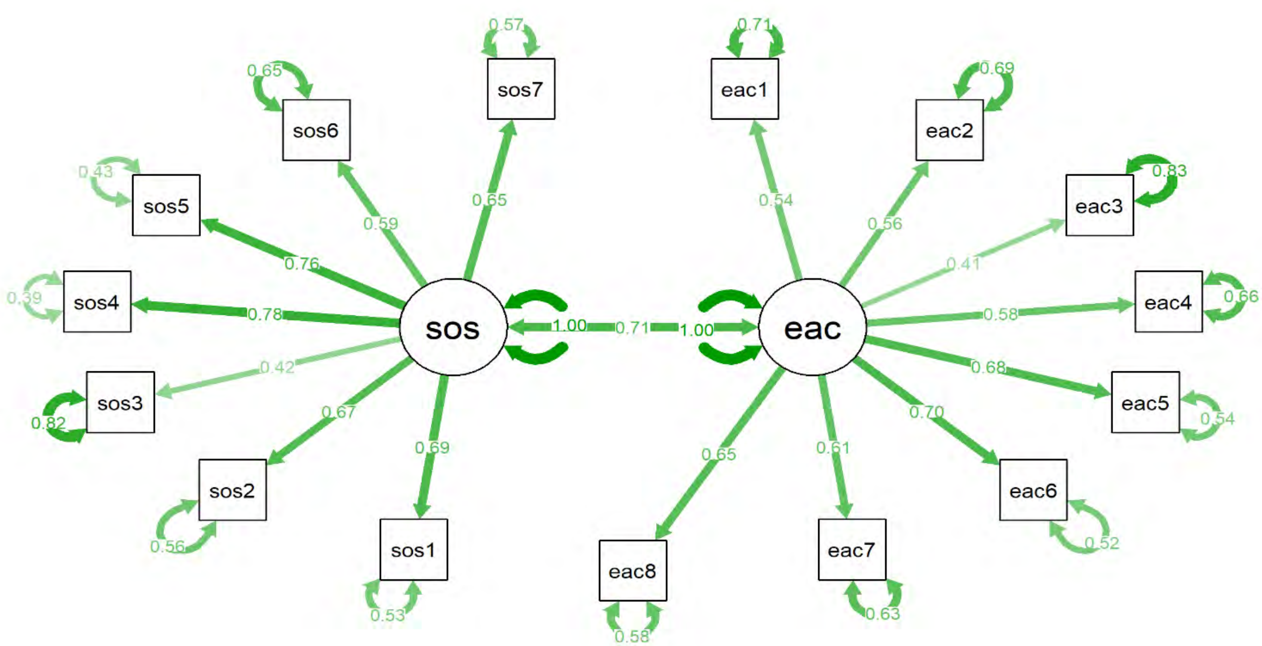

Figure 2. Standardized estimates for two-factor CFA model of EAC and SOS beliefs.

Broad Relevance (Assumption 2)

In addition to the theoretical argument establishing broad relevance and applicability of EAC and SOS across mathematics education settings (see section Theoretical Framework for EAC and SOS), the pre-distributions of PMI scale scores across the PD groups supports Assumption 2. Though each group differed in contextual variables, they had similar initial distributions of EAC and SOS belief scores (Table 1), this indicates the scales are likely to be broadly useful across different PD groups and settings.

Table 1. Distributions of EAC, SOS, and PMI Quadrants by PD Group

		EAC		*SOS*		*PMI Quadrants*			
Group	*n*	M	SD	M	SD	EAC&SOS	SMF&SOS	SMF&HSC	EAC&HSC
Program 1	107	0.1	0.8	0.0	0.9	41%	9%	36%	13%
Program 2	106	-0.2	0.8	-0.1	0.7	32%	16%	48%	4%
Other	432	0.0	1.0	0.0	1.0	43%	7%	40%	10%

Sensitivity to Group and Individual Changes (Assumption 3)

Figure 3 supports the potential for the PMI survey to detect change in teachers EAC and SOS beliefs. The chart illustrates how teachers in each PMI quadrant shifted in the post assessment, including a general pattern of small changes among teachers who began in the EAC&SOS quadrant, while teachers in the other quadrants showing increased variability in their post scores while generally shifting toward EAC&SOS. The ability to detect differential growth based on pre-PD scale scores indicates utility of the survey for detecting group and individual changes.

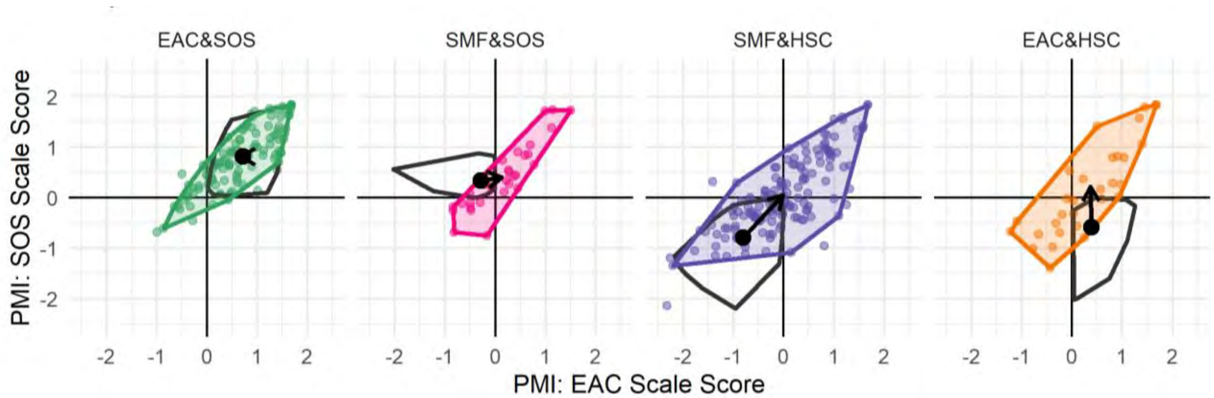

Figure 3. Post EAC and SOS scores, split by pre PMI Quadrant.
(Polygons capture the middle 90% of points by group, arrows indicate mean change.)

Figure 4 illustrates pre-post changes across the PD groups in EAC and SOS. The chart demonstrates substantial shifts toward prioritizing EAC and SOS among teachers in the Program 1 group. The ability to detect differential growth across PD contexts supports this proposed use.

Olanoff, D., Johnson, K., & Spitzer, S. (2021). *Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Philadelphia, PA.

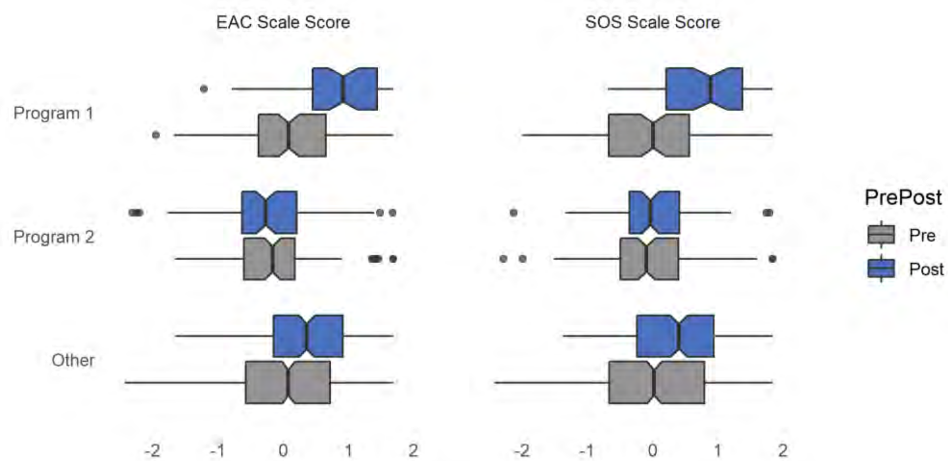

Figure 4. Pre and post distributions of EAC and SOS beliefs by PD Group. (Non-overlapping central notches indicate statistically different group medians.)

Social Desirability Response Bias (Assumption 4)

The paired pre-post EAC and SOS scores suggest minimal risks of social desirability response bias at the individual or group levels. Though teachers tended to shift toward the EAC&SOS quadrant after participating in PD (see Figure 5), the magnitudes and directions of those shifts varied greatly, with greater variability within groups than across. This variability supports the assumption that the social desirability of the response options is not obvious to respondents following PD that includes a focus on EAC and SOS.

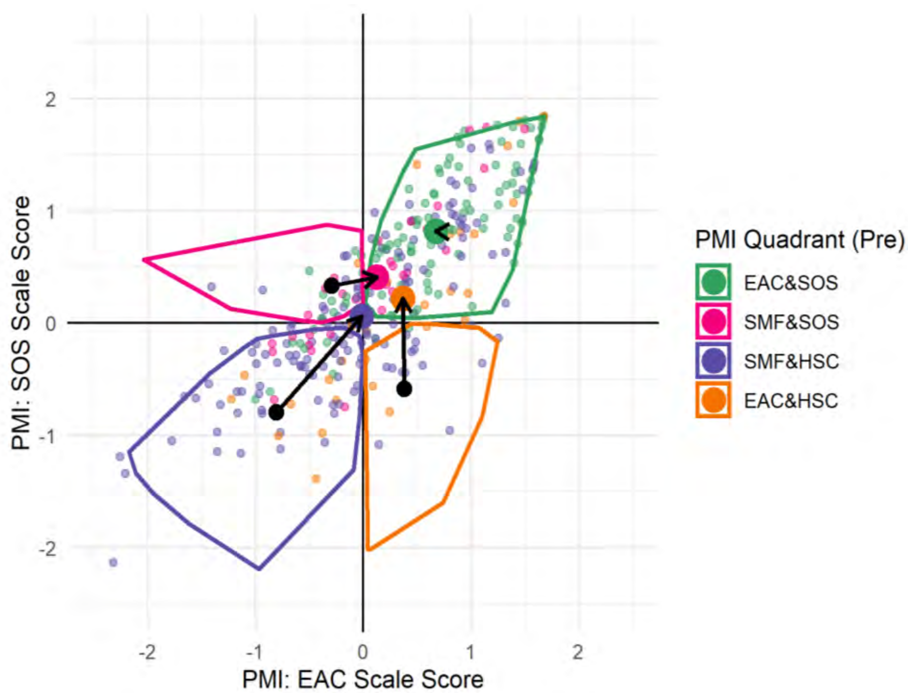

Figure 5. Post EAC & SOS Scores by Pre PMI quadrant. (Polygons capture the middle 90% of points in each group, arrows indicate mean changes.)

Olanoff, D., Johnson, K., & Spitzer, S. (2021). *Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Philadelphia, PA.

In addition, pre-post changes in EAC and SOS beliefs differed across the PD groups, with substantial changes in Program 1 (EAC: M = 0.8(SD = 0.9), SOS: 1.1(1.0)), insignificant changes in Program 2 (EAC: 0.0(0.8), SOS: 0.1(0.6)), and moderate changes in Other (EAC: 0.4(0.8), SOS: 0.5(0.8)). As shown in Figure 6, changes in PMI Quartiles differed significantly across the PD contexts. All three PD contexts included information about why EAC and SOS are important for classroom practice, and Program 2 in particular emphasized engaging in activities that make use of EAC and SOS in the classroom. However, there was significant variability in the amount of change in EAC and SOS across PD contexts with Program 2 having the least change and most focus on EAC and SOS. This evidence of variability across PD contexts supports the assumption that social desirability in favor of EAC and SOS is not impacting responses to the survey items. If it were, we would have expected the Program 2 scores to have shifted to reflect this bias.

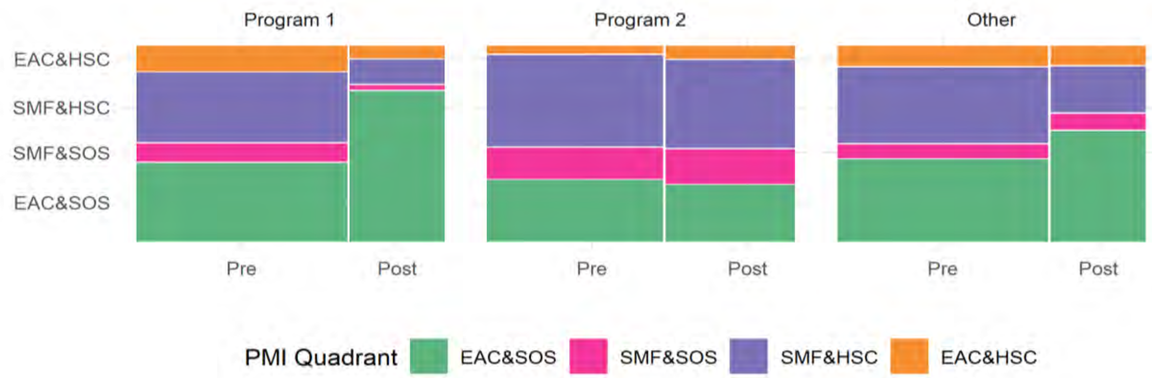

Figure 6. Pre-post changes in distributions of PMI quadrants across PD Groups.

Discussion

Instrument validation is an iterative process. This work presents an initial set of evidence for the interpretation and use of the PMI survey scale scores for *PMI: SMF-EAC* beliefs and *PMI: HSC-SOS* beliefs. Following the recommendations of the *Standards* (AERA, APA, NCME, 2014) we stated the interpretation and use of the two survey scales and identified the most critical assumptions to investigate. In particular, we investigated the following assumptions, and examined the associated evidence.

- The operationalization aligned with our theory (assumption 1). The CFA indicated a good fit which provides support for the unique operationalization of SOS to HSC and EAC to SMF as separate continuums of competing priorities.
- The survey scales scores are broadly relevant to the mathematics education community (assumption 2). The grounding of the scales in the work of Hiebert & Grouws (2007) and Stein and colleagues (2017), in addition to the finding of similar measures of center, variability and quadrant percentages across PD contexts, provide evidence in support of this assumption.
- The survey scales are sensitive enough to identify group and individual changes (assumption 3). The evidence of scale score changes from the perspective of both the pre-PD quadrant and three different PD contexts provides support for this assumption.

- Social desirability did not impact post survey responses (assumption 4). The strongest evidence in support of this assumption is that Program 2 participants – where the primary focus of the PD is EAC and SOS – had the least changes in pre-post scale scores.

Taken together the evidence in support of the four critical assumptions provides an important initial investigation into the interpretation and use of the PMI survey scale scores. We see this evidence as sufficient for recommending the use of the survey scales more broadly within the mathematics education community and hope that others will make use of the instrument and conduct additional validity investigations.

It is important to note a few key limitations. We did not complete a full investigation of the validity argument. There are additional assumptions that need to be examined and as the survey is used we anticipate others might have additional interpretation and use ideas that expand upon what was stated here. These would require further investigation. Finally, this work occurred during the COVID-19 pandemic, which likely impacted survey responses in complicated ways.

Note

[1] The Standards explicitly state "It is incorrect to use the unqualified phrase "the validity of the test" (p. 11)."

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IDENTIFYING AND RECONTEXTUALIZATING PROBLEMS OF PRACTICE IN LEARNING TO FACILITATE DISCUSSIONS WITH ARGUMENTATION

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Supporting teachers to facilitate discussions with argumentation is as complex as the practice of facilitating argumentation itself. In this paper, we describe how a community of teachers and one teacher within this community made sense of facilitating argumentation. We use the construct of problems of practice as an indicator of teachers' sensemaking and learning. We contend that problems of practice are highly situated within teachers' contexts. The teachers identified participation as a broad problem of practice for facilitating argumentation and we identify three aspects of participation salient for teachers. Moreover, we show how Amanda, one of the teachers recontextualizes these problems to her thinking and practice.

Keywords: Professional Development, Teacher Knowledge, Teacher Educators, Reasoning and Proof

Whole class discussions that focus on mathematical argumentation are central events of inquiry-oriented environments (Staples & Newton, 2016), and important for promoting conceptual understanding and developing mathematically proficient students (Osborne et al., 2019; Rumsey, 2012). Mathematical argumentation engages students collaboratively in a process where they make claims and justify them using reasoning that is based in disciplinary practices and in their existing knowledge and cultural and linguistic resources. Argumentation-based discussions, however, are uncommon in U.S. classrooms (Cazden, 2003). Studies of teachers' roles and responsibilities in these socially and intellectually demanding environments highlight various teaching practices that support mathematical argumentation (Lampert, 2001; Staples, 2007). However, this research also suggests that implementing these complex practices is not trivial. Teachers must judge how to elicit and respond to student thinking and how to facilitate students' engagement with each other's ideas around disciplinary content. Problems of practice are endemic to this work and how teachers understand them and what they do to address them reflect pedagogical reasoning that is deeply embedded in their context, students, and professional knowledge. In this paper, we investigate problems of practice emerging from a community of elementary teachers engaged in a practice-based professional development (PD) focused on facilitating mathematical argumentation. We also describe how one teacher in this community, Amanda, contextualizes these problems in her practice.

Problems of Practice

Problems of practice (PoPs) have been a focus of teacher education research. Lampert (2001) wrote extensively about problems of practice based on teaching mathematics over the course of a year in a fifth-grade classroom. Using a zooming metaphor, she described teaching as a complex web of relationships involving the teacher, students, and content on different levels. It is within these relationships she argued "problem spaces" lie. She elaborated, "the problem space in which the teacher works is full of ideals to be realized, full of worthy destinations… In the actuality of teachers' work, however, the practices intended to realize these ideals are often incompatible" (p. 447). Thus, PoPs arise through these contradictions. Ghousseini (2015) framed these contradictions in "the way a teacher holds and deploys knowledge and coordinates instruction can constrain or promote what students can do with the content. Similarly, students and content can both constrain and open up what the teacher can do to teach, hence rendering the relationships inside the instructional triangle as both sources of problems of practice and resources for solving them" (p. 338). In other words, when teachers draw on a set of knowledge, beliefs, and identities to manage their practice, most often, their visions are reshaped by contextual factors.

Building on this work, we view a problem of practice as teachers' perceived misalignment between idealized instruction and actual instruction as normalized by a community. This misalignment can be manifested through teachers' interactions expressed as "troublesome, challenging, confusing, recurrent, unexpectedly interesting, or otherwise worthy of comment" (Horn & Little, 2010, p. 189). This view is based on a number of assumptions. First, it foregrounds teachers' perceptions instead of the researchers given that problems of practice emerge from knowledge-in-practice where "teacher learning hinges on enhancing teachers' understandings of their own actions—that is, their own assumptions, their own reasoning and decisions, and their own inventions of new knowledge to fit unique and shifting classroom situations" (Cochran-Smith & Lytle, 1999, p. 267). Second, this view assumes that PD is geared towards developing and leveraging teachers' knowledge in practice-based contexts, where teachers may learn about and from practice, with support from "more knowledgeable others" such as university professors, PD facilitators, or cooperating teacher mentors. In fact, it is through such experiences that we can understand the genesis of teachers' idealized instruction. Third, this view acknowledges the social aspect of teaching—although teachers teach largely individually, problems of practice can be embedded in broader principles and visions of teaching when they are normalized by colleagues and other stakeholders (Horn & Little, 2010). Research on teachers' framing of problems of practice has painted a picture of teachers' reasoning as collaborative sense-making situated in contexts of particular schools and districts, rather than purely cognitive individual acts that educators deploy (Thompson et al., 2015).

Teacher Learning and PoPs

We view teacher learning through a situative perspective where teachers' engage in ongoing sensemaking and reasoning in a community of practice, using various types of available conceptual and practical resources and tools and/or representations. This view of teacher learning guided our PD structure which we elaborate in a later section.

Professional development, especially highly adaptive, collective professional development, supports teachers' sensemaking. Following Schwarz et al. (2020), we view sensemaking as "wrestling with ideas, language, experiences, and perspectives to figure out how and why the world works; sense-making means proactive engagement in understanding the world by generating, using, and extending scientific knowledge within communities" (p. 1-2). As teachers make sense of their practice, such as through discussing and resolving PoPs, "they can come to know their practice in a way that enables them to construct meaning, make inferences, and solve problems" (Lampert, et al., 2015, p. 349). When teachers work together to address PoPs, they develop their ideas, knowledge, practice, and identities as well as their evolving commitments about what it means to teach, learn, and engage in collective inquiry. Together they are able to generate knowledge including "how decisions are made, how strategies are selected, how disparate instances are connected to one another, how subject matter is conveyed, and how new occurrences are understood and framed" (Cochran-Smith & Lytle, 1999, p. 268). They consider problems of practice in particular situations, "intentionally and introspectively examining those situations, and consciously enhancing and articulating what is tacit or implicit."

When the design of professional learning allows for the coevolution of participation between classroom practice and PD (Kazemi & Hubbard, 2008), individual teachers have the opportunity to recontextualize PoPs normalized in a community to their personal practice. Recontextualization is "a process of disembedding, re-embedding, and change" of discourse from one social context to another (Ensor, 2001, p. 297). This is important within professional learning communities. Even though PoPs may bring the community into a shared problem space, when embedded into teachers' practice, they are likely to change when teachers render them as part of familiar practice or their vision of practice (Horn & Little, 2010). Recontextualization as a process affords insight into teachers sensemaking beyond the setting of PD.

In this paper, we examine what PoPs emerge in a professional learning community and how they get recontextualized in a teacher's practice. Specifically, we will address:

1. What are the problems of practice identified by a group of teachers learning to facilitate discussions with argumentation?
2. How are the PoPs recontextualized in one teacher's practice?

Professional Development Structure and Focus

We co-designed and co-facilitated a PD environment called Learning Labs (LLs) focused on facilitating students' practices of argumentation. LLs are organized to (1) be adaptive and responsive to local communities, (2) involve ongoing collaboration and inquiry, and (3) consist of cycles of investigation, enactment, and reflection (Lampert et al 2013; Kazemi et al 2018). Each LL included four phases: new learning, planning, enactment, and debrief (Kazemi et al., 2017). The new learning phase focused on an analysis and discussion of artifacts of practice, such as a video clip, to unpack the nature of productive argumentation. Lab members collaboratively prepared a lesson to enact in one of the teachers' classrooms during the planning phase. In the enactment phase, they facilitated the lesson. As the lesson progressed, teachers would pause instruction to discuss instructional decisions using a routine called Teacher Time Out (Gibbons et al., 2017). During the debrief, lab members discussed insights from the enactment and set goals for argumentation in their own classrooms. Such a model of PD falls within Koellner & Jacobs's (2014) description of adaptive models where we are responsive to "the goals, resources, and circumstances of the local PD context. These models are based on general and evolving guidelines rather than specific content, activities, and materials" (p. 51).

We designed LLs to support teachers to facilitate classroom discussions focused on argumentation. We frame argumentation as constructing a reasoned case for why a mathematical statement or claim is logical or true and deemed acceptable by the community, in this case, other students and the teacher(s) (Knudsen et al., 2018; cf., Toulmin 1969). Researchers have produced substantial reports on argumentation (and proof, a closely related concept) but most reports are focused on student conceptions and classroom-based research more than teacher knowledge and development around facilitating argumentation (Stylianides et al., 2016). In the sparse work on PD focused on learning to facilitate argumentation, researchers used models drawing on representations and in some cases approximations of practice (Grossman and McDonald, 2008).

For instance, Osborne et al. (2019) in a practice-based PD used classroom videos to support teachers in adopting a more dialogic approach to teaching and fostering argumentation from evidence. They identified a specific set of instructional practices to foster students' argumentation from evidence, which were modeled by PD leaders. They also tested an approach to support teacher learning in the form of a practicum where teachers had multiple opportunities to enact, collaboratively investigate, and refine their practices. While such PD models ground teacher learning in the context of practice, they do not clarify the nature of teachers' sensemaking and how the adaptive PD affords insight into how it unfolds over time.

We facilitated LLs at Lockwood Elementary which is situated in an urban area in the Midwest region of the United States. In 2018-2019, the school served 443 students. The demographic make up includes 10% Asian, 10% Black, 40% Hispanic, 32% White, and 7% identifying as two or more races. Moreover, 13% of students were labelled as "students with disabilities, 56% as "economically disadvantaged," and 41% as "limited English proficiency." We began our LLs in January of 2019. Across the LLs at Lockwood, two university-based mathematics educators worked with four third-grade teachers, a fourth-grade teacher, three fifth-grade teachers, two school-based mathematics instructional coaches, and a retired mathematics instructional coach who worked at Lockwood. Most of this group identified as white and the rest as people of color. We focus half of our analysis on one member of the LL. Amanda, a White teacher certified in bilingual and elementary/middle school education with more than 10 years of experience teaches fifth-grade mathematics in both English and Spanish as part of a Dual Language Immersion program (DLI) where native Spanish and native English speakers were placed in the same class with instruction in both Spanish and English.

Data and Analysis

We collected video and audio recordings and field notes from eight LLs spanning one and a half school years. In addition, we conducted teacher interviews at the beginning of the project (Fall 2018) and at the end of the first and second school year (Summer 2019 & Spring 2020) where we asked teachers about their conceptions of discussions, argumentation, and equity. Between LLs, instructional coaches conducted video-stimulated recall interviews (VSRIs) related to teachers' attempts to facilitate argumentation in their own classrooms.

Analyzing LL PoPs

Analysis consisted of multiple phases to identify PoPs relevant to argumentation and how they emerged. We first identified instances where teachers highlighted "classroom interactions experienced as troublesome, challenging, confusing, recurrent, unexpectedly interesting, or otherwise worthy of comment" (Horn & Little, 2010, p. 189). Two authors then condensed these instances into PoPs exhibiting similar concern or comments in the context or argumentation. We also distinguished between one-off comments and comments contributing to a PoP. In order for a comment to contribute to a PoP the contribution had to normalize an existing PoP (e.g., a teacher connecting another teachers' similar experiences or concerns in their own), discussing potential reasons for a PoP, or relating the PoP to a teaching principle (see Horn & Little, 2010).

Analyzing Amanda's PoPs

To analyze Amanda's recontextualization, we first needed to understand how Amanda made sense of teaching in general. To do this, we analyzed her three teacher interviews. We looked for general principles that Amanda held by finding instances where she made sense of a set of experiences (e.g., her instruction with multilingual students) or general principles of teaching. From this, we identified her idealized version of teaching. One researcher then analyzed her

VSRIs and participation in LLs and looked for experiences where Amanda expressed misalignment with her practice and idealized practice. These, similar to the previous analysis, were collapsed into PoPs and were mapped back to the set of PoPs constructed in LLs in order to see how Amanda's PoPs related to those discussed in LLs. This allowed us to describe what Amanda felt as familiar to the group's sensemaking. We acknowledge there is more complexity in recontextualizing than the directionality we are present i.e., Amanda recontextualizations PoPs constructed from LLs, but space limitations only allow us to present one direction.

Results

We found PoPs related to argumentation to be primarily focused on participation processes. Teachers generally agreed that argumentation and discussions are important mathematical practices; however, they viewed the ideal classroom discussion focused on argumentation entailed students participating in discussions. Thus, the three PoPs we identified in the LL were couched in social processes to get students to participate, mostly thought talk, and engage in other students' ideas. We also describe one PoP from Amanda and how it relates to her practice.

Learning Lab PoPs

As the teachers made sense of experiences of teaching with argumentation in LLs, they identified PoPs centered around the social aspect of argumentation—particularly participation and their role in it. Because of the adaptive nature of the PD, these concerns became a strong focus of the group and we structured experiences in the PD provided teachers the opportunity to investigate the relationship of participation and argumentation (see Kazemi et al, 2020).

Teacher involvement. Teachers were broadly interested in making sure their involvement in discussions were minimal. Ideally, teachers wanted students to have autonomous conversations where they would step back from directing the discussion; however, they found themselves constantly stepping in to ask questions and push on new ideas when students did not immediately contribute to the discussion. A desire to decrease teacher involvement related to a school-wide commitment to student-centered mathematical learning.

Decreasing teacher involvement was constantly normalized as teachers provided different accounts of similar sentiments. Christina recounted a few times she wanted students to carry on autonomous conversations by either moving to the back of class or averting her gaze while students were talking about each other's ideas. Others also provided accounts of how they decreased involvement when eliciting students' ideas. For instance, Alyssa described wrestling how she could revoice student ideas but also having students revoice. The teachers further specified the problem by indicating teacher involvement usually occurs when students are given free rein of the conversation, such as during turn-and-talks, where students may talk about tangential mathematical or off-task topics. Several teachers added more dimension to this problem by discussing revisions to the nature and causes of this PoP. Karla thought there is a difference in values—teachers viewed math talk communities as important, yet students did not see the relevance yet. Christina thought one potential reason is because she acknowledges how she tends to make assumptions about students' thinking.

Value of students' ideas. Teachers realized when students participate in discussions, they are put in an emotionally vulnerable position. Ideally, teachers wanted to make classrooms a space where students felt comfortable and confident in sharing their ideas. However, they found that many students were complacent to be silent, letting others share ideas they assumed would be correct. Some teachers described students who had something to contribute but were uncomfortable sharing their ideas with the whole class because instruction was provided in a language different from their home language, especially in DLI classrooms.

The teachers normalized the PoP by recounting when students felt confident sharing their thinking with each other and in whole group discussions. For example, teachers shared accounts of students who shared ideas with a partner were asked to share their thinking with the whole class, but showed reluctance or resistance to sharing with the whole class. Teachers revised this problem of practice to consider how the activity itself encouraged or discouraged students from seeing their ideas as significant and the risk involved in sharing with others. Karla shared how in choral counts, "more kids would participate because we were part of a whole… So if I wasn't sure if my next number was correct I could say it a little bit quieter and then I would notice it if I was not with the group". Teachers generalized the PoP to principles of teaching, based on what moves would support students to see their thinking as meaningful within class discussion. The teachers wanted students to connect to their feelings on what it means to share and create space for vulnerability, recognizing that they can make mistakes and sharing an idea does not have to be an entire solution to the problem. For example, Melissa and Lea highlighted that making mistakes is part of being human and that mistakes can often help others learn. Similar to how the teachers talked about a math talk community for the sake of decreasing teacher involvement, the teachers talked about how a math talk community can create an environment where students feel comfortable sharing ideas. Towards the end of the LLs, Christina talked about how the environment (or math talk community) can be both nourishing and nurturing to better bring about student talk.

Students' engagement with peers. Teachers wanted to know how they could support students to have productive conversations with their peers. In teachers' classrooms, students were used to mathematics lessons in small groups while LLs were conducted in whole groups. Teachers recognized the challenge in how students engaged with their peers in this environment. Building on the first PoP, teachers were interested in making sure autonomous conversations were productive. The teachers continually returned to revise the problem of student engagement with peers and its possible causes. Olivia highlighted how students want to take ownership and connect with each other's ideas, but teachers need to be explicit on what it means to share your work with others. Revisions of the question lent elements of complexity, considering the experiences and needs of individual students. Teachers discussed the power of listening to others as a form of engagement, the implications of diverse language needs, and what it means when students are not socio-emotionally ready to explain their thinking to a partner. Addressing this problem within principles of teaching heavily relied on the use of turn and talks and teachers being explicit in particular questions or expectations of what it means to share with others. Alyssa and Lea shared how teachers can ask questions about students explaining another student's thinking or adding on to an idea proposed by a different student as a form of engaging students with each other. The growth in how students connected with one another individually in turn and talks and eventually in the whole group showed a greater number of students listening to and learning from one another. Karla noted in LL6 that there was more accountability for students in large group discussions as they engaged with each other's ideas.

Amanda's Recontextualization

Amanda's view of discussions with argumentation, like her peers, hinged on having students being able to participate in discussions. In making these PoPs more familiar to her practice, she strongly envisioned these discussions to be "equitable" in order for argumentation to occur. She viewed equitable discussions as students viewing their ideas as valuable as others and autonomously participating in discussions equally; however, in her classroom, students are not contributing to discussions equally especially when she reflected on her experience with emergent bilinguals in the DLI classroom. Thus, one of her PoPs is to support these students to talk at the same rate as their peers in order to achieve equitable participation.

Amanda frequently stated equitable participation as the ideal discussion to facilitate argumentation. During interviews, Amanda expressed equity as "feeling that each student in the classroom matters equally and what they have to say matters as much as anybody else in the classroom." She also expressed her role as "responsible for keeping discussions equitable" and try to "draw out maybe students that aren't talking as much, um, to give, like to privilege their voice a little bit more because they don't have that experience all the time." She also said she used drawing sticks where students' names are on sticks and Amanda draws a random student's name. During LLs, Amanda said one of her goals for equitable participation is how she can support students to join in on discussions particularly during whole-class discussions where she notices that students are comfortable sharing with a partner but not to the whole class. She viewed whole-class discussion as important for empowering students, "what feels good is honoring each student's perspective and having them communicate their ideas about math in front of the whole class is empowering to every student."

Whenever Amanda talked about her class or looked closer at her class during VSRIs, she notes the misalignment between her goal of equitable participation. She notes those larger societal inequities at play. For example, she notes gender during a VSRI, "I can see it's still not where I want it to be in this lesson. It seems like the boys, when they're in mixed gender pairs, the boys seem to talk a lot more. So yeah, just being more conscious of that and finding ways that they can participate. In front of everybody." More importantly, she consistently drew from her experience as one of the bilingual teacher at the school who taught in DLI classroom and noted students' home language as an inequity feeding into discussions, "I think English dominant students tend, I don't know, tend to feel more power in the school system for some reason, and they just tend to dominate space wherever they are..Spanish speaking students in summer school [where majority of students speak Spanish]… become a lot more lively and active and participate in the discussion." Amanda noted that because of this inequity, she felt "responsible for keeping the discussion more equitable and trying to draw out maybe students that aren't talking as much—to give privilege to their voice a little bit more because they don't have that experience all the time." Her goals for future work centered around cultivating equitable whole class discussion and to build a math talk community where equitable participation happens.

We largely saw Amanda's PoP of equitable participation as a recontextualization of PoPs from LLs. Amanda expressed part of equitable participation is students' seeing their contributions as valuable. During LLs, Amanda shared that one sticking point for discussions is figuring out how Amanda can build up students' ideas as valuable in order for them to share during whole-class discussion. In order for students to participate, Amanda needed students to view their ideas as valuable, especially for emergent bilinguals. If they are valued, then students may participate more. She commented, during a VSRI, about how an interaction supported a student. She noticed student M, who she said does not normally participate, talking with her partner and nodding along as her partner built on what student M said. Amanda queried, "I'm curious about whether you know it's a language thing or she doesn't have that much language to explain it" but because her partner participates more in discussion, Amanda saw this interaction as reaffirming her and her ideas. Amanda's PoP also was a recontextualization of students' engagement with each other. Amanda noted students needed to view each other peers' as valuable, noting that if students engage with each other's ideas during discussions, students will build more confidence in sharing their ideas, in turn, participating more. She shared that she provided students with sentence stems in order for students, especially those needing language support, to be able to begin and carry on discussions with others.

Discussion and Conclusion

In this paper, we identified PoPs related to discussions with argumentation constructed by a group of elementary school teachers. PoPs are important for teachers' sensemaking because it provides them with a shared space to identify parts of practice they would like to work on. Because the shared space is tied to the contexts and experiences of the teachers, PoPs are also highly situated and are a reflection of the values of the community. At Lockwood, the teachers' saw an ideal classroom situation where students would almost independently carry deep mathematical conversations; however, in most teachers' classrooms, this was not the case. Thus, the three PoPs—reconsidering their position in discussions, valuing students' ideas, and supporting student-to-student interaction— allowed the teachers to map out how to improve their practice in order to better facilitate discussions with argumentation. We also described the relationship between one teacher's PoP and those brought up in LLs. Amanda saw equitable participation as the ideal space for discussions with argumentation, related to two of the PoPs brought up in the LLs. Her idea of equitable participation can be drawn from her values and her experiences as one of the bilingual educators teaching in a DLI classroom in thinking about equitable participation where she wanted to see all students talking at a similar rate.

There are some conclusions and implications we draw from this report. This work highlights the importance of the work of PD in supporting teacher learning. We structured LLs to be highly responsive to teachers' ideas and practice; thus, our PD design provided space for teachers to wrestle with supporting students' participation in discussions with argumentation. This does not mean teachers did not concentrate on components of argumentation as identified in the literature (e.g., claim-making, providing data, establishing warrants). In fact, teachers saw participation and argumentation as intimately tied together and were attuned to aspects of argumentation in LLs and designed tasks to support argumentation; however, the teachers elevated participation as a critical component for argumentation to occur. This marks a strong movement from knowing that an idea exists to knowing how to bring that idea to life (Mason, 2002). Such PD experiences not only support teachers in wrestling issues close to their practice but also provide teacher educators the opportunity to think alongside teachers and construct powerful experiences.

We also saw PoPs brought up in this group were situated. In a community of teachers, shared experiences provide material for teachers to construct PoPs rooted in these experiences and thus, they make sense and solve these problems together. In our case, teachers generally agreed with the work on participation that needed to be done to facilitate discussions with argumentation. They were all committed to honoring students thinking and creating communities where students can deeply talk to one other about mathematics. Having this group, can support teacher learning. As Mason (2002) asserted, "real change also requires the support of a compatible group of people whose presence can sustain individuals through difficult patches, and who provide both a sounding board and a source of challenge for observations, conjectures, and theories" (p. 144). Further, any work that needed to be done would be constrained and supported by the contexts the teachers were in as seen in Amanda's work. As a teacher attuned to the language needs of her students, Amanda saw some inequities, particularly with language, in her classroom connected them with the PoPs brought up in the group. Her recontexualization of the group's PoPs was framed through, what she called, equitable participation, making something in one context, familiar in another. Further work needs to be done to examine the other direction (i.e., how teachers contribute to community PoPs). It is through this collective sensemaking in context that teachers learn—that we cannot refer to teacher learning as acquiring declarative knowledge, but rather as sensemaking from practice to change practice.

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LESSON STUDY: SUPPORTING SECONDARY TEACHERS' PERSEVERANCE TO ENGAGE WITH STUDENT THINKING

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Teachers often struggle to attend to student thinking in ways that facilitate students' conceptual understanding of mathematics. This struggle is particularly evident in the context of teaching mathematical modeling. Lesson study has supported teachers' engagement with student thinking. Hence, this qualitative study investigated how lesson study and the "Five Practices for orchestrating mathematical discussions" (Stein et al., 2008) supported three secondary teachers' engagement with student thinking while they implemented mathematical modeling tasks. The findings revealed how the teachers: anticipated valid and emerging student responses, used the five practices to advance student thinking, and focused on student thinking to refine lesson plans. These findings indicated that the teachers were supported by an integration of the Five Practices and lesson study in the context of teaching mathematical modeling.

Keywords: Professional Development, Instructional Activities and Practices, Modeling

Increased awareness of student thinking can improve teaching (e.g., Fennema et al.,1996). When teachers attend to student thinking they can be prepared to facilitate rich discussions (Stein et al., 2008). However, mathematics teachers have struggled with attending to student thinking (e.g., Smith & Stein, 2018). As a further complication, teachers have struggled to attend to student thinking while implementing complex mathematical modeling tasks (e.g., Thomas & Hart, 2013). Due to these challenges with teaching mathematical modeling, researchers (e.g., Ang, 2013; Kuntz et al., 2013) recommended that teachers receive professional development (PD) on implementing modeling, such as lesson study (see Turner et al., 2014).

Lesson study consists of four essential activities: Curriculum Study, Lesson Planning, Teaching and Observing, and Debriefing. Outside of modeling contexts, researchers found the activities and goals of lesson study naturally supported teacher learning, and improvement of teaching (e.g., Lewis et al., 2009). These outcomes were influenced by a focus on student thinking (Murata, et al., 2012; Stigler & Hiebert, 1999).

Because each phase of lesson study is guided by student thinking, this study, situated in the context of teaching mathematical modeling, employs a framework to support teachers to engage with student thinking through student discussions. Stein et al. (2008) proposed Five Practices (5Ps) written about in *Five Practices for Orchestrating Productive Mathematics Discussion* (Smith & Stein, 2018). The 5Ps are as follows: 1. *anticipating* likely student responses to challenging tasks; 2. *monitoring* students' actual responses to the tasks (while students work in pairs or small groups); 3. *selecting* particular students to present work during whole-class discussions; 4. *sequencing* student responses to be displayed in a purposeful order; and 5. *connecting* different students' responses to each other and to key content ideas.

Hence, the aim of this study was to investigate how teachers engaged with student thinking as they worked to improve their teaching through lesson study in the context of teaching mathematical modeling. The lesson study team consisted of one university researcher, the author of this paper, and three secondary teachers who completed two cycles of lesson study. The following question guided the research: *In what ways does teachers' participation in lesson study support their engagement with student thinking while teaching modeling?*

Research Methods

The participants were three mathematics teachers who taught in a vocational high school with a diverse student population in the mid-Atlantic region of the United States (all names are pseudonyms). At the time of the study, Ms. Dain was a second-year teacher. Next, Ms. Maronis, a former engineer, had six years of teaching experience. Lastly, Ms. Denvers had 21 years of teaching experience. The researcher served as a "knowledgeable other" and facilitator by providing curriculum materials, relevant practitioner articles, and guiding discussions during meetings. During the summer, after an introduction to lesson study and a curriculum study, the teachers planned a two-day lesson. Each of the enactments took place in the fall, about six weeks apart in the following order: (1) Ms. Dain, (2) Ms. Denvers, (3) Ms. Maronis. The teachers observed each other teach the lesson and debriefed after enactments (2) and (3).

All meetings, lesson enactments, and debrief sessions were video and audio-recorded and transcribed. To maximize opportunities to analyze how the participants engaged with student thinking, and to reduce the data, only transcripts centered around specific tasks were analyzed. Specifically, the tasks were open-ended and could have a variety of valid responses. Figure 1 includes examples of tasks that were selected for the data analysis. Day 1 Task 1 provided opportunities for students to develop various state apportionment methods. Day 2 Task 1 allowed for exploration of Thomas Jefferson's state apportionment method. Using themes from the literature and data, the transcripts were analyzed using constant comparative methods (see Strauss, 1987). Then final codes were organized into categories aligned with the Five Practices (i.e., anticipating, monitoring, selecting, sequencing, and connecting).

Day 1 Task 1: For simplicity, imagine that a newly formed country wishes to copy the U.S. House of Representatives. This new country has just 100,000 people split up into only four different states, listed in the table below.

State	Population
A	15,000
B	17,000
C	28,000
D	40,000

A. If the new country plans on having 25 representatives in its House of Representatives, how many should each state receive?
B. What if they plan to have only 17 representatives?
How did you calculate how many representatives each state should receive? Did you use the same method for both 25 and 17 representatives?

Day 2 Task 1: Watch the video on the Jefferson method, apply the Jefferson method for 25 representatives. Link: https://tinyurl.com/SGJefferson Use the tables to apply the Jefferson method.

Jefferson's Apportionment for 25 Seats

State	Population	Calculation	Number of Seats
A	15,000		
B	17,000		
C	28,000		
D	40,000		

Why did Jefferson use this method?
What are the differences and similarities between the Jefferson Method and the Hamilton Method?

Figure 1: Example Tasks from the Lesson on State Apportionment Findings

The findings are organized according to teachers' engagement with student thinking through the 5Ps during each lesson study activity: Planning, Enacting, and Debriefing the Lesson.

Planning: Anticipating Multiple Responses; Advancing Thinking; Purposeful Sequencing

Because the teachers had solved the tasks and had considered multiple approaches to each task, they were able to anticipate *multiple student responses*. In the context of teaching mathematical modeling, it was also important for the teachers to acknowledge that modeling tasks can have more than one correct approach. For example**,** in addition to considering multiple student responses, Ms. Maronis acknowledged for state apportionment methods, "there's not really a right or wrong answer."

In their planning, the teachers intended to *advance student thinking* through judicious telling (Lobato et al., 2005). For instance, when Ms. Maronis and Ms. Denvers discussed how to support students with Day 1 Task 1. They both discussed how to "steer" students without "taking the steering wheel." In their efforts to avoid "telling," teachers also planned advancing questions such as these suggested by Ms. Denvers: "What's the purpose? What do you have to accomplish by doing this?; How do you know when to stop guessing?" By asking these questions, the teachers hypothesized that students might reflect on their work or think about next steps.

In planning to sequence various student responses and support student understanding, the teachers planned to show *simple strategies before complex strategies*. Also, the teachers added connecting questions for the whole-class discussion (see Figure 4). By adding these questions, they planned to support students in making connections and building on one another's thinking.

Student Response Sequencing	Questions to ask during share-out:
First: Guess and check weighting method Second: Percentage method & rounded to get too many or too few representatives Final: Rounded and ended up with the right number of representatives (compare two groups that rounded differently)	What do you notice about each method? What are the pros and cons of each method? What are your revisions for your initial responses?

Figure 4 Teachers' Planned Sequencing for Day 1 Task 1

Enacting: Advancing Student Thinking and Purposeful Sequencing

As the teachers monitored student thinking, they used student responses to *advance student thinking* by asking *assessing questions* and using *judicious telling*. Although the teachers did not explicitly plan assessing questions, the teachers planned to use judicious telling (Lobato et al., 2005). Thus, asking assessing questions was a natural first step. Then, based on the planning, the teachers chose their next move. For example, during the second lesson enactment, in a discussion between Ms. Denvers and her student, Sam, about Day 2 Task 1, Ms. Denvers started the discussion by asking the following assessing questions: "I would like to see is yours exactly the same? Tell me exactly what you did." Then Sam guided Ms. Denvers through his written work.

For whole-class discussions, the teachers selected and sequenced student responses from *simple to complex* and *connected* student responses. As an example, Ms. Dain sequenced the responses as planned. Then she *connected* student responses, as illustrated in the following transcript for Day 2 Task 1, by encouraging students to engage in discussions with each other.

Ms. Dain Why did your group decide to take away the representative from State A…?

Jamal …you can't have .55 for a person and we didn't know what to do. And we think that's what, we didn't know any other way to do it. If we add all this together it made 17.

Ms. Dain I think Alisha wants to add to the question I just asked.

Alisha Well, my group, did the [approach] that you said, but we just looked at the decimal numbers and we chose the lowest one that could be rounded…

Ms. Dain Excellent, I heard other groups saying that too…

In this case, Jamal was unsure of his group's methods, so Alisha jumped in to validate their method. Ms. Dain then connected Alisha's response to other groups' responses.

Another theme that emerged was that the teachers *adapted to unanticipated student responses* so that the responses would still be sequenced from *simple to complex* approaches. Ms. Denvers encountered an unanticipated and unique approach and chose to share the response as one of the complex responses. In the transcript, Ms. Denvers asked *connecting questions* and encouraged students from other groups to explain Jake's approach.

Ms. Denvers Anybody have any thoughts about what those calculations are able to achieve?...

Kelsey He has 100,000/15,000, uh 6.666.

Ms. Denvers … What would that mean? … What's'that 6 and 2/3 represent? Kelsey Is it because a certain portion out of the whole will go into each? Ms. Denvers And that will give you 25/6 and 2/3, right? Jake, can you tell us?...

Jake So, the uh, the size of the state compared to the size of the overall population.

Ms. Denvers So that sounds like what Kelsey was saying right?

Although this student response was unanticipated, Ms. Denvers was able to adapt how to sequence the approach and Ms. Denvers was prepared to facilitate a future discussion about it.

Debriefing: Modified Tasks and Enhanced Anticipated Responses

The lesson study team used evidence of student thinking collected during observations to *modify tasks* and revise the lesson plans. For instance, during the first debrief session, after Ms. Dain taught the lesson, the group acknowledged that the student responses for the Day 1 Launch were not aligned with a learning goal about students developing their methods for state apportionment. Thus, the teachers decided to *modify* the task (see Figure 5). As a result, the student responses in subsequent lessons were specific and better aligned to the learning goal.

Version 1: Launch/Warm Up (10 minutes)	**Version 2: Launch/Warm Up (10 minutes)**
How might you arrange a system so that each state is represented fairly? What obstacles do you think might be present?	If you were in charge of determining how many representatives each state in the United States should have, what information would you need. How would you use that information? What obstacles do you think might be present?

Figure 5 Modification to the Day 1 Launch

The teachers also used their observation notes to *refine anticipated student responses*. During the debrief meetings they used their observations of student response to add specific details to the student responses. In addition to refining anticipated student responses, the teachers *enhanced* anticipated responses by adding student responses that surfaced during each lesson enactment. For example, the teachers used detailed notes to add Jake's method, discussed earlier, to the lesson plan for Day 1 Task 1. This type of focus on student thinking provided opportunities for teachers to improve the lesson plans for future enactments.

Discussion and Conclusions

The findings of this study provided insight into how lesson study can influence teachers' engagement with student thinking. For one, contrary to previous studies the teachers in this study engaged with student thinking in productive ways such as anticipating multiple student responses (e.g., Thomas & Hart, 2013). Then to make use of multiple student responses, the teachers planned and executed judicious telling (see Lobato et al., 2005) and asked questions that could support student thinking. Finally, also contrary to previous studies (e.g., Stein & Smith, 2018), these teachers purposefully planned and executed the selecting and sequencing of student responses from simple to complex approaches. The participants in this study further planned and executed the connecting of student responses. By planning how to select, sequence, and make connections, these teachers were prepared to engage with student thinking in ways that could support students to engage in complex tasks. Finally, the debrief sessions, focused on student thinking, supported the improvement of the lesson plan. Also, a main implication, needing future research, from this study, suggests that as recommended by Turner et al. (2014), lesson study and the 5Ps have the potential to support teachers when implementing mathematical modeling.

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AN EXPLORATION OF COACH-TEACHER INTERACTIONS DURING MODELING

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Drawing on six coach-teacher dyads' interactions (n=295) across 25 lessons, this study examines the foci and depth of coach-teacher modeling interactions. Qualitative analyses revealed six types of coach-teacher interactions, and two levels of depth that take into account the extent to which reasoning about content, pedagogy, and students was made evident during coach-teacher interactions. Implications for future research as well as practice are provided.

Keywords: Professional Development, Instructional Leadership, Classroom Discourse

A growing body of research examines what coaching activities and coach-teacher interactions are most productive for supporting teacher learning of complex teaching practices (Gibbons & Cobb, 2017; Saclarides & Lubienski, 2021). One such coaching activity is modeling (Gibbons & Cobb, 2017), where the coach embodies the teacher role to demonstrate a pedagogical practice to support the observing teacher's learning. And yet, modeling involves far more than performing instruction while teachers observe; to support teachers in making sense of the modeled lesson, coaches must make visible the complex reasoning work involved in enactment through professional interactions. By providing teachers with a window into the ways that more knowledgeable others reason pedagogically in the moment, teachers' learning opportunities may be enhanced. Doing so is a central tenant of the teacher education pedagogy of modeling (McGrew et al., 2018). This type of talk during teaching is not unprecedented (Gibbons et al., 2019; Munson & Dyer, 2020), but how it might look during a modeled lesson has not yet been explored.

The focus of coach-teacher talk circumscribes the learning opportunities created for teachers. Talk that addresses student thinking, disciplinary content, or pedagogy could be venues for teachers to learn about professional practice (Russell et al., 2020). While talk that addresses logistics, such as timing or classroom management, may be less oriented to teacher learning (Horn et al., 2017). Our conception of modeling points not just to the importance of coach-teacher talk about content, pedagogy, and students being *present*, but to the role of its *depth*. Depth of talk, or the degree to which talk engages with pedagogical reasoning, reflects teacher learning opportunities (Horn et al., 2017). When teachers moved beyond reporting what did or will happen in classrooms (low depth) to addressing how or why those events might unfold (high depth), they opened learning opportunities by exposing their pedagogical reasoning for collective consideration. Prior research on depth examined this construct in the context of extended professional interactions, where the unit of analysis was often a teacher meeting (Horn et al., 2017). Depth in brief interactions, such as during modeled lessons, has yet to be characterized. In this study, we examine the coach-teacher interactions in six dyads during modeled lessons to explore: 1) What are the foci of coach-teacher interactions; and 2) During conversations about content, pedagogy, and students, what depth of coach-teacher interactions is possible?

Method

Setting and Participants

This study took place in two different public school districts, Midtown and Southampton, where all coaches had full-time release from teaching, did not evaluate teachers, and reported directly to their building principals. We partnered with four coaches (Beth, Jade, Meg, Latoya), and six elementary teachers (Barbara, Brianna, Lindsey, Michelle, Mackenzie, Jennifer).

Data Source and Analytic Technique

Twenty-five modeled lessons (24-75 minute range, 45 minute mean) were observed, audio recorded, and transcribed. All transcripts and accompanying field notes were read to identify coach-teacher exchanges during modeling, which we define as subsequent turns of talk. Then, complex exchanges that included multiple ideas were parsed into exchanges that focused on a single topic. This led to the identification of 295 coach-teacher exchanges about a single topic (mean of 12 exchanges per lesson) across the 25 modeled lessons.

An open coding process (Creswell, 2013) was used to inductively develop codes for the focus, or topic, of each exchange. Codes and definitions were refined until all exchanges were reflected in six focus codes. We then isolated exchanges that centered on content, pedagogy, and students (n=60), and inductively developed *depth* codes to attend to the reasoning that was evident. *Low-depth* interactions were characterized by observations without reasoning or evidence, while *high-depth* interactions included reasoning or evidence, making thinking public.

All codes were mutually exclusive and assigned at the exchange-level. After coding was complete, data were analyzed to tabulate percentages for all codes for each coach-teacher dyad, as well as across all data. Last, matrices were created to detect patterns within and across dyads.

Findings

Foci of Coach-teacher Interactions

Analysis of 295 exchanges across 25 lessons yielded six distinct foci of coach-teacher interactions during modeled lessons (see Table 1).

Table 1: Foci of Coach-teacher Interactions

Coach-teacher dyad	n	Logistics	Building Relationships	Performative Praise and Discipline	Joint Teaching	Discussing Content and Pedagogy	Noticing Student Thinking	Other
Meg-Michelle	83	50%	2%	35%	0%	1%	10%	3%
Meg-Mackenzie	65	51%	24%	8%	0%	10%	7%	1%
Jade-Jennifer	17	19%	12%	14%	15%	22%	20%	0%
Beth-Barbara	26	32%	8%	0%	0%	20%	41%	0%
Beth-Brianna	42	44%	25%	0%	0%	23%	8%	0%
Latoya-Lindsey	62	37%	2%	3%	4%	22%	31%	1%
Mean	49	39%	12%	10%	3%	16%	20%	1%

Logistics. When engaged in logistics conversations, the dyads discussed issues that arose from the coach teaching in a classroom not her own, such as materials (Coach Latoya: "Do they have markers?"), technology functionality (Coach Beth: "Does your eject button work?"), and attendance (Coach Beth: "Ms. Barbara, are you missing a student?"). Logistics conversations were the most prevalent topic across all data, making up 39% of the coach-teacher interactions.

Building Relationships. When building relationships, the dyads displayed their partnership through, for example, greeting one another (Coach Latoya: "Hey ladybug..how are you?"); apologizing (Coach Beth: "Sorry, it's just been kind of a crazy morning"); complementing one another (Coach Beth: "You're so good and I am a forgetter"); or commiserating (Coach Beth: "I even got here early this morning. I just can't seem to get it all together.") Across all dyads, building relationships conversations made up 12% of interactions.

Performative Praise and Discipline. Performative praise and discipline interactions featured public, emotive conversations between dyads for which students were the intended audience. The dyads either praised students' academic or behavioral efforts or expressed their frustrations or disappointment in students' behavior. For instance, at the end of her modeled lesson, Coach Jade publicly complimented the fourth grade students' behavior: "[Teacher Jennifer], you've done such a lovely job with them! This may be my favorite class ever!" Across all dyads, performative praise and discipline conversations made up 10% of interactions.

Joint Teaching. When engaging in joint teaching, the teacher shared moments of brief, public teaching with the coach that involved interacting with both the coach and students. For instance, Coach Jade asked students to find a spot on the rug so that they were seated in a square. Teacher Jennifer took this opportunity to incorporate the lesson's vocabulary; she interjected, raised her voice, and looked directly at Coach Jade while saying, "Or, make the *perimeter* of the rug." Acknowledging Teacher Jennifer's contribution, Coach Jade replied, "Thank you, Teacher Jennifer." Across all dyads, joint teaching interactions were rare (3%).

Discussing Content and Pedagogy. When discussing content and pedagogy, the dyads talked about the content featured in the lesson or pedagogical dilemmas that surfaced during instruction. For instance, while Teacher Jennifer was playing a subitizing game with a small student group, Coach Jade prompted the teacher to more quickly flash the dot cards to encourage automatic recognition: "You can flash it [cards] for like 1, 2, 3, and then turn it [over]. So, you don't want them [students] to count it. You want it to be instant." Overall, discussing content and pedagogy made up 16% of interactions.

Noticing Student Thinking. While noticing student thinking, the dyads shared their noticings about how students grappled with the content, made predictions about how students might engage with the lesson's content, discussed their perceptions of student affect, and set student growth goals. For instance, during one modeled lesson, Coach Beth shared her observations with Teacher Barbara about two students: "But I think that there was great success right here with Oscar and Juan. They are super clear about where their model and their problem…have the same link, right? So, they were actually able to write…this is the 10 times 6 part." Overall, interactions focused on noticing student thinking made up 20% of the data.

Depth of Coach-teacher Interactions

There was substantive variation in the depth of coach-teacher talk about content, pedagogy, and students (see Table 2).

Table 2: Depth of Coach-teacher Interactions

		Discussing Content and Pedagogy		Noticing Student Thinking	
	n	Low Depth	High Depth	Low Depth	High Depth
Coach Meg-Teacher Michelle	9	100% *(1%)*	0% *(0%)*	37% *(4%)*	63% *(6%)*
Coach Meg-Teacher Mackenzie	5	24% *(2%)*	76% *(8%)*	100% *(7%)*	0% *(0%)*
Coach Jade-Teacher Jennifer	5	29% *(6%)*	71% *(15%)*	100% *(20%)*	0% *(0%)*
Coach Beth-Teacher Barbara	7	0% *(0%)*	100% *(20%)*	0% *(0%)*	100% *(41%)*
Coach Beth-Teacher Brianna	9	42% *(10%)*	58% *(13%)*	0% *(0%)*	100% *(8%)*
Coach Latoya-Teacher Lindsey	25	21% *(5%)*	79% *(18%)*	28% *(9%)*	72% *(22%)*

Note. In each cell, the first percentage indicates the percent of coach-teacher talk at a particular depth for the foci of discussing content and pedagogy and noticing student thinking. The second parenthetical percentage indicates the percent of all the given dyad's talk at that depth.

Low-depth coach-teacher interactions about content, pedagogy, and students lacked reasoning and did not provide evidence to support statements made. Typically, one partner restated students' actions, named mathematical content, or stated mathematical procedures. In the absence of reasoning, we argue that fewer teacher learning opportunities were available.

High-depth coach-teacher interactions were marked by reasoning, as the coach and/or teacher elaborated on the process or the justification for their observations or decisions regarding content, pedagogy, or students. Typically, one partner justified their pedagogical decision making, provided reasoning behind their interpretations of student work, discussed their plans to implement pedagogical strategies or activities, or made conceptual mathematical connections visible to one another. In the presence of such reasoning, we argue that high-depth interactions open up rich opportunities for teacher learning. To illustrate, the following interaction took place while Coach Latoya modeled instruction in Teacher Lindsey's fourth grade classroom. Students had been independently working on a fair sharing fractions task, and the dyad came together during students' work time for the following 23-second, high-depth exchange:

Lindsey: So, even if they're not ready and they don't have it—

Latoya: But then they could talk with people at their [table], and that's the reason why I wanted to keep giving them independent time. Because I didn't want them to feel like somebody else is talking them through their thinking.

Lindsey: Right. And what if they're still, after five or 10 minutes they still don't have anything?

Latoya: Right. With the model, they should have something.

Above, Latoya provided a justification for her pedagogical decision to have students first grapple with the task independently before working in groups. Furthermore, when Lindsey raised concerns about students who may not have any work recorded, Latoya assured Lindsey that students should have something recorded, providing the reasoning that they were permitted to use manipulatives and pictorial representations.

Discussion and Implications

This study found six types of coach-teacher interactions during modeled lessons. We propose that four interaction types (joint teaching, logistics, performative praise and discipline, relationship building) lay the foundation for the model to occur by attending to logistics, materials, and the coach-teacher relationship. These kinds of interactions do not, however, open up opportunities for teacher learning about content, pedagogy, and students. Interactions focused on discussing content and pedagogy and noticing student thinking moved beyond discussing necessary mechanics to create learning opportunities about the work of teaching. Our analysis of the depth of coach-teacher talk found that when discussing topics of potential teacher learning, even brief interactions could include reasoning.

This study has implications for both practice and research. Our results serve as existence proof that coaches do not need to wait until after the lesson to engage teachers in high-depth interactions, and that such moments can happen as instruction unfolds in the presence of students. Furthermore, coaches need opportunities to learn how to engage teachers in high-depth interactions about content, pedagogy, and students during instruction. Future research is needed to explore the kinds of coach-teacher interactions that are possible against different disciplinary backdrops, as well as the contextual factors that enable and inhibit coaches as they seek to engage teachers in high-depth interactions during modeling.

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THE INFLUENCE OF POSITIONALITY ON COACHES' OPPORTUNITIES FOR PROFESSIONAL LEARNING

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Content-focused coaching is highly complex work, yet little is known about how coaches develop expertise needed to support teachers. This discourse analysis explored one group of elementary mathematics coaches' learning opportunities while collaboratively engaging in mathematics. Drawing on video and interview data from coaches and their district leader, our analysis highlights that coaches' discursive positioning influenced the types of coaching expertise they were able to develop while engaging collaboratively in mathematics. Implications for future research and practice are discussed.

Keywords: professional development, instructional leadership, classroom discourse

Effective coaching requires multiple forms of expertise, including disciplinary knowledge, pedagogical knowledge, as well as a professional coaching vision (Kane et al., 2018); however, little research has explored how instructional coaches develop these forms of expertise. Because *doing the math* has—for at least two decades—been recommended as a support for teachers' development of specialized disciplinary knowledge (Gibbons & Cobb, 2017), we were interested in how *doing the math* might support mathematics coaches' development of coaching expertise.

Because the research base on coaches' learning from *doing the math* is still in development, we extrapolate from findings about teachers' learning to better understand how *doing the math* might support coaches. In *doing the math,* educators are positioned as students in mathematics classes and asked to engage in inquiry with rich tasks that could be solved using multiple solution strategies. *Doing the math* has been found to support teachers to experience mathematics as a field in which knowledge is constructed—not received and reproduced—for the first time (Schifter & Fosnot, 1993). This is a foundational realization if one is to teach mathematics ambitiously (Windschitl, 2002). In addition, *doing the math* can support teachers to consider the mathematical strategies students might use and to discuss how they, as teachers, might guide and refine students' strategy use and selection (Borko et al., 2011).

Conceptual Framework

Building from Greeno and Gresalfi (2008), as well as a growing body of work on teachers' professional learning (e.g., Horn et al., 2017; Horn & Kane, 2015), we highlight that coaches' professional learning opportunities—like teachers'—are always influenced by the affordances and constraints of the contexts in which coaches learn. From this perspective, individuals' identities and positionalities within particular groups – that is, how they position themselves and how others position them – are an integral aspect of the ways in which individuals come to learn and to be accepted members of particular communities of practice (Lave & Wenger, 1991). Coaches' professional learning opportunities are thus shaped by the structure of professional development activities, how members are positioned within these activity, available tools and resources, histories of participation within particular groups, predominant topics and modes of talk, and the ways in which particular problems of practice are framed. Viewing coaches' professional learning opportunities through this lens allows us to analyze systematically how the design of the activity structure *doing the math* opened up particular foci for coaches' professional learning while constraining others.

Methods

Setting and Participants

Using a best case sampling logic (Yin, 2009), we selected one group of 12 elementary mathematics coaches from a district we call Hamilton. Hamilton coaches were a best case, because they met regularly (twice a month for 8-hour sessions) to participate in collaborative *doing the math* sessions. This group was especially notable because they worked together to solve cognitively demanding (i.e., "rich") tasks and held one another accountable for comparing and contrasting multiple mathematical solution strategies. All of these practices feature prominently in the literature on ambitious instruction in mathematics. These coaches reported to the building principal, were released full-time from teaching, and did not evaluate teachers.

Data Source and Analytic Technique

This analysis rests on two main data sources: transcribed video recordings of coaches' ongoing professional development sessions (n=6) and 15 semi-structured interviews conducted with a subset of coaches and the district administrator. We began by coding for two discursive moves, which have been identified in previous literature as central to educators' opportunities for professional learning (Horn & Kane, 2015): *epistemic claims* (assertions about what is true about students, teaching, mathematics, or coaching) and *representations of practice* (descriptions of classroom life focused on students, teaching, mathematics, or coaching). Coding was completed at the turn level, and utterances were coded multiple times as appropriate. This round of coding revealed qualitative differences in the ways that epistemic claims and representations of practice were used to make sense of mathematics, students, teaching, coaching, or any combination thereof, depending on whether participants spoke from the positioning, or footing (Goffman, 1974), of a student of mathematics, a teacher or mathematics, or a mathematics coach. Thus, in our second round of coding, we focused on which of those three roles participants took up. We used linguistic ethnographic methods (Rampton et al., 2015) to analyze the positionality from which coaches spoke, analyzing what opportunities for professional learning these positionalities made available about mathematics, students, teaching, and coaching. Last, we triangulated our findings using interview data from a subset of coaches.

Findings

When coaches spoke about mathematics, they were positioned as learners engaged in the process of doing mathematics over half of the time (435/791), which makes sense, since coaches were participating in *doing the math* sessions. Forty-three percent of the time, when coaches talked about mathematics, they did so from the positioning of a teacher. The frequency with which coaches spoke of mathematics from the positioning of teacher is notable, given that none of these coaches were teachers of record at the time of this study, although some coaches did teach an intervention course during the school day. Only six percent of the Mathematics codes were spoken from the position of coach, meaning that coaches were rarely positioned as coaches, despite that this data set is one in which, ostensibly, coaches met to learn about coaching.

From Student to Teacher of Mathematics. Hamilton coaches consistently shifted from a positioning as a participant in mathematics to a positioning as a teacher, which opened up opportunities for Hamilton coaches to make connections between their own mathematical thinking and instructional adaptations they might make in order to better support students. For instance, during a meeting in February of 2019, Hamilton coaches discussed a task which required participants to read a recipe and decide how they could cut a block of cheese into 36 pieces which were under half an ounce each. After the group had engaged with the task themselves, they had decided that "There are a lot of directions" (Facilitator Beth) and that the task would be a "reading comprehension activity" (Coach Lauren). Facilitator Beth then shifted the group's positioning from practitioners to teachers of mathematics, asking: "[W]hat might be something that you do to the problem itself to make it maybe more accessible?"

Coaches took up the positioning of teachers, discussing instructional adaptations such as breaking the problem up (Coach Maya), putting steps on index cards (Facilitator Beth), and giving students a "context for what was actually happening," since many students do not cook (Coach Lola). In this way, *doing the math* supported coaches to consider multiple instructional adaptations to use with rich tasks.

***Doing the Math* Supports Empathy for Students.** *Doing the math* also led coaches to take up an empathetic stance toward students. In the following excerpt, coaches discussed their own experience of engaging with the task and connected that experience to ideas for other instructional adaptations:

> Lola: 'Cause I cook, and I was reading this and finding myself checking out…You know, like I don't want to know all that. [CROSSTALK]
>
> Kaci: When I saw about cutting the cheese into 36 pieces I was like, yeah, next recipe.
>
> Beth: Yeah. Yes. Same… So that leads me to another thought. Simplify the wording, the numbers, the tasks. Simplify the amount of instruction for kids. Take away, like you said: "I'm a baker, I cook, and I was checking out."
>
> Lola: Right.
>
> Beth: You saw the number 36 and was like, nope! So already—we're adults.
>
> Lauren: So I checked out at dissolve yeast in warm water.
>
> Lola: I don't yeast. I don't yeast. [LAUGHTER]
>
> Lauren: I was like, I'm gonna go to another question.

Here, again, coaches presented their own thinking as practitioners of mathematics before making explicit connections between their own experience of *doing the math* and students' potential affective and academic responses to the tasks: "You saw the number 36 and was like, nope! So already—we're adults." In this way, Facilitator Beth introduced an empathetic stance toward students, and coaches then suggested a number of instructional approaches that could proceed from this understanding of students' thinking.

Interview data validates that being positioned as practitioners of mathematics supported Hamilton coaches to empathize with students. Three of the four coaches we interviewed highlighted *doing the math* as central to their learning, noting that it led them to more carefully consider their mathematical thought processes (i.e., specialized disciplinary knowledge) and to empathize with students. As Coach Lola pointed out in her year-end interview:

> I love the math, doing math, and I think it just is always good to get back in that being the learner…[It's] probably the most helpful thing that we do…because we're all learning the

> content part, but we're also putting ourselves in the, you know, position of students to remember, you know, what that feels like. (March 13, 2019)

Thus, *doing the math* supported coaches to empathize with students and to devise instructional adaptations in light of that empathy.

Coaching as a Rare Focus of *Doing the Math*. However, *doing the math* sessions rarely opened up opportunities for coaches to discuss issues related to the work of coaching. Across our data set, coaches discussed mathematics from the standpoint of a coach in only 6% of the total utterances devoted to mathematics (50/1961). When coaches spoke from the positioning of a coach, they most frequently discussed how they might support teachers, usually by sharing rich tasks and other resources with teachers (27 utterances); teachers' resistance to rich tasks (19 utterances); described teachers' success (2 utterances); and described school-level or administrative work (2 utterances). Interview data corroborated that coaches' participation in doing the math encouraged them to make rich tasks available to teachers. However, apart from these references, coaches did not elaborate in interviews about how *doing the math* influenced their coaching. This is perhaps unsurprising, given the infrequency with which Hamilton coaches spoke from the positioning of coaches or directly discussed coaching.

Discussion and Implications

Implications for the field include that coaches may value opportunities to collaboratively do mathematics, as it may support them to empathize with students' thinking and to think through instructional adaptations. Thus, our analysis demonstrates that findings about the usefulness of *doing the math* for teachers' professional learning also extend to coaches. Yet, findings also highlight that we must carefully attend to how coaches are positioned within activity structures (e.g., *doing the math*), since positioning coaches as teachers of record may truncate coaches' learning opportunities about coaching. Given the complexity of coaching, coaches need support not only in content and content-specific pedagogies, but also in developing a professional coaching vision. By attending to how coaches are positioned while *doing the math*, coaches might be supported to link their discoveries from when they were positioned as practitioners of mathematics to those that arose when positioned as teachers, and—finally—to those that arise when positioned as coaches. In this way, *doing the math* has the potential to help coaches develop empathy for both students and teachers. We look forward to future research that will undoubtedly clarify and refine the ideas we present here.

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TEACHER SELF REPORT FRAMEWORK FOR MAKING TEACHING VISIBLE

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The Whole Class Discussion Framework (Author, 2019; Author, 2020) was administered pre and post PD. Specifically, teachers self-reported their teaching practice as it related 1) the design of physical space, 2) classroom routines, 3) lesson planning and 4) the whole class discussions. The meaning of these categories was explicitly addressed in the professional development that teachers participated in. The pre and post data were compared, and the data revealed that there were shifts in every category. The greatest shifts took place on how they facilitated whole class discussions. Implications for using this tool for teachers to professionally notice their practices to refine teaching, and for professional developers to make decisions to adapt PD are discussed.

Keywords: Teacher Noticing, Teacher Education, Professional Development

Theoretical Framework

The challenge of teacher professional development is to ensure that teachers refine their classroom practice to improve student learning. Even though many teachers engage in professional development, teachers often struggle to implement these ideas into their teaching practice (Borko, 2004). This is because the new ideas that they learn in the professional development that align with the Common Core (CCSSM, 2019) Standards for Mathematical Practice, does not align with more traditional approaches in teaching. Therefore, teachers struggle to figure out how to implement standards-based approaches into their existing routines. This is because most of what teachers do in planning and organizing for teaching becomes routinized and invisible (Author, 2003).

Teaching is a complex endeavor and therefore removing the complexity of teaching into isolated parts such as learning how to ask great questions without addressing how asking questioning fits supporting student learning, limits teachers' ability to transfer PD into the classroom (Opfer & Pedder, 2011; Van Driel and Berry, 2012). This example, can be thought of as a car engine. If you remove the battery, the engine does not work.

Therefore, a consistent pedagogical framework to support teachers to connect theory and practice along with time to learn and refine practice is needed (Heller at. al, 2012; Opfer & Pedder, 2011). The Whole Class Discussion Framework (Author, 2019; Author, 2020) was used as a tool for teachers to self-reflect to make their teaching practices visible and document shifts in practice. This study investigated what teachers reported as their teaching practice after they participated in a content and pedagogy based summer institutes that explicitly discussed aspects of the framework with regard the design of classroom environment, classroom routines, lesson planning and discussions. In addition, we investigated what ratings teachers self-reported after they had a whole academic year to implement the ideas that they learned to determine shifts in practice.

Methodology

Twenty-seven teachers participated in a yearlong content and pedagogy based professional development aimed at supporting teachers to implement the Common Core Standards (CCSSM, 2010). This was part of a larger study that was implemented in a Western State. The professional development was designed using the PD design outlined in Author, (2020). The participants were in-service teachers from several school districts who taught grades K-8th. The Whole Class Discussion Framework (Author, 2019) was administered at the end of a weeklong professional development and at the end of the school year. The weeklong professional provided teachers with content and pedagogical knowledge and explicitly addressed the aspects of the Whole Class Discussion Framework. Twenty-seven teachers completed the pre-survey and 23 teachers completed the post survey. The mean for each category of the framework was calculated and the differences between the pre and post survey means was determined to identify the change that took place in the mean scores. The scale was between 0 - 4.0

Findings

The pre and post surveys revealed that the teacher practice shifted in all areas of the whole class discussion framework (See Table 1).

Table 1: Teacher Reported Shifts in Practice

	Pre-Survey	Post Survey	Change
Design of Physical Space	2.26	3.17	0.91
Classroom Routines			
Routines for Discussion	2.36	3.17	0.81
Routines for Communication	2.32	3.35	1.03
Routines for listening	2.21	3.04	0.83
Lesson planning			
First Level	2.21	3.04	0.83
Second Level	1.93	2.83	0.90
Third Level	2.00	3.00	1.00
Whole Class Discussion			
First Level (Making thinking explicit)	2.00	3.09	1.09
Second Level: Analyzing Thinking	1.79	3.78	1.99
Third Level: Making big ideas explicit	1.79	2.91	1.12

The pre and post survey data indicate that teachers changed how they designed the *physical space* in the classroom for productive discussions and thinking. The initial mean was 2.26 and shifted to a mean of 3.17. The classroom routines involve what teachers do to prepare for discussions. This involves giving student time to think, work in small groups, and using representations and tools. When teachers started the professional development, the *routines for discussion* was a mean of 2.26. Many teachers did not think about what they needed to do to get students to prepare for discussions by thinking through the problem and engaging in problem solving. The mean shifted to 3.17 in the post survey. The *routines for communication* involved creating a classroom environment where students felt safe to express their ideas respectfully. This involved communicating in a manner so that the whole class could understand what was being shared. This means explaining their thinking and reasoning using gestures and representations. Teachers became mindful of creating a classroom culture for communication after the PD. The mean shifted from 2.21 to 3.04. The *routines for listening* involved creating norms for student expectations to listen to each and reflect on what is being shared. The mean shifted from 2.21 to 3.04.

Teachers were exposed to *Three Levels of Lesson Planning* with a focus on interconnections and sequencing. *The First Level of Planning* involves thinking about the standards and the "big ideas" that students need to learn. This process involved looking at the big picture and getting a sense of what students were expected to learn throughout the year. The initial mean was 2.21 and the post survey mean was 3.04. The *Second Level of Planning* involved looking at the unit and planning how to implement a lesson. This involved taking into account prior knowledge and sequencing lessons and tasks to support mathematical connections. Many teachers did not think about the sequencing of lessons and considering how to take into account student prior knowledge when planning lessons. This category had a low score of 1.93 prior to implementing ideas from the professional development and a mean score of 2.83 after the professional development. The *Third Level of Planning* involves adapting the lessons while teaching based on student reasoning. The pre-survey mean score was 2 and the post survey mean score was 3. Lesson planning was an area that scored lower than the rest of the categories.

Teachers were exposed to *Three Levels of Sense Making* to facilitate mathematical discussions so that students can make mathematical connections. The First level involves *making thinking explicit*. In other, words getting students to share their thinking. Teachers indicated pre-survey a mean score of 2 which increased to a mean score of 3 post PD. The *Second Level of Sense Making* involves having students analyze each other's solutions. This process involves looking at the structure of mathematics and strategies students used to identify similarities and differences in ways of thinking. Many teachers reported that they did not do this. The mean pre-survey score was 1.79 and the post survey score was 3.78. This mean that many of the teachers were beginning to go beyond having students share their thinking but digging deeper to think about the similarities and differences. The growth was 1.99. This was a huge shift in their practice. The *Third Level of Sense Making* involved thinking about abstraction to make the big mathematical ideas and strategies explicit so that students could transfer what they learned to new situations. There was significant growth in this area. The pre-survey score was 1.79 and the post survey score was 2.91. There was a growth of 1.12 in the mean.

Discussion

Teachers identified areas of strengths and growth and rated themselves critically in the pre and post teacher self-reported survey. A non-threatening environment was created to support learning in the PD sessions. This was critical for teachers to feel comfortable to honestly rate themselves to reflect on their teaching. The goal was to make teaching visible to the teachers and to have a tool (Author, 2020) that the professional developers could use quickly to make decisions to focus on areas of emphasis. The PD design and the professional development that was implemented is outlined in Author, (2020).

Prior to administering the framework, it was important to ensure that teachers all understood what the framework measured. The weeklong PD explicitly focused on various aspects of the framework, that involved the physical design of space, classroom routines, lesson planning, and whole class discussions (Author, 2019). They got the opportunity to learn math themselves and reflect on their learning experiences as it related to the aspects of the framework. They got to watch and analyze videos on various aspects such classrooms that used the *Three Levels of Sense Making* against discussions that only sharing took place. Teachers were given an opportunity to plan as they learned content and pedagogical content.

The framework identifies the complexity of teaching (Opfer & Pedder, 2011; Van Driel and Berry, 2012) and makes it possible for teachers to pinpoint their strengths and weaknesses. Furthermore, the aspects of the framework are aligned with the process of organizing for teaching and facilitating lessons in the daily work of teaching.

Therefore, the Whole Class Discussion Framework (Author, 2019; Author, 2020) served as a useful tool for teachers to think about the complexity of teaching and to make practices visible through self-reflection. Understanding this complexity and how everything is connected is an important part of shifting practice (Opfer & Pedder, 2011; Van Driel and Berry (2012). This is an important part of making teaching visible to shift practice. In addition, the framework is a useful tool for professional developers to quickly assess areas of strength and growth and plan professional development to meet the needs of teachers. The whole class discussion framework integrated multi-dimensions of teaching as suggested by Bransford, Brown, and Cocking (2000). The pre and post survey made it possible to identify areas of shifts and where teachers struggled the most and grew the most. This framework served as a performance tool that teachers could continuously use independently or with others to make teaching visible to improve teaching.

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BUILDING "SMALL WORLDS" IN ONLINE PROFESSIONAL DEVELOPMENT WITH EVIDENCE-BASED NOTICING AND WONDERING

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Understanding how to design online professional development environments that support mathematics teachers in developing mathematical and pedagogical knowledge is more important than ever. We argue that productive social and sociomathematical (SM) norms have benefits for teachers learning mathematics in online asynchronous collaboration and that particular patterns in interactions can create context for the emergence of such norms. We employed social network analysis to compare the emerging social networks of two iterations of an online asynchronous professional development course focused on functions to understand whether particular scaffolds can support the emergence of specific patterns of interactions. Results suggest that evidence-based noticing and wondering can impact the "small world" properties of a social network and associated potential for the emergence of social and SM norms.

Keywords: Professional Development, Online and Distance Education, Teacher Knowledge, Noticing and Wondering

Objectives and Purposes

Our work focuses on the design of online professional development environments that support teachers in collaboratively developing mathematical and pedagogical knowledge. One challenge associated with such design endeavors is moving mathematics teachers from "show and tell" to collaboratively building mathematics knowledge together (Stein, et al., 2008) by participating in productive social and sociomathematical (SM) norms (Cobb et al., 2001). We argue that there can be a connection between the evolution in particular patterns of teachers' interactions in online asynchronous collaboration and potential for the emergence of social and SM norms. The current paper documents evidence-based noticing and wondering (EB-NW) scaffolding the emergence of these particular patterns of interactions in mathematics teachers' online asynchronous collaboration, where the focus of collaboration was on developing foundation reasoning skills for understanding the concept of function.

Theoretical Framework

Social norms and their mathematics-specific counterpart SM norms – accepted and expected regularities in mathematical dialogue – have benefits for collaborative mathematics learning in both face-to-face (Clark et al, 2008) and online mathematics teacher professional develop. Such norms can guide generative and collaborative mathematical activity that includes explaining and justifying one's reasoning, communicating the meaning of mathematical ideas, and critiquing colleagues' mathematical reasoning (Elliot et al., 2009; van Zoest et al., 2012). As such norms emerge, they create conditions for teachers learning to make contributions to collaborative mathematical activity that align with these generative forms of participation (Cobb et al., 2001). Further, teachers participating in productive norms provides them with experiences learning mathematics in a discourse-centered environment and these norms can become tools for building similar norms in their own classes (Clark et al., 2008; Tsai, 2007). Thus, it is important to understand how to support the emergence of norms in online professional development settings –

a setting that can be scaled to increase the impact of professional development on teachers' mathematics instruction.

Mathematics teachers accessing and engaging with their colleagues' mathematical reasoning is important for the emergence of social and SM norms in online settings. A key difference between building norms in face-to-face and online settings is how one gains access to or listens to their colleagues' ideas (Dean & Silverman, 2015). In face-to-face settings, teachers can listen to a mathematics conversation simply through proximity to others; in online asynchronous collaboration, researchers must define "listening" in a different way (e.g., see Wise et al., 2013). In our work, we define listening as explicit interaction with colleagues' mathematical reasoning by reviewing and responding to another's post. Because of the publicity and permanency of teachers' contributions to online asynchronous collaborative environments, reviewing and responding to another's post can include extended reflection on a specific way of reasoning in the post. Therefore, an individual's mathematical reasoning in an online environment can become a scaffold that supports others in learning to engage in generative contributions and/or interactions in the online space. Regularities in mathematical reasoning can emerge when mathematics teachers are reflecting on, taking up, and trying out their colleagues' mathematical reasoning. This process can result in specific ways of reasoning becoming more visible in an online space (Borba et al., 2018), which increases the potential influence of specific reasoning on collaborating teachers' future use of reasoning (Lave & Wenger, 1991) – if they are interacting with colleagues' in the online space.

Small world networks can create context for interaction and, ultimately, the emergence of social and SM norms in online professional development settings. The concept of a small world - what is commonly thought of as the "six degrees of separation" between any two people in the world - is often applied to studies of social networks. Formally, a small world is a sparsely connected social network - – set of nodes (people) and edges (an interaction between two people) - with both high local clustering and short paths of connections between individuals in the network (Watts, 1999). In the context of online asynchronous collaboration via discussion boards, a social network with a minimal average path length means that mathematics teachers are accessing and engaging with a large proportion of their colleagues' mathematical reasoning. We argued above that access and engagement with mathematical reasoning can create context for the emergence of social and SM norms because of the potential for specific ways of reasoning to diffuse through the network. Therefore, we argue that the "small worldness" of mathematics teachers' social network is an indicator of the potential for emerging social and SM norms in online asynchronous collaboration.

Further, we argue that EB-NW can scaffold the emergence of small worlds. Noticing and wondering is receiving increasingly more attention in the literature (e.g., Dobie & Anderson, 2020). We are currently engineering a virtual assessment environment that scaffolds a specific type of noticing and wondering – EB-NW, which is noticing and wondering that is explicitly connected to a colleagues' thinking. The environment enhances typical online asynchronous discussion forum conversations by scaffolding EB-NW with two key design features: a selection tool that allows teachers to highlight specific aspects of colleagues' mathematical reasoning and a commenting tool that supports noticing and wondering that is explicitly connected to the selections (the evidence). Our past work has documented the effectiveness of the environment to support teachers in engaging with the details of their colleagues' mathematical reasoning and providing generative feedback that moves beyond a focus on the correctness of their colleagues' solutions (Matranga et al., 2018). Further, we have found that teachers are less likely to provide one another evidence-based and generative feedback when online asynchronous collaboration is scaffolded by discussion forums (Matranga, 2017). Thus, we argue that technologically scaffolded EB-NW can increase the proportion of interactions in an online asynchronous collaborative setting that include mathematics teachers' explicitly engaging with colleagues' mathematical reasoning, thus enhancing the small world*ness* of a social network and associated potential for emergent norms.

Methods

We investigated the small world properties of mathematics teachers' evolving social network in two iterations (C1 and C2) of an online asynchronous professional development course focused on understand the behavior of functions. The course includes eight weekly problem-solving modules, each featuring a set of mathematics tasks and scaffolds to support participant engagement with the mathematics and interaction with colleagues. The modules included an initial period of individual problem solving and then a period specifically devoted to peer-to-peer collaboration. The two iterations of the course differed only by the collaboration scaffolds provided – the first utilized traditional discussion boards (C1), while the second utilized the virtual assessment environment designed to scaffold EB-N&W and mediate teachers' collaboration and interaction (C2). Our research question is: How does participants' engagement with and access to colleagues' mathematical reasoning differ between C1 (n = 16) and C2 (n = 23)? In particular, we seek to understand if one course and associated scaffolds more effectively support participants' engagement with and access to colleagues' mathematical reasoning.

Social Network Analysis (SNA), an analytical tool that can be used for quantifying patterns in interactions (Light & Moody, 2020), and statistical analysis was used to examine and compare the extent to which the networks exhibited small world properties. Accordingly, we modeled C1 and C2 as a set of nodes (participants) and directed edges connecting nodes (a response from one participant to another). We used the SNA metric of network efficiency to examine the small world properties of the network because this metric can provide insight into the extent to which network members are accessing and engaging colleagues' mathematical reasoning (Latora, & Marchiori, 2002). Specifically, network efficiency is quantified by counting the minimum number of edges required to connect one colleague to another. The individual degrees of separation for each pair is used to calculate the network efficiency by summing across all pairs and normalizing results. Network efficiency ranges between 0 and 1, where 0 is a minimally efficient network (a completely disconnected network) and 1 is the most efficient network (a fully connected network - the smallest possible world). We extracted participant interactions (358 for C1; 385 for C2) from the courses, generated cumulative interactional datasets for each week of the courses (e.g., the week two data set from C1 included interactions from week 1 and week 2 of C1), and then imported the data into UCINET to assess the network efficiencies.

SNA measures are highly sensitive to the number of nodes in the network (Wasserman & Faust, 1994). Therefore, in order to compare the two courses and interpret our results, following Opsahl et al. (2017), we modeled 50 different hypothetical networks with the same number of nodes (participants) as the courses under investigation but with edges (interactions) randomly distributed between pairs of nodes. The mean efficiency of these hypothetical networks, referred to as the *average random graph network efficiency* (RGNE), allowed us to compare the observed network efficiencies from each week of C1 and C2 to RGNE for each week of each course, where engagement with and access to mathematical reasoning was randomly distributed throughout the network. This included verifying that the network efficiencies of each set of 50 hypothetical networks were normally distributed and then calculating significance levels by comparing the observed network efficiency from each week of each course to the corresponding RGNE.

Results

Table 1 presents results for the observed network efficiency (O), the RGNE, and the corresponding p-values when comparing the observed network efficiencies to the RGNE. In both courses, the network efficiencies increased throughout the course, which is expected because participants had increased opportunities to access and engage with colleagues' reasoning as the course progressed. The network efficiency of C1 remained larger than C2 throughout the course. Further, the network efficiency of C1 was significantly lower than RGNE for weeks 3-8 *($p <$ 0.05)*, while the network efficiency of C2 was not significantly different from the RGNE for weeks 1-7. However, the network efficiency for C2 was significantly lower than RGNE after week 8.

Table 1: Week by week observed network efficiency, RDNE, and p-values

	Wk1	Wk2	Wk3	Wk4	Wk5	Wk6	Wk7	Wk8
O-C1	.140	.321	.431	.465	.528	.570	.613	.630
O-C2	.055	.138	.301	.366	.395	.444	.537	.548
RGNE-C1	.112	.336	.489	.534	.577	.623	.646	.664
RGNE-C2	.054	.113	.276	.377	.416	.472	.549	.564
***P* - –1**	0.129	0.374	0.015	0.001	0.003	1.1E-06	3.2E-06	3.2E-09
***P* - –2**	0.396	0.221	0.289	0.351	0.235	0.097	0.191	0.026

Discussion

The results of the analysis indicate that for the majority of the course (week 3-8), C1 had a significantly lower network efficiency than would be predicted by the RGNE, while the network efficiency of C2 was not significantly different than the RGNE. Watts and Strogatz (1998) note that a small average path length (i.e. higher efficiency) is one characteristic of randomly generated graphs and, as a result, the C2 network has small world characteristics. This result provides evidence that C2 (scaffolded by technologically supported EB-N&W) more effectively supported participants' engagement with and access to colleagues' mathematical reasoning throughout the "meat" of the course, increasing the likelihood for social and SM norms to emerge. Implications of this study include (1) the design of online teacher professional development environments with scaffolds that support teachers in connecting their N&Ws to evidence in their colleagues' reasoning when providing feedback, and (2) a methodology that can increase the scale of rigorous SNA studies of collaborative professional development, from examining single implementations of professional development to comparing multiple iterations of the same professional development as well as across professional development programs (Borko, 2004). Our plans for future research include expanding the current results to examine the specific social and SM norms that emerged in C1 and C2 as well as the specific role of EB-NW in scaffolding the emergence of norms.

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HIGH SCHOOL TEACHERS' THINKING ABOUT THE LIMIT CONCEPT

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Facilitating meaningful discourse is a component of supporting productive struggle. To facilitate meaningful discourse in their classrooms, teachers need to be aware of and reflect on their own mathematical discourses and communication. This study examines one pre-service and seven in-service high school teachers' thinking about the limit concept in a calculus content course they took as part of their professional development. The course focused on eliciting teachers' discourses on limits and make them explicit topics of discussion and reflection to support teacher thinking and communication. The results indicate that the approach has the potential to support teacher thinking and increase awareness of their mathematical discourses and communication. The study concludes that it is important for teachers to engage in productive struggle to use it as a practice in their own classrooms.

Keywords: Calculus, classroom discourse, communication, professional development

Introduction

Supporting productive struggle is a component of mathematics teaching and facilitating meaningful discourse is a critical aspect of supporting productive struggle (Boston et al., 2017). It is unlikely for teachers to facilitate meaningful discourse in the classroom unless they are aware of and reflect on their own mathematical discourses. This work focuses on high school teachers' discourses on the limit concept with a focus on their thinking in a calculus content course designed to elicit their discourses on limits and then make them explicit topics of discussion and reflection to support thinking and classroom communication.

Limit is a foundational concept of calculus that presents major challenges for students and teachers (e.g., Masteroides & Zachariades, 2004; Williams, 1991). Students often think about limit as a dynamic process rather than a mathematical entity (a number) obtained at the end of that process, leading to challenges in thinking about the formal aspects of limits. This issue is referred to as a process-product or a process-object duality inherent in limits through its dynamic and static realizations (e.g., Gray & Tall, 1994; Güçler, 2014). Some other student difficulties about limit include thinking about limit as a *bound* (Cornu, 1991; Williams, 1991), as *unreachable* (Williams, 1991) and assuming that *limit implies continuity* (Bezuidenhout, 2001). Students' realizations of limits can differ from their mathematical definitions of limit, leading to confusions about the concept and its representations (Güçler, 2014; Tall & Vinner, 1981).

Although teachers can flexibly move between different realizations of limits, the associated changes in their discourses can remain tacit for the students in the classroom (Güçler, 2013). Teachers can enhance classroom communication and facilitate meaningful discourse by explicating their discourses and talk about the different assumptions shaping different realizations of the limit concept (Güçler 2013; 2014). To create such a classroom environment, teachers need to be aware of different contextual realizations of limits and their own discourses about the concept. The study uses a discursive lens to answer the following question: How do eight pre- and in-service high school teachers think about limit in a calculus content course that promoted the elicitation, discussion, reflection, and explication of their discourses to support thinking and mathematical communication?

Theoretical framework

This study uses Sfard's (2008) communicational approach to cognition, which highlights the sociocultural origins of human development and views thinking as communicating with one's self. From this perspective, mathematics is characterized as a discourse – a form of communication that can be distinguished by its word use, visual mediators, routines, and endorsed narratives (Sfard, 2008). Word use refers to the ways in which participants use mathematical words in their discourses. Visual mediators refer to all the visuals generated and used for the purposes of mathematical communication. Routines are the meta-level rules that characterize the patterns in participants' discourses. Endorsed narratives are the utterances describing mathematical objects and their relationships the participants consider as true. In this study, the focus is on high school teachers' word use and endorsed narratives about limit in the form of definitions and the potentially implicit aspects of their discourses on limits. A fundamental assumption of the study is that teachers' awareness of and reflections on their own discourses on mathematics does not only support their thinking and communicating, but also generating meaningful discourse in their own classrooms.

Methodology

The focus of the study was to elicit teachers' discourses on limits and make their discourses explicit topics of discussion and reflection in the classroom. The activities used in the classroom were developed based on research on student and teacher difficulties about limits and the research highlighting the tacit aspects of the discourse on limits to expose teachers to different realizations of limits. The goal was to use the classroom activities to bring forth and reflect on teachers' discourses to support their thinking on limits, which could then help them support the thinking of their students.

The participants were one pre-service and seven in-service high school teachers taking a content course in calculus as part of their initial licensure program or professional development. Except for the pre-service teacher, who had no prior teaching experience besides calculus tutoring, the participants' teaching experiences ranged between 4 –12 years. The researcher was the instructor of the course. The data for this work consisted of an initial survey given to teachers at the beginning of the course and 3 video-taped classroom sessions (each lasting 2.5 hours) on limits. The classroom sessions were transcribed. The transcripts were examined with a focus on teachers' word use (particularly with respect to the process-object duality of the limit concept), endorsed narratives about limit (in the form of definitions), and the assumptions shaping their word use and narratives.

Results

In the initial survey, which was administered during the first week of the course, the teachers were asked to define what a limit is in their own words. Table 1 shows the teachers' responses to the initial survey. All the names used in the study and pseudonyms; Steve was the only pre-service high school teacher in the course.

Table 1: Teachers' definitions of limit in the initial survey

Carrie	[1] A value that is approached but not necessarily reached.
Fred	[2] As an input gets very large or very small, the function approaches a particular

	value.
Lea	[3] Restriction
Martin	[4] The point to which a function no longer adheres to the same rule.
Milo	[5] An interval of data options
Ron	[6] Looking at a function at a specific point from both sides of the function
Sally	[7] As the x values approach certain values, the function approaches the limit.
Steve	[8] As an equation or function approaches a specific point (or increases/decreases) the limit is a point in which the function approaches.

Table 1 indicates that not all teachers provided a mathematically accurate definition for the limit concept ([3], [4], [5]). Some teachers referred to limit as an object (a value [1], a point [4,8], an interval [5]), whereas others talked about the dynamic process of obtaining a limit ([2], [6], [7], [8]). Carrie's response indicated that she viewed limit as *unreachable* [1] and Lea seemed to refer to limit as a *bound* [3] demonstrating that some of the teachers showed difficulties about limit mentioned in the literature. It was unclear how Martin and Milo thought about limits. Martin referred to limit as a point where a function does not adhere to the same rule [4] whereas Milo seemed to think of confidence intervals in his response [5].

The elicited discourse in the classroom provided more information about how teachers thought about limit and clarified some of their responses in the initial survey. The classroom discussions about limits started with an activity where teachers were asked to define limit in their own words (a) using one word, and (b) using as many words as they wanted. They were then asked to elaborate on their responses and reflect on their definitions in terms of their similarities and differences. The one-word definitions of limit ("A limit is a….") generated by the teachers were: boundary, unreachable, constraint, value, approach, convergence, exists/does not exist. When teachers were asked to elaborate, Carrie said "I think of limit as the value you approach. We sometimes attain it; it is a value" indicating that she viewed limit as a value that can be attained and did not realize limit as unreachable, which was different than her response in the initial survey, where she argued that the value would not necessarily be reached [1]. Fred said "you can't go beyond it; it is like a bound or constraint" and "you don't ever get to the limit", indicating that he viewed limit as a *bound* and *unreachable* – views that were not apparent in his response in the survey [2]. Ron, when talking about the word approach, said "you approach the graph, the function and say where is your function? So, you approach from the left and right, where does the function approach? The function may or may not be defined at that point." When asked what a limit is, he replied "it is the approaching". Ron's discourse indicated that he realized that discontinuous functions could have limits and limit is the dynamic process of approaching. He also seemed to think about limit visually since he consistently used the word graph when talking about limits throughout the class. Steve, building on Ron's discourse, elaborated on his word convergence by saying "I see it as the same thing as approaching; it is just I am more used to hearing convergence. As you are approaching infinity, you are trying to see what value this function converges to". Here, Steve referred to limit as a value through a dynamic view of limit involving approaching and convergence. Sally elaborated on her words exists/does not exist by saying that "in algebraic problems, and depending on which way you are approaching, sometimes the limit exists and sometimes it doesn't".

The teachers then generated three definitions of limit using any number of words as they wanted:

Carrie: A limit of a function as x approaches a is the value that the function approaches as x gets very close to a. $f(a)$ may or may not be equal to the limit value.

Fred: The limit is the function output as the input value of x approaches a value resulting in a convergence of a specific value.

Lea: Limit is a specific value a function approaches but will never reach.

Carrie's definition indicated that she did not hold the view *limit is the function's value* and her realization was based on a process view of limit resulting in a value as the limit. Fred adopted the word "convergence" used by Steve in his definition and his definition was also based on a process view through dynamic motion while referring to limit as an object (function output). Yet, his views of limit as a *bound, unreachable,* and *constraint* were not evident in his definition. Lea's definition referred to limit as a value based on a dynamic process view while also retaining the realization that limit is *unreachable.*

Although these discussions did not completely reveal all aspects of teachers' discourses when thinking about limits, they revealed that many teachers showed the student difficulties about limit highlighted in the literature. The teachers' discourses indicated that—even when they referred to limit as a value—their realizations of the concept were mainly based on the process view of limit through dynamic motion, where the function approaches a value as the x approaches a specific point. The discussions showed that the teachers were not aware of the process-object duality inherent in the concept. There was also no indication in the teachers' discourses that they were aware of the static view about limits consistent with the formal definition of the concept. This activity showed that teachers' realizations of limit could be varied and fragmented, providing a mathematically correct process view definition, but at the same time having the many mathematically incorrect realizations of limit such as viewing it as a *bound* or *unreachable*. Teacher responses during the activity also suggested that the assumptions teachers hold in their realizations and definitions of limit may remain tacit, especially if those assumptions are not explicitly spelled out in their definitions. The discussion ended with the explication of and reflection on the assumptions and metaphors inherent in the teachers' discourses about the definitions as well as those not mentioned by the teachers (e.g., limit as a process, limit as an object, metaphor of motion through the dynamic view, metaphor of discreteness through the formal view).

Space constraints do not allow elaborating on all the classroom activities in detail or demonstrating the development of the teachers' thinking throughout the course, but those results indicate that activities tailored to elicit teachers' discourses on limits, encourage reflection on their discourses and the tacit assumptions shaping their discourses have the potential to support teacher thinking, learning, and enhance communication in the classroom.

Discussion and Implications

The results of the study suggest that making teachers' discourses explicit topics of discussion and reflection in the classroom supported their thinking and communication about the limit concept. The teachers learned about and increased their awareness of the tacit aspects of their discourses and such awareness is necessary for them to communicate mathematical ideas effectively and facilitate meaningful discourse (a component of supporting productive struggle).

The pedagogical approach used in the study helped teachers realize their own struggles with the limit concept, which can help them better anticipate student difficulties—another component of supporting productive struggle in the classrooms (Boston et al., 2017). The study demonstrates the importance of engaging teachers in productive struggle in the context of teacher education to prepare them to support productive struggle in their own classrooms. The discursive inquiry, explication, and reflection demonstrated in this study can be useful for teachers to model, adopt, or adapt similar approaches in their own teaching practice to facilitate meaningful discourse and support productive struggle.

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PRODUCTIVE DISRUPTION IN A AN ONLINE PROFESSIONAL DEVELOPMENT ENVIRONMENT

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The EnCoMPASS project (Emerging Communities for Mathematical Practices and Assessment) at Drexel University has produced a web-based software tool for the assessment of student work. This paper discusses research on the impact of this tool on teachers' attitudes toward engaging with students in the software environment. The tool supports teachers adopting a more dialogic perspective towards learning and teaching through cycles of problem solving, discussion and mathematical development. It is suggested that the tool aids teachers' transition toward this more interactive approach to teaching mathematics while also acknowledging and addressing concerns about the time it takes to engage in more detailed dialogue and thinking about mathematics with their students.

Keywords: Problem Solving, Professional Development, Technology, Teacher Knowledge

Introduction

EnCoMPASS (http://mathforum.org/encompass) is an NSF-funded project that is focused on creating an online community of teachers that supports and encourages the use of students' mathematical work – particularly detailed and structured analysis of students' work – to inform instruction. As a central part of the project, the EnCoMPASS tool, was developed to support teachers to shift from the assessment of the "products" of student work toward a process-oriented approach involving interaction with students around mathematics. The EnCoMPASS Tool is a web-based software environment focused on the assessment, analysis and support of student mathematical problem solving. We argue that the EnCoMPASS tool disrupts the normative tendency for teachers to focus on what students know and correct answers. Instead, it provides a scaffold for teachers to look carefully at student work, selecting specific evidence from that work, and using that evidence to begin the process of dialogue with students about mathematical thinking and ideas. Built into the EnCoMPASS tool is the Noticing & Wondering (N&W) framework that was developed and promoted by the Math Forum to encourage teachers to focus on the evidence from the work that student have produced. N&W also scaffolds teachers as they begin a process of dialogue with students about how students' ideas and understandings are developed and supported by those noticings and wonderings.

The EnCoMPASS tool is designed to collect student work and allows a teacher to identify and highlight potentially significant excerpts from the student work and comment on selected text or "selections". These selections can then be sorted and categorized into a number of folders that allow for quick categorization of the work of multiple students and for teachers to easily look at the aggregated work in the folders. In addition, teachers can craft feedback to a student, or the students in a particular folder, to push their mathematical thinking further. Through a process of iterative design testing of the tool with teacher collaborators, the N&W framework was integrated into the tool to scaffold teachers in their efforts to comment upon selections from the student work and organize the work into folders with the selections connected to their N&W comments.

In this segment of our research, we have attempted to understand how the EnCoMPASS tool helped to disrupt traditional norms mathematics teaching, as recognized by the participants, and further, helped teachers see the value of taking students ideas seriously and beginning a conversation with those students about the mathematics. The research questions guiding this work are: 1) How does the EnCoMPASS tool support teachers to engage in a processual approach to mathematics?, 2) What does the shift toward process look like in this context?, 3) What forms of teacher reflection are produced through the EnCoMPASS tool and the Noticing & Wondering scaffold?, 4) How does the tool support a continued dialogue with students?

Theoretical Framework

Researchers at the EnCoMPASS project see mathematics as part of people's everyday lives. Many of the math problems used come from everyday life and they remind students and adults that doing math is part of what we all do. Everyday problems remind us that math is about the practice of problem solving. And problem solving involves understanding situations, assumptions and includes conversations with others. Thus, mathematics is a communicative and social process. Finally, the result of working out problems and talking with others about that work results in mathematical thinking. The way for all people, students and teachers alike, to get better at math is to improve their mathematical thinking.

Central to this practice is the relationship between conversation, interaction and thought. Several philosophers and social scientists have pointed out that thinking, learning and knowledge production are social phenomena (Peirce, 1931; Bakhtin, 1981). Sfard (2008), bringing together the work of scholars such as Dewey (1938), Vygotsky (1978, 1986) and Wittgenstein (1953) claims that communication and cognition are flip sides of the same coin and that our traditional ideas about knowledge acquisition are incorrect. Knowledge and what we call learning, are communicative acts and necessarily social. She coined the term commogition to underscore that communication and cognition are social and intersubjective. To Sfard (2001), mathematical development involves being assimilated to a new discourse akin to the ways that Lakoff & Johnson (2003) talk about how metaphor is used to expand understanding in general.

These ideas are compatible with the notion of sociomathematical norms and mathematical identity (Cobb, Gresalfi,& Hodge, 2009; Boaler & Greeno, 2000). Teachers must be immersed in a discourse of mathematics, but they also need to be able to help students move to use these new signifiers before they are fully able to understand the mathematical objects they represent (Horn & Kane, 2015). Students and teachers must then find themselves in a discourse community where problem solving and mathematical practice is part of the norms of that social group (Gresalfi & Cobb, 2011). Being a member of that discourse community leads in a dialectical way to more conversation and more thought and deeper forms of knowledge and understanding (Bannister, 2015). Lave & Wenger (1991), for example, note that it is impossible to distinguish the learning from the context within which the learning takes place.

We can think of the traditional norms of math education, where students and teachers focus on using the right procedures and getting the right answer as a scaffold to this more complex process of building mathematical knowledge. The problem with that scaffold is that it reifies the process and the product and only aids the development of and only values mathematical thinking in a limited way. The EnCoMPASS tool, and the N&W model built into it, was designed to provide a more productive scaffold. For the purposes of this paper N&W allows teachers to notice things about student work, wonder what the student was thinking and where that thinking could go. This scaffold them allows the teacher to begin a dialogue with the student about their ideas and move both the teacher's thinking about the students' understanding and the students' mathematical thinking forward.

The EnCoMPASS tool not only allows N&W to move a conversation and a process forward, but it also productively disrupts the normative practices and assumptions in mathematics education (Figure 2 above). The online tool allows the conversation about mathematics to slow down and not move so quickly toward correct answers.

Disrupting traditional norms of the mathematics classroom (Yackel & Cobb, 1996) then makes room for a more process-oriented approach. The notion of productive disruption has been used in other contexts by other education researchers (Hall, Stevens & Torralba, 2002; Ma, 2016). What is consistent in those contexts and ours is the idea that we are disrupting existing norms and assumptions in order to have a productive effect on the conversation and thinking of a group of people. The scaffold the tool provides support for teachers and as a result they attempt to engage in a more organic conversation about the math centered on the student's thinking. In this way it mitigates against teachers' anxiety about not having enough time.

Methods and Data Sources

Because the goal of this analysis is to look closely at the ways in which teachers interact with student work in the EnCoMPASS environment discourse analysis was used (Gee, 2014). Data sources for the discourse analysis came from several teachers who were enrolled in a graduate education program at Drexel University and were using the EnCoMPASS tool as part of the work they did in a course focused on student problem solving and student thinking. There were a total of 18 middle or high school teachers enrolled in the course. Several different kinds of text-based data sources were subject to an interpretive and iterative analysis. In the course, teachers first highlighted selections from student work with the EnCoMPASS tool. They then commented upon these selections by making a noticing about the highlighted selection and/or a wondering about the selection. For our analysis we paid attention to what was highlighted, the kind of commentary the teacher made (noticing or wondering) and then the content of the comment.

Additional data included teachers' reflections upon their experience using the tool, using the noticing and wondering framework and the process of taking students ideas seriously. These comments were also analyzed using an interpretive and iterative analysis. At the point this data was collected, students had used the EnCoMPASS tool, as a sometimes option tool, in assignments/class for over 6 months.

Data Analysis

In our data, we see teachers who are working with the EnCoMPASS tool attending closely to the students' mathematical work. They are more likely to respond to specific aspects of student solutions. In follow up interactions with teachers, we observed three important characteristics. First, teachers noted the importance of slowing down their interactions with students and how the N&W approach allowed her/him to see things in the student work. Second, we saw evidence of teachers asking the student what they were thinking, wondering about connections the student was making. The teacher was moving toward a more dialogic approach to working with this student on mathematics, placing emphasis on continued and generative mathematical conversation and not just evaluation of correctness. Teachers also noted that much of this work was not part of their "initial instinct[s]" about this student's thinking. In addition, we saw that the EnCoMPASS tool and the N&W framework encouraged shifts in their practice. Teachers would compare their solutions to the problem with those of their students. The teachers also discussed how the N&W framework really forced them to think about what the students were doing and to ask questions about what the students were thinking. One teacher noticed that the value of the software tool was not only to help the process of dialogue with the student, but also the tool helped organize the teacher's thinking and perhaps aided efficiency.

Discussion

As we can see, from the brief summary of examples in the data, the EnCoMPASS tool allows teachers to use student work and student thinking as the starting point for pedagogically purposeful conversations with students. Evidence indicates teachers believing this work has enabled shifts in their instructional practice and the value they found in these student-centered instructional practices. Of course, not all students respond to these prompts. But the nature of the prompt is to disrupt a more normative response and reorient the teacher toward dialogue about mathematics and thinking rather than the assessment of correct strategies and correct answers. We can see further from some of the responses, that in order to reflect on what the student is doing it is natural to reflect on what one did to work with the problem – a process referred to as double reflection (Shumar, 2017). Double reflection can be a critical attitude and practice in the building of mathematical knowledge for both the teacher and the student. It enables teachers' changing orientations toward problem solving, student engagement and making the focus process not production, allowing the teacher to more naturally move toward the dialectical process of practice-talking-thinking.

Significance

The analysis here demonstrates that the EnCoMPASS tool and the N&W scaffold helps to move teachers toward an interactive stance with students around the doing and talking about mathematics. The tool has helped move them toward paying close attention to their own mathematical work and has transformed how the look at student work and student assignments. There are suggestions here for future research. We suggest that this tool and way of working will help teachers deal with unique situations and more unusual responses on the part of students. This should help teacher take advantage of opportunities to help students make insights and advance their thinking. The tool moves everyone from reified notions about being good at math toward genuine dialogue. Our contention is that this should make both students and teachers better mathematical thinkers. Looking at how to assess teachers and students as mathematical thinkers is a next step in the research.

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HOW ADJUNCT INSTRUCTORS' PERCEPTIONS ABOUT IMPLEMENTING RESEARCH-BASED MATHEMATICS CURRICULUM MAY INFLUENCE THEIR PROFESSIONAL DEVELOPMENT NEEDS

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When planning professional development (PD) for adjunct instructors the PD developers should be mindful of the diverse experiences and needs of adjunct instructors. Developing PD that is one-size-fits-all may not work for all adjunct instructors. In this study the diversity of 3 adjunct instructors' experiences is highlighted prior to and in the initial stages of implementing a new research-based Precalculus curriculum. This work draws from a larger study (Rahman, 2018) about adjunct instructors' experiences navigating a new mathematics curriculum.

Keywords: Undergraduate Education, Curriculum, Professional Development

Teachers play an important role in implementing research-based curricula that aim to prepare and retain STEM students (Ball & Cohen, 1999; Cohen & Ball, 1999; Ellis, 2014; Remillard, 2000). They often find implementing such curricula challenging (Thompson & Carlson, 2017) and must be supported (Ellis, 2014; Remillard, 2000; Thompson & Carlson, 2017). It must be noted that the needs of teachers varies based on their situation. For example, adjunct instructors, who are part-time, non-tenure track faculty at the college- and university-levels have their own sets of challenges and constraints (Pepin, 2014). In the United States, adjunct instructors are increasingly being employed by institutions of higher education (Mason, 2009; Curtis, 2014; Green, 2007), often receiving less pay than full-time faculty (Gerhart, 2004). Their increased presence in higher education classrooms emphasizes the need to understand their experience, especially as it connects to their students' learning (The Delphi Project, 2012). Specifically, it's important to understand the professional development (PD) needs of adjunct instructors.

Literature Review

PD for part-time faculty is important because regardless of their appointment type, all faculty members are valuable to an institution (Gappa, Austin, & Trice, 2007). With support, faculty can continue to be effective and strengthen the quality of their instruction, research, and outreach (Leslie & Gappa, 2002; Gappa et. al, 2007). Teacher learning takes place within a context and it is important to recognize the environment within which learning takes place (Putnam & Borko, 2000). Teachers benefit from PD when it is connected to their own contexts (Lave & Wenger 1991, Greeno et al. 1996, Borko 2004). Effective PD has four parts: design, teachers, facilitators, and context in which the teachers function (Borko, 2004). Participants must find the PD relevant to their context and connect with other members of their community (Desimone, 2009). Further, the design of effective PD must include teachers as co-designers of the learning experience to ensure that the PD is relevant to their learning needs (Timperley, 2011).

When designing programs to support adjunct faculty, it is important to keep their needs in mind. Adjunct faculty needs to feel like they are a part of the intellectual life of the institution (Gappa et al., 2007; Lyons, 2007). In addition, adjunct faculty require continued PD, recognition for good work (Lyons, 2007), and access to resources needed to fulfill their responsibilities (Gappa et al., 2005). Administrators should learn about adjunct instructors' specific PD needs

(Green, 2007; Gappa et al., 2007) as their working conditions and lack of PD opportunities can influence student learning (The Delphi Project, 2012). Research on mathematics adjunct instructors is scarce, in particular research about adjunct instructors implementing research-based mathematics curricula. To this end the research question for this study was: *How did three adjunct instructors' perceptions about a new research-based mathematics curriculum influence their engagement with professional development?*

Theoretical Framework

To answer the research question, the study used the teacher engagement, challenge, & opportunities for learning framework (Rahman, 2018). The framework allows for analysis of teachers' learning opportunities emerging from challenges they face while engaging with curricular resources. The framework was developed to analyze teachers' engagement with curriculum resources where engagement includes teacher actions such as planning, implementing, collaborating and reflecting. The engagement between teacher and tool can be viewed in light of their actions. This engagement and the challenges the teachers face have the potential for their learning. For this paper the engagement between the adjunct instructors and the curriculum as mediated through the professional development was analyzed.

Methodology

I used case study methodology (Yin, 2009) to answer the research question. This research took place at a midsized university in the northeastern United States within a department of mathematical sciences. The study was part of a larger course coordination effort for Precalculus. As part of this effort a new research-based curriculum was adopted (*Pathways to Calculus*). The instructors teaching the course participated in a 2 day summer PD prior to their first semester of implementation. The summer PD was led by the curriculum developers. The instructors also had access to hour long, weekly online PD sessions for continued support. These online sessions were facilitated by a faculty member or a doctoral student in the Mathematics Education program. During the meetings, the instructors had a chance to ask questions, share their classroom experiences and seek advice. The course coordinator for the course provided the instructors with a pacing guide, syllabus, and common assessments.

Data was collected during Fall 2016 and Spring 2017 semesters and included semi-structured interviews (Merriam, 2002), audio recordings of PLC meetings, and classroom observations. Data was transcribed and analyzed using elemental methods to develop an initial set of codes, and then pattern coding to further categorize the data (Salda a, 2009).

For all three participants, it was their first time teaching Precalculus using the *Pathways to Calculus* curriculum. The three participants are as follows:

Caleb (Pseudonym) taught at several institutions at the time of the study. He was a Ph.D. student in Mathematics Education at the institution where the study took place. He had 18 years of high school and college teaching experience, and had taught Precalculus 10 times prior to this study. He had an undergraduate (Accounting) and a master's (Mathematics Education) degree. Michael (Pseudonym) had only taught Precalculus at the college level with no experience teaching at the K-12 level. He worked as an accountant at the time of the study. He received his undergraduate (Mathematics) and graduate (Mathematics, Education concentration) degrees from the institution where the study took place and felt comfortable in the department. Justin (Pseudonym) had taught Precalculus at the high school level, working as a full-time teacher for over ten years. He had an undergraduate degree in Mathematics and received his master's degree (Mathematics, Pure and Applied concentration) from the institution where the study took place. He taught accelerated Precalculus at his high school at the time of the study.

Findings

Caleb – A Case of Self Professional Development

Caleb perceived the new curriculum as an opportunity to learn and was enthusiastic about improving his teaching practice. His engagement with the new curriculum was marked with proactive decision making and planning to challenge as well as support his students.

Caleb perceived the summer PD and the ongoing online PD as learning resources. He actively engaged in the summer and weekly PD, and also drew upon the curricular resources as a guide. For example, the design of tasks in the online homework portal that provided a piece of the problem first, then guided the students to build on their own responses. Caleb shared that he recognized what the curriculum developers were trying to achieve by having the students struggle through the problems and guiding their learning. Caleb was seeking his own PD and actively learned from all the resources available to him. Even though implementing a curriculum that emphasized student engagement was challenging, he embraced this challenge and shared, "My challenges are part of my own professional development" (Interview 1, Spring 2017)

As the semester progressed, his vision of the curriculum itself broadened. After implementing the curriculum for two semesters, he had a better sense of the curriculum's goals, the big ideas to be discussed in class, as well as the recommended pedagogy.

Michael – Using PD to Solve Problems

Michael's engagement with the PD focused on fixing immediate problems his students faced. He perceived the curriculum as beneficial for the students but he also had concerns about the emphasis on conceptual understanding. At the beginning of the first semester Michael shared that his students were struggling with the investigations. He attributed this difficulty to his students' prior experiences in a mathematics classroom. He reasoned that students were used to a classroom that required them to master procedures instead of developing conceptual understanding. One of his challenges was, "Getting the kids onboard!" (Interview 1, Fall 2016) He reported realizing early on in the first semester that his students were not used to investigating concepts in the classroom, or being attentive to the use of precise language. According to Michael, familiarizing himself with the new curriculum and getting his students on-board were his main challenges when implementing the new curriculum. His plan was to guide the students through their challenges in problem solving. In his experience, having a guided approach worked for students to understand concepts and develop problem solving strategies.

For Michael his collaboration with his colleagues during the weekly online meetings played a big role in supporting him. Michael asked questions, shared concerns about students or pacing of the course. He actively participated in the summer and weekly PD during both the semesters. Over the course of the two semester Michael's teaching practice and his challenges with the students remained the same but he had a better understanding of the curriculum and its goals.

Justin's Engagement – PD as Inspiration for Creating Instructional Resources

Justin's engagement with the curriculum, exhibited a focus on developing his own instructional materials. Justin saw the new curriculum as a learning opportunity for himself. He perceived the instructional materials themselves as possible tools to facilitate his students' learning and the problems in the curriculum resonated with Justin.

Participation in the summer workshop allowed Justin to get an overview of the new curriculum, the investigations, and pedagogical suggestions. Justin was receptive to the new curriculum and appreciated its focus on developing students' conceptual understanding of Mathematics. As he started to implement the curriculum, he experienced challenges such as, encouraging student discourse, student participation and engagement. He shared, "I think there is a little resistance..We're having kids do work together in a group that really aren't used to that so they naturally fight it!" (Interview 2, Fall 2016) Justin blamed his lack of experience teaching at the college level for his students' lack of engagement in the classroom. He explained that he spent a large portion of each class to motivate his students. He felt that his students were demotivated by the problems and he wanted them to be persistent.

Justin rarely relied on the weekly meetings to ask questions or share his concerns. He attended most of the meetings during the first semester but seldom attended them during the second semester. He found support in another adjunct instructor whom he talked to regularly. Justin knew him from outside of the university setting and reached out to him for advice. In terms of his engagement with the PD, after the summer workshop he found the weekly meetings beneficial during the first semester but not so much during the second semester.

Discussion and Conclusion

I described three adjunct instructors' experiences engaging with PD as they implemented a research-based mathematics curriculum. The instructors engaged with the PD guided by their own needs. While Caleb was motivated to learn from the curriculum, not all adjunct instructors might have the time or the motivation to struggle through their challenges like him. For both Michael and Justin their student challenges posed a more immediate need than a possible long-term goal of improving their teaching practice. Instructors' current or future goals might not be aligned with learning new teaching practices, especially if this means challenging their students.

When developing effective PD for adjunct instructors it is crucial to keep in mind the factors impacting their teaching experience. For all three adjunct instructors, being an adjunct instructor was not their only occupation. With other responsibilities and resulting time constraints, the online format of PD was accessible. It was a form of support during the first semester because all three instructors shared the need to learn about the curriculum. During the second semester their needs changed as they had implemented the curriculum once and knew what to expect. For example, Caleb continued to be proactive and reflected on his students' past experiences to develop more effective lesson plans. Michael played a supportive role in the weekly meetings and shared his experience, and Justin's attendance in the meetings dwindled during the second semester. Interview data revealed that Justin's needs were no longer being met by the weekly online PD. He wanted to develop instructional materials and actively collaborate to design learning resources but the PD focused on trouble shooting curriculum implementation challenges and sharing student experiences. PD is beneficial for teachers when it is aligned with their goals and needs. Further, the design of the PD did not include the adjunct instructors as active co-designers of the PD. Surveying the instructors about their PD needs and learning goals can aid the PD designers in aligning the sessions to meet the instructors' needs.

When designing PD for adjunct instructors it is essential to be mindful of the diversity of their learning needs, their conflicting schedules, their prior knowledge about mathematics pedagogy and experience teaching at the K-12 or college level. It is also important to include their voice in the decision-making process when deciding the type and focus of PD activities.

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MAKING SENSE OF MULTILINGUAL LEARNER PRINCIPLES AND MATHEMATICS LANGUAGE ROUTINES THROUGH STUDIO DAY PROFESSIONAL LEARNING

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We studied 11 high school teachers' reflections of their experiences in a professional learning project organized around five principles of effective mathematics instruction for multilingual learners and mathematics language routines (MLRs). We examined how teachers' ideas about these principles developed using the MLRs across four studio day professional learning cycles. Using open-coding, we analyzed pre-studio day and studio day reflection responses to understand how teachers' understanding of the MLRs and principles developed, looking for patterns in individual teachers, individual routines, and across all routines and teachers. We found that teachers discussed sentence frames less often as a method for attending to their multilingual learners; identified that the MLRs were an organizational method their students could use; and, finally, noted they could use multiple modes of communication with MLRs.

Keywords: Professional Development; Equity, Inclusion, and Diversity

The population of multilingual learners continues to increase in US schools, with over 10% of students in K-12 settings identified as multilingual learners (National Center for Education Statistics, 2020). Teachers often find it challenging to engage multilingual learners in rich mathematical work, such as that associated with the *Common Core State Standards for Mathematics* (National Governors Association Center for Best Practices, Council of Chief State School Officers [NGA Center, CCSSO], 2010), as few teachers have had professional learning experiences that are mathematics and multilingual learner-specific (Ballantyne et al., 2008). Additionally, the field is lacking in research around professional learning that attends to mathematics and multilingual learning (de Araujo et al., 2018). To fill both these practice and research gaps, we studied a two-year professional learning opportunity organized around mathematics language routines (MLRs) and five principles of effective mathematics instruction for multilingual learners. We examined how teachers' ideas of these principles developed using the MLRs across four studio day professional learning cycles (Von Esch & Kavanagh, 2018). This study answered the following research question: How did teachers' understanding of using MLRs to enact five principles of effective multilingual learner instruction develop as they participated in studio day cycles of professional development?

Theoretical Framework

This study is organized around two complementary theoretical ideas: key principles of reform-based instruction for multilingual learners and MLRs, both of which are meant to engage multilingual learners with content in meaningful ways. The five principles of reform-based instruction (see also Roberts, 2021) are understood as reinforcing and overlapping with one another. For the first principle, *build on and use multilingual learners' funds of knowledge and resources* (Moll et al., 1992; Moschkovich, 2002), teachers identify, celebrate, and use the knowledge and skills students, their families, and their communities bring to the classroom. With the second principle, *provide multilingual learners with cognitively demanding work* (Stanford Graduate School of Education, 2013), multilingual learners have the opportunity to engage in cognitively rich activities and assignments often reserved for English-only students (Iddings, 2005; Planas & Gorgorió, 2004). In the third principle, *provide multilingual learners opportunities for rich language and literacy exposure and practice* (Khisty & Chval, 2002; Lee et al., 2013), teachers engage multilingual learners in the language of mathematics by creating opportunities for students to receive comprehensible input through listening and reading and to produce comprehensible output through speaking and writing. The fourth principle, *identify disciplinary language demands and supports for multilingual learners* (Aguirre & Bunch, 2012), involves teachers' attending to language demands and implementing appropriate scaffolds so that students can read disciplinary texts, as well as share their ideas and reasoning. The fifth principle is *create a safe classroom* and allows for intellectual risk-taking (Luria et al., 2017), where everyone is part of a community of learners and free to learn (Hernandez et al., 2013).

These five principles provide the foundation for MLRs (Zwiers et al., 2017). These routines support students' productive engagement with cognitively rich mathematics content (Kelemanik et al., 2016). Routines empower students to focus on their learning, because they allow for sense-making of challenging mathematics and for building important mathematical thinking habits—providing more students with access to important mathematics ideas. Examples of MLRs include Stronger and Clearer and Co-Craft Questions. Teachers can use these routines specifically with multilingual learners to amplify, assess, and develop their mathematics thinking and language simultaneously (Kelemanik et al., 2016; Zwiers et al., 2017).

Methods

Our study was situated in a school district in California that included a substantial number of multilingual learners. As introduced above, teachers participated in a two-year professional learning program organized around multilingual learner mathematics studio days (Von Esch & Kavanagh, 2018), developing and studying lessons focused on one multilingual learner principle and one MLR for each of the four studio day cycles.

Context: Studio Days Enactment of Multilingual Learner Principles and MLRs

Teachers participated in three professional development meetings in each studio day cycle (in person during the first year and over Zoom during the second year). Each cycle paired one multilingual learner principle with one targeted MLR (e.g., *funds of knowledge* with Stronger and Clearer). During the pre-studio day, teachers learned about the multilingual principle and MLR and prepped a lesson to implement. Teachers then enacted this lesson at their schools during the studio day, with teachers observing each other's implementation. During the final day of the cycle, the post-studio day, teachers examined student work to assess studio day lessons, shared challenges and successes, and considered implications for their future practice.

Participants

Eleven high school teachers participated in the study. Two teachers were in their first year of teaching, four had 1-4 years of teaching experience, and five had 10-19 years of teaching experience.

Data Collection and Analysis

Teachers completed Google Form reflections about their understanding and implementation of the MLRs and multilingual learner principles as well as their general thoughts on the studio day experiences. For this paper, we focused on two questions from these forms: (1) "[MLR] will support emergent multilingual students engaging in [multilingual learner principle] in my class by…" from the pre-studio day; and (2) "[MLR] is intended to support multilingual learners in their work with [multilingual learner principle]. How well do you think the routine accomplished this today?" from the studio day. From our four studio day cycles, we collected four sets of these two questions. To analyze teachers' responses to the above identified questions, we used open-coding (Yin, 2016) and looked for patterns in the pre-studio day and studio day responses. We compared pre-studio day and studio day coding to make sense of teachers' understanding of the MLRs and the principles, looking for changes in individual teachers, individual routines, and across all routines and teachers.

Findings

Overall, we found that the principles and routines came into clearer view for the teachers after they were able to see the MLRs in each other's classrooms. In comparing teachers' responses to the pre-studio day and studio day, we found that there were three key themes that illuminated this development among teachers. First, teachers mentioned sentence frames less often as a method for attending to their multilingual learners. Second, teachers identified MLRs as a structure or organizational method that their students could productively use. Finally, teachers noted they could use multiple modes of communication with their students when implementing the MLRs.

Decreased Mention of Sentence Frames

Teachers described the use of sentence frames to engage multilingual learners in two of the four cycles of studio days. In both of these cycles, three teachers mentioned this as linked to a principle (*rich language opportunities*, *funds of knowledge*) and MLR (Clarify, Critique, and Correct; Stronger and Clearer) in the pre-studio day forms, but only one did so in the studio day forms. As a project, we would link sentence frames to the principle *disciplinary language support*, however, it is not surprising that teachers would see sentence frames as supporting other principles. Sentence frames are both a common multilingual scaffold that teachers use and we had built this scaffold into most of the MLRs, to model using specific sentence frames with specific content. In her initial pre-studio day mention of using sentence frames with *rich language opportunities* and Clarify, Critique, and Correct, Ms. Frasca noted, "Sentences frames…[are an] easy 'entry' point (just a list of what they have seen). [It] is a way to support these students." Here, providing these sentence frames was described as an entry to the content. Ms. Frasca was also one of the teachers who identified use of sentence frames in her *funds of knowledge* and Stronger and Clearer pre-studio day, sharing succinctly that she would leverage multilingual learners' funds of knowledge "using the sentence frames." It is possible this was a go-to method for attending to multilingual learners for Ms. Frasca.

While several teachers mentioned sentence frames initially in their pre-studio day reflections, in contrast, only one teacher mentioned sentence frames in their studio day reflections. Ms. Parker noted in her studio day on *rich language opportunities* and Clarify, Critique, and Correct, "I love the sentence frames and leading questions. They really help students begin to formulate coherent arguments." In this case, Ms. Parker's attention to sentence frames was closely connected to the routine itself: She was having students construct viable arguments using the frames (MP.3; NGA Center, CCSSO, 2010).

MLRs Provided Structure and Organization

The MLRs provided structure and a form of organization (i.e., *disciplinary language support*), according to the teachers, again in two of the four cycles of studio days (*disciplinary language support* with Three Reads; *rich language opportunities* with Clarify, Critique, and

Correct). For Three Reads, the teachers recognized this structure similarly both in the pre-studio day and in their studio day reflections, while more teachers mentioned this structure in the pre-studio day than in the studio day for the Clarify, Critique, Correct MLR. This might indicate that the role of this structure was more central to the actual implementation of the Three Reads MLR, than to the Clarify, Critique, and Correct MLR. In Mr. Bakula's pre-studio day reflection, he shared that this MLR helped multilingual learners access disciplinary language by "modeling the thought process they should be engaging in every time they approach a problem. It also helps de-construct the problem into more easily-digestible chunks." Similarly, Ms. Scott explained on the pre-studio day that Clarify, Critique, and Correct helped students "organize their thinking and break down the process of looking at another person's work and forming their own thoughts/critiques/compliments and then giving them the opportunity to implement these ideas mathematically." This MLR provided disciplinary language support structures for students to access text, although teachers noted the importance of actually making the routines routine in their classrooms for students to take advantage of these structures.

Potential Use of Multiple Modes of Communication

The teacher participants shared that three of the four routines provided opportunities for multiple modes of communication (i.e., *rich language opportunities*). The MLR missing, Three Reads, could also include modes of communication in theory, however, it was simply not one that teachers highlighted. For the Clarify, Critique, and Correct and the Stronger and Clearer MLRs, teachers mentioned multiple modes of communication more in their studio day reflections than in their pre-studio day reflections, possibly indicating that seeing the MLRs in action provided more context for eliciting *rich language opportunities* or more ideas for how to execute such opportunities. Teachers illuminated that the Clarify, Critique, and Correct MLR allowed for revoicing (i.e., *disciplinary language support*), as well as "moving students forward to read, write and talk more in class," according to Ms. Frasca. Mr. Bakula shared that the Co-Craft Questions MLR allowed students to use "math-specific academic language." Our final routine, Stronger and Clearer, Ms. Lacrosse explained, "Taught students how they could build on their communication skills they bring to class…. They can build on it by listening to other students' points of view and acquire language that might be richer, stronger, and clearer."

Discussion and Conclusions

Our 11 teacher participants came to understand the MLRs as structures that helped students organize their thinking mathematically and linguistically. They recognized they could use these structures to provide disciplinary language scaffolding and rich language opportunities during their instruction, supports that included and went beyond sentence frames, perhaps a nascent type of multilingual learner scaffold. While they initially noted familiar practices, like sentence frames, after being in each other's classrooms as part of the studio day cycles, teachers' frames of reference appeared to expand to include wider notions of how to engage their multilingual learners. Their understanding of using the MLRs appeared to grow clearer. In summary, the studio day cycles provided teacher participants opportunities to develop a broader repertoire of understanding how to implement the MLRs and multilingual learner principles. Providing meaningful instruction for multilingual learners and compelling professional learning for their teachers remains important; this study provides an example from which we can draw.

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THE IMPACT OF AN IMMERSIVE TEACHER PROFESSIONAL DEVELOPMENT PROGRAM BASED ON EXPLORATORY NUMBER THEORY

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In the literature around high quality professional development (PD) opportunities for teachers of mathematics, immersion, duration, collaboration, and a focus on content have emerged as some of the key factors which lead to impactful experiences. Despite knowing which attributes are associated with effective PD, only a small number of programs are rigorously studied or evaluated. This multi-year qualitative study investigated a professional development program – PROMYS For Teachers (PfT) – which immerses teachers in a 7-week intensive summer program in exploratory number theory. Through three case studies with participating teachers, the authors sought to understand the key dimensions of PfT and the impact that it has on participants. Findings show that while PfT can elicit changes in teachers' relationship to mathematics and teaching practice, a lack of clear program goals leads to varied outcomes.

Keywords: Professional Development, Advanced Mathematical Thinking, High School Education, Mathematical Knowledge for Teaching

Theoretical Framework

In the literature around high quality professional development (PD) opportunities for teachers of mathematics, immersion, duration, collaboration, and a focus on content have emerged as some of the key factors which lead to impactful experiences. Experiences which have these attributes ask teachers to engage in the process of actively doing mathematics with other professionals on a regular basis over an extended period of time (Loucks-Horsley & Matsumoto, 1999) (Darling-Hammond et al., 2017).

Of particular interest in this nexus is the content focus of PD opportunities. The field has, at best, a tenuous understanding of the kind of content that mathematics teachers should focus on in PD in order to improve their practice. This is particularly true for teachers of secondary mathematics and for content beyond the scope of their curriculum (Ball & Hill, 2009).

While some programs choose to focus on content which teachers might be expected to teach at some point in their career, other programs use more advanced content beyond the scope of the high school curriculum in an effort to engage teachers meaningfully in immersive, challenging, and deep mathematical experiences.

These programs – like many other professional development programs for teachers in the US – have not been rigorously studied or evaluated. (Borko, 2004; Hill, 2009) The present study is a multi-year qualitative investigation into the PROMYS for Teachers program at Boston University and the impact that it has on teacher participants. There have been two previous scholarly investigations into PfT since its inception in the 1990s, both resulting in doctoral dissertations. The first (Abel, 2010) asked how participation in PfT impacted teachers' conceptions of student learning. (Abel, 2010, p. 257). The second (Matthews, 2014) asked questions quite similar to those of the current study: what is the nature of the PROMYS experience and are teachers' beliefs and/or teaching practices influenced by the program? Ultimately, both previous investigations found that PROMYS had little impact on teachers' beliefs and practices. However, in both instances, the researchers brought measurement tools to the table which didn't necessarily align with the outcomes one might expect from the program. By taking a more open-ended and qualitative approach to data collection – and by looking for emerging themes from a mixture of recorded classroom observations, semi-structured interviews, and ethnographic field notes – the current study aims to more thoroughly document the potential impact of the PfT program.

Methods

For this study, the authors utilized a mix of ethnography and case-studies. An initial offer to participate in the study was made via email to the entire incoming cohort of first-year teachers. Four teachers responded and three of them ultimately agreed to be participants. Participants were compensated with a stipend of $500, half of which they received at the beginning and half of which they received once their participation was complete.

Ethnographic data was compiled by visiting the program site an average of 1.5 days per week throughout the summer. During these visits, the first author would sit through lectures, participate in activities, and often simply sit quietly and observe. Case study data was collected through one-on-one interviews and recorded classroom observations. The study design entailed interviews and classroom observations before the program to establish a baseline for each teacher, interviews and observation during the program to track participants' experiences, and follow-up interviews and classroom observations to see how, if at all, PfT impacted teachers' relationship to mathematics, ideas about teaching, and classroom practice.

PROMYS For Teachers: History and Program Structure

The PROMYS (Program in Mathematics for Young Scientists) organization was originally founded at Boston University in 1991. PROMYS engages high school students, pre-service teachers, and active teachers of mathematics in an exploratory course in number theory.

While the PROMYS program for teachers (PfT) has undergone shifts over the years, the content and fundamental approach has remained remarkably similar. Throughout 7 weeks in the summer, pre-service and in-service teachers come to the Boston University campus Monday through Friday for 7-8 hours each day to do mathematics. Largely, each day is organized around solving a set of open-ended problems in number theory. As a direct result of the fact that problem-solving sessions make up the vast majority of teachers' time in the program, their nature and structure comprise the vast majority of the PfT experience. The structure and ethos of these problem-solving sessions can be characterized by the following ideas: a) learning new ideas and content is best done through exploration and discovery, b) while there is no explicit demand for people to work in teams, an ethos of collaboration is encouraged, c) there is no inherent value in working on easier or harder problems, d) while hints and help are readily available from program admins, answers are not, and e) assessment is entirely intrinsic.

These structures and practices are, in the frame of education theory, oriented toward a constructivist, inquiry and/or discovery-based approach to the teaching and learning of mathematics. This kind of framing, though, is almost entirely absent from any documentation or official descriptions of the program. Furthermore, despite being a program for teachers, there are no workshops, classes, or structured discussions about teaching and teachers aren't directly instructed in any teaching techniques or curricula relevant to their classes. In the following sections, data from each case study will be used to grapple with precisely how a program with such loose goals might ultimately have an impact on teachers' beliefs and practices.

Results

Changes in Teachers Beliefs and Practices

The following two categories emerged thematically as areas in which all three teachers demonstrated noticeable changes during and after their summer at PfT.

Reigniting and Reforming Relationship to Mathematics. Although each of the three teachers came into PfT with widely diverging backgrounds in mathematics, all of them experienced shifts in their relationship to the discipline throughout and after the program.

For Grace, a teacher who came into the program with an unusual amount of skill and interest in mathematics, the most impactful changes in her relationship to the discipline were realized through the low stakes structure of the daily problem-solving sessions. In an interview midway through the summer, Grace mentioned that she hadn't previously seen mathematics as a thing that could be done without time constraints, assessments, and clear goals in mind. The process of working on problems without any due dates or consequences was new to her and made her realize that she could have a very different relationship to the discipline than she previously imagined.

Jordan, on the other hand, said that this was her first experience doing mathematics where she had to figure out how to solve problems on her own. Of her time as an undergraduate student, she said that she has never "experienced the maximum potential" of her brain and that with this experience, she was able to retain concepts and understand them on a deeper level. Ezra echoed these feelings, saying that while her mathematical work in high school and college could be mostly characterized by trying to replicate what she saw being done in class, online, or in a textbook, this was her first experience "not having anything but myself, my own mind." Both Ezra and Jordan also found the summer to be an empowering experience in terms of the way they viewed their own competency within the discipline. For Ezra, immersing herself in a challenging experience filled with a lot of unknowns was intense, especially because by the end of the summer, a lot of the content still felt muddy and uncertain. However, she found value in this struggle and the ownership that she was able to take over her ideas.

Particularly important about these observations is that while each teacher started in a very different place in terms of mathematical comfort and knowledge, they all had experiences which positively changed their relationship to the subject.

Attention to Inductive Learning Through Pattern Hunting. All three teachers who spent their summer at PfT came back to their classrooms and introduced problems and activities which had a common thread: asking students to learn content through induction and pattern hunting rather than through direct instruction or deductive reasoning. Prior to their summer at PfT, none of the teachers expressed in interviews that this was an important or useful pedagogical tool and there was no evidence of this kind of practice during observations. After the summer, none of the teachers explicitly mentioned this as a new instructional tool, but all three of them either referenced this kind of approach in an interview or demonstrated it during an observation.

In two of the three of Grace's classes which were observed after her summer at PfT, pattern noticing played a central role in how students were expected to understand the key objectives or theorems from the lesson. In the first lesson, Grace was teaching a lesson about the first derivative test for local extrema. Rather than tell students what to look for to confirm the results of the test, Grace would elicit student input about the sign of the derivative on either side of the point. During the first example, she said "keep an eye on this, how the sign is changing, it might be useful." As the lesson went on, Grace explicitly used the language of pattern hunting, telling students to "look for a pattern in how the sign is changing – how does that relate to the shape of the function?"

While the day that Jordan's classroom was visited didn't include an activity which involved an inductive or pattern-hunting process, when asked whether she took anything from PfT and applied it to her teaching, she described a lesson which fits into the pattern seen in Grace and Ezra's classrooms. In a lesson about graphing quadratics, Ezra described giving her students dozens of examples of different equations of the form $ax^2 + bx + c$. Rather than show students specific examples and explain to them how different values of a would impact the graph of the function, Ezra asked students to look at a number of graphs and try to find a pattern between the changing value of a and the way that the graphs looked.

This idea that one should learn mathematics by noticing patterns and formalizing them is heavily represented in the structure of the number theory problem sets that teachers worked on over the summer. It is therefore perhaps not surprising that teachers took this away from the program. While this attention to pattern hunting and logical induction is a noteworthy finding, it also belies another important part of the PfT experience and the ethos of the program: like learning mathematics, the program seems to suggest that learning to teach is a process that can be done through pattern hunting. The limitations of this are discussed in the next section.

The Limitations of Inductive Learning for Teacher PD

While the findings above represent real and important changes in teachers' beliefs and practices, interviews and observations also demonstrated a number of areas in which things either stayed the same or, in some cases, actually changed in a way that seemed in opposition to the ethos of PfT. For instance, observations after the summer revealed that other than in Ezra's Geometry classroom, traditional teacher-led classroom structure prevailed. More than the other two teachers, Grace took a more varied set of instructional practices from PfT. In addition to the themes mentioned above, Grace also allowed students to choose which problems they wanted to work on and only graded the ones they did – a practice taken directly from PfT. However, despite the strong ethic in PfT that ability and knowledge should not be hierarchically applied to how and what a student learns, Grace ultimately felt like most of the practices which she learned at PfT were only truly appropriate for more advanced students. Although Ezra completely redesigned her Geometry class to align with more constructivist and induction-based approaches, she also mentioned in an interview that participation in PfT had given her the confidence to give students who didn't complete assignments on time zeros in her gradebook without worrying about their reaction. This practice seems directly opposed to PfT's ideas about grading.

A common theme running through each of these stories that without being given specific ideas about how the pedagogy of PfT might relate to their classrooms – or even being given structured space or time in the program to contemplate this relationship for themselves – each of the teachers was left to their own devices to take what they could from the program.

Conclusions and Discussion

Through these case studies and our time observing teachers' experiences with PfT, it is clear that the program has an impact on the way that teachers relate to the field, their beliefs about themselves, their beliefs about teaching, and ultimately their teaching practice. In fact, some of the changes mentioned above could be described as transformative. However, it seems like PfT as an organization approaches the notion of teacher learning in the same way that it approaches the notion of mathematics learning: discovery, induction, and pattern hunting. In this context, it is difficult for teachers to take actionable steps back to their classrooms.

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SHAPING THE PROFESSIONAL GROWTH OF MATHEMATICS FACULTY WHO TEACH PROSPECTIVE SECONDARY TEACHERS

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Professional development (PD) that supports faculty in teaching courses for prospective secondary teachers, especially courses focused on mathematical knowledge for teaching, are largely absent from higher education, despite the need to improve instruction in these courses. This study examines a novel PD program whose structure was inspired by rehearsals (Lampert et al., 2013). We analyzed PD discussions throughout the year using an instructional triad framework, and we interpreted the PD structure using Clarke and Hollingsworth's (2002) Interconnected Model for Professional Growth. We suggest that a rehearsal-inspired pedagogy offered opportunities for faculty growth in attending to student contributions.

Keywords: Mathematical Knowledge for Teaching, Undergraduate Education, Professional Development

Recent policy documents agree that secondary mathematics teacher preparation must attend to developing teachers' mathematical knowledge for teaching (MKT) (Conference Board of the Mathematical Sciences, 2012; Association of Mathematics Teacher Educators, 2017). Although there is promise in simulating practice to develop teachers' MKT in ways that coordinate mathematical and pedagogical sensibilities (Biza et al., 2007; Stylianides & Stylianides, 2010), such tasks are uncommon in undergraduate content courses for pre-service secondary teachers (Lai & Patterson, 2017). To expand opportunities for developing MKT in content courses, the project Mathematics of Doing, Understanding, Learning, and Educating for Secondary Schools (MODULE(S^2)) has developed curricular materials for Algebra, Geometry, Modeling, and Statistics undergraduate content courses for prospective secondary teachers. To support enacting these materials, MODULE(S^2) provided a year-long professional development (PD) program for mathematics faculty. This PD included activities inspired by teaching rehearsals (Lampert et al., 2013; Ghousseini, 2017), and aimed to support attending to prospective teachers' thinking.

In this study, we drew on the MODULE(S^2) data to examine: *How does a rehearsal-inspired pedagogy shape the interactions among mathematics faculty during a professional learning experience?*

Theoretical Perspective

Throughout this paper, *student* refers to a prospective secondary mathematics teacher, and *instructor* refers to a mathematics faculty member. Following Lampert (2001) and Cohen, Raudenbush, and Ball (2003), we model instruction as attention to relationships formed between instructors, students, and content. In this view, learning occurs as students work to develop, understand, and strengthen their relationship with content and each other; meanwhile, instructors enact their practice through relationships with content and with students, in addition to their relationship with student learning whereby they attend to and are influenced by the student-content relationship.

Following Clarke and Hollingsworth (2002), we view professional growth as “an inevitable and continuing process of learning” (p. 947). They conceptualized teacher professional growth as an “Interconnected Model” of dynamics among four domains: the Personal Domain (i.e., a teacher’s individual knowledge and beliefs), the Domain of Practice (i.e., all forms of professional experimentation), the Domain of Consequence (i.e., inferred outcomes of instructional decisions), and the External Domain (i.e., entities outside the teacher’s self). They argued that professional growth, such as that shaped by PD, can be represented through “change sequences” of reflection and enaction.

Teaching rehearsals can shape teachers’ professional growth (Ghousseini, 2017; Lampert et al., 2013). Initially conceived to support novice K-12 teachers, rehearsals provide opportunities to “simulate and analyze manageable chunks of interactive teaching before enacting them with students in classrooms” under the guidance of knowledgeable practitioners (Ghousseini, 2017, p. 188), followed by a collaborative debrief discussion. We hypothesized that rehearsal-inspired experiences could provide opportunities for professional growth for mathematics faculty, particularly in developing capacity for attending to student thinking.

Data and Method

Design of Rehearsal-Inspired Experiences

The PD for the instructors teaching with the MODULE(S^2) materials spanned the summer prior to teaching and the academic year. A cornerstone of the PD was a series of *rehearsal lessons*: in the summer, each piloting instructor planned a lesson from the materials and then taught the other instructors, who took on the role of acting students. To assist the instructors in planning their rehearsal lessons, the facilitators provided a planning guide with prompts intended to support attending to student thinking. Facilitators video-recorded the lesson and immediately played the recording to all participants. During the viewing of the recording, facilitators provided instructors an observation guide framed toward noticing student thinking in the rehearsal. The video viewing was followed by a facilitator-led debrief.

During the academic year PD, the instructors continued to meet over Zoom with project facilitators. Prior to each meeting, one instructor shared a video recording of a MODULE(S^2) lesson that they implemented in their own classroom with the rest of the group. To begin a meeting, a facilitator prompted the group by asking open-ended questions (e.g., “How are things going?”) so that instructors shaped the focus of meetings. Conversations addressed debriefs of the video recording of the participant’s lesson, the materials themselves, and instructional experiences that the participants wished to reflect on. In contrast to the debriefs of recorded lessons during the summer portion of the PD experience, the debriefs of recorded lessons during the academic year did not follow particular structures or prompts.

Participants and Data Sources

We focus on one group of three instructors who implemented the Algebra strand of MODULE(S^2) materials. During the summer, they participated in three teaching rehearsals --one for each of the instructors to act as the instructor. During the academic year, they participated in five facilitated meetings, which included debriefs of recorded lessons. The three debriefs of the summer teaching rehearsals were video recorded and were each approximately ten minutes in length. The five meetings during the academic year were also video recorded and ranged from

30-90 minutes each. The transcriptions of these eight recorded debriefs and meetings serve as our data source for this study.

Analysis

The authors used an instructional triangle as an analytic tool. An example of this coding is shown in Figure 1. We have highlighted how these interactions would have been coded in the transcript for these portions. "Teacher" refers to instructors of content courses and "students" refers to prospective teachers.

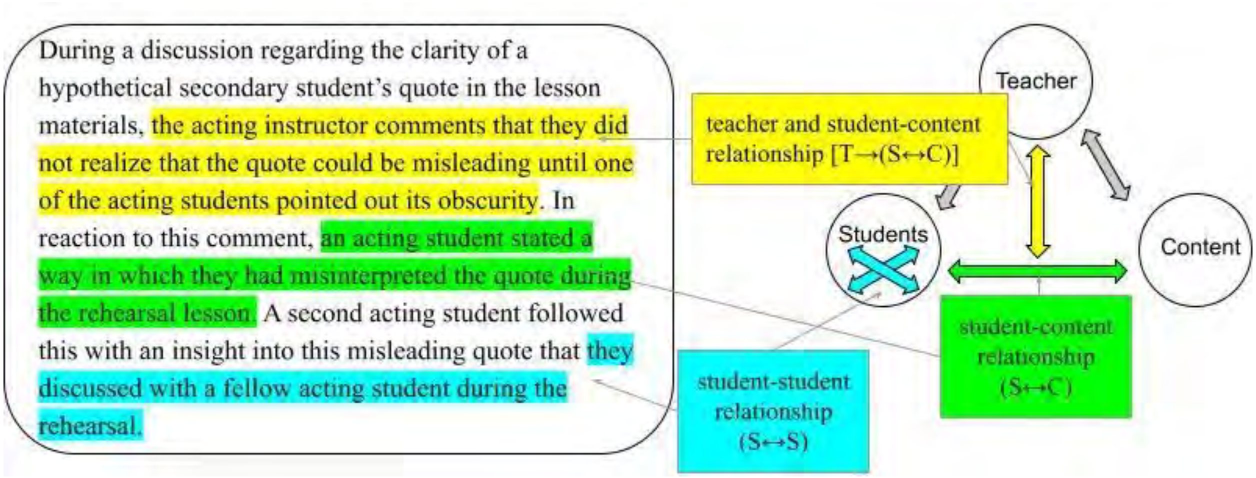


Figure 1: Vignette with categories of interactions highlighted

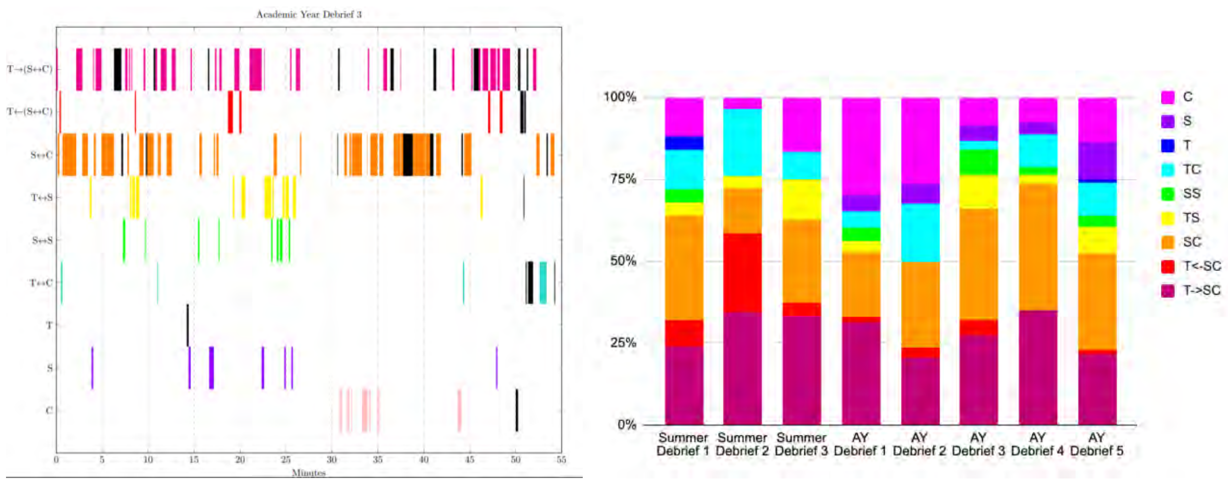
To understand the role of the PD structure in shaping the interactions between the participating faculty members, we utilized the Interconnected Model of Professional Growth. Specifically, we identified how the materials and activities utilized in the PD operated within the model's four change domains and provided opportunities for reflection and enactment. With these results, we produced a change sequence representative of the opportunities for growth offered by this PD.

Results

Mathematics Faculty Attention to Students and Content

We created visualizations for all eight debriefs and meetings, one of which is displayed in figure 4(a). The horizontal axis represents the time at which a statement referring to instruction is made, and the vertical axis represents to which particular instructional relationship a statement is referring. Statements about instruction made by instructor participants are shown as a colored block whose horizontal length indicates for how long that particular instructional relationship was being referenced. Statements about instruction made by project facilitators are colored black and indicate length similarly.

As is the case for all eight debriefs, this example displays a high density of statements referring to the student-content relationship (S↔C) and the relationship between the teacher and the student-content relationship [T→(S↔C)]. (Again, "teacher" refers to a mathematics faculty member, and "student" refers to a prospective teacher.) To further investigate this density of codes, the frequency counts for each instructional relationship code are plotted in a stacked bar graph in Figure 4(b). This stacked bar graph reveals that, for all of the eight debriefs and meetings, half or more of the instructional relationships being referred by instructors and facilitators were to the student-content relationship (S↔C) and the relationship between the teacher and the student-content relationship [T→(S↔C) and T←(S↔C)]. The frequency of references to instructional relationships related to the student-content relationship suggests that instructors maintained a focus on student thinking throughout the entire PD experience.



(a) **(b)**

Figure 2: (a) Visualization of the third debrief during the summer portion of the PD and (b) Stacked bar graph of the frequencies of each code during the eight debriefs

The PD Experience as a Change Sequence

In the space allowed in this brief report, we present only a summary of the results of our analysis of the PD structure using the Interconnected Model of Professional Growth. Our data suggested two change sequences: one for the summer PD structure and one for the subsequent academic year PD structure. Each change sequence used all four change domains and was based on numerous opportunities for enaction and reflection.

We offer two illustrative examples. First, during the summer portion of the PD experience, instructors were given rehearsal planning guides that focused on aspects of student thinking to scaffold the planning of their rehearsal lessons. Planning their rehearsals using these resources were instances of instructors enacting from the External Domain to the Domain of Practice. Second, throughout the academic year, instructors were encouraged to discuss both the practices of the instructors whose lesson has been video recorded as well as their own practice. Reflection on practice that supports prospective mathematics teachers' thinking, whether it be their own or another participant's, are instances of an instructor reflecting from the Domain of Practice to their Personal Domain.

Opportunity for Professional Growth

This study shows promise for adapting and translating the concept of rehearsals to the context of faculty who teach undergraduate mathematics courses. Because mathematics learning is supported in classrooms where there is attention to learners' thinking about the content (e.g., Learning Mathematics for Teaching Project, 2011), this could mean that leveraging an adapted rehearsal pedagogy during PD of mathematics faculty teaching content courses for prospective secondary teachers could support the future teachers' development of MKT. In the case of MODULE(S^2) materials, because the materials connect mathematics and teaching in ways that are designed to promote MKT, this means that centering teaching on student thinking means centering teaching on student development of MKT.

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COMPARING ELEMENTARY MATH SPECIALISTS' BELIEFS WITH THEIR PEERS

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This study's purpose is to explore how Elementary Math Specialists (EMS) teachers' practice-based beliefs differ from their peers, especially regarding situation-specific, self-reported practices. Likert-scale survey data were compared to teacher responses to situation-specific questions where teachers were asked to explain how they would respond to various mathematics classroom situations. This approach allowed the researchers to compare self-reported beliefs with practice-based beliefs, building on prior research. Two findings from this study are: (1) EMS teachers were more likely to believe that conceptual understanding should come before instruction and (2) teachers who exhibit more confidence about doing mathematics and more security in their teaching of mathematics are more likely to engage in practices like having students share and discuss multiple strategies and delaying the teaching of standard algorithms.

Keywords: elementary school education, teacher beliefs, affect, emotion, beliefs, and attitudes

Mathematics teachers hold beliefs about mathematics, learning mathematics, teaching mathematics, and their own ability to do mathematics, and these beliefs play an important role in how teachers plan, implement, and assess their instruction (Swars, Smith, Smith, Carothers, & Myers, 2016; National Council of Teachers of Mathematics [NCTM], 2014; Ambrose, Clement, Philipp, & Chauvot, 2004). For example, Peterson, Fennema, Carpenter, and Loef (1989) found that teachers who reported beliefs such as "the natural development of children's mathematics ideas should determine the sequence of topics used for instruction" (p. 7) were more likely to engage their students in mathematical word problems which required reasoning and problem solving. Based on these relationships, many preservice and in-service teacher education programs are designed, in part, to influence what teachers believe about mathematics teaching and learning.

One category of such programs is focused on developing Elementary Math Specialists (EMSs), content experts who serve in a variety of roles to support mathematics instruction (de Araujo et al., 2017). Research has found that EMS programs can have positive impacts on teachers' beliefs, such as shifts towards a cognitive orientation (i.e., implementing cognitively demanding tasks) and increases in levels of teacher efficacy (Swars et. al., 2016). Many such studies are based on Likert-scale survey data and produce general conclusions about clusters of beliefs, but some surveys, like the Integrating Mathematics and Pedagogy (IMAP), are designed to probe teachers' reasoning around their actions in specific situations. We find the IMAP to be a relatively untapped resource for exploring in more granular detail the ways that beliefs are enacted in practice (i.e., "practice-based beliefs"). In this paper, we focus specifically on some of the pedagogical situations contained within the IMAP and explore how general belief clusters might relate to decisions EMSs and their peers make regarding these situations. Our research questions are: (1) How are EMS teachers' practice-based beliefs different than their peers? (2) What relationships exist between teachers' attitudes and these practice-based beliefs?

Theoretical Framework

Beliefs are defined as "psychologically held understandings, premises, or propositions about the world that are thought to be true" (Phillip, 2007, p. 259). Some beliefs about teaching and mathematics are more beneficial for teachers and students than others. For example, in *Principles to Actions,* NCTM (2014) shares a list of productive and unproductive beliefs which impact effective mathematics teaching. Productive beliefs are related to engaging students in tasks that promote reasoning and problem solving (NCTM, 2014). Since beliefs can be productive or unproductive, it is important to find ways to help teachers shift toward more productive beliefs.

Teachers' beliefs are generally grounded in their prior experiences and evolve through reflection on new experiences (Beswick, 2012; Giboney Wall, 2018; Raymond, 1997; Vace & Bright, 1999), especially those experiences which challenge existing beliefs. For example, beliefs can change when teachers experience and reflect on unexpected results within their classroom (e.g., students providing rich mathematical explanations or exhibiting excitement about engaging in mathematics). Reflection is key because beliefs can be tacit; that is, teachers may have beliefs that guide their behavior they are not aware of, or that they are unable to articulate directly. Such beliefs can be identified by exploring how teachers respond to specific classroom scenarios (e.g., Joram, 2007). One way to help teachers transition to more productive beliefs is to encourage them to examine ways that their practice may be in tension with their stated beliefs (Boyd & Ash, 2018; Vace & Brith, 1999). In addition, the role of teachers' attitudes (how teachers feel about doing and teaching mathematics) can influence their practices (Philippou & Christou, 1998; White, Perry, Way, & Southwell, 2005/2006). Specifically, more positive attitudes towards a concept or action tend to be taken up more than negative attitudes towards a concept or action (White et. al., 2005/2006). Thus, understanding how teachers' beliefs relate to their attitudes may help us to understand why they enact certain teaching practices more than others.

Methodology

Context

The data for this paper were drawn from a funded study investigating differences between graduates of a 24-credit Elementary Mathematics Specialist program and their peers teaching at the same schools and grade levels. Previous findings showed that EMSs had significantly different beliefs than their peers (Webel et al., 2018; Webel et al., under review). An exploratory factor analysis (EFA) had reduced 38 items from a survey developed by White and colleagues (2005/2006) to 4 clusters of teacher beliefs: Constructing (10 items, mathematical knowledge is constructed), Computing (7 items, knowing mathematics is mostly about knowing how to compute the right answer quickly), Security (14 items, security in teaching mathematics), and Confidence (7 items, confidence in doing mathematics). Analyses had revealed significant differences between EMSs and their peers for each of those factors. Specifically, EMSs had higher scores for Constructing, Confidence, and Security, whereas nonEMS teachers had higher scores for Computing. In this paper, we explore relations between these scales and individual items in the 2012 Horizon and Integrating Mathematics and Pedagogy (IMAP) surveys to examine relationships between these general beliefs and attitudes and specific reported practices.

Data Collection

This study included 61 elementary teachers, of whom 28 were EMS. Data was collected from three different surveys. The first survey was a web-based assessment which originated as part of the Integrating Mathematics and Pedagogy (IMAP) project (Ambrose, Philipp, Chauvot, & Clement, 2003). The survey consists of 9 sections, each with multiple parts, for a total of 46 questions, which includes elementary level, open-ended questions set in classroom contexts, including videos. Those items were used to identify teacher beliefs in 7 areas. For the purposes of this study, we focus on IMAP item 3.3, in which participants were shown five student strategies for solving a multi-digit addition problem, 149 + 286. The strategies included 1) a manipulative approach using base 10 blocks, 2) the standard United States algorithm for addition, 3) left to right addition (sometimes called the "intermediate algorithm"), 4) combining the same units (with a minor computation error), and 5) a compensating approach in which 149 was temporarily rounded to 150. The teachers were then asked, "Consider the strategies on which you would focus in a unit on multidigit addition. Over a several-week unit, in which order would you focus on these strategies?" We chose to focus on this item because of the direct connection to practice and, when comparing EMS and nonEMS teacher responses, we found a significant difference in their ordering of the five strategies.

The second belief survey was designed by White and colleagues (2005/2006). This survey consisted of 18 Likert scale items from 1-strongly disagree to 5-strongly agree. The questions in the survey included statements like, "mathematics is computation" and "being able to memorize facts is critical in mathematics learning." Also from White et. al. (2005/2006) we utilized a teacher attitudes survey. The survey consisted of 20 Likert-scale items related to teachers' attitudes towards mathematics and teaching mathematics, such as, "mathematics makes me feel inadequate" and "I'm quite good at mathematics."

The final survey that was utilized to understand teacher beliefs was the 2012 Horizon Survey (Banilower, Smith, Weiss, Malzahn, Campbell, & Weis, 2013). We looked at teacher responses to items in three sections of the Horizon Survey. The first part of the survey, which focused on beliefs, consisted of 11 items which participants responded to on a 5-point Likert scale ranging from a 1 (strongly disagree) to a 5 (strongly agree). Questions included "teachers should explain an idea to students before having them investigate the idea." The second section asked participants to rate their emphasis on eight instructional goals ("Learning test taking skills/strategies") on a 5-point Likert scale ranging from 1 (none) to 5 (heavy). The third section, related to frequency of current practices, included 15 5-point Likert scale items ranging from 1 (never) to 5 (all the time), with items such as, "explain mathematical ideas to the whole class."

To address RQ1, we started by identifying significant differences between how EMS and nonEMS teachers responded to IMAP survey item 3.3. We then looked for significant differences on the White and colleagues (2005/2006) and Horizon (2012) survey items. This process allowed us to limit which items we wanted to examine further; we only examined items that had a significant difference between EMS and nonEMS teachers. We then explored relationships between beliefs and specific reported practices for each group, including the situation described in the IMAP item. To address RQ2, we calculated correlations between each item on the pedagogical goals and self-reported practices scales (Horizon) and the factor scores on the attitudes survey (White et al., 2005/2006) to compare how teachers who exhibited more or less confidence in doing mathematics and security in teaching mathematics answered questions about specific teaching practices (Horizon and IMAP). We also looked at how teachers' attitudes related to their responses on Item 3.3 (see below) from the IMAP survey, comparing means via *t*-tests for teachers who responded differently.

Results

Differences in Teacher Beliefs

One primary theme from our analysis was that EMSs were more likely than their peers to believe it is important for students to develop conceptual understanding of mathematical concepts before they learn mathematical procedures. On IMAP question 3.3, when asked to say which of the five strategies they would focus on first in a unit on multi-digit addition, 100% of EMSs chose a strategy which employed the use of base-10 blocks to model the place value of the digits in the problem. When asked to justify their reasoning, one EMS teacher wrote, "I would first have students use manipulatives to make sure they understood the concept of adding and had a good grasp on place value." This teacher's ordering of student representations, along with most EMSs, moved from concrete to abstract with 60% of EMSs choosing the standard algorithm as the last approach they would show to students. In contrast, 33% of nonEMS teachers chose to share the standard algorithm last and 12% chose to share it first. A focus on conceptual understanding prior to teaching procedures was not only represented in teachers' practice-based, situation-specific beliefs but also in their self-reported beliefs. In the Horizon (2012) survey none of the EMSs agreed with the survey question, "Teachers should explain an idea to students before having them investigate the idea" as compared to 34% of nonEMS teachers. Additionally, on the White et. al. (2005/2006) survey 11% of EMSs agreed with the statement, "mathematics is computation" as compared to 42% of nonEMS teachers. These examples taken together show that EMSs believe conceptual understanding should serve as a foundation for procedural fluency and that their self-reported beliefs aligned with their practice-based beliefs.

Teacher Attitudes and Reported Practices

In general, correlations between attitudes and reported practices revealed that teachers who expressed more confidence in their mathematical ability were significantly more likely to report engaging in class discussions ($r = 0.26$, $p = 0.06$) and to focusing on a wider variety of mathematical representations and solution strategies when responding to the situation on IMAP item 3.3. Teachers who expressed more security with regard to their teaching of mathematics were similarly more likely to report having students compare and contrast their strategies and solutions ($r = 0.25$, $p = 0.06$), and justify their reasoning ($r = 0.30$, $p = 0.03$). On IMAP question 3.3, teachers who felt less secure in their mathematics teaching were more likely to share the standard algorithm with their students first, second, or third (such teachers had an average security rating of 3.77). Teachers with higher security scores were more likely to share the standard algorithm later or not at all (such teachers had an average security rating of 4.12, a significant difference, ($t(29) = 1.84$, $p = .03$). This suggests that teachers who feel less secure in their mathematics teaching might feel less comfortable with nonstandard solution methods.

Discussion and Implications

In this study we sought to build on previous research that focused on EMS teacher beliefs by closely examining how teachers held beliefs were represented their practice-based and situation-specific, self-reported beliefs. Not only did the researchers find that EMS and nonEMS teachers' beliefs about mathematics, teaching, and learning were different but it was also confirmed that teachers self-reported beliefs were aligned and reflected in their practice-based beliefs. This adds to the previous literature, which only shared generalized results about teacher beliefs without exploring how those self-reported beliefs were enacted in situation-specific scenarios.

Additionally, limited research has been done comparing EMS and nonEMS teacher beliefs. Finally, this study connects teacher attitudes with teacher beliefs and self-reported practices.

Future research might take this work a step further by comparing self-reported and practice-based beliefs with classroom observations to understand the more tacit beliefs that teachers might hold. The findings of this study present a way of exploring teacher beliefs through a variety of lenses which allows for a more robust understanding of teacher beliefs. This understanding could help teacher educators to present elementary teachers with any disparities between self-reported beliefs and practices-based beliefs, which could lead to changing beliefs.

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(MIS)ALIGNMENT BETWEEN TEACHERS' IN-THE-MOMENT NOTICING AND POST-INSTRUCTION NOTICING

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Being able to notice students' mathematical thinking during teaching (in-the-moment noticing, IMN) impacts the quality of instruction. Also, noticing students' mathematical thinking and reflecting on the activities of teaching after instruction (post-instruction noticing, PIN) is important for teachers' long-term professional development. We explored the relationships between IMN and PIN by examining the data from seven elementary in-service teachers engaged in a professional development program. By analyzing 33 coaching videos and post-coaching conversations, we found that teachers' IMN did not align with their PIN, and PIN tended to be of lower quality than IMN. We discuss implications for future research and practice.

Keywords: Teacher In-the-Moment Noticing, Teacher Post-Instruction Noticing, Teacher Professional Development

Introduction

Noticing students' mathematical thinking during instruction is an essential component for "highly individualized and responsive instructional tactics" (Thomas et al., 2017, p. 6). We refer this kind of noticing (noticing during instruction) as in-the-moment noticing (IMN) (Cross Francis et al., 2021). Additionally, reflecting on instruction with a focus on students' mathematical thinking after the lesson helps teachers identify meaningful and problematic aspects of students' thinking that can inform subsequent instructional practices (Sherin et al., 2011). We refer to this kind of noticing, which occurs after instruction, as post-instruction noticing (PIN) (Cross Francis et al., 2021).

Research findings support that high level IMN helps teachers in making strategic choices about adapting their instruction, hence, acts as a central factor in high-quality instruction (Thomas et al., 2017; Walshaw & Anthony, 2008). As such, high quality teaching would be, at least in part, the result of high-level IMN. Subsequently, if a teacher demonstrates high level IMN, then they would have access to relevant knowledge and skills, which theoretically would inform high PIN. However, despite this apparent connection, the link between IMN and PIN has been understudied. In this study, we examined the alignment and misalignment between teachers' IMN and PIN. We aimed to answer: What are the alignment patterns between elementary teachers' in-the-moment noticing and post-instruction noticing levels?

Theoretical Framework

In-the Moment Noticing (IMN)

High-level IMN supports meaningful instructional decision-making that facilitates students' learning (Ayalon & Hershkowitz, 2018; Thomas et al., 2017). The skills encompassed in IMN – attending, interpreting, and responding to students' mathematical thinking – are essential for teachers to differentiate their instruction and support students effectively. However, these noticing skills can hardly be accessed directly. As Jacobs et al. (2010) proposed,

> We suggest that, before the teacher responds, the three component skills of professional noticing of children's mathematical thinking – attending, interpreting, and deciding how to respond – happen in the background, almost simultaneously, as if constituting a single, integrated teaching move… (p. 173).

Additionally, as Schoenfeld (2011) argued "what you see and don't see shapes what you do and don't do" (p. 228), so we can infer teachers' noticing activity from observing students' and teachers' in-the-moment interaction. Therefore, strong facilitation of students' engagement with high frequency of use of students' thinking, close attention to students' learning difficulties, and facilitation of student substantive contributions, serve as indicators of high-level IMN.

Post-Instruction Noticing (PIN)

Researchers (e.g., Amador et al., 2017; Sherin & Van Es, 2009) have focused on investigating PIN to understand what and how teachers observe, how they interpret the gathered information, and how professional developers can support teachers in this process. Research on noticing found that teachers' PIN skills help teachers in identifying significant student contributions (Leatham et al., 2015). Furthermore, PIN skills can be improved through professional development (PD) (Sherin & Van Es, 2009; Star & Strickland, 2008). Different from IMN, PIN via watching videotaped lessons offers teachers fruitful opportunities to observe and attend to students' thinking as they can pause the video, replay interesting and meaningful instances, and spend time making sense of students' mathematical thinking. This suggests that teachers will have more opportunities to notice students' thinking than during actual instruction.

Methods

Context

This study was conducted within a PD program designed to collaboratively work with elementary teachers on improving their MKT and instructional practices. The data were collected from the coaching intervention, termed as Holistic Individualized Coaching (HIC) (Cross Francis, 2019), implemented during the second year of the PD. During each coaching cycle, teachers engaged in conversations with the coaching team to plan their lesson (pre-coaching conversations) and to discuss the taught the lesson (post-coaching conversations). For the post-coaching conversation, the teachers were asked to select three videoclips they considered useful for improving instruction from the videotapes of their taught lessons to discuss with the coach.

Participants

Seven elementary teachers participated in this study. They taught across three different schools that served high populations of minoritized students with over 50 percent of students qualifying for free/reduced lunch. Table 1 includes additional information about the teachers.

Table 1. Demographic Data on Participants

	Bo	Liv	Avi	Mia	Kai	Joy	Sky
Gender	Male	Female	Male	Female	Female	Female	Female
Grade Level	3rd	4th	6th	2nd	1st	2nd	K
Years of Teaching	5	9	15	10	19	5	6

***** All names are pseudonyms.

Data Sources and Analysis

Instructional videos. Instructional videos were used to determine instructional quality and the level of instructional quality was used as a proxy for IMN. Each teacher was involved in five HIC cycles. We analyzed 35 (seven teachers times five cycles) coached lesson videos using the Mathematical Quality of Instruction instrument (MQI) (see Hill et al., 2008). We specifically used three IMN related items from the *Whole Lesson Codes* section: i) Teacher Uses Student Ideas (USI); ii) Teacher Attends to and Remediates Student Difficulty (RSD); and iii) Lesson Contains Common Core Aligned Student Practices (CSP). The items were scored from 0 to 5 (0 – not present, 1 – low, and 5 – high) and the average scores of items were used to indicate teachers' levels of IMN. We assigned the average score of 0 to 1.6 as low level, 1.7 to 3.3 as medium level, and 3.4 to 5 as high level.

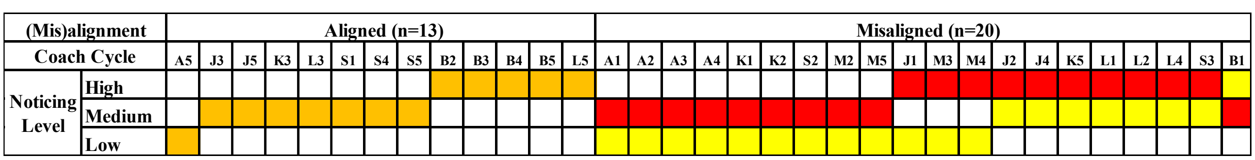
Post-coaching conversations. We examined 33 post-coaching conversations (two teachers were not available) to identify the clips the teachers selected. Then we analyzed the conversation around these videoclips by using Van Es' (2011) noticing framework to determine PIN. We assigned a score of 1 to 4 for (1: low and 4: high) *what* and *how* they noticed for each of the videoclips, and we took the average scores to show their PIN level. We defined the average score between 1 to 1.3 as low level, 1.4 to 2.6 as medium level, and 2.7 to 4 as high level.

To answer the research question, we examined both the IMN and PIN levels per coaching cycle (Figure 1) to determine the alignment patterns. We further grouped the seven teachers into three pattern categories based on the (mis)alignment features.

Findings

Overall (Mis)alignment Pattern

We compared each teacher's IMN level with the corresponding PIN level to determine the nature of alignment between them (Figure 1). The IMN and PIN comparison showed both alignments and misalignments. Specifically, in 13 of total 33 lessons teachers' IMN aligned with their PIN (the orange squares: one L-L; seven M-M; five H-H); in 19 lessons, teachers IMN levels were higher (the red squares are above the yellow squares: seven H-M; nine M-L; three H-L) while only in one lesson, the teacher's PIN level was higher (the yellow square is above red square, B1). Overall, there were more misalignments than alignments and teachers IMN levels were higher than their PIN levels, except for one case, B1. B1 means Bo's first coaching cycle.



RED: IMN level; **YELLOW**: PIN level; **ORANGE**: Alignment between IMN and PIN. Coaching cycle: Teacher's initial and the order of coaching cycle. These norms also apply to Figures 2 and 3.

Figure 1: Alignment Patterns Between IMN and PIN

Three (Mis)alignment Pattern Categories

Cases of mixed alignment. Mixed alignment cases appeared to be more common among all cases. Four teachers' cases (Kai, Sky, Joy, and Liv) belonged to this type (Figure 2). Their IMN and PIN levels did not show a regular pattern of alignment. The teachers' IMN levels varied from medium to high, while PIN levels ranged from low to high. Their alignments clustered at medium level. For all the cases PIN was at an equal or at a lower level to IMN.

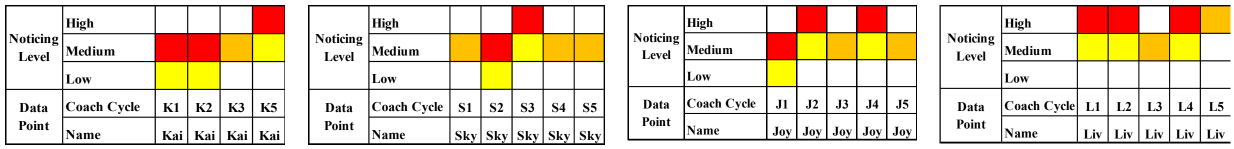


Figure 2: Four Cases of Mixed Alignments

Cases of low reflective noticing. Figure 3 shows two teachers' cases (Avi and Mia) whose PIN levels were consistently low although their IMN levels varied. Avi's IMN was medium across the five coaching cycles, while his PIN levels were consistently low. Mia's IMN levels ranged from medium to high while her PIN levels were also consistently low. These two cases show that for some teachers PIN level are stable regardless of their IMN level. Also, IMN was higher or at the same level as PIN.

Case of H-H alignment. Figure 3 includes a special case where the teacher's (Bo) IMN and PIN levels were aligned at the high level (four over five instances). In the first coach cycle, the PIN is higher than the corresponding IMN.

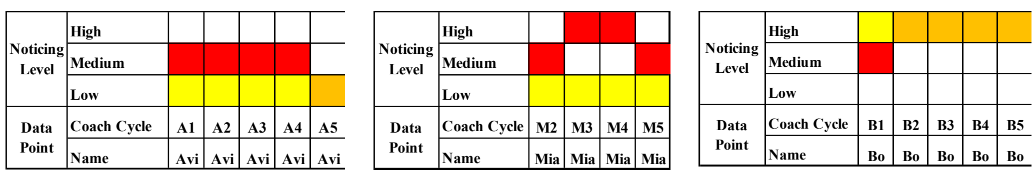


Figure 3: Three Cases of Consistent PIN Level

Discussion and Implication

In this study, we found that the teachers tended to notice in the moment of instruction, but they were not always able to recall or re-notice those moments when they watched their own teaching videos. This situation indicates that as Jacobs et al., (2010) stated, the nature of noticing involving "attending, interpreting, and deciding how to respond – happen in the background, almost simultaneously, as if constituting a single, integrated teaching move" (Jacobs et al., 2010, p. 173), may render IMN "invisible" even for the person who enacted that noticing during their instruction. It may be that in the heighten activity of the classroom, more focused attention is placed on the more salient aspects of teaching. However, when watching the video post instruction, other aspects of the classroom activity become more visible, for example classroom management as in the case of Mia. In this regard, more research is needed to understand the distinctive nature of both types of noticing and how to support teachers in better focusing their attention after instruction. In this study, we found Bo was the only teacher who showed H-H alignment. We observed that Bo had a strong teacher identity that foregrounded students' thinking (Cross Francis et al., 2021). We believe such identity might have contributed to his attentiveness to students' mathematical thinking in both IMN and PIN. We inferred that Bo's identity might have played a key role. Supporting teachers in developing their identity and to change their beliefs about student learning to be in accordance with a student-centered teaching approach would make a difference.

Practically, as Mason (2002) explained, noticing is "an opportunity to act appropriately" (p. 1) and teachers' IMN impacts opportunities for effective instructional decisions in the classroom while PIN influences teachers' opportunity to learn from their own practice for growth. Therefore, as teacher educators we must focus on developing both IMN and PIN among the prospective and in-service teachers. By doing so, teachers would be more consistent in foregrounding students' thinking as a central part of their math teacher identity and instructional practices.

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TEACHERS' ROUTINE AND ADAPTIVE EXPERTISE THROUGH MATHEMATICAL MODELLING INSTRUCTION IN REMOTE LEARNING

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As part of an ongoing study, this paper reports on teachers' use of instructional routines for modelling prior to and during remote teaching. We problematize the use of instructional routines through notions of adaptive and routine expertise and pose questions to explore how routines can be adapted to attend to instructional responsibilities in remote learning environments. This study contributes to modelling instruction literature and extends research on use of instructional routines in innovative teaching.

Keywords: modelling, instructional activities and practices

Mathematical modelling is an opportunity to engage students in real-world, meaningful, mathematical activity (Cirillo, Bartell, & Wager, 2016; Kaiser, 2017; NCTM, 2018). Modelling instruction has the potential to engage students in mathematics relevant to their interests and asks teachers to support students' non-routine solution pathways (Aguirre et al., 2019). This instruction aligns with ambitious mathematics teaching, which centers supporting all students to see themselves as capable of solving authentic mathematics (Jackson & Cobb, 2010). With the move to remote learning, teachers have been pressed to modify instruction with new modes of student interaction, (re)focused mathematics content (e.g., Achieve the Core, 2020), and the need to support students' learning as holistic human beings (Horn & McGugan, 2020). Modelling has the potential to engage students in key mathematics and humanize mathematics such that students see themselves as capable mathematicians.

Classroom mathematical modelling asks students to grapple with authentic and ill-defined problems in a cycle where they make decisions, defend those choices, and revise their thinking as part of a classroom community (Consortium for Mathematics and Its Applications (COMAP) & Society for the Industrial and Applied Mathematics (SIAM), 2016). To support students, teachers may recruit instructional resources, such as modelling routines, to navigate the complex and ill-defined mathematical demands across the modelling cycle (Elliott, Stoddard, & Brunner, 2019; Lucenta & Kelemanik, 2020). Instructional routines are sequences of interaction that, when used regularly, scaffold student learning and teachers' instructional decision-making toward specific goals.

As part of an ongoing research project, this paper investigates teachers' self-reported instructional shifts regarding use of existing modelling routines prior to the pandemic and during remote teaching. We explore the patterns that arose around teachers' self-reported modelling instruction via their responses to a questionnaire and interviews. We discuss our questions for further research based on observed trends. This paper adds to the scholarship on pandemic impacts for ambitious instruction with a focus on modelling instruction and routines.

Framework

Instructional routines allow teachers to flexibly engage with students and content, relying on structures to support risk-taking or adaptation of practice (Lampert & Graziani, 2009). They are *routine* in their regular use and flexible in application to support teaching and learning across contexts. Researchers posit that instructional routines build familiarity for students and teachers and can shift the cognitive work to the mathematical content under study (Lampert & Graziani, 2009). In this paper, we examine the use of teacher designed modelling routines that support specific modelling processes. Two routines, henceforth referred to as Routines 1 and 2, focus on the practices of making mathematical assumptions (Routine 1) and developing processes for revising a model (Routine 2); these elements of the modelling cycle have been documented as challenging for both students and teachers to navigate (Galbraith & Stillman, 2006).

Remote learning calls teachers to develop new instructional practices or adapt existing ones to fit new contexts. As a result, teachers are pressed to apply their knowledge and skills of teaching to flexibly modify instructional tasks and interactions to meet student learning goals. We frame how teachers deploy their expertise in teaching via the interaction of routine and adaptive expertise (Hatano & Inagaki,1986). Routine expertise is the efficient and accurate expertise that emerges in stable contexts. Adaptive expertise is leveraged towards innovation when teachers must effectively attune instruction to shifting situations, such as remote learning. Further, because modelling routines offer *both* predictable sequences of activity *and* tools to responsively and flexibly elicit students' use of modelling practices (Lampert et al., 2013), we were interested in how teachers in the study might draw upon routine and adaptive expertise to enact modelling instruction across their transition to remote learning.

Context and Methodology

The data in this paper are part of a broader study on the use of instructional tools in mathematics classrooms in a mid-sized suburban school district. We coordinate data from a questionnaire on ambitious instruction and modelling, completed by secondary mathematics teachers who had participated in a district-led professional development series on modeling instruction and incorporating instructional routines, and interviews on similar topics with a subset of these teachers who had long term commitments to designing modelling routines.

Here, we draw upon a subset of questions from the Qualtrics (Provo, UT) questionnaire, where teachers recorded an approximate number of times they had used Routines 1 and 2 prior to and during remote learning. Ten teachers from the district completed the questionnaire. While this sample size is limited, it represents the majority of teachers from one mathematics department and from the set of teachers participating in the professional development, therefore providing insight on general department trends. Four teachers completed a semi-structured interview, providing detailed descriptions of their use of Routines 1 & 2 prior to and during remote learning. After reading all data multiple times, the research team identified and discussed themes from teachers' reported use of the routines and coordinated them across data sources and participants. These analyses allowed us to frame a problem space for further inquiry. We present findings from our analysis and identify an area for continued exploration in light of current literature.

Findings and Discussion

We wanted to understand how teachers drew upon their expertise around the modelling routines in their instruction. Teachers were asked to identify if they never used the routine, if they used it 1-2 times an academic year, quarterly, monthly, or at least weekly. *Prior to remote learning*, every teacher responded that they had enacted Routine 1 at least once, and all but one teacher had used Routine 2 at least once (Table 1).

Table 1: Reported use of the instructional routines prior to and during remote teaching.

Routine Use	Routine 1(Pre)	Routine 1 (During)	Routine 2 (Pre)	Routine 2 (During)
Weekly or more	2	0	1	0
Monthly	4	4	2	2
Quarterly	2	2	3	2
1-2 times a year	2	1	1	2
Never	0	3	1	4

Prior to remote learning, all ten teachers stated in their use of the modelling routines helped students make real-world connections with mathematics. A majority of teachers noted the routines fitting into their curriculum (n =8), guiding the flow of their lessons (n=9), and supporting work on future modelling problems (n=9). Additionally, almost all the teachers (n=9) reported that the use of the routines promoted student critique of models and explicit metacognitive reflection on learning. During an interview, one of these teachers reflected on the regularity of Routine 1 use in their classroom, saying, “I would say it is definitely becoming more of a regular routine for me in my teaching…kids really seem to enjoy the experience. They’re just getting used to not being able to all get the same answer in the end, and they’re getting more comfortable with that.” This teacher saw a relationship between routine use and student engagement in mathematical modelling processes that could invite students into doing mathematics, a challenge teachers may experience during remote instruction (Horn & McGugan, 2020). Another teacher, however, reflected on the challenge of feeling routine expertise with the modelling routines, even prior to the shift to remote learning:

> I was reflecting back on when and when and how often I use the routine that we built, and it was sporadic and random. And its always felt like it had to be forced in some way. Like, it wasn’t natural to want to use the routine, because [they were] still in this developmental stage and it never really has landed in a place where it just is automatic.

After the transition to remote learning, we recognized that teachers’ use of both routines had decreased. In particular, four teachers had never used Routine 2, three teachers had never used Routine 1, and no teachers were using either routine weekly. While this is not surprising due to the variety of needs and expectations teachers have had to balance over the last year, we were interested in continuing to learn more about why this particular resource was impacted. Even though these teachers identified benefits of the routines that could attend to needs for innovation and practice in remote learning, the use of the routines still decreased. We saw clues as to why this might be in teachers’ questionnaire responses describing challenges they faced. Half of the teachers noted time as a leading challenge prior to and during remote learning. They also noted limitations of find “good [modelling] tasks” that don’t feel like “field trips” outside of the typical curriculum and uncertainty in how to assess progress on modelling. We found these responses especially interesting in light of teachers’ regularity of use of modelling-routines prior to remote learning. We wondered how the use of modelling routines had supported the majority of teachers’ *routine expertise* prior to remote learning, given the comment that they were still in development. To this end, we are currently exploring how the modelling routine may need to be adapted to be an effective resource for both teachers and students in a virtual environment so that teachers may develop *both* routine and adaptive expertise. Further, is it possible to have a routine for modelling that can attend to mathematical challenges as well as support the ever-evolving needs and responsibilities for teachers and students in a remote learning environment?

Conclusion

This study explores how teachers enacted modelling routines in pursuit of ambitious instruction and considered features of the instructional environment that impacted their use during remote learning. Teachers recognized benefits to using the routine and these benefits aligned with some of the central goals for remote learning. However, teachers' reported use of the routines declined since the shift to remote learning which may indicate a need for more routine and adaptive expertise to support flexible use amongst the group of teachers. We continue to advocate for the use of routines and other resources to support modelling instruction and invite the research community to engage in our shared inquiry: what features of classroom instruction and expertise might support the adaptation of routines to serve different needs and scenarios? How might routines be useful in teachers' development of adaptive expertise?

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SYNCHRONOUS ONLINE VIDEO-BASED PROFESSIONAL DEVELOPMENT FOR RURAL MATHEMATICS COACHES

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Keywords: Inservice Teacher, Professional Development, Research Methods, Coaching

In this project, we have designed, implemented, and started to research an innovative fully online video-based professional development model for mathematics coaches in rural contexts. Coaches in rural areas often lack access to professional development available in more populated areas, fueling the need for an online model that bridges geographic barriers (Howley & Howley, 2005; Maher & Prescott, 2017). The intent of the poster will be to share the professional development model and describe the research processes that are currently in progress.

The goal of the project is to support mathematics coaches in rural contexts to improve their ability to (a) facilitate productive planning and debriefing conversations with teachers; (b) notice salient coaching practices and their impact on teachers' thinking; and (c) use evidence of teacher learning to make decisions about their own coaching practices. Focal research questions include: (1) How do the nature and content of coaches' contributions evolve across the multiple components of the model? (2) How are the contributions in one part of the model associated with those from other parts of the model? (3) What resources (e.g., time, logistics, skills) are required to enact the three-part model?

We are currently engaging coaches in a three-part professional development model that includes (a) an online course on content-focused coaching, (b) one-on-one video-based coaching cycles with a Mentor Coach, and (c) an online video club in which practicing coaches reflect on dilemmas of practice and the impact of coaching moves, using evidence from their own coaching. Three cohorts, each comprised of 12 Coach Participants (supporting rural teachers), will engage in a two-year professional development model, each supported by a Mentor Coach (project personnel recruited for their expertise in content-focused coaching). The first cohort is completing Year One. In Year One, participants engage in all three parts of the model, and in Year Two they engage in online video clubs only. The study follows a design-based research model (e.g., Barab & Squire, 2004) with iterative cycles of design and revision of the three-part online model. These cycles are being used to test and revise our theory of action and our conjectures about the affordances and constraints related to online professional development for coaches. We follow Sandoval (2014) in developing and using *conjecture maps* to articulate our model, guide our research, and build theory. There are two types of measures: (1) baseline and outcome data related to the perceptions and practices of the Mentor Coaches and Coach Participants; and (2) measures related to the content and nature of interactions across all three components of the professional development model.

The focus of the poster presentation will be to share the overall project design, including descriptions of the three-part model as well as the data collection process and initial analysis processes, as related to the research questions.

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LEVERAGING TEACHERS' COMPLEX PERCEPTIONS OF STUDENTS

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Keywords: Teacher Beliefs, Instructional Activities and Practices

Researchers have reported that mathematics teachers hold unproductive views of students (e.g., Jackson et al., 2017; cf. Lambert, 2015), or adjust their instruction in unproductive ways (Wilhelm, 2014). *Unproductive* views attribute students' struggle to factors characterized as internal to the student such as ability or motivation; *unproductive* instructional adjustments are aimed at decreasing the rigor of learning opportunities. Here, we bring together findings from two studies that, in some ways, reiterate these trends. However, when we further unpacked participating teachers' talk, we found nuance in teachers' perceptions of their students, which led us to ask how we might leverage teachers' ability to understand their students in complex ways and shift their discourse to be more productive.

The first study, *Teachers' Views of Students with Disabilities as Mathematically Capable,* found that mathematics teachers explained students' struggle in unproductive ways and gave unproductive rationales for instructional adjustments (Mason, 2019). Also, there were differences between explanations and rationales for students with and without disabilities. The second study, *Unpacking K–3 Teachers' Stories of their Students as Mathematics Learners*, found that teachers combined their perceptions of students' performance and aptitude in mathematics with other factors including personality, effort, engagement, behavior, and family context (Altshuler, 2019). Also, teachers drew on different sources of evidence when supporting their descriptions of students in relation to their perceptions. Taken together, these studies echo the narrative that some teachers describe students' mathematics learning as outside of their instructional control. This is an important finding, and one that is problematic for students' opportunities for learning. However, data from both studies also revealed that teachers talked about their students in multi-faceted ways, including students' assets. Even when teachers' descriptions of students were ultimately coded as unproductive, we noticed nuggets of productivity, which, upon reflection, complicates these findings and suggests the need to identify ways to support teachers in leveraging these views.

One approach is supporting teachers to reconceptualize the nature of intelligence as malleable rather than fixed (e.g., Boaler, 2013). This may lead to more productive instructional responses because when teachers attribute students' struggle to innate traits, they may be more likely to absolve themselves of instructional responsibility (e.g., Horn, 2007). By reframing intelligence as malleable, teachers can leverage their agency to support learning. Further, as teachers work to shift their discourse about students, we should similarly shift research practices and analyses. We frequently ask teachers to notice, collect, and share data about their students; what would it look like to identify moments of productive talk and use such instances as a springboard for discursive (and instructional) change? Together, shifting teachers' discourse and perspectives to be more productive and aligning analytical tools to do the same may be one step toward creating richer mathematics learning opportunities for all students.

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MASTER'S PROGRAM FOR IN-SERVICE TEACHERS WITH A FOCUS ON IMPROVING MATH TEACHING AND LEARNING

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In-service teachers need sustained professional development that influences practice to positively impact student achievement (Even, 1999). A master's program for in-service teachers is a good context for providing teachers with sustained professional development. Unfortunately, getting a master's degree does not necessarily translate into shifting teacher practice and student achievement. Therefore, it is important to consider what criteria are needed to develop a high-quality master's program that impacts teacher knowledge, beliefs, and practice. Some key criteria that can impact teacher learning and practice are professional learning communities (Graven 2004; Hill, 2007) and rich pedagogical and content knowledge (Basista & Mathews, 2002; Darling Hammond, 2000; Borko & Putman, 1995; Lamberg et al., 2020; Swackhamer et al., 2009). In this session, we will present the design (Lamberg et al., 2020) and data from our master's program that is currently in its seventh iteration. Twenty-five participants participated in our study. The data collected include teacher reflection on the impact of the program, a survey of teachers' teaching practices framework (Lamberg, 2019) and MAP test student scores. The teachers' written reflections were analyzed using Strauss and Corbin's(1998) constant comparison method to identify themes. Mean pre and post test scores were calculated, with the differences used to assess the growth of individual teachers and to compare with national normed scores.

Findings

The analysis revealed that teachers' teaching practices were positively influenced as indicated in the teachers' self-report survey. Student achievement pre and post scores were positively impacted, and the gain scores were above national norms. The following respondent comments was representative of how teachers were impacted:

- When I began the first math class of our program, I could figure out everything from an algorithm, but that did not get me very far when I had to prove my answers. It was a steep learning curve to not only change my thinking process on what mathematics is, but I had to change everything I had believed for so many years. Along with mathematics concepts, we were learning progressions and digging deeply into the common core standards. Digging into those standards and looking at exactly what students need to know and how we can get them there was "eye-opening".
- What I learned from this program is that teaching does not necessarily equal learning. I was teaching my students procedures that I was familiar with in an environment where nothing was making sense to a lot of kids. The teaching was there and I was going through all the motions. However, the learning was not evident the way it should be. This

program has helped me evolve from being a teaching to being a facilitator of professional discussions.

Conclusion

This master's program design has been effective in supporting shifts in teachers' classroom practice and impacting student achievement. It meets the criteria outlined on teacher quality and student achievement by Darling-Hammond (2000). A master's program's effectiveness is dependent on whether it impacts student practice. The program discussed here considers teachers' mathematical and pedagogical content knowledge and develops a professional learning community of practice as outlined by Hill (2007). The poster presentation will include details on the design of the master's program.

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THE STRUGGLE OF COVARIATIONAL REASONING ABOUT EXPONENTIAL GROWTH: EMBRACE IT OR ERASE IT

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Exponential functions are an important part of secondary school mathematics (NGACBP & CCSSO, 2010). Yet not much is known about the link between a teachers' knowledge and student learning with respect to exponential functions (O'Bryan, 2018). Covariational reasoning has been identified as one important component important in learning and understanding exponential functions (Ellis et al., 2016; Strom, 2006).

As part of a larger study, we engaged several secondary mathematics teachers in analyzing a task designed to support the development of covariational reasoning. The task had several nonstandard features that supported covariational reasoning such as not asking for an expression or equation, not providing an initial value associated with time $x = 0$, and asking about fractions of a growth period. Here we focus on two teachers' responses that highlight their thinking as they struggled to make sense of a fractional growth period. Kathy and Ben represented a spectrum of views on the Growing Rabbits Task and evidenced different ways the teachers engaged in covariational reasoning concerning non-integer multiples of the growth period.

Both Ben and Kathy completed the task by writing an exponential equation. Ben initially used reasoning about growth periods but quickly abandoned this reasoning in favor of an equation-oriented approach. He recognized the existence of alternative approaches but saw procedurally using an equation as the best approach to the problem. His focus was on producing a solution as opposed to engaging in covariational reasoning of exponential change.

Kathy initially used an equation but then extended her inquiry to make meaning of what that equation was doing. She embraced the relationship between additive and multiplicative change, grappled with the effects of estimation on an exponential function, and used ratios to compare interval lengths of the dependent variable. She struggled to identify how that relationship worked with non-integer multiples of the growth period but did not let her struggle derail her inquiry.

The two teachers shifted between covariational and correspondence perspectives (Ellis et al., 2016) in different ways and each encountered different difficulties extracting a rich interpretation of a fractional exponent or partial growth cycle. Neither teacher demonstrated the full covariational reasoning underlying the fractional growth periods in the task, though Kathy's work demonstrated that she sought rich meaning. Additional research on teachers' covariational reasoning with exponential functions, particularly fractional exponents, can inform teacher training and professional development to prepare teachers to embrace these topics more deeply.

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THROUGH THE STUDENTS' EYES: MATHEMATICS TEACHERS' CURRICULAR NOTICING IN A PROBLEM-BASED CURRICULUM

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Although problem-based mathematics curriculum materials have the potential to provide opportunities for students to develop mathematical understanding (Harris et al., 2001; Reys et al., 2003), teachers who did not learn mathematics through an inquiry approach can find it challenging to implement (Edson et al., 2019). If they do not notice student learning opportunities in the curriculum, teachers are less likely to promote students' mathematical understanding (Amador & Earnest, 2019). This study investigated teachers' curricular noticing when they solved textbook problems from a student's perspective. The following research question was addressed: How does solving problem-based textbook problems in the process of planning lessons engage mathematics teachers in curricular noticing?

Theoretical Framework

This study draws on the Curricular Noticing Framework (Dietiker et al., 2018) and an active learning approach to teacher learning (Darling-Hammond et al., 2017). Focusing on problem-solving experience as leverage for curricular noticing, we utilized the Curricular Noticing Framework, which consists of *Attending, Interpreting, and Responding*, to describe how teachers make sense of a curriculum and how they find opportunities in it for students' understanding.

Data Collection and Data Analysis

Data sources included both audio-visual recordings of interviews and the written responses from two middle school mathematics teachers. Both participants used *Connected Mathematics Project* (*CMP3*) (Lappan et al., 2003) as a school designated curriculum. For each participant, we conducted two interviews and asked for one written response regarding their problem-solving process and curricular attending. Data analysis was guided by a qualitative method (Saldaña, 2016), applying both a structural and a holistic coding approach. The teachers' curricular noticing phases (Attending, Interpreting, Responding) were identified, followed by an analysis of how their problem-solving experience related to their noticing phases.

Findings and Implications

When solving textbook problems from a student's perspective, the teachers spontaneously engaged in curricular noticing from a teacher's perspective as well. Further, as they planned lessons, they attended to resources within the teacher's guide that could support students' various problem-solving strategies; they interpreted guidelines based on their problem-solving experiences and their own solutions; and they responded by creating lesson plans that would ensure their students would be able to develop agency while doing mathematics. For example, one teacher came up with a unique strategy to solve a problem, while the other teacher struggled to generate two equivalent expressions. Based on their problem-solving experiences, both teachers decided to encourage students to create various strategies. These teachers reported that problem-solving from a student perspective did not take them very much time, and further helped them better understand curriculum materials and thus be well-prepared for the lessons.

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THE EFFECT OF RESEARCH EXPERIENCES ON TEACHERS' BELIEFS AND INSTRUCTION: PILOT STUDY

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Research suggests many benefits of adopting Discovery Learning (DL) methods for teaching mathematics (Herdiana, Wahyudin, & Sispiyati, 2017). The question then becomes, how do we prepare teachers to use DL in their classrooms? One potential avenue is to provide them opportunities to experience DL themselves. Professional Development (PD) programs are one of the ways that teachers can gain knowledge and skills for teaching by DL methods. In this study, 14 volunteer mathematics teachers participated in a PD program in which they had the opportunity to participate in an eight-week authentic research experience and then taught DL in a one-week Math Camp (MC) for high school students. The theoretical perspective supporting the design of the PD was Professional Learning Communities which is "a group of people sharing and critically interrogating their practice in an ongoing, reflective, collaborative, inclusive, learning-oriented, growth-promoting way" (Stoll et al., 2006). This study explored how this PD and DL practice in the MC changed participants' beliefs and teaching methods? Furthermore, this study provided suggestions to improve the measurement tool for future studies.

This study was a mixed-method. Participants answered an online survey (consisted of 16 items including open-ended and scaled questions about demographic information, teachers' beliefs, teaching methods, and MC effects). After data collection, the authors looked for patterns in responses to open-ended questions for emergent themes, did open coding separately, and then compared the results and agreed on the coding method.

Responses to the scaled questions showed that more than ⅔ of participants believed that this PD caused moderate or significant change in their beliefs (78% mentioned changes in their beliefs about teaching mathematics, and 84% of them mentioned changes in their beliefs about doing mathematics). Furthermore, results showed that after this PD, participants use more exploration methods in their classroom (77 % use exploration at least 1-2 times each week, and 22% of them use exploration three or more times every week). Furthermore, 64% of them found the MC moderately or extremely beneficial.

Responses to the open-ended questions revealed changes in participants' beliefs about their own ability and their students' ability to learn mathematics, and also teachers' expectations from students had changed. Factors mentioned as affecting these changes were MC experience and the DL situation of PD. They mentioned that DL "enhances and deepens students' understanding of mathematics", it "makes mathematics more accessible to students", and "it is a great way to introduce new topics" and "increase student engagement". Also, they mentioned that they more frequently use DL and problem-solving methods after this PD. Learning classroom management skills, designing a week-long lesson and implementing it, and a chance to talk to students one on one and support their conjectures were mentioned as beneficial features of the MC.

This pilot study helped the authors identify some themes about changes in teachers' beliefs and teaching methods that will help in developing questions for interviews. However, it did not show a relation between the effect of change on specific beliefs and their consequent effects on teaching methods. Several modifications were implemented to the research questions (they will be mentioned in the poster presentation) to get more detailed responses to answer this concern.

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MODELING EQUITABLE PRACTICES MATH TEACHER EDUCATORS' REFLECTION AND PRACTICE

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The purpose of this study is to explicate how a group of Texas Mathematics Teacher Educators (MTEs) used the professional development tool of Lesson Study to examine use of an equity framework, namely the Torres' Rights of the Learning (RoTL; Torres, 2020), into their practice. This lesson study served as a peer-professional development in which each of the authors brought their own ideas to the work as a means of collectively improving our practice. The research will outline aspects of Lesson Study, share the development of a task, and the results from the analysis.

The study used a Design Experiment methodology (Cobb et al., 2003) in whIthe group of MTEs refined and revised the lesson while reflecting on their own practice and needs of the students. Lesson Study process encourages teachers to study, plan, do, reflect, and refine (Lewis & Perry, 2014; Lewis et al., 2006). The researchers focused on how teacher candidates can learn about the Torres' Rights of the Learner which are the right to be confused, to claim a mistake; to speak, listen, and be heard; to write do and represent what makes sense (Torres, 2020) through a groupworthy task (Lotan, 2013) about non-standard units of measurement. The lesson first introduced the Torres' ROTL, then showed Cognitively Guided Instruction (Carpenter et al., 2015) of a video with a child solving a mathematics problem and exercising his Torres' ROTL. The lesson took three iterations that varied slightly (e.g., limiting unnecessary dialogue by the MTE and making transitions in and out of the breakout rooms to the main room more smoother). Data collected through the various iterations along with experiences of the MTEs were analyzed to explore and examine MTEs own practice in incorporating equity framework for teacher candidates. The research results will not only describe the experiences of the lesson and how the teacher candidates experience the Torres' RoTL, but also how the process of Lesson Study and collaboration supported our own practices as MTEs, especially during a challenging time as COVID 19.

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THE RELATIONSHIP BETWEEN TEACHING EXPERIENCE AND TEACHERS' CAUSAL REASONING ABOUT POSITIVE AND NEGATIVE STUDENT EVENTS

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Researchers have suggested that how teachers understand the connections between teaching practice and student learning can develop over time through their everyday classroom experience (Hiebert et al., 2003; 2007), which could explain why teaching practice improves as teachers gain more experience (Papay & Kraft, 2015). Teachers' explanations of student events have been shown to influence teaching quality in the classroom (Jackson et al., 2017; Wilhelm, 2017). Furthermore, teaching experience has been linked to improved teaching ability (Copur-Gencturk, 2015; Santagata & Yeh, 2016).

This study explores how teachers use causal reasoning (CR) to explain student events. CR is characterized by "identifying explanatory factors (i.e., causes) that lead to particular outcomes (i.e., effects) in the classroom related to students" (Dyer, 2016). We hypothesize (*H1): Teachers with more experience (ten years or more) will be more likely to explain student events with teacher-oriented explanations (productive causal reasoning) than teachers with less experience (less than ten years).* Additionally, we hypothesize (*H2): Teachers will be less likely to use teacher-oriented explanations (productive causal reasoning) to explain negative student events.*

We analyzed the causal reasoning of ten high school math teachers in four interviews (40 total) from point-of-view observations (Sherin et al., 2011), in which the teachers discuss moments they tag in real time during a lesson. We qualitatively coded instances of causal reasoning ($n=587$) for whether the explanatory factor included the teacher (*teacher-related factor*; $K = 1.0$) and the outcome was negative ($K > .90$). Comparisons between teachers with greater than 10 years of experience to those with less were done using multilevel mixed effects logistic regression models to account for clustering in the data by lesson and moment discussed.

Our results indicate that teachers with more than ten years of experience were more likely to use explanatory factors about teacher-related factors ($\alpha = .7143$, $SE = .3107$, $z = 2.299$, $p = .0215$), and overall teachers were less likely to explain negative outcomes with teacher-related explanations ($\alpha = -0.6493$, $SE = 0.2070$, $z = -3.137$, $p = .00171$). Thus, *H1* and *H2* are supported.

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MDISC FOR GTAS: A DISCOURSE-ORIENTED TEACHING PROFESSIONAL DEVELOPMENT FOR GRADUATE TEACHING ASSISTANTS

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Keywords: Classroom Discourse, Doctoral Education, Professional Development

Many mathematics graduate students receive little or no support for their teaching. As the *Progress through Calculus* (PtC) survey (Apkarian & Kirin, 2017) showed, more than half of the 223 responding institutions did not offer a university-wide teaching professional development (T-PD) to their graduate teaching assistants (GTAs) and a third had no department-specific GTA T-PD. When a T-PD is offered to GTAs, it is typically confined to GTAs' first year of teaching: Fewer than 20% of the institutions who responded to the PtC survey (Apkarian & Kirin, 2017) offered GTAs T-PD beyond their first teaching year. The problems presented by this lack of continued T-PD are three-fold: (a) a yearlong course is insufficient for learning how to teach; (b) even if one could cram all there was to learn about teaching into a yearlong course, one's priorities while teaching for the first time may not be in line with topics that go beyond classroom management (e.g., equity and active learning); and (c) when one looks at available data on GTA T-PD activities (Apkarian & Kirin, 2017), it appears as though topics beyond classroom management may indeed be receiving little attention.

To address the lack of support that mathematics GTAs receive, particularly beyond their first year of teaching, I adapted and offered the "Mathematics Discourse in Secondary Classrooms" (MDISC) (Herbel-Eisenmann et al., 2017) T-PD in spring 2021 to three GTAs who were teaching undergraduate mathematics and who were no longer in their first year of teaching. The MDISC T-PD, among other things, introduces participants to the mathematics register (Pimm, 1987), six teacher discourse moves (Cirillo et al., 2014; Herbel-Eisenmann et al., 2013) (i.e., waiting, inviting student participation, revoicing, asking students to revoice, probing a student's thinking, and creating opportunities to engage with another's reasoning), and positioning. The goals of this poster presentation are to share details of the adapted MDISC T-PD's implementation and findings of a research study involving this GTA T-PD. Regarding the latter, I present the discourse-oriented T-PD I offered as an instrumental case study (Stake, 1995).

As Stake (1995) noted, an instrumental case study is driven by issues rather than the case itself, and issues evolve and emerge as the study progresses. Currently, my focal issues can be summarized through the following questions:

1. Which changes in discourse (as described by the usage of teacher discourse moves and dimensions of the EQUIP [Reinholz & Shah, 2018]) and positioning (of mathematics, oneself, and students) occur in the participants' classrooms over the course of the T-PD?
2. Which aspects of the adapted MDISC T-PD—a T-PD originally developed for secondary school teachers—are perceived relevant by the T-PD participants and why?

The data I draw on to answer these questions consist of: (a) video-recordings of thirteen weekly 2-hour T-PD sessions; (b) reflective memos written by me after every T-PD session; (c) three video-recorded semi-structured interviews with each of the three participants before, during, and after the T-PD; and (d) audio-recordings of all classes taught by the participants in spring 2021. Findings from the data analysis will be shared at the conference.

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DEVELOPING SUSTAINABLE MATH INSTRUCTIONAL LEADERSHIP IN A NETWORK OF UNDER-RESOURCED SCHOOLS

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Keywords: Instructional Leadership, Professional Development, Sustainability

Over the last three years, the Responsive Math Teaching Project (2021) has been developing and refining a model for the development of mathematics instructional leadership in a network of 14 urban under-resourced elementary schools. The project is a research-practice partnership with the School District of Philadelphia, where more than 80% of students live below federal poverty levels, in a state with some of the largest gaps in the country between wealthy and poor districts. The goal of the project is to develop a sustainable model for instructional improvement and improve outcomes for students by increasing opportunities for professional learning situated in and around practice (Ball & Forzani, 2009) and fostering a networked community across schools (Coburn et al., 2012; Penuel et al., 2006).

Drawing on research on professional learning, teacher instructional capacity is built through developing new visions of mathematics teaching and learning (Ebby et al., 2020; Munter, 2014; Wilhelm, 2014), practice-based pedagogies of enactment and reflection (Grossman, 2018; Grossman et al., 2009), and mentored engagement in collaborative lesson design and enactment (Hiebert & Morris, 2012). Teacher leaders first learn what high quality math instruction and instructional leadership looks and feels like, then develop the necessary teaching and coaching skills and practices through practice-based professional development (representation, decomposition and approximation) and support for enactment in the classroom through collaborative planning and coaching. Over time, and with ongoing opportunities for practice, teacher leaders take over professional development facilitation and peer-coaching roles in their own schools, and across the network.

In this poster, we will present our current model which has been refined through cycles of development, implementation, and revision. We will also show how we are continually analyzing data and responding to local conditions to improve and refine our model. Through qualitative analysis of interviews, we trace the development of teacher leaders over time along several dimensions of leadership capacity, highlighting three key shifts that take place in their *instructional leadership vision*. We also show shifting from a school-based coaching model to a virtual cross-school model of collaborative lesson design, enactment, and reflection provided more equitable and sustainable opportunities for teacher learning and instructional support across the network, particularly for those schools that face the greatest challenges around resources.

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Chapter 9:

Policy Leadership & Miscellaneous

MATHEMATICAL MOTHERS: INVESTIGATING SHIFTS IN PERSPECTIVE AROUND WHAT COUNTS AS MATHEMATICS

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Bridging the gap between mathematical learning at home and school has been an issue for education research for decades (Galindo & Sheldon, 2012). Expectations for mathematics do not often align for teachers and parents (Posey-Maddox & Hayley-Lock, 2016) and a limited view of what counts as mathematics persists. What needs more attention is the meaningful mathematical learning that happens at home but is rarely seen as mathematics. Parents frequently struggle in supporting their children's mathematical learning, but that struggle becomes productive when parents are recognized as mathematically capable. This paper shows how two mothers shift their perspectives of what counts as mathematics and recognize the rich content in their current interactions with young children. Making such connections between mathematics and parent action can strengthen the relationship between at-home and school learning.

Keywords: early childhood education, informal education

Many parents and teachers struggle to connect mathematics learning for children at home and in school. In many cases, parents feel that they do not understand school mathematics and struggle to help (Jackson & Remillard, 2005), while teachers wish that parents would be more involved in their children's learning (Wilder, 2017). Parents and teachers are often talking past each other, with different goals for children in mathematical learning. Previous work has been done to help parents engage with the mathematical activity that happens at school (e.g., Blevins-Knabe, Whiteside-Mansell, & Selig, 2007; Starkey & Klein, 2000). However, attention to connecting parents to mathematics is frequently school-centric (Jackson & Remillard, 2005) and ignores the meaningful mathematical interactions that may already be happening at home. This element is often missed because parents do not often see these interactions as mathematical (Goldman, 2005), further exacerbating tensions in parents about their mathematical ability. As such, supporting at-home mathematics research would benefit from helping families recognize the mathematics that already happens in their everyday lives. I explore this issue of at-home mathematics recognition in a case study analysis of two mothers of young children by responding to the question: How do parents begin to change their perception of interactions with mathematics when made aware of the mathematics they already do? Making home connections to mathematics may not be a re-teaching of school mathematics for parents, but eye-opening to what happens already in the everyday that is mathematical and supports meaningful connections to children's mathematical learning.

Literature Perspective

In addressing the disconnect between school and family mathematics learning and considering perception changes in parents, this paper highlights literature around parental engagement, parents' current perceptions of at-home mathematics through Funds of Knowledge literature, and common activities rich with early mathematical context. Together, this literature perspective shows how helping parents recognize the mathematics they already do can support their intentionality and confidence in such engagement.

Many studies investigating parental engagement indicate a difference in the expectations of student learning for parents and teachers (Quaylan & White-Smith, 2018; Wilder, 2017). Some teachers want parents involved in learning by being physically present at the school, but this is not always possible (Posey-Maddox & Hayley-Lock, 2016). Beyond the disconnect of parents' and teachers' expectations of mathematics is a large body of work around at-home mathematics that shows parents are interacting with mathematics in meaningful ways but are not seeing it as mathematical (Pea & Martin, 2010; Skwarchuk et al., 2014). As Goldman (2005) recommends, "getting parents to recognize their life skills as mathematics is a first and necessary step for building more connections for students with mathematics" (p. 71). Goldman's recommendations, and the continued trends around a lack of recognition of at-home activity as mathematical, frames the intent of this study and its results, that parents of even very young children can be highly engaged with mathematics, and with attention to their interactions as mathematical, they may become more intentional in the mathematical learning for their children.

The literature on at-home mathematics that centers Funds of Knowledge (Moll, Amanti, Neff, & Gonzalez, 1992) is one area that recognizes the value of the mathematical activity that occurs in the everyday lives of families. Funds of Knowledge work has shown the significance in validating parents' mathematical skills by increasing engagement in parents, which in turn can support children's mathematical engagement (Gonzalez, Andrade, Civil, & Moll, 2001; Whyte & Karabon, 2016). The present study acts to enhance the intent of Funds of Knowledge literature on parental engagement in mathematics. The existing literature stresses the impact on children and creating stronger school and family relationships. This study highlights the specific impact on parents' mathematical identity when shifting perspectives of themselves as more mathematical thinkers.

Given this study's focus on parents with young children (i.e. pre-kindergarten age), it is important to identify what kinds of activities are likely to occur in families that are rich in mathematics. Significant literature has explored types of mathematics in authentic activity that can be done (or have been done) by parents for the sake of school readiness (e.g., Anderson & Anderson, 2018; Leyva et al., 2017) and include skills such as numeracy, shapes, spatial reasoning, and measurement. At its core, Cannon and Ginsburg (2008) argue that "mathematics education should be fun, be relevant to young children's lives, and build on their fledgling mathematical understandings" "p. 242). This means activities that parents do with their children that are connected to their interests and building on beginning math understandings are meaningful forms of mathematics education. The activities present across the literature in early childhood mathematics become touch-points for activity in the work of the participants of this study, drawing attention to similar activity that they already do, as mathematical activity.

Theoretical Perspective

Supporting parent recognition of their actions as mathematical is tied to concepts of mathematical identity. Drawing from Bishop (2012) and Martin (2000), I frame mathematics identity to mean that how a community recognizes certain actions as mathematical or not will influence an individual's perspective of their own mathematical identity. Mathematics identity is about beliefs of ability, constraints, opportunity, and positioning of self and others in what it means to do mathematics (Martin, 2000). As Bishop (2012) summarizes, "a mathematics identity is dependent on what it means [to] do mathematics in a given community, classroom, or small group" (p. 39). I understand this perspective to mean that how a community recognizes certain actions as mathematical or not will influence an individual's perspective of their own mathematical identity. It is centered around how people see themselves and others, particularly its limitations, as mathematical thinkers, doing work that is mathematical. Previous literature has already indicated that many parents do not see their interactions with children as mathematical (Goldman, 2005) because those actions have not been given recognition as mathematical within their community, whether that be family, schooling experience, or messages from the media. Because identities are malleable (Bishop, 2012), if people are influenced by perspectives around the self, identities can change and grow. If the actions of an individual are given credence as mathematical, then their perspective of themselves as mathematical can strengthen. Mathematical identity is a combination of positioning and action, such that affirming people's actions as mathematical can enhance their mathematical identity. As such, a parent's acceptance of their mathematical identity through recognition of their actions can support more intentional and positive interactions with their children around mathematics.

Study Details and Methods

The results reported for this paper is a case study of two participants from a larger study on the impact of past experience on current mathematical interactions for mothers. While all participants from the larger study demonstrated transitions in their understanding of what counts as mathematics, this paper focuses on the experiences of two participants, given the limited space. The two focal participants, Kelsey and Elizabeth (pseudonyms), are white middle-class mothers living in the Midwest with challenging past math experiences. These mothers are the focal points of this paper because of the interesting transition of their perspectives of what counts as mathematics over the course of the study. The results of Kelsey and Elizabeth shared here are not meant to act as a generalization of what all parents experience, as there are certainly limitations in the range of their perspectives, but the results show what may be possible in shifting parental identity and ultimately engagement with mathematics for families.

The overall study involved a series of three interviews and two observations with debriefs of mothers' past mathematical experience and their current interactions with their children in mathematical activity. All interviews and debriefs were audio recorded and transcribed. Observations were captured with written field notes. The design of the interviews as well as analysis of the data came from a narrative inquiry perspective (Clandinin & Connelly, 2000; McAdams, 1993). Analysis began with reading and rereading the words of the participants, ultimately focusing on themes of the stories told by participants (McAdams, 1993). Further analysis for capturing the participants' meanings and stories was through strategies of meaningful qualitative analysis suggested by Corbin and Strauss (2012): asking questions of the stories told, making comparisons within and across participants, and looking at the emotions expressed in their stories.

The first interview focused on past experience and drawing on memory of what mathematics was like for the participants growing up, in both in-school and out-of-school experiences. Stories shared about past experiences were attached to particular feelings, locations, and individuals. The second interview focused on current practices with mathematics, both for themselves and in interactions with their children. In order to achieve depth in responses that connect to a story-telling model, I related initial questions within Clandinin and Connelly's (2000) three-dimensional space of inquiry, where "stories have temporal dimensions and address temporal matters; they focus on the personal and the social in a balance appropriate to the inquiry; and they occur in specific places or sequences of places" (p. 50). Questions were directed to consider experiences in mathematics at different times within the lives of the participants, consider the personal emotions attached and the social context of the moment, as well as the location and its context to the moment. The final interview acted as a form of authenticating what participants shared across the study, with review of particular transcript excerpts and making changes to make sure the excerpts best reflected the participant.

The observations were an opportunity for participants to show activity that could be mathematical and the debriefs of the observation were a time to reflect on how it went and what mathematics was actually present. My aim was to draw attention to the positive forms of interaction that were discussed in their interviews and carried out in meaningful interactions, to show that the mathematics at home can be rich and meaningful (Pea & Martin, 2010). The debrief was also a time to expand on what is recognized as mathematical. After participants discussed what they saw as mathematical in the observation, I would point out actions that I noticed they did and how it was connected to mathematics based on early childhood mathematics literature (Anderson & Anderson, 2018). In both participants and at every observation, there was at least one moment that involved mathematics that was not initially mentioned by the participants that was mathematical. Participants were then given an opportunity to reflect on how the newly proposed mathematical connection to an activity happened in other interactions they had. The time in the debrief sessions that drew attention to, validated, and allowed for reflection of previously unrecognized mathematical action allowed the participants space to shift their perspectives of what counted as mathematics and their perceptions of themselves as mathematical.

Results

Kelsey and Elizabeth showed throughout this larger study how they were developing new understandings of the mathematics they did on a daily basis, building their confidence and their intentions to engage with their children. Kelsey was a confident and effervescent mother in her late 20s, with a 15-month-old daughter, Amelia (pseudonym). She lived with her family in a medium-sized village in the Midwest. Kelsey worked from home, teaching violin lessons, but was a speech language pathologist before Amelia was born. Kelsey consistently spoke of how hard math was for her to understand, and the extra effort she had to put in to make the good grades she wanted. Elizabeth was a deeply empathetic mother in her late thirties living in the suburbs of a city in the Midwest. She stayed at home with her two daughters, Talia who was almost two and Luna (pseudonyms) who was seven years old, both adopted as infants. Talia was part Hispanic and Elizabeth was incredibly intentional in the experiences she provided to make sure that Talia was surrounded by people, books, and toys that looked like her. Elizabeth identified a constant lack of confidence in understanding mathematics that made her more hesitant to engage in mathematics with her daughters at the start of this study.

Math Disconnection

Early experiences in mathematics for both Kelsey and Elizabeth were challenging and often confusing. Both participants identified issues with understanding mathematics and believing that math was not for them. Kelsey found math frustrating because she had to work significantly harder to understand it than any other subject. Her frustration translated into a constant desire to avoid the subject because she felt math was not for her. Elizabeth struggled with confidence in doing the work, feeling that to be considered a good math student she needed to figure it out on her own. Elizabeth thus avoided math classes she felt she would not excel in on her own. Mathematics did not come easily and was a subject both participants avoided when possible.

Kelsey and Elizabeth did not first believe that what they did with their children was mathematical, and that mathematics was strongly associated with what happened in school. As Kelsey reflected, when she did not feel she did anything with mathematics, "well it's not a worksheet and not a test so I guess I just don't associate math with it" (Interview 2). Elizabeth made similar remarks that indicated an understanding that mathematics was something that happened at school, and for older children, when she shared "my daughter is in first grade so she started doing math already. Um, I don't remember doing math that young but maybe we did" (Interview 1). Although this study was centered on the interactions of mothers with pre-kindergarten aged children, Elizabeth's connections to mathematics frequently returned to her 7-year-old and her school work, struggling to identify what types of mathematics she might be doing with her youngest child. Kelsey and Elizabeth's statements at the start of the study reflected particular perspectives of what mathematics is (problems on worksheets or tests) and when it happens (in school and at later grades) and paralleled their own early experiences with mathematics.

In addition to their remarks about what mathematics is and where it occurs, Kelsey and Elizabeth pointed out how they did not recognize their actions as involving any kind of mathematics. Kelsey's first response to ways she interacted with her young daughter Amelia and mathematics was to laugh and say she did not think she did anything with math. As my study was focused on the interactions of mothers with their pre-kindergarten aged children, I was most interested in Elizabeth's actions with her youngest, Talia. However, Elizabeth's connections to mathematics were almost always centered around her 7-year-old and her school work, struggling to identify what types of mathematics she might be doing with her youngest child. In the debrief of the first observation, Elizabeth claimed "I think, there's a lot that happens that I don't' I don't realize is math." This claim was the first indicator from Elizabeth that her perspective of what math is and what she thought she did with math was limited. Kelsey and Elizabeth's perceptions of mathematics tied to school activities and later grades paralleled the lack of mathematical activity they saw in the interactions they had with their younger children.

Transition of Perspectives

The asking of pointed questions about mathematical activity and affirming actions seen in observation led to a shift in perspective. Following the debrief meetings that pointed out particular instances of mathematical activity that the participants already did were reflections on how they saw those interactions with new eyes, and how many other practices contained meaningful mathematics. It was the participants' reflections on recognizing current activity as mathematically rich that framed their transition of perspective about what mathematics is and their mathematical identity.

Kelsey, as a speech language pathologist, initially talked about how she did not do math but was intentional to connect to language and reading for her daughter. After the second observation, Kelsey reflected on the change in the math language she thought she used before the study:

> I feel like I've noticed a lot more since doing this [study]. Um, just different words…and initially I didn't really, when you were like 'what experiences with math do you have with Amelia?' and I'm like 'hm, nothing, she's 15 months old' [laughs] but now, having thought about it more and you asking questions I've realized how much those, how often those words

> come up and I feel like they come up in everything. In bath time, in meal time, in playtime, in story time in…like everything. So it's the same kind of vocabulary, but in lots of activities (Debrief 2).

Her recognition of the language modeling she did as mathematical helped her to realize that math learning was possible and already occurring with her young daughter. She used language that supported size comparison, amount, compared shapes, and described quantity, in the activities she did every day. This study prompted Kelsey's attention to her actions and word choice as mathematical and already occurring in her interactions.

Elizabeth showed similar recognition of the mathematics she was already doing with her youngest daughter, when before she did not believe that mathematical learning could really happen for a toddler, starting instead in school. During the observations, Elizabeth showed engagement with her youngest in a number of activities that used mathematics: making patterns out of blocks, comparing the size of towers, fitting toy people in a train, and counting sheep in a farm book. For Elizabeth, this growth was in reflecting on ideas of what mathematics could be possible and also prompting from observations the particular interactions that were viewed as mathematical. In taking more time to think about her involvement in mathematics after the second interview, Elizabeth shared how she went to bed thinking of many more activities that they did as a family that involved mathematics but she had never thought of before. Time for reflection on what math happened brought up many other affirming actions that Elizabeth recognized in herself. Additionally, the debrief of observations for Elizabeth prompted numerous interactions that were pointed out by the researcher as mathematical. For example, in pointing out her use of patterns in a play scheme with Legos, Elizabeth explained "you said 'oh do you see patterns in other activities that you do?' and I was like 'wait patterns? This is about patterns. Oh that would be math.'" (Final Interview). Affirming a small activity involving mathematics helped her see it as a mathematical situation and prompted her to make connections to other ways she did or could in the future use mathematics with her children.

Discussion

How Kelsey and Elizabeth understood what mathematics is shaped their perspectives of themselves as doing (or specifically not doing) math. Their past experiences with mathematics shaped their perspectives of what mathematics must be and how they fit into the narrative of mathematical experience. How they framed themselves and mathematics before the study is not new for parents. Their perceptions of mathematics from their past and schooling experiences is pervasive across the United States, with mathematics viewed as doing algorithms and mental calculations (Stevens, 2013b). However, Kelsey and Elizabeth were doing mathematics with their children, in meaningful and important ways. The perspective of what counts and how Kelsey and Elizabeth saw themselves as supporting mathematical learning was already adapting with the naming and recognition of their activities as mathematics. Their initial experience reflected a continued problem in the disconnect between at-home and at-school mathematics, that the meaningful mathematics learning that was already happening in the home was not recognized as mathematics (Anderson & Minke, 2007). The transition in perspective highlights two key features for understanding parents' mathematical engagement: that what counts as mathematics is much broader than what many people would recognize and that much mathematical learning is already happening outside of school even for very young children.

Kelsey and Elizabeth's past experience in school mathematics and initial understanding of what counts as mathematics reflects a limited view of what counts as math: higher level work, tests, and worksheets. These earlier perspectives are built from what their surrounding culture has taught them that math should be, and for them that was centered on what happened in school. As Stevens (2013a) highlights, "what counts as math or science depends on how the culture represents them, and school is but one setting where math and science are represented" (p. 4). Mathematics that happens at home may not involve worksheets or algorithms but can still build important skills in mathematical thinking. What counts as mathematics includes both formal and informal activity. As Skwarchuk et al. (2014) emphasize, formal and informal mathematical learning is important for early mathematical development, and those informal activities that happen at home are part of that learning. Much of the activities that Kelsey and Elizabeth engaged in paralleled activities in existing literature about early math skills (Leyva et al., 2017).

While what counts as mathematics is much broader than what happens in a school setting, this does not mean parents need to change or incorporate specific activities to engage in real world mathematical activity. The recognition of more activity as mathematical shows how those skills are already being learned at home. There is so much mathematics that can and already is happening at home with families. For Kelsey and Elizabeth, this included describing numbers of objects and comparing them in books, talking about objects fitting into other objects during play, and describing patterns in building blocks. Each of these interactions was related to a foundational mathematical concept that before this study, neither participant would have indicated is mathematical. These sample activities of Kelsey and Elizabeth are also part of a larger list of activities parents have been shown to do with their children, engaging in informal mathematical connections (Anderson & Anderson, 2014; Leyva et al., 2017). Much of the interactions with children parents already have are mathematical, even if they do not initially recognize them as such (Goldman, 2005).

Research using a Funds of Knowledge approach (e.g., Whyte & Karabon, 2016) does center families as mathematical knowers. Similar to what occurred in this study, Funds of Knowledge research draws attention to parents' activity and mathematical skills in the context of their lives. For example, in Gonz lez and colleagues' study (2001) they found "that household knowledge is broad and diverse, and may include information about, for example, ranching, farming, and animal husbandry, which are associated with households' rural origins" (p. 117). These activities have embedded mathematics that is not often highlighted or validated as mathematical in a school setting. The body of work within Funds of Knowledge research highlights the value of parents' skills in mathematics, while the current study pushes further to consider the impact of valuing parents' skills on their changing mathematical identities.

The final link between these themes of mathematics as a broader field than what happens in school and recognition of the mathematics that parents already do with their children is shaping parents' mathematical identities. As framed earlier, mathematical identity is about what it means to do mathematics in different settings (Bishop, 2012). In the case of this study, when the participants were positioned as mathematical through the activities they engaged in with their children, it strengthened their own mathematical identities. Kelsey and Elizabeth's past experience with mathematics shaped a particular perspective of who they were in the subject: bad at mathematics. However, reflections and time in the study helped them recognize other ways to engage with mathematics and build confidence in what they did with their children in the subject. Kelsey and Elizabeth's changing perceptions of themselves and mathematics was similar to what Esmonde and colleagues (2013) found in families engaging with mathematics activities at home, that it is more than just "someone who was good or bad at mathematics" (p. 18). The current analysis of Kelsey and Elizabeth digs deeper into Esmonde and colleague's (2013) results by looking at the evolution of perspective for parents with a history of mathematics aversion. The difference being made for these participants shows that pointing out the mathematics that was already happening in their homes helped them recognize their actions as mathematical and feel more confident to encourage types of play that would incorporate that mathematics later.

Significance and Future Work

The experiences of parents and families interacting with mathematics can be significantly different and more diverse than what happened in the lives of Kelsey and Elizabeth. The point of this paper was not to generalize the experiences families have with mathematics and mathematical identities. Instead, it was meant to show what changes are possible in the parent perspective in mathematics and spark more research that engages exploration of parents' mathematical identities and activities with their children. Creating connections between the everyday activity of parents and that of mathematical thinking has the potential to further support children's mathematical development. As Goldman (2005) argued, "getting parents to recognize their life skills as mathematical is a first and necessary step for building more connections for students with mathematics" (p. 71). Recognition and action in mathematics has ties to Bishop's (2012) perspective of mathematical identity. Mathematics identity is built off of the actions of an individual and how those actions are accepted as mathematical by the community. In the case of this study, Kelsey and Elizabeth were already doing mathematics, but needed the recognition that their actions were mathematical. Funds of Knowledge research grounds families' experiences as legitimate mathematics, but can be pushed further to consider the change in mathematical identity of parents when their activity is validated. This change in mathematics perspective for families can build confidence and create further engagement with mathematics, ultimately supporting children's learning.

Mathematics learning is often studied as school-centric, privileging the knowledge and structure of learning that happens in school (Jackson & Remillard, 2005). However, rich and meaningful mathematical learning can and does happen outside of school (Pea & Martin, 2010). Researchers can create better connections between schools and families in mathematics by recognizing and encouraging the mathematical learning that happens from the parent perspective. Stevens (2013b) proposes this call for the research community to "build a conceptual vocabulary that does not take school mathematics as the exclusive reference frame for understanding mathematical work across society and that can follow mathematical practices in and across time and place, including school" (p. 81). This study focused specifically on parents, to recognize the authentic and contextual mathematical experiences as a bridge to children's mathematical learning. Broader vocabulary, broader understanding of experiences and what counts as mathematics has the potential to validate the mathematical activity that happens at home.

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OPPORTUNITIES TO DEVELOP STATISTICAL LITERACY: A COMPARISON OF STATE STANDARDS TO GAISE II

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In today's polarizing political climate, there is a need to build citizens' statistical literacy to combat misinformation and support data-based arguments. To that aim, we investigate K-12 standards documents for their alignment with the American Statistical Association's Guidelines for the Assessment and Instruction in Statistics Education (GAISE II). We found that the states that explicitly reference GAISE or had standards that explicitly addressed the statistical investigative process did not offer consistent opportunities for students to engage in each element of the investigative process and at each developmental level. We discuss the implications of the findings and provide recommendations for policy makers and standards writers.

Keywords: Data Analysis and Statistics, Policy, Standards

The current political climate of the United States is polarizing, often fueled by inflammatory rhetoric. Messaging that attempts to use data-based evidence for proposed policies or statistics to explain the spread of a deadly disease is met with disbelief and the cry of "fake news!" Some politicians have pushed the public to mistrust data and have been aided by people leaning on partisan trust instead of statistical literacy. This has created a heightened sense of urgency among data scientists, journalists, and educators to foster statistical literacy in the citizenry across all age groups. In schools, statistics is generally embedded as a content strand within the larger K-12 mathematics curriculum and may be offered as a separate course for students at the high school level (National Governors Association Center for Best Practices [NGA Center] & Council of Chief State School Officers [CCSSO], 2010). However, many states do not include statistics as a formal area of study within their mathematics courses until students reach the middle grade levels. This has been met with some pushback by researchers (Confrey, 2010) and there has been a concerted effort in providing educators and policymakers with resources to improve access to quality statistics-related content. For instance, the American Statistical Association (ASA) developed and released the *Guidelines for Assessment and Instruction in Statistics* (Franklin et al., 2007), which has had some impact on the grades 6-12 content in the Common Core State Standards for Mathematics. To give further guidance to educators and policymakers on statistics literacy, the ASA has recently published an updated report entitled the *Pre-K-12 Guidelines for Assessment and Instruction in Statistics Education II (GAISE II)* (Bargagliotti et al., 2020). In spite of such guidance, the opportunities to learn statistics through state mathematics standards vary from state to state and many are not aligned to the GAISE reports (Dingman et al., 2013; Newton et al., 2011; Weiland & Sundrani, under review).

Objective

With the current reality of statistical literacy in the U.S. and the resources now available to aid in creating opportunities to learn statistics, policymakers have the opportunity to update and improve state mathematics standards with statistics learning goals in mind. To support such efforts, in this study we investigated present efforts at purposefully incorporating the GAISE framework into state standards. Building on previous findings (Weiland & Sundrani, under review), we utilize state standards documents that explicitly reference the GAISE framework to answer the following research question: *To what extent are state K-12 statistics standards aligned to the GAISE II Framework for states that considered the GAISE report in their revision process?* To answer this question, we will investigate states' current efforts at building statistical literacy through incorporating suggestions from the GAISE framework into standards and to provide policymakers and standards writers with recommendations for future standards work based on our findings.

Background

When the Common Core State Standards for Mathematics (CCSSM) were developed in 2010, they were adopted by 48 states. This move signaled a shift towards a national set of standards designed to provide students with equal opportunities to succeed in mathematics. However, some politicians and citizens felt that this was the federal government's attempt at taking control of K-12 education from states (Orrill, 2016), in spite of it not having a role in the writing of the standards. Further, educators and families were unclear on how to enact the new standards, leading to frustration with the implementation and assessment of the standards. As a result of political pressures, a multitude of states have revisited their mathematics standards. Furthermore many states have revised their standards because of policies that require them to pass new standards after a set number of years (Achieve, 2017). The process to rewrite standards differs from state to state and involve a variety of constituents.

In a larger study, Weiland and Sundrani (under review) found that some states have referenced external sources in their standards revision process. Of the states that have mentioned the use of additional documents, five specifically cite the GAISE framework in their K-12 mathematics standards document – Louisiana, Massachusetts, Ohio, Virginia, and Wyoming. Louisiana, Massachusetts, and Wyoming all include the GAISE report in their updated standards document references, but their standards still do not differ from the CCSSM statistics standards in any substantial way. Ohio has made changes to their statistics standards to explicitly include the GAISE report. Virginia is the only state that did not adopt the CCSSM and has therefore incorporated the GAISE report differently from the CCSSM. Additionally, while Kentucky does not reference the GAISE report in their standards document, they do explicitly use the four-step statistical investigative process referenced in the GAISE report in their grade 1-6 standards. This inclusion of the investigative process may come from textbook *Elementary and Middle School Mathematics Teaching Developmentally*, which makes use the GAISE report (Van de Walle et al., 2019), and is used as an external reference document within the Kentucky state standards (Kentucky Department of Education, 2019). The only other state to explicitly name the statistical investigative process in their standards is Ohio, but only in grades 6 and 7.

GAISE II Framework

The GAISE reports were developed by the statistics education community to support the development of statistical literacy at the K-12 level (Bargagliotti et al., 2020; Franklin et al., 2007). The reports emphasize the need for students to understand statistical concepts and reasoning. The reports differ from other policy documents as they do not detail standards to cover at each grade level, rather they provide three levels of development (i.e., level A, B, and C) around the statistical investigative process. The investigative cycle includes four steps: formulate question, collect/consider data, analyze data, and interpret data. Though the three levels seemingly follow a grade band trajectory, the GAISE authors clarify that a student cannot progress to Level B unless they have mastered Level A skills regardless of their age or grade. The second iteration of the report, GAISE II keeps the core of the original report, adds on to the framework, updates the language, and provides more recent examples. We use the GAISE II report in our standards analysis because it has been developed by the statistics education community, provides more detailed guidance than other standards documents, and has been recommended by the National Council of Teachers of Mathematics (2020).

Methods

Data Sources and Collection

The data for this study includes official standards documents from states in the U.S. that reference the GAISE report. The states were identified one of two ways, searching for "GAISE" or "Guidelines for Assessment and Instruction in Statistics Education" within the state's standards documents , or from a larger analysis, where we noticed a close alignment with GAISE framework, though not explicitly stated. The only states that met the criteria of referencing the GAISE framework and differing from the CCSSM were Ohio and Virginia. Louisiana, Massachusetts, and Wyoming referenced the GAISE report, but did not meaningfully alter their standards from the CCSSM, so we considered them together as a case using the CCSSM standards. Kentucky was also identified for this study. Although Kentucky did not mention the GAISE report within their standards document, the standards incorporated the statistical investigative process in a way that was clearly aligned to the GAISE framework.

Identifying Learning Expectancies

Because states use different structures to organize their standards, we decided to analyze what we call learning expectancies (LEs). Learning expectancies represent the lowest unit of standard designation that provide a unique learning objective within the official standards documents analyzed. Virginia's standards only incorporate one level of standards, so these were taken as the LEs in that state. Kentucky and Ohio standards may include two or more sub-standards that elaborate on the top-level statement, so the sub-standards were taken as the LEs, in place of the top-level statement (see Figure 1).

~~*KY.7.SP.2:* Use data from a random sample to draw inferences about a population with an unknown characteristic of interest.~~

> *KY.7.SP.2.*a. Generate multiple samples of categorical data of the same size to gauge the variation in estimates or predictions.
> *KY.7.SP.2.*b. Generate multiple samples (or simulated samples) of numerical data to gauge the variation in estimates or predictions.
> *KY.7.SP.2.*c. Gauge how far off an estimate or prediction might be related to a population character of interest.

Figure 1: Example of Standards from the grade 7 Kentucky Mathematics Standards

At the elementary level, many states do not include a formal statistics strand, but do include a Measurement & Data strand that include statistics-related content. Therefore, we included any standards from the Measurement & Data strand that were statistical in nature. At the middle and high school grade bands, we included standards from the Statistics & Probability strand. However, we did exclude a number of probability LEs that focused on the mathematical aspects of theoretical probability (Bargagliotti et al., 2020).

Analytical Framework

In order for the findings to be useful to policymakers, standards writers, statistics education researchers, and educators, we analyzed the data utilizing the GAISE framework as our lens. We used a binary coding to identify which process element(s) each learning expectancy addressed and also determined what developmental level was appropriate for each LE. It is possible that a single LE could encompass multiple process elements and developmental levels, depending on the language used. We discussed and agreed upon all coding to ensure inter rater reliability.

Results

The number of statistics LEs vary by state and differ substantially by grade level (see Table 1).

Table 1: Number of Statistics LEs by State and Grade Level

	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**A1**	**A2**	**HS**	**Total**
CCSSM	2	1	2	2	1	1	8	7	4			31	59
Kentucky	2	4	4	7	3	1	13	10	3			30	77
Ohio	2	1	2	2	1	1	11	11	4			31	66
Virginia	2	2	3	2	3	7	5	5	6	2	3		40
Total	**8**	**8**	**11**	**13**	**8**	**10**	**37**	**33**	**17**	**2**	**3**	**92**	**242**

Kentucky and Ohio utilize a great deal of the CCSSM language in their LEs, but have added, edited, or deleted some of the verbiage. Additionally, while it seems that Kentucky includes the greatest number of statistics LEs, many of them are smaller, discrete concepts covered in a single LE in the CCSSM. For instance, the CCSSM includes the following standard, "Generate measurement data by measuring lengths using rulers marked with halves and fourths of an inch. Show the data by making a line plot, where the horizontal scale is marked off in appropriate units— whole numbers, halves, or quarters." This LE is broken up into two in Kentucky – "Generate measurement data by measuring lengths using rulers marked with halves and fourths of an inch" and "Show the data by making a dot plot where the horizontal scale is marked off in appropriate units – whole numbers, halves, or quarters." Virginia, being the only state to not adopt the CCSSM in this study, greatly differs in content and number of LEs in each grade level. It is also important to note that Virginia does not combine their high school level standards into one grade band. Instead, their standards are separated by mathematics course. For the purpose of this study, we only included LEs Algebra 1 and Algebra 2, which are required for graduation in the state (see Table 1).

Alignment to GAISE II

The data were coded with respect to the statistical investigative process elements and developmental level. *Formulate question* LEs (5%) lead students to create or verify questions that are statistical in nature. *Collect and consider data* LEs (31%) focus on data collection strategies, bias, and simulations. *Analyze* LEs (74%) ask students to make meaning of a data set's variability and distribution and create visualizations. *Interpret* LEs (45%) concentrate on summarizing, drawing conclusions, and making predictions based on the context in statistical problems. Statistical process elements are not mutually exclusive; one LE may be coded as one element or as many as all four elements. Because of the large overlap in LE language, Kentucky, Ohio, and the CCSSM all include a similar proportion of collect and analyze LEs in their state standards documents (see Figure 2). Ohio and the CCSSM also include similar proportions for the formulate question and interpret LEs. Kentucky deviates slightly – 10% of this state's LEs incorporate the formulate question element compared with 3% and 5% in the CCSSM and Ohio respectively and include 13% less interpret LEs than the CCSSM and Ohio. Virginia's LEs differ significantly, as the state does not include any formulate question LEs and incorporates the analyze element in almost every statistics-related LE.

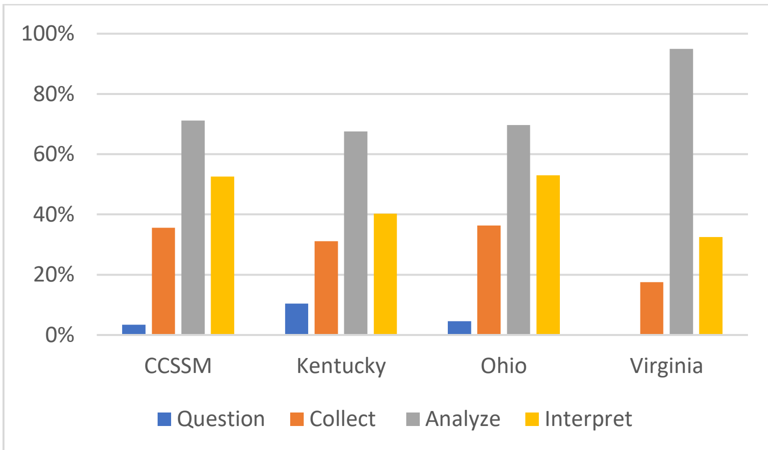

Figure 2: Bar Graph of Proportion of LEs for Each Statistical Investigative Process Element by State

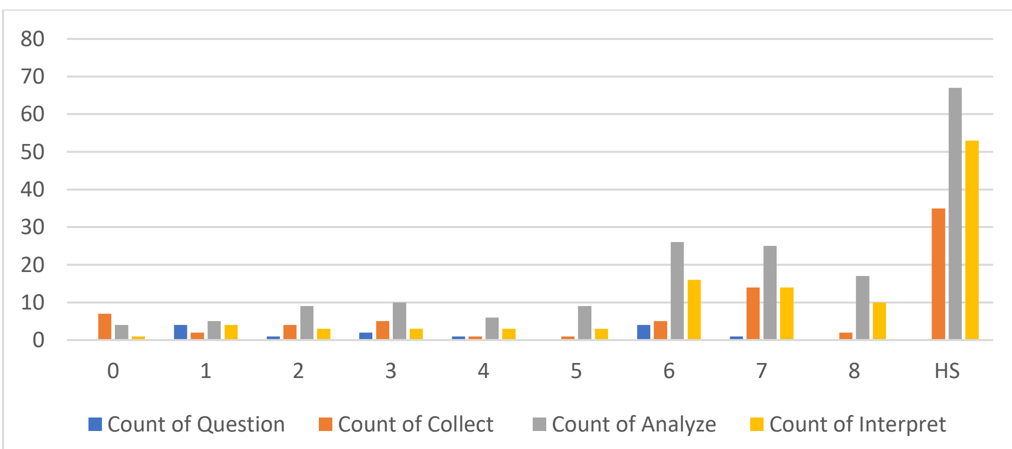

Figure 3: Bar Graph of Total Number of LEs by Statistical Investigative Process Element and Grade Level (N=242)

As the standards documents progress from grade band to grade band, the number of LEs generally increases as well (see Figure 3). In the CCSSM, Kentucky, and Ohio, as the standards progress from elementary to middle school grade levels, students have more access to the collect, analyze, and interpret elements. In the middle grade levels, formulate question LEs stay the same in the CCSSM, decrease in Kentucky, and increase by one in Ohio as compared to elementary grades. Virginia's statistics LEs provide students with more opportunities to analyze data in the middle grade levels, equal opportunities to interpret data in elementary and middle schools, no opportunities to formulate questions in either grade band, and less collect data LEs at the middle grade levels. The transition to high school adds more LEs on the collect, analyze, and interpret elements of the statistical investigative process in the CCSSM, Kentucky, and Ohio. It is important to note that while the number of statistics LEs increased in the high school grade band, this is potentially distributed over four grade levels. Again, Virginia differs and only provides a handful of experiences in the Algebra I and Algebra II courses offered to students. Overall, analyze data LEs make up the majority of students' experiences with data in the K-12 setting in all states, followed by interpret data, and then collect and consider data. Lastly, formulate question LEs are absent at the high school level in all four standards documents, continuing the pattern that as students move to higher grade levels, they have fewer opportunities to engage in the statistical questioning. Therefore, students have almost no experiences with formulating questions in the K-12 setting.

Approximately 26% of all LEs are at developmental level A, 34% are at level B, 41% of LEs are at level C, and 1% of LEs' developmental level is unclear typically due to vague wording. Kentucky, Ohio, and the CCSSM provide similar opportunities at each developmental level, whereas Virginia substantially differs (see Figure 2). Level A LEs are clustered in the elementary grade levels in all four state standards documents, while level B are mostly present in middle grade levels, and level C LEs are clustered in the high school grade levels. Grade six serves as a transition year as students move from level A LEs to level B. Virginia is the only state that does not provide any learning opportunities at level A for grade six students. During the middle grade years, most LEs are at level B, with a few experiences at level C. In high school, students gradually move from level B to level C LEs. Again, Virginia differs from the other three sets of standards, as it only includes level C LEs at the high school level. The CCSSM, Kentucky, and Ohio seem to align their LEs' developmental level with the recommendations from the GAISE II report. Formulate question LEs are only present at levels A and B in the CCSSM, Kentucky, and Ohio and as previously mentioned, formulate equations LEs do not appear in Virginia's standards. Collect LEs appear throughout the grade levels and cover all three developmental levels in all standards documents, except Virginia which does not include any collect LEs at level B. Analyze LEs are also split between levels A through C, with gradually more LEs at each level; again, the only exception is Virginia. This state provides equal opportunities for students to engage in statistical reasoning at levels A and B, but fewer opportunities at level C. Interpret LEs greatly increase as the developmental level increases with LEs almost doubling for as the level progresses in the CCSSM, Kentucky, and Ohio. This may mean that standards developers place increasingly more importance on the interpret element of the statistical investigative process but may not have provided enough opportunities at the earlier levels to support this move.

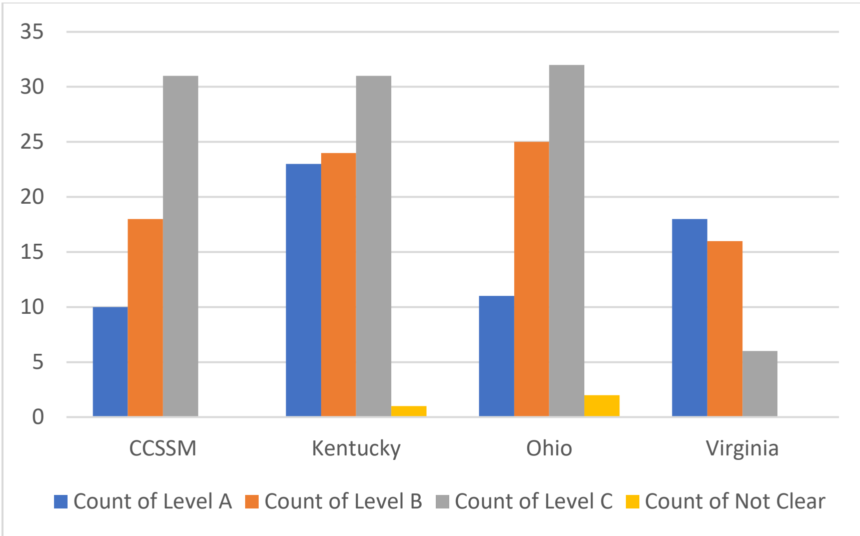

Figure 4: Bar Graph of Statistical Investigative Process Element of Total LEs by Developmental Level and State (N=242)

The GAISE II framework also details the need for students to experience different elements of the statistical investigative process in tandem. Therefore, it is also important to identify how many of the process elements are linked through the LEs in each state. The CCSSM and Ohio provide similar linkages, though Ohio's percentages of overlap between different process elements are slightly more spread out across the grade levels (e.g., the CCSSM provides no linkages in grade five, while about 2% of Ohio's overlap occurs in grade five). Both of these standards documents include some overlap between process components for over 80% of their LEs. The greatest proportion of connected elements is between analyze and interpret, followed by collect and analyze. Kentucky, as discussed above, has made a concerted effort to separate CCSSM language into smaller, discrete LEs. As a result, approximately 62% of Kentucky's LEs have some elemental overlap, compared with the CCSSM's 88% overlap. This has also created no opportunities for students in Kentucky to engage in multiple aspects of the statistical investigative process in grades K, one, and four. However, just like the CCSSM and Ohio, most overlap occurs between the analyze and interpret elements and then collect and analyze. Virginia offers the least proportion of linkages between process components, with only 55% of LEs combining two or more elements. Virginia also follows suit with the other three states in providing the most linkages between analyze and interpret and then collect and analyze.

Discussion and Implications

Overall, the CCSSM, Kentucky, Ohio, and Virginia offer some alignment to the GAISE II framework supporting student's development of statistical literacy to take on the demands of our data rich society. The most developed area is the support for students to move between developmental levels. Also, each standards document incorporates LEs that span the statistical investigative process, but do so in varying ways. Virginia, as the only state in this analysis that did not adopt the CCSSM, significantly veers from the statistical content covered and provides fewer opportunities for students to engage in each process element throughout K-12.

Through our analysis, we found few formulate question LEs. Virginia is the only state that did not include any LEs of this type, and Ohio and the CCSSM only included two and three LEs, respectively. These documents provide one LE at level A and the others at level B. Additionally, the first two opportunities to formulate questions is separated by four grade levels, making it difficult for students to build on prior knowledge. Kentucky was the only state to provide multiple opportunities for students to ask and identify statistical questions in grades one through six but did not incorporate any opportunities at the high school level. This element is crucial in teaching statistical literacy, as it is central to the statistical investigative process. Although all four states utilized the GAISE framework in designing their standards, there is still a need to purposefully include this element within the LEs and to connect it to other process elements across grade levels. Standards writers should consider including formulate questions LEs throughout K-12 and explicitly connect this element to the collect, analyze, and interpret elements to give students opportunities to engage in the entire statistical investigative process.

Another important consideration is the deliberate incorporation of all four elements in each grade level. Students need to have experiences with each process component throughout their K-12 careers to fully understand the purpose of each. Further, through repeated experiences with all four components at different grade levels, students will gain more consistent instruction at each developmental level. Kentucky creates the most consistent experience for students to engage in each component of the statistical process throughout K-12, with Virginia, Ohio, and the CCSSM following close behind. The CCSSM creates opportunities for students to engage in the entire statistical investigative process but does so sporadically. Ohio does not include opportunities for students to collect or consider data in grades one, four, and five. While the state does include more collect LEs overall, students do not receive instruction on collecting and considering data between grades four and five and are then expected to learn content at developmental level B in grade six. In addition, while each process component plays its role in developing statistical reasoning, students need to be exposed to the connections between each. Currently, there is significant overlap between process elements in the CCSSM and Ohio standards documents. Kentucky and Virginia do not include as many links, but do still connect at least two elements across their K-12 statistics LEs. However, most of the linkages offered in each state's LEs exist between the analyze and interpret elements. Without a solid grasp of how analyzing and interpreting data begins with statistical questioning and data collection, students will have difficulty developing statistical literacy. Therefore, we recommend that standards writers consistently incorporate LEs that provide students opportunities to experience each process element in isolation and together throughout K-12. Further, we recommend that researchers explore the impact standards on the instruction students receive at the classroom level.

One type of LE that was included in state standards documents that was missing from the GAISE II report is establishing the difference between correlation and causation. While this is an important statistical concept that every student should have the opportunity to learn, it is all but absent from the GAISE framework. Additionally, there is no formal mention of the normal distribution within the framework, despite the central importance of this concept within the statistics field. Therefore, it is important to note that the GAISE framework is an invaluable document, but should be supplemented by other statistics-related guidance.

While Kentucky, Ohio, and Virginia have used the GAISE report while rewriting their mathematics standards, each state has taken a different route to achieve this goal. Ohio's writing teams explicitly used the GAISE report when updating the mathematics standards and did not reference any other external resource in their standards document (Ohio Department of Education, 2017). Virginia's standards writing committee developed their mathematics standards using a number of impactful resources, such as the National Council of Teachers of

Mathematics's (NCTM) *Principles and Standards for School Mathematics* and the GAISE report (Virginia Board of Education, 2016). Kentucky did not include the GAISE report, but the standards writers in the lower level grades utilized the textbook *Elementary and Middle School Mathematics Teaching Developmentally*, which explicitly makes use the GAISE report in their tenth edition (Kentucky Department of Education, 2019; Van de Walle et al., 2019).

Each of the states analyzed referenced the GAISE framework when developing their standards. Despite this, each varied in their alignment to the statistical investigative process at each developmental level. This is due to a multitude of factors. Of note is the CCSSM, as Ohio and Kentucky have kept much of the language from this set of standards, influencing how much statistics could be incorporated into the standards documents. There also seems to be a disconnect between grade bands, particularly at the high school level. Additionally, statistics LEs are embedded within the larger mathematics LEs, restricting its space to one content thread among many others. Therefore, utilizing the GAISE framework does not seem to provide enough guidance to ensure appropriate alignment with the statistical investigative process. Policymakers and standards writers should aim to include teachers and university faculty with a background in statistics education in the standards writing process at all grade bands.

It is through the deliberate and consistent inclusion of the statistical investigative process throughout K-12 schooling and accompanying statistics concepts that students may develop their statistical literacy to become well-informed citizens, capable of interrogating data and the sources they come from.

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POSITIONING OF NOVICE ELEMENTARY TEACHER LEADERS IN ADVICE AND INFORMATION NETWORKS FOR MATHEMATICS

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In this paper we examine how teachers who are pursuing their Elementary Mathematics Specialist certification—Elementary Mathematics Specialists in Training (EMSTs)—are positioned in their advice and information networks for mathematics. We analyzed the instructional networks of six elementary schools in one Midwestern school district. Our analysis suggests that EMSTs did occupy central positions in their networks. EMSTs were sought out by more individuals compared to other teachers, and when sought out by others, provided advice and information at a greater frequency than formal leaders. We also considered the school's informal and formal structure, finding that EMSTs' positioning was related to the broader school's information seeking behavior and whether there is a math-specific formal leader.

Keywords: leadership; elementary mathematics specialists; social network analysis

Teacher leaders have the potential to play an important role in supporting instructional improvement. While some teacher leaders occupy formal, full-time positions (e.g., as coaches), many continue as full-time classroom teachers. Though they are not afforded dedicated time for leadership, teacher leaders with full-time classroom responsibilities may be more likely to engage with other teachers about classroom instruction and viewed as more credible sources than formal leaders (Spillane & Kim, 2012). Therefore, a potentially productive type of informal leadership that teacher leaders can enact is offering advice and information about mathematics teaching and learning. The extent to which teacher leaders can engage in such leadership, however, depends on the context of their schools. A collegial and collaborative school culture, for example, supports teacher leadership, while hierarchical and formal designations can increase distance between teachers (York-Barr & Duke, 2004). In this paper, we examine how novice elementary mathematics teacher leaders who maintain full-time classroom responsibilities are positioned in the instructional and advice networks for mathematics, and how their positioning might be related to the school's informal and formal leadership structures.

Theoretical Framings & Related Literature

In line with research that takes a distributed perspective (Spillane et al., 2004), we recognize leadership as extending to those with no formally designated position, and as the product of *interactions* between leaders, followers, and their situation. The situation shapes teacher leaders' interactions with others and includes, for example, school norms, structures and routines (e.g., grade-level teams), and formal positions (e.g., presence of an instructional coach) (Diamond & Spillane, 2016). Thus, teacher leaders' social influence interactions—providing advice and information about mathematics teaching and learning—constitute a form of leadership. We focus on advice- and information-giving because such leadership activities improve mathematics teaching through a variety of professional supports, including increasing teachers' knowledge about the learning and teaching of mathematics (Gigante & Firestone, 2008). Social network analysis allows us to examine social influence interactions while simultaneously attending to the school's formal structure and how it constitutes said interactions (Moolenaar & Daly, 2012).

Previous research on advice and information networks in mathematics suggests that, while principals do not figure prominently in their school's networks, formal leaders with subject-specific positions are the most central, with teacher leaders offering advice and information to more people than other teachers (Spillane & Kim, 2012; Spillane & Hopkins, 2013). This research has primarily focused on *centrality*—the extent to which an individual is connected to others—and less so on the nature of those interactions. A notable exception is the study by Coburn and Russell (2008) which investigated how district policy shapes teachers' social networks, including the frequency of interactions (*strength*), the substance of those interactions (*depth*), and the extent to which those interactions span different functional areas (*span*).

Research has identified factors of the situation that support and constrain teacher leadership. In general, the literature suggests that inadequate time for collaboration and traditional top-down structures can inhibit teacher leadership, while cultural norms of openness and trust, positive working relationships, and support from school administration support it (Wenner & Campbell, 2017; York-Barr & Duke, 2004). Because leadership, particularly exercised through social influence interactions, is grounded in authority and legitimacy (Diamond & Spillane, 2016), the positioning of teacher leaders as knowledgeable and expert is crucial. Therefore, our study builds on this literature by investigating how novice teacher leaders are positioned—in terms of centrality, span, strength, and depth—in their networks for mathematics instruction, and how their positioning might be related to their situation. Specifically, the research questions that guided our investigation are: 1) How are novice teacher leaders positioned in their advice and information networks for mathematics, especially compared to teachers and formal leaders? 2) How is their positioning related to school level factors?

Methods

Study Context

The data analyzed for this paper is part of a larger project in which 24 teachers in a Midwestern state received funding to complete Elementary Mathematics Specialists (EMS) certification and serve as informal leaders in their schools. Data was collected in Fall 2019, the first year of teachers' participation in their EMS programs. Because teachers were not formal leaders, nor necessarily identified by school administration or colleagues as experts, we consider them novice teacher leaders, or Elementary Mathematics Specialists in Training (EMSTs). In this paper, we focus on survey data from six elementary schools (Briar, Palm, Reed, Rowan, Thorn, Woods) in one participating district. Thirteen EMSTs worked together in school-based teams, ranging in size from 1-3 EMSTs in each of the six schools. As part of their graduate coursework, each team was asked to distribute a survey to the teachers in their school, analyze the results, and use the results to inform a plan for improving support for mathematics instruction at their school.

Data

The survey included items related to advice- and information-seeking interactions in mathematics, which were based on those developed and validated in other studies (Pitts & Spillane, 2009). In particular, we asked "During this past school year, is there a person in your building or district you have turned to for advice or information about teaching mathematics?" (Middle School Mathematics and the Institutional Setting of Teaching, n.d.). Respondents listed up to three individuals, and for each of those individuals, were also asked "how often do you seek advice or information from this person" and "what type(s) of advice or information do you seek from this person? Please check all options that apply." The options for these questions are described in the analysis section, which we turn to next.

Analysis

For each individual that responded or was named, using the school and district websites, we collected data for the individual's role (e.g., leader, teacher), associated site (e.g., school, central office), and, if applicable, grade level. Using the social network data, we calculated centrality, and span, strength, and depth of relationships (ties) of each individual. *Degree centrality* measures how well connected an actor is in a network (Freeman, 1979), and can be broken into in-degree—the number of people who sought out that actor for advice and information—and out-degree—the number of people that actor sought out. *Betweenness centrality* measures brokering and the extent an actor connects two other actors in the network (Freeman, 1979). Specifically, betweenness measures the number of shortest paths between two other actors that go through a given actor.

In addition to centrality, we also calculated measures to describe ties actors had with others. For each, we considered whether the tie *spanned* outside the actor's grade level (1 = yes, 0 = no). For ties with teachers that taught multiple grade levels, if the two teachers had at least one overlapping grade, we considered this as not spanning grade levels. For *strength*, we considered the frequency of the interactions, with four options: a few times a year (1), once or twice per month (2), once or twice per week (3), and daily or almost daily (4). For *depth*, we based our definitions on those of Coburn and Russell (2008), with three options: low (1), medium (2), and high (3) (see Table 1). Because respondents were able to select multiple options, we calculated an average depth, in addition to whether or not the interaction included at least one high-depth activity (1 = yes, 0 = no). For any relation between two actors, there are two possible ties, one from actor A to actor B, and the other from actor B to actor A. For example, if actor A responded that she asked actor B for advice daily, then the strength of actor A's out-tie with actor B and strength of Actor B's in-tie with actor A would be 4. For individuals that were named but did not respond to the survey (e.g., formal leaders), we only computed measures related for in-ties, including in-degree centrality and associated strength and depth; span was not relevant since respondents (teachers), by definition of role, were outside leaders' functional area.

Table 1: Depth of Interactions

Depth	Types of Advice and Information
Low	• Discussing pacing • Sharing materials or activities • After a lesson, sharing whether students "got it" • Updating one another on a student or students' progress in mathematics
Medium	• Discussing what materials to use for a lesson • Analyzing student work to see if students "got it" • Discussing why some students didn't learn as expected in a lesson in order to plan for future success • Doing mathematics problems together with discussions of different solution strategies
High	• Discussing different ways students are likely to solve tasks • Analyzing examples of student work to understand the different ways that students solve problems • Analyzing examples of student work in order to adjust instruction • Discussing how to make use of student solution strategies in whole class mathematical discussions

To examine the positioning of EMSTs in their networks, for the first research question, we compared the measures previously described (centrality, span, strength, depth) for EMSTs, teachers, and formal leaders and tested differences for significance using analysis of variances with permutation tests. Because social network data are not independent, we used UCINET software (Borgatti et al., 2002) to conduct a random replication procedure with 5000 permutations (Carrington et al., 2005; Spillane & Kim, 2013). Because respondents had ties with those outside their school, including district leaders, we did not limit networks to those of the school.

For the second research question, to examine the relation between EMSTs' positioning and school factors, we limited networks to those of the school. First, we explored to see if there were between-school differences in how EMSTs were positioned. To account for the size of the network, we normalized centrality by expressing it as a percentage of the maximum possible centrality an individual could have had. The school factors that we investigated included the school's formal (e.g., whether there was a mathematics-specific formal leader) and informal (e.g., advice- and information-seeking behavior) structure. Regarding the latter, we calculated network *density* for each school. Network density is the total number of ties divided by the total number of possible ties. In addition, we compared the average span, strength, and depth of ties between schools and tested differences for significance using analysis of variances with 5000 permutations. To illustrate our findings, we share network diagrams of three schools, selected based on contextual variation. Some of this variation included the size of schools, the nature and density of school networks, and whether there was a mathematics-specific formal leader.

Findings

First, we describe the positioning of EMSTs in their advice and information networks for mathematics, especially compared to other teachers and formal leaders. Then, we turn our attention to school networks and how EMSTs' positioning might be related to school factors.

EMSTs' Positioning in District Network

Overall, the EMSTs in our study occupied central positions in their advice and information networks for mathematics (see Table 2). Specifically, EMSTs were sought out by more individuals than other teachers (in-degree, $p < 0.01$), and were more often positioned as brokers for advice or information (betweenness, $p < 0.001$). All of the EMSTs had at least one tie, while 19.35% of teachers had no ties. There were no significant differences in advice-seeking behavior (out-degree), nor differences in span, strength, or depth of ties.

Only three formal leaders were named as individuals whom teachers sought out for advice and information, and none of those included school principals. The three formal leaders named were the district mathematics coordinator, an instructional mentor in the district special education department, and the Title I Math teacher at Woods (Title I is a United States government program in which schools with high levels of low-income students receive federal funding which can be used to hire additional teachers or instructional aides (United States Department of Education, n.d.). While these three formal leaders did have the highest average in-degree, because we did not ask formal leaders to complete the survey, we were not able to compare centralization between formal leaders and EMSTs. We were, however, able to compare the strength and depth of the ties that were reported. When sought out by others, EMSTs provided advice and information at a greater frequency than formal leaders (strength, $p < 0.05$), but the depth—average and whether or not the interaction included at least one high-depth activity—did not differ significantly.

Table 2: Means and Standard Deviations of Centrality and Tie Dimensions by Position

	EMSTs	Teachers	Formal Leaders
N	13	124	3
Betweenness	5.231 (10.892)	0.548 (1.876)	
In:			
Degree	2.077 (1.979)	0.694 (0.785)	7.333 (4.643)
Tie Span	0.193 (0.267)	0.086 (0.273)	
Tie Strength	2.686 (0.682)	2.924 (0.620)	1.389 (0.550)
Tie Depth (Avg)	1.902 (0.215)	1.808 (0.269)	1.675 (0.139)
Tie Depth (High)	0.765 (0.377)	0.737 (0.413)	0.398 (0.308)
Out:			
Degree	1.385 (1.003)	0.944 (1.117)	
Tie Span	0.533 (0.420)	0.290 (0.413)	
Tie Strength	2.518 (1.011)	2.522 (0.913)	
Tie Depth (Avg)	2.008 (0.273)	1.825 (0.368)	
Tie Depth (High)	0.900 (0.200)	0.674 (0.434)	

Note: As a reminder to the reader, tie span refers to whether it extended beyond a teacher's grade level, strength refers to frequency, and depth refers to the substance of an interaction.

EMSTs' Positioning in School Networks and Related Factors

There was great variation in the centrality among our EMSTs. We found that, while the normalized in-degree averaged 0.098, it ranged from 0 (EMST was not named as a provider of advice or information) to 0.333. Similarly, normalized out-degree averaged 0.045 and ranged from 0 to 0.136, and normalized betweenness averaged 0.095 and ranged from 0 to 1.183. Because of this variation, we wondered how differences might be related to school-level factors. In particular, we looked at the school's advice- and information-seeking behavior (see Table 3) and the formal structure—whether there was a formal mathematics leader. Regarding the former, we found significant between-school differences for span ($p < 0.05$) and strength ($p < 0.01$) of in-ties, and span ($p < 0.01$) and strength ($p = 0.08$) of out-ties, suggesting that some schools had more frequent sharing of information, particularly across grade levels. The only school in our sample with a formal mathematics leader was Woods. To illustrate our findings, we focus on and share network diagrams for three schools: Briar, Rowan, Woods (see Figure 1). Individuals were labeled by role and grade, with those teaching multiple grades labeled as "Other."

Briar was one of the larger schools in our sample but had the lowest network density. A significant number of teachers (31%) had no relationships with others, though the ties that were present were quite frequent (second highest strength). The Briar network also had more substantive interactions (an above average depth rating), but they were only within grade levels. We see these school-level patterns repeated in the ties EMSTs had with colleagues at their schools. Only one EMST sought out colleagues for advice and information, with the other two being sought out by others. And, all EMST ties were with peers teaching at the same grade.

By contrast, the network at Rowan was the densest (i.e., had the most total ties relative to possible ties). However, teachers' ties were not as frequent or deep, though this might be because of the higher proportion of interactions that spanned grade levels (often teachers seeking EMSTs for advice). In addition to being sought out, EMSTs at Rowan also went to colleagues for advice and information. Because of this, the EMSTs at Rowan connected and brokered advice and information about mathematics across the first and 3-5 grade levels. Though the kindergarten and

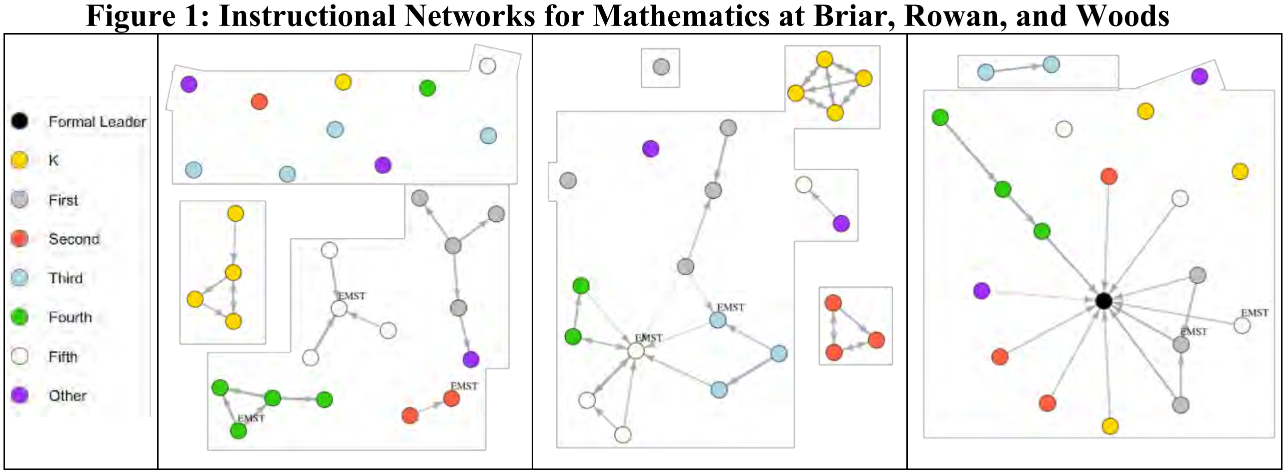
Table 3: Density and Means and Standard Deviations of Tie Dimensions by School

	Briar	Palm	Reed	Rowan	Thorn	Woods
N	29	32	23	23	10	20
Density	0.021	0.029	0.043	0.065	0.044	0.045
In Ties:						
Tie Span	0	0.015	0.028	0.214	0.333	0.200
		(0.071)	(0.118)	(0.385)	(0.471)	(0.447)
Strength	3.083	3.121	2.694	2.702	1.500	3.033
	(0.633)	(0.517)	(0.518)	(0.717)	(0.707)	(0.650)
Depth (Avg)	1.992	1.825	1.843	1.754	1.667	1.694
	(0.186)	(0.216)	(0.308)	(0.224)	(0.923)	(0.232)
Depth (High)	0.875	0.894	0.639	0.659	0.500	0.607
	(0.311)	(0.255)	(0.479)	(0.439)	(0.707)	(0.487)
Out Ties:						
Span	0	0.036	0.083	0.287	0.500	0.750
		(0.133)	(0.289)	(0.399)	(0.577)	(0.380)
Strength	2.917	3.000	2.778	2.546	1.750	2.392
	(0.793)	(0.784)	(0.641)	(0.885)	(1.500)	(0.738)
Depth (Avg)	1.941	1.883	1.907	1.815	2.000	1.774
	(0.250)	(0.328)	(0.326)	(0.347)	(0.816)	(0.365)
Depth (High)	0.833	0.857	0.667	0.667	0.750	0.607
	(0.389)	(0.363)	(0.449)	(0.424)	(0.500)	(0.487)

second grade teachers were isolated from those in other grade levels, they had fairly reciprocal relationships as teachers reported ties with one another.

Woods was the only school with a mathematics-specific formal leader. Though the school had an average network density, a large majority of ties were to the formal leader. Ties spanned outside the grade level, though this was, again, only to the formal leader. So, similar to the ties at Briar, teachers were isolated from those outside their grade level, and sometimes, even from those in the same grade. Ties were somewhat frequent but were less substantive (relatively low depth). Both EMSTs only sought the formal leader, and only one had others seeking her for advice and information.

Figure 1: Instructional Networks for Mathematics at Briar, Rowan, and Woods



Discussion

There is little research on the positioning of (novice) teacher leaders with full-time classroom responsibilities, and this research primarily considers centrality. Our findings align with this literature—that EMSTs were more central than teachers, but not as central as formally designated leaders with subject-specific positions—and also adds to it by examining the nature of social influence interactions. We found that, when sought out by others, EMSTs in our study provided advice and information at a greater frequency than formal leaders. This is important because interactions with greater frequency facilitate the learning of complex knowledge (Coburn & Russell, 2008), which teachers need to improve their instruction. For example, researchers have found that interactions with colleagues who have developed more ambitious instructional visions can support improvements in teachers' own visions, particularly in cases where interactions were more frequent (Munter & Wilhelm, 2020).

Findings from our study also add detail regarding how teacher leaders' positioning is related to school structures. First, EMSTs' interactions with their colleagues were similar to the overall school advice- and information-seeking behavior (e.g., density and if interactions spanned grade levels). Second, similar to prior research that identified subject-specific formal leaders as the most central actors in school networks (Spillane & Kim, 2012; Spillane & Hopkins, 2013), at Woods, the majority of interactions, including those of the EMSTs, went to the formal mathematics leader. One interpretation of these findings is that school norms of collaboration and views of expertise shape teachers' advice- and information-seeking behavior, particularly who they turn to (Wenner & Campbell, 2017; York-Barr & Duke, 2004). And the extent teachers at a school interact and communicate regularly, particularly with those outside their grade level, influences whether and how they seek information from teacher leaders. Also significant is whether colleagues perceive teacher leaders as knowledgeable. For schools with a mathematics-specific formal leader, like Woods, EMSTs' expertise might be undervalued by their colleagues.

Implications

One of the implications of our findings is related to the division and coordination of leadership between formal leaders and teachers who exercise leadership through informal means. Because effective professional development typically includes sustained learning opportunities over time and sensitivity to local contexts (Sztajn, Borko, & Smith, 2018), it seems that there are opportunities for formal leaders to enlist novice leaders with mathematical expertise in change efforts. That is, teacher leaders like the EMSTs in our study could serve as brokers for efforts initiated at the district level, as well as sources of information with regard to teachers' perspectives and impressions of these efforts. Formal leaders could explicitly position teacher leaders as resources for ongoing conversations about mathematics teaching and learning, including serving as leaders of professional learning teams, book studies, video clubs, etc.

Our findings also highlight the limited nature of some of the information networks that exist in schools, limitations that could be explicitly attended to by school leadership. For schools with only grade-level connections, like Briar, it might be helpful to leverage teacher leaders as agents for promoting across grade-level collaborations. The presence of a formal mathematics specialist at Woods seemed to promote advice-seeking, but such interactions were dominated by the formal leader. Positioning teacher leaders with expertise and authority could support more collaboration among teachers and teacher leaders. And, for schools with a robust network of within and across grade-level connections like Rowan, teacher leaders can be mobilized to support bottom-up change across a school by, for example, creating additional opportunities for teachers to share their practice, visit classrooms, and talk with colleagues teaching at different grade levels.

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MATHEMATICS DOCTORAL STUDENTS' PRIORITIES: WHAT THEY LOOK FOR WHEN CHOOSING A PH.D. PROGRAM

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Mathematics doctoral programs have high attrition rates, however, the research examining students' experiences in doctoral programs is limited. The work presented in this paper examines the priorities doctoral students describe when transferring to a new Ph.D. program in mathematics. Although our participants provide both climate and programmatic desires for a program, the two most influential reasons for choosing a new program were wanting a good fit mathematically and professionally between the program and their career goals and wanting more opportunities to do research. These results have implications for the design of mathematics doctoral programs.

Keywords: Affect, Emotion, Beliefs, and Attitudes; Doctoral Education

The work presented in this paper looks at mathematics graduate students' priorities when choosing and transferring to a new doctoral program in mathematics. This work has implications for understanding what mathematics graduate students are looking for in a doctoral program and reasons for why they might leave.

Theoretical Lens

When considering the experiences of mathematics doctoral students, we draw on the theory of *communities of practice* (Lave & Wenger, 1991; Wenger, 1998). Mathematicians, as a cultural community, have developed their own common activities, practices, language, thought processes, and beliefs and therefore can be considered a community of practice. New members are able to join such a community through *legitimate peripheral participation*, a process in which *newcomers* are educated and transformed into *full-members* and eventually *old-timers*, who must then draw in more newcomers (Lave & Wenger, 1991). This transformation process comes from being accepted as a legitimate member of the community, like an apprentice, who focuses on participating in small parts of the practice within the community. As the newcomer's experience increases, they are given further tasks that are more central to the practice to learn, also providing a more central perspective from which to observe the practice as a whole. We view the graduate school experience as playing this role.

Literature Review and Purpose of the Study

Although only a small body of research has been conducted on the experiences of mathematics doctoral students, specific themes can be found within the literature. Most notable is the role that faculty-student relationships play in the success and perseverance of graduate students in mathematics (Borum & Walker, 2012; Herzig, 2002, 2004). When supported and encouraged by faculty, mathematics graduate students are more likely to feel a part of the department and overcome programmatic and personal obstacles. However, when graduate students feel ignored by their faculty or lack the mentoring they desire, it leads to high attrition rates (Borum & Walker, 2012; Herzig, 2002, 2004; Sumpter, 2014a). Furthermore, one's choice of advisor and dissertation topic also seem to be influential in graduate students' success and satisfaction with their program. Graduate students describe wanting an advisor who is approachable, shows an interest in them, and encourages them (Earl-Novell, 2006; Herzig, 2004). Similarly, having a manageable dissertation topic, and ownership of the topic, also influences success within the program (Earl-Novell, 2006; Morton & Thornley, 2001).

In addition to support from faculty, experiencing social and scholarly support from peers is crucial to the success of graduate students in mathematics (Borum & Walker, 2012; Morton & Thornley, 2001; Sumpter, 2014a, 2014b). Unfortunately, due to narrow research interests and small programs, graduate students in mathematics often feel isolation from their peers, at least with respect to research interests (Morton & Thornley, 2001).

A lack of resources and financial hardship are also hinderances to the success of graduate students in mathematics. Although many graduate students in mathematics are funded through their programs as graduate teaching assistants, they often experience mixed emotions regarding the need to teach while completing their programs. While they appreciate the opportunity to teach, they also want to be able to do so in small amounts, so that it does not interfere with their studies (Earl-Novell, 2006; Morton & Thornley, 2001).

While some research has been conducted to examine graduate students' experiences in mathematics doctoral programs, more work remains to be done. The work presented here addresses the following research question: What do Ph.D. students in mathematics prioritize when choosing and transferring to a new graduate program?

Method

The work presented here is part of a larger study examining the experiences of graduate students who left a Ph.D. program in mathematics to either transfer to a doctoral program at another university or to leave graduate mathematics altogether. In this paper we will focus on the participants who transferred to an alternate program.

Participants

For this subset, we identified 10 graduate students who transferred from one mathematics Ph.D. program to another and invited them for interviews. Eight (80%) agreed to participate. All eight were male; four were international students. To protect participant identities, the international pseudonyms we use do not necessarily match with the international participants.

Interviews and Data Analysis

To avoid bias, a researcher outside of mathematics administered the interviews. The interviews were conducted electronically and ranged in length from 20 to 45 minutes, with an average of 30 minutes. The interview protocol focused on questions such as why they chose their original graduate program, why they left, why they chose their next graduate program, and what they believed were the strengths and weaknesses of any graduate program they had enrolled in.

After the interviews were transcribed, we coded the data following Campbell and colleagues (2013). We discussed all coding together and reconciled any disagreements. After coding was complete, we examined all data instances for each code, looking for commonalities across participants and identifying themes addressing our research question.

Results

Participants believe several priorities are important when choosing a graduate program and choosing whether to remain in a graduate program. In particular, the following themes were identified as important: a good fit mathematically and professionally, a positive and friendly peer climate, approachable faculty, more academic options, and the opportunity to do research.

Participants also identified practical concerns such as location, funding, and acceptance as reasons for picking or remaining in a graduate program, though only if other priorities were also met. In this paper, we will be focusing only on the two most influential themes mentioned above.

A Good Fit Mathematically and Professionally

All participants in the study said they were looking for a good fit mathematically when choosing a graduate program. Generally, this mathematical fit was described as a fit between them, their mathematical interests, and the faculty who were available to do research within that mathematical sub-field. Ajit described leaving his previous program because he did not believe he had this fit. He said it “didn't feel like I was a good fit… with the few faculty that were in my area.” When Hayden was asked if he would recommend his previous program, he said:

> It depends on which area they are working… suppose somebody wants to work in algebraic geometry and, currently, they don’t have any people in algebraic geometry, I would not recommend them..So, it depends on which area they want to work in.

Participants expressed a similar concern about the shortage of graduate students working in a specific sub-field as well. For example, Hayden said of his previous program that there was a “very limited number of graduate students in a certain area of the mathematics. So… I could not find a big group to work with, work in my area.”

Participants also described the importance of having a good fit between the school and the level of challenge provided by classes. For example, Eric said he left his previous program partially due to the lack of challenge, which he found appropriate at his new school:

> I probably said this before, that [at my previous program] I wasn't being challenged much. My thing is that if I can do better, I really want to push to do better and I could say that. I felt that at some point I was not doing the best I can, and I think that, in retrospect, I think that is true because after coming here [to my new program], I pushed myself and now… I'm functioning on a higher level. Like I have a better understanding of math and things like that.

Participants also wanted a good fit between the program and their other professional goals. For example, Fareed explained that a lack of fit between his professional goals and his previous program was the main reason he chose to switch programs:

> There was a trend that I noticed about graduating PhD students [at my former university]. Not a lot of them were applying for further research, as in, for post docs or anything. A lot of them were going for, like college jobs, which is OK, but like I wasn't’ I'm an international student. I want to go back home, and I want to do a job there. So, I need to have more research experience before I can start the job. That's a requirement. So, I wanted to do a postdoc.

Fareed explained that because other doctoral students in his former program did not pursue postdocs after earning their degrees, he would not know how or where to apply. Instead, he wanted a program that was designed to transition doctoral students into postdocs.

Opportunities to Do Research

Six participants said they wanted or liked having a graduate program that gave them the opportunity to do research, meet outside researchers, or attend outside seminars or conferences on research. Eric described a strength of the program he transferred to as:

> There's a lot of activity going on in the Department in terms of people visiting from outside, there's a place… right next to our University that hosts people from all over the world and we

> get to attend a lot of lectures and talks going on. There is so much more activity in just in terms of seminars and talking to people.

In contrast, about his previous program, Eric said, "I guess, to put it succinctly, would be that I'd probably needed a lot more exposure and interaction with people from other, other universities and things like that." Cameron also described the exposure to outside researchers as a benefit of his new program, "I think also that there, ah, the number of Departmental seminars and talks and all of these things. They're also much more common in my current Department, so we get to interact more with researchers in the field."

Discussion

Similar to the previous literature on this topic, our study found that graduate students provided both programmatic as well as climate reasons for what they deemed as important in a doctoral program in mathematics. Interestingly, although the participants in our study stated that having a positive departmental climate, including approachable faculty and a positive peer environment, was an important characteristic for a graduate program, they prioritized programmatic features. In fact, many of them left a program that they found welcoming and supportive for a program that better fit their programmatic needs. In particular, one participant even suggested that it was a necessary trade-off.

While there were many programmatic criteria that students mentioned as important belonging to a program that was a good fit for them either mathematically or professionally seemed to be the most important criteria. In particular, mathematics graduate students found it crucial that there was a cohort of faculty and graduate students in the department working in their particular sub-field of mathematics. Even when their previous program had one or two faculty members specializing in their area of expertise, they felt that their options were limited and opted to transfer to an institution that provided them with more choices for their future dissertation advisor. This seems to be a common difficulty that many smaller graduate programs in mathematics face (Morton & Thornley, 2001).

Another programmatic feature that graduate students found important was having direct exposure to research and outside members of the mathematics research community. These graduate students wanted the opportunity to work with faculty on original research, even prior to embarking on their own dissertation project. Furthermore, they believed that being able to attend conferences or seminars with external speakers was critical to their success as future research mathematicians. We view this as the mathematics graduate students expressing the desire to become *legitimate peripheral participants* within the community of mathematicians by being exposed to tasks central to the work of an academic mathematician (Lave & Wenger, 1991).

Overall, the results of our study suggest that graduate students in mathematics prioritize programmatic features that they believe will provide them with a research community and research opportunities while they are in their doctoral program. We find these results to be promising, even for smaller doctoral programs that might have a limited number of faculty within their department. For example, to provide their graduate students with such opportunities, departments can redesign their curricula or program requirements to provide research opportunities early on for their graduate students. Furthermore, while department budgets may be tight, smaller departments may find it worth the investment to fund graduate students to attend national conferences or to bring external speakers to the university to present research colloquia. Modest investments such as these may pay off in the retention of doctoral students in the program, as well as provide the students with experiences that will assist them in their transition into *full-members* of the community of mathematicians.

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CAREGIVER-CHILD INTERACTIONS: INFORMAL WAYS OF DOING MATHEMATICS DURING ENGINEERING TASKS

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Caregiver's involvement in their child's engagement of everyday mathematical practices is foundational to children's learning and doing of mathematics in out-school contexts. The purpose of this study is to understand how applications of math concepts and practices unfolded between children and caregivers during making-engineering activities in their home environments. Through the analysis of approximately 5.65 hours of video collected from four families, we observed caregivers and children involved in three mathematical practices – informal measurement, spatial reasoning, and curiosity. In this paper, we present how informal measurement unfolded differently among two child-caregiver dyads within one making-engineering activity – creating a rain gauge. We demonstrate how physical objects and caregiver guidance afforded children a way to externalize their mathematical thinking.

Keywords: Everyday mathematical practices, Caregivers, Making

Purpose of the Study

Regardless of caregivers' own experiences as mathematics students and negative feelings and emotions towards mathematics, they are able to act as mathematical educators and support their children's learning and doing of mathematics at home (e.g., Berkowitz et al., 2015; LeFevre et al., 2010; Sheldon & Epstein, 2005). Some of these opportunities to engage children as mathematical learners are not grounded in what may be considered more formal ways of learning and doing mathematics, but framed within everyday family experiences such as gardening, cooking, budgeting, and sports (e.g., Esmonde et al., 2012; Goldman & Booker, 2009; Jay et al., 2018). Engaging in engineering design processes may also serve as an everyday experience to support children's application of math concepts and practices (e.g., Berland et al., 2014); yet, little is known how this may unfold between children and caregivers in their home environments.

We intend to look for alternative ways to acknowledge caregivers as math educators by addressing the following research question - –ow might caregivers and children engage in everyday mathematical practices through participation in making-engineering kits in their home environments? These kits were developed as part of a larger project aimed at integrating engineering design practices into home environments of families with children in grades 3-6. Hence, these kits were not developed with increasing caregiver-child interactions specific to mathematical concepts and practices. Yet, in this paper, we demonstrate how caregivers and children can engage in making-engineering tasks that provide opportunities to utilize everyday mathematical practices in their home environments. The significance of this study lies within new possibilities for engaging caregivers and children in the doing and thinking of mathematics within their home environments. The development and use of making-engineering kits may provide opportunities to legitimize ways of doing mathematics outside of school contexts (Masingila et al., 2011), opportunities aligned with humanistic and work-based perspective of mathematics (D'Ambrosio & D'Ambrosio, 2013; Masingila, 1993).

Perspective

This study is informed by mathematical practices for making (Simpson & Kastberg, under review). This framework builds upon Civil's (2002) everyday mathematical practices and situated within any learning environment (e.g., kitchen, garage, museum) and activity that includes creating, designing, transforming and/or repurposing materials and objects into a new physical or digital product (Vossoughi & Bevan, 2014). As an example, within this study, pasta noodles, pipe cleaners, rubber bands, and paperclips were used to construct a container of any shape and size. Simpson and Kastberg (under review) characterized mathematical practices for making as (a) mutual opportunities to teach and learn from one others (e.g., caregiver-child) through social interactions (Bevan et al., 2018, Petrich et al., 2013); (b) imaginative play and/or exploration of mathematics within contextualized activities (Featherstone, 2000, Petrich et al., 2013); (c) personalization of the process and/or object through mathematical strategies, tools, and/or risk-taking (Lindsey et al., 2018); and (d) hidden mathematics or mathematics that is not apparent to those involved as it does not resemble mathematics in school settings (Smith III et al., 2011). These four characterizations were utilized to identify opportunities that caregivers and children engaged in everyday mathematical practices in their home environments.

Methods

The context for this study was a community-based program that invited families with at least one child in grade 3-6 to engage in engineering design processes as part of their home environment. The program consisted of two phases and this study utilized data collected from the second phase the program - the incorporation of take-home engineering kits developed by our team. Five engineering kits were distributed; these are Package for Delivery, Rain Gauge, Trendy Tennies, Joystick, and Blooming Flower. Guided by a set of instructional cards, these kits engaged families in different engineering design stages. The kits also included reusable materials that ranged from "no tech" (e.g., straws, felt) to "low tech" (LED lights, battery pack).

Participants

A total of four families that lived in the Midwest region of the United States were included in this study. Together, these family participants included 1 female and 6 male child participants between 6-12 years of age and 4 female and 1 male caregiver participants. The self-identified ethnicity of the child participants included 2 African American, 4 Caucasian, and 1 self-identified as two or more ethnicities. Caregivers' educational backgrounds ranged from a bachelor to a doctoral degree and two caregivers had a career in a STEM field and/or some experience related to STEM.

Data Source

The data source for this study was video recordings of caregiver-child dyads interacting with one another in their home environment. The data was collected through a tablet distributed to families together with the kits. Caregivers were instructed to record their interactions from the beginning to the end of their engagement with each kit. They were also informed that finishing the kit in one sitting is not required. The duration of each video depended on families' progress with the tasks and ranged from 25 min to 96 min with an average of 48 min. A total of 5.65 hours of video data were collected and analyzed.

Data Analysis

The first and third authors watched each video, identifying events that were characteristic of the mathematical practices for making. We each identified 15 events and agreement for these events were 100%. These events were then transcribed and included both verbal and non-verbal acts of communication (Ochs, 1979), as well as descriptive and analytical memos regarding the mathematical concepts that unfolded between caregiver and child as experienced within the context of the engineering kits (Birks et al., 2008). These memos informed the specific mathematical practices for making observed in the videos and were discussed in more detail and depth between the first and second authors.

Results

Across the video data, we observed children involved in three mathematical practices for making while engaging in the various making-engineering kits in their home environment – informal measurement, spatial reasoning, and curiosity. In this paper, we describe and share examples for one mathematical practice for making – informal measurement. Informal measurement in this study was broadly defined as intuitive approaches grounded in cultural experiences and meaningful contexts in out-of-school settings (e.g., Owens & Kaleva, 2007). The goal is to illustrate the ways that two children-caregiver dyads engaged in the doing and thinking of mathematics during the creation of a rain gauge to measure the amount of rain fall over a period of time. They were challenged to create a simple circuit so that LED lights would light up when a certain amount of rainfall had accumulated (e.g., 1 cm=green light, 2 cm = blue light). See https://youtu.be/MRv1VsA7RBM for an example (Maltese, 2020). One of the steps in the kit instruction asked them to use the provided ruler to measure and mark the vertical distance (e.g., cm, mm, or inches) on the outside of a clear cup (i.e., rain gauge). Specific to the following two examples, the children were observed using visual intuition (Cox, 2013; Owens & Kaleva, 2007) to reason through how the shape of the cup (i.e., rain gauge) affected how the amount of rain fall should be measured.

Example 1

The first example is from Sara (child) and Amanda (caregiver). Sara was curious and questioned why they were being asked to measure the vertical distance of a cup and not volume. She reasoned, "Because if it's a cup, the volume will change through it, so it wouldn't really be accurate if you measured it in centimeter." Amanda, Sara's caregiver, supported this curiosity as they further explored different-size measuring spoons to determine an appropriate and visual amount of water to "mark" as a place to insert a LED light. Sara stated, "I'll do 30 [mL]." The transcript below occurred as Sara marked 120 mL on the cup as a line, while Amanda adds another 15 mL of water into the cup. Actions are italicized and first names abbreviated to their first initial.

S: *Looks closely at the cup.* See right now, it's barely going up from 15, and 30 is only going to be a little.
A: Why do you think that is?
S: Because like…It's because it's spacing out a little. *Takes two fingers – thumb and pointer finger – and seemed to place along an upper and a lower line previously marked.* So these two are the same. *Moved two fingers down to next upper and lower lines.* These two are the same. No, those two are…
A: So the cup gets wider as it goes up?
S: Yeah, and so it's going to get smaller and smaller and smaller. *Pinching motion with two fingers.*
A: So you wanted to measure something and have more visual height, what would you want to do for your cup?

S: I would want to make it…like a tube.

In the first line of the transcript, we observed Sara visually noticed that the space between the lines of 120mL and 150 mL is "barely going up." She further described this as the space between any two lines as getting smaller and smaller as the cup gets wider and wider as it goes up. Sara used her fingers as a way to informally compare the difference in vertical distance between the lines and considered this in relation to another measurement, volume. We further acknowledge how engagement in this mathematical practice for making was supported through Amanda's knowledge of Sara and in the questions posed.

Example 2

The second example is from Roberto (child) and Jared (caregiver). This dyad spent time thinking through how to appropriately mark the vertical distance on the cup. Roberto was observed placing the ruler vertically on the inside of the cup, turning the cup on its side and laying the ruler alongside the top, and placing the ruler across the diameter of the bottom of the cup as three examples. The transcript begins as Roberto is adding lines to the cup to represent the vertical distance for the LED lights.

J: Now you do realize that…as we go up, what changes as we go up with this, which would make it slightly less accurate.

R: Because it like curves up. I mean like at the bottom it's the smallest and then it starts getting bigger and bigger and bigger. *Moved hand from bottom to top of the cup.* I have an idea. We can measure the bottom…*Picked up the ruler and laid across the bottom of the cup*...and see how much it is and then take like all the of the way. And then when we know that number…*circled his hand around the top of the cup*…and when we get the answer, we can subtract that much from it.

Similar to Amanda, Jared noted how the shape of the cup would lead to an inaccurate measurement. Using everyday language, Roberto noted how the circumference gets bigger from the bottom to the top of the cup. Roberto understood this inaccuracy of the vertical distance between each line or LED light in relation to the circumference of the cup.

Conclusion

In this paper, we illustrated how caregiver-child dyads engaged in a specialized form of everyday mathematical practices (Civil, 2002) - mathematical practices for making (Simpson & Kastberg, under review) - during a making-engineering kit within their home environments. We provided two examples above in which the shape of the cup and the support of caregivers through questioning afforded the children a way to externalize their mathematical thinking within an authentic and "worldly" problem (Civil, 2016). We further contend that this study highlights new possibilities for mathematics learning and teaching in out-of-school contexts while positioning caregivers in the role of educator based on their own humanistic and work-based perspectives of mathematics (e.g., Bartlo & Sitomer, 2008; Jay et al., 2018). Specifically, the two children intuitively conceptualized the relationships between the vertical distance between lines on the cup and either the volume or the circumference as a quantity (Thompson, 1994). "A person is thinking of a quantity when he or she conceives a quality of an object in such a way that this conceptual entails the quality's measurability" (p. 184). It is an indirect relationship, which may serve as a seed for the children's understanding of inverse proportional relationships.

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"FIRST TEACHERS TALK, TALK, NOW THEY LISTEN, LISTEN": GHANAIAN TEACHERS' PROFESSIONAL READINESS FOR MATHEMATICS REFORM

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This study examined one Ghanaian teacher's, Eugenia, professional readiness to make significant curricula and pedagogical changes within the context of a nationwide educational reform initiative. Using Scaccia et al.'s (2015) organizational readiness framework, we examined the alignment among Eugenia's professional readiness, what the reforms were asking of her, the supports provided to her to fulfil these demands, and what she was actually doing in her classroom. We found that Eugenia's beliefs, emotions, and efficacy primed her for implementing the reform. We discuss the importance of teachers' knowledge, beliefs, emotions, and efficacy as indicators of professional readiness and how they bode well for successful reform implementation.

Keywords: Curriculum, Instructional Activities and Practices, Policy, Systemic Change, Professional Development

Education reform initiatives often fall short of intended effectiveness due to insufficient capacities or mediocre stakeholder commitment (Elliot & Mihalic, 2004; Gottfredson & Gottfredson, 2002). Theoretical or empirical grounding for an initiative is insufficient to ensure a positive outcome. The availability of relevant intellectual and physical resources coupled with the motivation to implement the initiative is essential (Markle, 2016). For educational reform that foregrounds curricular and pedagogical changes, effective implementation requires teacher capacity and buy-in. In this regard, teachers' professional readiness serves as a good indicator of success potential. In this study, we draw on Scaccia and colleagues' (2015) readiness framework to explore one Ghanaian teacher's professional readiness for nationwide mathematics education reform and the ways in which her interpretations of the reform were translated into her instruction.

Ghana Education Context

In September 2019, Ghana underwent its third national education reform initiative (National Council for Curriculum and Assessment, 2019). With respect to math, the reform involved major shifts in teaching emphasizing the importance of applying inquiry-based approaches (GhanaWeb, 2019). This focus meant a shift in the roles of teachers, moving from lecturer or knowledge bearer to serving as facilitators, prioritizing students' cognitive engagement to support the development of their critical thinking and problem-solving skills. With respect to content and teaching, these articulated changes align very closely with the content of educational standards documents in the U.S. [e.g., Principles and Standards for School Mathematics [2000] *Principles to Action*, (NCTM, 2014)]; however, significant differences exist in the educational landscape across countries. First, Ghana's new curriculum includes a strong focus on cultural identity and global citizenship that is explicitly articulated in the math standards documents (Aboagye & Yawson, 2020). Second, the reform requires a reconceptualization of the roles of teacher and student. However, cultural expectations of Ghanaian elders and youth (i.e., teachers and students in the context of the classroom), tends to elevate the voice of the elder – an expectation that shapes teacher-student interactions.

Teachers' Professional Readiness

We draw on Scaccia et al.'s (2015) organizational readiness framework which posits that readiness is a combination of general capacities, innovation-based capacities, and motivation. We foreground aspects of innovation-specific capacities and motivation. One central component of innovation-specific capacity is stakeholders' knowledge, skill, and abilities that align with the needs of the initiative. Within the context of reform implementation, teachers' knowledge and skill capacity are critical for ensuring effective assimilation of newer ideas into their existing practices. There is a strong correlation between mathematical knowledge for teaching [MKT; see Ball et al., 2008)], which combines pedagogical and content knowledge and skills, and teachers' mathematics quality of instruction (Hill et al., 2008). We consider teachers' professional readiness to implement mathematics reform to be their investment in, and motivation to, implement the reform for which their cognition (math knowledge), psychology (beliefs, efficacy), and affect (emotions) are key indicators. In particular, strong efficacy beliefs enhance teachers' abilities to navigate challenging situations, thus, making them more likely to take professional risks and to persevere and persist (Guskey, 1988; Pintrich & Schunk, 1996; Stein & Wang 1988). Thus, teachers with strong efficacy beliefs are more likely be open and willing to implement reform. Emotions is an important construct in teachers' relationships, instructional decision-making, and overall well-being (Zembylas & Schutz, 2016). Positive high-arousal emotions (e.g., excitement), tend to focus attention on events that are desirable or rewarding (Tamir & Robinson, 2007). Thus, if an event or experience elicits positive emotion, then the teacher will be inclined to increase the frequency of that experience. A plethora of research (Cross, 2009; Cross Francis 2015; Ernest, 1989; Fives & Gill, 2015) suggest that teachers' math-related beliefs influence their instructional practices, teacher-student relations, and how they organize their classrooms. In this regard, there will be greater likelihood of implementation success, if teachers' beliefs are aligned with core reform mandates.

We consider these factors to be very influential in teachers' professional readiness. In this regard, teachers who exhibit reform-supportive beliefs, emotions, and knowledge, to be primed, and highly motivated to implement the reform, thereby increasing the likelihood of positive outcomes. Our study is guided by the following questions*: (i) In what ways were Ghanaian primary teachers primed to implement national mathematics education reform? (ii) How do Ghanaian primary teachers translate reform mandates in their instructional practices?*

Methods

Data were collected during Fall 2019 - seven weeks after reform implementation began. This study is situated within a larger study including eight primary teachers. In this case study, we focus on Eugenia, to develop an in-depth understanding of how her readiness, coupled with reform-based supports, were translated into practice.

Participants. Eugenia was a fourth-grade teacher at a public school. She was identified by one of the district's educational supervisors as one of the best math teachers to assimilate the practices outlined in the reform documents into her instruction. Additionally, intermediate analysis of data showed Eugenia's strong enthusiasm for the reform, exuding confidence and enjoyment related to teaching mathematics. In-depth analysis of her interview and survey data confirmed our observations showing that she was primed for the transition.

Data Sources and Analysis

Interview. Eugenia completed one semi-structured, 30-minute interview. Questions focused on her knowledge, thoughts, and feelings about the reform specific to mathematics. We analyzed her responses by developing a codebook based on Scaccia et al.'s (2015) framework. All three authors coded the same transcript, then discussed any discrepancies in our coding until an agreement was reached, leading to refinement of the descriptions or addition of new codes.

Surveys. The participants completed the *adapted* Self-Efficacy Teaching and Knowledge Instrument for Science Teachers (SETAKIST) survey (Roberts & Henson, 2000), to capture teaching efficacy measuring confidence related to the teaching of mathematics and knowledge efficacy measuring teachers' confidence about their knowledge of math*;* Teacher Emotion Scale (TES: Frenzel et al., 2016), to capture teachers' discrete emotions in relation to their teaching; the NCTM published scale *Principles to action: Ensuring mathematics success for all* (PtA), to capture teachers' mathematics-related beliefs.

MKT Survey. The participants responses on the multiple-choice items from the Learning Mathematics for Teaching Instrument (Hill et al., 2008) were used to get a measure of teachers' content knowledge and knowledge of content and students. As the LMT instrument is designed to be used to determine MKT changes in groups over time, we used the scores as an indicator of MKT relative to other participants and not as an absolute measure of MKT.

Instructional Videos. Each participant taught a 45 – 60 minutes mathematics lesson which was videorecorded. Videos were analyzed using the Mathematical Quality of Instruction (MQI) rubric to determine the quality of instruction (see Hill et al., 2008; https://cepr.harvard.edu/mqi). Acknowledging a different cultural context, we excluded items that we deemed required Ghanaian linguistic and cultural knowledge (e.g., *Imprecise language and notation*).

Findings

Analyses of the survey results showed that Eugenia was primed for reform implementation. With respect to MKT, Eugenia had the highest score (of the 8 teachers) for knowledge of content and students (KCS = 0.65), and the second highest score for mathematical content knowledge (CK = 0.54). She was very confident about her mathematical knowledge (4.25) and had medium level confidence about her ability to teach in ways that supported students' learning (3.38). She experienced high levels of enjoyment (4) and had strong beliefs related to teaching and learning (4). She supports the use of mathematical tools to help students communicate their ideas (4.17). Eugenia expressed she was eager to assimilate the reforms into their instruction because they promoted a pedagogical shift supportive of students' development. Although the reform included changes to mathematics content and what should be taught at each grade, she emphasized that the most significant change was to how they (teachers) were teaching,

> *… the new one is more child centered, yeah. The target is the child, and the child should be allowed to do more activities. The child should be helped to discover things for themselves. So, the teacher is just serving as a guide or a facilitator, helping the child to achieve what he wants to achieve within that period.* (interview)

She noted that the reforms mandated the teachers to engage students in learning so they could explore mathematics. She recognized that teachers should not "tell, tell, tell", rather they should be a guide to support students' meaning making. Eugenia excited and surprisingly positive, about the reform. Eugenia explained that "the children we have today want to explore … so this [curriculum] will help them to have a firsthand context". Eugenia's knowledge, beliefs, emotions, and efficacy indicated that she was highly motivated and primed for reform implementation. Eugenia seemed to be an advocate for the reforms because it facilitated student engagement so students could learn, discover, and find mathematics interesting.

Translating Reforms into Practice

Given Eugenia's high level of professional readiness, we examined the ways the mandates were translated into her instructional practices. The results of the MQI showed that although her instruction was error-free and demonstrated medium levels of mathematical richness, teacher talk was still prominent in her instruction. We noted the contrast between her enthusiasm about, and knowledge of the specifics of the reform, with the instructional practices we observed. Our analyses highlighted three possible barriers including minimal professional support, lack of mathematical tools and resources, and activation of cultural filters.

Lack of professional support

Eugenia stated the training organized by education authorities was short and while helpful in communicating the goals of the initiative and expectations of teachers, there was minimal substantive information to assimilate the reform in their existing practices. She suggested that "once a while they should organize workshops, service training, just to keep us abreast with..new things, yes. Different way of doing it, we'd be happy to learn". Eugenia scored a score of zero for *Errors and Imprecision* on the MQI scale, showing that she had a strong grasp of the content. However, score of one for the dimension *Working with Students and Mathematics* shows that she struggled to respond productively to students' mathematical contributions

Lack of Mathematical Tools and Materials

The challenges of incorporating reforms into her teaching practices were further compounded by the lack of *mathematical tools and materials.* Eugenia reported that they were required to adopt these reforms from the start of the school year onward, but they did not receive any physical resources (e.g., textbooks or manipulatives). She voiced the concern that the teachers cannot devise productive lessons in the absence of instructional materials.

Activation of filters

We observed the interpretation of child-centered pedagogical strategies through existing filters, that may have been cultural. Eugenia foregrounded physical (e.g., singing a song) over cognitive engagement (Cross et al., 2012) in selecting her starter activities. This shows an interesting case of how the meaning underlying suggested reforms can be misinterpreted. Second, interpretations of "more student talk" and "group work" were skewed. Eugenia interpreted "more student talk" as having students respond more verbally and not necessarily on the content of their productions, hence, she asked a lot of direct structured questions. Although she did put students in groups, the groups comprised many students which inhibited productive interaction.

Discussion

Eugenia perceived the reforms as a catalyst to shift the ways roles were assigned to both students and teachers and expressed sheer enthusiasm to accept these changes. This perspective contrasts teachers' typical levels of motivation to implement mathematics reform, which is often lackluster (see Choi, 2017; Ling, 2002). Eugenia was primed for implementation and at seven weeks into implementation was working hard to teach in these new ways. Despite this professional readiness, Eugenia experienced multiple barriers: lack of pedagogical support, lack of mathematical tools and materials, and the activation of filters. Change of any form tends to be challenging and arduous. Thus, teachers, who are the "boots on the ground" for educational reform and tend to carry the brunt of its burden, often pushback or implement changes only under compulsion. Eugenia was willing, even enthusiastic to engage in the productive struggle characteristic of reform implementation despite minimal support. In this regard, we draw attention to the important role of productive math-related beliefs, positive math-related emotions, and strong efficacy in gauging teachers' professional readiness, which bodes well for investment in the reform, thus, increasing the likelihood of sustained change when appropriate supports are provided.

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THE SOCIAL COMMUNITY OF A MATHEMATICS SUPPORT PROGRAM

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The purpose of this report is to share the facets of a mentoring program set in a mathematics department in a mid-sized western university. To examine the efficacy of the program, I use the social community framework (Mondisa & McComb, 2015) to compare the students in the program to a control group of students. Results indicate that the program is helping the students develop connectedness and a community of practice.

Keywords: Undergraduate Education, Systemic Change, Equity, Inclusion, and Diversity

Increasing the amount of people of color and women in mathematics is imperative (i.e., PCAST, 2012). While most would not argue with that statement, the question of how to help support students in mathematics majors remains open (Karp, 2011). Studies about the leaky pipeline indicate that we lose students from high school to college (Snyder & Dillow, 2011), and also lose them throughout their undergraduate careers, but specifically within their first year of college (Chen 2009; Higher Education Research Institute 2010). Some students leave mathematics majors because they want to pursue a different subject, and for those students, we wish them well. However, research indicates that many students who are capable of achieving in mathematics, leave the major because they have a bad experience or feel unwelcome (Seymour & Hewitt,1997). Both people of color and women are more likely to leave mathematics majors than white or male students (e.g., Anderson & Kim 2006; Hill, Corbett, & Rose, 2010; Griffith, 2010).

With these disparities in mind, I worked with a team of mathematicians to design a support program for mathematics majors at a mid-sized western 4-year university. While our program was open to all students, the majority of the students we recruited were people of color, and more than half were women. The purpose of this presentation is to describe the parts of our support program, and then to share the second year results of studying the students' reactions to the program.

Literature Review

Karp (2011) in conducting a review of non-academic supports that are necessary for students found there were four main categories: building *social relationships*, educating students about *career options*, illuminating the *college structure*, and supporting students through *life issues*. Italics have been added to indicate the shortened names that will denote each of these categories in this paper. *Social relationships* means helping find a sense of belonging within a community at school. Students who built communities at college were more likely to remain in college (Crisp, 2010). Karp (2011) found that educating students about *career options* helped them to see how college was useful to their future goals. Illuminating the *college structure* is necessary because many students do not understand how to register for classes, or apply for financial aid. The last category is *life issues*, which Karp (2011) explains that students experience and leave college without realizing that there are resources they make use of, or alternate ways to make up work. Karp's (2011) metanalysis indicated that while the studies showed the helpfulness of these four categories, many of the studies did not ask students why or how the supports were helpful.

Estrada et al. (2017) noted there is a further need for documentation of what works to help support underrepresented students in STEM.

Mondisa and McComb (2018), in studying a minority mentorship program, described how many other studies of mentorship programs focus on comparisons of students' GPA or attrition rates without examining how mentorship programs affect individual students on a social level. "Evidence is needed identifying what social elements contribute to the positive experiences of program members, how these elements influence an array of participant outcomes (i.e., not simply academic achievement), and what characteristics may explain differences across member experiences (Mondisa & McComb, 2018, p. 94).

The Support Program

The study was conducted at a mid-sized western comprehensive university. The university has a student population including 67% first generation students, and 61% Pell-eligible, indicating a financial need. The overall student population demographics are: 3% African American, 14% Asian (mainly Southeast Asian: Hmong and Cambodian), 49% Hispanic, 6% non-resident students, 3% two or more races, 5% unknown, and 20% white.

Two focus groups of mathematics majors were held to gather information so that we could create a program that was most beneficial for the students. The themes that came out of these focus groups were: (1) students had multiple off-campus jobs and were often care-givers for family members, (2) students were unaware of programs offered by the university that might support them, (3) students struggled to find a community of other students, (4) students did not see faculty members as mentors (i.e., individuals they could go to for help with *life issues*), and (5) students were unaware of career options in mathematics besides teaching.

The results from the focus groups reinforce Karp's (2011) four supports: *social relationships* (1), (3), (4); *career options* (5); *college structure* (2), (4); *life issues* (1), (3). With this grounding, I worked with a team of three mathematicians, who I will call mentors, to create supports to help address these four categories. The program, which began in Fall 2018, consisted of four main supports: scholarships, advising, workshops, and problem solving challenges.

Social relationships were addressed through scholarships, weekly meetings, and required use of office hours or tutoring. The scholarships are in this category because as noted by (1) above, the students' jobs to support their tuition limited the amount of time they were able to be on campus, and thus limited the amount of relationships they were able to build on campus. Additionally, the scholars met with the mentors and each other on Fridays for either problem solving, workshops/speakers, or tutoring. *Career options* were addressed through advising meetings (at least one each semester) and through the guest speakers on Fridays. *College structure* issues were addressed through workshops on Fridays when various campus offices (financial aid, health center, mental health center, etc.) were brought in to talk with the scholars. Lastly *life issues* are difficult to address directly, but the hope was that through their connections to the mentors, the students would communicate when they needed help.

Theoretical Framework and Methods

Mondisa and McComb (2015) define mentoring programs as being comprised by the following elements: "(a) program values, (b) access to faculty and peers, and (c) formal and informal group activities (Ehrich et al., 2004; Gershenfeld, 2014; Hrabowski & Pearson, 1993; Maton et al., 2000; Treisman, 1992)" (p. 3). To measure outcomes from mentoring programs, they proposed using the framework of Social Community (Mondisa & McComb, 2018). In this framework, the researchers measure the connectedness, resiliency, and communities of practice level of each participant. Communities of practice is defined as “collections of like-minded individuals sharing similar experiences and social resources as they interact with and support each other (Eckert, 2006; Wenger, 2000)” (Mondisa & McComb, 2018, p. 98).

Thus, to measure the social community outcomes of our support program, I used this framework to design interview questions for the students. An example of a connectedness question was: Do you feel you have a mentor (peer or faculty member or otherwise)? An example of a resiliency question was: What was an obstacle you faced this year, and how did you overcome it? An example of a communities of practice question was: Do you feel like you belong as part of the math department?

In total, there were 16 scholars in the second year of the program. Of the 16 scholars, 6 were second year students from the first cohort, and 10 were first year students from the second cohort. There were 12 students recruited in the control group: 5 second year students, and 7 first year students. All of the scholars and control group students were mathematics majors, and the control group was chosen to be the same level in the program, and both groups were majority people of color. All of the participants were interviewed for approximately 30 minutes in October-November of 2019. The interviews were audio recorded, transcribed, and were coded according to the three themes of the social community framework: connectedness, resiliency, and community of practice (Mondisa & McComb, 2018).

Results

The results will be divided according to the three themes from the social community framework: connectedness, resiliency, and community of practice.

Connectedness

The Scholars were more likely to identify as feeling connected than the control group. In particular, in answer to the question of if they had a mentor, 12 out of 16 of the scholars versus 3 out of 12 students stated that they felt like they had a faculty member as a mentor at the university. The general feeling from the scholars is captured in this expression: "*For the first year I did. So it’s really easy to talk to [one of the mentors] because he was the first person that I really talked to and kind of felt since I already knew him…But through the course of that year and a little bit of this semester, I can talk to anybody now.*" Many of the scholars also mentioned that they relied on one another for help in courses, or just support in school in general. In contrast, the control group had mentors, but those people were either a family member outside of the university, or one other student in the program. This indicates that the control group were less connected to faculty members, and also to less other students in the department than the Scholar group.

Resiliency

The biggest challenge that all of the students faced (Scholar or control group) was time management. It was less evident in the second year students, but because the schedule change from high school to college can be extreme, it was difficult for the students to try to schedule their days at first. Two women from each of the control group, and from the scholars mentioned their biggest challenge was being “too shy”. All of them said it was difficult for them to make connections with other people in college. The two scholars mentioned that it became easier because they got to know people through the Friday meetings, so even if they were not in the same classes, they saw each other around campus. Out of the two control group women who mentioned shyness, one of them said she was getting past that by using a first generation college student group, and the other one said it was still something she was working on. Both groups also mentioned that the classes were much harder in college than in high school, but that they were working through the challenge: "*When I fail, I have to beat it. That's how I've gone through most of my life*." In summary, all of the students displayed resiliency. Resiliency might also be a topic that comes up more later on in their academic careers when more of the students start reaching the proof courses in the mathematics major.

Communities of Practice

The main interview question that I used to code this aspect of the social community framework was: Do you feel like you belong at (university) in the math department? Out of 16 scholars 15 said they felt like they belonged, and out of 12 of the control group, 10 of them said they felt like they belonged. However, the specificity of the group's answers varied quite a lot. Here is a representative answer from the control group: "*I would say yes. There was a point where I just felt like, a little lost in my math class. Okay, so I felt like I didn't belong, but then I got the ball rolling in.*" Now, in comparison, here is a representative answer from the scholar group: "*I feel like I am home. Like, I have so many professors and advisors and now, like I walk in math department and like, I always encounter someone like hello*." In comparing these two responses, both students stated they belonged, but notice that the control group is basing this belonging on their relationship with mathematics, and not with other people in the mathematics department. In contrast, the scholar group focused on their interpersonal connections with other scholars (through the Friday meetings), with faculty members, and with other students in the mathematics department. Their responses indicated more of a social community and network than the control group students.

Conclusions

The results indicate that while all of the students expressed resiliency, the scholar students had much greater connectedness and communities of practice developed within the mathematics department. This builds off of research shared from the first year of the program. In the first year, themes indicated that the scholars were more likely to feel comfortable seeking out a faculty member for help with academic or life issues (6 out of 7 versus 1 out of 6) (Tague, 2019). That has continued to grow with the first cohort, and has held consistently for the second cohort as well. The control group students seemed to define their belonging and their community of practice as their relationship with mathematics rather than their relationship with people within the mathematics department. That is troubling because as mathematics courses get more difficult, it could lead to a drop in a feeling of belonging to the community of practice. The support program has helped the students in the first two cohorts develop a community of practice amongst one another, and also amongst others in the mathematics department. More research in the following years will indicate if the patterns will continue in this way, however, preliminary it seems this model of mentorship program could benefit mathematics students.

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CONCEPTUALIZING PRACTICES IN INTERDISCIPLINARY GROUPS FROM A MATHEMATICS EDUCATION RESEARCHER'S PERSPECTIVE

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Mathematics Education Researchers (MERs) contribute to the growth of mathematics education when joining interdisciplinary groups. However, little is known about the ways of work within such groups (i.e., practices). We aimed to enhance our definition of practices by exploring practices reported by an MER in her interdisciplinary group. A transcript from a semi-structured interview comprised the data. Our grounded theory analysis resulted in an enhanced definition of practices informed by the participant's descriptions of interdisciplinary work. We argue that practices in interdisciplinary groups involve ways of being, operating, and interacting.

Keywords: Interdisciplinary STEM/STEAM, Research Methods, Sustainability

Williams et al. (2016) defined disciplinarity as a "phenomenon" involving "specialization" of work and discourse (p. 4). Interdisciplinary group members are practitioners of different disciplines who have been socialized and encouraged to exhibit specialized forms of work and discourse, also called practices (Hyland, 2004; Williams et al., 2016). Researchers define practices in various ways (e.g., Cobb & Yackel, 1996; MacIntyre, 1984; Schön, 1983; Wenger, 1998). Most definitions of practices include descriptions of methods for functioning within a group or community. Based on existing definitions and Williams et al.'s (2016) description of disciplinarity, we described practices as "established or emergent ways of *being*, *operating*, and *interacting* [italics added] with others" during collective activity (Suazo-Flores et al., 2021, para. 4). *Being* referred to how an individual conceptualizes thIs part of a group. *Operating* referred to ways of doing within a community. *Interacting* referred to developing and communicating standards for group discourse (Suazo-Flores et al., 2021). In this study, we explored: In what ways can our original definition of practices be enhanced using empirical data from a mathematics education researcher's (MER's) reported experience in an interdisciplinary research group?

Mathematics Education and Interdisciplinary Group Practices

Work in interdisciplinary groups has become common for MERs since integrated expertise is an effective way to solve large-scale problems (Bruce et al., 2017) and enhance education (e.g., National Governors Association, 2007). Given that researchers use disciplinary-based practices, MERs are likely to face challenges when interacting with researchers from other disciplines. Bruce et al. (2017) described the challenge of framing a research idea within an interdisciplinary research team "in ways that permit all potential research collaborators to identify and situate themselves" (p. 158). Goos and Bennison (2018) described physical and institutional challenges during interdisciplinary work like traveling to different disciplinary meeting venues and ensuring that interdisciplinary work contributed to achieving tenure. These examples suggest that MERs need to navigate and negotiate new practices in interdisciplinary groups.

Although interdisciplinary groups can present challenges, they offer opportunities for MERs to address complex problems and develop new knowledge for mathematics education. Four examples from published articles in mathematics education are provided here. Bruce et al. (2017) explored different disciplinary understandings of spatial reasoning to illustrate the discipline-specific educational significance of spatial reasoning. Goos and Bennison (2018) documented collaborations of mathematicians and mathematics educators designing teacher education curricula. The authors described ways group members created curricula integrating mathematics content and pedagogy. Biology education researchers and a MER studied undergraduate biology students' graphing practices to address the complexity of assessing graphing knowledge using auto-scored question formats (Gardner et al., 2021). Krummheuer et al. (2013) used a socio-constructivist approach and psychoanalytic perspective to contribute new understandings of children's creativity in mathematics problem-solving. The researchers used attachment theory to explore a child's activity in mathematical situations.

To better understand interdisciplinary groups that include MERs, Suazo-Flores et al. (2021) identified three examples of interdisciplinary practices reported in published research articles: "working towards research interests," "cultivating trust and open-mindedness," and "understanding of institutional support" (para. 9). "Working towards research interests" referred to researchers establishing shared research interests. "Cultivating trust and open-mindedness" referred to researchers finding ways to express themselves freely and consider the perspectives of others. Finally, "understanding institutional support" referred to researchers operating within existing structural and institutionalized norms. Since the published articles communicate findings from empirical studies rather than describe interdisciplinary group practices, the examples we identified were initial and tentative. To extend this research, we enhanced our definition of practices drawing from the experiences of one MER who participated in an interdisciplinary group.

Methods and Analysis

This study is part of a larger project exploring MERs' lived experiences working in interdisciplinary groups. We conducted semi-structured interviews (Kvale, 1996) with MERs and sought to understand practices reported by participants in such groups. Amelia (pseudonym) was an MER who led an interdisciplinary group. Evidence of practices reported by Amelia in the form of an interview transcript constitutes our data.

We used grounded theory (Charmaz, 2005) to analyze Amelia's descriptions of working in an interdisciplinary group. The analysis involved three phases. First, we used our original definition of practice to identify instances of practices from Amelia's reported experience in interdisciplinary group activity. Each instance was coded as *being*, *operating*, or *interacting* and assigned a short phrase to summarize the practice. Second, we refined our descriptions and definitions of *being*, *operating*, and *interacting* and identified examples of practices for each category from the transcript. The third phase of analysis involved using the revised definitions and collections of practices to review and update the codes for our data corpus, resulting in new enhanced definitions of *being*, *operating*, and *interacting*.

Findings: Ways of Being, Operating, and Interacting

Our findings comprise three parts. First, we developed a more robust definition of practices as they relate to ways of *being*, *operating*, and *interacting*. Second, examples are provided that illustrate the need for the elaborated definition. Third, although each practice was coded as *being*, *operating*, or *interacting*, we found that these categories were not mutually exclusive. We provide an example of the ways the practices are not mutually exclusive.

The initial definition of *being* practices referred to how an individual identifies as part of a group. Based on Amelia's experience, ways of *being* also referred to how other group members were identified. For example, Amelia described members as being "champions" who supported innovation. "When you do something innovative, to get started you need champions. So these, these people were champions." Another refinement to the definition of *being* includes specific roles such as leading and providing disciplinary expertise. For example, Amelia described her role as a leader and the roles of other members, "Yes, I am the leader of this. I do have a lot of experience. I do have things to offer, but I don't know everything and there's various ways of thinking." Amelia describes that other members contribute "various ways of thinking" referring to their disciplinary expertise. Based on the data, ways of *being* are defined as how an individual identifies these others as part of the group, including specific roles such as leader or providing disciplinary expertise.

The initial definition of *operating* practices referred to ways the interdisciplinary group worked. Amelia's experience helped us clarify what working in an interdisciplinary group entailed. For example, Amelia described a practice for the group of making small changes rather than having a grand design that could only be enacted with permissions from outsiders.

> Amelia: So I think our project was not so much about, "Let's have this grand design" [...] it was about thinking about the small changes that we can actually make that are not going to require layer upon layer upon layer of [institutional] approvals which we know is going to take a long time. [...] And it becomes doable and possible because big changes are frightening and threatening to people.

Operating practices are now defined as the group members' understandings of the group's work, organizing ideas that enable the work, and ways of navigating institutional policies to get the work done.

The initial definition of *interacting* practices referred to developing and communicating standards for group discourse. Amelia described experiences that helped us clarify that *interacting* also involved negotiating the meaning of ideas, frameworks, or representations. For example, Amelia described how the mathematicians and scientists who were part of the interdisciplinary group had to develop new ideas and language.

> Amelia: It was much more challenging for my mathematician and scientist colleagues to do this work because the project was located within teacher education. That's my world. So they were the ones who really had to step into this world and learn new ways of thinking and new language.

Amelia's examples further illustrated ways she advocated for the value of the group's work. Instances like this example were identified as *interacting* because the interdisciplinary group had to find ways to communicate findings to people outside of the group.

> Amelia: Talking about getting promoted to me [...] [was one of] the best outcomes out of this project. [...] [Two] cases independently of each other, they asked me if I would be one of the referees for their research portfolio. Their teaching was fine, their service was fine, but research you know the things we're looking at this stuff and saying "What is this? What are these journals? What's this project about? Why are you doing this? What is your

> mathematics research?" So I was able to be a referee for both of them and explained and advocated for this project. Talk about the conferences I go to, the journals like publishing and why it matters. And they both got promoted.

Interacting practices are now defined as ways members of the interdisciplinary group: (1) develop and communicate standards for group discourse, (2) negotiate the meanings of ideas, frameworks, or representations to develop common understandings (i.e. taken-as-shared meanings), and (3) communicate findings to people outside of the group.

Beyond redefining the practices, Amelia's experience highlighted the ways the practices were not mutually exclusive. For example, Amelia explained that how the group pursued goals and learned to work together evolved over the course of the interdisciplinary work.

> Amelia: What did change over time was an evolution in the way that we pursued those goals. And there were other off-shoots and new things that were generated because of the fact that we had twenty-five or so people from six different universities learning how to work together.

Amelia described her understanding of the group pursuing goals, a way of *operating*. She also described that the way the group worked together evolved over time, suggesting that the group negotiated the meanings of ideas and developed new understandings, a way of *interacting*. Amelia's example illustrates how *operating* practices such as an understanding of the group's work and *interacting* practices such as negotiating new ideas are mutually informing.

Discussion and Conclusion

We have enhanced our original definition of practices (Suazo-Flores et al., 2021) and provided examples of ways of *being*, *operating*, and *interacting* from Amelia's interview. *Being* now refers to MERs describing their view of themselves and others in the interdisciplinary group including specific roles taken on by group members. *Operating* has been expanded to mean members' ways of doing in the interdisciplinary group and acknowledging institutional policies and actions in order to complete the work. *Interacting* is now understood as developing communication standards, negotiating the meaning of ideas that allows the group to collaborate, and explaining work to people outside of the group.

Practices in interdisciplinary groups comprise ways of *being*, *operating*, and *interacting* (Suazo-Flores et al., 2021). As members of interdisciplinary groups exhibit different practices based on their disciplinary backgrounds (Hyland, 2004; Williams et al., 2016), MERs can benefit from being conscious of how they see themselves as part of the group in addition to how they see others in the group (i.e., *being*). Amelia recognized that she needed to understand and identify ways to operate within the group and under institutional policies (i.e., *operating*). Also, similar to the researchers in Bruce et al. (2017) who strived to frame a research idea within the interdisciplinary research team while allowing the collaborators to maintain their identities and situate themselves relative to the ideas of others, Amelia understood the group's need to negotiate and reconstruct meanings of ideas (i.e., *interacting*).

Amelia's experiences allowed us to enhance our definitions of ways of *being*, *operating*, and *interacting*. To feel sustained in interdisciplinary interactions, MERs need to be aware of their personal and disciplinary identity, acknowledge the personal and disciplinary identity of other group members, negotiate and develop standards of discourse, and identify ways to collaborate with respect to institutional and group norms. Given the limitations of personal accounts as evidence of practices, research explorations of practices should include observations of group members' interactions to situate practices in the *being*, *operating*, and *interacting* in activity.

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A THEORETICAL FRAMEWORK FOR DESCRIBING COMMON MANIFESTATIONS OF MATHEMATICAL ANTHROPOCENTRISM

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Keywords: Equity, Inclusion, and Diversity; Ethnomathematics

This paper uses the constant comparative methodology (Glaser, 1965) to categorize the most common mathematically anthropocentric manifestations of mathematicians and mathematics educators. The literature is replete with statements that mathematics is a “human creation”, “human activity”, “human construct”, “human construction”, “human endeavor”, “human enterprise”, “human invention”, “human potential”, “human social activity”, or their human-only equivalents. I call such anthropocentric mathematical perspectives that place humans as the central element in mathematical development and operation *mathematical anthropocentrism*. In the literature, I distinguish three main types of mathematical anthropocentrism: (1) *absolute*, (2) *strong*, and (3) *weak* mathematically anthropocentric perspectives. In particular, the *absolute* mathematically anthropocentric position maintains that mathematics is exclusively a human domain. An example of an absolutist would be fler (2007), who considers it a “trivial fact mathematics is a human activity.... Under all circumstances mathematics is done and produced by human beings. …Thus, mathematics is deeply and genuinely human” (p. 105). The *strong* mathematically anthropocentric position considers mathematics to be essentially a human domain; Núñez and Marghetis (2014) manifest this perspective by at least acknowledging some rudimentary mathematical ability in animals, although they limit these examples to rare exceptions. The *weak* mathematically anthropocentric position (e.g., Denahue, 1997) gives broader latitude to non-human mathematical ability, but still centers human mathematics as superior. I conclude by challenging mathematical anthropocentrism as anachronistic (in lieu of recent scientific developments in a wide range of scientific fields, such as animal cognition, plant behavior, bacteriology, genetics, etc.; see Howard, 2018 [Fig. 1, p. 12] for a brief introduction to animal mathematics), significantly limiting the interpretation of what counts as legitimate mathematics (and hence, limiting the paradigms of mathematics education). I posit that mathematical anthropocentrism is a culturally reproduced phenomenon that can be troubled through adequate education about non-human Other mathematics. I recommend that mathematicians and mathematics educators consider post-anthropocentric mathematical perspectives legitimizing non-human Other mathematics by elevating this Other mathematics onto an equal plane with the mathematics that humans do.

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Chapter 10:

Pre-Service Teacher Education

RETHINKING HOW UNITS COORDINATION IS ASSESSED IN PRESERVICE TEACHER POPULATIONS

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Fraction proficiency continues to be a challenge for many learners of mathematics. Valid and reliable methods for assessing fraction understanding are critical tools in the pursuit of meeting this challenge. Written assessments have been widely used with K-12 students to assess fraction understanding, including units coordination. However, using these types of assessments with a preservice PreK-8 teacher population has proved difficult and inconclusive. Preservice PreK-8 teachers have a variety of algorithmic techniques at their disposal, which has resulted in the need to reexamine how units coordination is assessed in this population. This paper shares the subsequent reconceptualization of assessing preservice PreK-8 teachers' units coordination.

Keywords: Mathematical Knowledge for Teaching, Rational Numbers, Preservice Teacher Education

For decades, proficiency with fraction concepts and computations has been a bane to many students and teachers alike (e.g., Ball, 1990; Bentley & Bosse, 2018; Borko et al., 1992; Izsák et al., 2010; Menon, 2009; Olanoff et al., 2014; Rathouz, 2010; Rizvi & Lawson, 2007; Schneider & Siegler, 2010; Stafylidou & Vosniadou, 2004; Tirosh, 2000). In a previous study of preservice PreK-8 teachers' (PSTs') fraction knowledge (Busi et al., 2015; Lovin et al., 2018), we found evidence that many PSTs struggled with the more sophisticated reasoning needed for fluency with rational numbers. Subsequently, we investigated ways to improve PSTs' fraction content knowledge through changes in our pedagogy (Stevens et al., 2020).

The framework we have used to guide our work in assessing and making sense of PSTs' conceptions of fractions is based on a trajectory of fraction schemes and operations (Norton & Wilkins, 2012; Steffe & Olive, 2010; Wilkins & Norton, 2011). A key component of moving through this trajectory relies on the number of levels of units the learner can coordinate simultaneously. Specifically, to reach the higher levels of reasoning in the trajectory, the learner must be able to coordinate three levels of units simultaneously (3UC) – meaning they can anticipate the outcome of this coordination before they do it. Having this anticipation is known as interiorizing the ability to coordinate units. If someone is unable to anticipate the outcome of the coordination, they may either not have acquired this coordination or may solely coordinate units in action, in the midst of solving a fraction task.

Throughout our work, we have experienced a productive struggle with confidently assessing PSTs' ability to coordinate three levels of units. Our initial study identified 13 cases in which it appeared PSTs had developed a fraction scheme in the developmental trajectory beyond coordinating three levels of units before they had acquired 3UC. This is contradictory to the validated theory in which each step of the developmental trajectory requires the acquisition of the previous scheme or operation.

One confounding fact is that the written assessment used in this first study was initially developed for use with upper elementary and middle school students (Norton & Wilkins, 2012; 2013; Wilkins & Norton, 2011). When used with PSTs, PSTs' overreliance on procedures to find common denominators or to do fraction computations masked evidence of whether they had interiorized the operation of coordinating three units. Since this first study, we have been exploring alternative tasks and strategies to better assess PSTs' ability to coordinate three levels of units. Our ensuing productive struggle led us from solely written assessment tasks to observations of PSTs completing written tasks to structured interviews and has helped identify issues with our tasks that can be used to create improved assessments. Our goal is to share some observations from this process.

Theoretical Framework

An existing developmental trajectory of fraction schemes and operations serves as our framework. This trajectory was validated for upper elementary and middle school students (Norton & Wilkins, 2012; 2013; Wilkins & Norton, 2011) and later validated for PSTs (Busi et al., 2015; Lovin et al., 2018). These schemes and operations can be grouped into three bands of developmental knowledge of fractions with each subsequent band relying on an increasing number of levels of units the learner can simultaneously coordinate: fractions as solely part-whole concepts (only requires the coordination of one level of unit); fractions as measures (requires the coordination of two levels of units); and fractions as numbers "in their own right" (requires the coordination of three levels of units) (Hackenberg, 2007, p. 27; Hackenberg et al., 2016). Our previous work discovered a majority of PSTs were not proficient in being able to reason about fractions as numbers "in their own right" (Hackenberg, 2007, p. 27). This finding corroborates existing research (e.g., Chinnappan, 2000; Olanoff et al., 2016; Son & Crespo, 2009; Son & Lee, 2016). The catalyst for developing this reasoning is being able to simultaneously coordinate three levels of units (Steffe & Olive, 2010), which is the part of the trajectory we focus on in this paper. (For more information about this developmental trajectory of fraction schemes and operations, please see Norton & Wilkins (2009), Norton & Wilkins (2012), Norton et al. (2018), Steffe (2002), Steffe & Olive (2010), Wilkins & Norton (2011).)

Methods

Participants and Instrument

Participants in the study comprised seven undergraduates enrolled in one of three required mathematics content courses for PSTs at a southeastern university. The first in this sequence of courses focuses on number concepts and operations, with significant time dedicated to developing fraction understanding. Four of the participants were enrolled in the first course and participated in the study prior to fraction instruction. The other three participants were enrolled in one of the subsequent courses.

Because the motivation for the study was to investigate strategies that may impact or mask PSTs' ability to demonstrate the interiorization of 3UC, a written 3UC assessment was developed that paralleled assessments used in previous studies (Busi et al., 2015; Lovin et al., 2018). Previous studies showed some evidence of PSTs' reliance on algorithms such as dividing fractions or finding equivalent fractions as masking evidence of 3UC interiorization. This study sought to further investigate these potentially confounding algorithms by combining clinical interviews of the participants with the written assessment so PSTs' approaches and reasoning could be explored further.

Data Collection

The participants completed a twelve-item assessment designed to determine whether or not they were able to coordinate three levels of units. PSTs who have interiorized the ability to do this have an immediate, productive plan to solve a 3UC fraction task and can anticipate the results irrespective of context, denominator choice, or representation; they do not rely on their written work to discover a productive strategy in action (Hackenberg et al., 2016).

The assessment began with four items with no accompanying representations or context. For example, PSTs were posed the following question: "Envision 2/3 of a whole. Now consider 1/12 of the same whole. How many 1/12s are in the 2/3 you originally envisioned?" The remaining eight items were written within a specific context (e.g., an amount of pizza or the length of a jump rope) and provided a specific representation (e.g., a portion of a circle or a line). The fractions used in both sections were varied in structure; the denominators either allowed for halving strategies (e.g., relating 3/¾nd 1/8) or required strategies other than halving (e.g., relating 3/5 and 1/15).

For each item, the PSTs were asked to provide both a solution and a demonstration of their reasoning. For the first four items, the PSTs provided no written documentation of their thinking; their explanations were verbal. For the remaining eight items, the PSTs were asked to use the provided representation to diagram their thinking, and the researchers asked them clarifying questions about their diagrams. The PSTs were observed and video-recorded while completing the entire assessment. The observations and clinical interviews were an essential portion of the study because they enabled the researchers to watch the participants' approaches in action, rather than solely evaluating written evidence of their strategies after they submitted the assessment.

Data Analysis

All four researchers independently rated the written responses for each item and then compared the documentation to the video recordings of the verbal explanations to evaluate each participant's interiorization of 3UC. The researchers then discussed their ratings and came to a consensus based on the evidence provided by the comparisons.

The video recordings allowed the researchers to look for discrepancies in the written documentation, the participants' observed approaches, and the participants' verbal descriptions of their strategies. Participants may show evidence for 3UC in their written work, but then describe their reasoning in a manner that indicates otherwise. In this case, looking solely at written work would result in a *false positive*. Participants may also show counterevidence for 3UC in their written work, but then describe their reasoning in a manner that indicates otherwise. In this case, looking solely at written work would result in a *false negative*. Based on previous findings, the researchers hypothesized these discrepancies would exist between some participants' written evidence and their verbal descriptions of their approaches.

Results

We will share one illustrative example of a *false positive* assessment of 3UC and one illustrative example of a *false negative* assessment of 3UC.

False Positive Example

The PST was given this written question: "The candy bar shown below (represented by a rectangle) is 5/6 of a whole candy bar. If each person wants 1/24 of a whole candy bar, how many people can share the amount shown below?" In her written work (Figure 1), the PST seemed to create the whole by partitioning the given diagram into five 1/6 pieces and adding on one additional 1/6 piece to the given diagram to create six 1/6 pieces. From there, it seemed like she further partitioned each 1/6 piece into four smaller pieces, essentially cutting each 1/6 piece into fourths. There are now 24 pieces within the whole candy bar. Each group of four 1/24 pieces (contained within each of the original 1/6 pieces) is marked with four symbols to show they make a group. For example, there are four x's above four 1/24 pieces in one 1/6 piece and four +'s above four 1/24 pieces in another 1/6 piece. (Note: the symbols above the four 1/24 pieces in the added-on 1/6 piece are difficult to decipher.) The PST seemed to be coordinating the 1/24 pieces within each of the 1/6 pieces within the whole and showing she has five groups of four 1/24 pieces in the given amount, indicating she was coordinating three levels of units.

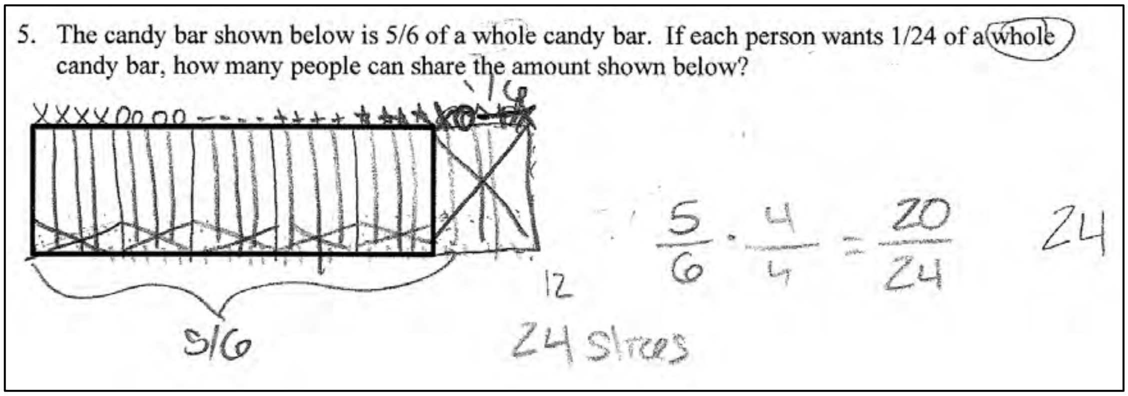

Figure 1: PST Solves Candy Bar Problem

The same PST was given this written question: "The length of rope shown below (represented by a line) is 3/5 of a whole length of jump rope. If each jump rope requires 1/10 of the whole length of rope, how many jump ropes can you make from the length of rope shown below?" In her written work (Figure 2), the PST again seemed to create the whole by partitioning the given diagram into three 1/5 pieces and adding on two additional 1/5 pieces to the given diagram to create five 1/5 pieces. This can be seen with the dotted lines. From there, it seemed like she further partitioned each 1/5 piece into two smaller pieces, essentially cutting each 1/5 piece in half. This can be seen with the solid lines. She then labeled each piece as 1/10 in size and circled the six 1/10 pieces that were in the given diagram. The PST seemed to be coordinating the two 1/10 pieces within each 1/5 piece, five of which make the whole, again indicating she was coordinating three levels of units.

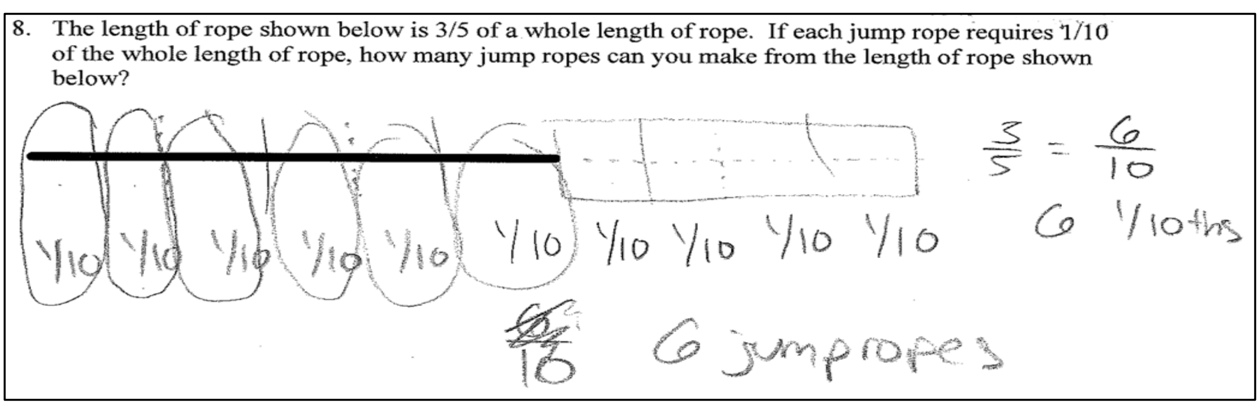

Figure 2: PST Solves Jump Rope Problem

Based solely on her written work, it would seem this PST was coordinating three levels of units. She identified four 1/24 pieces within each 1/6 piece, five of which were given and six of which make the whole. She also identified two 1/10 pieces within each 1/5 piece, three of which were given and five of which make the whole. However, listening to this PST answer the first four questions of the interview protocol, it was clear this PST is not coordinating units and is instead using a generalized procedure to create equivalent fractions.

For example, she was asked: “Suppose you have 3/¾f a whole, can you explain to me how many 1/8 pieces of the whole you have?” She almost immediately answered correctly that she would have six 1/8 pieces of the whole because she “converted [3/4] into eighths.” She described a procedure she named the “giant one” (see Figure 3 for a visual representation of the “giant one”) which tells her she has to multiply the numerator and the denominator by the same factor so she is “not changing the value… just changing the representation of it.” In this example, she explained she multiplied the four by two (in the denominator) so she must multiply the three by two (in the numerator) to get six 1/8s.

In another example, she was asked: “Suppose you have 2/3 of a whole, can you explain to me how many 1/12 pieces of the whole you have?” The PST used the same process of “multiplying by the giant one, or four-fourths” to know there would be two times four or eight 1/12 pieces of the whole. With this strategy, this PST claimed, “I don’t change the value of the original fraction, I’m just changing the way it looks.” When posed with a third, similar question, the PST asked the interviewer, “Is it okay if I use the same explanation?” indicating the “giant one” strategy is what she is most comfortable with and most confident in.

This PST’s strategy for each conceptual problem at the beginning of the interview protocol was to create an equivalent fraction. She could clearly articulate her strategy of multiplying the numerator and the denominator of the given fraction by the same number. She could also clearly articulate that by doing this, she is not changing the value of the fraction, she is just changing “the way it looks” or the “representation” of it. However, this strategy does not give any evidence of coordinating three levels of units; there is no indication she sees two 1/8 pieces in each 1/¼iece or four 1/12 pieces in each 1/3 piece.

When listening to this PST explain her thinking about her written work, there was further evidence she is not actually coordinating three levels of units. The PST was given this written question: “The pizza shown below is 2/3 of a whole pizza (represented by 2/3 of a whole circle). If each person wants 1/9 of a whole pizza, how many people can share the amount shown here?” In her written work (Figure 3), the PST initially performed the “giant one” procedure to get an answer of 6/9, which she correctly interpreted as six people eating pizza. Then she moved to the diagram. She split the given amount (2/3 of a whole pizza) into thirds and then split each of those into three smaller equal pieces, making 1/9-sized pieces relative to the given amount (2/3 of a whole pizza). When she did this, it seemed like she might be coordinating three 1/9-sized pieces within each 1/3 piece, even though she is ignoring the size of the whole pizza, giving some indication of coordinating units. However, when she verbally described her thinking, she explained, “I have three parts of a pizza and if each person wants 1/9, I must split up the thirds…I must multiply by something to get nine and I know three times three is nine. So, I split it up again into three equal pieces…so then it was a total of nine pieces and when I multiplied by the numerator it was six, so I know that six people could eat pizza.” When asked to identify the six 1/9 pieces in the diagram, she could not find them. She went on to say, “I was just thinking about it numerically. I was thinking about whatever I multiply by the denominator, I must multiply by the numerator.” This PST did not seem to be coordinating units and was instead attempting to use the diagram to explain the "giant one" procedure. In reality, conceptualizing equivalent fractions requires one to see there are two groups of three 1/9 pieces within the given amount and three groups of three 1/9 pieces within the whole, which does require the coordination of units.

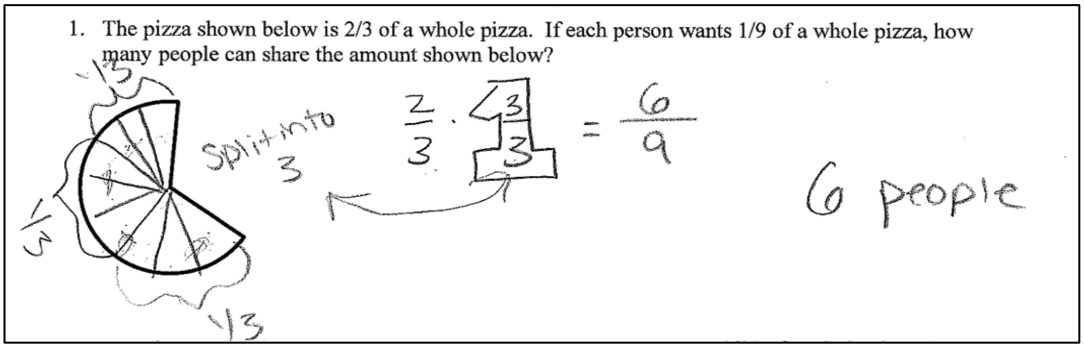

Figure 3: PST Solves Pizza Problem Using a "Giant One"

This situation was thus labeled as a false positive. After the clinical interview, it was determined the PST was in fact completing and trusting the multiplication algorithm to find an equivalent fraction and then translating that number onto the diagram. As static work, it appeared multiple levels of units had been coordinated, but after hearing from the PST, it became clear she was not seeing units within units. Rather, she was retroactively placing the units onto the diagram without any coordination of unit size. It was ultimately concluded this PST has not interiorized the operation of coordinating three levels of units, even though her written work seemed to provide evidence that she had.

False Negative Example

The PST was given this written question: "The pizza shown below is 2/3 of a whole pizza (represented by 2/3 of a whole circle). If each person wants 1/9 of a whole pizza, how many people can share the amount shown here?" In her written work (Figure 4), it seemed as though the PST is trying to figure out how to partition a whole circle into nine relatively equal pieces. She made a few attempts, including a familiar cut-in-half, cut-in-half method, before achieving her goal. But that was where she stopped; she did not provide an answer to the question. There is no evidence of coordinating units in this written work.

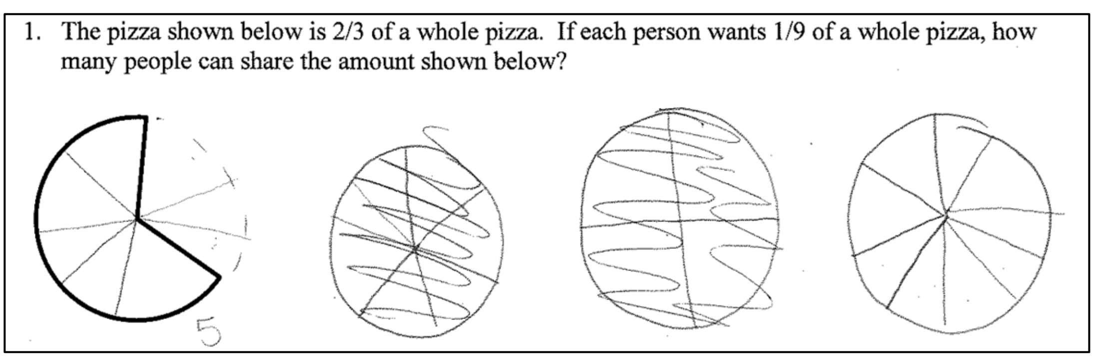

Figure 4: PST Attempts to Partition a Circle into Nine Pieces

The same PST was given this written question: "The length of rope shown below (represented by a line) is 3/5 of a whole length of jump rope. If each jump rope requires 1/10 of the whole length of rope, how many jump ropes can you make from the length of rope shown below?" In her written work (Figure 5), the PST initially labeled the given amount as 3/5, extended the line to represent the whole length of rope, and drew four larger pieces, creating 1/¼ieces, with six smaller pieces within each of the larger pieces, creating 1/24 pieces. She labeled each set of two of the larger pieces as 1/10, making a total of 2/10. The PST abandoned this attempt and started again below it. She drew a second line that has five clear larger pieces, creating 1/5 pieces, with three smaller pieces within each of the larger pieces, creating 1/15 pieces. She labeled three of the larger 1/5 pieces as 3/5. Like the previous question, the PST stopped and did not provide an answer to the question. Considering both attempts on this question, there is no evidence this PST was coordinating multiple levels of units.

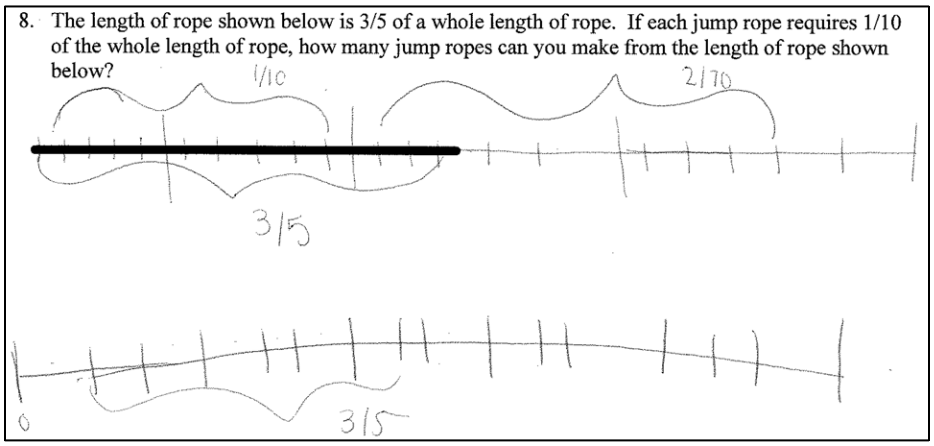

Figure 5: PST Makes Two Attempts at Jump Rope Problem

Based solely on her written work, it would seem this PST was not coordinating three levels of units. She struggled to make connections between the relative sizes of the pieces of the pizza, which was represented with a circular area model, or the pieces of the jump rope, which was represented with a linear model. However, listening to this PST answer the first four conceptual questions of the interview protocol, it did seem like she is able to coordinate units.

She was first asked: "Suppose you have 3/¾f a whole, can you conceptually explain to me how many 1/8 pieces of the whole you have?" She responded with, "If you have a pizza and you cut it into four slices and you shade in three of them, then you can divide all of the fourths into half again and that will give you eighths. So then the section of the three-fourths that isn't shaded, you would have two-eighths not shaded and the rest…you would have six-eighths shaded." The PST was able to confidently describe cutting each 1/¼iece in half to create eight 1/8 pieces within the whole pizza.

Next, she was asked: "Suppose you have 2/3 of a whole, can you conceptually explain how many 1/12 pieces you would have in the whole?" Her answer was, "You would take a pizza and divide it into thirds and then shade two of those thirds. And then you could divide each slice into fourths, each third into four additional sections. And then you would have the section not shaded; there would be four pieces not shaded of the one-twelfths, so four-twelfths not shaded. And then you would have the remaining part of the pizza would be the shaded twelfths…so you would have eight-twelfths shaded." The PST was able to confidently identify the number of 1/12 pieces within each 1/3 piece of the whole.

This PST gave a very similar response to the remaining two questions included in this portion of the interview protocol. In each response, this PST gave a clear articulation of the number of fractional pieces within another. Furthermore, she was able to describe the pieces within the pieces of the shaded part of her diagrams as well as the pieces within the pieces of the unshaded part of her diagrams. For example, she was able to coordinate the number of 1/12s not shaded (four 1/12s) as well as the number of 1/12s shaded (eight 1/12s). This shows a coordination of units within units within the whole (i.e., 3UC).

It is interesting this PST initially asked if she could draw a visual representation to help her solve these four problems. Even though the interviewer asked her to share her thinking without drawing a visual representation, the PST still visualized and described exactly what she would have drawn. This PST clearly favors visualization, but struggled with the visual representations of any type in the written work portion of the interview.

This situation was thus labeled as a false negative. After the clinical interview, it was clear this PST was in fact able to confidently coordinate units even though she struggled to show it in writing. Her static work appeared void of unit coordination. But, when given the opportunity to talk about the problems, it became clear she was very capable of this coordination. It was ultimately concluded this PST has interiorized the operation of coordinating three levels of units.

Discussion

In our previous work with written assessments, many PSTs used computational procedures to solve 3UC tasks, masking evidence of coordinating three levels of units (Busi et al., 2015; Lovin et al., 2018). Through this previous work, it became evident that intentionally designed assessments were necessary to help unpack the masking issue. Originally, the new written assessments aimed to vary contexts, (e.g., candy bars), denominator choices (e.g., allowing for halving strategies) and representations (e.g., rectangular area) to further explore PSTs' true ability to coordinate three levels of units. However, it quickly became apparent PSTs were still exhibiting inconsistencies with how they solved these written problems. We again noticed algorithm use and incomplete diagrams caused us to be inconclusive in our attempts to determine if 3UC was evidenced in the work.

To help guard against the inconclusive nature of the written work, a clinical interview protocol was also created. Striking observations were made in terms of the differences between looking at a PST's static work and hearing a PST talk about her reasoning. As described in the results section above, there were some PSTs whose written work showed evidence of coordinating three levels of units, but when listening to their reasoning during the clinical interview, it became clear that seeing units within units within the whole was not occurring. This indicated they had in fact not interiorized 3UC. On the other hand, there were some PSTs whose written work indicated they could not coordinate three levels of units. But when they described their thinking about the problems during the clinical interviews, they could clearly and confidently talk about units within units within the whole. This showed evidence that they in fact had interiorized 3UC.

The additional interview data is providing evidence that PSTs' written work as a single artifact of evidence is not sufficient to determine the presence of the interiorization of 3UC. This is a significant finding given that many previous studies (e.g., Busi et al., 2015; Caglayan & Olive, 2011; Lovin et al., 2018; Son & Lee, 2016; Ubah & Bansilal, 2018) have relied on written assessments to determine PSTs' ability to coordinate units. This begs the question: how do we best assess 3UC in PSTs? The clinical interviews we conducted seem to be effective. By listening to a PST reason about 3UC problems conceptually and by listening to a PST describe her thinking about a specific problem in context, we felt confident about our assessment of whether or not that PST had interiorized 3UC. Although clinical interviews are time consuming, our findings indicate they must be conducted to develop and validate interventions for developing PSTs' 3UC.

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ELICITING DISPOSITIONS FOR TEACHING IN THE CONTEXT OF A VIDEO-BASED INTERVENTION FOR SECONDARY TEACHER CANDIDATES

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For years, teacher education programs have focused considerable effort on teacher knowledge and how to develop the types of knowledge that matter in teacher education candidates. Meanwhile, candidate dispositions for teaching have received little attention, particularly in mathematics courses for candidates. As developers and practitioners of a curriculum intervention designed to support candidates' mathematical knowledge, we are beginning to see how much disposition towards teaching mathematics matters in a candidate's ability to attend to students' ideas. In this paper we share results from a pilot study investigating the dispositional characteristics elicited in an online video-based curriculum focused on students' ideas on a figural pattern task. Results indicate that efforts to cultivate secondary candidates' disposition for teaching may have payoffs with respect to both dispositions and knowledge.

Keywords: Affect, Emotion, Beliefs, and Attitudes, Preservice Teacher Education, Teacher Beliefs, Teacher Noticing

Video-based instructional interventions have been used in mathematics teacher education for decades (Lampert & Ball, 1998; Philipp, 2008; Seago et al., 2004) and can have a positive impact on teachers and teacher candidates' (TCs') mathematical knowledge (Jacob et al., 2009), professional noticing skills (van Es & Sherin, 2008), and knowledge of students' conceptions of mathematics (Powell et al., 2003). Influenced by others' success with video-based interventions, two of the authors embarked upon a design-based research (DBR) project, VCAST (video case analysis of student thinking). From the beginning of the VCAST project, we hypothesized that engaging TCs in analysis of video and written evidence of student thinking could serve as a meaningful way to structure candidate engagement with a) key ideas of the secondary mathematics curriculum and b) a range of productive ways students might interact with those same key ideas. And while candidate data do support our initial hypotheses, themes related to candidate dispositions emerged. As a result, we recently turned our attention to how we might cultivate particular dispositions for teaching mathematics in TC in the context of attending to students' mathematical work.

An important part of DBR is the involvement of practitioners--those responsible for implementing the intervention (Amiel & Reeves, 2008). To that end, a team of practitioners (partner instructors) and VCAST curriculum developers (developers) collaborated to investigate the evidence of dispositions for teaching mathematics elicited as secondary mathematics TCs engaged in a curricular module focused on student thinking on figural pattern tasks. Our research question is: *What do partner instructors and developers learn about candidate disposition towards teaching the mathematics of figural pattern tasks?* In this paper we share a summary of the results of our collective analyses, including interpretations from each partner instructor, along with implications for module revision and implementation.

Background

This is the fourth year of VCAST, a four-year DBR project funded by the National Science Foundation (Award #1726543) focused on designing video-based curriculum to improve secondary mathematics TCs' ability to attend to student thinking. In this section we provide an overview of the project and make connections to the literature relevant to the current study.

Mathematical Education of Teachers

The mathematical preparation of teachers has received significant attention over the last couple of decades, with mathematicians and educators collaborating on the educational expectations for beginning teachers at various levels of the school curriculum (CBMS, 2012). A repeated theme is the importance of being able to elicit and interpret students' ideas (NCTM, 2014). For instance, a TC's ability to complete a mathematical task, recognize the potential mathematical complexities for students, make inferences about a particular student's understanding based on the evidence students produce, and then decide on an appropriate response that builds upon, as opposed to simply redirecting or correcting, that student's thinking all rely upon various subdomains of candidate knowledge.

Dispositions for Teaching Mathematics

We think about dispositions for teaching mathematics as a set of interrelated habits of mind that teachers embrace to carry out their practice. Interestingly, professional standards documents for teacher education programs rarely address these habits of mind explicitly. Rather, they are implicit in the sets of knowledge and skills TCs are expected to acquire by the time they enter the teaching profession. However, one can readily see the influence of various fields of study with standards that advocate for 1) the use of culturally relevant pedagogy (Ladson-Billings, 1995), 2) the application of educational ethics and caring (Noddings, 2003), 3) attending to students' mathematical thinking (Author, 2017; Jacobs et al., 2010), and 4) understanding power and privilege in the history of mathematics education (Gutierrez, 2013).

Given the focus of VCAST, this study is centered upon dispositions associated with attending to students' mathematical reasoning. That is, we are concerned with the habits of mind needed to put a candidate in the best position possible for analyzing and interpreting student thinking. We have tentatively identified two such habits of mind: *awareness of differences in reasoning*, and *adaptability in one's own thinking*. Awareness of differences in reasoning is about acknowledging that individuals will necessarily have different ways of reasoning in sensible ways about mathematics. Such awareness involves considering any evidence of student reasoning on its own merits and places great value on individual student perspectives. With awareness, the expectation is that students' ways of reasoning are sensible and it is up to the TC to identify how the student is making sense of the ideas. Adaptability in one's own thinking is about being willing to revise one's knowledge and assumptions when presented with additional evidence that warrants such a change. Adaptability involves recognizing that all knowledge is tentative and the active pursuit of additional information and evidence in an attempt to more fully understand. In the context of attending to student mathematical reasoning, TCs demonstrate adaptability when they recognize that additional information about a student's reasoning may alter their perceptions about what the student understands and is able to do. These habits of mind provide the foundation for our coding framework.

The Intervention

From a design perspective, we focused on featuring nonstandard mathematics tasks, collecting evidence of secondary students working on those tasks, and then selecting artifacts of student evidence that revealed a range of productive approaches and strategies to solving those tasks. The intent was to introduce TCs to new ways of thinking about the featured tasks and to support the development of attentiveness (Carney et al., 2017). Developers also purposely selected artifacts to illustrate how students' productive struggle can lead to important insights.

Design and context. The intervention consists of four modules, each of which features an asynchronous online component, a synchronous in-class component, and an asynchronous exit ticket. TCs engage with the asynchronous components via the project's digital platform and instructor support materials are made available through the project's website. The in-class component can be completed in a variety of synchronous formats and leverages social learning through group activities. The modules are designed for use in the Functions & Modeling course, a mathematics course for secondary mathematics TCs taught at replication sites of the UTeach teacher preparation program. The case studies reported on here involve partner instructors from the third and fourth year of implementation.

The hexagon task module. The Hexagon Task (see Figure 1) is a figural pattern task designed to encourage far generalizations which can be determined using a variety of approaches. For example, a student might choose to focus on how the configuration of hexagons contributes to the perimeter, on how the perimeter increases from one figure to the next, or perhaps a combination of these and other approaches. The range of approaches that can be productively leveraged while completing the Hexagon Task afforded multiple opportunities for TCs to examine a variety of students' mathematical reasoning (Cavey et al., 2018). The task also requires students to attend to three interrelated quantities: the figure number, the perimeter of the figure, and the number of hexagons in the figure.

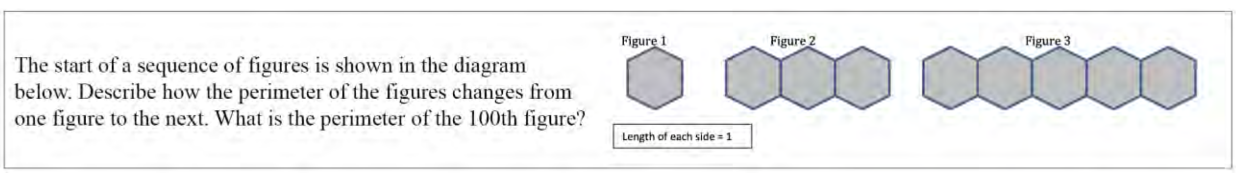

Figure 1: Adapted Hexagon Task; Hendrickson et al. (2012)

The module features video and written evidence produced by three students who approached the task differently and exhibited productive struggle in a range of ways. The pseudonyms and images of each student, along with their final written work included in the module, are provided in Figure 2. Ashley focused on geometrical aspects and developed a function for the perimeter of a figure based on the number of hexagons in the figure. Maria began her work on the task using an approach similar to Ashley's but then switched to an approach that focused on using the increase in perimeter from one figure to the next to determine the relationship between perimeter and figure number. Brandon, like Ashley, remained focused on the relationship between perimeter and the number of hexagons. He made a more explicit assumption that there are 100 hexagons in the 100th figure, recognized his error, and then tried to correct his final answer by using recursive reasoning to determine the number of hexagons in the 100th figure.

The in-class component features written student evidence from an additional six students. This work was selected for candidate group analysis and discussion. Providing this broader range of student thinking affords social construction of candidate knowledge related to the mathematics of figural pattern tasks and how students think about and reason with that mathematics (Franke and Kazemi, 2001). By highlighting areas of secondary student struggle, our intent was to help TCs gain an appreciation for the complexity of figural pattern tasks and to foster empathy for how each individual student navigated that complexity.

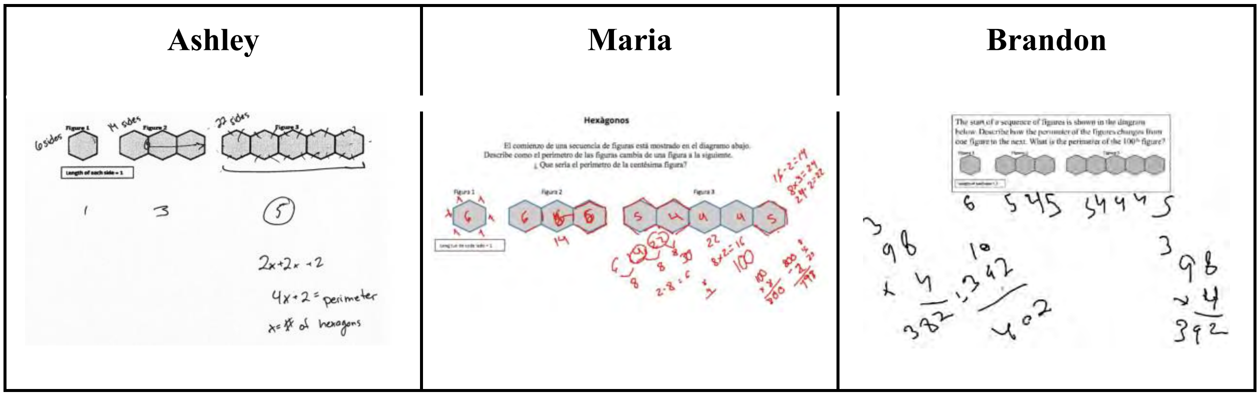

Figure 2: Student evidence featured in the asynchronous components.

Methods

Participants and Settings

Partner instructors for project X were recruited using the UTeach email listservs and during annual conference presentations for Functions & Modeling instructors at the UTeach conference. Both instructors for this study are faculty at mid-size, public university UTeach replication sites located in the United States. Instructor K implemented the year 3 version of the VCAST curriculum materials in fall 2019, whereas Instructor N implemented the year 4 version in fall 2020. As such, Instructor K was in the first group of partner instructors and was able to meet with their students in a standard face-to-face classroom setting for the in-class component. Instructor N was in the second group of partner instructors, who had the benefit of improved materials based on the lessons learned in year 3, but implemented the in-class component remotely due to the ongoing COVID-19 pandemic. Both instructors used the same order of modules during their implementation, with the Hexagon Task as the second. Candidate participants who elected to participate in the study were undergraduate students enrolled in the partner instructors' courses. See Table 1 for summary information about each instructor.

Table 1: Partner Instructors and Their Candidate Participants

	Participation Year	# of TCs	In-Class Format
Instructor K	Year 3 (fall 2019)	8	face-to-face
Instructor N	Year 4 (fall 2020)	13	remote

Data Collection

Data for these case studies were collected using observations, instructor reflection surveys, digital captures of candidate work produced during the in-class component, and the online platform designed specifically for VCAST's delivery of asynchronous module content. Instructor K's in-class session was recorded by a VCAST research team member. Instructor N's class was captured via Zoom, as its synchronous enactment occurred online during the COVID-19 pandemic. Following implementation, each instructor submitted reflection feedback via a Google Form. Each reported their perceptions regarding candidate engagement with the module content and uploaded candidate artifacts produced during the in-class session. Candidate data consisting of responses to asynchronous module prompts were collected digitally via the VCAST digital platform and then downloaded for analysis.

Data Analysis

Candidate data were analyzed using a coding framework derived from the literature on professional noticing (Jacobs et al., 2010; van Es, 2011) and attentiveness (Carney et al., 2017; Carney et al., 2019) that focuses upon the two habits of mind outlined earlier for dispositions associated with attending to student thinking. With respect to awareness, we looked for evidence that the candidate was able to focus explicitly on a student's way of reasoning with the Hexagon Task rather than imposing their own ideas or that the candidate engaged in making sense of student reasoning. With respect to adaptability, we looked for evidence that the candidate was receptive to new information about a student's reasoning and for evidence that the candidate was willing to acknowledge when their own original ideas were proven incorrect.

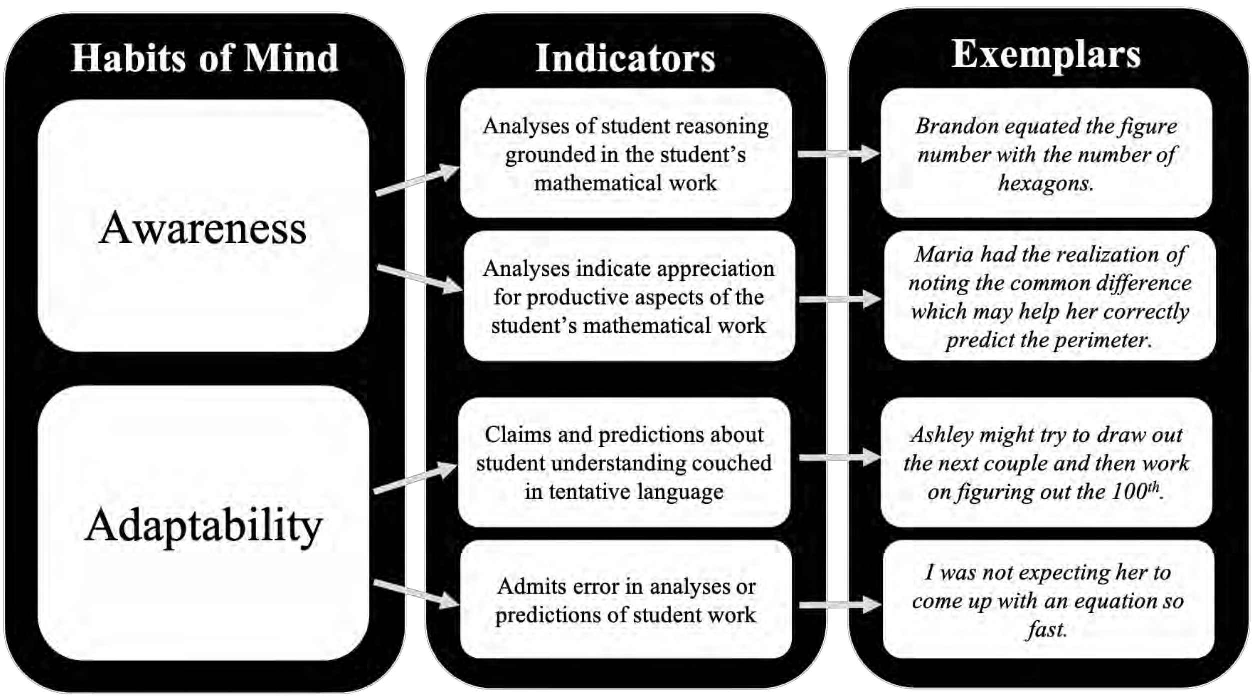

Figure 3: Selected Indicators of the Candidate Dispositional Coding Framework

To start, researchers decided on the unit of analysis to be coded using a methodology similar to that used by van Es and colleagues (van Es et al., 2014). Because we were interested in analyzing evidence of candidate disposition elicited through sequences of student work analysis, we first identified the particular segments of data, or units of analysis, we felt were most likely to provide this evidence. Pairs of researchers then applied the coding framework to a selection of data for each instructor, then met to calibrate codes and reach consensus on the meaning of framework indicators. Following these conversations, all four researchers met to share results of calibration conversations and to collectively refine indicators and interpretation of the framework. The original researcher pairs then coded and calibrated the remaining data for their assigned instructor. Each unit of analysis was independently coded by two researchers and calibrated until consensus was met.

In the following section, partner instructors present emerging findings from their case studies. We deliberately use a first-person narrative so as to enable readers to gain deeper insight into individual perspectives and lenses that instructors used to interpret candidate data.

Instructor K's Results

As a mathematics educator who prepares TCs to teach students with diverse backgrounds and experiences, my goal is to implement strategies that increase their mathematical knowledge, skills, and dispositions for teaching all students. Thus, when approached to participate in VCAST, with its focus on student thinking, I saw it as an opportunity for TCs to develop an awareness of various student approaches and openness towards multiple forms of reasoning. Additionally, I hypothesized that engaging in the cognitively demanding tasks and productive struggle could lead to an increase in TCs' content knowledge and an appreciation of their future students' struggle. My experience as an instructor, coupled with the recent analysis of the Hexagon Task data, indicates growth in the TCs' awareness of student thinking and evidence of their ability to predict student moves and admit when those predictions were incorrect.

Growth in Awareness of Student Thinking

Initially, when TCs reviewed written students' responses to the task associated with the first module, they focused their analyses on whether students' work was correct or incorrect. However, to develop their attentiveness, I prompted my TCs to look beyond correctness by focusing on the students' explanations. By the second module, 4 out of 8 TCs commented on how the module made them aware of the multiple solution paths, and all of the TCs were describing, discussing, and making comparisons between the various students' approaches or comparing their approaches to those of the students. For example, TC4 wrote "In my process, I did not use the number of hexagons to calculate the perimeter. [Ashley] took the number of hexagons times 2 for the bottom perimeter and did the same for the top then added the other sides." TCs were also able to analyze and interpret student thinking based upon evidence. Consider the following statements by TC3 and TC9:

- TC3: Maria wrote 6, 14, 22. Then drew a line in between them to show the increase of 8. This is the increase per figure …, to find out the perimeter per figure.
- TC9: So, by multiplying 4 by 98 and adding the extra sides for the end [h]exagons [Brandon] is assuming there are 100 Hexagons in the 100th figure.

Evidence of Adaptability

When making predictions about students' thinking, TCs demonstrated adaptability in both their tentative language and their willingness to revise their assumptions. Most of their speculations began with the phrases "I think …", "I believe …" or "She may or might..." indicating the TCs' were trying to identify how the student is making sense of figural patterns. The data analysis revealed that 7 out of 8 TCs admitted to incorrectly predicting Ashley's next move, while only 3 out of 8 admitting errors with Maria, possibly because their approach was more similar to hers. When the students' actions did not match the prediction, the TCs would either acknowledge the error or adjust their interpretations. The TCs made comments like "I'm surprised!", "My prediction was completely wrong with what Maria actually did," and "[Ashley] … pulled in the number of hexagons in the figure …, which I did not anticipate."

As the Hexagon Task was only the second of the four modules, the TCs were already showing evidence of important skills necessary for effective mathematics teaching. They were able to shift their focus from the correctness of students' answers to an awareness of diversity in students' mathematical reasoning. They were able to assess, compare, and make predictions about students based on video and written work, and became open and willing to learn and revise their assumptions when presented with new evidence. Even the two TCs who initially incorrectly solved the task themselves exhibited these skills. This suggests that giving TCs opportunities to examine, discuss, and predict student thinking may help them develop effective teaching practices to use in their classrooms.

Instructor N's Results

As a mathematics teacher educator who focuses on equitable teaching and culturally responsive pedagogy, I emphasize the importance of treating all students as capable learners. Thus, I was interested in how the VCAST materials, with their emphasis on the analysis of student evidence of mathematical reasoning, would provide opportunities for me to surface and support the dispositional development of my TCs. Analysis of their data from the Hexagon Task module illuminated several interesting areas of potential insight and growth for my TCs. I discuss two themes in particular that emerged from my TCs' engagement with student work analysis: (1) TCs appeared to grow in their own mathematical understanding and (2) TCs appeared to develop a more empathetic stance toward the students whose work they analyzed.

Growth in Mathematical Understanding

Data analysis indicates that 6 out of 13 students exhibited growth in their own mathematical understanding, either by improving the quality of what they noticed and described in student strategies, articulating that a featured student strategy was something they had not initially thought about, or by solving the adjusted version of the task correctly after submitting an incorrect answer for the Hexagon Task. For example, TC3 responded, "I realized that Maria's way of thinking also works and makes a lot of sense, even though I hadn't initially considered thinking the way she did" and TC13 observed, "Some of the strategies used by different people for this task surprised me because I did not think of the problem in those ways." For TC5 and TC14, both of whom initially solved the Hexagon Task incorrectly, analysis of student thinking not only appeared to reinforce their dispositional traits, but also appeared to enable them to correct their own mathematical errors and solve a related task, presented later in the module, correctly.

Growth in Empathetic Stance

Data analysis indicates that 10 out of 13 students exhibited growth in their ability to empathize with students' struggle with the task. For instance, TC13 acknowledged, "Pattern tasks are really easy to get confused on if you do not know what to look for," while TC10 noticed, "Brandon is focusing on how many hexagons each figure has, and he is struggling to find the pattern to find the number of hexagons in the 100th figure." Developing more empathy towards students' mathematical reasoning also allows for more flexibility in their interpretations and helps TCs recognize that students' mathematical thinking is fluid. As TC13 notes, "Predictions are predictions, they are not factual. Always be ready for any reaction or questions asked by the students."

TCs across the board agreed that 'brief isolated episodes' may not portray a complete picture of students' thinking and that it is necessary to initiate and engage in an ongoing mathematical discourse to gain insight into their thinking. This is evidenced by TC5, who observed, "It helped me make sure that I try to understand each student's thinking and why they did [a] certain mathematical process" and TC6, who realized, "Students need to be given time to show their mathematical process, thinking, and reasoning before making assumptions."

During my implementation of the Hexagon Task module, I noticed my TCs shift from attending to students' ideas for the purpose of evaluating student work to a desire for understanding students' mathematical ideas. The analyses of TCs' responses to the Hexagon Task module not only support my impressions during implementation but also highlight other areas of growth. From an equity standpoint, I am excited about the potential to cultivate TCs' dispositions for teaching mathematics while also supporting TCs' mathematical knowledge.

Discussion and Implications

Instructors and developers, alike, observed shifts in the evidence of TCs' dispositions as TCs engaged with the VCAST materials. Our curiosity about this phenomenon led to a shared interest in investigating the extent to which TCs' dispositions for teaching mathematics were elicited with a single module. Since none of us had previous experience researching dispositions for teaching, one primary aim was to settle on a framework that would allow us to capture the nuances we observed in the language TCs used when analyzing and reflecting on their analyses of student evidence for reasoning. By doing so, we not only have a framework for future analyses, but we also uncovered several potential directions for future research across all partner institutions as well as implications for module revisions.

For one, we did not expect to see marked shifts in evidence of TCs' dispositions within a module. Moreover, we observed shifts in TCs' dispositions in two distinct ways. The data from Instructor N's TCs showed impressive gains in disposition from the beginning to the end of the online component, with all TCs demonstrating evidence of awareness and adaptability by the end. For Instructor K, we observed shifts in evidence of TCs' dispositions in relation to the students featured in the module. Naturally, we wonder, *In what ways does the evidence for TCs' dispositions for teaching mathematics shift when engaging in a video-based intervention focused on student thinking on figural pattern tasks?*

Second, while the focus of this pilot study was on TCs' dispositions, our work has led to a hypothesis about the relationship between disposition and the ability to learn from students' mathematical work. Of the TCs who started the module with an incorrect solution, those who exhibited multiple indicators from both habits of mind were more likely to correct their mathematical errors by the end of the module. As a result, we wonder, *How are TCs' disposition for teaching mathematics related to their ability to learn mathematics from students?*

Lastly, our analyses revealed a gap in module questions about Brandon's reasoning evidence. In particular, the current questions are not structured to elicit evidence with respect to one of the indicators for adaptability. Thus, the developers must now decide whether that type of evidence is desired and how to restructure the questions to elicit that evidence.

In summary, what began as a trend in TCs' module responses about student thinking has evolved into a list of potential lines of inquiry into TCs' disposition for teaching mathematics. And while we have more questions than answers at the end of this study, we hope this work sparks interest from the larger field of professional noticing.

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PROSPECTIVE MATHEMATICS TEACHERS' DESIGNED MANIPULATIVES AS ANCHORS FOR THEIR PEDAGOGICAL AND CONCEPTUAL KNOWLEDGE

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Positioning teachers as designers of curricular resources invites opportunities for exploration at the intersection of content, pedagogy, and design. As researchers accepting greater responsibility for preparing teachers to maintain a commitment to their pedagogical vision in practice, this work seeks to cultivate the imagination of humanistic forms of mathematics teaching and learning by supporting these explorations. Toward that end, this paper reports on research that examines connections between the pedagogical/conceptual knowledge that prospective teachers embed in the designs of original manipulatives and how those designs mediate the pedagogical moves they make in teaching situations. The promise of this work is that these connections may reveal a viable means to support bolder connections between teacher preparation and practice. Implications of our findings for teacher preparation are considered.

Keywords: Teacher Knowledge, Technology, Preservice Teacher Education

It is an unfortunately perennial problem that teachers often experience considerable challenges in transferring their theoretical knowledge into practice (Ünver, 2014). While teacher education programs that explicitly link teacher preparation coursework to field experiences tend to be more effective than those that do not (National Academy of Education, 2005), colleges and universities have often been criticized for implementing teacher education programs that do not sufficiently engage their students in actual and ongoing practice situated in authentic education settings. Although future teachers tend to craft their pedagogies as they learn about research-supported instructional methods, teacher educators also stress the importance of developing one's practice in real classrooms with real students (Kazemi, et al., 2009). It is with this critical concern in mind that the field seeks to determine the means by which teachers can transform teacher knowledge from theory into practice through approximations of practice (Grossman, et al., 2009) that simulate the work of teaching.

Our work connects with the body of literature that frames teachers as *designers* (e.g., Brown, 2009; Svihla et al., 2015) of teaching and learning experiences and the material resources that mediate them. We conceive of design broadly to include the "intentional activity of transforming ideas and knowledge" (Carvalho et al., 2019, p. 79) into "tangible, meaningful artifacts" (Koehler & Mishra, 2005, p. 135). Our purpose in doing so is to present a novel Making experience within mathematics teacher preparation that we hypothesized would inform their conceptual and pedagogical thinking. Making in this sense is conceived as the creative production of artifacts via activities that include designing, building, and innovating with tools and materials to solve practical problems (Halverson & Sheridan, 2014). Thus, the experience tasks prospective mathematics teachers (PMTs) with digitally designing (using Tinkercad; Autodesk Inc., 2016), 3D printing, and evaluating original manipulatives that are responsive (Akuom & Greenstein, 2021) to the curricular (Dewey, 1990; Pinar et al., 1995) needs and interests of actual learners.

While there is a considerable body of research on *students'* mathematical Making (e.g., Bower et al., 2020; Valente & Blikstein, 2019), research is only beginning to uncover the benefits that *teachers* experience in Making contexts (Greenstein & Seventko, 2017; Greenstein & Olmanson, 2018; Greenstein et al., 2019). Our prior research (Akuom & Greenstein, 2021) addressed this gap by exploring the conceptual, social, and material resources that mediate (Vygotsky, 1978) the design decisions of prospective teachers' Making of mathematical manipulatives. This paper reports on research that extends that work by discerning whether connections can be made between the pedagogical/conceptual knowledge that prospective teachers construct in teacher preparation and how that knowledge is enacted in their teaching. Specifically, this work seeks to address the question: *As prospective teachers Make new manipulatives for mathematics teaching and learning, can connections be made between pedagogical/conceptual resources for their design decisions and how those designs mediate the pedagogical moves they make in practice?* If connections can be made between the knowledge that prospective teachers construct in teacher preparation, how that knowledge materializes in their designs of physical manipulatives, and how those knowledge-embedded designs mediate their teaching interactions, we propose that these findings can illuminate and subsequently strengthen the relationship between instructional intention and enactment in particular (see Remillard, 2018), and teacher preparation and practice more broadly.

Theoretical Framework

Fundamentally, this research is about the mediating role of conceptual, social, and material resources in design activity. In particular, we seek to extend prior research on the resources and rationales that mediate design decisions when designing a tool by exploring the mediating role of those tools in teaching situations. Accordingly, we take a sociocultural perspective and ground this work in the notion of mediated activity, derived from Vygotsky (1978) and advanced as instrumented activity by Verillon and Rabardel (1995). In terms of instrumented activity, an artifact is a material object that becomes an instrument (e.g., tool, sign) for the subject (e.g., actor, learner, teacher) when the subject has integrated it with their activity. Thus, an instrument is a psychological construct (as opposed to a material one) that "results from the establishment, by the subject, of an instrumental relation with an artifact" (p. 85). What the distinction between artifacts and instruments reveals is the possible range of actions one might take with an artifact and what those actions might implicate about a subject's knowledge. For our purposes, we are specifically interested in PMTs' pedagogical and conceptual knowledge and how their practice is mediated by such knowledge as it is intentionally embedded in their designed artifacts.

In our prior research, we analyzed PMTs' design decisions – and the rationales they gave for those decisions – as they made original manipulatives to teach a mathematical concept. As they designed these manipulatives, it was the PMTs' intention (Malafouris, 2013) to embed their tools with particular affordances (Gibson, 1977) for utilization schemes (Verillon & Rabardel, 1995) that they hypothesized would enable the child to form abstractions, through their sensorimotor engagement (Kamii & Housman, 2000; Piaget, 1970), of the perceptual elements that are the groundings (Nathan, 2014) for target concepts. As this *learning by design* (Koehler & Mishra, 2005; Koehler et al., 2004) process invites occasions for their active inquiry, PMTs made a host of design decisions for a variety of reasons; they drew on a range of conceptual, social, and material resources to mediate them. In order to characterize and organize these resources, we appealed to Sc (1992) design-centered notion of "knowing in action" (p. 2). Sc considers knowledge to be in action as "the designer sees what is 'there'… draws in relation to it, and sees what [they have] drawn, thereby informing further designing” (p. 5). This thought-revealing (Black & Wiliam, 1998) process of seeing-drawing-seeing is what Schön means by the phrase “designing as a reflective conversation with materials” (p. 3).

For this phase of the research in which we analyze PMTs’ usage of tools in practice, we use the term *embedding* to connote an intentional design element that embeds a PMT’s pedagogical and/or conceptual (i.e., mathematical) knowledge. As an example, a PMT named “Moira” designed a fraction tool with a variety of fractional pieces of a whole. She was concerned that if each piece had its own unique color, that might “take away reasoning from children. If a student believes that a yellow ring represents sixths, they will immediately reach for yellow the second that they hear sixths.” By giving the pieces the same color and leaving them “unmarked,” she intended for children to construct their own meanings for each of the [pieces]. Thus, we say that pedagogical/conceptual knowledge mediated this design decision and refer to the corresponding design element as an *embedding* of that knowledge. In addition, when we infer from a PMT’s use of the manipulative in a teaching situation that the tool served as a resource for (e.g., a reminder of) pedagogical and/or conceptual knowledge embedded in the tool, we will refer to that as an *anchoring* phenomenon, as in, “Moira’s fraction tool served as an *anchor* for her attention to the pedagogical practice of implementing tasks that promote mathematical reasoning.”

Methodology

This study is part of a larger project that aims to test and refine the hypothesis that a pedagogically genuine, open-ended, and iterative design experience centered on the Making of a mathematical manipulative would be formative for the development of PMTs’ inquiry-oriented pedagogy. The larger project took place across two semesters of a graduate-level specialized mathematics course for PMTs at a mid-sized university in the northeastern United States. Forty students comprised thirty-four groups. For the study reported here, we took an exploratory case study approach (Yin, 2009) in order to determine what connections could be made between pedagogical and conceptual rationales for PMTs’ design decisions and how those designs mediated the pedagogical moves they made in enactment. We did so by taking as the unit of analysis *instances* in PMTs’ teaching when the use of their manipulative implicated the pedagogical and/or conceptual knowledge underlying their design rationales. The locus of these particular research efforts among the broader research project is depicted as the arrow from “Design Decision” to “Enactment” in Figure 1. In addition to the PMT’s designed manipulative and a video recording of problem-solving interviews with them and their elementary-age focus student, four written project components comprised the data corpus: a “Math Autobiography,” an “Initial Idea Assignment,” a “Project Rationale,” and a “Final Paper/Reflection,” which includes findings from their problem-solving interviews.

We took a grounded theory (Corbin & Strauss, 2008) approach to analyzing the data. We began by collectively analyzing the written and video components of one PMT’s design case to identify instances in their teaching from which we could infer that the PMT leveraged a particular embedding of a design decision in their manipulative to enact a teaching move that was consistent with aspects of their purported pedagogy, which they shared in the written artifacts of their Maker projects. These inferences constitute our conjectures that their designed manipulative served as an anchor for the pedagogical/conceptual knowledge they had been constructing in the course. We generated codes for this design case to characterize connections between embeddings of design decisions and their mediating role in the PMTs’ teaching. Next, we collaborated to identify additional instances of anchoring in other design cases. Analysis involved the constant comparison of data to ensure coherence is maintained across the generated codes and to get a good sense of the variety of ways in which affordances of the designed manipulatives that were either intended (those that PMTs intended to embed in their tool) or unintended (those that PMTs hadn't intended but realized in practice) could be leveraged to support a PMT's pedagogy.

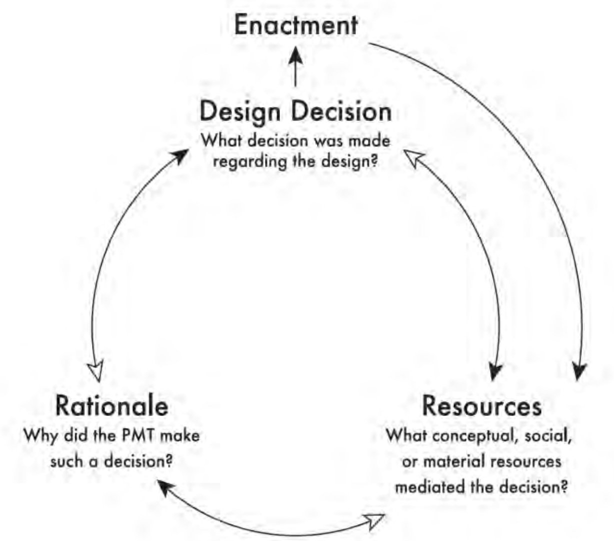

Figure 1: Conceptual resources inform rationales for design decisions and may also be evoked in enactment. Open arrows acknowledge that feedback is reciprocally informing.

Results

Here we present just three excerpts from among the thirty-four task- and tool-based problem-solving interviews that PMTs conducted with the intended user of their manipulative. Findings from our analyses of these excerpts suggests that they are instances in a PMT's teaching when a pedagogical move they made was mediated by the instrumental leveraging of a design affordance whose rationale was explicitly linked by the prospective teacher as designer to their pedagogical and/or conceptual knowledge. In short, these are instances in which a design embedding served as an anchor for a PMT's pedagogical and/or conceptual attention. The manipulatives mentioned in these results are shown in Figure 2.

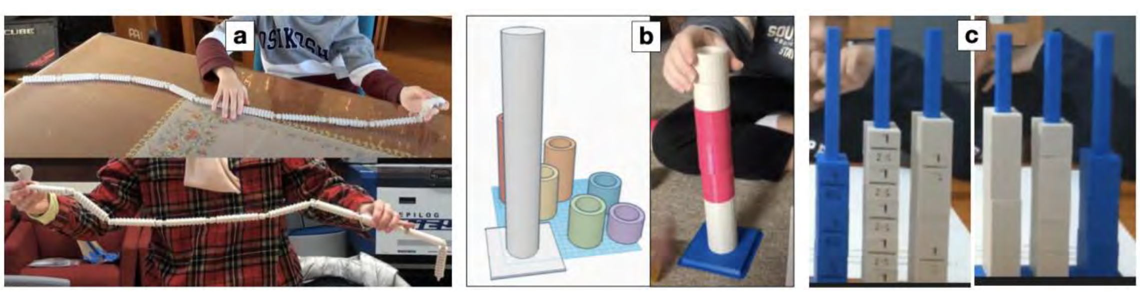

Figure 2: (a) Roda's decimal tool; (b) Kerina's fraction tool; (c) Anyango's fraction tool.

Reasoning about the unit whole

Roda designed a "Decimal Snake" in order to teach a child about decimals and decimal comparison. As shown in Figure 2a, her tool consists of ten connected pieces. Each of these pieces is equally partitioned into ten parts. Thus, the decimal snake can be used to represent tenths of tenths, or hundredths, of a whole, that is, any value between 0.01 and 1 to two decimal places. These design features are Roda's embeddings of the concepts of the whole and its decimal parts.

At one point in the interview, we observe Roda asking the child to compare 5.5 and 5.47. [Note that it would not be possible to represent 5.47 if the entire snake represented 1 or even 10.] The child responds, "5.47 is 5 and 47 hundredths, because it's 3 hundredths away from 5 and 5 tenths." Perhaps because Roda is interested in how her tool can support the child's reasoning, she then asks him to "Use the tool to show me?" Over the next sixty seconds, we witness the child struggling to locate 5.5 and 5.47 on the tool. Finally, he locates 5.5 at (what we would identify as) 0.55 (if the entire snake represented 1), and 5.47 at 0.47. Given that several minutes earlier the child established that entire snake is the "whole" and that each piece of the snake is one tenth of a whole, we infer from his solution – locating 5.5 at 0.55 – that he had unintentionally designated each piece of the snake as 1 (as opposed to 0.01) and each partition of a piece as 0.1 (as opposed to 0.01). In doing so, he changed his designation of the entire snake from the whole (1) to 10, and consequently, each piece of the snake now represented 1. Thus, 5.5 would be presented as the 5th partition of the 5th piece.

Roda's next move aims to help the child identify and resolve this confusion. When she asks him to "Show me one tenth," he points to one of the tenth pieces. When she asks for, "Two tenths," he points to the second piece. Then she asks, "Where is 5 and 5 tenths?" And in doing so, she perturbed his thinking and provoked disequilibrium. Soon thereafter, he resolves it and declares, "Oh, wait! This [entire snake] is one whole! 5 and 5 tenths, you can't even make it out of the snake!" In response to this unanticipated move in the child's activity, Roda leverages an affordance of her tool – namely that each piece of the snake could represent either a tenth of a whole or one of ten wholes – and she exploits it to support new ways of thinking for the child as he resolves his confusion about the representational capacities of the tool.

R(oda): You need how many snakes to make 5.5?

C(hild): You need 5– No, 6 snakes!

R: How can we compare [5.5 and 5.47] using 1 snake? Is that possible?

C: We can pretend that each piece is one snake.

In this instance, Roda leverages the embedding of a conceptually resourced design decision that enabled the snake's user to engage in conversations about the unit whole. Specifically, she leveraged a design decision that allows for flexibility in naming the unit whole in relation to the snake and its pieces. And her rationale for leveraging that affordance was a pedagogical one. Rather than correct the child's interpretation, she sought to help him reason through his interpretations in order resolve the confusion himself. In this respect, the tool's capacity for flexible interpretations of quantities (a conceptually resourced design decision) served as an anchor for pedagogical knowledge about the value of revealing student thinking and posing purposeful questions to advance their mathematical reasoning. Worth noting, Roda did not plan for this conversation about the unit whole, nor had she anticipated it. Regardless, her tool mediated activity that made it possible to do so.

Generating a space of inquiry

The second instance we present is from the problem-solving interview that Kerina conducted using the fraction tool she designed for conversations about the meaning of a fraction's denominator (see Figure 2b). Kerina's tool features "*a variety of rings which each represent different fractions (from 1/½o 1/8) that are scaled in relation to the pedestal [whole] that they go on top of.*" Each set of like fraction pieces is a *"different color, so it's easy to determine which pieces are the same size."* When fraction pieces are stacked on the pedestal, the tool provides feedback to the child that they can use to determine whether that combination is equivalent to a whole.

Kerina's fraction pieces have no identifying attributes other than color, so if a child wanted to determine what fraction of a whole is represented by a pink piece, for example, they would make that determination by seeing how many pink pieces it takes to "fill" one pedestal. If 6 pink pieces fit on a pedestal, then each pink piece would represent ⅙. This finding would give meaning to the 6 in the denominator of fractions of the form $n/6$. As she designed her manipulative, Kerina was mindful that students tend to struggle with symbolic representations of fractions, particularly in the context of adding fractions and "finding least common denominators." As an alternative, she proposed that "students' brains will work in more creative ways than we can anticipate." Accordingly, she wanted to design her tool that would accommodate such diversity and enable students to "visualize" concepts and avoid the "frustration" that purely symbolic approaches to fractions often cause.

With these intentions in mind, Kerina embeds a particularly salient feature of her pedagogy in the design of her tool that is made evident in one task that challenges a child to use the tool to "Find three different ways to make a whole." Operating in tandem with a tool that requires its users to construct their own meanings for each of its pieces, the task generated a space (Stroup et al., 2004) for the child's active, creative, and playful inquiry and insight into fraction meanings and relationships. Indeed, Kerina designed her tool for such an imagined utilization scheme in which the child, at least initially, uses trial and error to stack different pieces onto the pedestal and then "see how much space is left" before adding on more pieces to make the whole. These accomplishments would be seen as groundings (Nathan, 2014) for connections she would subsequently help the child make to symbolic representations of their tool-based activity.

In practice, we observed Kerina's commitment to her design intentions. At one point, when she posed her "Find three ways" task, the child selected pieces of the same size to place on the pedestal in order to form a whole. Kerina notices this strategy and asks the child to "Try to use ones that have different denominators." Note her use of "different denominators" as opposed to "different sizes," even though she's referencing physical objects. In doing so, she is cultivating a connection between physical and symbolic representations of fractions. At the same time, it's also important to note that Kerina had written the symbolic names of each fraction piece on its interior where they could be concealed from the child's view. Thus, she seems to have a trajectory in mind for the meaningful development of fraction proficiency from physical to symbolic representations of collections of different unit fractions. Her tool and tasks are anchoring pedagogical and conceptual knowledge that mediate her response to the child's initial activity at this moment as she supports his construction of procedural fluency on a foundation of conceptual understanding. Specifically, design elements of her tool embed conceptual knowledge relevant to that trajectory (e.g., a "complete" stack of pieces represents a sum of unit fractions equal to 1), and design elements of both the tool and the task embed pedagogical knowledge about the value of enabling multiple solution strategies in order to generate a space for open and productive inquiry.

Noticing in action

Anyango designed a fraction tool “to help the student visualize and deepen their understanding as they explored fraction relationships.” Her tool looks similar to Kerin’s and appears in Figure 2c. In contrast, however, Anyango emphasizes a different purpose for a similar affordance. She explained that her design decision to stack fraction pieces on vertical pegs rather than lining up those pieces horizontally would enable her to use those pieces to represent “height as value and amount.” “What was most important to me,” she wrote, “was having all the fractions mounted on one platform with the 1 (whole) always being visible, so that the student could begin to grasp how all the smaller parts can equate and compare to the whole.” Also in contrast to Kerin’s design, Anyango engraved the name of each piece on one of its lateral faces.

In practice, Anyango posed the following task to an intended user of her tool: *Jack and his two friends each had the same size pizzas for lunch. Jack ate 5/8 of his pizza. Judy ate 2/3 of her pizza. And Sam ate 3/6 of his pizza. Who ate the most pizza? Who ate the least?* In response, the child stacks five one-eighth pieces, two one-third pieces, and three one-sixth pieces, each on their own pedestal with their labels facing him (as shown in the image on the left of Figure 2c) and says nothing further. Following up on the child’s activity, Anyango asks, “So, if we just look at this, who ate the most?” We interpret this pedagogical move by Anyango as one that leverages her design decision to represent fractional values in terms of height by directing the child’s attention to the relative heights of the three fraction pieces. In other words, she’s prompting the child to decide which person ate the most pizza by choosing the fraction piece that is the tallest, and which person ate the least by choosing the piece that is the shortest. Counter to her expectations, the child attended exclusively to the symbolic representations engraved on each piece and not their heights. This led him to decide that, “It’s Jack” (represented by the ⅝ piece) who ate the most. He justifies his answer by saying that “5 out of 8 is the biggest of all of them… 2 out of 3 is smaller and 3 out of 6 is… kind of small.” When Anyango asks, “What makes you think it is small?” he explains that, “The top is two and the bottom is three.” We infer from this response that the child is basing his comparisons on interpretations of fractions not as parts of a whole but as two separate whole numbers. This would explain why, for the child, ⅝ is greater than 3/6, which is greater than ⅔.

We interpret Anyango’s next move as a noticing one (Sherin et al., 2010) that leverages her pedagogical knowledge about the efficacy of attending to, interpreting, and responding to student thinking. Indeed, the design of her tool embeds this knowledge, as a primary rationale for its design was to enable a child to compare fractions without having to rely on the overhead of a symbolic representational infrastructure. In a move that we interpreted as unplanned and that was therefore striking for each of the researchers to observe, Anyango turns her tool around (see Figure 2c, right) in order to hide the symbolic labels on each piece.

A(nyango): If I turn this [*pedestal*] around [*so that the child’s gaze can no longer be restricted to the fraction labels on the pieces*], who has the most?

C(hild): This one [*points to the stack of two one-third pieces, which corresponds to Judy’s share*].

A: Who has the lowest?

C: This one [*points to the stack of three sixth-pieces, which corresponds to Sam’s share*].

What we find remarkable is that while Anyango made the intentional design decision to label each of her pieces, this "flipping" move leveraged an *unintentional* design affordance, that the opposite face of each piece is *not* labeled. In this regard, we suggest that Anyango's tool served as an anchor for a pedagogical *knowing* in action mediated by that affordance. Translating Sc oncept of knowing-in-action as a noticing-in-action, we suggest that in this instance, Anyango sees what is there, makes a move in relation to it, and sees what that move accomplishes, thereby informing her next steps. In those next steps, she returns the tool to its initial, label-facing orientation so that she can connect the physical representation of amount to the symbolic one, and asks the child, "Who ate the most?" "Judy," he says with a smile, and points to her stack of fraction pieces.

Concluding Discussion

This work set out to explore teacher learning at the interface between theory and practice by discerning whether connections can be made between the pedagogical/conceptual knowledge that prospective teachers construct in teacher preparation and how that knowledge is enacted in their teaching. The following question framed the inquiry: "*As prospective teachers Make new manipulatives for mathematics teaching and learning, can connections be made between pedagogical/conceptual resources for their design decisions and how those designs mediate the pedagogical moves they make in practice?*" We pursued this inquiry by analyzing approximations of practice in order to identify instances in PMTs' teaching when their manipulative served as a mediating anchor for pedagogical and/or conceptual knowledge acquired in teacher preparation and subsequently embedded in their designs.

Findings from previous work that explored the conceptual, social, and material resources that inform the rationales for PMTs' design decisions suggest that engagement in an open-ended and iterative design experience centered on the Making of a mathematical manipulative can be formative for their conceptual and pedagogical thinking. Findings from this work extend the value of that experience by considering the use of made manipulatives in practice. Specifically, the identification of instances of anchoring phenomena suggest that the experience can also yield material epistemic scaffolding (in physical manipulative form) that supports teachers and their commitments to the models of knowing and learning they construct in teacher preparation. Relative to theory, these findings suggest the analytic value of our design, rationale, resource, and practice (DRR-P) framework for revealing the promise of such an experience. Relative to practice, they suggest that the experience offers a viable means by which more robust connections between teacher preparation and practice can be nurtured.

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IMPACT OF MEDIATED FIELD EXPERIENCES ON TEACHER CANDIDATES' SELF-REPORTED LEARNING: A MULTI-INSTITUTIONAL DESCRIPTIVE PILOT STUDY

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Mediated Field Experiences (MFEs) provide teacher candidates (TCs) structured opportunities to unpack and enact core teaching practices, gain mathematics content and pedagogical knowledge, and reflect critically on mathematics teaching and learning. In this paper we present findings from a multi-institutional descriptive pilot study that investigates the impact of MFEs on TC learning. TCs reported that they (1) deepened their understanding of the targeted core teaching practice(s), (2) developed a vision of ambitious mathematics teaching, (3) recognized the importance of cultivating a positive classroom learning community, and (4) increased their confidence when teaching after their completion of a one-term course implementing MFEs.

Keywords: Preservice Teacher Education, Instructional Vision, Teacher Beliefs

Efforts to improve the preparation of teachers are not new, and recent publications have begun to define the knowledge, skills and dispositions of a "well-prepared" beginning teacher of mathematics to provide ambitious mathematics instruction (AMTE, 2017). By definition, ambitious teaching facilitates a learning environment that is accessible to all students because it requires that teachers teach in response to students' thinking and actions (Kazemi et al., 2009; Cawn, 2020). Beginning teachers are increasingly expected to teach ambitiously from day one (Anagnostopoulos et al., 2020), which requires teachers to engage deeply with each student's thinking and adjust their instruction accordingly to promote student learning—actions that are predicated on creating an environment that is accessible, strengths-based, and community oriented (Yeh et al., 2017). Creating such equitable spaces and enacting such practices is challenging. In particular, learning to listen effectively and respond to the variety of factors specific to students' thinking "is surprisingly hard work" (Empson & Jacobs, 2008, p. 257) requiring immense amounts of support during teacher preparation. We, along with others (Ball & Forzani, 2011) argue that attending, interpreting, and responding appropriately to students' mathematical thinking is a specialized pedagogical skill that needs to be explicitly taught within teacher preparation programs.

To meet this challenge, we re-envisioned our own mathematics content and methods courses, modeling our approach on the practice-based "third spaces" (Zeichner, 2010). In each of our respective initial certification programs, we as mathematics teacher educators (MTEs) now accompany our teacher candidates (TCs) into authentic classroom settings to prepare, enact, and reflect on practice in shared classroom spaces. These *mediated field experiences* (MFEs) have provided incredible opportunities for TCs to learn with and from children (Billings et al., 2021; Billings & Swartz, 2019; Campbell, 2012; Campbell & Dunleavy, 2016; Horn & Campbell, 2015; Knapp et al., 2018; Lynch et al., 2019). MFEs are intentionally structured opportunities for beginning teachers to (1) learn about core practices in teacher preparation coursework, (2) implement those practices during a facilitated K-12 classroom experience, and (3) debrief the classroom experiences as a whole group with teacher educators and at times with the partner teachers to build a shared vision of ambitious and accessible mathematics instruction. MFEs address a critical need for supporting TCs to develop ambitious teaching practice through partnerships with local K-12 schools, while embracing the power of appropriate struggle as an opportunity for learning and growth. When MTEs support and engage TCs in productive struggle to make sense of and develop the skills of ambitious teaching through their work with K-12 students, an opportunity for TCs' development occurs that they would not experience in a university classroom setting alone. This paper presents our findings from a descriptive pilot study investigating TCs' perceived learning from our MFE courses that supported TCs in this productive struggle of learning about and enacting ambitious mathematics teaching.

Conceptual Framework

This research team joins a growing body of educational research (e.g., Ball et al., 2014; Ghousseini & Herbst, 2014; Lampert et al., 2010; McDonald, et al., 2014; Santagata & Yeh, 2014) that explores what has been characterized as "the turn to practice-based education" (Zeichner, 2012). The premise of these studies is based on the belief that in order to better support TCs to learn to *do* ambitious teaching we, as MTEs, need to teach them both the interactive skills required to engage students in meaningful mathematics work, and flexibility to use this knowledge in particular moments of practice. Like others, we argue preparing TCs for doing the complex work of ambitious mathematics teaching requires we implement *different* pedagogies of teacher education in *deliberate* ways that make the practice of teaching a central focus.

Practice-based learning describes types of field experiences that situate TCs' learning in K-12 classrooms coupled with coursework focusing explicitly on the work of teaching (Forzani, 2014). Grossman et al. (2009) describe how a core-practice approach in teacher education necessitates organizing coursework and fieldwork around core practices of the teaching profession while simultaneously providing TCs ample opportunities to "practice" enacting these teaching practices in structured and supported ways. Other research within the teacher education community has identified "core" or "high-leverage" teaching practices that effective teachers use while teaching (i.e., Ball & Forzani, 2009; McDonald et al., 2013; NCTM, 2014) and we draw on these in this paper. By purposefully designing teacher preparation coursework to include the pedagogies of enactment that have MTEs side-by-side with TCs in a K-12 school setting, we are working to develop TCs' understanding of such core teaching practices and know how to enact them skillfully.

An important factor in redesigning our teacher preparation courses, we drew on McDonald et al.'s (2013) learning cycle (Figure 1) to illustrate how core practices are embedded into the MFE design and developed across the four phases. The four phases of the learning cycle provide structured supports to develop TCs' understanding and enactment of such teaching practices by: (1) learning about the instructional activity (including envisioning the practice), (2) preparing for enacting the activity (including rehearsing), (3) having opportunities to enact that practice via the activity in authentic classroom settings and (4) analyzing those enactments as a way to connect the educational theory to the classroom practice. The learning cycle puts core practices into conversation with a vision of professional learning (McDonald et al., 2013) and gives a structure for MTEs to support TCs learning to understand and skillfully enact core practices.

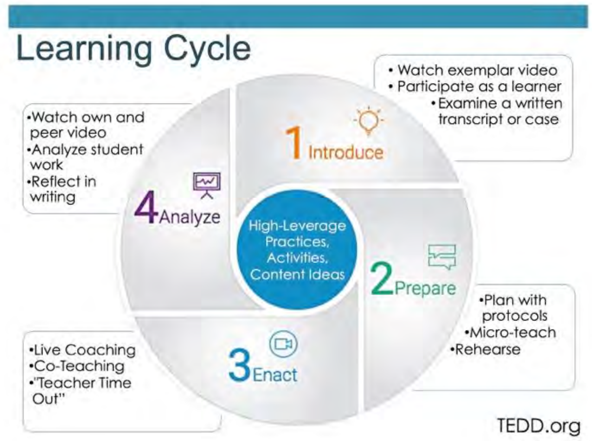

Figure 1. Cycle for learning "core" or "high leverage" teaching practices

To support TCs' understanding of ambitious teaching in each of our courses, we focused on at least one of the National Council of Teachers of Mathematics (NCTM, 2014) eight research-informed effective (core) teaching practices. These practices represent "a core set of high-leverage practices and essential teaching skills necessary to promote deep learning of mathematics" (NCTM, 2014, p. 9), examples include: facilitating meaningful discourse, supporting productive struggle, and eliciting and using evidence of student thinking.

Methods

In this paper we present the findings from a multi-institutional descriptive pilot study investigating the impact of MFEs on TCs' learning. This work is a collaboration between five different colleges and universities where MTEs teach either integrated content-pedagogy or methods courses implementing MFEs. To document the shared learning outcomes of the TCs enrolled in our courses implementing MFEs, we analyzed TCs' written reflections. The research question guiding this study was: What do TCs report in their end-of-course reflections, where MFEs were enacted, as most impactful for their learning to teach mathematics?

Context and Participants

Data collected included TCs' written reflections of the MFE experience. Each institution had a different context for the course and grade level focus for the MFE. Two institutions situated MFEs in elementary methods courses, while one university incorporated the MFEs in an integrated content and pedagogy course for elementary teachers. The other two universities placed MFEs in middle-grades methods courses. Additionally, the MFE at each institution focused on a different subset of core practices. This data collectively represents 97 TCs and their responses to culminating course assignments related to MFEs.

Data Collection and Analysis

The data collection spanned one to three terms of the courses across the five institutions, comprising a collective repository of written artifacts from each course. These end-of-term written assignments asked TCs across all institutions to share the most impactful aspects for their learning during the term and what aspects of this learning they plan to bring to their future classrooms. We aggregated all TC responses across institutions to have a reasonable sample size (97 TCs) and to look for themes across the MFE and independent of the institution/instructor.

We utilized the six phases of thematic analysis (Nowell et al., 2017) to create and apply codes to the entire data set. For the first phase of thematic coding, familiarizing ourselves with the data, we met weekly to discuss theoretical perspectives of the data set and brainstorm potential themes and codes by looking at the common reflection prompts in course assignments as well as elements of the NCTM effective teaching practices. After examining our own data sets and drafting initial themes of each individual data set, we used peer debriefing to generate initial codes (phase two) and utilized our theoretical framework to organize our codes into logical clusters. Using elements of analytic induction (Erikson, 1985), we used confirming and disconfirming evidence to verify the existence of each emerging theme. In phase three, we searched for themes, documenting any excerpts from the data that seemed problematic for the next phase of peer debriefing. At this point the group entered phase four and five, reviewing themes and defining and naming themes. In this process, we cross examined data sets both collectively and individually, discussing examples from all data sets to determine working definitions to accurately depict the data and align with our theoretical framework. We then divided the data by codes and individually re-examined individual code applications. The group continued to reconvene for peer debriefing addressing any anomalies in the data and reconciling codes as needed. In this iterative process, code application was triangulated by individual researchers taking on different codes in each analysis. Finally, in phase six of thematic analysis we present our final analysis. For the purposes of this paper we are using a subset of our codes utilizing findings related to TCs' reporting related to ambitious teaching and the number of TCs referring to these practices within our data set. The coding scheme was further expanded as other themes and elements emerged from the data (e.g., TCs' attribution to distinct phases in the learning cycle).

Findings

In this descriptive pilot study, we sought to identify any common learning outcomes identified by the TCs in our courses due to the common implementation of MFEs. TCs reported that they deepened their understanding of the targeted core teaching practice(s), developed a vision of ambitious mathematics teaching that aligns with NCTM's vision, recognized the importance of cultivating a positive classroom environment, and identified the importance of knowing students and building on students' knowledge when planning and/or teaching along with an increase in confidence when teaching. The common learning outcomes identified by a threshold of at least one-quarter of all TCs in the study are presented in Table 1.

Table 1: TC's Self-Reported Most Impactful Aspects of Learning in Courses with MFEs

TCs' Learning as Reported in Open-Ended Reflection Question	**# of TCs**	**%**	**Representative Quote**
Core teaching practice that was a focus of MFE	82	85%	*This class has taught me the importance of asking purposeful questions to students to elicit deeper thinking and to help connect the math.*
Teaching mathematics in ways that align with NCTM's vision	56	58%	*Another one of the eight practices that really change[d] my mind about math was facilitating meaning for mathematical discourse. Instruction that is focus[ed] on mathematical discourse engages students as active participants and making sense [of] mathematical ideas and raising of a mathematical relationship.*

Importance of creating positive classroom environment	55	57%	*..I will be important to create a community among my students, sharing the idea that it's okay to not understand, and we can support each other in the process of learning through respectful discussion, and sometimes debate.*
Importance of knowing students as people/learners	34	35%	*I learned that in order to understand your students, you must get to know them. I had the opportunity to get to know my math buddy well, along with several other students we worked with when co-teaching some of the math lessons. Getting to know these students gave me insight to how they each learned and what I could anticipate from them in order to prepare in a way that would support their learning.*
Importance of building on students' knowledge in planning or teaching a lesson	34	35%	Planning: *I feel as though my lesson planning has improved. Anticipating student strategies and misconceptions has helped me figure out in advance the kinds of questions I might ask, what student work might look or not look like, and what group conversations might sound like.* Teaching: *Another important aspect of math that I have learned throughout this semester is allowing students to work through their thoughts before jumping in and correcting/helping them. I have a tendency to interrupt students as soon as they start to make a mistake, but this course has taught me that these mistakes are crucial to student thinking.*
Confidence in teaching or teaching mathematics	27	28%	*Reading the textbook and putting the concepts into action at* [site of MFE] *has raised my own confidence about teaching math as I have watched my math buddy overcome his own hurdles and enjoy problem solving.*

Core Teaching Practices and Additional Learning

Across each of our courses, all five MTEs focused on a subset of NCTM's (2014) eight effective teaching practices as our "core practice." Given the variety of the courses and MTE's learning goals for each of their respective courses, no particular core practice emerged as key. However, it is notable that 83% of the TCs highlighted the importance of at least one core practice for students' learning or self-reported they improved in their enactment of the practice: the specific core practices named corresponded to those that were a focus of the iterative MFE cycle in their respective course. For instance, in a methods course that focused on the connections across all eight ETPs, a TC wrote:

> Often, you can utilize multiple teaching practices by doing one thing. Using mathematical representations helps elicit thinking, which they then use to engage in discourse. While this is all being done, they might be engaging in productive struggle. Seeing how everything ties in together makes it far less intimidating.

Whereas in a course that emphasized a subset of NCTM's core practices, and one core practice was the focus of multiple iterations of the MFE so TC could have multiple ongoing opportunities to hone and enact this core practice, TCs reflected about learning specific to that core practice. For example, where posing questions was the targeted core practice, this representative quote highlights TCs' self-reported learning:

> My observation of [our partner classroom teacher], along with my new found knowledge of the power of questioning, has encouraged me to implement questioning into my own practice. I have put an emphasis on using questions as a tool to uncover student understanding, much like [partner classroom teacher]. This small change has dramatically changed my interactions with students. Instead of telling students how to fix their mistakes, I ask students to take me through their thought process.

In addition to reporting on their deepened understanding of the targeted core teaching practices of the course, TCs wrote that they learned about the importance of the classroom environment, knowing their students, and incorporating their students' thinking into their lessons. They also envisioned a supportive and productive mathematics classroom that included positively framing students as capable mathematicians, providing access for all students to learn mathematics including choice, and incorporating a growth mindset where mistakes are viewed as essential aspects of learning. Almost 60% of the TCs across all five institutions described their vision for teaching mathematics in ways that align with NCTM's vision (when they chose to describe that vision; not all TCs described their vision in this open-ended question). In these end-of-the-term reflections, many TCs described what a positive mathematics classroom environment looks like, sounds like, and feels like for every student and reflected on how important these aspects are for ambitious mathematics lessons. Lastly, almost 30% of the TCs mentioned an increase in their confidence in teaching after just one term of a course implementing MFEs.

Impact of the MFE Structure

85% of the TCs specifically named at least one aspect of the learning cycle structure as impactful for their learning, and roughly one-quarter explicitly pointed to the specific feedback received, including peer, partner classroom teacher, and MTE feedback, as reported in Table 2. Collectively, TCs specifically named each phase of the learning cycle as important for their learning. For approximately one-fifth of the TCs, the impact of experiencing the lesson themselves, before enacting with students during the *Introduce Phase*, was highlighted. Approximately one-third of the TCs pointed to the *Prepare Phas*e, and co-planning with peers and/or receiving feedback from the MTE in preparation for teaching, or *Analyze Phase*, where TCs reflected about their experiences and debriefed about their experiences working with students both through discussion and individual writing assignments facilitated by the MTE, as essential. The phase identified the most often as impactful was the *Enact Phas*e of the learning cycle. TCs attributed their experiences of teaching and working directly with students as key for connecting theoretical (course) learning with the practice of teaching.

Table 2: TC's Self-Identified Impact of the MFE Structure on their Learning

TCs' Attribution of Learning to MFE Structure	# of TCs	%	Representative Quote
TCs attribute learning to at least one aspect of the learning cycle	82	85%	*Having the opportunity to put my teaching into practice within [partner classroom teacher's]class has helped me to grow and adapt to become a better teacher in the future. Three areas that I have grown throughout this semester are planning and organizing lessons, creating*

			and implementing student-centered learning experiences and creating a positive learning environment.
Introduce Phase	18	19%	*I found it useful putting ourselves in the student position first in doing the lesson we were about to teach to the students, it helped us become flexible, intuitive, and help to predict student reactions and areas of struggle.*
Prepare Phase	29	30%	*During our own planning of the Three Reads Lessons, my team was able to determine what materials to provide students that would be the most helpful. For example, one of our problems had numbers that were too big to use cubes like we had done the week before. If we had not planned out the lesson, then we might have given students manipulatives that made the problem harder for them instead of helping them make sense of the problem. In short, a solid plan makes a solid lesson.*
Enact Phase	53	55%	*I think the aspect of being able to apply what we learned every week by physically teaching it to our students had a heavy impact on my learning.*
Analyze Phase	30	31%	Live: *We were able to debrief after every lesson … and that really helped bring everything we did and learned together.* Written: *As much as I didn't honestly enjoy doing math reflections every week, they were actually very helpful. They helped me not only analyze what students were understanding and how they came about the answers that they had gotten, but it also helped me know what things I should add to the next lesson in order to ensure better understanding and more successful instruction. It helped me to see what parts of my instruction were successful, unsuccessful, or what I needed to include in my next lesson to better reach the students.*

In addition to attributing their learning to the specific phases of the learning cycle, 27% of the TCs identified feedback during at least one aspect of the learning cycle as impactful. Most notable, TCs highlighted feedback during the *Enact Phase*, when a teacher time-out was called and the TC conferred with either their TC partner, the MTE, or the classroom teacher to solicit support and confer about what to do next, in that moment. Others pointed to the specific feedback received during the *Analyze Phase*, either (1) during the debrief as they shared instructional moves and received verbal feedback from their partners/peers or the MTE or (2) written feedback on assignments given at the conclusion of the cycle for TCs to continue reflecting and analyzing their learning. Because MFEs provide that *shared* and authentic teaching experience, TCs are able to receive feedback across all phases of the learning cycle (e.g., on their planning, teaching, *and* reflecting) from the MTE, the partner classroom teacher, and their peers, to which many credit their deepened understanding and improved enactment of these core teaching practices and other important aspects of the course (e.g., positive classroom community).

Discussion and Implications

Working in authentic classroom environments within the MFE structure, where TCs' productive struggle was supported through the iterative structure of the MFE and feedback across all phases of the learning cycle, provided TCs with a working vision of ambitious mathematics instruction. In addition to identifying how their understanding of core teaching practices and their impact are essential for teaching and learning, they reported this approach impacted their appreciation and aspiration of a productive mathematics classroom environment where students are central, viewed as capable learners and where the teacher actively plans and teaches lessons to build upon their students' ideas.

The MFE structure offers unique means of supporting the development of beginning teachers' practice and awareness of ambitious mathematics teaching (Campbell, 2012; Campbell & Dunleavy, 2016; Horn & Campbell, 2015). The TCs' self-reported understanding or growth in enacting targeted core teaching practices and cited how various phases of this iterative learning cycle supported their understanding and enactment of these practices. This suggests the MFE structure is effective for developing TC's knowledge and awareness of the importance of these core practices for teaching mathematics. Further study is needed to ascertain how effective this instructional approach is for their proficiency in enacting these teaching practices.

The MFE provides a promising pedagogical approach for preparing beginning teachers. TCs, through this structured learning experience, identified at the end of the course, without explicitly being prompted, specific core teaching practices and characteristics of a classroom that provide access, support and challenge for their students, an essential standard in the preparation of math teachers (AMTE, 2017, p. 13-Indicator C.2.1). In addition, the learning cycle and structure of the MFE provided iterative opportunities for TCs to: "plan for effective instruction" (p. 14, Indicator C.2.2), "use a core set of pedagogical practices that are effective for developing students' meaningful learning of mathematics" (p. 15, Indicator C.2.3) and "analyze teaching practice" (p. 16, Indicator C.2.4) as the MTE provided feedback and support to TCs at all phases of the learning cycle. The MTE served to mediate tensions arising as TCs were asked to apply and integrate their theoretical learning about mathematics and ambitious teaching through the practice of teaching (Billings et al., 2021). What we are asking TCs to do is a challenging way of teaching: it is notable that more than one-quarter of TCs self-report they developed confidence to teach this way. We attribute this in part to the highly supportive MFE environment.

A limitation of this study is that TCs self-identified aspects of their learning, and thus areas of learning may not have been reported. The open-ended nature of the questions did not guide the TCs to reflect on specific course learning objectives. Consequently, TCs may have acknowledged growth in an area, such as confidence, if directly asked, but may not have reported a growth in this area due to the nature of the questions posed. Revision of the data collection tools, explicitly asking TCs to report about specific areas of learning, are needed for future study about the impact MFEs have on TCs' learning. Looking forward, we hope to document TCs' perceptions of the impact MFEs have on their learning during their preparation and follow TCs into the field to investigate their enactment of the ambitious teaching practices into their own classrooms.

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PROSPECTIVE SECONDARY MATHEMATICS TEACHER RESPONSES AND THE STRUCTURE OF APPROXIMATIONS OF PRACTICE

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To support productive struggle, prospective secondary mathematics teachers (PTs) need to elicit and respond to students' mathematical ideas in ways that focus on those ideas and that position students to build on those ideas. Using the Teacher Response Coding framework (Van Zoest et al., 2021) we analyzed PTs' responses in three rehearsals of instruction. We identified significant differences in the natures of responses in one rehearsal compared to the other two. Using a Levels of Constraint framework based on findings of Kavanagh et al. (2020) we compared structures of the rehearsals and developed hypotheses regarding which aspects of structure might account for differences in PTs' responses.

Keywords: Classroom Discourse; Instructional Activities and Practices; Preservice Teacher Education

Over multiple decades, mathematics education researchers have identified mathematics teaching practices that aim to use and build on student thinking. Particularly, practices that invite students to share and elaborate their thinking (Franke et al., 2009) and that build on student thinking by asking students to connect their thinking to others' ideas (Smith & Stein, 2018) have been shown to support student learning (Webb et al., 2014). One goal of teacher education is to support prospective teachers (PTs) in developing such practices. Yet there is often dissonance between what prospective teachers learn about teaching in their professional coursework and the teaching that they have opportunities to enact and witness in field placements (Grossman et al., 2009). In response, MTEs have developed approximations of practice (Grossman et al., 2009) in which PTs to interact with simulated or authentic students (e.g., Arbaugh et al., 2019; Lampert et al., 2013). Our use of *approximations of practice,* in this context, is meant to describe events that occur in a context that is similar to authentic classroom teaching but less complex, less authentic, and more controlled. In theory, such opportunities provide contexts for PTs to engage in teaching practices, such as eliciting and responding to student thinking, in settings that are less complex than those in which teaching typically occurs. However, approximations of practice differ with regard to multiple features, including how students are embodied and by whom, and the duration of the experience. The purpose of this study is to examine how differences in the structure of approximations of practice relate to differences in how PTs engaged in teaching within those approximations of practice.

We report our empirical investigation into the following research questions:

1. How do the ways that PTs respond to student thinking within approximations compare across approximations of practice that differ in structure?

2. What aspects of structure of approximations of practice might account for differences in the ways that PTs respond to student thinking?

Theoretical Perspectives

Our study is situated at the intersection of research and scholarship on how teachers respond to instances of student mathematical thinking and research and scholarship related to the use of approximations of practice in mathematics teacher education. In the next two sections we share the perspectives informing this study.

Perspective on Teacher Responses

In a thorough review of literature on researchers' ways of examining teacher responses, Van Zoest et al. (2021) identified three core aspects that describe productive teacher responses: the "who, what, and how" of a response. In other words, *who* in the classroom is asked to publicly interact with a student's mathematical contribution (SMC)? *What* actions do classroom member(s) get to take with respect to the SMC? And *how* does the teacher's response relate to the substance of the SMC? For students to engage with each others' SMCs, it is important for teachers to attend to SMCs in their responses and for teachers to position students to take action on the mathematical idea contained in the SMC (Bishop et al. 2020; Robertson et al. 2016).

Perspective on Approximations of Practice

Grossman et al. (2009) described the nature of professional learning in human improvement professions, including teaching, in terms of three constructs: Representations of practice, decompositions of practice, and approximations of practice. We define an approximation of practice as an activity that is similar to, but not identical to, typical activities of a professional teacher, such as participating in a simulation of an interaction with students around mathematics content. Engaging in approximations of practice provides PTs with opportunities to act as teachers in a context designed to resemble an instructional interaction. Approximations of practice aim to support PTs to learn in and from practice (Lampert, 2010). Based on the positions taken by others who have worked in this area, we argue that the use of approximations of practice in mathematics methods courses is guided by three principles. First, PTs need repeated opportunities to engage in practices that are challenging for novices (Grossman et al., 2009), such as responding to student thinking. Second, rehearsals must increase in complexity across consecutive rehearsals by involving more authentic classroom interactions, broader components of instruction, and less intervention from the teacher educator (Boerst et al., 2011; Grossman, 2009). The third principle is that a teacher educator (TE) must mediate PTs' experiences in an approximation. This includes representing teaching in terms of significant practices (e.g., Grossman et al., 2009) to support the PTs' "development of situationally appropriate knowledge and skill" (Lampert et al., 2013). Mediation is significant because it can shape what teachers attend to within an approximation and how they respond (Kavanagh et al., 2020).

Methods

Context of The Study

We report the results of an analysis of prospective teachers' responses to SMCs within three approximations of practice conducted across a single semester in a secondary mathematics methods course at a large public university in the mid-Atlantic region of the United States. Each approximation occurred within a learning cycle (Lampert et al., 2013) in which PTs engaged in a mathematical activity, examined representations of teaching the activity, planned to teach an activity, enacted that teaching within the methods course, and reflected on their experiences (see Arbaugh et al., 2019). These activities were all focused on a set of communication moves (Freeburn & Arbaugh, 2017) that served as a decomposition of eliciting and responding to student thinking. We focus our study on the planning and enactment within each cycle in which PTs focused on developing those communication moves. We refer to a *rehearsal* as an approximation of practice in which PTs, in the role of teacher, interact with one or more individuals in the roles of students, whom we call Enacted Student(s) (ES). We designed each rehearsal to align with the three guiding principles described in the theoretical perspectives. Figure 1 provides a summary of the intended variations in structure across the rehearsals.

Element	**Rehearsal 1**	**Rehearsal 2**	**Rehearsal 3**
Ratio Teachers: Students	2:1	1:2	1:7
Intended Duration	6 minutes	3 minutes	15 minutes
Enacted Student(s)	Elif or Lewis	Elif and Lewis	Small Group of Peers
Student Behavior	The ES is partially finished with the task at the start of the rehearsal	The ESs believe they are finished with the task at the start of the rehearsal	ESs begin the task at the start of the rehearsal
Student Work	Student work constructed prior to and during the rehearsal	Student work constructed prior to and during the rehearsal	Student work constructed during the rehearsal
Mathematical Task	Textbook task focused on linear functional relationships	Construct an argument for a given number theoretic claim	Textbook task focused on exponential functional relationships or statistics

Figure 1: Intended Variations Across Rehearsals

Prior to each rehearsal PTs received a copy of the task that would frame their conversation with one or more ESs. PTs did the task and then planned in small groups to enact the task in their rehearsals. In RH2, PTs received copies of partially-complete student work before the rehearsal. We did not provide partially-complete work prior to RH1 or RH3. In Rehearsal 1 (RH1) and Rehearsal 2 (RH2), PTs interacted with Elif or Lewis, who were graduate students portraying ESs based on protocols that we designed to guide the representations and student misconceptions that the graduate students would enact during the rehearsal. In Rehearsal 3 (RH3), peers were the ESs that were engaging with the mathematical task for the first time.

Data Collection and Analysis

We analyzed transcripts of video-recorded rehearsals for three pairs of PTs. Each pair participated in RH1 together, and then in RH2 and RH3 as individuals, which resulted in a total of 5 rehearsals per pair and 15 rehearsal videos as data for the study. We also analyzed artifacts of class sessions (e.g., boardwork, handouts, and other such artifacts) as well as statements made verbally or in writing by the research team during their planning sessions.

We analyzed data in three phases. In Phase 1 we analyzed recordings of our planning sessions, video-recordings and artifacts from the class meetings preceding each rehearsal, and video records of rehearsals using a framework based on the findings from Kavanagh et al. (2020) which delineated four elements of rehearsals that can vary in degrees of scaffolding:

- *disciplinary content* (e.g., task, learning objective)*,*
- rehearsing teacher's *instructional routine,*
- *texts/tasks* used in the rehearsal*,*
- and TE *facilitation moves.*

Kavanagh et al. further delineated each element along a dimension of constraint ranging from *loosely constrained* to *tightly constrained,* depending on the extent to which decisions related to that element are made by the TE or left open for the PT to decide. We analyzed the design and enactment of each rehearsal to characterize levels of constraint with respect to each of the elements from Kavanagh et al. (2020). Our analysis led us to extend to two additional elements, which we describe in the findings.

In Phase 2 we applied a subset of the Teacher Response Coding Scheme (TRC; Van Zoest et al., 2021) to video transcripts to analyze PTs' responses to each ES mathematical contribution. The subset of the TRC is composed of three categories of codes that align with the *who*, *what,* and *how* facets of teacher responses described in our perspectives section. We coded a total of 304 PT responses across the three rehearsals and compiled contingency tables that reported the frequency of each code in each category of the TRC across rehearsals.

Category Description	**Code Descriptions**
Actor (Who?): The person publicly given the opportunity to consider the instance of SMC.	**teacher, same student(s), other student(s), whole class**
Action (What?): What the actor is doing or being asked to do with respect to the instance of SMC.	**check-in** (elicits self-assessment or understanding of a SMC), **clarify** (asks an actor to make an SMC more precise), **collect** (asks an actor to contribute a new or alternate SMC), **connect** (asks an actor to connect an SMC to a previously introduced idea), **develop** (asks an actor to build on an SMC)
Student Ideas (How?): The extent to which the student who contributed the instance of SMC is likely to recognize their contribution in the teacher response.	**core** (the response explicitly references the SMC), **peripheral** (the SMC is implicit but recognizable in the response), **other** (the SMC is not recognizable in the response), **not applicable** (SMC can not be inferred or the teacher response is too vague)

Figure 2: Subset of TRC Categories and Codes

In Phase 3 we compared differences in proportions of codes in the three categories of Actor, Student Idea, and Actions pairwise between the rehearsals. Because our goal was to examine the development of practices that engage students in sharing ideas and in building on their own and others' ideas, we grouped codes for Actor into student-centered codes (e.g., student, other student(s), and whole group) and teacher codes, and grouped codes for Student Idea into two groups: Core/Peripheral versus Other/Not Applicable. We used Chi-Square Tests of Independence (p=0.05) to test for significance of differences related to Student Idea and Actor. Because there are 15 distinct codes for Actions in the TRC, the Chi-Square test becomes inappropriate (28 degrees of freedom with many small expected values. Therefore, we identified those Actions that accounted for 10% or more of the codes in each Rehearsal and compared the results across Rehearsals.

Results

Finding 1: RH2 Was Constrained Differently From RH1 and RH3

Our analysis revealed that RH2 was constrained at a different level from RH1 and RH3 in five elements: *Disciplinary Content, Text/Tasks, Student Work, Instructional Routine,* and *Pedagogical Learning Objectives*. We found the *Facilitation Moves* element had the same level of constraint across all three approximations. We elaborate in the next few paragraphs.

Disciplinary Content: Learning Objectives. In RH1 and RH3, PTs constructed the mathematical goals for student learning based on mathematical content and practices that the PTs identified while working through their respective mathematical tasks. In contrast, for RH2 the TE identified and communicated the mathematical learning goal to the PTs during the planning class session prior to the rehearsal.

Texts/Tasks: Mathematics of the Tasks. The TE scaffolded PTs' engagement with the mathematics of the tasks differently in RH2 than in RH1 or RH3. In preparing for RH1 and RH3 the TE did not constrain PTs' activities to focus on the mathematical content or practices involved in the task. Mathematical ideas in the tasks surfaced when PTs asked questions during their planning. For example, one group shared the difficulty they were having with creating questions in preparation for RH1 because they were uncertain of student approaches. The TE responded, "So, one way you may approach those is to think about what the mathematical goal is. Then, go back to your assessing questions and examine how well these questions help me to understand where the student is with respect to that goal." However, in preparing for RH2, the TE pushed PTs to stipulate criteria for acceptable student responses and to explicate multiple ways that students might engage in the task. The TE also engaged PTs in constructing arguments and critiquing arguments.

In addition, there was a difference in the nature of the mathematical tasks for the rehearsals. The mathematical tasks in RH1 and RH3 were both selected from units in the *Connected Mathematics Project* (Lappan et al., 2002) that presented questions about a realistic scenario in order to develop the mathematical content, such as linear versus exponential growth or measures of center in data sets (see Figure 1a for one example). While mathematical content was certainly involved in the RH2 argumentation tasks (Figure 1b), the nature of the task was oriented more towards mathematical practices than content.

a.cerpt from *Moving Straight Ahead Task* (Lappan et al., 2002b, pp. 24–25) from RH1.

1.

Cycling time (hours)	Distance (miles)		
	José	Mario	Melanie
0	0	0	0
1	5	7	9
2	10	14	18
3	15	21	27

b.) **Perfect Squares Task from RH2.**

Write an argument for the claim, "the product of perfect squares is a perfect square".

1. a.) How fast did each person travel for the first 3 hours? b.) Assume that each person continued at this rate. Find the distance each person traveled in 7 hours.	

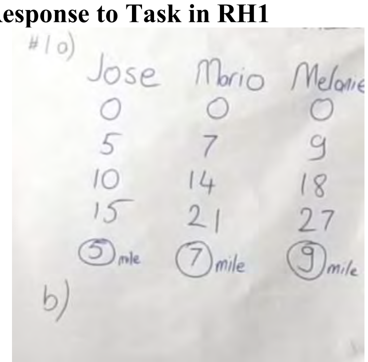
Figure 3: RH1 and RH2 Tasks

Instructional Routine: Discussion Structure. In RH1 and RH3 the TE did not stipulate a prescribed discussion structure for PTs' interactions with ESs. TE instructed PTs to ask questions to elicit ESs' thinking and to support their progress towards the mathematical goals of the activity, but otherwise left it to PTs to decide the structure of their interactions with ESs. However, in the planning session for RH2, the TE directed PTs specifically to facilitate a conversation between two ESs to support their progress in constructing a valid argument. The TE directed,

> You are to work with these two students to get them to talk to each other in a way that takes the arguments they've crafted and help them make progress . . . in writing a valid argument by connecting the representations together. . . . Questions you are asking are not just for you to get information . . . but also to get them to give information to each other . . . . The arguments don't have to be the same [by the end of the rehearsal] but they need to be coordinated.

Student Work: Availability and Amount. RH2 was more constrained than RH1 or RH3 with respect to 1) how PTs accessed student work and 2) the extent to which student work represented a "finished" product. In RH2 the TE provided PTs with ESs' arguments the day before the rehearsal as a resource for their planning (see Figure 2a). The work was presented as each ESs' "finished"''rgument for a given claim. In RH1 and RH3, however, PTs' first encounters with ESs' work was during the rehearsals, and the presented work was that of a student(s) still in the process of completing the task (see Figure 2b).

a.**if's Partial Response to Task in RH1**



2.

b.) **Lewis's Argument in RH2**

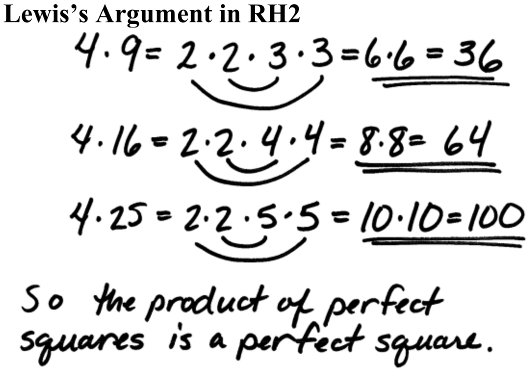


Figure 4: Samples of Student Work from RH1 and RH2

Finding 2: PTs Positioned Actors Differently in RH2 than in RH1 or RH3

The contingency table shown in Table 2 presents the number of PT responses that we coded from each rehearsal in the Student Idea category and in the Actor category. The extent to which responses explicitly incorporated student ideas does not appear to depend on whether the responses occurred in RH1, RH2, or RH3 (X^2 (2, N = 304) = 2.42, p = .30). However, the proportion of PTs' responses that positioned students as actors was significantly different across rehearsals (X^2 (2, N = 304) = 25.9, p <0.05). Closer examination revealed that the proportion was significantly different in RH2 than in either RH1 (X^2 (1, N = 111) = 12.15, p < 0.05) or RH3 (X^2 (1, N = 247) = 29.44, p <0.05), but not between RH1 and RH3 (X^2 (1, N = 250) = 4.365, p = 0.113). We interpret this as evidence that some aspects of RH2 must have supported a different kind of response pattern, not with respect to student ideas, but in terms of the extent to which students were invited to consider and to build on their own and others' ideas.

Table 1: Contingency Tables of Codes in Student Ideas and Actor Across Rehearsals

Coding of Response for Student Ideas	RH1 (n=57)	RH2 (n=54)	RH3 (n=193)
Student Idea: Core or Peripheral	44	37	128
Student Idea: Not applicable or other	13	17	65
Coding of Responses for Actor			
Teacher	26	10	111
Student(s)	31	44	82

Finding 3: Responses Involve A Broader Set of Actions In RH2 than in RH1 or RH3.

As shown in Table 3, PTs' responses in RH2 had the greatest variety of *actions* with relative frequencies of 10% or above. We found this somewhat surprising, given that PTs had the least amount of time (three minutes) to interact with Elif and Lewis in RH2 than with students in RH1 or RH3 (6 minutes and 15 minutes respectively). The *develop* action was the most frequent action across all three rehearsals. These actions were ones in which the PT expanded or requested ES(s) to expand on an ES's contribution.

Table 2: Most Frequent Actions Across Rehearsals

	Check-in	Clarify	Collect	Connect	Develop
RH1 (n=57)	**	8 (14.04%)	**	**	26 (45.61%)
RH2 (n=54)	6 (11.11%)	7 (12.96%)	9 (16.67%)	9 (16.67%)	12 (22.22%)
RH3 (n=193)	**	**	25 (12.95%)	**	55 (28.50%)

Discussion

If we characterize PTs' capacity to build on student thinking in terms of their observed patterns of response via the TRC scheme, our findings suggest that progressive increase in complexity and authenticity across approximations is insufficient to account for differences in the patterns of their responses within the approximations --in fact, variations in constraints across other elements of the approximations seem to better explain those differences. We found that

PTs' responses in RH2 were directed more at students and involved a greater repertoire of actions than in either of RH1 or RH3. When we examine the structures of approximations through the lens of levels of constraints with respect to some elements (Kavanagh et al., 2020), we find ways that RH2 was more constrained than RH1 and RH3--namely, in terms of *disciplinary content, texts/tasks, instructional routine,* and *student work*. Our findings further support the findings of Kavanagh et al. (2020) which suggested that more tightly constrained rehearsals might explain differences in how PTs engage in those rehearsals.

Grossman et al. (2009) used the metaphor of "learning how to kayak in calm waters" (p. 2076) to describe the notion of learning to engage in complex practices in environments of decreased complexity. Kavanagh et al. (2020) used this metaphor to describe how various constraints serve to calm the waters for novices by constraining the complexity of the context in which they engage in approximation of practice. We also find this metaphor helpful for thinking about how some of the constraints may be related to differences in the PTs' responses.

Truly learning to kayak involves learning to navigate toward a destination while considering and choosing among multiple possible routes. The constraints in *disciplinary content, texts/tasks, and student work* that we identified in RH2 may have served to support the PTs by making explicit the destination (the learning goals and the criteria for evaluating students' arguments) and anticipated routes (potential ways that students might approach the task, and the specific discursive structure to use in the rehearsal) that they might experience within the rehearsal. For *disciplinary content*, the MTE established a well-defined and common mathematical destination for the PTs to focus their students towards in RH2. Additionally, they had more scaffolded experiences working with mathematics related to the *texts/tasks* and *student work* of RH2 than in RH1 and RH3. In preparing for RH2, PTs were able to become more familiar with the mathematics of the rehearsal tasks and the ways students might approach the task. Having a greater understanding of the mathematical goal and of the multiple routes students may take may have freed PTs to invoke a broader repertoire of actions to help copilot students in ways that positioned the students as the ones to consider each other's thinking.

The constraints in *instructional routine* related to the PTs' discussion structure in the RH2 may also have contributed to calming the waters in the rehearsal. In preparation for RH2, the MTE and PTs discussed how a teacher may support students in connecting their thinking and analyzed the way a teacher in a narrative case positioned the students to consider each other's thinking. Understanding how a teacher is able to enact this discussion structure seems to have supported the PTs with the repertoire of *actions* in our findings that one would use to get students to notice and respond to each other's contributions during a classroom discussion. Additionally, the discussion structure itself in RH2 naturally oriented the PTs to position the students (*actor* category) as the ones who needed to make sense of each other's work.

Preparing PTs involves supporting them to develop knowledge about teaching alongside their emergent skills as teachers. Lampert et al. (2013) and others suggest that the exchanges that TEs have with PTs during rehearsals affect the opportunities for PTs to develop knowledge of concepts and ideas related to teaching. Our study extends that understanding by illustrating how differences in rehearsal design, along with TEs' pre-rehearsal actions, also may shape PTs' opportunities to engage in core practices within rehearsals. Our study reinforces the findings of Kavanagh et al. (2020) on teacher responses to student thinking in rehearsals and extends that work by characterizing responsiveness in ways that allow for clear examination of differences across rehearsals.

Although our study contributes to understanding how differences in structure may relate to PTs' enactment of core practices within approximations of practice, our rehearsals share important features with each other (e.g., use of semi-scripted ESs in a live interaction) that are not necessarily characteristic of all approximations of practice. Further research will be needed to explore other aspects of design and enactment of approximations of practice and their relationships to the ways that PTs engage in core practices within those approximations.

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ELEMENTARY PRESERVICE TEACHERS' VIEWS AND ENACTMENTS ON FOSTERING PERSEVERANCE

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The purpose of this study is to investigate three elementary preservice teachers' (PSTs') conceptions of and reflections on the role of perseverance in mathematics. This study presents findings regarding the PSTs' perseverance conceptions, student interactions, and the PSTs reflections from a two-week summer math camp in which they assisted in teaching elementary student campers. Additionally, this study used an analytic framework based on a corpus of literature to capture how the preservice teachers' conceptions, interactions, and reflections aligned with each other and with current and relevant research recommendations.

Keywords: Preservice Teacher Education; Affect, Emotion, Beliefs, and Attitudes

Research has connected student perseverance to many constructs, including productive struggle, self-efficacy, motivation, mindset, locus of control, and grit (Bettinger, Ludvigsen, Rege, Solli, & Yeager, 2018; Dweck, 2006; Pajares & Miller, 1994; Warshauer, 2015), all of these aid in students learning mathematics. How then can teachers help support students in their perseverance? How are practices of perseverance conceived, viewed, and developed for future teachers? More specifically, as elementary teachers "develop the foundation of mathematical understanding, beliefs, and attitudes among young learners that start children on their mathematical journeys" (Association of Mathematics Teacher Educators [AMTE], 2017, p. 48), how do they conceive, viewed, and develop the practice of perseverance?

Literature Background

Conceptions

Elementary preservice teachers (PSTs) often enter teaching programs with preconceived conceptions based on their own experiences as learners (Stohlmann et al., 2014). Thus, it is critical to understand PSTs' conceptions about teaching mathematics early in their educational program. These conceptions have been known to change through content courses that use artifacts of children's' mathematical thinking (Thanheiser et al., 2013) and are taught in ways that align with content standards for doing mathematics (Conference Board of Mathematical Sciences [CBMS], 2012). Thus, by changing PSTs conceptions to align with teaching standards, there is reason to believe that these newly-formed conceptions may influence teacher practice (Ambrose, Clement, Philipp, & Chauvot, 2004; Stohlmann et al., 2014; Thompson, 1984, 1992).

In order to better focus PSTs on the content, Philipp (2008) suggested centering the content around children's thinking. However, certain positive conceptions regarding teaching and mathematics should be maintained to optimize the benefits of this focus, as conceptions "play a significant role in shaping the teachers' characteristic patterns of instructional behavior." (Thompson, 1992) Therefore, mathematics teacher educators must understand what PSTs conceptions are, how they and other experiences influence their teachings, and how we as a mathematics community can help the PSTs develop.

Perseverance

In this study, I follow Dweck's (2006) notion that perseverance is related to having a growth mindset. Students with a growth mindset pursue goals to attain a deeper understanding (Sun, 2018), see success as expanding their capabilities, thrive on challenges, and don't give up easily (Dweck, 2006). Bettinger and colleagues (2018), in agreement with many other researchers, note that "growth mindset interventions shape students' beliefs in their ability to learn and cause lasting improvements in school outcomes" (p. 2). Dweck (2006) maintained that confidence is not always needed to persevere in a task. In other words, confidence and perseverance do not automatically hold a bidirectional relationship, in that perseverance instills confidence but confidence is not necessarily needed for perseverance. In order for teachers to impart a growth mindset to their students, the teachers must take care that their praises are of the child's learning process and not ability, that mistakes are not met with anxiety or concern for the child's ability but should not be glossed over either, and teaching should be focused on understanding and not memorization of facts, rules, or procedures (Dweck, 2006). In addition, teachers should supplement textbook material with curricular tasks that incorporate opportunities for collaboration and sensitivities toward student autonomy (DiNapoli, 2016). Thus, "the aspects of classroom culture that seem to support student willingness to engage with challenging tasks are those related to the ways that the lessons are conducted and the expectations set for the students not only in terms of the mathematics but also the ways of learning it" (Sullivan, Aulert, Lehmann, Hislop, Shepherd, & Stubbs, 2013, p. 621).

Encompassed within the idea of a growth mindset, Russo, Downton, Hughes, Livy, McCormick, Sullivan, and Bobis (2020) note that further study on the topic has informed and altered Australian teachers views and beliefs about struggle. Moreover, "[i]n the United States…creating opportunities for students to persist in problem solving is a tenet of effective teaching that is often described as creating the condition for productive struggle." (Sengupta-Irving & Agarwal, 2017, pp. 115-116)

Productive struggle ensues when "students expend effort in order to make sense of mathematics, to figure out something that is not immediately apparent" (Hiebert and Grouws, 2007, p. 387). Warshauer, Starkey, Herrera, and Smith (2019) found that preservice teachers (PTs) in a mathematics content course, were unfamiliar with the ideas of productive struggle and generally saw struggle as something negative. Additionally, "PTs placed the responsibility of productive struggle on the student, not the teacher, when learning mathematics…and had not considered it as a teacher-driven educational tool for learning mathematics (Hiebert and Wearne,2003)" (Warshauer et al., 2019, p. 26). Although the semester was not long enough to fully develop "robust mathematical interpretations" of productive struggle, most PTs were able to indicate at least one teaching strategy notated from Warshauer (2015):

> (1k questions to help students focus on their thinking and identify the source of their struggle, (2) encourage students to reflect on their work, (3) give time and support for students to manage their struggles, and (4) acknowledge that struggle is an important part to learning and doing mathematics (Warshauer, 2015; Warshauer et al., 2019, p. 25).

3. Furthermore, there is evidence that shows mixed results regarding teachers' comfort with pedagogies that lead to students engaging with struggle, especially low-performing students (Russo et al., 2020). Although beliefs often differ from what is incorporated into practice, Russo and colleagues (2020) found that most teachers in their study (n=93) held positive beliefs about the value of struggle, citing "benefits of struggle were the

opportunities it provided students to persist through challenge, take risks, build autonomy, build confidence, foster self-efficacy, learn through mistakes, and acquire a growth mindset" (Russo et al., 2020, p. 6), and only nine of the 93 teachers in this study held descriptive beliefs that contained neutral or negative ideas.

In an effort to illuminate teaching moves that could be made in the daily-classroom that help foster perseverance, Lewis and Özgün-Koca (2016) shared five categories of teacher moves to foster student perseverance in problems solving:

1) Selecting Mathematical tasks that require and support perseverance,
2) Talking about strategies for problem solving,
3) Demarcating phases in problem-solving process,
4) Naming feelings attendant to problem solving, and
5) Narrating internal processes.

Thus, research on these teacher moves is relatively new, and has not yet made its way into the teacher preparation work. Therefore, similar to other research about mathematical practices, Warshauer and colleagues recommend that teacher educators "introduce opportunities to connect PTs mathematical content knowledge to practices like understanding productive struggle in mathematics early in their teaching continuum" (Warshauer et al., 2019, p. 26).

Theoretical Orientation

This study operationalized a social constructivist perspective of collaborative learning (Vygotsky, 1978). In using this approach, the study conformed with the Vygotskian ideals of learning, meaning that people learn as they work to form understandings and create meaning through their shared experiences in any given situation. Therefore, the researcher acknowledges that the participants in this study were learning due to a multitude of factors from the social environment, such as their experiences from this study's interviews and the daily camp classroom and accompanying professional development. Thus, the study draws on these factors whilst operationalizing and tracking the categories from the study's perseverance framework.

Methods

Settings & Participants

This study, which is part of a larger project, follows three typical PSTs majoring in interdisciplinary studies, who had completed both content courses but had not yet completed elementary observations, and were participating as Fellows (teaching assistants) in a two-week research-based summer math camp program for elementary and middle school students and professional development. Specifically, the PSTs assisted in the classrooms focused on Integers & Algebraic Modelling (Grades 3-4). At the time of selection, one of the PSTs had two years of prior experience as a Fellow, two had one year experience, and one had none. The PSTs were chosen based on their applications, camp administrator recommendations, and selected degree plan. The PSTs are referred to typical PSTs in the sense that they could not explain why the basic algorithms of addition and subtraction of integers worked before learning the models used in this setting or from their content course. The PSTs mentioned that they were excited to work with students and hoped to learn how to teach math in helpful and engaging ways for their future students.

Data collected consisted of pre-surveys, PST-student (individual/group) interaction recordings, stimulated-recall interviews, post-survey reflections, and clinical interviews. During the camp, whole class discussions were led by an experienced middle school teacher, but group and individual work were frequently facilitated by the PSTs. The PSTs helped answer questions, provided feedback, and assisted with other classroom needs and management. These PST-student interactions were recorded, with a select number used for the stimulated-recall interviews. Clinical interviews and post-survey reflections were conducted on the last day of the two-week camp.

Analytic Framework

The unit of analysis for the stimulated-recall interviews consisted of a daily interview. The interviews could not be separated into clear distinct segments since PSTs would often refer back to previously watched interaction recordings from the daily interview and blend their reflections regarding the different interactions together. Thus, this study looked at each interview holistically. However, the clinical interviews, which were task-based, were analyzed per task. Therefore, the framework was applied to the recorded interactions used in the stimulated-recall interviews and the clinical interviews. Moreover, the pre- and post-surveys supplied additional information, along with the clinical interviews as to the conceptions the PSTs held regarding perseverance. This allowed for a triangulation between what was observed by the researcher and how the PST reported and reflected upon their supports of perseverance.

In each of the selected interactions and reflections, instances of PST moves, or lack of moves, to support the students' mathematical perseverance were noted. These fell into one of three categories: (1) praise for unsuccessful efforts to answer a question (2) praise for the process, or (3) fosters perseverance. The first two categories stemmed from Dweck's (2006) growth mindset ideas, while the third category was based on Lewis and Özgün-Koca's (2016) ways of fostering perseverance. Based on Dweck's approach, the PST could have chosen two routes: (1) the route which can produce a fixed mindset and decreased perseverance by praising the student's unsuccessful effort or answer, or (2) they could have chosen the direction of a growth mindset and praised a productive process that yielded understanding. Adapting some of the teaching moves from Lewis and Özgün-Koca's whole class orchestration to a small group or individual conversations surrounding pre-determined problems, five moves similar to their five themes were established: (1) attending to students' emotional needs, (2) focusing the discussion on the strategy or different strategies, (3) changing the participation format of the conversation, (4) creating opportunities for students to reflect on their work, stuck points, or the language of the problem, and (5) creating an opportunity for the students to extend their knowledge beyond the problem.

After assigning the categories, I conducted individual member checks with two of the three participants about their conceptions and views to verify the accuracy of the coding and interpretations. Additionally, an external reliability check was made for a random 25% of the stimulated-recall interviews and 40% of the clinical interviews were checked. Resolution discussions were had and adjusted the framework, which was then reapplied to all remaining data.

Results

This section focuses on describing how the PSTs conceptualized, used, and reflected on supporting young students' mathematical perseverance. The PSTs will be known henceforth as Amy, Becky, and Linda. The conceptualizations were primarily based on the pre- and post-surveys, but also included information from the clinical interviews. Since the camp structure focused on using the word persistence, the surveys asked the participants to describe and define the teaching practice of fostering persistence, and to address what they thought to be valuable in the practice.

Amy

Amy was the most experienced PST in the study and had been a Fellow for the previous two summers. In conceptualizing perseverance, Amy was asked to think about persistence as this was the term used by the camp. She wrote the following:

> Allowing students to have enough wait time. Asking guiding questions instead of giving direct answers. The value is creating a growth mindset which gives students endurance to work on hard problems longer.

Amy's definition aligned with the ideas of *fostering perseverance* because it focused on providing time for the students to work on the problem, while focusing the student on the process and strategies through questioning during the problem-solving process. Additionally, Amy attributed the value of perseverance to the amount of time spent working on a problem and establishing a growth mindset. At the end of camp, when asked to reflect on what she had written about perseverance, Amy said, "I think like if I would add something, something that we talked about in seminar was asking purposeful questions and so I think that's more important than just like guiding questions, … purposeful questions would be like asking questions for understanding". This addition, although clarifying what type of questions Amy would use to foster perseverance, still did not add or alter Amy's conception of perseverance.

Throughout the camp, Amy was observed not only supporting the students both by *fostering perseverance*, and by *praising the process*. When reflecting on her interactions during the stimulated-recall interviews, Amy was able to point out some instances of both of these supports, noting her remarks about emotional states, focusing on the wording of the problem, changing the participation format, reflecting on the problem, extending the problem, and praising a productive process. Most of her reflective efforts regarding perseverance were spent toward ideas of *fostering perseverance*, with only one observed instance from the stimulated-recall interviews reflecting *praising the process*. The reflection seemed almost an oversight to Amy, who recalled more of her excitement for the student's discovery than her actual turn of praise by saying, "I got excited when she came to the conclusion that you had to do 7 minus 5 in the other problem. So I was like, "yeah you do, dang"…" In fact, Amy did not acknowledge her own efforts in the students' perseverance. Amy recognized when students persisted but did not attribute any of her own supports to students' persistence even if it meet her definition. She noted how the students persevered in the problem by responding to Amy's prompt to reflect and explain their process and answer.

> I really liked their responses, that they didn't give up. … And both [students], too, didn't go straight to thinking that they were wrong. So I liked that. That wasn't necessarily anything I put in them, but whoever they had in the past, teachers and stuff, they've given them that sense of confidence.

From this statement, it is clear that Amy attributes parts of perseverance and "endurance" to levels of confidence.

Linda

Linda had been a Fellow for the camp the previous summer and was about a semester behind Amy in their educational program coursework. When conceptualizing perseverance, she wrote the following:

> I would see this as an environment where students feel comfortable not getting the right answer on the first attempt. Instead its viewing problems as a journey that takes multiple attempts and you don't give up. These [sic] is extremely valuable when learning topics to truly understand the material.

Linda's definition aligned with the ideas of fostering perseverance because it focused on providing time for the students to work on the problem while focusing them on the process and strategies instead of the answer during the problem-solving process. Additionally, one can see that Linda attributed the value of perseverance to learning and understanding mathematical concepts. During the clinical interview, Linda noted that follow-up questions served to foster perseverance in that it made the student continue to think about a problem.

Like Amy, Linda although conceptualizing perseverance in terms of *fostering perseverance*, also supported and reflected on both *fostering perseverance* and *praising the processes*. During her stimulated-recall interviews, Linda reflected on her supports for perseverance by noting instances when she changed participation formats, praised students' processes, extended the problems, had students reflect on the problem, and prompted students to try different strategies. However, there was no observed instances of emotional supports for fostering perseverance, and Linda primarily worked in a one-on-one environment except on a rare occasion. Although her interactions were not typically group interactions, Linda frequently thought about wanting to have included other students in her conversations, noting that she thought this would have been beneficial. Linda also pondered the idea of using different strategies for the same problem. However, Linda reflected on the students using a model or strategy they were unsure of after they were already confident in their answer. Lastly, Linda made several moves to *praise processes* but did not always reflect on them. Linda would frequently and explicitly praise students during and after a productive conversation by saying things like, "Nice. Okay, so let's go look at our paper again", "Right. Awesome", or "Cool". Additionally, Linda would also show praise implicitly by becoming more excited and animated when a student began a productive argument using vocabulary and descriptive words. This was also evident in her reflection on the interaction when she noted, "I think it shows they're understanding when they start using that in their vocabulary… that's why I got excited when she said operator". This also highlights that Linda viewed student understanding as being tied to the student's processing and use of vocabulary words to describe their processes.

Becky

Becky was new to the camp program but was at a similar rank in her educational program to Linda. Although Becky's reflections and descriptions were often very detailed, they would sometimes double-back and re-examine things in a different way. Thus, her conceptions were constantly assimilating to her current experiences and she would often bring these ideas up during her reflections. When asked at the beginning to conceptualize perseverance, Becky wrote her definition from the prospective of the teacher by saying the following:

> Persistence to me is defined as the continuing to push through something with determination. So I believe fostering persistence would be able to grow/develop the ability to push through

negative behavior, confusion, and other obstacles that teachers may face in the classroom and turn it into positivity so the student can overall learn.

Becky's definition discussed how teachers persist in the classroom to help their students learn, but also spoke about helping the students persist. Becky added that you shouldn't just give up on students and that it is important for the students to persevere; however, she also stated that she wasn't sure how to "make them be persistent…if they don't want to". Thus, Becky spoke about pushing through negative behavior and confusion to foster perseverance.

In the clinical interview, Becky noted that the interviewers were persistent in asking the student questions, which encouraged the student to persevere in problem-solving. Becky said that by asking questions, the interviewers encouraged the student to keep thinking about the problem. Therefore, although unclear in her definition, Becky alluded to the importance of questioning related to fostering perseverance in students.

Becky's conceptualization of perseverance, in terms of *fostering perseverance*, was unclear as to how exactly she would support students, but when she reflected on her interactions, she noted supports for both *fostering perseverance* and *praising processes*. Becky attended to changing participation formats, emotional needs when students became frustrated, wanting to bring in multiple strategies, praising processes and productive efforts, problem extensions, and having students reflect on completed problems. Although a few of these interactions did successfully include a change in participation format and multiple strategies, these were two supports of fostering perseverance that Becky continually echoed wanting to include more. In fact, Becky noted that exploring ideas students are uncertain of after being confident in their answer would result in a more answer-driven process, whilst the opposite ordering, although less confident, would instill a sense of growth. Additionally, when trying to support students, especially those who were becoming frustrated, Becky would rely on praising their productive efforts. Becky notated this by saying, "I felt like I needed to give her some validation and that she was doing something right, she was in the right direction, she just kind of got confused on something or tripped up, … to help her not get so discouraged and still want to participate because she had turned her body away and gave her pencil away." Moreover, Becky was observed praising another student every step of the way by saying "good" or "good job", but didn't reflect on the praise she gave but reflected more on the questions she was using to have the student explain and reflect on their process.

Cross-Case Analysis

The conceptualization of perseverance seemed to be viewed uniquely across the PSTs, but all thought of it as a way to overcome struggles and confusion. Amy viewed perseverance as being synonymous with a growth mindset, which is similar to Linda's definition of "viewing the problems as a journey" and not being afraid of getting the wrong answer. Becky's definition was slightly different from Amy and Linda's, but this may be because Amy and Linda's perception of perseverance had been affected by the camp since persistence is a key component of the camp structure. Becky's ideas revolved around turning a negative situation into a positive one, and not giving up on the problem.

In addition, all of the PSTs have a conceptualization related to fostering perseverance, they all enacted moves to support this categorization and praising the students' processes. However, Linda was the only PST to not be observed supporting students' emotional needs for fostering perseverance but was more likely to praise the students' productive processes in a more purposeful way. In contrast, Becky used praise as a form of student support when the student's emotional needs came to the forefront.

Discussion and Future Research

The PSTs mostly conceptualized perseverance in terms of productive struggle and a growth mindset but also included remarks that aligned with the five ways of fostering perseverance in the analytic framework that was adapted from Lewis and Özgün-Koca (2016). The PSTs intentions to define perseverance in terms of growth mindset (Dweck, 2006) and productive struggle (Hiebert & Grouws, 2007; Warshauer, 2015) was not surprising, given that their content courses and the camp were structured around these ideas. Moreover, this suggests that introduction to and involvement in such work altered the PSTs beliefs to include components of perseverance. This finding agrees with that of Russo and associates, who found that recent emphasis on growth mindset helped shift "teachers' willingness to embrace struggle and view it as a necessary aspect of learning mathematics" (Russo et al., 2020, p. 8). The PSTs views on perseverance fit mostly into Russo and colleagues (2020) ideas of conditionally positive responses, in that they held positive beliefs, but mentioned teacher involvement in the struggle. Linda's conception was the only one that fell into the perspective of a positive belief; however, in her reflections, she would often structure struggle with questions.

A possible explanation for why the PSTs viewed supporting perseverance in terms of fostering perseverance instead of both fostering perseverance and praising productive process could be influenced by how the PSTs perceived their role in students' perseverance. Evidence suggests that the PSTs saw perseverance as something that the students were responsible for, and the PSTs often had difficulty noticing their supports of perseverance as related to the practice. This was especially evident for Amy, who described students' confidence and willingness to explain ideas from previous problems as attributed to past teachers. This finding agreed with what Warshauer and colleagues (2019) found in their preservice teachers' understandings of productive struggle. Furthermore, the PSTs viewed perseverance as connected to confidence, however, it is unclear as to the direction of this relationship the PSTs imparted between the two ideas when research suggests a clear directional connection. Dweck (2000) noted that students with a growth mindset associated with their mathematical ability are more likely to have greater confidence that they will succeed; however, Amy noted that confidence allowed the student to persevere. This conceptualized connection to confidence and perseverance merits further study, to see if it aligns with current literature or if it disagrees, and if so, how.

Additionally, the PSTs noted that having students reflect on their work served the purpose of having students become confident in justifying answers. The thought of this as a move to foster perseverance seemed to be an afterthought or an accompanying outcome. Overall, the PSTs did not view perseverance as something supported by the teacher, but rather as something internal to the student. Although the PSTs' moves aligned with research, these moves were not all recognized as noteworthy, or for the purpose of supporting perseverance. Thus, consistent with the literature recommendations mentioned here (Lewis & Özgün-Koca, 2016; Warshauer et al., 2019), university coursework should include an awareness of perseverance and ways of fostering it. The PSTs in this study were already supporting perseverance to some extent, and when made more aware of perseverance and ways of fostering it, these existing supports could potentially be used more purposefully and provide better support for students' learning. Thus, future studies would benefit from providing PSTs with techniques to better support student perseverance and observing how these supports are taken up and could potentially alter the supports they use with students. Implications from this study suggest a particular need for university coursework to emphasis ways of promoting student-to-student talk as a way to foster a productive and persevering learning environment. Similarly, reflective or metacognitive questions would be a beneficial addition to not only model but include as a topic for discussion in university coursework, as these skills not only foster perseverance but are valuable mathematical reasoning habits (NCTM, 2009).

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OPPORTUNITIES TO LEARN IN CYCLES OF ENACTMENT AND INVESTIGATION

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In recent decades, scholars of teacher education have suggested that teacher educators (TEs) should integrate the development of prospective teachers' (PTs') knowledge with their skills for enacting teaching, characterized in the literature as pedagogies of practice. One way to operationalize pedagogies of practice is through engaging PTs in cycles of enactment and investigation (CEIs). Using an opportunity to learn (OTL) lens, this study investigated one CEI enacted in a secondary mathematics methods course. Analyzing course artifacts and final interviews, we found that the PTs had OTL in all six nodes of the CEI, that OTL differed across the nodes, and that OTL in later nodes depended on knowledge built in previous nodes. Implications include the importance of PTs engaging in all nodes of a CEI to maximize OTL about mathematics teaching practices, mathematics, students, and learning.

Keywords: Preservice Teacher Education, Teacher Knowledge

In recent decades, scholars of teacher education have suggested that teacher educators (TEs) should integrate the development of prospective teachers' (PTs') knowledge with their skills for enacting teaching, which Lampert et al. (2010) described as using a *pedagogies of practice* perspective. TEs who design learning opportunities from a pedagogies of practice perspective focus on specific *decompositions of practice* (Grossman, Compton et al., 2009), which "[break] down practice into its constituent parts for the purposes of teaching and learning" (p. 2058). PSTs interact with *representations of practice* (e.g., narrative cases or video-recorded teaching episodes) and engage in *approximations of practice*, which are "opportunities for novices to engage in practices that are . . . proximal to the practices of a profession" (p. 2058). Theoretically, novices can learn complex practices by engaging in learning opportunities designed from a pedagogies of practice perspective (Grossman, Compton, et al., 2009; Grossman, Hammerness et al., 2009; Kazemi, et al., 2016; Lampert et al., 2010). Some mathematics teacher educators in the United States have taken up this perspective to design learning activities for pre-service teachers that include some form of engagement in approximation of practice (e.g., Lampert et al., 2013; Campbell & Elliot, 2015). Much of the early research regarding these designs has been descriptive in nature; now the field needs research to examine how such pedagogies relate to PTs' understandings and skills.

Theoretical Perspective

Opportunity to Learn (OTL) emerged in the 1960s as a construct for characterizing instructional environments by input variables that might predict student learning as an output (Elliott & Bartlett, 2016). Early works used variables such as instructional time spent on specific content, content coverage, and instructional quality indicators as proxies for OTL (Elliott &

Bartlett, 2016). Gee (2008) explained that from what he called the mental representations perspective,

> learners have had the same OTL if they have been exposed to the same [content] . . . . If they have been exposed to the same content, then, according to this view, they have each had the opportunity . . . to "learn it." (Gee, 2008, p. 77)

As an alternative to the mental representations perspective on OTL, Gee argues that we should conceptualize OTL from a sociocultural perspective by conceptualizing learning as learning how to act in specific kinds of situations in ways that are aligned with the normative practices of some community. From this conceptualization, acting in some particular situation involves identifying objects in one's environment that one could use or act upon to achieve a desired result. The actor identifies *affordances,* which are defined as the possible actions that the individual can envision carrying out on, with, or in response to those objects. The actor then selects and operationalizes one of those affordances. To do so, the action must fit within the actor's understanding of which possible actions would be consistent with the accepted practices of some community that they identify with, which Cobb et al. (2009) describe as a *normative identity* that the actor has co-constructed with other members of that community. Further, the actor must have *effectivity* with respect to the selected affordance, defined as the capacity to operationalize a possible action (Gee, 2008).

Methods

The context for this study was a semester long methods course for secondary mathematics PSTs at a Mid-Atlantic university. Two mathematics teacher educators (MTEs) taught the course, which met two times a week for fifteen weeks. Seventeen of the 18 PTs in the course participated in this research. The MTEs designed the course from a pedagogy of practice perspective. Specifically, the course involved three cycles of enactment and investigation (CEI) (Lampert et al., 2013; Arbaugh et al., 2020) as described in Figure 1. The focal decomposition of practice in all three CEIs was a set of communication moves: Asking assessing and advancing questions, and using judicious telling (Freeburn & Arbaugh, 2017).

Figure 1: The CEI

In general, PTs begin a CEI in Node 1 by doing and discussing a mathematical task. In Nodes 2 and 3, PTs analyze and discuss a representation practice (e.g., narrative case) through the lens of a focal decomposition of practice. In Node 4, PTs use the focal practice and mathematics discussed in previous nodes to frame their planning for enacting teaching in Node 5, where they rehearse a teaching episode with simulated students. In Node 6, PTs analyze their rehearsal videos through the lens of the focal practice. This study focused on one of the three CEIs, which occurred mid-way through the course. In Figure 2, we give a brief description of the events that took place in this specific CEI.

<table>
<tr><th colspan="4">Node 1: Doing the Mathematics</th></tr>
<tr><td>Goal: Defining the mathematical learning goal for CEI focal task by developing criteria for determining if a mathematical argument is a proof.</td><td>PTs completed the Odd + Odd = Even Task (Blinded)</td><td>PTs analyzed student work for Odd + Odd = Even (Blinded) task to judge whether argument is a proof or not.</td><td>Group reached a consensus for criteria to use for when an argument counts as a proof</td></tr>
<tr><th colspan="4">Nodes 2 and 3: Individual Analysis and Collective Analysis of the Narrative Case</th></tr>
<tr><td>Goal: Applying PTs' understanding of focal practice to analyze a representation of practice.</td><td>PSTs individually coded the narrative case through focal practice</td><td colspan="2">PTs discussed their analyses of the narrative case in small group and whole-class discussions.</td></tr>
<tr><th colspan="4">Node 4: Planning for the Rehearsal</th></tr>
<tr><td>Goal: Learning to plan instruction using a focal practice in rehearsal.</td><td>PTs completed rehearsal task.</td><td colspan="2">PTs used focal practice to frame their planning guided by a modified "Thinking Through the Lesson Protocol" (TTLP) (Smith, Bill, & Hughes, 2008).</td></tr>
<tr><th colspan="4">Node 5: Rehearsal</th></tr>
<tr><td colspan="2">Goal: Developing skills for engaging in the focal practice and developing deeper understandings of the teaching practices addressed in previous nodes and course activities.</td><td colspan="2">The PTs individually enacted their plan from Node 4 by engaging a "student" in moving towards achieving the mathematical goal of proving a number theory conjecture.</td></tr>
<tr><th colspan="4">Node 6: Collective and Individual Analyses of Rehearsal</th></tr>
<tr><td>Goal: Developing skills of analyzing teaching</td><td>Using StudioCode©, small groups of PTs collectively coded each PTs' rehearsal.</td><td colspan="2">PTs individually reflected on what</td></tr>
</table>

through the focal practice.	Analysis included writing rationales for coding choices as well as an assessment for if the focal practice "worked" or not based on student response.	they learned from engaging in rehearsal and analyses.

Figure 2: Goals for PT learning and descriptions of PT activities in each node of the CEI

Data Collection

Data analyzed for this study include data collected during the multi-day enactment of the CEI. Audio-recordings were collected of whole-class discussions during Nodes 1, 3, 4, and 5 and small-group discussions in Nodes 1, 3, 4, 5, and 6. PTs' rehearsals in Node 5 were video-recorded. Students' written artifacts - notebooks, assignments, reflections that occurred during the CEI were collected. In addition, post-course interviews were captured by audio-recordings; the interview data analyzed for this study focused on responses to questions that asked PTs to reflect on how the CEI activities supported their learning.

Data Analysis

Our unit of analysis was a segment of communication, which we define as a series of turns of talk with a common focus (Bishop et al., 2016) and with a consistent form of participation (whole-class, paired work, individual work, or group work). We analyzed data sources in three phases. In phase one, we randomly chose three participants' data corpus and used the four dimensions of Ghoussieni and Herbst's (2016) Framework for Learning to Teach (FLT) as a priori codes: *Knowledge of Students and Content*; *Repertoire of Practices and Tools*; *Dispositions for Teaching and Learning*; and *Professional Vision* (see Table 1, Column 1 for definitions). At the same time, we began to develop subdimension codes using constant comparative analysis (Miles, Huberman, & Saldaña, 2013) and inductive analysis, and wrote analytic memos that detailed commonalities across the data. Once satisfied with the secondary coding scheme with the limited set of data, we began coding additional participants' data, refining the secondary codes (e.g., renaming, collapsing similar codes) until all coding was complete. Table 1 contains the resulting subdimensions (Column 2). In phase two, we coded segments for appropriate CEI node (1-6). In phase three, we organized our coded segments (*n*=414) and related analytic memos into a data table that allowed us to sort the instances by CEI node, dimension, or Subdimension. We used the sorted table to identify themes (Miles, Huberman, & Sa , 2013) in the data that allow us to describe the PTs' opportunities to learn during each CEI node and in each dimension across the nodes. We constructed frequency counts of the coding in each dimension across the CEI's nodes. Within each dimension, we created frequency counts of the subdimensions identified during phase 2. We then examined the frequency counts and made profiles of the OTL in each node. We then examined OTL across the nodes to arrive at the claims we present next in the findings section.

Dimensions & Descriptions (Descriptions are excerpted from Ghousseini & Herbst, 2016)	**Subdimensions Identified through Data Analysis**
Knowledge of Students* & Content: Understandings of the subject matter, of students as learners, and of ways to support their engagement with this subject matter . . . Teachers not only need to know the content but also understand the kind of reasoning that is entailed in doing mathematics. They should be able to interpret student work in light of what students already know and the tools at their disposal. (p. 82–83)	**Criteria for Valid Arguments -** PTs articulate criteria for a valid argument or criteria for why an argument is not valid.
	Type & Components of Arguments - PTs describe a type of an argument (e.g., proof by induction) or components of proof and reasoning (e.g., identifying a pattern).
	Representations in Arguments - PTs recognize a type of representation and address how the representation is incorporated into an argument.
	Mathematical Ideas and Practices - PTs describe a particular mathematical concept(s) or practice(s).
	***Student difficulties -** PTs identify students' errors in arguments or suggest ways a student could improve an argument.
	***Students think differently about the same task.** PTs explain similarities or nuances among student arguments as well as attributes among student arguments.
	***Validity of Arguments vary depending on grade level -** PTs share ideas about validity or appropriateness of student arguments as a consequence of grade level.
	***Student mathematical understanding -** PTs address student's understanding of mathematical content or capabilities to engage in mathematical practices.
Repertoire of Practices and Tools: Support teachers' beginning enactment of important aspects of instruction. Tools . . . can help teachers translate abstract conceptual tasks into more concrete steps and objectives (p. 83).	**Recognizing practices and tools -** PTs recognize a teaching move or routine in a segment of classroom instruction.
	Attributes of practices and tools - PTs articulate features, definitions, purpose, or characteristics of a teaching move or routine.
	Engaging in Practices - PTs engage in or reflect on their engaging in a teaching move or routine.

Dispositions for Teaching and Learning: Teachers' dispositions to see students as sense makers and learn the intellectual and professional stance of inquiry are important aspects of teachers' learning in and from their practice (p. 83).	**Honoring Student Thinking -** PTs communicate their stance that mathematics instruction should recognize, incorporate, or build on student thinking.
	Learning mathematics for understanding - PTs communicates their stance towards learning mathematics for sensemaking or understanding.
Professional Vision: The ability to notice and interpret features of practice in ways that are valued by a particular professional group. . . . A vision of practice may also delineate what is possible and desirable in teaching . . . it gives teachers a sense of direction (p. 82)	**Visions of instructional practice -** PTs notice and interpret a component of instruction as desirable based on the PTs' interpretation of a community's considerations for teaching and learning.

Figure 1: The A Priori Dimensions and Emergent Subdimensions of the FLT

Findings

One adaptation of the Gousseini and Herbst (2014) framework that resulted from our analyses is a need to separate *Knowledge of Students and Content* into two distinct dimensions: *knowledge of students* and *knowledge of mathematical content*, which more closely reflects a pedagogical content knowledge (Grossman, 1990) perspective. Subdimensions for *knowledge of students* are indicated by asterisks in Figure 1. We present three claims in this paper (see Table 1), and then, due to limited space, we expand only upon Claim 3 to show how the OTL in later nodes depended upon knowledge developed in previous nodes.

Table 1: Three Claims

Doing the mathematics (Node 1) and planning for teaching (Node 4) created opportunities to develop knowledge of content.
Analyzing the narrative case (Nodes 2&3), engaging in the rehearsal (Node 5), and reflecting on the rehearsal (Node 6) created opportunities to develop a repertoire of practices and tools.
Doing the mathematics (Node 1), planning for teaching (Node 4), and reflecting on rehearsals created opportunities to develop knowledge of students.

Doing the Mathematics, Planning for Teaching, and Reflecting on Rehearsals Created Opportunities for PTs to Develop Knowledge of Students

Evidence of opportunities to develop knowledge of students was much more prevalent in Nodes 1, 4, and 6 than in Nodes 2, 3, or 5. Further, the number of PTs providing that evidence is higher in Nodes 1 (n = 12), 4 (n = 11), and 6 (n = 7) than in Nodes 2 (n = 3) or 5 (n = 2). For those reasons, we conclude that OTL in the *knowledge of students* dimension are primarily accounted for in Nodes 1, 4, and 6 and we next describe the progression of the OTL across these nodes in two subdimensions: *student difficulties, errors, and areas for improvement* and *differences* and *students have different ways to think about the same task.*

Student difficulties, errors, and areas for improvement. The discussion of the student work samples in Node 1 provided OTL for PTs to develop their knowledge of the kinds of errors that students might make when attempting to engage in argumentation. For example, as PT10 read the Student Work Sample A, she noticed a similarity between the argument from Student A and the argument that PT10 had constructed, namely that both Student A and PT10 had used one variable (n) to represent two different odd integers. The instance is evidence of OTL for identifying an error that might occur as a student engages in constructing an argument. Similarly, as PT10 and PT13 examined Student F's argument, PT10 stated, "I don't know how you would judge what they know from this." T13 stated, "Well, they have some errors." The PTs agreed that the algebra of the argument is wrong, and that the argument lacked coherence. As PT13 said, "[“he] statements are not connected to the ones before it." In Node 4, as PTs responded to the elements of the TTLP, they discussed possible errors that students might make or misconceptions that they might have when attempting to argue for the given claim. These errors fell into two broad categories: Errors that were general to argumentation, and errors that were specific to possible approaches to arguing for a particular claim. For example, PT7, PT8, and PT9 suggested that a student might not understand definitions of *even* and *odd*, that a student might consider examples sufficient justification for a general claim or might not use enough evidence to justify the claim. These potential areas of difficulty are more general across claims. However, they also examined errors for each of five different possible approaches to proving the specific claim that they were assigned.

In addition to opportunities to consider both difficulties at the general level and at the specific level, PTs’ statements again gave evidence that their OTL was mediated by their experiences in previous nodes. Specifically, in Node 4, while discussing potential errors related to how students might argue for their claim, PT3 anticipated that students might use a table of values to present examples in support of the claim that the product of two squares is a square. PT3 connected that anticipated response to their experience in Nodes 2&3: "Kind of like the students in [the case narrative]. Using examples to prove, but we need to get them to do a general case. Not just use examples."

Students have different ways to think about the same task. In Node 1, PTs had the OTL to see that students (themselves, their peers, and the students represented in the student work samples) had different approaches for writing an argument that the sum of two odd numbers is always even. In Node 1, PT12 and PT15 briefly discussed their arguments for the task - –T15 explained that his argument involved stating that odd numbers are even numbers plus 1 and that the sum of the two odd numbers then will be an even number plus 2. PT12 replied that she had argued for the claim using the same approach. Each of the groups noted strong similarities between the arguments from Student B and Student C and interpreted the differences as a distinction between a valid argument (Student B) and a not valid argument (Student C). The activity also included opportunities to recognize similarities between the arguments in Student Work Samples and their own attempts at proving the claim--for instance, while reviewing Student D's argument, PT10 stated that the argument was valid because it was similar to an argument that she and PT13 had previously identified as valid (though she did not clearly indicate which Student Work Sample she was referencing) as well as to the argument that PT10 had made in her own attempt to prove the claim.

The main areas of OTL within the Node 1 activities were in the dimensions of Knowledge of Content and Knowledge of Students. Given that the activities were explicitly intended to engage PTs in conversations about criteria for valid arguments, it is unsurprising that the plurality of instances coded in Knowledge of Content dimension were related to the domain of Criteria. However, in the context of those conversations about criteria there was also OTL about types of arguments, the components of arguments, and to compare and contrast students' arguments toward a claim as well as the errors or areas for improvement in students' arguments. These domains of knowledge are important for establishing learning goals for students' argumentation, for anticipating the kinds of arguments that one might encounter in a secondary learning environment, and for framing how one determines, of the affordances he or she recognizes as possible actions to take in response to student argumentation, which affordance to attempt to transform into action.

In Node 4, the OTL about students' mathematical understanding was primarily a consequence of PTs anticipating students' solutions in response to an element of the TTLP. Drawing on the PTs' arguments in Node 1 as well as the Node 1 student work, PTs discussed arguments students may make for their assigned number theory task. PTs also raised questions about what knowledge students might be expected to already have, and whether that would change which parts of the argument would need to be supported rather than assumed. For example, PT8 and PT9 wondered whether students could be expected to know that N^2 is even if and only if N is even, and if so whether that would mean that students could draw on that fact without justifying it. Evidence in Node 4 indicates that PTs drew upon their experiences in Node 1 as a resource to support their anticipation of student thinking. For example, PTs referred back to the student work samples from Node 1 for ways to represent even numbers and odd numbers.

Connecting to the theoretical perspective. Viewed through the sociocultural perspective, the PTs had OTL in Nodes 1 and 4 about how to act as a teacher who knows how students think mathematically in ways that are aligned with what it means to prove, which is a normative practice of the mathematics community. In doing the mathematical task and analyzing student work in Node 1and then anticipating student responses in Node 4, the PTs had the opportunities to build the kinds of knowledge that will allow them to preplan possible actions they can choose from in Node 5 (rehearsal).

Discussion and Conclusion

This study contributes to the field's understandings of what PTs and the opportunity to learn from engaging in pedagogies of practice, extending the work of Arbaugh et al. (2019; 2020), Ghousseini and Herbst (2016), Tyminski et al. (2015), and Baldinger et al. (2016) and adding to current evidence of the impact of CEIs on PTs' building of knowledge about teaching mathematics. What makes this research unique is that we studied OTL through the content of what PTs took up and discussed in small groups, large groups, and in reflective interviews. Much OTL work has been done from a researcher-down perspective – what we, as researchers, intend for PTs to learn. Considering OTL through a PT lens provides powerful indicators of the possible impacts of engaging in CEIs. One implication from this research is that it is important to engage PTs in a whole CEI – not just choose to do parts of it (e.g., planning and then enacting practice in rehearsals). Our findings indicate that opportunities to learn occurred in all CEI nodes and, perhaps more importantly, OTL in latter nodes depended upon knowledge built in previous nodes. We have also come to understand the power of having PTs analyze student work samples in Node 1. Simply doing the mathematical task itself would not have offered the same kind of OTL about student thinking that doing the task and analyzing the work samples did.

This study joins very few others who are examining (possible) outcomes for PTs who learn to teach through engagement in pedagogies of practice and learning cycles. Much work is to be done before the field has a solid understanding of this kind of pedagogy in ways that are convincing about its effectiveness.

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EXPLORING CONNECTIONS BETWEEN PROSPECTIVE TEACHERS' VIEWS OF AUTHORITY AND EXPERIENCES IN JUSTIFICATION

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The purpose of this project was to understand how implicit views of authority support or limit prospective elementary teachers' (PTs) mathematical activity of justifying and to understand how the experience of justifying might support a development of an internal source of authority. In this case study of 18 PTs, we coordinate an analysis of 1) their responses to two justification tasks and 2) interview transcripts in which they discuss their experiences in learning to justify. Preliminary results indicate ways in which their views of authority limited their reasoning about mathematics by not recognizing their own sense-making and supported a sense-making exploration of mathematics that was freeing and empowering. These results provide mathematics teacher educators with insight to help them identify and address limiting views of authority and leverage productive views of authority.

Keywords: Preservice Teacher Education, Reasoning and Proof, Teacher Beliefs

Purpose of the Study

Attending to this year's theme of "persevering through challenges", we address a familiar challenge encountered in mathematics teacher education. When prospective elementary teachers (PTs) enter content courses, they typically hold limiting views of mathematics as memorizing procedures. They often are not aware that mathematics makes sense and that procedures can be justified (Ball, 1990; Feiman-Nemser, 2001; Ma, 1999; Spitzer et al., 2010; Thanheiser, 2009), nor do they generally view themselves as a source of authority for reasoning about mathematics as mathematical sense-makers (Cady et al., 2006; Perry, 1970; Povey, 1997). As mathematics teacher educators (MTEs), we want our students to take ownership of their mathematical sense-making. Explaining and justifying one's thinking are activities that support the vision of sense-making and argumentation described in national documents: "By developing ideas… justifying results, and using mathematical conjectures… at all grade levels, students should see and expect that mathematics makes sense" (NCTM, 2000, p. 56).

When student generated mathematical contributions are validated through collaborative reasoning, students are supported in developing the skills of explaining and justifying their thinking along with assessing the validity of that thinking (Gresalfi & Cobb, 2006; Reinholz, 2012). Such support is necessary for students to develop an internalized authority that is based on sense-making through their own reasoning, rather than relying on an external source of authority represented by experts such as the teacher or textbook (Boaler & Selling, 2017; Engle & Conant, 2002; Lampert, 2003; Reinholz, 2012; Schoenfeld, 1994). To better understand how to support students in taking ownership of their sense-making, we use the context of a justification-feedback-revision cycle to explore the connections between PTs' justifications, their process of justification, and their views of authority and sense-making in the mathematics classroom.

Understanding how PTs view authority to reason about mathematics in the classroom provides valuable insight into how MTEs can uncover these views, support PTs in interrogating their views, and then help PTs learn how to use their authority to contribute ideas and evaluate the reasonableness of contributed ideas. In this proposal we share how one class of 18 PTs engaged in a justification-revision cycle and argue that attending to PTs' views of authority provides insight into (1) potential barriers to learning to justify and (2) productive views of authority that support learning to justify.

To support this argument, we seek to answer the following questions:

1. How do PTs' descriptions of their justification process provide insight into their views of authority?
2. How do PTs' views of authority support or limit their justification?

Perspective/Theoretical Framework

As justification and authority are central to our study, we unpack these two constructs and explain how the relationship between them frame our work (see Figure 1).

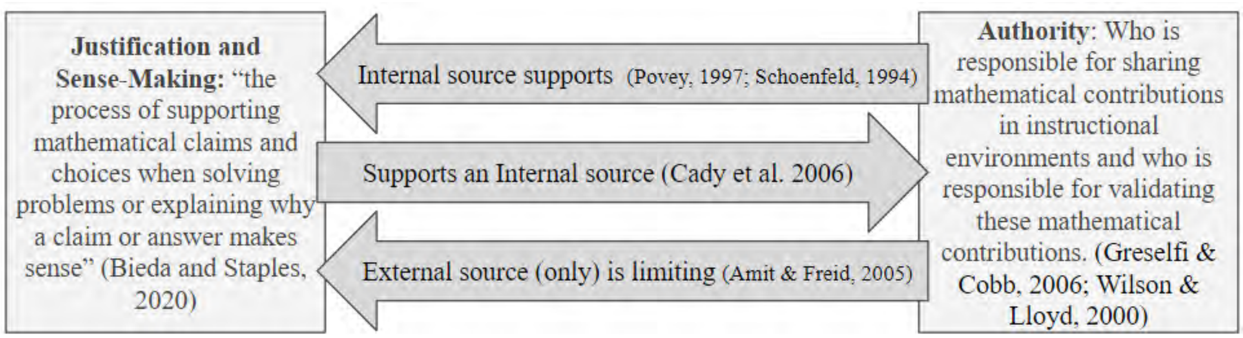


Figure 1: Framework that connects authority and justification

Authority

We define authority as who (or what) is responsible for sharing mathematical contributions in instructional environments and who (or what) is responsible for validating these mathematical contributions (Gresalfi & Cobb, 2006; Wilson & Lloyd, 2000). In instructional environments, students engage with a web of authority that includes instructors, their peers, themselves, textbooks, and other authorities in their life (Amit & Fried, 2005). The development of an internalized source of mathematics authority in which students view themselves as an authority is associated with the development of mathematical sense-making abilities (Povey, 1997; Schoenfeld, 1994). This view of the self as an authority supports sense-making because it carries the expectation that the students take on responsibility for reasoning about what makes sense – first through sharing their own ideas, and then through providing an explanation that justifies their thinking and solution to their peers and instructor. This view is in contrast to the expectation that the teacher or other external sources is responsible for telling them what makes sense and what is correct, a view that limits students' sense-making.

Viewing mathematics authority as residing in the teacher negatively impacts students' conceptual mathematical thinking "by turning always from one figure to another, and never to themselves, the students …fail to develop their own mathematical thinking," (Amit & Fried, 2005, p. 165). In addition, they position "themselves as outsiders with respect to mathematical discourse" (ibid). Undergraduate students develop an internal source of authority as they learn to think critically and develop persuasive arguments (Povey, 1997). A shift in authority from external to internal becomes apparent when students "support their opinions with reason and logic" (Cady et al., 2006, p. 296), in other words, when they justify.

Justification

Students are sense makers (Ernest, 2000) and justification is essential to sense-making (Bieda & Staples, 2020). We adopt Bieda and Staples' (2020) definition of justification as "the process of supporting mathematical claims and choices when solving problems or explaining why a claim or answer makes sense," (p. 103). (See left box in Figure 1). We think of justification as a way of communicating understanding (Jaffe, 1997), and as distinct from mathematical proof, which is a final product, in that a justification does not have to be logically complete (Melhuish et al., 2020). A justification should seek to convey structure and generality if applicable. For example, for their first justification task, PTs were asked to determine if the sum of two odd numbers is always odd, always even, or sometimes odd and sometimes even, and then justify their thinking. While we did not expect them to use a number theoretic approach to the justification, they were encouraged to attend to the structure of an odd number as a collection of groups of two and one left over, or as two groups of the same size with one left over and use one of these definitions of odd to argue a general case (see Table 1 in the next section for examples).

PTs experience challenges in learning to justify and in supporting children in learning to justify. (G. J. Stylianides et al., 2013). PTs often conflate justification with providing/checking multiple examples rather than viewing justification as a general argument based on mathematical properties and definitions of terms (Harel & Sowder, 2007). Teachers (including PTs) need to develop a common language and understanding of justification so they can understand what justification and proving look like in an elementary classroom and can support their students in this activity (Harel & Sowder, 2007; Staples & Lesseig, 2020; A. J. Stylianides, 2007). We add to the literature by building on our understanding of PTs' justifications. We coordinate this understanding with their views of authority through examining the limiting/supporting relationship between views of authority and justification (see Figure 1).

Methods

Participants

The 18 PTs participating in this study were enrolled in their first mathematics content course for prospective elementary teachers. The study was conducted at an urban public university in the Pacific Northwest of the United States. PTs at this university were required to complete a sequence of 3 mathematics content courses to enter their teacher education program. This course was taught by one of the authors.

Context

This first course in the sequence was inquiry-based with the goal to develop students' mathematical knowledge for teaching (Ball et al., 2008; Hill et al., 2008) and the expectation that students share their reasoning as a learning community. The main topics of the course were number and operation with an emphasis on sense-making through justifying, representing ideas in multiple ways, and making connections between these multiple representations.

For the present study, the course was taught asynchronously online via an online learning platform and shared Google slides. Emphasis was placed on the value of reviewing and reflecting on previous work and providing feedback to their classmates with the intention of positioning PTs as sense-makers and as sharing responsibility for theirs and their classmates' learning as a part of a community. To support PTs in sense-making and justification, they were asked to complete multiple cycles of (a) sharing a rough draft (Jansen, 2020) of a justification, (b) reviewing and providing feedback on other PTs' justifications, and (c) revising their initial draft based on the feedback they received from classmates and the instructor. In this study, we focused on their first two justification tasks, in which they completed the following statements and then justified why the statements were true:

- The sum of any two odd numbers is [always odd, always even, sometimes odd/even].
- The sum of three consecutive numbers is [always, sometimes, never] divisible by three.

Data Collection

Data includes (a) PTS' written rough draft and revised justifications provided via Google slides, and (b) transcripts of an hour-long interview with PTs conducted via Zoom. Interviews were conducted by the lead author during week six of a ten-week term after PTs had completed both justification-revision cycles. The semi-structured interviews included questions asking PTs to describe the process they went through when creating their justification, how confident they were that their response was a valid justification, and how (and to what extent) they utilized classmates' 'ork and the instructor's and classmates' feedback.

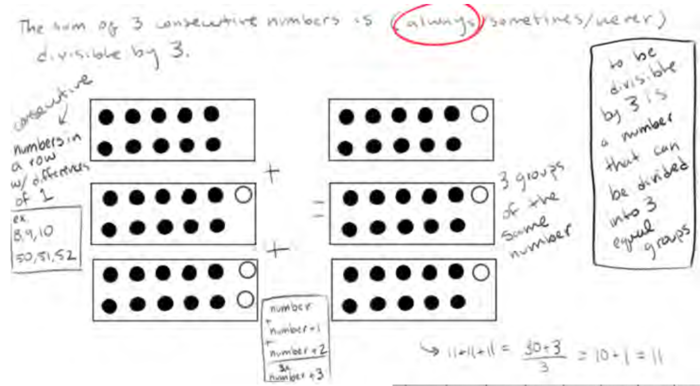
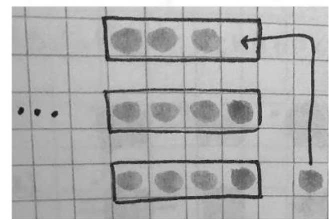
Data Analysis

To analyze our data, we used an inductive approach (Thomas, 2006). We started with reading through the PTs' responses to the justification tasks (provided in our shared Google slides) and their interview transcripts and recording observations about (1) how PTS justified, (2) key moments in the interviews in which PTs described their experience in justifying, and (3) statements from the transcripts that provided insight into their views of authority. After an initial pass through our data, we then developed categories as described below.

Justification Data. We analyzed PTs' draft (D) and revised (RIesponses to the two justification prompts (see above). Since the PTs were introduced to the construct of convincing yourself, convincing a friend, and convincing a skeptic to characterize PTs' levels of justifications in the class (Mason et al., 1982), we leveraged these categories in our analysis to code their justifications (see Table 1). Note that the code *Misinterpreted* was used to code PTs' justifications that misinterpreted the prompt. Two of the authors coded each of the justifications individually and met to resolve disagreements through debate.

Table 1: Coding Scheme for Justifications

	Description	Example for: The sum of three consecutive numbers is [always, sometimes, never] divisible by three.
Self	PTs relied solely on specific examples	Example: 4+5+6 = 15 Justification: 15/3 = 5
Friend	PTs described general structure and didn't yet use this description of structure to argue for the general case,	If you add any 3 same groups of #s together, that large sum can always be separated again back into 3 equal parts. Because consecutive numbers are 3 of the same number plus an additional 3, that additional 3 that is added into the numbers to make them consecutive, can therefore be separated into 3 equal parts. (PT 1)

	or their argument was unclear	
Friend/Skeptic	PTs written justification is a general argument based on structure and the visual representation uses specific examples	
Skeptic	PTs written justification and visual representation that build a general argument based on structure.	The sum of three consecutive numbers is ***ALWAYS*** divisible by 3 because when you add 3 consecutive numbers, it is equal to adding the first number in the sequence times 3 plus 3. 3 consecutive numbers added together can be grouped into 3 equal groups. 

Interview Data

To create an initial analysis of the interview data, the first author wrote brief descriptions of each interview summarizing each PTs' general approach to justification and how their discussion of their justification process provided insight into their view of authority. To further this analysis, we reviewed the interview transcripts and identified moments the PTs talked about sources of authority they looked to while writing their justifications (self, peers, instructor, previous experience, etc.) and how they determined that their justification was valid. For example, in describing their experience with justification, PT 1 said she did not understand "what you guys wanted" and that she didn't "know what you need." This indicated to us that the student was seeking verification from an external source of authority. In contrast, PT 2 shared, "For some reason, especially math, when I learned something that I can connect something on my own. I feel so much more accomplished!" This indicated that this student was starting to view this responsible for their own sense-making, indicating an internal source of authority. After all PTs' responses were read with such moments indicated, we categorized the PTs as having a primarily external source of authority, internal source of authority, or mixed. The mixed source of authority emerged when a student alternated between describing internal and external sources of authority. It is important to note that these interviews are snapshots of one moment during a 10-week term. Thus, their discussions of justification give us insight into their views of authority at the moment of their interview, but do not yet provide a comprehensive description of their overall orientation toward authority.

Results

Justification

Results from our analysis of the PTs' justifications are shown in Table 2. In general, the PTs did attend to the mathematical structure of the concepts they were justifying, as shown by the prevalence of Friend and Friend/Skeptic justifications. However, their justifications demonstrated mixed success with developing logic to link the structure to the concept. For example, for the first justification, 9/18 students' justifications were Friend level with missing or unclear logic preventing the justification from being coded Friend/Skeptic. Including a generalized visual representation of their justification proved to be the challenge that limited many of the justifications to Friend/Skeptic rather than reaching the level of Skeptic. While students demonstrated some success with written generalizations of their justifications, with few exceptions, visuals were limited to "dot drawings" of one or two examples.

Table 2: Results of Justification Coding

Justification	Misunderstood	Self	Friend	Friend/Skeptic	Skeptic
Sum of 2 Odds (D)	2	3	9	4	0
Sum of 2 Odds (R)	0	2	9	7	0
3 Consecutive #s (D)	5	5	3	5	0
3 Consecutive #s (R)	1	3	4	8	2

Authority

In characterizing PTs' views of authority (as seen during their week 6 interview) we found that of the 18 PTs, 8 provided evidence of viewing themselves as an authority (internal), 6 PTs primarily looked to an external source of authority, and 4 provided evidence of a mixed view of authority. From the analysis of interview transcripts, we selected two PTs that represent contrasting examples of views of authority. The two examples are contrasting because one PT's descriptions indicated more external views of authority, whereas the other was more internal. We used the following figure (Figure 2) to represent the distribution of how we categorized PTs' views of authority and how each of 18 PTs' justifications were coded in our analysis. We then trace PT 1's and PT 2's level of justification.

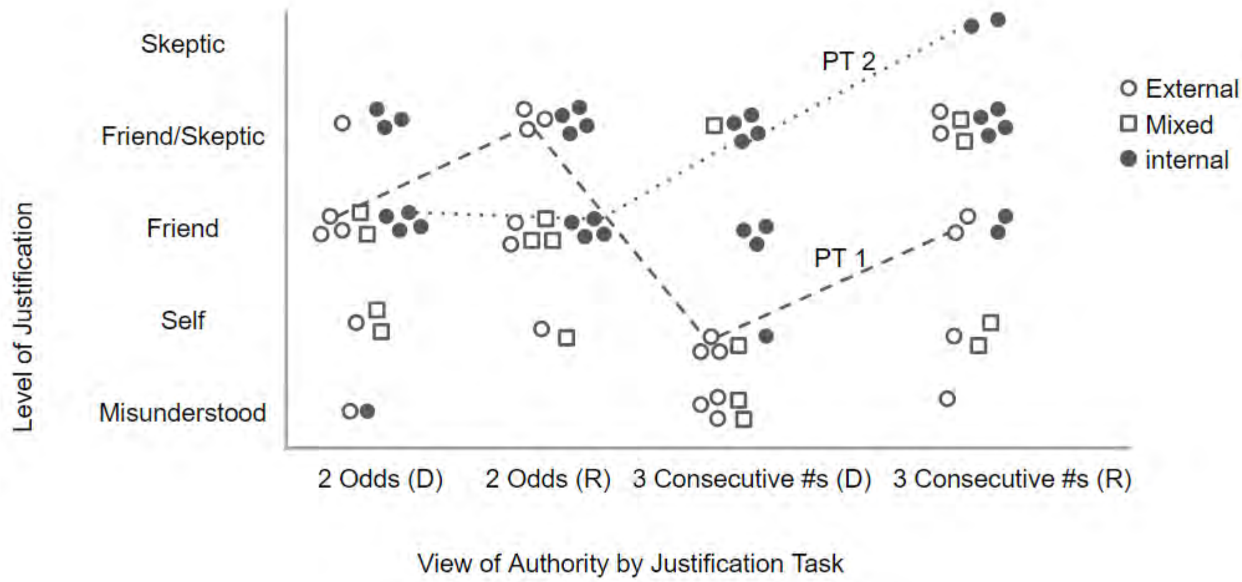


Figure 2: PTs Views of Authority and Level of Justification

PT 1. We categorized PT 1 as primarily looking to external sources of authority. Overall, in terms of justifying, PT 1 struggled with providing a clear argument. For example, for the first justification PT 1 revised her justification from the level of Friend to a Friend/Skeptic, but in the second justification, her justification started at Self and improved to the level of Friend after revision. In her interview she shared a desire to explain the math behind a concept while also wanting to know what "we," as the instructors, wanted for a solution. She explained that she did not understand "what you guys wanted" and that she did not "know what you need." She also shared that she found this difficult because the instructor did not provide an outline of an expected solution. At times, PT 1 did say that her explanation made sense to her: "I'm just like, I think this works. I hope it works. I don't know if it works, but it makes sense to me." This suggests that PT 1 is experiencing tension between viewing herself as responsible for determining what consists of a valid justification and wanting to meet the external standards of what the instructor wants for a justification to be considered valid.

PT 2. In contrast, we categorized PT 2 as having an internal source of authority. PT 2 clearly articulated coming to understand mathematics through her own sense-making. Throughout the interview, she mentioned first wanting to remember "math I was taught" but then recognizing that she could create her justification based on her own understanding, saying "for some reason, especially math, when I learned something… or like I can connect something on my own… I feel so much more accomplished!" In addition to viewing herself as an authority, she also mentioned looking to the instructor as a source of authority in sharing her uncertainty about "what [the instructor] was looking for" and looking to her classmates' slides as a source of authority for contributing mathematical ideas. She stated that she used her classmates' slides to help her know what to do on the first justification. While PT 2 viewed herself as a primary source of authority, she also recognized there is value in seeking help from the instructor, and her classmates. This illustrates how PTs with an internal source of authority incorporate, but do not replace, other external sources of authority with their own internal source of authority.

Discussion

PTs in this study struggled to provide justifications that were general both verbally and/or visually and the prevalence of the Friend and Friend/Skeptic codes indicate that PTs struggled with providing generic examples through a visual representation. This is consistent with current literature in our field (Lo et al., 2008; Martin & Harel, 1989; Rø & Arnesen, 2020). This analysis gives us context as we examine PTs' views of authority. The interview analysis indicated that nearly half (8 of the 18) of the PTs viewed themselves as primary sources of authority, while the others (10 of the 18) looked primarily to external authorities or had mixed views of authority. The purpose of this study is to explore potential limiting and supportive views of authority as evidenced through their discussion of their justification process. We turn now to a comparison of their justifications and their views of authority.

To address our first research question, "How do PTs descriptions of their justification process provide insight into their views of authority?" we analyzed PTs' interview transcripts as described in the results section above, finding that 7 of 18 PTs' first justifications were at the level of friend/skeptic and 10 of 18 PTs' second justification were at or above the level of friend/skeptic. We use this analysis to now address our second research question, "How do PTs' views of authority support or limit their justification?" and look back to PT 1 and PT 2 as illustrative examples, comparing their justifications with our findings about their views of authority.

PT 1

During the interview, PT 1 explained that she primarily looked to external sources of authority for validation. If we look at PT 1's justifications, we see that she did not develop a robust understanding of justification with her second justification remaining at the level of friend (see Table 1). However, several times in her interview she explained that "it makes sense to me, I don't know why." This is a productive place to start – recognizing the need for mathematics to make sense – to develop a robust understanding of justification. However, the need to "do what the instructor wants" appeared to limit her reasoning about each justification task. Instead of reasoning about the examples she had tried out, she expressed that she felt lost because she did not have an outline to follow, saying "Like there's no outline so I'm like, I don't know what you need." PT 1's description of this tension in determining whether or not she had a valid justification indicates that her need for external validation limited her efforts to produce a valid justification by way of making sense.

PT 2

In her interview, PT 2 described a process of discovery as she reflected on her experience in learning to justify. Her growing awareness that she can make sense of mathematics and does not need to rely on rules that she was taught, supported her exploration and reasoning. Several times in her interview she expressed that "When I figured it out, I was so glad!" and "It was like, just a cool connection to make!". PT 2 developed an understanding of justification that aligns with the course goals, i.e., she describes justification as a process of making sense, and we see the result of this in the improvements that were observed across justifications 1 and 2. Her work for her initial justification 1 and revision were coded at the same level of Friend/Skeptic and her justification 2 was coded as Skeptic (see Table 1). PT 2's view of herself as someone who could make sense of mathematics supported her experience in learning to justify.

Conclusion

In this study we observed PTs developing sense-making and mathematical reasoning skills through justification. We saw how students whose ideas about learning mathematics were focused on remembering what they had learned or trying to "do what the instructor wanted" limited their exploration of these tasks and contributed to their sense of frustration. PT 1's story illustrates this experience. In contrast, we saw students excited about their growing awareness that they can reason about mathematics for themselves, that they could contribute ideas in the instructional space through our shared Google slides, and could learn from their classmates' work. PT 2's experience illustrates this freedom in exploring mathematical ideas. Our PTs' descriptions of their experiences in learning to justify provides insight into views of authority PTs hold and provide evidence that what we see in their justifications do not tell the entire story. On the surface, reviewing their justifications does not explain the reasons for their incomplete justifications. Understanding how their views of authority set up barriers to reasoning about mathematics informs our work as mathematics teacher educators (MTEs). For example, noticing when PTs may view justification as writing "what the teacher wants" helps us identify this barrier and then address it through dialogue about a) the value of sharing one's initial understanding of a task and b) how to build on this understanding to reason about mathematics and "justify *your thinking"*. Furthermore, identifying moments when PTs use their authority to reason about mathematics helps us to support and leverage these moments and to alleviate uncertainty PTs may experience about sharing their reasoning. It is when we identify these barriers and address them, and when we identify productive views of authority and encourage these views, that MTEs will be better able to support PTs in justifying their reasoning and make sense of mathematics. Future studies can build upon this understanding and examine the impact of different teaching practices and tasks that are designed to support PTs in developing their internal source of authority.

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ENHANCING K-12 PRE-SERVICE TEACHERS' EMBODIED UNDERSTANDING OF THE GEOMETRY KNOWLEDGE THROUGH ONLINE COLLABORATIVE DESIGN

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In this study, we devised research design that provides pre-service teachers to effectively experience embodied geometric thinking with the goal that it will impact teachers' instruction to students in their classrooms. Using a motion-capture video game and design tool, we offered opportunities for pre-service teachers to experience of performing mathematically related movements as well as creating their own directed actions for given conjectures. We hypothesize that these gameplay and co-design activity will reinforce not only teachers' understanding of the embodied nature of geometric thinking, but also their abilities to transfer their understanding to classrooms and the activities and assessments they design for their students. The results showed that after experiencing the interventions including embodied gameplay and co-design activity, teachers' awareness of students' 'sage of gestures was changed and they had better ability to understand and interpret students' gestures as a means of teachers' formative assessment practices.

Keywords: Geometric thinking, embodied learning, gestures, pre-service teacher education

Geometry knowledge is a core component of mathematics curriculum in secondary education. Unfortunately, the formalisms and current pedagogical approaches for instruction in math classrooms are dominated by abstract and a-modal curricula that are often barriers for many students as well as their teachers. It is crucial that geometry instruction connects with students in ways that are intuitive and easier to understand and leverage the inherent, embodied ability of individuals to reason about space and shape.

In order to address such a challenge, our research team created a motion-capture video game, *The Hidden Village* (THV), to both direct and elicit learners' mathematically related movements and enhance geometric understanding. In effect, THV facilitates players' actions (i.e., *directed actions,* ref. Nathan, 2014) that are emblematic of geometric representations and transformations while also providing prompts that elicit gesture production during the process of geometric conjecture reasoning. The current study addresses a practical problem that many teachers often have naïve views of the role of the body in geometric thinking (Walkington, 2019) and fail to understand the cognitive connections that embodiment reinforces in geometric thinking. There is a recognized need for providing empirical guidance for when and how teachers can implement body-based learning activities in curriculum, and evidence-based principles for understanding the usage of gestures in embodied math activities.

Researchers had devised an in-person research program that brings pre-service teachers together to effectively teach embodied geometric thinking with the goal that it will impact teachers' instruction to students in their classrooms. Unfortunately, COVID-19 impacted *in situ* research such that it must be done remotely. The interactions between teachers, the actions they perform, the rationales they provide, the gestures they express, and the designs they co-create must occur virtually. Thus, providing teachers with embodied perspectives on geometry instruction in consideration of how to integrate these practices into their formative instruction and assessment. In the current study, teachers play THV to understand how the embodied curriculum connects to geometric thinking, after which they engage in reflective group discussion and co-design new conjectures with directed actions to help learners conceptualize geometric transformations.

Theoretical Background

Embodied mathematics specifically integrates multi-sensory experiences into conceptual understandings. Embodiment grounds (Barsalou, 2008) multi-sensory perceptions of mathematical structures and patterns that may not be accessible from symbolic representations (Gerofsky, 2007; Sinclair, 2005). With math knowledge grounded in body-based and spatial metaphors (Lakoff & Núñez, 2000; Roth, 2011) and action-oriented language (Nathan et al., 2014), learners often express their conceptual understandings in the form of gestures (Alibali & Nathan, 2012; Edwards, 2009; Ng & Sinclair, 2015a, b). In prior work, students' mathematical intuitions and judgments of conjecture veracity (i.e., *always true* or *false*) were reliably predicted by their dynamic gestures (Nathan et al., 2018).

Creating body-based systems for geometric thinking that employ directed actions offers opportunities for learners to physically mimic the spatial dimensions, relationships, and transformations of geometric conjectures (Nathan & Walkington, 2017). In turn, teachers can translate these experiences into effective embodied instruction (Alibali & Nathan, 2007; Roth, 2001) to connect concepts for students. For the current study, the intersubjectivity (Matusov, 2001) of teachers' collective embodied reasoning is distributed (Walkington, Chelule, Woods, & Nathan, 2019) during our collaborative design intervention.

We hypothesize that this co-design activity will reinforce not only teachers' understanding of the embodied nature of geometric thinking, but also their abilities to transfer their understanding to classrooms and the activities and assessments they design for their students. Thus, our main research questions are: (RQ1) How does embodied video gameplay and co-design activity eliciting teachers' gesture production enhance teachers' awareness of students' use of gestures? (RQ2) How do these interventions improve teachers' ability to interpret students' gestures and develop formative instruction and assessment practices?

Methods

Participants

Participants were K-12 pre-service teachers (N=16) enrolled in math courses at a midwestern research university. Participants received a $150 e-gift card for their participation. COVID-19 health precautions instituted by the chancellor of the university forced this *in situ* research on embodied group collaboration to take place entirely online, using Zoom, a securely private platform approved by the university's IRB. Fortunately, Zoom enables participants to interact and collaborate with each other in a virtual group setting while recording audio and video of each participant.

Materials

We observed teachers' cognitive processes during THV gameplay. This included their performances of directed actions to embody geometric conjectures as well as their reasoning and explanations. Given the constraints of conducting remote research in the time of COVID-19, both direct and indirect scaffolds for each step of the intervention were developed to ensure adherence to research protocols and preserve fidelity of data collection.

The Hidden Village (THV). THV delivers interactive math geometry curriculum in the form of online motion-capture video game in which each player mimics movements of in-game characters and then reads a geometry conjecture to determine its veracity. Levels of the game are each comprised of 6 parts (*see* Figure 1): (A) Meeting members of the hidden village, (B) the village people implore players to mimic movements (e.g., mathematically relevant directed actions), (C) players receive a math conjecture, (D) players indicate if the conjecture is *always true* or *false* and provide explanation, (EIultiple choice, and (F) players receive rewards and progress game achievement.

Prior to gameplay, participants were provided an introductory tutorial containing instructions for setting up for research participation, a practice trial to familiarize participants with THV gameplay. For gameplay, students were paired into dyads with one person performing the directed actions (i.e., the actor) from the game while the other player observes (i.e., the observer). Midway through the game, players switch roles.

After gameplay, participant dyads rejoined their group to discuss any connections between the directed actions and the conjectures they were proofing. Discussion was guided by three prompts: (1) how in-game directed actions affected teachers' understanding of geometry, (2) how directed actions can be applied in the classroom to support geometry learning, and (3) how teachers can interpret students' spontaneous gestures as a formative assessment of students' understanding to improve their instruction.

Figure 1: The overall structure of THV gameplay

The Hidden Village Conjecture Editor. THV conjecture editor is a tool for participants to create new content for THV. This includes the creation of directed actions for each conjecture in the co-design activity. Since the co-design activity was conducted virtually, a researcher served as a proxy to manipulate the conjecture editor at the participants' behest. The teachers were given a tutorial outlining the features of THV conjecture editor and explaining the mechanics for manipulating the in-game avatar to create directed actions for a given conjecture (see Figure 2). The entire co-design activity was video recorded including participants' gestures and discussions.

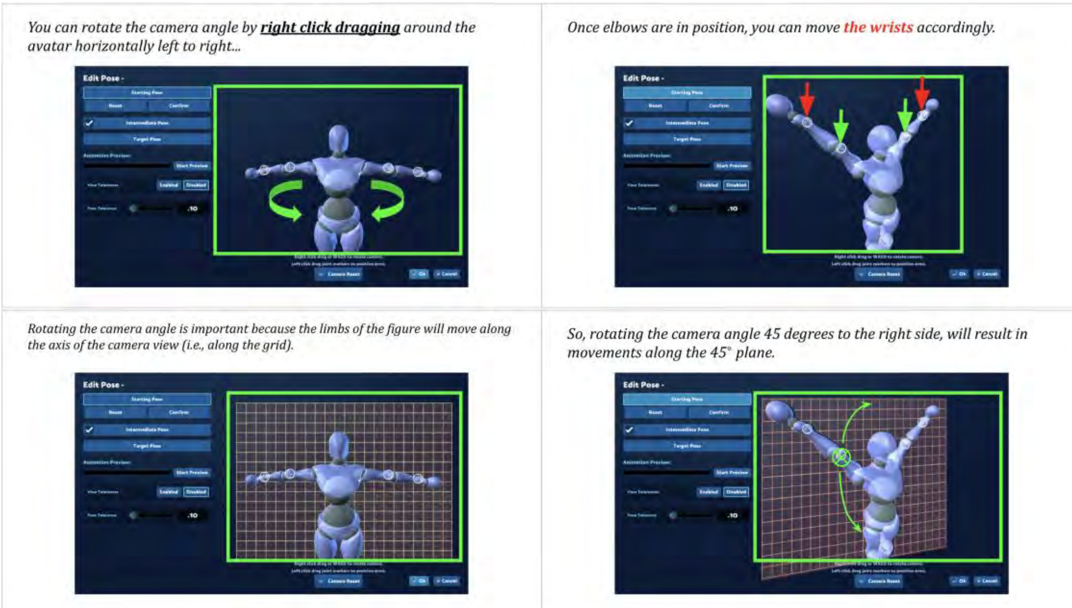


Figure 2: The slides of the design tutorial to inform how to use THV conjecture editor. (e.g., The avatar is posable using computer mouse. The figure can be rotated using a right-click drag or reset back to its origin.)

Online videos

In the beginning and the end of the intervention, teachers watched short 1-minute-long videos in which a student reasoned why a certain geometric conjecture statement is either always true or false. The scenes in the videos depicted high school students reasoning geometric conjectures (re-enacted for confidentiality reasons). In a semi-structured interview, teachers were prompted to comment on the videos and asked to explain (1) how they interpreted or assessed the student's understanding of the mathematical concept, (2) what evidence they observed, and (3) what practices they could employ in their classroom.

Online surveys

One week prior to the intervention, participants received online links to provide consent and complete surveys. These survey measures included: (1) the *Diagnostic Geometry Assessment* (DGA; items clustered in three areas: properties of shape (14 items, $\alpha \geq 0.86$), transformations (10 items, $\alpha \geq 0.795$), and measurement (11 items, $\alpha \geq 0.81$)); (2) a truncated version of the *Spatial Reasoning Instrument* (SRI, 15 items clustered around three constructs of mental rotation, spatial visualization, and spatial orientation, $\alpha \geq 0.88$); and (3) a survey of *Teachers' Attitudes about Gestures for Learning and Instruction* (TAGLI; 40 items, Cohen's $\alpha > .70$).

Procedure

This study is a mixed-methods, repeated-measures, within-subject design. The day of the intervention, participants took part in a number of activities over the course of 3.5 hours. First, participant groups were banded together by grade levels (i.e., grades K-5 and 6-12), after which they began a series of activities, including: (1) each teacher previewing and commenting on a 1-minute video of a student considering whether a geometric conjecture is either always true or false (30 minutes), (2) paired gameplay of four conjectures in THV in which players are encouraged to perform the directed actions by mimicking an in-game avatar's movements (30 minutes), (3) a whole-group discussion and co-design activity in which four pre-service teachers co-create directed actions for given conjectures using THV conjecture editor (90 minutes), and (4) watching and commenting (i.e., interpreting the relationships (if any) between students' gestures and geometric thinking) on two new 1-minute videos of a student considering whether a geometric conjecture is either always true or false as well as retaking the TAGLI survey (30 minutes).

Data analysis

The activities and measures provided researchers with data to assess changes in teachers' awareness and interpretation of students' understandings. The design of the study included pre- and post-intervention measures, teachers' gameplay and co-design activity. As an ongoing study, all videos of gameplay and co-design will be transcribed, segmented, and coded for teachers' speech and gesture usage articulating their intuition, insight, and proof production.

Results

Our major predictions were that gameplay and co-design activity that promote teachers' use of gesture will affect (1) teachers' awareness of students' 'sage of gestures and (2) teachers' ability to understand and interpret students' gestures as a means of teachers' formative assessment practices. In order to find out how the interventions we designed affect teacher's ability to interpret students' gestures, we compared the pre-intervention interviews and post-intervention interviews. The interviews were conducted individually.

Below are the examples of teachers' gameplay (see Figure 3) and co-design activity (see Figure 4) that teachers experienced. After the gameplay, teachers had a debrief of their experiences, discussing how in-game directed actions affected teachers' understanding of geometry. Half of the teachers said that they did make the connection between the conjectures they played and the movements they made during gameplay whereas the rest of half said they honestly did not notice the linkage. In common, the teachers who realized the connection during gameplay mentioned that they were able to get the ideas about the veracity of the geometric conjectures by performing the directed actions and they used that information in their reasoning processes.

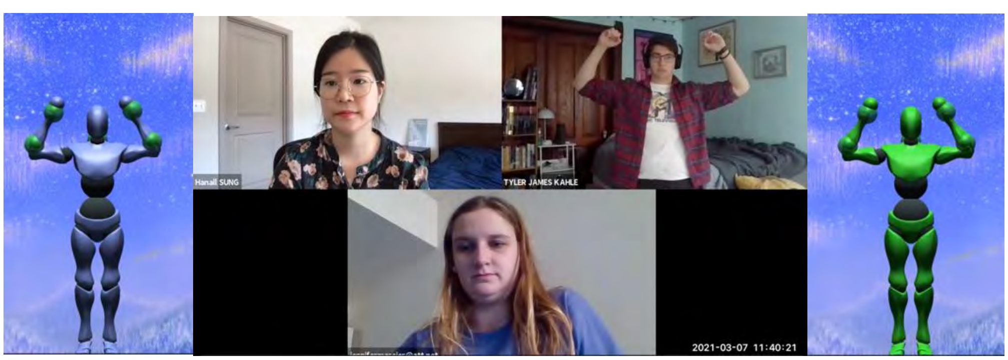


Figure 3: A screenshot of performing a directed action for a conjecture during THV game play

For the co-design activity (see Figure 4), participants were prompted to discuss geometric conjectures and consider how creating body-based directed actions enactive of geometric transformation could foster student learning.

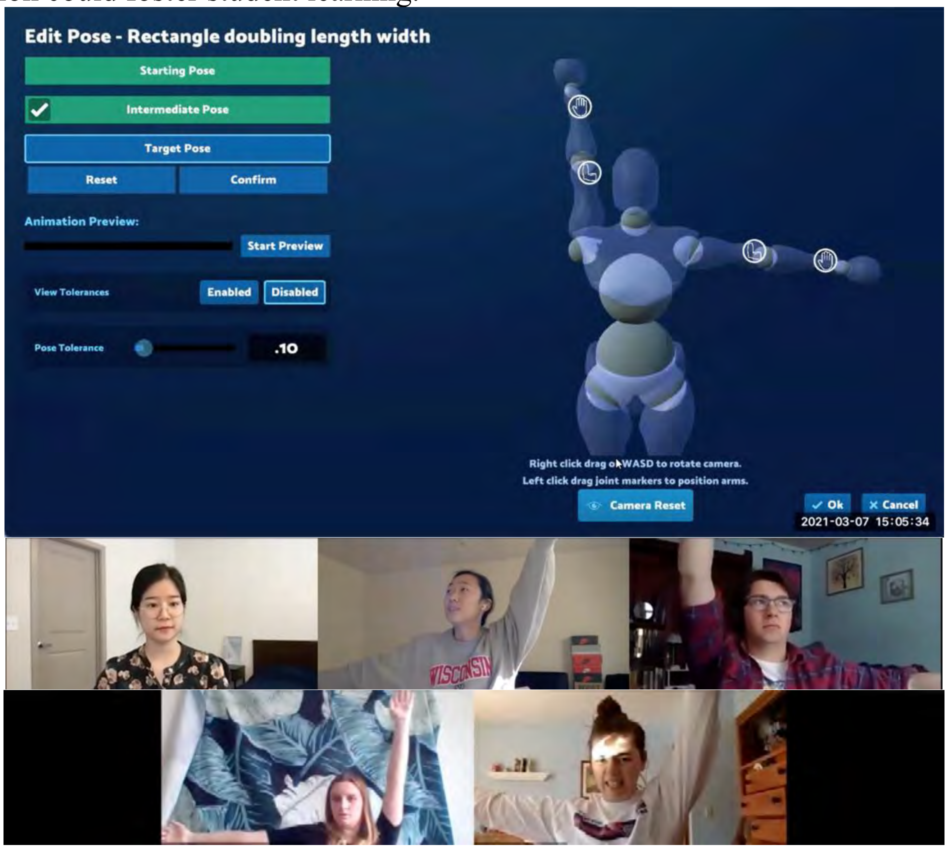
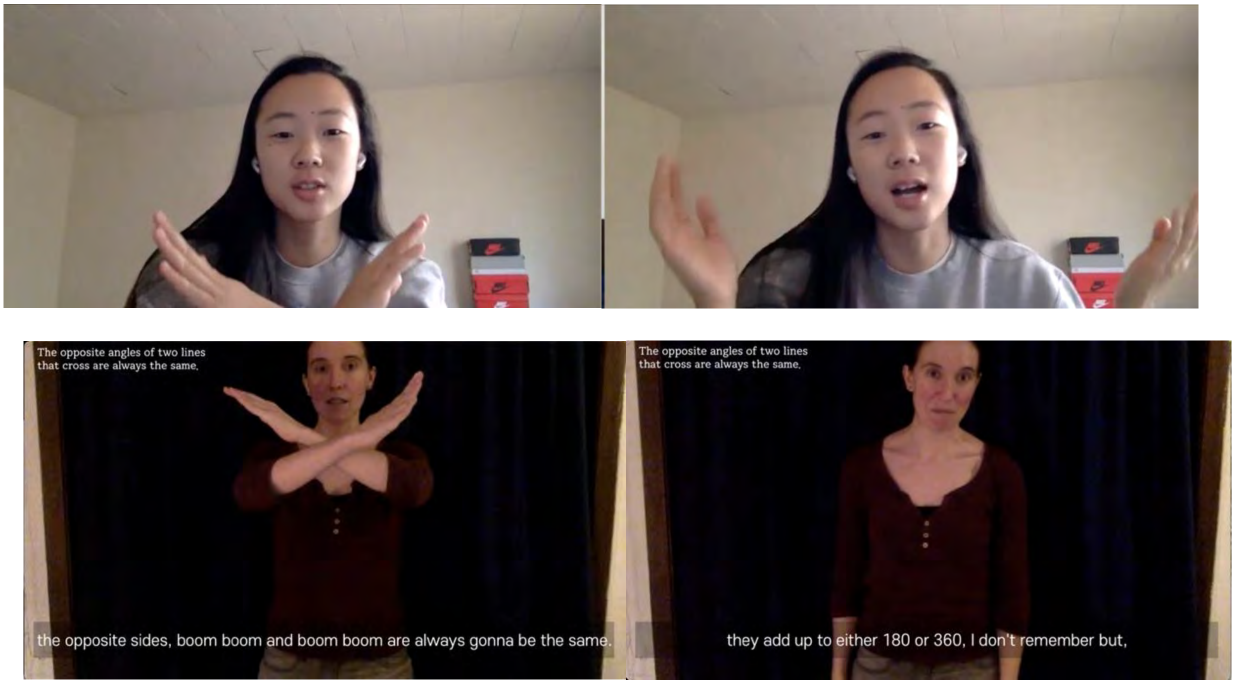


Figure 4: A screenshot of teachers' discussion to create new directed actions for conjectures during co-design activity

In the pre-intervention interviews with teachers, we found that teachers provided both gestures and verbal utterances of the student in the video as their evidence to assess the student's mathematical understanding of a certain conjecture (i.e., *The opposite angles of two lines across are always the same*). However, teachers' interpretation was limited—they were only able to share their superficial impressions of whether the student has the correct idea of the geometric conjecture, speculating from the student's tone, attitude, and vocabularies.

Presented here is a representative example of the pre-interviews with teachers (see Figure 5). Included is a photo transcript of Teacher 1's gestures and speech while explaining their interpretation of the student's mathematical understanding.

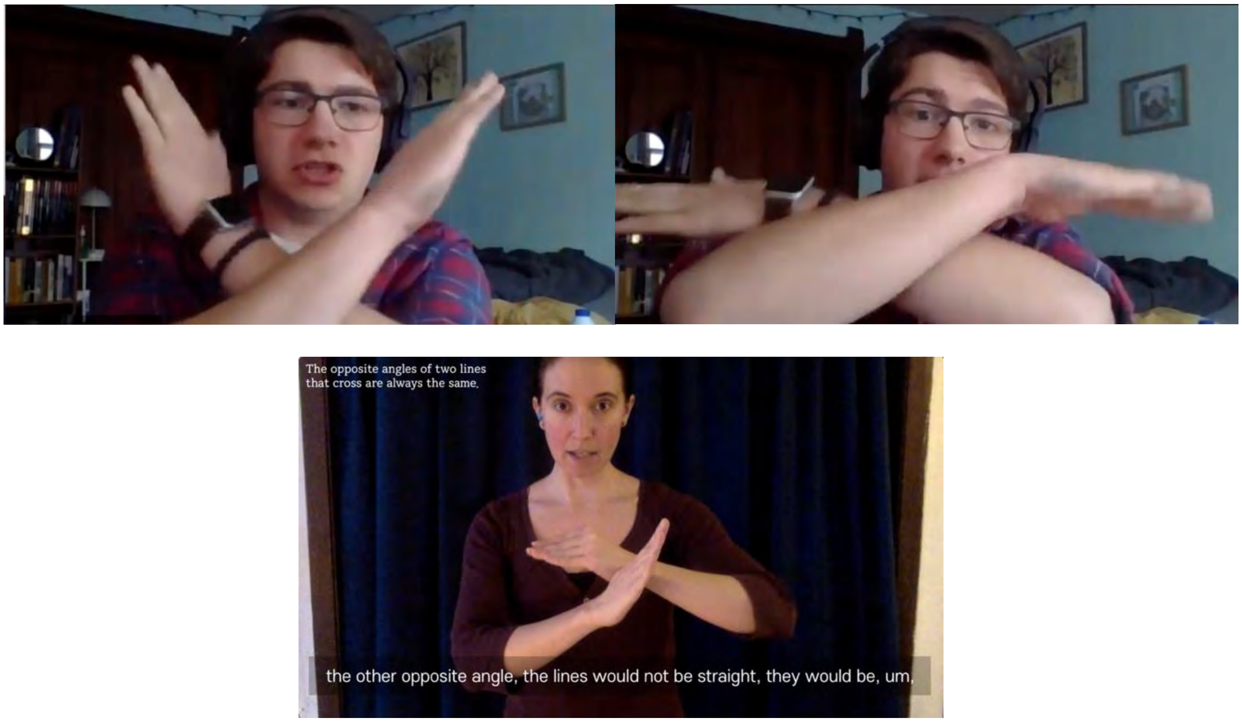


Teacher 1: She understood the opposite rule [mimicking the student’s X pose], but when she said “it either adds up to 180 or 360” and then was like I don't know really [making a pose to portrait ‘I don’t know’] at the end that just tells me that she doesn't fully understand the role, because if you understood the rule that would be like the two side by side angles would equal 180, so she doesn't have like the full understanding, but she has like the very basic core understanding of what the rule should be.

Figure 5: An example of pre-interviews
(top row: Teacher 1, bottom row: the pre interview video clip)

By mimicking student’s particular gesture representing vertical angles (Figure 5, top row), Teacher 1 showed that they perceived the student’s basic understanding of the geometric conjecture, but their focus quickly moved to the utterance and interpreted the student’s level of understanding based on how the student verbally described the utterances (“It either adds up to 180 or 360”).

On the contrary, we found that teachers in the post-interviews were more focused on the connection between the student’s gesture and their reasoning processes. For example, Teacher 2 interpreted that the level of the student’s mathematical understanding of the conjecture is poor and provided several rationales. First, by integrating the information from student’s gesture and speech (Figure 6, bottom row), Teacher 2 noticed that the student in the video was trying to provide reasoning with proof by contradiction (Transcript in Figure 6, “The logic she was trying to use was almost like a contradiction, like a proof by contradiction I felt like”). Next, Teacher 2 pointed out that the ineffectiveness of the student’s gesture in their reasoning process by saying “the gesture, she kept using, which is this [mimicking the student’s poses], but like she didn't’really do anything with it” (Figure 6, top row). That is, Teacher 2 paid attention to not only the meaning of the student’s gestures but also the function of the gestures in their geometric understanding.



Teacher 2: I thought she did a kind of poor job on that one. You very much could see the problems. The logic she was trying to use was almost like a contradiction, like a proof by contradiction I felt like. She was trying to say “okay, well, if the lines aren't straight than the angles won’t work”. And the gesture, she kept using which is this [mimicking the student’s sequence of poses], but like she didn't really do anything with it.

Figure 6: An example of post interviews (top row: Teacher 2, bottom row: the post interview video clip)

Discussions

The qualitative results of this study demonstrated that the interventions that include gameplay and co-design activity facilitated teachers’ use of gesture had impact on changes in teachers’ awareness of students' ’sage of gestures. Although teachers mimicked the student’s gestures that they observed in the videos in both pre- and post-intervention interviews, the level of information that teachers were able to extract from the gestures were different—they were more likely to focus on how and why students used gestures in their proofings in the post-interviews. Moreover, the results showed that the embodied interventions affected teachers’ ability to understand and interpret students’ gestures as a means of formative assessment practices. After experiencing the interventions, teachers were more likely to assess students’ geometric thinking better by focusing on the function of gestures in their reasoning.

This study has several limitations. First, as an ongoing study, we do not provide any quantitative results yet. Second, the recordings of gameplay and co-design have not been fully analyzed, so we were not able to articulate teachers’ learning processes during the interventions. Third, the sample size is pretty small because it was conducted as pilot study. We plan to deal with these limitations in our future work.

Despite these limitations, the preliminary results of the study suggest that the embodied intervention has reasonable potentials to change teachers’ belief and attitudes toward gestures.

Considering an average 14.2-year career of teachers, we expect the potential changes in these 16 teachers could influence over 15,000 students. In a broader consideration, the findings of this research are for the benefit of pre-service and in-service teachers who provide instructional practices at the forefront as well as professors of mathematics education and can be extended to professional development.

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TASK DEVELOPMENT TO ADDRESS ERROR PATTERNS IN PROSPECTIVE ELEMENTARY TEACHERS' POSING OF MULTI-STEP WORD PROBLEMS

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National and state standards in the US have emphasized the importance of solving and posing word problems in students' mathematics learning for decades. Therefore, it is essential for prospective teachers (PTs) to have the mathematical knowledge necessary to teach these skills to their future students. Unfortunately, little research has investigated how PTs develop problem-posing skills. By employing thematic qualitative text analysis, the researchers identified nine distinct patterns in errors identified in K-8 PTs' posing of two-step addition and subtraction word problems, in the context of a collegiate teacher education course. These results were used to inform the initial design of an interventional task to bring awareness of common errors to PTs.

Keywords: Preservice Teacher Education, Number Concepts and Operations, Elementary School Education, Instructional Activities and Practices

Introduction

The Standards for Preparing Teachers of Mathematics put forth by the Association of Mathematics Teacher Educators (AMTE; 2017) recognize that well-prepared beginning teachers of mathematics "regard doing mathematics as a sense-making activity that promotes perseverance, problem posing, and problem solving. In short, they exemplify the mathematical thinking that will be expected of their students," (p. 9). Research focuses heavily on teaching prospective elementary teachers (PTs) to persevere and problem solve (Alibali, et al., 2014; Bruun, 2013; Crespo, 2003; Green & Emerson, 2010; Jitendra et al., 2013; Polya, 1945) but, until recent years, has put little emphasis on the problem-posing skills PTs are expected to develop.

The Common Core State Standards (CCSS) Initiative in the United States (NGA & CCSSO, 2010, Table 1; NGA & CCSSO, 2010, Table 2) has presented taxonomies of common addition and subtraction and common multiplication and division situations that can be used to introduce PTs to the intricacies of the operations. These taxonomies can function as a steppingstone for helping PTs develop single-step problem-posing skills. However, the CCSS for Mathematics suggest that children "solve one- and two-step word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions" (NGA & CCSSO, 2010, p. 19) already starting in the second grade. Thus, we argue that once an introductory understanding of problem posing has been developed in teacher education coursework, attention should be drawn to developing PTs' skills for posing multi-step word problems.

When it comes to solving multi-step word problems, Heffernan and Koedinger (1997) found that solvers may experience what they called a "composition effect," meaning that "a two operator problem is harder than both of the parts that make it up put together" (p. 310). Alibali et al. (2014) extended this idea to posing, noticing that middle school students who were able to pose single-step problems also showed increased difficulty in posing two-step problems. As such, the researchers investigated two research questions:

1. What patterns emerge in the errors that arise when PTs write two-step addition and subtraction word problems?
2. What instructional interventions may help PTs to notice and potentially avoid making common errors when posing multi-step problems?

The researchers evaluated a corpus of 282 two-step addition and subtraction word problems posed by PTs enrolled in a problem-solving course for undergraduate education majors to identify common error patterns. With the results of this analysis, the researchers then created an error-analysis task for K-8 PTs to help draw attention to common two-step problem-posing errors with the hope that it might help them prevent making such errors in their own problem-posing experiences. This paper reports on the results of the thematic qualitative text analysis completed to answer the first research question and shows how these findings were used to develop an interventional task. The task was implemented in this course the semester following the word problem analysis. We will share our preliminary findings from this first implementation and discuss how they informed modifications. A second version of the task is currently being implemented and data is being collected to analyze the effectiveness of this task.

Literature Review

While problem posing research has been growing in popularity over the past several decades (Cai, Hwang, Jiang, & Silber, 2015; Einstein & Infeld, 1938; English, 1998; Kilpatrick, 1987; Silver, 1994; Silver & Cai, 1996; Singer et al., 2013), only over the past 15 years have researchers turned their attention to the preparation of PTs as problem posers (Cai et al., 2020; Crespo & Harper, 2020; Crespo & Sinclair, 2008; Ellerton 2013; Lavy & Shriki, 2007; Leikin & Elgrably, 2020). U.S. national and state standards have been encouraging teachers to implement problem-posing activities in their K-12 classrooms for over three decades (NCTM, 1989, 1991, 2000; NGA & CCSSO, 2010; TEA, 2012), but in order for those problem-posing tasks to be implemented well, PTs must be well-prepared to "manage the complexities of such contexts" (Singer et al., 2013). The Standards for Preparing Teachers of Mathematics put forth by AMTE (2017) indicate that effective mathematics education programs "develop positive dispositions toward mathematics, including persistence and a desire to engage in posing and solving problems," (p. 70). Our attention as mathematics educators then turns toward how we can efficiently prepare PTs to pose a variety of word problems and how the skills they learn can be applied in their future classrooms. For the sake of this study, the researchers will be referring to free problem posing, which occurs before problem solving, where problems are generated using a "contrived or naturalistic situation" (Stoyanova & Ellerton, 1996, p. 519).

Problem posing has shown to illuminate areas of conceptual misunderstanding in students (Alibali, 2014; Sharp & Welder, 2014), so it is important that introductory problem-posing activities involve mathematical content with which PTs are familiar. The CCSS Initiative has broken down addition and subtraction scenarios into 15 clearly distinguished categorical structures (NGA & CCSSO, 2010, Table 1). It is important that children are exposed to all 15 types of addition/subtraction problems or they may develop limited conceptions of the meanings of the operations (Van de Walle et al., 2019) or limited solution strategies (Carpenter et al., 2015). Therefore, PTs must not only be aware of the 15 different structure types of additive word problems, they must also be skilled in solving and posing every type. Once familiar with the various one-step scenarios, teacher educators can discuss how to connect multiple scenarios together to form multi-step problems. At this point, PTs can be engaged in developing their skills in posing two-step problems by connecting a variety of problem types.

In mathematics, the literature has recommended the use of error analysis as a means for gaining a deeper understanding of student knowledge (Fleishchner & Manheimer, 1997; Luneta & Makonye, 2010; Raghubar et al., 2009; Seng, 2010). By analyzing the types of errors made by PTs when posing two-step problems, we can design intentional instruction to support PTs in recognizing common errors and understanding why such errors occur. We utilized the skill of error analysis to form the basis of our task design, which asks PTs to analyze a set of flawed, or negative, problems. Research shows that the use of negative examples, in addition to positive examples, can be helpful in teaching good writing habits (Grow, 1987). Instructional psychologists also recommend using negative examples to "prevent certain classification behavior errors" (Ali, 1981). As such, the researchers of this study used the results of our analysis of PTs error to create an interventional task containing a set of negative examples for PTs to analyze, prior to posing their own multi-step problems.

Methods

A team of researchers at a tier one research institution in the southern United States worked over several years to create and incorporate a variety of instructional activities that could effectively support PTs in developing problem posing skills. This work was in the context of an undergraduate mathematics problem-solving course with the purpose of helping PTs understand how to teach mathematics through problem solving (Alwarsh, 2018; Bostic et al., 2016; Chapman, 2017; Fi & Denger, 2012), which puts word problems at the forefront of the lesson and introduces new content through the problems themselves (Alwarsh, 2018). Therefore, problem-posing activities were integrated into this course to support PTs in developing the skills they will need for posing problems for their students – specifically single- and multi-step. The writing and analysis of these word problems creates a productive dialogue between the PTs and the course instructors that allows PTs to develop a deeper understanding of the meanings for the mathematical operations and prepares them for posing problems for their future students to solve.

In this course, PTs are originally introduced to the previously mentioned taxonomy of common addition and subtraction structures as identified by the CCSS (NGA & CCSSO, 2010) as a basis for discussing structural differences between addition and subtraction word problems. The PTs then work on posing one-step word problems to match each of 14 possible problem structures (the fifteenth structure, part-part-whole – both parts unknown, involves multiple unknowns and, for the sake of one-step solvability, is left out of posing instruction). Class activities then turn to focus on categorizing and solving two-step addition and subtraction word problems. PTs are then given a series of assignments in which they practice posing a variety of two-step word problems . Increasingly more complex pairs of structures are assigned to guide PTs' posing to push them beyond using the same or easiest structure types (e.g., PTs tended towards posing change – add to – result unknown and part-part-whole – whole unknown problems, two structures introduced in Kindergarten (NGA & CCSSO, 2010)). Table 1 shows the culminating task for this set of lessons, in which PTs are assigned to pose four two-step addition/subtraction problems to match the four provided pairs of structures.

The data included in this report was collected from a single instructor's course across two semesters (including 74 total PTs). All PTs were enrolled in teacher certification programs in the areas of EC-6 (Generalist) or grades 4-8 mathematics and science or English and history. The assignment in Table 1 resulted in 282 PT-posed, two-step word problems that were analyzed in this study.

Table 1: Two-Step Addition and Subtraction Word Problem Prompts

Prompt	Assigned Pairs of Structures
1	change – subtract from – change unknown; compare – fewer – bigger unknown
2	change – add to – start unknown; compare – more – bigger unknown
3	part-part-whole – addend unknown; compare – more – difference unknown
4	change – subtract from – start unknown; part-part-whole – addend unknown

The researchers used thematic qualitative text analysis to analyze the submitted word problems resulting in 124 correct problems that matched the assigned structures and 158 problems that showed at least one error. As each error was identified, a temporary category was created based on the number of steps required to solve each incorrect problem and whether the structures used matched the assigned prompt. Two researchers independently coded the corpus of 282 word problems and discussed any disagreements until 100% of the analyzed word problems had been assigned an agreed upon code.

Results

Phase 1: Error Analysis

As previously mentioned, 124 of the submitted word problems correctly posed a question that required a two-step calculation utilizing two scenarios that both matched the assigned structures. The remaining 158 PT-submitted problems were coded as having one or more errors. The analysis of these errors led to the identification of nine distinct categories of error patterns (see Table 2), dependent upon the number of steps required to solve the problem, the appropriateness of the question(s) asked, and the use of the assigned structures. Below we will introduce each category of error pattern by providing examples of PTs' work and highlighting where the error occurred.

Two-step – Incorrect structure(s). Fifty-six of the problems correctly posed a question that required two connected steps that required addition or subtraction and were only deemed to be flawed because they did not entirely match the assigned prompts. For example, one such problem submitted in response to the first prompt read:

> Lauren has five apples on the table and Henry has two apples on the table. Lauren gives some of her apples to Henry. Lauren now has three apples. If Lauren now has less apples than Henry, how many apples does Henry have now?

Although the first step correctly matches the first assigned structure (change – subtract from – change unknown), the second step is a change – add to – result unknown scenario (instead of compare – fewer – bigger unknown). The question requires the solver to take the two apples that Lauren lost and add them to Henry's original two apples. There is no comparative structure included in this problem.

Table 2: Frequency of Error Type Exhibited by PTs

Category	Sub-Category	Frequency
Two-step	Incorrect structure(s)	56
One-step	One question: Two correct structures	31
	One question: Incorrect structure(s)	32
	Two questions	5
More than two steps	Including two correct structures	8
	Incorrect structures	4
Zero-step		3
Algebra-style		4
Unanswerable		15
	Total	158

One-step problems. With 43% of the total errors, the most common error group resulted from PTs who could build up two addition/subtraction structural situations but could not properly combine the unknown values from each scenario into a question that required a two-step calculation. These problems can be separated into three subgroups: problems that used scenarios matching both of the assigned prompts , problems that used at least one incorrect scenario, and problems that asked two independent questions.

About 50% of the 63 one-step, one-question problems built contextual scenarios that correctly matched both of the requested problem structures. The remaining 50% of the one-step one-question problems were flawed in the sense that at least one scenario did not fully match the requested structures. However, due to a lack of connection between the two scenarios, in both of these groups, the question posed only required one calculation to be solved. This tended to result from a known from the first scenario being used in the second calculation. One example of such a problem was submitted to Prompt 2. The PT wrote:

> Sarah had some pieces of candy. Four more pieces were given to her, so she had ten pieces of candy total. Amanda had five more pieces of candy than the amount of candy Sarah was given. How many pieces of candy does Amanda have? (Problem 1)

In order to answer the question that was posed, the solver would use the four pieces of candy that Sarah was given and add the comparative difference of five to reach Amanda's total of nine. This only requires one calculation, i.e. step. The PT did correctly set up the structures from the prompt but did not connect the unknown from the first scenario to the second.

A second example of a one-step problem was submitted to the first prompt. It read, "Some balloons were blown up. Two of them were popped. Only three remained. Two of them were red and the rest were blue. How many were blue?" (Problem 2). Both scenarios match the prompts, but notice that with the three remaining balloons, "two of them were red and the rest were blue," only requires one calculation to solve (3-2=1). This PT developed a scenario where the two unknowns could have been connected by referring back to the original set of balloons, but instead posed a question that didn't require the solver to utilize the unknown information from the first step (the original number of balloons) in the second step. The use of a known from one scenario to set up the structure of a second scenario was not specified as an error in the table but was by far the most common issue in one-step problems. PTs could create scenarios matching the given prompt structures independently but experienced the "composition effect" (Heffernan & Koedinger, 1997) of not being able to merge two unknowns into a single question.

In the remaining five of these 68 problems, the poser wrote two separate questions, each only requiring a single step to solve. Four of these problems included the correct assigned structures, one did not. The PTs who wrote these problems knew that two steps were necessary to satisfy the given task but showed difficulty in connecting their unknowns into a single question. One such example is, "Tommy had seven envelopes, then he lost some of them. Now he has four envelopes. Tommy has five fewer envelopes than his friend Milton. How many envelopes did Tommy lose? How many envelopes does Milton have?" (Problem 3). The two, single-step questions here are clear, but a well-developed, two-step problem must require the solver to complete two connected calculations by only asking one question.

More than two steps. Twelve of the remaining problems successfully posed a valid multi-step addition/subtraction problem but asked a question that required three or more steps to answer. Eight of these problems correctly included both of the assigned structures, but the PTs were unable to formulate a question that only used the information found in those two steps. An example of this that was submitted to Prompt 1:

> Lucy has 3 fewer cookies than Julie. Lucy has two cookies. If Lucy and Julie both put their cookies in a jar but someone takes some cookies and leaves only 3 cookies in the jar, how many cookies were taken? (Problem 4)

The first step in solving this problem is a compare-fewer-bigger unknown (which matches the second assigned structure) where the solver calculates Julie's amount. The second step is a part-part-whole where Lucy and Julie merge their cookies together. Once the solver finds the joint number of cookies, some of those cookies are taken out. A third step then asks the solver to find how many cookies were removed, a change – subtract from – change unknown scenario (which matches the first assigned structure).

Zero-step problems. Three problems posed by PTs provided the answer to the posed question within the problem statement. These were labeled at zero-step problems since no computation was necessary to answer the posed question. In one example of a zero-step problem, the PT wrote, "I had some cookies. I got seven more cookies. Now I have 13 cookies. Courtney has some cookies. I now have seven less cookies than Courtney. How many cookies do I have?" (Problem 5). There are no steps required to solve this problem as the answer lies in the statement ("now I have 13 cookies"). This was the least frequent of the error patterns.

Algebra-style problems. A similarly small group of four problems offered scenarios for two unknowns that were not able to be solved by simply computing two sequential one-step arithmetic calculations. Instead, these problems provided information that connected the two unknowns in two different ways making the question posed answerable but requiring the solver to apply more complex algebraic thinking. For example, the following problem was submitted to the Prompt 2:

> Sally had some pencils. Then her mom gave her three more pencils to take to school. When she got to school Ruby had 3 more pencils than Sally. Altogether they had 13 pencils. How many pencils did Sally have when she got to school?

The given information could be represented with two numerical equations, but, in each equation, both quantities are unknown (i.e., $S + 3 = R$ and $S + R = 13$).

Unsolvable problems. The remaining 15 problems posed were deemed unsolvable as there was no clear way to actually answer the question that was asked. However, this error occurred due to multiple reasons three examples are given below. The first problem was unsolvable because no question was asked. This scenario was submitted for the second prompt, "Santa wrapped some presents. His wife helped him wrap 6 more, now there are 10 presents. Santa's elf helped too, he wrapped 2 more presents than Santa did initially" (Problem 6). The context was set up perfectly for the prompt, a question just needed to be asked.

The second example of an unsolvable problem was submitted for Prompt 4. The PT wrote, "Some oranges were on the plate, 3 were big and the rest were small. I ate 3 oranges. Then there were 2 oranges left on the plate. How many small oranges are left?" (Problem 7). This problem is missing contextual clarification. The poser chose to distinguish small oranges from big oranges, but then went back to discussing "oranges" in general. In order to solve the problem, the reader would need to know what size of oranges were eaten and what size of oranges remained at the end.

The final unsolvable problem example was submitted to Prompt 2. The PT wrote, "Nine bananas were on the table. Five were green and the rest were yellow. Then my mom came and took 2 of the yellow bananas. How many more bananas do I have than my mom?" (Problem 8). In order to confidently solve this problem, the solver needs more information about who each set of bananas belongs to. It is not clear whether all nine of the beginning bananas belong to the problem poser or the family in general.

Phase 2: Task Design

The results of the error analysis provided the researchers with valuable information regarding the challenges faced by PTs when posing two-step addition and subtraction problems. The researchers utilized this information to develop an interventional task for use with future PTs. The task was designed to bring awareness of common errors made when posing two-step problems to PTs prior to having them pose their own two-step problems. The task offered a set of flawed, or negative, problems that PTs would be asked to analyze. To create authenticity, we used example problems from our error analysis instead of constructing our own. The PTs were presented with a set of word problems and the following instructions:

> Imagine that you asked your students to write a 2-step addition/subtraction word problem and below are some of the word problems they wrote. For each word problem, explain to the student why their problem is incorrect or incomplete and briefly discuss how it could be fixed.

To create authenticity, we used the results from our Phase 1 analysis to identify key examples of PT-written word problems exhibiting the identified errors and used these on the task instead of constructing our own flawed problems. In developing this task, the researchers chose to focus their attention on PTs' ability to create well-written, two-step problems rather than on their ability to match a specified set of structures. For this reason, none of the 56 PT-submitted questions that correctly required two steps to solve (but were deemed flawed because the structures were not entirely matched) were included in this error analysis task. Additionally, in this initial version of the task, the researchers chose to focus only on problems that were written arithmetically. At that time, algebra-style problems were not being covered in the course, so those types of errors were excluded from the error analysis task as well. The task included eight questions previously coded as having a structural error (labeled as Problems 1-8 above).

Discussion

Implementation of Task

The error-analysis task created as a result of this analysis was implemented in the following semester. The task was strategically placed between two lessons that ensured PTs had already been exposed to analyzing two-step addition and subtraction word problems but had not yet attempted posing their own two-step word problems. Leading up to the error-analysis task, PTs had spent time learning about the 15 categories of addition and subtraction word problem types, practiced posing one-step addition and subtraction word problems, and categorized and written number sentences to solve two-step addition and subtraction word problems. The error-analysis task was assigned for PTs to complete individually outside of class time. Each problem on this task and PTs' responses to this task were discussed in the following class meeting. After this work, PTs began posing their own two-step addition and subtraction problems. As they practiced posing two-step word problems, first with addition/subtraction and later with multiplication/division and mixed operations, the errors described in this task continued to be identified and discussed.

Preliminary Task Results

Initial analysis of the task's first implementation showed promising results in terms of PTs being able to accurately identify most of the intended errors on the task and to identify common errors later when reflected in their own and others' posed problems. However, the researchers found that four of the problems were not demonstrating the errors strongly enough. During a preliminary qualitative analysis of the results, researchers aimed to identify the cause for PTs' difficulties in identifying the intended errors in the word problems on the error-analysis task. The researchers observed that Problem 2 was missing contextual information which distracted PTs from the intended error of it being a one-step problem. On Problem 4, a three-step problem, PTs focused on the tense of the verbs in the word problem and the order of which the numbers appeared; for example, some thought the result would be a negative amount of cookies. Similarly, on Problem 1, PTs seemed confused by the verb tense used throughout the problem and this took focus away from the intended issue that it was only a one-step problem. Finally, Problem 3, an example with two individual questions, also has the error of one of the knowns from the first step being used in the second step, making it a one-step problem. Although it was useful for PTs to point out all of these issues, these four problems were not adequately focusing PTs' attention on the type of error we intended the problem to highlight. Therefore, all four of the problems were traded for other PT-submitted problems that were coded under the same error category during the original data analysis. The instructions were also adjusted to reflect the fact that PTs' responses tended to offer fixed versions of the problems instead of identifying the error. The PTs were more explicitly asked to identify the error made, without fixing it, and to provide an explanation that would help a peer understand why the problem is not a two-step arithmetic addition and subtraction problem.

Future Research

The modified version of this task is being implemented in current sections of the course. Data is being collected and the researchers will analyze results from the implementation of the updated error-analysis task for future use and improvement. Additionally, researchers will perform a comparative analysis of PTs' posed word problems from the semesters in which the interventional task was implemented to previous semesters to investigate the possible implications the task may have had towards supporting PTs' problem-posing abilities through raising their awareness of common error patterns.

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FORMS OF NASCENT POLITICAL CONOCIMIENTO LEVERAGED IN A CONTENT COURSE FOR PRESERVICE ELEMENTARY TEACHERS

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This study examines the nascent forms of political conocimiento demonstrated by elementary preservice teachers before and after a series of activities designed to engage them in thinking critically and quantitatively about the impacts of different grading systems. In reflections about their learning, the preservice teachers most frequently raised considerations related to knowledge with students and communities, followed by political knowledge. Many of the preservice teachers anticipated using their new knowledge in the near future to understand the implications of their professors' grading systems and in the more distant future to design grading scales in their own classrooms. A few reflections showed awareness of the tensions involved in grading systems, potentially a sign of developing Nepantla, while others evidenced more binary thinking.

Keywords: Social Justice, Preservice Teacher Education, Affect, Emotion, Beliefs, and Attitudes

Introduction

Teachers, especially those working in urban environments, need political *conocimiento* for teaching—a nuanced, situated form of political knowledge-in-practice—to effectively advocate for their students and themselves in the multi-layered, complex political environments of schools (Gutiérrez, 2013). From renewed controversy around standardized testing (Barnum, 2021) to conflicts over school reopening plans (Ludlow, 2021) to debates about student needs and claims of learning loss (Gabriel, 2021), the effects of and societal response to the novel coronavirus in 2020 have made teachers' need for political *conocimiento* even more apparent and more urgent than before the pandemic. Teacher preparation programs must do everything possible to help preservice teachers (PSTs) develop political *conocimiento*. While many researchers and teacher educators whose work focuses on helping PSTs develop political and cultural understanding have distinct but interrelated goals including social justice, equity, culturally relevant/sustaining pedagogy, and political *conocimiento*, there is broad agreement that to effectively prepare PSTs, teacher preparation programs must integrate these goals as broadly as possible throughout multiple courses and program components (e.g. Garii & Appova, 2013; Gutiérrez, 2013; McDonald & Zeichner, 2009; Nieto, 2000; Wiedeman, 2002; Xenofontos et al., 2020).

Math content courses are a perhaps unexpected but potentially powerful category of courses into which political *conocimiento* could be integrated. Math content courses are often among the first courses that elementary PSTs take, and they rarely focus on social justice topics (Felton-Koestler, 2020), so using a math content course to foster political *conocimiento* can create new opportunities for PSTs to start to develop and integrate mathematical and political understanding early in their programs. PSTs can then build on that understanding in future courses such as math methods courses in which they must connect mathematical and social justice goals, with the additional complexity of creating lesson plans (Meyers, 2019). Exploring topics related to political *conocimiento* in content courses could also potentially increase PST investment in content courses which are sometimes perceived as "relearning" elementary mathematics (Zazkis, 2011) by linking them more closely with PSTs' other courses.

The current study explores PST learning in one such content course. It attempts to make two main contributions to the existing research. First, the course design—discussed further below—focuses specifically on integrating political *conocimiento* in a math content course rather than drawing on a more general understanding of social justice as previous work has done (e.g. Bateiha & Reeder, 2014; Felton-Koestler et al., 2016; Martinez & I, 2019). In particular, the unit that is the focus of this study explored the quantitative and political implications of using different possible grading systems in a class. Second, while the framework of political *conocimiento* is based on extensive field research with practicing teachers (Gutiérrez, 2012, 2013, 2017) there is no empirical work describing its early stages of development in elementary PSTs or in the context of a math content course. Therefore, this research report uses PST reflections before, during, and after a unit on grading systems in an elementary math content course to examine the following research questions:

- What forms of political *conocimiento* underlie PSTs' reflections about their own learning, questions that they have, and what teachers need to consider?
- What roles does mathematical reasoning play in PSTs' reflections?

Theoretical Framework

Both this study and the design of the course from which the data are drawn are rooted in the intersection of two theoretical frameworks: Gutstein's (2006; 2016) framework for Reading and Writing the World with Mathematics (RWWM) and Gutiérrez's (2012; 2013; 2017) framework of political *conocimiento* for teaching mathematics. Gutiérrez's framework was used to conceptualize the goals for the course, while Gutstein's framework was used to inform the design of mathematical activities intended to facilitate those goals. Both frameworks were used to inform this study's data analysis, with Gutiérrez's framework structuring the response to the first research question and Gutstein's framework structuring the response to the second.

Gutiérrez (2012; 2013) describes political *conocimiento* as the situated knowledge-in-action that successful teachers of Black, Latinx, and low-income students use to successfully advocate for their students in the multi-layered, political environments of schools situated within the larger neo-liberal education system. The word *conocimiento* indicates a form of knowledge that is developed through subjective personal experience and grounded in solidarity and praxis. The use of "political" in political *conocimiento* refers to the political implications of actions and systems at multiple levels of society, from the micro to the macro, and the ways in which power and identity operate across and between those levels. For example, the choices that individual teachers make about how to set up grading in their classrooms are shaped and constrained by the requirements and expectations of their fellow teachers, school and district administrators, families of students, and schools that students may attend in the future. In turn, teachers' grading choices may significantly affect students' academic and extracurricular opportunities and mental health, and these effects may have complex predictors based on characteristics including students' racial and socio-economic backgrounds, family structure, disability status, and physical and mental health.

Gutiérrez's (2017) framework of political *conocimiento* for teaching mathematics builds on Shulman's (1986) framework of teacher knowledge. Like Shulman, Gutiérrez acknowledges the importance of developing content knowledge and pedagogical knowledge. She emphasizes the importance of developing knowledge related to students and their communities through interdependent relationships rather than abstract study, and therefore describes it as "knowledge with students/communities." To these three categories she adds the form of political knowledge described above and emphasizes that these four forms of knowledge are all interconnected. She grounds the forms of knowledge in the context of what she calls "community en el mundo zurdo," evoking the necessity of communal action and solidarity with those who have been disenfranchised. Gutiérrez situates developing political *conocimiento* in "histories in society" emphasizing the situated, non-generalizable nature of *conocimiento* and the ways in which it influences and is influenced by the histories of a particular context. Finally, Gutiérrez (2012) highlights the role of *Nepantla*—a liminal "third space" that entails constant tensions and an awareness of multiple, potentially contradictory realities and possibilities for the future—as a necessary state for opening the possibility of the development of new *conocimiento*. In the case of grading described above, *Nepantla* could entail a teacher's awareness that adjusting particular category or assignment weights could help some students and hurt others, a tension that could lead to a desire to fundamentally change the underlying system. The teacher would then need to integrate their various forms of knowledge to work to enact change: they might explore different alternatives with students and communities, consider the pedagogical implications of their options, and use their understanding of the political pressures inside and outside their school to join with others in solidarity and figure out the most effective ways to make change happen.

As referenced above, I used Gutstein's (2006; 2016) framework to make sense of how to integrate these goals for preservice teachers into a math content course. Gutstein describes his pedagogy as teaching students to read and write the world with mathematics (RWWWM). When he teachers RWWWM he builds on a Freirean framework, working to use students' experiential community knowledge as a foundation on which they can develop classical knowledge and critical knowledge. In my context, the PSTs' community knowledge was based on their own experiences with grading systems as students and their common goal of becoming teachers who would use grading systems in their future classrooms. The classical knowledge that PSTs needed to develop was their math content knowledge related to ratios, percentages, and weighted averages, and the critical knowledge was the political *conocimiento*—the integration of content, pedagogical, and political knowledge with knowledge with students and communities to consider the constraints and implications of different grading systems. Gutstein further divides critical knowledge into reading the world with mathematics (using mathematics to understand the world) and writing the world with mathematics (leveraging mathematics to change the world). In the context of this study, PSTs could read the world with mathematics by using what they learned to better understand the grading systems that their instructors used and act accordingly, and they could anticipate writing the world by advocating for instructors to change their systems or by planning to make considered choices about grading when they themselves become teachers.

Methods

Positionality Statement and Context

An important piece of context to provide for any math education research, and especially for research situated at the intersection of math, politics, and education, is my positionality as a researcher and an educator (D'Ambrosio et al., 2013). I grew up white, middle-class, and female identified, identities that I share with the majority of in-service and preservice teachers in the United States. I began to develop a more critical political consciousness during my own teacher training in a program focused on preparing teachers to work in urban environments, and my political *conocimiento* developed further through over seven years working in multiple roles in two public schools. I spent the last four of those years working as a school's data coordinator, a role in which I often collaborated with teachers and administrators to analyze and address problems related to different data systems, including intervention data, grades, standardized tests, and school rating results. Since moving from an elementary setting to a university setting I have had the opportunity to explore more theoretical critical perspectives that have helped me connect my personal experiences to an understanding of the larger, neo-liberal education system.

I currently work at a public research university in a large city in the Midwest. The university's undergraduate teacher preparation program focuses on preparing students to teach in urban settings, and many of the faculty specialize in critical theories, including some of the faculty who frequently instruct the undergraduate math methods courses. PSTs are required to take multiple courses related to the politics of education, including an introductory course on urban education and a course on race, ethnicity, and education during their first year in the program. Before the current study, the elementary math content courses focused on broadening PSTs' understanding of mathematics, developing their problem solving skills, deepening their conceptual understanding of elementary mathematics, and fostering the mathematical knowledge for teaching; there was not a focus on political understanding. Until the coronavirus pandemic, all courses were taught in-person; the data for the current study were collected during a semester in which the content course was taught entirely online.

Course and Activity Description

The data for this study are drawn from a series of activities around grading systems that took place during two class periods of an introductory math content course for elementary PSTs. The course met remotely twice a week for 110 minutes. The original intention was to include multiple activities connected to political *conocimiento* throughout the semester. Unfortunately, the course moved much more slowly than anticipated, so there was only time to do two mini-units, and the second mini-unit (which explored the local school district's school rating policy) occurred in an abbreviated form in the final class period and was not followed by student reflections. The course centered collaboration and problem-solving, though both were more challenging in the virtual context. The course was also designed to help students develop a supportive community, giving PSTs a variety of ways to share their experiences and emotions. One key form of communication was brief reflections that PSTs completed after every class session, for a total of 26 reflections. These reflections had a consistent format: upload a picture of your work from class, explain what you learned or figured out in today's class, and ask the instructor a question. The intention was to normalize regular communication and the value of asking questions—questions could be about anything, including creating an extension question based on a problem from class.

The grading system activities were designed to help familiarize PSTs with different ways of weighting assignments and grades. The activities alternated between PSTs collaboratively reflecting on the potential implications of and problems with different forms of grading and solving quantitative vignettes based on problems that I had seen arise during my work in schools when there was a mismatch between the design of a grading system and how teachers were using it. The goal was to support PSTs in taking the perspectives of others with different experiences, and connecting the implications of those differences with the quantitative choices made within the grading systems.

Participants

The participants in this study were drawn from PSTs enrolled in two sections of the relevant content course. I was the instructor for one of the sections of the course, and I co-planned with the instructor of the other section. There were 70 students enrolled in the two sections, and 29 consented to have their coursework analyzed for the purposes of this study. I suspect that the low rate of consent was related to the stress of the pandemic and the sense of surveillance that PSTs expressed as part of remote instruction. The majority of PSTs were freshmen or transfer students. Three participants did not provide demographic information. Of those who did, 3 were male, and 23 female. One identified as Middle Eastern, 2 as Asian, 5 as white, and 17 as Latina/o.

Data Sources and Analyses

The data for this study are drawn from four reflections that PSTs completed as class assignments before, during, and after the unit on grading systems. Before the activity PSTs reflected on their personal experiences with grading ("Have you ever had an experience as a student in which you felt like the grading for a class was unfair? What was unfair about it?") and their initial beliefs about teacher obligations ("What do you think teachers need to consider when they plan the grading system for their class?"). After each of the class sessions focused on grading systems, PSTs completed reflections describing what they learned and what questions they currently had. At the end of the unit they answered the prompt about teacher obligations again and commented on whether they felt that their answer had changed.

The reflections were first analyzed with respect to the first research question, looking for examples in which PSTs' reflections implicitly valued or applied any of the components of political *conocimiento*: content knowledge, pedagogical knowledge, knowledge with students/communities, political knowledge, community en el mundo zurdo, histories in society or *Nepantla*. Some reflections valued or applied multiple components, while others did not clearly reference any, so some reflections were given multiple codes and others received none. For the second research question, PSTs' reflections were analyzed to find instances in which PSTs referenced explicitly mathematical activities, and then those examples of referencing mathematics were coded for the roles for which they used mathematics: classical content knowledge, reading the world with mathematics, or writing the world with mathematics. Classical content knowledge was used to code reflections in which PSTs focused on calculations specifically as a course assignment or intellectual puzzle. Reading the world with mathematics was used to code reflections in which they described or anticipated making sense of an instructors' grading system or finding what score they needed on an exam to get an A in a course. Writing the world with mathematics was used to code reflections in which PSTs discussed quantitative reasoning that was important for either teachers in general or that they anticipated using when they became teachers.

Findings

Nascent Political *Conocimiento*

A total of 66 codes related to political *conocimiento* were assigned. Every code was used at least once except for histories in society. Examples and descriptions of the characteristics of each code follow:

Knowledge with students/communities. With 25 examples, this was the most common form of political *conocimiento* that PSTs referenced and valued. It emerged across the different reflections both before and after the activities. Some PSTs showed increased specificity and a shift away from their individual perspective in their reflections from after the activity. For example, "I used to think teachers should be more flexible with grades in case students don't have enough time to do homework, or don't have help at home, but now I think there are many other factors such as an unstable living situation, health concerns, and many others."

Political knowledge. This was the second most frequent form of political *conocimiento* referenced, with 13 examples. Many of the relevant reflections indicated a concern for fairness and for the implication of different grading systems (responses to the prompts that specifically asked about fairness were not included under these criteria). Some reflections also indicated an awareness of the complexity of roles and pressures that teachers need to navigate with grading: "My question is are all teachers required to grade a certain way depending on their school or do they each get to decide. I know for high school, teachers usually get to decide, but I wonder if it's'the same for elementary school teachers," and "Do you see there being a big change to standard based grading in the near future that we might have to implement with our own students?"

Community en el mundo zurdo. PSTs did not explicitly describe themselves as oriented in solidarity with social change, but there were 9 examples in which the positioned themselves as part of a community and emphasized how important the group collaboration was to them, both in the mathematical problem solving and the critical discussions. For example, "I also enjoyed being able to share out our different ideas on grading," and, "When answering and doing the grading problems I remembered how we talked them through in class and it really helped me to solve them."

Content knowledge and pedagogical knowledge. These codes were the least frequently used, with 3 and 7 examples respectively. Identifying cases of valuing content knowledge had many of the same challenges described below for identifying classical mathematical knowledge—there were relatively few examples where the PSTs were clearly valuing the mathematics for its own sake rather than for its potential use in RWWWM. One exception was, "We decided that the best way to show the grades equally weighed was to see what the fraction would look like if the final project was worth the same amount of points as the classwork assignments." A concern for pedagogical knowledge in this context was generally reflected as concerns that hypothetical students understand a class's grading system and considerations about how to make that happen—"I think whatever grading system they choose, they need to make sure to explain it to their students well to make sure they know what to expect if they miss a class or do poorly on an assignment."

Nepantla. Some PST reflections (7) demonstrated awareness of *Nepantla* when they described the inherent tensions in designing grading scales, where often the same choice can improve one student's grades and hurt another's. For example

> When teacher plan the grading system for their class teachers need to consider what will benefit students and what seems fair. This, however, I realized it can be very difficult to achieve because no matter what, not all students will benefit. In the excel page, I was playing around with how much each category should be weighted, but even when 2 students' 'cores would increase, the third student's score would stay the same or decrease so that student was not benefitting from the changes. (Kaila)

Importantly, for these PSTs the tension was something to be grappled with but not necessarily resolve. In contrast, there were 5 examples of reflections in which PSTs moved away from *Nepantla* and tried to identify clear-cut binaries in which one form of grading was always superior to another as in the claim "Teachers should also be grading things equally rather than proportionally because proportionally, in my opinion, only affects the student more."

Reading and Writing the World with Mathematics

In the reflections, there were a total of 49 codes for the different forms of using mathematics. The most common was reading the world with mathematics—it was used to code 24 responses. Five of those responses were based on student reflections about their personal experiences that referenced quantitative reasoning such as

> [W]e only have writing assignment every two weeks. Although that sounded fun at the beginning not having quizzes or tests, these writing assignments are worth 10 percent and can affect our grade if we do not get a good grade. (Julie)

The majority of the rest were descriptions of how useful it was to be able to do calculations about grades such as:

> This was a moment that made me realize how useful it is to know math. I always wished I knew how to calculate my grade or knew how and why a single score affected my grade so much. It was an Aha! moment both as a student and as a future teacher. (Imani)

Seventeen responses focused on mathematics specifically in the context of class assignments or in the abstract, and 8 referenced writing the world from a teacher perspective when designing grading systems.

Implications

This work provides some initial examples of what forms nascent political *conocimiento* can take for elementary PSTs in the context of a content course, and how PSTs may integrate mathematical content knowledge as part of their *conocimiento*. It found that for this particular set of activities and prompts, PSTs tended to value knowledge with students and communities and reading the world with mathematics. It indicated some potential for even relatively sparse PST reflections to show preliminary evidence for the acceptance or rejection of *Nepantla*. One somewhat surprising aspect of the findings was the relatively small number of references to pedagogical knowledge—surprising because questions related to “How would you teach this?” were one of the most common types of reflection questions during the rest of the course. It is possible that this finding is related to Myers’s (2019) description of the struggles that PSTs experienced when planning units that integrated mathematical and social justice standards. The cognitive complexity of balancing the mathematical and critical thinking may make it more challenging for PSTs to also consider pedagogy, at least this early in their teacher preparation programs. Another aspect of the findings is the lack of reflections that connected to history in society. This may be due to the design of the activities, which were specific in terms of individuals’ situations, but were not situated in a specific historical or geographical contexts. This is a weakness that was addressed in the abbreviated activity around school rating systems, and is something that I plan to address more thoroughly in future work.

There are a number of other ways in which I hope to be able to build on this work in the future. One is to more explicitly address race and various forms of racism in future activities—the individual focus of this set of activities afforded an unfortunately race-evasive discussion in which students discussed characteristics of students and communities such as income and linguistic background but did not explicitly talk about race. This has been identified in previous research as a common problem with integrating mathematics and social justice (Harper, 2019; Larnell et al., 2016), and I plan to address concerns of race and racism more directly and intentionally in the future. Follow-up research in which I am currently engaged will also integrate data from local schools and political lenses more thoroughly throughout a content course and will collect more robust forms of data to hopefully enable a richer description of PSTs' development of political *conocimiento*.

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PROBLEM SOLVING IN TEACHER PREPARATION: TWO CONTEXTS, ONE GLOBAL SCENARIO

RETOS MATEMÁTICOS EN LA FORMACIÓN DE MAESTROS: DOS CONTEXTOS, UN ESCENARIO GLOBAL

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We describe an investigation on the use of mathematical tasks in mathematics methods courses in teacher preparation programs in two contexts: the United States and Colombia. Specifically, we elucidate how mathematical tasks influence teacher training. This research supports a larger project studying how identities as a teacher and global citizen intertwine with the contribution of mathematics as a science to the development of nations. Preliminary results indicate that mathematical tasks strengthen a positive attitude towards the teaching of mathematics in both contexts. Likewise, we find differences between the contexts that align with characterizations of individualistic and collectivist societies described by different researchers.

Keywords: Preservice Teacher Education, Teacher Educators, Problem-Based Learning, Affect, Emotion, Beliefs, and Attitudes

Our article describes research developed in collaboration with two higher education institutions, one in the United States and the other in Colombia. In this article we document the planning, development, and initial implementation of a mathematics teaching methods course in the two contexts. This documentation forms only the initial phase of a research project that seeks to understand, from a critical point of view, what it means to be a teacher and global citizen, a discussion frequently absent from the school curriculum and sometimes from teacher education programs (Subedi, 2013). The research question guiding our study is: *what do pre-service teachers report having learned in their mathematics methods courses after engaging in mathematical tasks specifically designed to impart instructional practices for teaching mathematics in elementary school?*

Background and Theoretical Framework

We focused on preparation for teaching mathematics in elementary school for two main reasons. First, this provides the context most familiar to us and the focus of our work as teacher educators. Second, a global perspective underlying current teacher preparation is that both the mathematical content and the pedagogies of mathematics teaching inform the teaching practice that will prepare future teachers to be active citizens with knowledge and skills specialized for competing in the global economy (National Board for Professional Teaching Standards, 2016). This perspective is inclined to promote consumption; locate, distribute, and use resources; in addition to offering services for production and marketing throughout the world (Spring, 1998). To develop these capacities, according to Spring's (1998) explanation, it is essential for developing certain types of specialized knowledge, skills, and values. Mathematics lies at the center of this knowledge (Mochón & Morales, 2010). In contrast to this capitalist perspective, another critical perspective encourages teachers to be public intellectuals with a moral vision and analytical tools to form future citizens who participate in a critical democracy (Bates, 2008). We also see mathematics at the center of discussions of equity in society and therefore in the classroom. Either way, these three positions place mathematics in a place of importance.

Local Contexts

United States. This context focuses on the use of students' mathematical thinking as a way to inform instructional practices for teaching mathematics (Carpenter et al., 2014). In this context, the mathematics methods class focuses on providing opportunities for the pre-service teachers to share, justify, connect, and extend mathematical ideas. In this class, pre-service teachers learn to design, select, and implement instructional activities with a high degree of mathematical content.

Colombia. In the Colombian context, the class, Didactics of Mathematics, targets those preparing to become teachers at the preschool level. In this context the class places particular importance on the development of logical thinking, and problem solving (Pachón Alonso et al., 2016).

Global Context

Having identified the theoretical bases that guide the design and implementation of each local context, we proceed to identify how to study both contexts simultaneously. Based on studies previously carried out by the first author, we decided to implement the Mathematical Autobiographies as an initial instrument to be used in common in the two classes (Krause & Maldonado, 2019). The first author has used this task throughout her years preparing teachers and has found that pre-service teachers are not infrequently “afraid” of mathematics or feel that at some point during school they stopped understanding it (Krause & Maldonado, 2019). Naturally we wondered whether in the Colombian context we would find the same commonality. When analyzing the responses of the pre-service teachers, we found that a most of the students in the Colombian contexts described having little understanding of mathematics (Avila et al., In preparation). Through this process, we were able to identify “fear” of or “dislike” of mathematics as a problem we shared in our practice as teacher educators. We point it out as a problem because, by identifying mathematics as an area that they do not like, future teachers can pass the same impressions on to their students (Lee, 2005).

Once this common problem of practice was established, we proceeded to the next step: the design of activities that might improve these feelings about mathematics. We call such activities *math tasks*. For its design, we used as a basis the idea of equilibrium and disequilibrium (Van de Walle et al., 2013) and the concept of productive struggle (Franke et al., 2001). The fundamental idea was based on designing mathematical tasks that (1) were at the academic level of the pre-service teachers, that is, whose mathematical content accorded with what they should learn within the curriculum established for them, (2) had a certain level of difficulty so that they could experience productive struggle, and (3) whose solution could be found through different processes or strategies. These three components of what we call mathematical tasks are based on the three components that Hiebert et al. (1997) have described as components of a problem. In accordance with these theoretical foundations, we define four essential components that a mathematical task must include: (1) the mathematics that students already know, (2) the mathematics that students must learn, (3) different ways of reaching the solution, and (4) generation of a change in the way pre-service teachers perceive how they understand or how they feel about their ability to engage with mathematics.

Methods

Data

The mathematical tasks were implemented at the beginning of each class during the Fall semester of 2020. In the context of Colombia there were a total of 14 tasks, in the context of the United States 11. In addition, to recognize the impact of the tasks in component (4) mentioned above, at the end of each class the pre-service teachers documented their in-class experience in writing. Each of us designed and adapted the tasks according to the class topic of the day. As an example, we describe two of them below. The first we used in the class in the United States and the second in the Colombian class.

Example 1

Which is bigger, the height or the circumference? The activity begins with the teacher holding a jar of tennis balls. At the same time the teacher asks: which is greater, the circumference or the height? Almost immediately, students responded that the height is greater, because it is “obvious” just by looking at it. Once the question is asked, pre-service teachers were given a few minutes to write an explanation of their reasoning. Then we measured individually the height and circumference of the can with a separate pieces of string. In this case we used different colored strings to facilitate comparison of the lengths. The circumference of the can naturally turns out to be larger. Finally, the pre-service teachers are asked to find a way to explain why this is the case.

Example 2

Which is the impostor? The activity begins with the instructor showing the numbers 7, 9, 16, and 25 organized as shown in Figure 1. The pre-service teachers were told to look at the numbers and identify which one was the "impostor".”The pre-service teachers had 3 minutes to make their decision. Then, as a class, we took a survey on who had selected 7, 9, 16, or 25. At first only their choice was indicated, without an explanation. Then, as a class, we discussed why they had selected a particular number as the impostor. The second part of the task consisted of an adaptation of the first part. Here the pre-service teachers had to identify the impostor figure (Figure 2).

7	9
16	25

Figure 1: Impostor Number

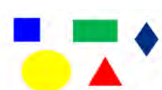

Figure 2: Impostor Figure

Results

After analyzing the responses of the pre-service teachers, we found 4 aspects in common in the two contexts: (1) the pre-service teachers tried different solutions. For example, in the tennis ball challenge, some made a list of formulas that they remembered (the formula for the circumference and even the formula for the area of a triangle), even though they weren't sure if they were going to use them. Others imagined cutting the can of tennis balls open completely, yielding a rectangle. In the impostor number task, the students presented their arguments for their answers by touching on concepts such as odd and even numbers, prime numbers, perfect squares, and addition and subtraction operations allowing them to identify relationships between the numbers. (2) Pre-service teachers reflected on understanding mathematics beyond a rote application of formulas and memorization. For example, in the case of the United States the pre-service teachers shared that “today I learned that memorizing formulas is not so useful if I do not understand how I should apply them. At the same time, I was able to reflect that the processes and making connections with ideas and concepts that we have already learned are fundamental to understand the solution of a problem." (3) The pre-service teachers expressed how they realized that they understood more about mathematics than they thought. In Colombia a student shared that "many of us found it somewhat complex even though the problem was not difficult, when we went to share our solutions, it was easier because of collaboration." "4) Pre-service teachers shared their love and motivation for our classes. However, we did not find any specific reference to how their mathematical knowledge has improved. In this data set we also found two emerging ideas that were not shared in the two contexts. In the Colombian context, for example, the pre-service teachers reported surprise when they saw the diversity of thoughts among their own classmates and how, thanks to the group work in class, it was easier to understand each challenge. In addition, they shared that they "really liked" discussing all the ideas. In the context of the United States, on the other hand, we found that students reflected on the importance of allowing time to think, analyze, and make connections when students are asked to find solutions to mathematical problems.

Discussion and Conclusions

In our first attempt to understand our local contexts to then understand the global context, we encountered a common problem in our practice: pre-service teachers, in both of our contexts, tend not to feel confident in their mathematical knowledge nor do they have a passion for this area. Although our decision regarding how to tackle this common problem did not produce a significant change, it nevertheless did produce a series of results that informs our practice both locally and globally. In particular, we found that pre-service teachers felt more comfortable facing solutions to mathematics problems given in each context after engaging in the mathematical task. We conjecture that this might have an impact on their comfort level teaching mathematics in the future. We make this conjecture based on the preliminary results of this study; our next step is to see if this is actually what happens.

On the other hand, we were able to identify some aspects unique to the local context. For example, in Colombia the pre-service teachers referred to the mathematical thoughts and ideas of their classmates and how they learn with others. Whereas in the context of the United States, the students did not refer to anyone other than themselves in their reflections. This result aligns with studies described by Triandis (2001), who describes individualistic and collectivist societies. According to Triandis (2001) the US is listed as one of the most individualistic societies, whereas Colombia is listed in 4^{th} place on the list of collectivistic ones. Finally, pre-service teachers in the United States made reference to the importance of giving more time to think about solutions, while in Colombia this aspect was not specifically mentioned. Time or the lack thereof in educational contexts is a widely discussed topic in educational research in the United States. Our conjecture in this regard is that pre-service teachers can identify this stress while working with their cooperating teachers in their practicum placements.

The use of mathematical tasks and the respective reflection around them has allowed us to identify common problems of practice in both contexts. We plan to further study these problems of practice as we continue to learn about how to best inform our instructional practices in a global setting.

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ETOS MATEMÁTICOS EN LA FORMACIÓN DE MAESTROS: DOS CONTEXTOS, UN ESCENARIO GLOBAL

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Describimos una investigación sobre el empleo de retos matemáticos en cursos de métodos de enseñanza de las matemáticas en la formación de maestros de preescolar y primaria en dos contextos: Estados Unidos y Colombia. Nuestro enfoque principal es describir cómo los retos matemáticos influyen en la formación de maestros. Resultados preliminares indican que estos inciden en el fortalecimiento de una actitud positiva hacia la enseñanza de las matemáticas en los dos contextos. Así mismo encontramos diferencias en cada contexto alineadas con caracterizaciones de sociedades individualistas y colectivistas descritas por diferentes investigadores.

Olanoff, D., Johnson, K., & Spitzer, S. (2021). *Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Philadelphia, PA.

Palabras clave: Preservice Teacher Education, Teacher Educators, Problem-Based Learning, Affect, Emotion, Beliefs, and Attitudes

Nuestro artículo describe un trabajo de investigación desarrollado en colaboración con dos instituciones de educación superior, una en Estados Unidos y la otra en Colombia. En este artículo documentamos la planeación, desarrollo e implementación inicial de un curso de métodos de enseñanza de las matemáticas en los dos contextos. Esta documentación es sólo el proceso inicial en un proyecto de investigación que busca entender, desde un punto de vista crítico, lo que significa ser un maestro y ciudadano global, discusión que está frecuentemente ausente en el currículo escolar y algunas veces, en el currículo de formación de maestros (Subedi, 2013). Nuestro estudio tuvo como pregunta orientadora *¿qué reportan los maestros en formación haber aprendido en las clases de métodos de la enseñanza de las matemáticas después de abordar retos matemáticos específicamente diseñados para cada clase?*

Antecedentes y Marco Teórico

Nos enfocamos en la formación de maestros en matemáticas; primero, porque este es el contexto que más conocemos y el enfoque de nuestro trabajo como formadores de maestros. Segundo, porque una perspectiva global subyacente a la formación actual de docentes es que, tanto el contenido matemático como las pedagogías de enseñanza de las matemáticas, dan forma a la práctica docente que va a preparar a los futuros maestros como ciudadanos activos con conocimientos y habilidades especializados para competir en la economía global (National Board for Professional Teaching Standards, 2016). Esta perspectiva se inclina en fomentar la capacidad para desarrollar el consumo; localizar, distribuir y usar recursos; además de ofrecer servicios para la producción y comercialización en todo el mundo (Spring, 1998). Para poder competir y desarrollar estas capacidades, según lo que explica Spring (1998), es fundamental desarrollar ciertos tipos de conocimientos, habilidades y valores especializados. Las matemáticas se encuentran justo en el centro de estos conocimientos (Mochón & Morales, 2010). En contraste a esta perspectiva capitalista, existe una perspectiva crítica que promueve que los maestros se preparen para ser intelectuales públicos con una visión moral y herramientas analíticas para formar futuros ciudadanos partícipes de una democracia crítica (Bates, 2008). Esta perspectiva pareciera no contemplar las matemáticas, sin embargo, como Moses and Cobb (2002) lo han expresado en el mundo de hoy la ciudadanía depende fundamentalmente de sus conocimientos en matemáticas para el acceso a beneficios económicos. Para Moses and Cobb (2002), este conocimiento de las matemáticas en las comunidades urbanas y rurales (en los Estados Unidos) es un tema urgente. El caso colombiano no es diferente, según lo resalta la Misión Internacional de Sabios (2019). Nosotros también vemos las matemáticas en el centro de las discusiones de equidad en la sociedad y por ende en el salón de clase. De cualquier manera, independientemente de nuestra perspectiva, estas dos posiciones ubican a las matemáticas en un lugar de importancia.

Contextos Locales

Estados Unidos. Este contexto se centra en el uso del pensamiento matemático de los estudiantes como centro del aprendizaje (Carpenter et al., 2014). En este contexto la clase de métodos se enfoca en proveer oportunidades a los estudiantes para compartir, justificar, conectar y extender ideas matemáticas. En esta clase los maestros en formación aprenden a diseñar, seleccionar e implementar actividades de instrucción con un alto contenido matemático.

Colombia. En el contexto colombiano, la clase, Didáctica de las Matemáticas, está orientada a quienes se están formando para ser maestros del nivel preescolar. Este enfoque da especial importancia al desarrollo del pensamiento lógico matemático, la resolución de problemas, y el planteamiento de nuevas situaciones en diversos contextos (Pachón Alonso et al., 2016).

Contexto Global

Identificadas las bases teóricas que guían el diseño e implementación de cada contexto local, pasamos a identificar de qué manera estudiar los dos simultáneamente. Basados en estudios realizados anteriormente por el primer autor, decidimos implementar las Autobiografías Matemáticas como instrumento inicial para usar en común en las dos clases (Krause & Maldonado, 2019). El primer autor ha usado esta tarea a través de sus años preparando maestros y ha encontrado un factor que parece emerger en los datos, los maestros en formac e "temen" a las matemáticas o sienten que en algún momento en sus años de estudio dejaron de entenderlas o de apreciar su belleza (Krause & Maldonado, 2019). Este factor común nos llevó a preguntarnos si en el contexto colombiano llegaríamos a encontrar lo mismo. Al analizar las respuestas de los maestros en formación encontramos que un número alto se describió como poco entendedor de las matemáticas (Avila et al., en preparación). A través de este proceso pudimos identificar "el temer" o "no sentir gusto por las ma ticas" como un problema común en nuestra práctica como formadores de docentes. Lo señalamos como problema porque al identificar las matemáticas como un área que no les gusta, los futuros maestros pueden transmitir tal idea a sus estudiantes (Lee, 2005).

Una vez establecido este problema de práctica común, seguimos con el siguiente paso: el diseño de actividades que puedan cambiar esta identidad matemática. Llamamos a estas actividades retos matemáticos. Para su diseño usamos como base la idea de equilibrio y desequilibrio (Van de Walle et al., 2013) y el concepto detrás del conflicto productivo (*productive struggle*) (Franke et al., 2001). La idea fundamental se basó en diseñar actividades que (1) estuvieran al nivel académico de los maestros en formación, es decir que el contenido matemático estuviera de acuerdo con lo que deben aprender según el currículo establecido para ellos, (2) que tuvieran cierto nivel de dificultad para que pudieran experimentar un conflicto productivo (*productive struggle*), y (3) que la solución pudiera encontrarse a través de diferentes procesos o estrategias. Estos tres componentes de lo que llamamos retos matemáticos están basados en los tres componentes que (Hiebert et al., 1997) ha descrito como componentes de un problema. De acuerdo con estos fundamentos teóricos nosotros definimos cuatro componentes esenciales que un reto matemático debe incluir: (1) la matemática que ya saben los estudiantes, (2) la matemática que los estudiantes deben aprender, (3) diferentes formas de llegar a la solución, y (4) generar un cambio en la forma como el maestro en formación percibe cómo entiende o cómo se siente con respecto a su capacidad de entender las matemáticas.

Métodos

Datos

Los retos matemáticos fueron implementados al inicio de cada clase durante el semestre del otoño de 2020. En el contexto de Colombia fueron en total 14 retos y en el contexto de Estados Unidos fueron 11. Además, con el fin de reconocer el impacto de los retos en el componente (4) antes señalado, al finalizar cada clase, los maestros en formación documentaban de manera escrita su experiencia en la clase. Cada uno de nosotros diseñó y adaptó los retos de acuerdo con el tema de clase. Como ejemplo, a continuación, describimos dos de ellos. El primero lo usamos en la clase de Estados Unidos y el segundo en la clase de Colombia.

Ejemplo 1

¿Qué es más grande, la altura o la circunferencia? La actividad comienza con el profesor sosteniendo un tarro de pelotas de tenis. Al mismo tiempo el profesor pregunta: ¿cuál es más grande, la circunferencia o la altura? Casi de inmediato, los estudiantes responden que la altura es mayor porque es "obvio" con solo mirarla. Una vez hecha la pregunta, el profesor da unos minutos a los estudiantes para que escriban una explicación de por qué creen que la altura es mayor. Luego el profesor mide la altura de la lata con una cinta de lana. En este caso usamos cintas de diferentes colores para poder comparar las longitudes. La circunferencia de la lata resulta ser mayor. Finalmente, el profesor pide a los maestros en formación que encuentren una manera de explicar por qué eso es cierto.

Ejemplo 2

¿Cuál es el intruso? La actividad inicia con el profesor mostrando los números 7, 9, 16 y 25 organizados como los muestra la Figura 1. La indicación para los maestros en formación era ob944anderbios números e identificar c l era el "intruso". Los maestros en formac enían 3 minutos para tomar su de944anderbi Luego, como clase hacíamos un sondeo sobre quiénes habían seleccionado el 7, el 9, el 16 ó el 25. En este primer momento só944ande indicaba su elección, no se explicaba el por qué. Después, como clase, discutíamos sobre el por qué la elección del número intruso que fue seleccionado por la mayoría. La segunda parte del reto consiste en una adaptación de la primera parte. Acá los maestros en formación tenían que identificar la figura intrusa dentro de 5 posibles (Figura 2).

7	9
16	25

Figura 1: Número Intruso

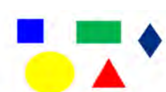

Figura 2: Figura intrusa

Resultados

Después de analizar las respuestas de los maestros en formación encontramos 4 aspectos en común en los dos contextos: (1) los maestros en formación intentaron diferentes soluciones. Por ejemplo, en el reto de las pelotas de tenis algunos hicieron una lista de fórmulas que recordaban (la fórmula de la circunferencia e incluso la fórmula del área de un triángulo) aunque no estaban seguros si las iban a usar. Otros hicieron una representación imaginando que cortarían el tarro de pelotas para abrirlo completamente y así tendrían un rectángulo. En el reto del número intruso los estudiantes expusieron sus argumentos respecto a sus respuestas tocando conceptos como números pares e impares, primos, cuadrados perfectos y realizando operaciones de suma y resta que les permitieran ubicar relaciones entre los números. (2) Los maestros en formación reflexionaron sobre entender las matemáticas más allá de un proceso que sigue fórmulas y memorización. Por ejemplo, en el caso de los Estados Unidos los maestros en formación compartieron que "hoy aprendí que memorizar f rmulas no es tan útil sino entiendo c aplicarlas. Al mismo tiempo pude reflexionar que los procesos y el hacer conexiones con ideas y conceptos que ya hemos aprendido son fundamentales para entender la solución de un problema". (3) Los maestros en formac esaron que se dieron cuenta que entiende e mate s de lo que en realidad pensaban. En Colombia un estudiante expre "a muchos de nosotros nos pareció algo complejo a pesar de que el problema no estaba difícil, cuando fuimos a compartir nuestras soluciones fue cil por la colaborac (4) Los maestros en formación compartieron su gusto y motivación por nuestras clases, sin embargo, no encontramos ninguna referencia específica a cómo ha mejorado su conocimiento matemático. En este set de datos también encontramos dos ideas emergentes que no fueron compartidas en los dos contextos. En el contexto colombiano, por ejemplo, los maestros en formación reportaron sorpresa al ver la diversidad de pensamientos entre sus propios compañeros y cómo gracias a los momentos de socialización era más fácil comprender cada reto. En sus respuestas, durante las reflexiones, incluso dieron crédito con nombre propio a estas ideas. Además, compartieron que “ ucho” discutir todas las ideas. En el contexto de los Estados Unidos, por el otro lado, encontramos que los estudiantes reflexionaron sobre la importancia de dar tiempo para pensar, analizar, y hacer conexiones cuando se les pide a los estudiantes que encuentren soluciones a problemas matemáticos.

Discusión y Conclusiones

En el primer intento de entender nuestros contextos locales para así poder extenderlos en un contexto global, encontramos un problema común en nuestra práctica: los maestros en formación tienden a no sentirse seguros de su conocimiento matemático ni tener gusto por esta ciencia. Aunque nuestra decisión sobre cómo afrontar este problema común no arrojó un cambio significativo, en el primer punto, sí produjo una serie de resultados que informan sobre nuestra práctica tanto local como global. Particularmente encontramos que los maestros en formación se sienten más seguros en la manera de afrontar soluciones a problemas de matemáticas dado el contexto en el que son presentados, lo cual, muestra su proyección como docentes de matemáticas. Esto es sólo una conjetura que hacemos basados en los resultados preliminares de este estudio, nuestro paso siguiente es tratar de comprobar si en realidad esto es lo que pasa.

Por otra parte, pudimos identificar aspectos característicos únicos al contexto local. Por ejemplo, en Colombia los maestros en formación hicieron referencia a los pensamientos e ideas matemáticas de sus compañeros y cómo se aprende con otros. Mientras que en el contexto de los Estados Unidos los estudiantes no hicieron referencia a nadie diferente a ellos mismos en sus reflexiones. Este resultado se alinea con estudios descritos por Triandis (2001) quien describe sociedades individualistas y colectivistas. Por último, los maestros en formación en los Estados Unidos hicieron referencia a dar más tiempo para pensar en las soluciones, mientras que en Colombia este aspecto no se mencionó específicamente. El tiempo o la falta de tiempo es un tema bastante discutido en investigaciones educativas en los Estados Unidos. Nuestra conjetura al respecto es que de alguna manera los mismos maestros en formación ven en sus prácticas este estrés por la falta de tiempo. En nuestra práctica formando maestros con una perspectiva global, que también incluya lo local, el empleo de los retos matemáticos y la respectiva reflexión en torno a ellos nos ha permitido identificar problemas de práctica comunes en los dos contextos.

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MODIFYING AND ACCOMMODATING INSTRUCTION OF MULTIPLICATION FOR STUDENTS WITH MATHEMATICAL LEARNING DISABILITIES (MLD)

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This study investigates preservice mathematics teachers' instructional approaches to teach multiplication to students with mathematical learning disabilities (MLD). 17 preservice teachers' lesson design were qualitatively analyzed. Findings revealed that the PTs modified mathematical strategies by providing a variety of multiplicative concepts and various types of representations. PTs set their expectations based on individual needs, manage lesson structure, and adjust cognitive demand level of lesson tasks. Results suggest that PTs need opportunities to consider deeper responsive teaching so their modification and accommodation for students with MLD enable quality learning opportunities.

Keywords: differentiated instruction, students with disabilities, preservice teacher education

Teaching mathematics for *all* students means that teaching should support each student's access to high quality learning opportunities. While previous research has proven that there are unequal distributions of high-quality learning opportunities within mathematics classrooms (Jackson & Wilson, 2012), how instructional strategies should be modified to maximize the quality of learning opportunities for each student has been an urgent need (NCTM, 2014; Urick et al., 2018). Modifying and accommodating mathematics instruction for all students can achieve this. Modification means a change that is being taught to or what students are expected to learn, and accommodation means a change that helps a student overcome or work around the disability. Our study aimed to let preservice teachers prepare to teach mathematics while including students with mathematical learning disabilities (MLD). PTs in our study were asked to develop lesson ideas on multiplication with single digit multipliers for an average performing student and a student with MLD. We investigated the PTs' lesson approaches for providing access and opportunity for those learners.

Literature Review

Teaching Students with MLD

While Mathematical Learning Disabilities (MLD) has multiple aspects, we consider students with MLD to be those that create unique patterns or different kinds of errors from typical or low achieving peers (Lewis, 2010). Following this perspective, our study centers on the idea that some students have special needs and teachers should modify and accommodate instructional strategies to respond to their special needs. Successful teaching approaches for students with MLD include explicit teaching (Fuchs & Vaughn, 2012; Kroesberfen & Van Luit, 2003; Leach, 2016; Stein, Carnine, & Dixon, 1998), Antecedent-Behavioral Response-Consequence (ABC) teaching sequences (Leach, 2016), **scaffolding (**van Garderen, Scheuermann, & Jackson, 2012), and Concrete-Representational-Abstract (CRA) (Fyfe & Nathan, 2019; Gibs, Hinton, & Flores, 2018).

Teaching Multiplication

Multiplication in elementary school is one of the fundamental operations across grades in school mathematics, and it is the basis for early algebra (Anghileri & Johnson, 1992; Otto et al., 2011). Greer (1992) classified four situations that can be modeled by multiplication, which include (1) equal groups, (2) multiplicative comparison, (3) rectangular area and rectangular array, (4) the Cartesian product. Models are external representations of those multiplicative concepts, and we identified four multiplicative models: grouping model, number-line model, array model, and combination model (Table 1).

Table 1: Multiplicative Strategies

External Representations of Multiplications		Multiplicative Situations and Structure			
		Equal groups	Multiplicative comparison	Rectangular array/area	Cartesian product
Grouping Model	A model of grouping of objects having the same number in each group.	Can be represented[1]	Can be represented		
Number-line Model	A number line model equally divided by constant intervals and drawing regular arcs of each interval.	Can be represented	Can be represented		
Array Model	A model that uniformly arranges several objects in a rectangular shape of rows and columns.	Can be represented	Can be represented	Can be represented	Can be represented
Combinations	A model that identifies possible ordered pairs made between two or more sets.				Can be represented

Research Questions

In our study, mathematical approaches including multiplicative concepts and its representation were used as *mathematics strategies*; non-mathematical approaches including lesson goals, structure, flow, and task were used as *instructional strategies*. Research questions that guided our study were:

1. How do preservice teachers modify and accommodate *mathematics strategies* to teach multiplication to students with different backgrounds and needs?
2. What *instructional strategies* do preservice teachers use for teaching multiplication in their lesson design for students with different backgrounds and needs?

Methods

Participants

Participants (N = 17) were undergraduate PTs enrolled in the early childhood licensure program (ECE, grades PreK-3) at a university in the Midwest United States. They had learned general educational theories and practices and special educational theories in their junior years. In their senior year, PTs continued to learn theories and applied their learning in practice through methods courses and fieldwork. Data for our study were collected from a mathematics methods course in their senior year.

Data Collection

Artifacts from lesson design activity were analyzed. The lesson design activity consisted of two phases: (1) PTs brainstormed and developed a plan to teach a mathematics concept to typically performing 3rd graders, including *Jose* and then (2) they were asked to modify their lessons to address the needs of a student with MLD, called *Liam*. Artifacts collected were mainly documents that described in detail their lesson plans and how they would modify and accommodate the lesson plans.

Data Analysis

Using open coding (Strauss, 1987; Strauss & Corbin, 1994), we categorized the meaningful strategies for responsive teaching revealed in PTs' lesson design documents as following: (1) mathematical strategies, involving concepts of multiplication in teachers' instruction and/or support of students' better understanding of the multiplicative concept; and (2) lesson design elements, including general pedagogical supports, task selection, lesson structure and process of instruction.

Table 2. Data Analysis Categories and Codes

Categories	Properties	Codes
Mathematical Strategies	Multiplicative concept	Situation modeled; External representation
	Ways of representing the multiplicative concept	Visual aid; Physical experience; Symbolic objectification; Verbal expression; Contextual transition
Lesson Design Elements	Articulation of expectations	Mentioned the standard; Stated learning objectives
	Instructional activity	Task selection and development; Instructional structures; Instructional Progress
	Formative assessment	Gathering/sharing; Attending/interpreting; Supporting/feedback

Results

Modification and Accommodation in Math Strategies

The PTs' approach to introducing the multiplication concept to both learners (*Jose* & *Liam*) was grounded in the same modeled situation, the equal group situation. For the average performing student, all PTs modeled multiplication based on equal group situations and used external representations. To represent the multiplicative concept, PTs used the grouping model most frequently followed by the array model. It was noted that the modification in the instruction of the multiplication concept occurred most often for Liam. This decision stemmed from the assumption that students with MLD generally have difficulty in skip counting and must derive the product of multiplication through one-by-one counting. Some used the number-line to prevent the omission of numbers or double counting in counting with manipulatives, which is a responsive teaching strategy for students with MLD by enabling them to visualize repeated addition of discrete quantities with directionality.

Most PTs provided physical experiences with visual aids and manipulatives for Jose, and physical experience was the most frequently used for Liam. PTs judged that students who have difficulty in understanding abstract concepts need a physical experience such as counting familiar objects or manipulatives. Discrepancies in the type of representation used for different learners were most clear in the use of symbolic objectification. Eleven out of 17 PTs provided a multiplication formula ($a \times b = c$) to Jose, which is an objectified symbol of multiplication learned by visual and physical reification, whereas no PT provided such objectified symbols to Liam. These results show that the PTs make instruction more responsive by encouraging Liam with physical experiences expressed by manipulating concrete materials.

Modification and Accommodation in Instructional Strategies

PTs discussed their plan for how they would execute their lesson in three areas: task/activity selection and development, instructional structure (how they would manage students' actions), and instructional progress (beginning and flow of the instruction). Only four PTs mentioned distinct mathematics tasks according to student profiles. PTs used the exact same number standards in their mathematics task for both Liam and Jose without any modification. PTs provided tasks with the same or a similar level of cognitive demand for Jose, while they provided tasks with a lower level of cognitive demand for Liam indicating such modification was necessary to reduce Liam's math anxiety and provide emotional support.

Most PTs implemented a combination of whole group, small group, and/or independent activities for Jose. In their modification for Liam, more than half of PTs addressed Liam's need for additional instructional time with tutoring during or after the lesson. However, there was no discussion of how Liam could join small group activities or contribute to the whole group discussions. Almost all PTs started their lesson with the teacher-direct instruction approach, such as modeling, demonstrating, or explaining to teach typically performing students. The lesson plan had progressed with a combination of concrete examples and representations, concreteness fading, and abstract representation. All PTs used concrete examples and objects at some point in their lesson plan, but the progression of the lesson varied.

Regarding assessment, approximately half of the PTs used formative assessment geared toward gathering and sharing; 35% of PTs gathered information on students' mathematical thinking and used the information to attend to their learning; and 15% of PTs gathered information, attended to the information, and supported and provided feedback. There was very limited discussion on assessments in PTs' plans (only two) for Liam.

Discussion and Implication

The results in this study highlight the need to reconsider responsive teaching with regard to preparing teachers who can maximize quality opportunities for *all* students. PTs indicated more emphasis on deficits than the strengths of the individual student. With Liam there was only discussion of teacher support, while peer support may have lessened his anxiety. Another point worth noting is that unlike Jose, there was no PT who provided symbolic objectification support to Liam. And too many of the PTs did not seek to challenge Liam with high level thinking requirements. It means that differentiated assistance provided to students with special needs could have the possibility to limit the chances of approaching the abstract mathematical concepts. For students with special needs, PTs did not seek ways to "ensure shared power" through inviting them to engage in whole-class or small group discussion and by encouraging them to share their ideas or respond to one another's ideas (Van de Walle et al., 2019).

Taking a deep dive into what PTs did allowed us to identify gaps in the skills needed for applying research-based responsive teaching. The PTs need more instruction in peer tutoring and explicit teaching for MLDs. They all need more understanding of what it means to aim for high levels of cognitive demand for *all* students. As PTs should learn to modify and accommodate instruction for diverse learners with explicit instructional strategies, teacher educators need to support PTs to construct and provide high-quality learning opportunities for improving their knowledge and practice. Another implication of this study is that the analytical tools for responsive teaching in elementary level multiplication can be useful in capturing equitable learning opportunities. By merging the two different perspectives of mathematics and special education, we enhanced the analytical tool to cover both mathematics and instructional strategies. We believe that this analytical tool contributed not only to the methodological lens of responsive teaching for *all* students but also to the theoretical lens. It allows both the mathematics and special education fields to make real progress in persistent challenges in the teaching and learning of mathematics.

Note

[1]This means "Multiplicative Structure of Equal groups can be represented with Grouping Model."

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HOW PRESERVICE TEACHERS GRAPPLE WITH THE CONCEPT OF MATHEMATICAL MODELING

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Mathematical modeling is an important process, concept, and practice in solving real-life problems but a source of concern for the preparation of preservice teachers (PSTs). To investigate this concern, we examined 31 PK–8 PSTs' conceptualization of mathematical modeling. We collected data using both qualitative and quantitative methods. Results indicated that most PSTs had minimal understanding of mathematical modeling. Most of our PSTs had misconceptions of mathematical modeling, and perceived mathematical modeling as an exclusive action reserved for only teachers. Based on our study results, we believe that infusing or integrating modeling courses or modules into existing methods or content courses can be an effective way to elevate PSTs from unfamiliarity with mathematical modeling practices and standards as emphasized by the Common Core State Standards for Mathematics.

Keywords: preservice teachers, mathematical modeling, conception, modeling, standards

The attention on mathematical modeling in teacher preparation programs is relatively low in the United States of America (U.S.), and this suggests that most teachers have not experienced mathematical modeling processes, standards, and practices (Asempapa & Sturgill, 2019), and learned it efficiently and consistently (Phillips, 2016). Looking back, Blum (2002), explained the rare use of modeling standards and process in mathematics education courses in teacher preparation programs. Moreover, Hamson (2003) further clarified as to why this occurred by arguing that "including more modeling in mathematics education has been a slow process" (p. 222). Before teachers implement modeling-based practices, they must not only have the required content knowledge but also the experiences with mathematical modeling standards. The luck of mathematical modeling practices in teacher preparation programs possess a problem for developing students' problem-solving skills (Borromeo Ferri, 2018; Kaiser et al., 2010; Paolucci & Wessels, 2017). Therefore, the purpose of this study was to exploratorily increase our understanding and describe PSTs' conception of mathematical modeling.

Few teacher preparation programs address mathematical modeling, its standards, and practices at the PK–8 level (Borromeo Ferri, 2018; Matthews & Reed, 2007; Zbiek, 2016). Additionally, the majority of teachers who are now expected to teach mathematical modeling have neither encountered modeling themselves nor studied it systematically (Borromeo Ferri, 2018; Phillips, 2016). According to Baumert et al. (2010), "the repertoire of teaching strategies and the pool of alternative mathematical representations and explanations available to teachers in the classroom are largely dependent on the breadth and depth of their conceptual understanding of the subject'' (p. 138). Furthermore, many teacher preparation programs rarely explore issues surrounding mathematical modeling practices, and these issues are often considered after teachers are on the job during professional development programs (Borromeo Ferri, 2018; Paolucci &Wessels, 2017). Thus, teacher preparation programs must develop modeling proficiencies and competencies in PSTs. Such goals are needed so they can experience the Common Core Mathematical Practice–*model with mathematics*–during their teaching.

The research question that guided this study was: What are preservice teachers (PSTs) conceptions of mathematical modeling?

Background and Relevant Literature

Mathematical Modeling

Historically, mathematical modeling has meant the use of mathematics to solve problems arising in everyday life (Blum & Boromeo Ferri, 2009; Lesh & Doerr, 2003). Mathematical modeling cuts across all levels of education, even teacher education. There is no question that teachers play an important role in motivating students to study mathematical modeling and the application of mathematics through engaging in mathematical modeling practices. Tremendous efforts have been made by the mathematics education community in studying mathematical modeling (Blum & Borromeo Ferri, 2009; Borromeo Ferri, 2018; Galbraith, 2012). Additionally, mathematical modeling, its applications, and the learning of it in schools have become prominent topics in mathematics education which have been discussed and advocated intensely in teacher preparation programs (Borromeo Ferri, 2018; Cai et al., 2014; Zbiek, 2016). Nevertheless, there exists a lack of attention to mathematical modeling in some U.S. mathematics teacher preparation programs (Borromeo Ferri, 2018; Newton et al., 2014).

Mathematical modeling “is the art or process of constructing a mathematical representation of reality that captures, simulates, or represents selected features or behaviors of that aspect of reality being modeled” (Cai et al., 2014, p. 150). Additionally, modeling involves an iterative process of interpreting a situation, constructing representational descriptions, and developing through revision cycles (Jung & Newton, 2018; Lesh & Doerr, 2003). Moreover, modeling requires a well-connected set of mathematical concepts and skills that, used flexibly, enables individuals to solve problems and better understand the real world, which goes beyond mere computational proficiency. On the contrary, learners’ success with routines or traditional tasks do not imply competencies or proficiencies with mathematical modeling because the process of moving from the givens (assumptions) to the goals (outcomes) may not be obvious (Lesh & Doerr, 2003; Lesh & Lehrer, 2003).

Preservice Teacher Education and Modeling

There is broad consensus that teachers need to have strong and sound knowledge of the subject content, as these influences both what they teach and how they teach it (Ball et al., 2008; National Council of Teachers of Mathematics [NCTM], 2014; Ponte & Chapman, 2016). Both elementary and secondary PSTs’ content knowledge of teaching mathematics have received attention in most recent studies in mathematics education (Borromeo Ferri, 2018; Ponte & Chapman, 2016). Studies on PSTs’ mathematical knowledge, including mathematical modeling practices, indicate serious issues that teacher preparation programs ought to address (Borromeo Ferri, 2018; Ponte & Chapman, 2016). Issues include misconceptions and deficient competencies for different topics of modeling standards, such as modeling process, modeling competencies, and modeling tasks. Therefore, it is no surprise that PSTs’ mathematical modeling content knowledge continues to be a central research theme for preservice mathematics teacher education.

Knowledge and experiences do impact individuals’ actions as well as form the foundation from which PSTs build upon to become high-quality, impactful teachers. To better prepare future teachers for the current classroom, they need to be engaged in mathematical modeling to support student learning and help students develop related competencies (Borromeo Ferri, 2018; Philips, 2016). Research has found that PSTs’ understanding of mathematical modeling varies and is impacted by the engagement in this modeling as well as through the investigation of modeling resources (Jung & Newton, 2018; Zbiek, 2016). Most research on the intersection of teachers and mathematical modeling focuses on inservice teachers or secondary preservice teachers. Yet, these populations are not the only ones who need to fully understand this modeling and integrate into their practice. To better prepare all mathematics teachers, more research is needed to understand elementary PSTs' knowledge of and experiences in mathematical modeling.

Methodology

Study Setting and Participants

The study participants were 31 PK–8 PSTs enrolled in mathematics methods courses. They were recruited from two, four-year universities located in the northeastern and midwestern U.S. Most participants were sophomores and juniors who enrolled in teacher preparation programs to gain licensure to teach elementary or middle grades. All PSTs were enrolled in mathematics methods courses whose objectives included the development of pedagogical knowledge and practice regarding mathematical modeling. The PSTs in this study have not taken any course in mathematical modeling.

Data Collection and Analysis

We used both qualitative and quantitative methods to examine how PSTs grapple with the concept of mathematical modeling. Data consist of participants' responses from a questionnaire that drew upon their conception of mathematical modeling. Their written responses were collected and converted into PDFs. We developed codes inductively and deductively to analyze the data. The process for coding transitioned from initial coding to focused coding (Saldaña, 2013). During initial coding, we collectively read participants' responses discussing potential themes and making memos. Initial codes were adapted and combined so distinctions among them were recognizable and applicable. These actions flowed into a focused coding of the data in which HyperResearch© was used to record and organize our results.

Validity was maintained due to the cyclic nature of coding, memo writing, collective agreement, and attributes of our expertise in mathematical modeling, representations, and qualitative and quantitative analysis. Data analysis began with creating and refining a rubric used to support the thematic analysis of participant's conception of mathematical modeling. Due to the nature of the data, Intraclass Correlation Coefficient (ICC) was chosen to measure our rating consistency of the rubric we created. The single measures ICC was used in this study because it is the appropriate coefficient (LeBreton & Senter, 2008; Liljequist et al., 2019) and has the ability to determine that the judgment of one rater is the same as that of the others. The ICC measures were acceptable and all above 0.89 (Liljequist et al., 2019).

Results and Discussion

All but 1 of 31 participants responded to the question: "What is mathematical modeling?" Twenty-six (87%) PSTs had a poor or fair understanding of mathematical modeling, while only four (13%) had a good understanding. Quotes from the respondents depicting fair and poor definition was "*using manipulatives to represent math/thinking*" and "*I believe it will be like the graphic organizers we made for the article review*" respectively. For instance, one PST described modeling as "showing examples to students of similar problems and then to solving the[se] problems" (Participant 6). Nonetheless, there were 4 PSTs whose conception or understanding of mathematical modeling were rated as good. For example, participant 29 stated, "mathematical modeling is taking concepts, formulas, etc. … and applying them to the real work." Table 1 below summarizes the categorized responses of the participants to the definition of mathematical modeling based on the rubric.

Table 1: Rated Definitions of Mathematical Modeling

Category	# of Count	%
Excellent	0	0
Good	4	13
Fair	15	50
Poor	11	37

n = 30

Given the nature of participants responses, we next summarize the common themes that emerged. The common themes include (a) *who uses mathematical modeling*—most of participating PSTs placed the act of modeling on the teacher; (b) *how and why use mathematical modeling*—majority of the PSTs explained that teachers use mathematical modeling during instruction to solve examples, show strategies, or both; and (c) *representations of mathematical modeling*—most participants described the act of mathematical modeling as either a visual element (n =10) or as a symbolic representation (n = 6). Per anecdotal evidence, we were not surprised that no participating PST had an excellent understanding of mathematical modeling.

The results indicate that almost all the participants in the study had minimal understanding of what mathematical modeling entails. Additionally, most PSTs perceived mathematical modeling as an exclusive action reserved for only teachers. This perception is contrary to what the Common Core modeling standards expect of teachers: "mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace" (National Governors Association Center for Best Practices [NGA Center] & Council of Chief State School Officers [CCSSO], 2010, p. 7, 2010). Moreover, PSTs described the act of mathematical modeling to include a visual element or symbolic representation. Again, this notion carried by our participants about mathematical modeling shows their inadequate understanding of what mathematical modeling is about—the translation between mathematics and the real world—as described and discussed in standards and modeling education literature (Association of Mathematics Teacher Educators [AMTE], 2017; Borromeo Ferri, 2018, COMAP & SIAM, 2016; Hirsch & Roth McDuffie, 2016; NGA Center & CCSSO, 2010).

Conclusion and Implications

The purpose of this study was to examine how elementary PSTs grapple with the concept of mathematical modeling standards and practices. Because modeling standards are relatively new in the U.S. Common Core mathematics curriculum, but an important concept in mathematics education, it was necessary and essential in undertaking this research study with PSTs. The results from this study show that elementary PSTs' conceptions of mathematical modeling standards were minimal. Notwithstanding the suggestions by researchers that students should engage in mathematical modeling early in their mathematics education, PSTs need more training and support in meeting this challenge. It is worthwhile for teacher preparation programs to reexamine their curricula and consider mathematical modeling so PSTs conceptual models could be revised and extended. We believe that infusing or integrating modeling courses or modules into existing methods or content courses can be an effective way to elevate PSTs from unfamiliarity with mathematical modeling practices and standards.

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EXAMINING THE VALUE, IMPORTANCE, SELF-EFFICACY AND PRIOR EXPERIENCES OF MATHEMATICAL WRITING FOR PRESERVICE TEACHERS'

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Writing in mathematics is critical to students' learning, yet few teachers assess students' mathematical writing (MW) or incorporate MW into their instruction. In an effort to increase preservice teachers (PSTs) attention to MW, we examined PSTs' experiences with MW and the impact of a content module focused on MW embedded in mathematics methods courses at four institutions. Findings indicate PSTs had limited experiences with MW in K-16, yet saw it as valuable to students' learning. After completing the module, the PSTs' self-efficacy related to MW grew, indicating a greater likeliness of incorporating MW into their future practice.

Keywords: Preservice Teacher Education, Curriculum, Communication

Since the National Council of Teachers of Mathematics (NCTM) published their *Principles and Standards for School Mathematics* (2000) over 20 years ago, communicating mathematical ideas clearly, coherently, and effectively to teachers, peers, and others, has been a distinct goal of mathematics instruction. One way students are expected to communicate their mathematical ideas is through writing. Writing plays a critical role in advancing students' mathematical learning by promoting reflection and clarification of ideas through explanations, descriptions, definitions, and critiques (Freeman et al., 2016; Marks & Mousley, 1990; NCTM, 2000). In addition, mathematical writing (MW) can support students' 'evelopment of productive mathematical identities (Boaler, 2002; Cobb & Hodge, 2002; Ivanič, 1998; Murphy & Hall, 2008). Yet, writing in mathematics classrooms is often overlooked even though it plays an important role in developing mathematical thinking.

Elementary teachers in the U.S. are typically responsible for teaching every content area to the same group of students, thus they are well positioned to foster students' MW and draw on students' competencies in mathematics and English Language Arts. Yet, many teachers report they do not provide instruction on MW nor provide many opportunities for students to write in mathematics (Banilower et al., 2018; Powell et al., 2017). Thus, in an effort to draw increased attention to MW and support preservice teachers (PSTs) in their ability to assess and craft instruction that facilitates students' MW, we created a module focused on MW that was embedded within mathematics methods courses across four institutions. In this paper, we report on early findings from this study. Specifically, we sought to answer the questions: *What are preservice teachers prior experiences with MW? In what ways, if any, does completing a module focused on MW impact preservice teachers' self-efficacy and their perception of the uses and benefits of MW to students' learning?*

Mathematical Writing

Mathematical writing is "a writing activity in which students write about mathematics concepts or procedures" (Powell et al., 2021, p. 418) that can vary based on purpose, level of formality, audience, structure, and required language (Chval et al., 2021). Although there are multiple MW genres (e.g., explanatory, argumentative; Casa et al., 2016), explanatory writing is the primary focus of teachers (Gillespie et al., 2014; Swinson, 1992), curriculum (Casa et al., 2019), state assessments (Powell et al., 2020), and research (Powell et al., 2017).

Although teachers recognize the value and importance of MW to students' learning (Powell et al., 2021), few teachers ask their students to write in mathematics and, even less, incorporate instruction on how to write in mathematics (e.g., model; Powell et al., 2017). Importantly, whether teachers incorporate activities and/or instruction related to MW is tied to their self-efficacy for MW, regardless of grade level (Powell et al., 2021). Consequently, if teachers have higher levels of self-efficacy, they are more likely to incorporate MW into their instruction.

Assessing students' MW is critical to craft instruction that is responsive to student learning. Yet, teachers who incorporate MW may not assess students' writing or encourage students to self-assess (Powell et al., 2021), which raises the question of how to effectively support future teachers in assessing students' MW.

Methodology

This study is part of a larger project examining the impact of a MW module on PSTs' MW and their ability to assess elementary students' MW. Data for this project was collected at four universities and involved 116 PSTs.

Participants

All participants in this study were junior or senior undergraduate PSTs enrolled in a mathematics methods course at one of four universities in the U.S. Each methods course was designed to prepare elementary PSTs seeking initial licensure in elementary education, special education, or elementary and special education.

Mathematics Writing Module

All participants completed a week-long asynchronous MW module as a part of their mathematics methods course. The module was composed of (1) a pre-survey, (2) MW content, and (3) a post-survey. The pre- and post-survey collected similar information and included PSTs' demographic data (e.g., age, race/ethnicity); self-efficacy related to MW instruction and assessment; prior experiences with and beliefs about MW; and asked PSTs to provide written responses to two mathematics tasks. The MW content included: an explanation of what MW is and why it is important to students' learning, a description of how to use a rubric to assess students' writing, and asked PSTs to score two student's MW using a rubric.

Data Analysis

To answer the research questions for this study, we drew from PSTs' pre- and post-surveys. This data included PSTs responses to items measuring self-efficacy in MW (see Table 1) and open responses. PST's indicated strongly disagree (-2), disagree (-1), neutral (0), agree (1), or strongly agree (2) to each Likert scale item. To analyze the quantitative data, we used descriptive statistics and Cohen's d for individual questions as well as t-tests with Cohen's d for general self-efficacy in MW (overall mean of the four Likert scale questions). To analyze PSTs open responses, we used a constant comparative method (Patton, 2015). We began with open coding and then moved to axial coding after themes had been identified. All qualitative data was coded by two of the authors who met to resolve disagreements and reach consensus.

Findings

General findings indicated that PSTs saw benefits to MW despite limited MW experiences prior to the module, but demonstrated improved self-efficacy in MW and expanded conceptualizations of how MW could be used in instruction after engaging with the module.

PSTs' Experiences Related to Mathematical Writing

The PSTs in the study had limited prior experiences related to MW. When describing their K-12 experiences, a third (31%) of PSTs stated they were never asked to write in their mathematics classes and just over a half (52%) were asked only once to several times a semester. Across their K-12 experiences, the majority (77%) of PSTs received no formal instruction on how to craft a written response to a mathematics task. Of those PSTs who did receive instruction, this was commonly described as a need to provide their answer in a complete sentence or to explain their problem-solving procedures. As one PST stated,

> In math when we were taught to write our answers out, we were simply taught to explain how we got it. For example, our sentences would look something like this, "I got the answer ______, I found this answer by _____."

The majority (83%) of PSTs also received minimal education or instruction in their teacher preparation program related to writing in mathematics, even though the PSTs had completed at least one elementary mathematics content course prior to their methods course.

PSTs' Self-Efficacy about Mathematical Writing

PSTs' mean responses across the four self-efficacy questions in MW were neutral on average prior to the module but moved in a positive direction toward an average of agree after the module, with medium to large effect sizes. Means and standard deviations are reported for each of the self-efficacy questions at pre- and post-survey as well as the effect sizes (see Table 1). The largest growth was seen for the self-efficacy questions with the "I know how to" stem as opposed to the "I feel confident" and "I can" stems, which suggests PSTs felt they improved the most in their pedagogical knowledge as opposed to their confidence in their ability to employ pedagogical skills once they learned them. The highest self-efficacy score at the post-survey was in relation to MW assessment, which aligns with the focus of the MW module. A paired sample t-test was used to evaluate growth in general self-efficacy. General self-efficacy in math writing increased from pre- to post-survey, which was a significant increase of 0.82, $t(112) = 14.34$, $p < .001$, for a large effect size of $d = 1.35$. Therefore, the MW module had a significant and large positive impact upon PSTs' general self-efficacy in MW instruction and assessment.

Table 1: Preservice Teachers' Self-Efficacy

Question	Pre M (SD)	Post M(SD)	Effect Size Cohen's *d*
I feel confident in my ability to teach math writing to the grade I currently teach.	0.26 (.86)	0.84 (.71)	0.69
I can effectively teach math writing.	0.21 (.60)	0.77 (.64)	0.84
I know how to teach math writing.	-0.31(.90)	0.69 (.73)	1.08
I know how to assess a student's math writing.	-0.12 (.94)	1.03 (.57)	1.23
General self-efficacy in mathematical writing	0.01 (.68)	0.83 (.57)	1.35

PSTs' Perceptions about the Uses and Benefits of Mathematical Writing

Although the PSTs expressed limited prior experiences related to MW, nearly all the PSTs reported that writing in mathematics is beneficial to students' learning. The PSTs also expressed they would ask their students to write in mathematics on a monthly, weekly, or daily basis. The differences between pre- and post-survey responses to this question were minimal, indicating the PSTs had already intended to incorporate this as a part of their practice. However, the PSTs conceptualization of what MW is appeared to have been rather limited at pre-test (e.g., writing out answers to word problems in complete sentences) and to have expanded at post-test (e.g., writing about multiple strategies or critiquing the ideas of other students).

When asked to identify MW activities for fourth-grade students, the PSTs consistently identified explaining a problem-solving procedure before and after completing the module. For instance, a PST stated students would be asked to write "A step by step explanation on how to do certain problems, mainly word problems and multiple step problems." The PSTs also identified other types of writing activities, but the focus of these shifted after the module. Before the module, the PSTs identified asking students to identify connections or applications of mathematics to their daily lives or write a story problem to a small degree. After the module, the PSTs identified asking students to write about their reasoning, justifications, or critique the reasoning of others to the same degree as writing explanations. The PSTs described such activities like, "I might have my students write about how they might approach a problem in contrast to another student" or "I would ask them to explain their answers, create problems, and explain how students in example problems may have made a mistake." Although the module focused on assessing students' 'W, it seemed to also expand PSTs' views of how MW could be used in instruction (i.e., beyond procedural descriptions).

At the completion of the module, the PSTs reflected on the evolution of their thinking about MW. To support this reflection, we offered the optional stem, "I used to think ____, but now I think ______" that PSTs could use in their responses. Nearly all the PSTs noted their thinking had changed and had a greater understanding of the importance and benefits of MW to students' learning. Although some PSTs spoke in broad terms, like "I used to not know much about mathematical writing, but now I think mathematical writing is a crucial aspect of math instruction", others spoke in specifics. For instance, one stated,

> I used to think that math writing was just the process of using words to relate math to the real world, but now I know there are many different types of math writing and levels. I think now that this process allows students to think deeply about course material and it's a good alternative to quizzing for understanding.

Whereas another PST said, "I used to think writing and math combined was silly and just extra work on students, but now I think it can benefit students in many ways including their understanding of mathematical concepts." In summary, while PSTs perceived value of MW was generally high even prior to the module, their perceptions of what MW is and its potential uses expanded.

Discussion and Conclusion

Like other studies have found (Gillespie et al., 2014; Powell et al., 2021; Swinson, 1992), the PSTs in this study experienced few opportunities to write in mathematics as a K-12 student and, when they did, it was often explanatory writing (e.g., write an explanation of procedure). Moreover, the PSTs had received minimal instruction in their teacher preparation program related to MW; yet they expressed intent to regularly incorporate MW into their future mathematics instruction and saw MW as critical to students' learning. However, PSTs grew in their self-efficacy in MW and expanded their conceptualizations of what MW entails in response to the module. Although these findings show promise, additional research is required to refine the module and examine its impact on actual classroom practices and student performance.

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DEVELOPING PRESERVICE TEACHERS' NOTICING OF EQUITABLE PRACTICES TO EMPOWER STUDENTS ENGAGING IN PRODUCTIVE STRUGGLE

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In our study, we used video analysis to examine how preservice elementary teachers (PSTs) noticed and described the equitable teaching practices that leverage students' multiple mathematical knowledge bases (MMKB) including prior mathematics knowledge, cultural, community, family, and linguistic knowledge and experiences; student interests; and peers, to support productive struggle. The PSTs (n=40) in their final mathematics content course analyzed a video episode of a classroom with a teacher and students engaged in productive struggle about a task. Frameworks of teacher noticing and productive struggle were incorporated in their assignment to guide the PSTs in their analysis. We report on the levels of connections PSTs made of the MMKB to the support and resolution of productive struggle.

Keywords: Pre-Service Teacher Education; Equity and Diversity; Teacher Noticing; Mathematical Knowledge for Teaching

Introduction

In this study, we focused prospective elementary and middle school teachers' (PSTs') attention to particular resources teachers use to support students in productive struggle. "Student resources included: prior mathematics knowledge; cultural, community, family, and linguistic knowledge and experiences; student interests; and peers as supports for learning," (Roth McDuffie et al., 2014, pp. 246-247), which is also referred to as students' multiple mathematical knowledge bases (MMKB). We used video clips showing classroom interactions between teachers and students to introduce the PSTs to how teachers draw upon and leverage resources and knowledge bases that students bring to the classroom and that support and empower students' mathematical understanding and productive struggle in inclusive settings (Lynch, Hunt, & Lewis, 2018; Santagata and Guarino, 2011). We are also informed by the Five equity-based practices to support mathematics learning as articulated in *Taking Actions* (NCTM, 2017), and focused the PSTs on their noticing of two practices in particular: 1) Leverage multiple mathematical competencies and 2) Draw on multiple resources of knowledge.

Theoretical Perspective and Related Literature

Studies suggest introducing frameworks to PSTs in aspects of teaching such as teacher noticing and productive struggle provide support and structure more effectively when discussing and analyzing important teaching practices (Warshauer, et al., 2019; Roth McDuffie et al., 2014; Santagata and Angelici, 2010; Stockero et al., 2017; Walkoe, 2015). The PSTs were introduced to the skills of noticing including the components of attending to, interpreting, and deciding on actions based on student thinking (Jacobs, Lamb, & Philipp, 2010). By learning to notice children's mathematical thinking and their cultural funds of knowledge, teachers can leverage these resources to better support students equitably and inclusively in their classrooms (Turner & Drake, 2016). The *Productive Struggle Framework* (Warshauer, 2015) provides a way of examining elements of a productive struggle episode from the initiation of a student struggle when students engaged in a task to the interaction between the teacher and student(s) with an action or response to support the students' struggle. In addition, we focused PSTs on ways that the resources were implemented by teachers in support of students' struggle and understanding. Studies on teacher noticing of equity suggest that teachers who notice equitably perceive mathematical and interpersonal activity as inextricably linked and notice individual student participation (van Es et al., 2017; Hand, 2012; Turner et al., 2012, Wager, 2014). Thus, including the use of resources and its connection to student struggles and teacher actions can inform how teaching can support student struggles productively or not.

Our research questions for this study are:

3. How do PSTs notice and describe the equitable teaching practice of leveraging students' MMKB (resources), to support student struggles and teacher actions during a video episode of students engaging in productive struggle?
4. How do PSTs interpret the role resources play in equitably supporting the productive level of students' mathematical struggle, as viewed in a video episode of a class discussion?

Methodology

This study was part of a larger study conducted in 2017 at a public, four-year, Hispanic Serving Institution in a rural area in the Western United States over a 14-week period. The original study consisted of 40 PSTs enrolled in one of two sections of their final mathematics content course for elementary teachers, taught by the same instructor. The PSTs completed three productive struggle writing assignments (WA), in which they reflected on a video episode of productive struggle with the use of resources. The PSTs were asked to connect the mathematical content of the video to the components above and decide how productive the struggle was for the student(s). The writing assignments provided opportunities for the PSTs to apply and develop their understanding of the mathematics and student's mathematical thinking from the content course. These assignments also served as approximations of practice to initiate PSTs learning key practices for engaging with children's multiple mathematical knowledge bases (Turner et al., 2011).

We coded all 40 WAs using qualitative content analysis (Hsieh & Shannon, 2005). The four researchers agreed on expert codes and the WAs were coded inductively to compare and identify themes in the PSTs' responses based on the MMKB. A sample of the WAs were analyzed by all researchers to ensure inter-rater agreement. This study focused on WA3. The video used for WA3, which we refer to as the *Equality* video, includes a dynamic class discussion about the concept of equality. In the video, the students demonstrate a misconception with regard to the equal sign. The sequence of events builds on multiple students' perspectives, actions, knowledge bases and misconceptions, as well as, the interpretations of the teacher's actions and practices she implemented to support student learning. Based on open coding, we developed our coding scheme and analyzed at levels 1, 2, and 3 of noticing of resources. At level 1, the PST only identified the resource, at level 2, the PST noticed how the resource to support learning in general, and at level 3, the PST noticed how the teacher leveraged a resource to support students engaging in productive struggle. Level 3 students particularly identified a student struggle, student interaction, teacher action, or mathematical interpretation of the struggle. The four researchers inductively coded all WAs focusing on the PSTs' noticing and description of the resources leveraged to support student struggles and teacher actions.

Findings

The PSTs noticed that the teacher drew upon students' mathematical knowledge, linguistic knowledge, and peers to support students' math understanding to solve the equation $8 + 4 = \_ + 5$. In addition to noticing how the teacher leveraged the students' mathematical thinking and funds of knowledge, PSTs identified visual aids and manipulatives as resources the teacher utilized to support students learning. One PST identified non-verbal communication (gestures) being used as a resource.

Table 1: Resources noticed by PSTs

	Mathematical Knowledge	Translanguaging	Peers	Visual Aids	Non-verbal Communication
Number of PSTs	27	25	28	8	1

Mathematical Knowledge Resource

Of the 27 PSTs who noticed the teacher leveraging the students' prior mathematical knowledge, two PSTs only identified without elaboration the teacher's use of this knowledge base. Ten PSTs discussed scaffolding problems that build upon prior concepts as a general, good teacher practice that supports students in understanding new ideas. Their analysis was generalizable to a general learning situation, whereas, 15 PSTs identified the specific mathematical concepts the teacher tapped into, such as prior knowledge of addition, equations and what an equal sign means. Additionally, the PSTs linked how specific problems were intended to address students' misconceptions, thus connecting the teacher action, prior mathematical knowledge and the mathematical struggle, as seen in this PST's observation:

> She uses what they already know about equations to push their thinking into how they could solve the problems on the board. The teacher writes $5 = 5$ to set a foundation for the fact that a number equal to itself is true. This helps the teacher to demonstrate that $5 = 4 + 1$. The students are able to use their newly gained knowledge to solve all the more complicated problems written on the board. (PST1240, Level 3)

Linguistic Knowledge Base

Twenty-five PSTs noticed the teacher drawing upon students' linguistic knowledge bases by translanguaging between Spanish and English. Four PSTs with a level 3 analysis connected the teacher's action of translanguaging to a particular mathematical struggle. These PSTs described how translanguaging engaged more students in the mathematical struggle, helped build a community of learners, familiarized students with math terms in English, and was an indication of the teacher's knowledge of each of her students and their needs. One PST discussed how peers (another resource) translanguaged amongst themselves to make sense of the equation. Nineteen PSTs with level 2 analysis discussed translanguaging as a general teacher action to support learning but did not connect translanguaging to any particular student struggle or student thinking. Three of these PSTs connected linguistic and peer resources and one PST with a level 2 analysis gave a negative evaluation of the teacher translanguaging in the video.

Peers as a Support for Learning

Of the 28 PSTs who noticed the teacher using peers as a resource, five did so with a level 3 analysis and 23 with a level 2. The PSTs with a level 3 analysis connected the use of peers to the teacher actions and the mathematical struggle of the students. The PSTs described the use of peers as a way for students to discuss their understanding of the problem and to learn from each other. The 22 PSTs with level 2 analysis connected peers as a resource to teacher actions as a general classroom learning tool. One PST discussed classroom community and another discussed zone of proximal development, which we categorized as a peers resource. These PSTs did not connect peers to the mathematical struggle or student thinking.

Findings Related to Resolution of Productive Struggle

Our second research question examined how PSTs interpreted the role student resources played in the productive level of students' mathematical struggle, as viewed in a video episode of a class discussion. Our analysis showed that 33 of the 40 PSTs agreed that the student struggles observed in the video episode resolved productively. Eight of the PSTs described individual students' struggles as being productive at a low level (6) or not productive (2). Two of these PSTs had also indicated a class resolution at a productive level. One PST did not state a resolution. Of the 37 PSTs who viewed the episode as productive or productive at a low level, 22 mentioned specific resource(s) that were leveraged in support of productive struggle while 15 made no mention of any resources tied to the resolution. Resources most often mentioned were students' mathematical knowledge (11) and peers (14). While mentioned by only six PSTs, PST1250 connected the translanguaging resource used to support the productive struggle of student S who, "showed a high level of productive struggle because even though she stumbled to say the answer to one of the problems, she did give the correct answer in Spanish." PST1221 noticed that both translanguaging and peers as a teacher action were being used to support the students' struggle, "The teacher was constantly asking questions [direct and probing guidance], responding and translating in Spanish to help students better understand the content. I thought it was great that when she saw the students begin to struggle she allowed them to speak in small groups …". PST1156 attributed three resources and the teacher's knowledge of their students that appeared to support productive struggle, "The students were able to struggle with a math problem and the teacher was able to scaffold their understanding using their previous knowledge, speak their language, and draw upon peer discussion to help the students understand equality."

Conclusion and Implications

Our analysis suggests PSTs are able to notice equitable teaching practices and resources and use their observations to inform their determinations of the productiveness of students' struggles, but do not concretely develop *how* equitable practices are implemented to support productive struggle. The PSTs overall made deeper connections about students' prior mathematical knowledge than peer and linguistic knowledge, with 15 PSTs connecting specific examples used to leverage students' previous mathematical knowledge. This discrepancy may stem from this video analysis occurring in a mathematical content course that places an emphasis on developing PSTs' understanding of students' mathematical thinking. Perhaps providing more context about the students in the video, i.e. if they are English Learners and if their proficiency level is emerging, expanding or bridging, would encourage PSTs to interpret more non-mathematically-based resources such as linguistic or peers. Providing them with authentic student information, student work and interactions may be key to help PSTs connect equity-based practices to effective practices.

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DISCIPLINARY CONTENT KNOWLEDGE ABOUT SOLVING PROBLEMS WHERE THE FRACTION ACTS AS OPERATOR

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This research tries to investigate the disciplinary knowledge of recently qualified teachers of Primary Education (PE), in its different mentions, about the fraction when it acts as an operator. For this purpose, a test was carried out in which two problems had to be solved, both arithmetically and graphically. The common characteristic of the problems is the fraction as an operator, on a natural number, and on a fractional number. The results reflect higher levels of success when the fraction operated on a natural number, and, in general terms, the graphical solution was more complex than the arithmetic one. As for the different mentions, the best results were obtained by mathematics teachers.

Keywords: Problem Solving, Fraction, Disciplinary Content Knowledge, Teacher Education

Introduction

However, at present, the training of mathematics teachers is still considered a topic of latent importance within mathematics education. This fact is determined by: a) unfavorable results in the development of cognitive abilities of students in national and international exams, and b) dissatisfaction of teachers in carrying out their work, together with the different curricular reforms that require a certain renewal of teachers (Godino, Batanero and Flores, 1999).

The question is that, faced with this panorama, the idea persists as to whether it is possible to do something more in initial teacher training in order to develop specific professional knowledge (of mathematics, in this case). The fact is that, as Lappan and Theule-Lubienski (1992, cited in Godino et al. 1999) state, an exclusively mathematical or psycho-pedagogical teacher training, of a generalist nature, does not seem to be sufficient given the cognitive and didactic complexity of specific mathematical concepts and methods.

Within this framework, this paper seeks to investigate the knowledge that recently qualified primary school teachers have about a specific mathematical concept, the fraction as an operator, in a problem-solving context.

Many authors highlight the importance of teaching fractions through problem solving (Llinares, 2003), however, a large part of the studies in this area show that the concept of the fraction as an operator, as well as its application, is not fully understood or mastered. In their day, authors such as Kieren (1976) and Freudenthal (1983) alluded to this idea, which is still present in works such as that of Egodawatte (2011), where they detect that students make mistakes about the whole on which the fraction acts, or even in teachers, as in Livas (2004), who studied the mathematical conceptions of eight PE teachers in relation to the fraction as an operator, with the result that their knowledge of the fraction is incomplete and, therefore, the teaching they can give about it will be deficient.

Purpose of the work (Objectives)

In the present study, the aim was to investigate the skills and disciplinary knowledge of PE teachers (with different mentions) about fractions as an operator in problem solving. In this way, the conjectures that are raised are:

1. The majority of PE teachers do not have enough level of knowledge of fractions, with Science and Mathematics (SM) teachers standing out as having a higher level of knowledge of the content.
2. The greatest success will be obtained when the operator acts on a natural number because it involves a simpler mathematical content, with SM teachers obtaining the best results.

Method

Sample

Forty teachers (16 men and 24 women) recently graduated from a primary school teaching degree at a Spanish Public University participated. Respondents ranged in age from 22 to 26 years, with a typical age of 22 years. The sample was a non-probabilistic sample of convenience and included participants from different mentions, with 13/40 being teachers of Science and Mathematics.

Instrument

In order to address the two objectives outlined in this paper, one battery of questions is required. It is a pencil and paper questionnaire on solving two problems; one where the fraction acts on a natural number (P1) and the other where the fraction acts on a fractional number (P2).

> *P1. A well with 20 liters of water was emptied three-fifths (3/5) parts to water the plants. How many liters have been emptied?*
>
> *P2. Half (1/2) of a well is full of water. If we empty one third (1/3) for consumption, how much of the initial amount of water has been emptied?*

The teachers' answers are coded in terms of the study variables according to the value of 0 (incorrect answer) or 1 (correct answer) and through the distribution of the answers in different variables established as follows: arithmetic and graphic resolution of a natural number (P1A and P1G, respectively), arithmetic and graphic resolution of a fractional number (P2A and P2G, respectively) or the mastery of both resolution methods in each of the cases. The aim of this tool is to explore possible difficulties of teachers with this disciplinary content, considering the two methods of resolution (arithmetic or graphical) and the two types of context depending on the problem (natural or fractional).

The analysis of the results will be a) descriptive, where the frequency of each of the coding values for each of the variables will be observed, and b) inferential, where the relationship between the different pairs of these variables will be studied according to the proposed objective. For this purpose, Fisher's test (TF), Cramer's V (VC) and Proportions Test (TP) will be used.

Results

Problem Solving Questionnaire

For P1, where the fraction acts on a natural number, all teachers answer (or try to) both ways. More successful results were recorded for arithmetic resolution (38/40) than for graphical resolution (28/40). Of the 28 who do well in graphical solving, none do poorly in arithmetic solving. This indicates that, out of 40 respondents, only 28 have solved P1 correctly. See Figure 1 for more detail.

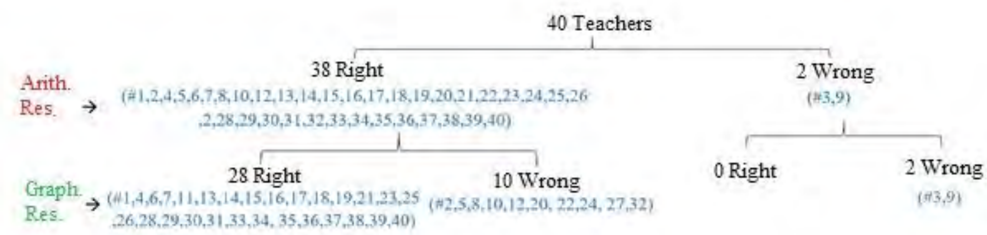


Figure 1: Analysis of results of Q1 of questionnaire 1

For the mentions, in P1 we obtain that all SM teachers have good arithmetic resolution, but 5 have bad graphic resolution. On the other hand, in the rest of the mentions, 2/27 and 7/27 have bad arithmetic resolution and graphic resolution, respectively.

Figure 2 shows the results obtained with P2. It can be seen that all teachers respond to both resolution, with more successful results in arithmetic resolution (29/40) than in graphic resolution (11/40). Of the 11 who do well in graphical resolution, none do poorly in arithmetic resolution. This indicates that, out of 40 respondents, only 11 have correctly solved P2.

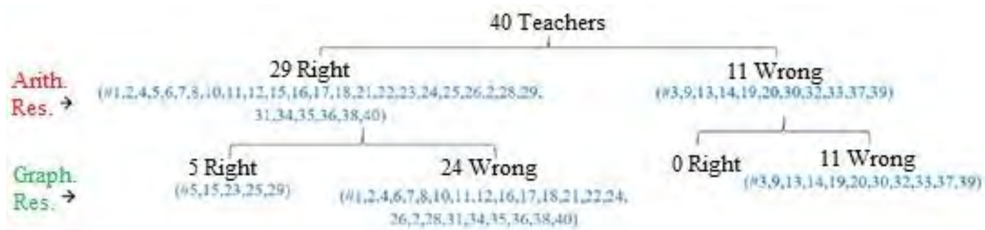
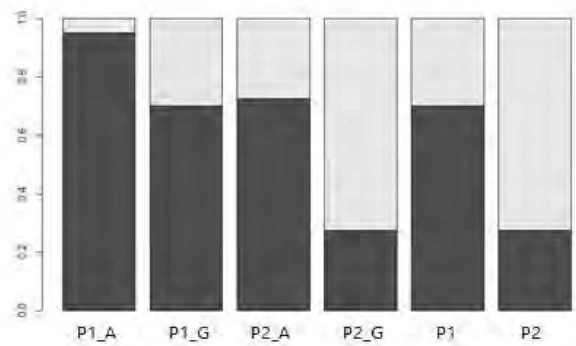


Figure 2: Analysis of results of P2 of questionnaire 1

On analyzing this P2 by mentions, we find that 2 of 13 SM teachers have poor arithmetic resolution and 7 have poor graphic resolution. On the other hand, of the 27 teachers of the other mentions, a) 9 have poor arithmetic resolution, and b)13 have poor graphic resolution.

The comparative analysis between the success rates is shown in Figure 3a). The numerical results (Figure 3b) reflect a median association (VC<=0.45) and differences between the successes in solving the problems (p-value<=0.015).



a)

	TF (p-value)	VC	TP (p-value)
P1_A vs P1_G	8.658(0.006)	0.329	7.013(0.008)
P2_A vs P2_G	16.797(<0.001)	0.45	4.45(<0.001)
P1_A vs P2_A	7.440(0.013)	0.305	5.878(0.015)
P1_G vs P2_G	14.459(<0.001)	0.425	12.808(<0.001)
P1 vs P2	14.459(<0.001)	0.425	12.808(<0.001)

b)

Figure 3: (a) Success rates by problem and type of resolution; (b) association and difference in rates

If we consider the different mentions, there are no statistically significant differences. However, there are significant differences within the mention itself, depending on cases: a) for SM teachers, P1 in the type resolution, arithmetic or graphic (TF=4.452, p=0.040; VC=0.488 > 0.3; TP= 3.962, p=0. 046); b) for non-SM, in P1 differences in the type of arithmetic or graphical resolution, (TF=8.761, p<0.001; VC=0.487 > 0.3; TP= 10.906, p<0.001); and c) for non-SM, in the resolution of P1 and P2, (TF=9.463, p<0.001; VC=0.557> 0.5; TP= 14.599, p<0.001).

Discussion and conclusion

If we recapitulate the different contributions of our research, we can see that with respect to the experimental PAEVs, there are two main elements that generate relevant differences in the final consideration of the results. By this we refer, on the one hand, to the quantity or whole on which the fraction acts as an operator, depending on whether it is a natural (discrete) or rational (continuous) number and, on the other hand, to the method of resolution requested, depending on whether it is arithmetic or graphical.

The results of this research point to a record of better grades when the fraction operates on a natural number, and not on another fraction. This fact may be obvious because it can be justified by the fact that natural numbers are a simpler content and introduced in school education earlier. On the other hand, Ríos (2007) points out that, in the teaching of fractions, the most dominant context is the discrete one, since the part-whole interpretation usually taught through graphical representations of discrete objects or quantities tends to predominate. Moreover, a recent study (Sanz et al., 2020) on the relationship between complexity -measured through the reading comprehension of the statement itself- and success in solving problems with the same characteristics as those present in this study, shows that they are related, with those whose whole is a fraction being more complex.

Considering each resolution method, the majority of participants obtain by far a higher level of success in arithmetic resolution than in graphical resolution. One of the important references is the role played by the curricular materials based on the provisions of Royal Decree 126/2014 of 28 February, since, given that in this document graphical representation is included as one of the key contents in relation to fractions up to 3rd year of Primary School, subsequently, this type of representation takes second place, being replaced by the prioritization of the arithmetical treatment of operations with fractions.

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PROSPECTIVE MATHEMATICS TEACHERS' GEOMETRIC DEFINITIONS AND CONCEPTIONS ABOUT PROPERTIES OF TWO-DIMENSIONAL SHAPES

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This qualitative study sought to investigate the understanding of middle grades prospective mathematics teachers' (PMTs) geometric definitions and conceptions about properties of two-dimensional shapes. Hierarchical classification systems for two-dimensional shapes created by 18 small groups of PMTs (n=52) were analyzed. Preliminary findings document inconsistencies for the definitions of kite and trapezoid. Implications concern *the role participants' concept images and their experiences using dynamic geometry*.

Keywords: Geometry and Spatial Reasoning; Instructional Activities and Practices; Preservice Teacher Education

Purpose of the Study

Although children typically learn about shapes prior to attending kindergarten, teachers play a significant role helping them form definitions of and relationships between two-dimensional shapes (Türnüklü et al., 2013). It has been documented that students experience difficulties with identifying quadrilaterals (Currie & Pegg, 1998; de Villers, 1998; Pratt & Davison, 2003; Vinner, 1991; Zaslavsky & Shir, 2005) and their hierarchical classification (Erez & Yerushalmy, 2006; Fujita, 2012; Fujita & Jones, 2007; Monaghan, 2000; Okazaki & Fujita, 2007; Pickreign, 2007). Researchers report that prospective mathematics teachers (PMTs) also have difficulty identifying types of shapes based on properties, rather than visual recognition (Burger & Shaughnessy, 1986; Türnüklü et al., 2013), and struggle to perceive properties of figures, notice relationships among figures, or understand class inclusion (McCammon, 2018; Ozdemir Erdogan & Dur, 2014). These limited understandings about shapes were likely later embedded in their own teaching, possibly causing their students to construct limited conceptions about geometric properties of shapes. Therefore, it is critical that teachers have clear understandings about properties of shapes in order to assist their students in constructing these understandings. Using proactive dragging (Hollebrands, 2007) and examining non-prototypical shape examples with dynamic geometry software (DGS), PMTs have the opportunity to deepen their concept definitions and form relationships between classes of shape. The purpose of this qualitative research study was twofold: (1) to investigate the understanding of middle grades PMTs' geometric definitions and conceptions about properties of two-dimensional shapes; and (2) to analyze the geometric language and reasoning PMTs employ after participating in dynamic geometry activities to explore definitions of and relationships between classes of shape.

Theoretical Frameworks

Two theoretical constructs guided this study. The first was Tall and Vinner's (1981) concept definition and concept image. The concept definition is "a form of words used to specify that concept" (Tall & Vinner, 1981, p. 152). The concept image is "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes … built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures" (Tall & Vinner, 1981, p. 152). Cunningham and Roberts (2010) documented that PMTs demonstrate inconsistencies between their geometry concept images and concept definitions.

The second theoretical framework was the van Hiele levels of geometric thinking (van Hiele, 1985). The van Hiele levels are commonly used to describe students' development of geometric reasoning. In general, level 0 students rely on visual prototypes to identify polygons, level 1 students use properties to describe shapes, and level 2 students use properties to describe relationships between classes of shapes. Therefore, students at level 0 use visual strategies to describe individual shapes, students at level 1 use attribute strategies to describe a class of shapes, and students at level 2 use property strategies to describe inclusive relationships between classes of shapes. These levels were used to characterize the language PMTs employed during a Hierarchical Classification Diagram Activity (see Figure 1), and provided information about each PMT's geometric reasoning about definitions and conceptions of shapes and their properties. This framework guided the data collection and analysis. Together, these frameworks provided a lens to examine the understanding of middle grades PMTs' geometric definitions and conceptions about properties of two-dimensional shapes.

Activity: Using the provided paper and pen, organize the following geometric shapes in a hierarchical classification diagram.

Circles, equilateral triangles, hexagons, isosceles trapezoids, isosceles triangles, kites, octagons, parallelogram, pentagons, polygons, quadrilaterals, rectangles, rhombi, scalene triangles, squares, trapezoids, triangles

Figure 1: Hierarchical Classification Diagram Activity (adapted Sowder et al., 2017, p. 410)

Methods

This study investigated the understanding of middle grades PMTs' geometric definitions and conceptions about properties of two-dimensional shapes. This study was enacted in two sections of a Geometry for Teachers mathematics content course with 52 PMTs seeking licensure to teach mathematics in grades 4-9 at two Midwestern universities. While the sections were taught by two different professors/researchers, the study was enacted similarly for both sections. The data collected at the beginning of the course (and part of a larger study) included: a van Hiele Geometry pre-assessment (Usiskin, 1982); a researcher-developed quadrilateral pre-assessment with adapted research tasks from Burger and Shaughnessy (1986) and Razel and Eylon (1991); and a Geometry Beliefs Survey (Utley, 2007).

Over a period of 4.5 weeks, topics from chapters 16 & 17 in *Reconceptualizing Mathematics for Elementary School Teachers* (Sowder et al., 2017) were discussed. During this time, the researchers shared the same class activities and homework problems to ensure both sections were introduced to the same mathematical concepts and topics. After completing in-class activities using DGS, the PMTs were asked to organize shapes (e.g., circles, various types of triangles and quadrilaterals, pentagons, hexagons) in a hierarchical classification diagram, in groups of 2-3 students (see Figure 1). The small-group activity was recorded using a Livescribe™ smartpen, recording both the PMTs' written work and their audio discussion. The PMTs individually completed a researcher-constructed quadrilateral post-assessment and the Geometry Beliefs Survey (Utley, 2007) seven weeks after the hierarchy group assessment task.

The van Hiele (1985) levels of geometric reasoning were used as a framework to assess each PMT's mathematical responses to the hierarchical classification diagram small-group activity. Using these levels to analyze responses provided information about each PMT's geometric reasoning about definitions and conceptions of shapes and their properties and helped document any changes that occurred in each PMT's understanding of shape properties.

Results

More results and implications will be shared during our presentation. Here, we present an overview of available findings to the first research question, what are PMTs' definitions and conceptions about geometric properties of shapes?

The 52 PMTs were placed into groups of 2-3 students to create hierarchical classification diagrams, resulting in 18 total diagrams (see Figure 2). Of the 18, there were 14 unique hierarchical classification diagrams for the set of quadrilaterals. It was common for PMTs to begin with the set of quadrilaterals at the top of the diagram. Many groups made single branches from quadrilaterals to kites (n = 16), trapezoids (n = 13), and parallelograms (n = 14), but only nine included all three single branches. Although one group placed trapezoids as a branch under polygons and did not connect trapezoids to any other set of quadrilaterals (i.e., parallelograms, rhombuses, etc.), all 18 groups included isosceles trapezoids as a subset of trapezoids. Only two groups permitted rectangles to be a subset of isosceles trapezoids. These two groups also identified squares as rectangles; therefore, for them, squares were also isosceles trapezoids. One other group identified squares as isosceles trapezoids.

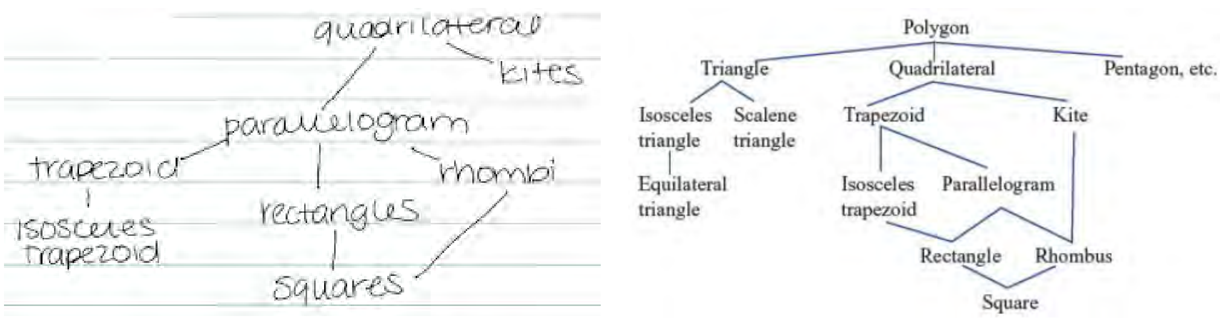

Figure 2: Sample Hierarchy (left) and Inclusive Hierarchical Classification Diagram (Sowder et al., 2017, p. 410) (right)

Zero groups placed parallelograms below trapezoids in the hierarchical classification diagrams, yet four groups did the reverse and placed trapezoids below parallelograms (e.g., see Figure 2 - –eft). Two groups correctly placed rhombuses underneath trapezoids, but they did not connect rhombuses to parallelograms. Four other groups drew a branch from trapezoids down to squares, parallelograms down to rectangles, and rectangles down to squares. Twelve groups drew two different branches from parallelograms down to rectangles and rhombuses; they also drew one branch each from rectangles and rhombuses down to squares. Four other groups drew a branch from rectangles to squares; therefore, 16 of 18 groups correctly identified squares as a subset of rectangles.

As for the hierarchical classification diagram of kites, seven groups drew a branch from quadrilaterals to kites and did not connect kites to any other shapes (e.g., see Figure 2 - left). Four groups drew a branch from kites down to rhombuses and then a branch from rhombuses to squares. These groups also drew a separate branch from kites down to squares even though they had connected rhombuses to squares. Four different groups placed squares underneath kites and separately placed squares underneath rhombuses; therefore, they did not link rhombuses to kites. One group placed rhombuses beneath kites but did not connect rhombuses to squares. Two groups correctly identified rhombuses as kites and squares as rhombuses.

Discussion

No groups correctly completed the hierarchical classification diagram, which is evidence that no group was working together completely at the van Hiele level 2. For all PMTs, kite was a new shape, and the results indicate that some students constructed the inclusive definition of kite which includes both rhombuses and squares as kites. This is evidence of van Hiele level 2 geometric thinking since they were using properties to analyze classes of shapes. Some groups demonstrated that they were still working with developing the relationship between kites, rhombuses, and squares or between classes of shapes. These groups connected kites to rhombuses and rhombuses to squares, yet also connected kites directly to squares; or the group connected kites to rhombuses but not rhombuses to squares. Many groups, however, were not thinking about the interrelationships among kites, rhombuses, and squares when creating their hierarchies.

Creager and Zeybek (2018) recommend that students have opportunities to evaluate alternate definitions of shapes to “help students make connections between terms they see as distinct, and to see distinctions between terms that are similar” (p. 258). Zaslavsky and Shir (2005) found that asking students to consider a variety of definitions is a powerful learning environment wherein personal concept definitions could be gradually refined along with conceptions of definition in general. The results of this study suggest that this refinement is definitely gradual since zero groups identified parallelograms as a subset of trapezoids. PMTs’ concept image (Tall & Vinner, 1981) of a trapezoid was the exclusive definition of a trapezoid with exactly one set of parallel sides, and the results indicate that they did not transition to the inclusive definition of a trapezoid with at least one set of parallel sides. There was some evidence that PMTs were making this gradual transition since four groups placed trapezoids below parallelograms in the hierarchical classification diagram and two groups placed rhombuses underneath trapezoids. Furthermore, four groups drew a branch directly from trapezoids to squares, although bypassing parallelograms and rectangles. These students were demonstrating both van Hiele level 1 geometric level of thinking, as they used properties of shapes to describe their location in the diagram and van Hiele level 2 geometric thinking as they used properties to describe relationships between classes of shapes. Since all 18 groups included isosceles trapezoids as a subset of trapezoids, it is evident that they had a correct definition of what it means for a set to be a subset. This is further evidence that not allowing parallelograms to be a subset of trapezoids directly relates to their difficulty with transitioning to a new definition for trapezoids. These results align with Bharaj and Francis’s (2020) results who found that preservice elementary teachers’ “reasonings seemed to be fixed towards certain images rather than mental manipulation of the attributes and concept definitions” (p. 727). In addition, these results align with implications from Miller (2019) who suggests that “the relationship between the complex collection of attributes which are true about a particular shape family and the concise concept definition which names that shape family is not always obvious to learners” (p. 429). Examining alternative definitions (e.g., the inclusive definition of trapezoid, a quadrilateral with *at least* one pair of parallel sides) provides students “with an opportunity to work in an authentic mathematical environment, and see that some, but not all, properties meet necessary and sufficient conditions" (Creager & Zeybek, 2018, p. 258). This suggests that it is extremely important for students to experience activities that include the inclusive definition of trapezoids during their PK-12 experiences.

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THE DEVELOPMENT OF PRESERVICE TEACHERS' THINKING ABOUT ENGAGING CHILDREN IN MATHEMATICAL ARGUMENT

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This study examined the development of 41 preservice elementary teachers' understanding of what it takes to engage children in mathematical argument. Findings indicate that when preservice teachers are provided opportunities to struggle with generating and justifying claims themselves, they more effectively recognize the value in having children discuss the same ideas. In addition, their characterization of argumentation becomes more honed when they are provided multiple opportunities to analyze teaching (their own and others') to discern when children's productive struggle is being undermined or utilized to enhance understanding.

Keywords: Pre-Service Teacher Education, Elementary School Education

Many studies have established that argumentation is an important component of developing mathematically proficient students simply because it requires higher levels of reasoning (e.g., Forman et al., 1998) and encourages productive struggle with valuable outcomes (Jager, 2017). Mathematics educators have contended that encouraging children to engage in argumentation helps them develop the willingness to try out ideas without prejudice and to share work that might be wrong (Russell et al., 2017). Therefore, many policy documents encourage teachers to create opportunities where children construct and critique mathematical argument (NGA and CCSSO, 2010; NCTM, 2014, 2020).

The call for increased student participation in argumentation provides a particular challenge for mathematics teacher educators working with preservice teachers (PTs). It is likely that most PTs have had little experience constructing arguments in mathematics class themselves, or if they have, are not aware of how to engage children in this work. Therefore, we designed a series of activities for our elementary mathematics methods course to engage PTs in considering how to encourage mathematical argument with children. The purpose was to determine 1) how preservice teachers characterize children's full engagement in argumentation; and 2) to what extent do deliberately-designed activities in a methods course help develop and strengthen that characterization.

Theoretical Framework

Researchers have found that PTs do not have a robust understanding of what is entailed in mathematical argumentation nor a vision of how to enact argumentation in (e.g., Kosko et al., 2014; Wagner et al., 2014). They have identified the following challenges for PTs: recognizing the value of extensive work on articulation of claims, distinguishing between specific and general claims, and recognizing valid justifications for those claims.

In their study with secondary preservice teachers, Wagner et al. (2014) describe an instructional sequence used in their methods course to develop PTs' understanding of argumentation. They began by engaging the PTs in geometry tasks to develop their characterizations of argumentation as involving claims (what is being argued for), referents, and a narrative link that describes how the referents support the claim. They followed this with observations and discussions of field experiences where they had to identify and reflect upon one episode of argumentation. Wagner et al. found that the PTs' characterizations of argumentation became more robust as a result of the sequence of activities they experienced.

Russell et al. (2017) provide a framework to engage students in argumentation that is focused on the elementary school level. They suggest that teachers should provide children opportunities to (1) notice patterns or regularities in problems; (2) articulate claims about the regularities they discover; (3) use representations to justify why the regularity must be true for specific cases; and (4) generalize for any case. The teachers they worked with found this sequence helped to elevate classroom discussions beyond typical sharing of strategies; children deeply considered the generalizability of strategies. Furthermore, productive struggle came from encouraging children to wrestle with wording in the articulation and justification phases as it takes time to develop solid arguments that are convincing to classmates.

As a result of these studies, we designed activities for the methods course that would move the PTs through a three-part instructional sequence beginning with doing mathematical work themselves, including making claims based on generalizations and justifying those claims in order to convince their classmates they were true. Next, we asked them to analyze videos of classroom discourse and discuss what they noticed using the argumentation framework by Russell et al. (2017). Finally, they taught lessons to small groups of children where they focused on encouraging argumentation using that same argumentation framework.

Methodology

Forty-one preservice elementary teachers from a large Midwestern university participated in this study. Students were either in their third or fourth year of college and were enrolled in a methods course that addressed the teaching of elementary school mathematics. In the beginning of the semester, the PTs analyzed a video of an elementary school classroom where children were engaged in argument in order to provide baseline data on their thinking. Then throughout the rest of the semester, PTs often worked on problems, creating and discussing claims and justifications, followed with analysis of videos of children working on similar problems.

This paper focuses on one three-part sequence about subtraction equivalence. This idea was taken from Russell et al. (2017) where they used subtraction equivalence to effectively engage second graders in argument. For their own work, the PTs were asked to generate three subtraction expressions that all had the same answer, produce a claim based on what they noticed about the patterns, and provide a representation-based justification for why they thought this was happening with subtraction. These ideas were discussed and the class collectively revised their claims and justifications. This was followed by an analysis of a video clip where second-grade students worked on the same idea using a story context. The PTs completed a writing reflection examining the children's claims related to the specific story context along with their attempts at generalizing their claims. The last step in the sequence involved a field experience where PTs worked with small groups of children on the same subtraction equivalence idea and encouraged the children to engage in argument. The PTs summarized with a written analysis of the children's claims and justifications as well as their own reflection on their teaching.

The aforementioned challenges identified from research (articulation of claims, specific and general claims, and valid justifications) informed the first level of data coding and analysis across all three parts of the instructional sequence. The PTs' own generated subtraction expressions and claims were sorted into categories according to the nature of their claims and the extent to which they were able to generalize their claims. The written reflections on the elementary classroom video and the field experience were analyzed for identification of specific and general claims and assessment of thoroughness and clarity of articulation of those claims.

Results

PTs' Initial Characterization of Argument

Early in the semester, the PTs discussed the meaning of CCSSM Standard of Mathematical Practice 3 (which focuses on argument) and were asked to use this practice to analyze a video of elementary children discussing a story problem that had more than one solution. As the children discussed possible solutions, they began to provide brief claims about what counts as a solution. However, they were not encouraged to justify their claims and the teacher intervened to resolve the issue herself. On their written reflections, the PTs were asked to use Mathematical Practice 3 as a lens to discuss the extent to which the teacher encouraged children to engage in argument about the solutions. Despite the teacher's intervention that circumvented the children's opportunity to justify their claims, about two-thirds of the PTs stated that they believed the students were fully engaged in argument. This confirmed the findings of Wagner et al. (2014) that a deliberate sequence of activities with the PTs would be needed to develop their understanding of effectively engaging children in argument.

First Activity in Sequence: PTs' Own Work on Argument

To begin our work on subtraction equivalence, the PTs were asked to generate three equivalent expressions for subtraction, (e.g., 22 – 8, 20 – 6, and 24 – 10). They were then asked to list what they noticed about their three expressions and write a claim about subtraction equivalence. While we did not expect an algebraic expression such as (a + x) – (b + x) = a - – or (a – x) – (b – x) = a - –, we considered an articulation of this relationship to be something like the following: "If you increase or decrease the minuend and subtrahend by the same amount you will get the same answer." Results from the PTs' work are shown in Table 1.

Table 1
PTs Claim Statements About Subtraction Equivalence (n=41)

Category of Claim	% Student Response
Correct Generalized Claim (Both Increasing/Decreasing)	37%
Generalized Claim (Only Increasing or Decreasing)	17%
Partially Generalized Claim using Specific Examples	15%
Other Correct Claims Not Addressing Equivalence	24%
Incorrect Claims/No Response	7%

As expected, this articulation was challenging for the PTs. It was interesting that 17% were close to a generalization, but only referenced either increasing or decreasing seemingly depending on the examples they generated. The partially generalized claims were also interesting as they highlighted the challenge PTs had in navigating between specific examples and generalizations. These PTs tended to use "number" to represent any starting number but used specific amounts for increases or decreases. For example, one PT said, "As you increase one number by 2, you have to increase the other number by 2." While promising, these findings showed that doing the mathematical work themselves was not sufficient for helping PTs fully appreciate distinctions between specific and generalized claims.

Second Activity in Sequence: PTs' Analysis of Classroom Video

After these ideas were discussed as a class so that students could come to an understanding of the fully generalized claim, they analyzed a video from Russell et al. (2017) of second-graders working on subtraction equivalence. The second graders worked on the following problem in pairs: *Alyssa had 15 cookies but she needs only 12, so Kussita ate 3. If Alyssa gets more cookies, how many cookies could Kussita eat?*, created representations, and the teacher facilitated a whole-class discussion of their findings. They discussed 16 cookies/4 eaten, 17 cookies/5 eaten, and 18 cookies/6 eaten. Then the teacher asked, "Okay, so what happens if we add any amount of cookies? What do we have to do so that Alyssa still will only have 12?"

Two children responded to this prompt with specific claims while the third student had a partially generalized claim using specific examples but then finally saying "As the numbers go up, the second number is going up too." The PTs were asked to analyze the three student contributions and consider the extent to which they were answering the teacher's question about what would happen if any amount of cookies were added. Almost all of the PTs were able to recognize that none of the students had articulated a satisfactory generalization of equivalence. However, only two-thirds were able to identify the student who provided the partially-generalized claim as being closest to articulating an accurate generalization.

In addition, the PTs reflections on what might be needed for the students to develop an accurate generalization highlighted two themes. One common misconception was that the PTs believed the students needed more work with specific examples in order to articulate the generalized claim. Another common misconception was that providing a context like Kussita's cookies would hinder students' ability to generalize, a direct contrast to the argument by Russell et al. (2017) of the importance of children using representation and context as a vehicle for proof.

Third Activity in Sequence: PTs Field Experience

While it was clear that the PTs were inching towards a more robust understanding of argument, we believed that working with children themselves would provide new opportunities to address expectations around both articulation and justification. Thus, we designed a field component where PTs worked with children to develop arguments for both addition and subtraction equivalence. Due to space constraints, we are not able provide detailed findings from the field experience. In general, on their written reflections the PTs did show a more fervent appreciation of the effort needed to articulate claims. They also demonstrated a deeper understanding of the distinctions between specific and generalized claims and the appropriateness of different representation-based justifications.

Discussion and Conclusions

This analysis has provided a beginning picture of the knowledge PTs bring to and the issues they have with understanding the important components of mathematical argumentation. Our intervention using the three-part sequence (PTs create and discuss mathematical claims, watch classroom video of children discussing claims, and then work with small groups of children on claims) encouraged more nuanced understandings, but also illuminated further challenges PTs have with understanding how to encourage children to use representation and context as a vehicle for articulating generalization.

In concurrence with other research (e.g., Borko et al., 2011) using video of actual elementary school classrooms proved beneficial. It allowed PTs to think more deliberately about how argument might play out with children and how valuable this is for developing understanding. One finding from that part of the sequence that was particularly enlightening was the PTs' inclination to work in a space between specific and generalized claims. While we expected a proclivity towards using specific cases to support claims given prior research (e.g., Adams et al., 2016/2017), the inability of some PTs to identify children's claims as close to a generalization suggests this is an important area to study further. In the end, working with children themselves to help PTs productively navigate through their struggles to articulate generalized claims is likely the mitigating factor to bridge the space between specific and general claims.

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RACIAL RECKONING IN TEACHER EDUCATION: USING CASES TO GET PRESERVICE TEACHERS TALKING ABOUT RACE

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Preservice teachers must have opportunities in teacher education to critically reflect on race and racism in mathematics education. Engaging preservice teachers in such conversations during methods courses can be challenging. This study uses a case specifically designed in a digital context to encourage preservice teachers to talk about race, including their understanding of race and racism in an authentic mathematics classroom situation and how a teacher might respond to the situation. Findings show that preservice teachers talk about race in terms of differing perspectives of discrimination (social versus racial exclusion) and how a teacher might respond (reactive and proactive initiatives). Recommendations and future research considerations are shared to address ways mathematics teacher education can shape preservice teachers' racial reckoning and critical consciousness.

Keywords: Preservice Teacher Education, Instructional Activities and Practices, Equity, Inclusion, and Diversity

With the teaching profession characteristically White, female, and middle-class and a growing public-school population composed of children with diverse cultural and racial backgrounds (National Center for Education Statistics, 2020a, 2020b), preservice teachers (PTs) must have opportunities to talk about race and racism within the context of teaching. The mere mention of these topics can often be difficult to discuss; however, research suggests that PTs who engage in such conversations can begin to critically reflect on race, including their racial identities and ideologies in mathematics education, and use racial noticings to examine lived experiences of students of color (Martin et al., 2017; Shah & Coles, 2020). Additionally, PTs can use conversations of race and other educational inequities to challenge spaces of marginality and create equitable learning environments in the mathematics classroom (Association of Mathematics Teacher Educators, 2017; National Council of Teachers of Mathematics, 2020).

One way for mathematics teacher educators (MTEs) to foster spaces for PTs to explore race and racism is by using case-based instruction during methods courses to promote prompt-based discussions (Gonzalez & Moldavan, in press; Gorski & Pothini, 2018; Kavanagh, 2020; Moldavan & Gonzalez, 2020). With COVID-19 forcing MTEs to rethink their instructional strategies in the context of a digital space, these discussions have also had to adapt with the aid of digital tools (e.g., asynchronous modules, discussion boards) to facilitate reflections about the discriminatory culture of mathematics education and how to disrupt racial inequities facing students of color. This study uses a case specifically designed to encourage PTs to identify race and racism in an authentic mathematics classroom situation and reflect on how a teacher might respond. Through the context of critical race theory, we examine the research question: *How do PTs leverage a case in a digital context to recognize race and racism in the mathematics classroom and a teacher's response to such situations?*

Theoretical Framework

To guide this inquiry, we referenced the work of critical race theory (CRT), which aids in examining issues of race in educational research (Ladson-Billings & Tate, 1995). CRT offers a theoretically grounded approach to explore aspects of race and racism, including the experienced discrimination and inequities from people of color and the counter-stories that undermine deficit narratives (Creswell & Poth, 2018). In the context of mathematics education, CRT can be used to examine how race and racism are embedded in school structures and practices and how racism has negatively impacted minorities and vulnerable communities, specifically their access to high-quality mathematics instruction (Davis & Jett, 2019). According to Berry and colleagues (2014), Black learners are "routinely given the least access to advanced mathematics content, the fewest opportunities to learn through methods other than memorizing facts and mimicking teacher-modeled procedures, and the least well-prepared teachers" (p. 541). Deficit discourse around mathematics learning for students of color perpetuates ideologies that problems exist with students' abilities to learn rather than with the responsibility of teachers to recognize how social and cultural experiences in school are shaped by racism and inequities (Gutiérrez, 2009). We engage PTs in case-based experiences, through a lens of CRT, to educate PTs in understanding the race-related experiences of students of color and how teachers can respond to such situations for purposes of advancing equity-oriented pedagogy.

Research Methods

We conducted a collaborative action research-based study across three online mathematics methods courses at two different universities. One university is situated in an urban northeastern city, while the other university is situated in a rural southern town. Both universities had instructors who used case-based instruction to incorporate cases depicting authentic mathematics classroom situations in their respective methods courses. The cases were used to engage PTs in conversations around equity and justice in mathematics teaching. One of those cases was designed to specifically elicit discussion related to race and racism and how it may present in mathematics classrooms. That case, reported here, depicts a Black student that was rejected from her peers based on her physical appearance and the deficit notions that her racial background could potentially cause disruption within a collaborative group activity. This case leveraged the national attention of racially motivated media rhetoric that unjustly depicts Blacks for disruptive behavior in local communities (e.g., riots), which students may have associated with as they organized their groups. The case provided the opportunity to examine the impact discrimination in the classroom can have on students of color, as well as the role of the teacher in responding to the situation to create an inclusive learning environment for all students.

The case was assigned to the PTs at the beginning of the Fall 2020 semester as an online discussion forum. The digital aspect of the instructional activity required the instructors to be methodical in their facilitation, given that the PTs would use the forum to host a "digital conversation." The PTs were asked to read the case and use the forum to respond to the prompts that elicited reflections on the racial injustices observed and how a teacher might respond to the situation. Their peers then reflected on the initial posts and provided feedback and suggestions. The forum posts were coded using in vivo and descriptive coding techniques (Saldaña, 2016). We then interpreted those codes using Kvale's (1996) meaning-making methods to capture the PTs' shared perspectives. Analytic memo writing was performed to specifically look for aspects of CRT that then moved our codes to themes (Grbich, 2013).

Findings

Two themes emerged from the data through the lens of CRT. The first theme addressed the PTs' awareness of race and racism in the mathematics classroom, particularly in the context of exclusion and how it stems from a perspective of racial discrimination. In the second theme, the PTs acknowledged teachers' roles in taking action with observed racism in the mathematics classroom. A close look at how teachers assume reactive and/or proactive initiatives provides insights into how students of color experience mathematics teaching and learning. It is notable that both themes depict PTs' recognition, and strong reactions at that, to developing critical consciousness of injustices in mathematics education and how reflection on racial dispositions and biases can prompt change in one's professional practice.

Recognizing Race and Racism in the Mathematics Classroom

In the PTs' responses to the case's prompts, the PTs were quick to write about how a group of students alienated a peer from participating in an assigned task. This observed exclusion presented itself in two ways: social exclusion versus racial exclusion. In the former, we define social exclusion as an observed injustice of a student being prohibited from working in a group without an explicit mention of race. While race may be a factor in the act of exclusion, the mention of race fails to be explicitly identified in the response. A handful of PTs who reflected in this way refrained from calling out the racial make-up of the students and justified potential blame for a student's actions on others (e.g., media, parents). For instance, a PT posted: "In our world today, there have been a lot of problems with diversity." The PT then went on to discuss how the media can influence one's action, which influenced why a student was targeted for exclusion. Another PT shared, "She [the White student Susan who made the comment to the Black student Dominique] may have misunderstood the context… She isn't trying to intentionally create harm. She is doing it because that's what she learned and doesn't know any better." This PT recognized that an injustice occurred but seemed to assume that being uneducated about one's racial disposition and biases could excuse one's actions. Likewise, another PT said, "Because I know that it is not Susan's fault for thinking this, I would contact her parents and explain to them that Susan's comments are hurting others in the class." The PT deflects blame away from the student, which is similarly noted by another PT who posted: "Susan may say this because Dominique looks different than her and she may not live in a diverse community or be around different races and ethnicities often." In several of the PTs' responses, the conversation of exclusion is sustained at a surface level response about race rather than digging deeper into the rhetoric of racism in school.

In the latter observed exclusion, we define racial exclusion as the observed discrimination based on the color of a student's skin. In this case, there is recognition of race-based privilege from a dominant group and how power can be used to marginalize or "other" a person who does not fit the color of the majority race. This notion of racial exclusion is noted by the majority of PTs and can be exemplified by the following shared responses from the PTs. For instance, a PT posted: "Susan obviously made a prejudice statement about Dominique based off of the color of her skin." Likewise, another PT said, "Susan, a White student in the class, may have said Dominique is dangerous due to her implicit (and quite obviously explicit) bias against African Americans. It is clear that Susan referred to Dominique's skin when she made that comment." This comment was followed by similar responses from PTs that said, "Susan has pushed false and negative stereotypes toward people of color, hurting Dominique in the process." Another PT posted: "Susan could not comprehend due to a lack of understanding of her White privilege," while a peer responded saying, "White students have plenty of time and space where they do not think about race. Black students do not have that privilege." These PTs consider the racial dynamics of the students and explicitly note the students' skin color and privilege when examining the situation. Some PTs also considered the intersectionality of other oppressions (e.g., gender, sexual orientation, language) to recognize the complex dynamics of exclusion and how discrimination impacts the participation and success of students of color in similar settings. One PT posted: "Maybe since Black women are portrayed as being aggressive, Susan thought it was okay to place that judgement on a student." The PTs used the case to leverage conversations about skin color and how racism is composed of racial prejudice, power, and privilege.

Teachers' Roles in Taking Action

The power and responsibility of the teacher in advocating for her marginalized student is demonstrated through these words from one PT: "The teacher has the authority to help Susan develop more truthful and less destructive beliefs about people of other races and to help Dominique feel safe in the classroom." This "authority" and responsibility was echoed in many of the PTs' responses and led to the second theme that addressed the role of the teacher in taking action to stop racism in the mathematics classroom. The PTs' responses considered reactive and proactive initiatives that could be made by the teacher. We define reactive initiatives as those that a teacher could do in the moment to intervene, while proactive initiatives refer to those that a teacher could do outside of the current situation to create an inclusive classroom environment.

The reactive initiatives the PTs described were either public or private in nature. Some PTs described how the teacher would need to "address the class as they were a witness to this event." These public discussions ranged from "shutting down what Susan said" to "discussing how the behavior is not acceptable." However, most of the PTs posted about how the teacher should have private conversations with Susan and Dominique. Although these private conversations could assist in helping those involved in the situation recognize the ramifications of their actions, private conversations could potentially restrict open dialogue about race and may not use the teachable moment to model how to be advocates for peers in similar situations.

The PTs also described proactive initiatives that focused on the various ways a teacher could create an inclusive classroom environment to prevent situations like that described in the case. The proactive initiatives targeted both racial representation directed at the classroom environment as well as at the student level. The PTs who targeted racial representation directed at the classroom environment wrote about "hanging diversity posters on the wall" or "having diverse children's books" in the classroom to honor narratives about people of color and challenge deficit notions. Other PTs placed more emphasis on the direct cultural and racial assets of the students in the classroom and provided opportunities for them to "get to know each other" as a way to "create mutual respect." Many PTs also mentioned having the teacher involve students in conversations related to "stereotypes" and "diversity" and then develop classroom norms for how to interact with one another. These proactive initiatives suggest the need to create space for marginalized students to be included so their stories counter deficit viewpoints.

Conclusion

Efforts to prepare PTs to identify and address race is crucial to examining the racial dynamics of mathematics classrooms. This study used a case specifically designed in a digital context to encourage PTs to discuss race and racism in mathematics classrooms. The PTs noted differing perspectives of discrimination (social versus racial exclusion) and how a teacher might respond (reactive and proactive initiatives) when racism arises in classroom situations. The responses provide insight into how instructional activities, such as case-based instruction, can be used to shape PTs' racial reckoning and critical consciousness (Freire, 1970). Further, the lens of CRT can be used to examine the ideologies of race to understand how race impacts the lived experiences of students in mathematics education (Berry et al., 2014). Teacher education must assist PTs in developing their awareness of race and the power they hold in advocating for students of color so such students are recognized as contributors in the mathematics classroom.

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MIDDLE GRADES PRE-SERVICE TEACHERS' STRUGGLE TO CREATE PRODUCTIVE STRUGGLE

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Task selection is a critical element of mathematics teaching because mathematical tasks differ in the learning opportunities made available to students. This study examines the tasks selected by pre-service teachers (PSTs) in a field based middle grades mathematics methods course. In this brief research report, I examine how PSTs' understanding of productive struggle informs their selection and implementation of tasks. Overall, PSTs sought to avoid unproductive struggle but were unsure how to create and sustain productive struggle. Implications for teacher education include supporting PSTs in developing a robust understanding of productive struggle and the types of tasks and instruction to support it.

Keywords: Preservice Teacher Education; Teacher Beliefs; Teacher Knowledge

Mathematical tasks are a critical component of mathematics instruction and continue to be an object of research interest (Tekkumru-Kisa et al., 2020). Research over the past 30 years demonstrates that the types of mathematical tasks students engage with heavily influences their learning opportunities (Stein et al., 2000; Tekkumru-Kisa et al., 2020). Thus, it is critical to continue to examine what tasks teachers select and why they select those tasks (Pimm, 2009). In this brief research report, I explore one influence on pre-service teachers' (PSTs) task selection and implementation by asking, *How do PSTs understand the relationship between productive struggle and mathematical tasks?*

Relevant Literature

Hiebert and Grouws (2007) identify engaging students in productive struggle as one of the key elements of mathematics teaching that promotes conceptual understanding. Productive struggle occurs when students are engaged in problem-solving, grappling with mathematical concepts that are within reach (Hieber and Grouws, 2007). In these instances, students are challenged to apply their mathematical knowledge in new ways. The notion of productive struggle is encompassed by the first Standard for Mathematical Practice in the Common Core State Standards: Make sense of problems and persevere in solving them (CCSS; 2012).

One way teachers can engage students in productive struggle is through the use of high cognitive demand tasks (e.g., Smith & Stein, 1998; Stein et al., 2000). The two higher levels of cognitive demand, *procedures with connections* and *doing math*, require students to link procedures with mathematical concepts as they engage in tasks that are open-ended. Tasks classified at the level of *doing math* engage students in problem-solving and complex thinking, ideally resulting in productive struggle. When students are engaged in a higher cognitive demand task, teachers can further support productive struggle by asking purposeful questions and encouraging students to explore multiple solution paths (Lynch et al., 2018).

PSTs often struggle with selecting and implementing high cognitive demand tasks that foster productive struggle. When beginning to design lesson plans, PSTs often select mathematical tasks that are procedurally focused, and rely heavily on memorization and reproduction (Anhalt et al., 2006; Nicol & Crespo, 2006). Even when provided with rich math tasks, PSTs modified the language of the tasks to prevent student struggle, not considering how their modifications altered students' opportunities to engage in mathematical discourse and productive struggle (de Araujo et al., 2021). When monitoring PSTs' problem-posing to elementary students during a methods course, Crespo (2003) found that initially PSTs tended to create or select tasks that were uncomplicated, unambiguous, and easy to solve. PSTs posed leading questions and strove to avoid student confusion. Over time, PSTs became more adventurous in their problem-posing, inviting students to solve open-ended tasks, which promoted productive struggle. Crespo attributes these changes in problem-posing to both PSTs' sustained interactions with elementary students and PSTs' reflections on those interactions. In this study, I highlight how PSTs' understanding of productive struggle informs their problem-posing, or task selection.

Methods

This brief research report presents findings from a larger study that investigates middle grade mathematics and sciences PSTs' task selection during a middle grades mathematics methods course (Anthony, 2021).

Middle grades mathematics methods course

The participants in this study are 10 middle grades mathematics and science undergraduate PSTs who completed a mediated field experience mathematics methods course, instructed by the author (Fall 2018). The field experience component of the course was an after-school enrichment program, which took place at a local middle school. Bumblebee Middle School (all names are pseudonyms) is a large Title 1 school with a student population that is majority Latinx (86%) and Black (10%). Each week PSTs prepared 90 minute lessons for their assigned small group of two to five middle grades students. PSTs had autonomy over the content of their lesson plans.

The concept of productive struggle permeated the mathematics methods course. Given the importance for teachers to regularly select and implement tasks that provide students with opportunities to actively partake in reasoning, sense-making, and problem solving (NCTM, 2014), the assigned readings and class discussions regularly addressed engaging students in productive struggle. In particular, PSTs read Lynch, Hunt, and Lewis' (2018) article about strategies for sustaining productive struggle through differentiated instruction.

Data sources

The data sources include PSTs' discussion board posts, final course papers, and individual interviews. Each week PSTs responded to discussion board prompts that asked them to reflect on their lessons at Bumblebee Middle and to connect course readings to their field experiences. The final course paper asked PSTs to reflect on what they learned in the methods course and how the course supported their growth as a learner and doer of mathematics. Given the open nature of the prompt, if PSTs wrote about productive struggle, it was voluntary.

PSTs participated in two semi-structured one-on-one interviews, which were audio recorded. The first interview was conducted the semester following the methods course (Spring 2019). PSTs were asked to reflect on their process for selecting tasks and preparing lesson plans for the after-school enrichment program. The second interview was conducted a year after the methods course (Fall 2019). PSTs were asked about their teaching internship and their perspectives on the essential roles and responsibilities of teachers.

Analysis

I used NVivo to code data sources for the phrases "struggle" and "productive struggle." With each instance, I examined the context of the phrase to look for patterns in how PSTs were describing productive and unproductive struggle or how they were attending to productive struggle in their lessons.

Findings

In their final course papers, six of the ten participants wrote about productive struggle as a necessary and beneficial aspect of learning mathematics. For example, Vincent wrote, “While seemingly counter-intuitive initially, it is this productive struggle that challenges students to ask questions, think critically, and solidify their understanding. As a teacher, it is my job to support this productive struggle.”

PSTs identified productive struggle as something to cultivate and unproductive struggle as something to avoid. According to PSTs, unproductive struggle can be avoided by selecting tasks that leverage students’ prior knowledge or by lowering the cognitive demand of the task. There was a shared concern among some PSTs (Briley, Carson, Grace) that when the cognitive demand of the task is “too high” students will descend into unproductive struggle. For example, Carson stated, “keeping the demand high but also not too high. Like, if it's too high then they won't get it at all. So, making sure it's productive struggle rather than non-productive struggle” (Interview 2). Thus, while Carson identifies maintaining high cognitive demand as important, this should not supersede the goal of students “getting it” through productive struggle.

PSTs identified supporting students in accessing and applying their prior knowledge to the task as a way to maintain the cognitive demand. Briley concluded that her students’ lack of prior knowledge led to unproductive struggle with tasks: “This lesson plan had a high cognitive demand, but it did not go as planned because the students were struggling past a productive level; they didn’t have the background knowledge necessary to complete the task” (Final Paper). In their reflection posts, several PSTs wrote about being surprised that students did not have the prior knowledge that they expected them to have, like Claire, who wrote, “I was in panic mode when they said that they had no experience working with percents since the entire lesson was literally on percents” (Discussion board post, Oct 25). When students lacked the prior knowledge necessary to engage with the planned task, PSTs often resorted to direct instruction, which lowered the cognitive demand of the tasks. In some cases, the PST’s subsequent lesson was of lower cognitive demand.

In their reflective writings, PSTs wrote about the benefits of productive struggle. For example, Carson wrote, “The students are willing to struggle through concepts instead of just giving up right away. This allows for more questioning and learning which is really cool” (Discussion Board, Oct 25). However, PSTs did not articulate how to create or sustain productive struggle. A few PSTs were able to identify when students did engage in productive struggle (Carson, Claire, Jessica, Grace). For example, Grace showcased a moment when her students demonstrated productive struggle:

> I could see that the students were struggling with reaching an answer, but they were not stuck, they were constantly coming up with new methods and ideas on how to solve this problem. After a while they got the answer of how much each item cost and they both were so proud of themselves. Then after I had a conversation with one of the boys who said that this had been the hardest lesson yet, but it was his favorite! He said that he really liked being challenged in math and it made him think more and that it was a lot of fun to figure out the prices. (Final Paper)

Undergirding PSTs' statements about struggle is the belief that productive struggle will lead to students successfully solving the task, while unproductive struggle results in students giving up, or otherwise being unable to complete the selected task. Interestingly, although PSTs write about the importance of supporting students in productive struggle, they still seem to value the successful completion of the task (arriving at a solution) over what they can learn about student thinking while working on the task (regardless of completeness).

Discussion

In both their reflective writings and interviews, PSTs communicated that they valued productive struggle as an important part of mathematics teaching and learning. However, PSTs had difficulty cultivating productive struggle through task selection. Regardless of the cognitive demand of the selected task, PSTs' interpreted students' unproductive struggle as a misalignment between students and the cognitive demand of the task. As shown in previous studies, PSTs sought solutions to alleviate struggle, such as direct instruction or choosing tasks of lower cognitive demand.

One explanation for the presence of unproductive struggle could be a failed launching of the task. How a teacher introduces the task to students is a crucial element in sustaining high cognitive demand tasks (Cobb et al., 2018). In a successful launch, the teacher supports students' development of a common language around the task features and cues students to key mathematical concepts without suggesting a solution method (Cobb et al., 2018; Jackson et al., 2012). When Grace's students showed signs of unproductive struggle, she found it helpful to "take a break and be like, 'Okay, so maybe, what do you guys remember about this? What are some aspects that you do know?'" (Interview 2). Grace's questions to students mirror the questions modeled in the methods course when learning how to effectively launch a task (Jackson et al., 2012). Her suggestion is to revisit those launch questions when students start to devolve into unproductive struggle while working on the task. Grace's solution to students' struggle implies that perhaps the initial launch was not successful at activating prior knowledge and creating a common language. Thus, revisiting the elements of a successful task launch throughout the methods course may be beneficial to PSTs.

Teacher educators can support PSTs' understanding for how to cultivate productive struggle through increased opportunities to observe students engaged in productive struggle. These observations will equip PSTs to not only identify productive struggle, but hopefully identify teaching practices and types of mathematical tasks that sustain productive struggle.

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TEACHER NOTICING OF STUDENTS' MATHEMATICS AS STUDENT CENTERED

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Attending to students' actions and mathematical thinking is an important aspect of professional teacher noticing. In this paper, we used 360 videos as a medium to examine the relationship between preservice teachers'(PSTs) observed attending behaviors and their written noticing. Findings suggest that PSTs focusing on students, instead of the teacher, during class discussions provide more specified descriptions of children's mathematical thinking.

Keywords: Teacher Noticing; Technology; Preservice Teacher Education

Professional teacher noticing involves "honing in on a key aspect of or instance that occurs during a lesson and engaging in reasoning to make sense of it" (Stockero & Rupnow, 2017). Experienced and knowledgeable teachers generally make sense of such instances by unpacking the mathematics that students engage in detail (Jacobs et al., 2010; Mason, 2017). By contrast, more novice teachers, such as many preservice teachers (PSTs), initially focus on the teacher's actions or on students' non-content related behaviors (Barnhart & van Es, 2015; Huang & Li, 2012). The differences in content-specificity of teachers' noticing corresponds with how and where teachers look when viewing a classroom scenario (Cortina et al., 2015; Dessus et al., 2016; Kosko et al., 2021). Scholars using eye-tracking have found that teachers with more specific descriptions of content focus on fewer students in a recorded classroom, whereas teachers with less specific descriptions of their noticings attempt to focus on multiple students (Dessus et al., 2016). Examining PSTs' teacher noticing while viewing a 360 video with a VR headset, Kosko et al. (2020) found that where and how PSTs turned their head and focused corresponded with how they described events within the recorded scenario. Such findings provide useful evidence of how teachers' physical actions of attending correspond with their verbal and written descriptions of what is noticed.

In this paper, we use the 360 video medium to study PSTs' tacit choices of where and what to attend with a focus on how such choice informs their articulated professional noticing. Contrasting standard video recorded from camcorders and Swivel cameras, 360 video records omnidirectionally so that the viewer can choose where to look within the classroom. This facilitates a sense of being in the classroom, as it more closely approximates standing in the classroom (Ferdig & Kosko, 2020; Walshe & Driver, 2019). Recording PSTs' 360 video viewing sessions provides useful data to examine their tacit choices of what, where, and when to attend (Gold & Windscheid, 2020; Kosko et al., 2020). We used such data to examine the nature of PSTs' attending in relation to the specificity of students' mathematics described in their noticing.

Classrooms where such student-centered actions can be observed are sometimes perceived as chaotic. Often, students are engaged in different content-specific actions that are happening simultaneously. Teachers must be able to make sense of what they notice in the moment and respond accordingly (Luna, 2018; Sherin, 2011). Less sophisticated noticing is evidenced by attending to superficial aspects of the classroom environment such as class management rather than focusing on student learning (Mitchell, 2015). More nuanced professional noticing is evidenced by attending to more specific student actions (Huang & Li, 2012; Jacobs et al., 2010).

Sherin (2007) describes two interrelated subprocesses of professional noticing: selective attention and knowledge-based reasoning. Selective attention, what we have referred to as attending, is when the teacher "selects certain stimuli of a perceived scene for detailed analysis" (Scheiner, 2016, p. 231), where knowledge-based reasoning, or interpreting, is the act of using one's professional knowledge and prior experience to unpack what was attended (Sherin & van Es, 2009). Studying the interrelationship between these subprocesses is complex, and video has traditionally been used in examining this phenomenon (Rosaen et al., 2008; van Es, 2002).

The use of video can help PSTs to refine their descriptions of students' actions to be more content-specific reflections that shift from more general, to procedural, and then to conceptual descriptions of children's actions (Barnhart & van Es, 2015). Early evidence suggests that 360 videos provides a more immersive viewing experience to study teacher noticing (Kosko et al., 2021; Walshe & Driver, 2019). Particularly, different scholars have begun to record where PSTs attend in a 360 video and relate those attending behaviors to their pedagogical decisions and reasonings (Huang et al., 2021; Ferdig et al., 2020; Gold & Windscheid, 2020). Examining where and how PSTs look *at* a scenario, such as with eye-tracking data with standard video (Dessus et al., 2016), is useful. However, examining where and how they look *within* a scenario provides an added dimension of data regarding what Sherin (2007) describes as PSTs' selective attention. In this paper, we present a preliminary analysis of PSTs' attending behaviors (via recorded 360 video sessions) in relation to their interpreting acts of the recorded scenario.

Therefore, the purpose of this paper is to examine the relationship between where PSTs chose to attend in a 360 video and the specificity of their descriptions of children's mathematics.

Method

Sample and procedure

Participants included 21 preservice teachers enrolled at a Midwestern U.S. teacher education institute in Spring 2020. Most participants identified as white (91.7%), and female (76.1%). After completing consent and basic demographic questions, participants engaged in a brief tutorial describing how to watch 360 videos on a laptop and how to screen record their 360 video viewing sessions. Analysis of participants' screen recordings enabled us to identify their field of view (FOV) (Huang et al., 2021), where FOV includes the *location* and *time* a viewer looked at a specific point. After the tutorial, PSTs watched a 360 video (5 minutes and 49 seconds) of fourth grade students explored equivalent fractions using fraction strips. Within the video, students were asked to use their fraction strips to find an equivalent fraction to 5/6. Midway through the video, the teacher engages students in a brief class discussion where two students describe not being able to reduce the fraction because 5 is a prime. Students are then asked to find an equivalent fraction to 3/8. The video ends after a brief discussion of how students needed to use an algorithmic approach, instead of fraction strips, to find an equivalent fraction. After viewing the 360 video, PSTs were asked to describe all pivotal moments they had noticed in their viewing (i.e., any moment you (PSTs) believe is important for the teaching and/or learning of mathematics). Then, PSTs selected one of these moments as the "most informative for them for teaching and/or learning of mathematics" and describe it in further detail.

Analysis and findings

In order to analyze participants' written noticing, Systemic Functional Linguistics (SFL) was used (Halliday & Matthiessen, 2014; Eggins, 2004). SFL is an approach to linguistics that examines how grammar functions to convey meaning. This method allows "the detailed and systematic description of language patterns" (Egging, 2004, p.21).

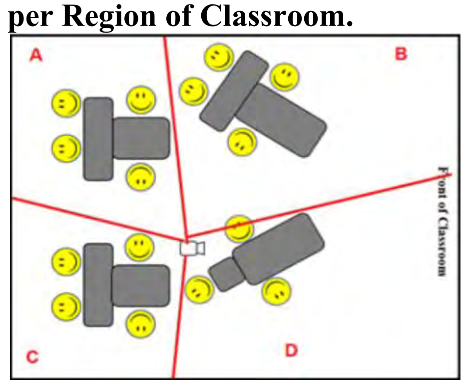
PST's math-specific written noticing	PST's generic written noticing
The teacher **challenged** her students **to use** the fraction strips for the second fraction (3/8). I noticed// the students closest to the 360 camera quickly **realized** that // the second problem (3/8) *could not be* **demonstrated** using the fraction strips.	When the students **shared** what answers they got // and [students] **worked** together, [students] **talking** through the problem. *Sometimes* it is helpful for students **to discuss** their thought process // and [students] **work** with their peers to come to a conclusion.

Figure 1: Examples of a specific (left) and nonspecific (right) written noticing.

In the current study, we examined the grammatical resource of reference. *Reference* refers to "how participants are introduced and 'managed' 's the text unfolds" (Mehler and Clarke, 2002, p. 160). The repeated patterns of referencing builds reference chains, which can also convey how a particular referent is operationalized by an individual. Figure 1 illustrates two participants' excerpts with coded reference chains. The PST on the left introduces the referent "fraction strips" which is then connected to "the second fraction (3/8)." As the text continues, these two referents are conveyed as not being the same, since "(3/8) could not be demonstrated using fraction strips." By contrast, the student on the right references "answers" and builds a reference chain that identifies students conveying their answers along with their "thought process" to come to a "conclusion." Although this PST incorporates discourse in how the referents are conveyed, the reference chain clearly ends with a focus on a final answer (i.e. "conclusion"). After examining PSTs' written noticing using reference chains, reliability of whether the theme was observed or not was examined by the first and second author (0=fractions not referenced; 1=referenced fractions). The Kappa coefficient (.857) indicated near perfect agreement, with 52.3 % of PSTs attending to fractions in their written noticing and 47.7% not doing so.

Table 1: Contingency Table for Seconds Focusing per Region of Classroom.

	A	B	C	D	n/a*	Total
Not Attend	**166** **219.9**	**1157** **1092.3**	**460** **440.7**	**1622** **1655.3**	**47** **43.8**	**3452**
Attend	**296** **242.1**	**1138** **1202.7**	**466** **485.3**	**1856** **1822.7**	**45** **48.2**	**3801**
Total	**462**	**2295**	**926**	**3478**	**92**	**7253**



**Indicates a region could not be identified (i.e., scanning or moving back-and-forth).*

To examine variations in PSTs' specificity of noticing equivalent fractions across four regions of classroom we analyzed their recording videos second-by-second. A total of 7253 seconds across 21 participants were examined for which region of the classroom PSTs' FOV included at any given second (see Table 1). We estimated a chi-square statistic to determine whether where PSTs focused during the video was independent from whether they attended to fractions in their written noticing. Results indicated a statically significant chi-square statistic (χ^2 (df=4)=35.85, p<.001), suggesting PSTs' written noticings and where they attended in the video were not independent from chance. To better understand this finding, we conducted a post hoc chi-square analysis using z-scores to compare observed and expected counts within cells of Table 1. In particular, PSTs who attended differently in mathematical noticing, spent different amount of time in region A and B. Next, we created graphical representations of each individual PSTs' viewing patterns across the length of the 360 video (see Figure 1 for a cumulative example). Based on the chi-square analysis, we focused our attention on variations between participants' attending for regions A and B. Notably, a specific time frame [2:28-2:44], indicated in Figure 2 by a yellow rectangle, illustrates considerable traffic in the fraction-specific group (blue) for region A. This prompted a review of PSTs' screen recordings to better understand what was happening in this 16 second interval. Essentially, PSTs who attended to students' fractions in their written noticing were looking back-and-forth between one student in region A describing their math and the teacher in Region B writing on the board. By contrast, PSTs who did not attend to fractions in their written noticing focused almost exclusively on the teacher during this timeframe (only one PST looked at the student, and did so for 1 second).

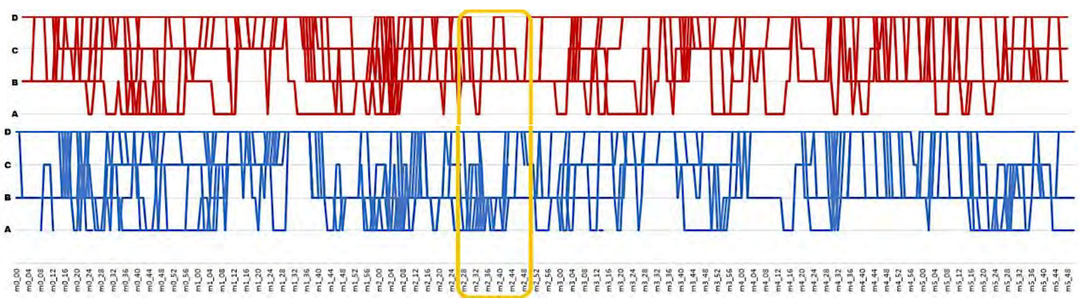


Figure 2: Region PSTs focused by second for not specific (top) & specific (bottom) noticing.

Discussion

The study described the relationship between PSTs' attending within their FOV and the specificity of their written noticing. PSTs selective attending as well as their reflection on what they attend are considered as key elements of professional noticing (Sherin, 2007). Using 360 videos allowed us to understand how PSTs' content-specific descriptions of students' thinking related to their FOV being, literally, student-centered. Thus, PSTs with student-centered attending behaviors provide more specific descriptions of students' mathematical thinking. This corresponds with prior research on teacher noticing (Jacobs et al., 2010). Future study is needed to confirm trends observed in this paper, as well as applied to different contexts to provide additional empirical evidence of how PSTs' develop their professional noticing.

Acknowledgments

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CONFRONTING COLORBLINDNESS: THE IMPACT OF CRITICAL MATH MODULES ON PRESERVICE TEACHERS' 'ONCEPTION OF RACE AND RACISM

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How do undergraduate preservice teachers (PSTs) incorporate their conceptions of race and racism into their understanding of math methods and their teaching philosophies? We report on an intervention conducted in an introductory math methods course. PSTs experienced course content that was explicitly reframed around issues of critical mathematics, social justice, and systemic racism. Data from final assignments was analyzed to determine students' conceptions of race. While the majority of conceptions remained colorblind, students more critically analyzed race and racism when course topics explicitly presented and supported their importance.

Keywords: Preservice Teacher Education, Social Justice, Teacher Beliefs

Purpose of Study

Sociopolitical issues of equity, race, and social justice have been central in challenging the paradigm of math education over the past two decades (Gutierrez 2013; Gutstein, et. al, 2005; Martin, 2009 are just a few of these extensive works). The call to build explicitly antiracist, socio-politically conscious, equitable and critical mathematics teacher education programs has reverberated throughout the math education world (Wilkerson & Berry, 2020; PMENA, 2020; TODOS, 2020). It is necessary for teachers to understand how race and racism affect teaching as part of their preparation to become effective educators**.**

Responding to this call, we, two mathematics teacher educators (MTEs), redesigned the first math methods course for undergraduate preservice teachers (PSTs) at a large public university in the southeastern United States by adding topics with an antiracist, critical lens on mathematics teaching and learning. In doing so, we sought to understand the impact of these changes on PSTs' experience in the course, their beliefs and their future teaching practice.

To understand how our shift impacted students, we sought to answer the question: Which resources and tasks presented to new pre-service mathematics educators were most impactful in shaping their identity and philosophy towards antiracist teaching?

Theoretical Perspectives

Critical Race Theory (CRT) has become essential in helping education scholars explore the centrality of race in classrooms (Tate, 1995; Ladson-Billings, 1995, 1998). CRT proceeds from the understanding that racism is normative in U.S. society, and that these normative structures reinforce a dominant Whiteness throughout society and education. Employing this CRT lens has shown math education to be a White-institutional space in both visible and invisible, central and peripheral ways (Battey & Leyva, 2016; Larnell, et. al, 2016; Martin 2013). In order to address this reality, MTEs must develop students' explicit awareness of race in the mathematics classroom, "to counteract the mechanisms and institutional ways in which White supremacy in mathematics education reproduces subordination and advantage" (Battey & Leyva, 2016, p. 51). Mathematics education is situated in the larger, overwhelmingly White field of teacher education (Sleeter 2001, 2017; Matias & Lackey, 2016), which points to a need for greater understanding of how PSTs are learning to enact antiracist and critical pedagogies.

If teachers are to deconstruct the otherwise invisible White "frame" of math education, they must escape from a "colorblind" or "race-neutral" perspective that dismisses the racism embedded within these systems (Ladson-Billings, 1998; Milner, 2006) and supports the racial ideological status quo (Battey & Leyva, 2016). Milner (2006) proposes that PSTs' conceptions of race change when they: 1) move past pervasive colorblindness and to become racially and culturally aware, 2) engage in critical reflection about their own positioning and 3) explore both theory and practice to help them make sense of their awareness and positionality. Gutstein (2016) notes that this antiracist approach can support critical mathematics pedagogies.

Particular resources and pedagogical moves can be utilized to change pre-service teacher beliefs around race (Matias & Lackey, 2016). However, the study of teachers' beliefs and practices is replete with evidence that change is difficult (Pajares, 1992), particularly shifts towards antiracist practices (Haaviland, 2008; Lawrence & Tatum, 1997). Given this added challenge, continued research is critical to better understand how PST beliefs and practices can begin to shift, at the time when PSTs are developing knowledge from course experiences and incorporating their learning into their beliefs and future practices.

Methods

Participants + Context

The course was adapted from a previous introductory methods course, which had a focus on lesson planning, mathematical tasks, and adopting mathematical mindsets. The authors made the decision to adapt the course: maintaining a focus on these methods while infusing each week with an antiracist and critical lens on math education. (Following Gutstein (2016), we used the terms critical mathematics and social justice math interchangeably.)

Including these lenses explicitly throughout the course, was an attempt to overcome the apprenticeship of observation (Lortie, 1975) through the principle of overcorrection—a pedagogical move where pre-service teachers experience ideal learning situations (Grossman, 1991). To that end, the instructors made certain to include issues of critical mathematics explicitly each week, including within topics previously covered without this lens. For example, classroom activities around assessment explored the racist and inequitable effects of standardized testing (see Table 1 for a full list of topics covered).

The class was taught synchronously online. Students and instructors met once weekly as a group to do math together and discuss course material for 75 minutes. Three to five further hours were budgeted per week for asynchronous work. The authors adopted the stance of "texts, texts and more texts" (Matias & Mackey, 2016) to allow students to engage with these critical and antiracist philosophies and practices not only through research articles but podcasts, videos, newspaper articles, lesson plans, and mathematical tasks.

Participants were new secondary mathematics education undergraduate majors. This was their first course in mathematics education. 19 of the 27 students in the course consented to the research. Based on self-reported information, the students were majority White women, with a small group of students of color across genders, and a few of White men. Exact numbers are not reported to retain anonymity of participants. Both authors were instructors. The first author is a White, male PhD student, teaching the course for the first time. The second author is a White, female assistant professor of math education, teaching the course for a fifth time.

Data Collection and Analysis

The data analyzed for this qualitative study was the two-part final assignment. The first part of this assignment was an annotation of the course syllabus. Students selected nine of the 15 course topics and explained how those topics influenced their vision of math teaching. The second part of the assignment was a teaching philosophy, written as a personal statement highlighting important learnings from the semester. A priori codes were applied at the paragraph level based on module topics (Table 1) and race-discussion-type (Table 2). Codes for race-discussion-type were generated from Milner's (2006) discussion of 1) colorblindness ("colorblind"), 2) racial/culture awareness ("race-aware," acknowledging race, often in a list of ways students are diverse) or 3) critical reflection on race and its societal importance ("race-critical") (see Table 2). Assignment paragraph was chosen as the unit of analysis because students usually discussed one course topic, and their understanding of race in relation to it, per paragraph. During the initial coding process, we observed that students engaged in more than one race-discussion-type at different points in the assignment. Data analysis focused on the co-occurrence of critical racial reflections with course topics.

Table 1: Course Modules Taught

Topics kept from previous iteration of course		**Critical / sociopolitical / antiracist topics added**	
Norm Setting	Assessment	Carceral Pedagogies	Caring Classrooms
Mathematical Mindsets and Valuing Mistakes	Heterogeneous Math Groupings	Learning Pods and Inequity	Antiracist Teaching Through Mathematics
An Introduction to Mathematical Tasks	Math Tasks & Classroom Environment	Teaching Math in Diverse Classrooms	The Math Teacher Community
Technology	Launching Tasks	Creativity, Beauty in Math & Number Talks	

Table 2: Race-Discussion-Type Codes & Examples

"color-blind"	This week's topic really encouraged me to rethink the purpose of a math classroom. Traditionally, math has been thought of as a class with a narrow scope that should avoid all controversial issues. However, … math is among the most powerful tools we have as humans to understand and change the world; thus, teaching math in a vacuum to students is essentially denying them the preparation they need …
"race-aware"	I learned about the inclusion of mathematics in a process called culturally responsive-sustaining education. Focusing on diversity in a math classroom with CR-SE, I learned that students use their own identity to get education. They learn using aspects of their race, social class, gender, language, sexual orientation, nationality, religion, or ability.
"race-critical"	I would like to learn more about anti-racist teaching practices … I have learned, after the events of the past year, that you cannot simply proclaim to be *non-racist*, but must actively declare *anti-racism.* As America's political landscape has started to shift after George Floyd's death, my students' perceptions about race in society will have shifted and I would like to know how to better react.

Results

Across all paragraphs in both annotated syllabi and philosophies (n=286), 84% were colorblind, 8% were race-aware and 12.5% were race-critical. Of the 28 total race-critical comments, 54% (n=15) were discussed in the context of the Antiracist Teaching module of the course. Eight more were spread across four other "critical" topics (Teaching Math in Diverse Classrooms, Carceral Pedagogies, Caring Classroom, Inequity/ Learning Pods), while only five were in relation to modules taught in the previous iteration of the course (Tasks, Classroom Environment, Mathematical Mindsets). As one student noted about learning pods: "I had never realized or understood the racial segregation that created and the increase in the opportunity gap between my classmates and me."

Antiracist teaching was one of three topics most frequently cited as significant. However, of the 15 teachers who cited antiracist teaching as impactful, eight discussed the module in a colorblind way. For example, PSTs avoided the term "race" or "racism" by using "social issues," "culturally relevant issues," "diversity," or by shifting the discussion to class or ability level. Teachers who discussed antiracist teaching practices in aware or critical ways were frequently concerned with how they would implement those practices consistently. As one noted:

> Equity is definitely something that should be discussed in schools, though it is hard for math teachers. I think that introducing racism in regular conversation in class when talking about how to apply mathematics would be beneficial. Getting to know each of my students and building a relationship with them…will help me to introduce active conversations about race and where math applies in each of their lives.

Four students indicated explicitly that discussions of antiracist teaching and culturally responsive and sustaining education were new ideas to them, and that these discussions had revealed the importance of incorporating these pedagogical stances moving forward. For example, one student named her own growth from the course:

> I have now adopted anti-racist teaching, specifically social justice learning, as one of the main pillars of my teaching philosophy. I had never seen social justice be incorporated into a math lesson before in my entire academic career, so I was very excited to create my own lesson! ..It made me feel hopeful that I will be able to teach my students about the world around them and math at the same time, which I hadn't felt confident about before.

Discussion

We are excited that this intervention impacted student understanding and beliefs, and it is clear there is more work to be done. First, we believe that the shift in course content partially opened critical reflection space for teachers. In answering the research question of which topics were most impactful, we find particularly suggestive the connection between explicitly race-conscious modules and PSTs' critical reflections, particularly for the antiracist teaching module. When assigned an explicitly race-aware context, students were more able to reflect critically.

This intervention also made visible the White institutional frame in documenting how most conceptions of math teaching remained colorblind, even as course modules stated the importance of racism. This is not unique to our course. In courses specifically oriented towards social justice in White-majority PST classes, there is still an extreme reticence to talk about race explicitly (Lolkus, 2021). We also hope more research follows Shah & Coles (2020) on racial noticing, to provide insight into how to create critical reflective space for PSTs.

We recognize the limitations of our own planning for this course in the Fall of 2020. Efforts to live and create an antiracist, humanized, critical and rigorous course for students understandably will require edits and updates in future iterations. Our lack of a pre- and post-survey around concepts of race in math classrooms prevented us from measuring more directly the impact the course itself had on teachers' understanding of their knowledge and beliefs. In this study, racial and gender demographics were understood based solely on how students described themselves and their experiences in class discussions and assignments. Moving forward we must collect this demographic data explicitly: there can be no question that PSTs' racialized lived experiences will impact conceptions of race in any coursework.

Some of our students ended the course less hesitant to openly discuss issues of race and its impact on math classrooms than the start. However, discussions of race, power, oppression, and systemic injustice, even when intentionally and thoroughly centered in a methods course, will not take hold for more students without these ideas being reinforced systemically throughout

PSTs' coursework and fieldwork (Sleeter, 2017). Seeds planted within this first methods course may take years to flower.

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DESIGNING A VIDEO-BASED INTERVENTION TO ELICIT TEACHER CANDIDATES' MATHEMATICAL KNOWLEDGE: THE HEXAGON TASK

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Researchers increasingly take a design-based research approach to iteratively design, implement, and revise interventions. In this paper, we describe how our iterative design led to improvements in design principles aimed at supporting secondary mathematics teachers' attentiveness development. We describe issues we encountered and insights for developing video-based interventions to improve attentiveness.

Keywords: Design Experiments, Mathematical Knowledge for Teaching, Preservice Teacher Education, Teacher Noticing

Video-based interventions can lead to growth in teachers' and teacher candidates' (TCs') knowledge of mathematics (Jacob et al., 2009), professional noticing skills (van Es & Sherin, 2008), and knowledge of students' mathematical ideas (Powell et al., 2003). Lacking in the literature, however, are detailed accounts of how iterative curriculum design processes are used to improve video-based interventions (Cavey et al., 2020). We are in the fourth year of a design-based research (DBR) project (VCAST) focused on developing video-based modules to improve secondary mathematics TCs' ability to attend to student thinking (Carney et al., 2017). Data analysis revealed limitations in TC responses to prompts about student thinking. In this paper, we describe how the iterative design process led to improvements in our ability to elicit better evidence of TCs' mathematical knowledge associated with figural pattern tasks.

Background

For decades educators have leveraged online instructional materials to maximize in-person class time (Graham, 2006). We began this project interested in developing an intervention with online videos and supporting materials for mathematics courses, similar to earlier work (Goldman & Barron, 1990; Lampert & Ball, 1998). Instructional interventions are rarely iteratively designed, implemented in authentic environments, and improved over time (Amiel & Reeves, 2008). DBR focuses on solving educational problems by connecting research, theory, and practice through iterative theory-driven development focused on making an intervention effective in authentic settings (Anderson & Shattuck, 2012). We took a DBR approach when developing our intervention.

Attentiveness is the ability to analyze and respond to a student's mathematical ideas in ways that build upon student understanding towards formal mathematics and its conventions (Carney et al., 2017) and is grounded in professional noticing (Jacobs et al., 2010), mathematical knowledge for teaching (Ball et al., 2008), and progressive formalization (Freudenthal, 1973). It narrows the focus of professional noticing to an individual student and provides a lens through which to theorize how a teacher's mathematical knowledge and pedagogical stance are elicited in the analyses of student work. Two key aspects of this lens are the ability to recognize, articulate, and connect (1) a student's productive reasoning along with ideas that may be in need of refinement, and (2) the key ideas associated with a mathematical task.

The Intervention & Design Principles

Each VCAST module centers on a mathematical task and features video clips and written artifacts produced by secondary students during their engagement with the task. Each module has an online, in-class, and exit ticket component. In each module, TCs complete the online component, then take part in class discussions on the task and artifacts, and then complete the exit ticket. Each module is typically implemented over a week. Design principles guided our iterative development.

Design Principle 1 (DP1): Solve non-routine mathematical tasks. The use of non-routine tasks creates opportunities to exhibit mathematical reasoning and can be more accessible to students with a range of background knowledge (Schoen, 2003). Each module begins with TCs solving the same task as the secondary students. The Hexagon Task (Figure 1) elicits a range of approaches and is ideal for analysis of student reasoning (Cavey et al., 2020).

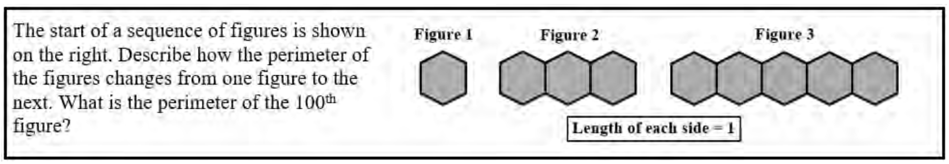

Figure 1: Adapted Hexagon Task; Hendrickson et al. (2012)

Design Principle 2 (DP2): Analyze a range of student evidence. Directed analysis of student evidence can help TCs improve their ability to notice students' mathematical reasoning (Sherin & van Es, 2009; Star & Strickland, 2008). To target DP2, we selected three students who used a range of different strategies and approaches to solving the Hexagon Task (Figure 2).

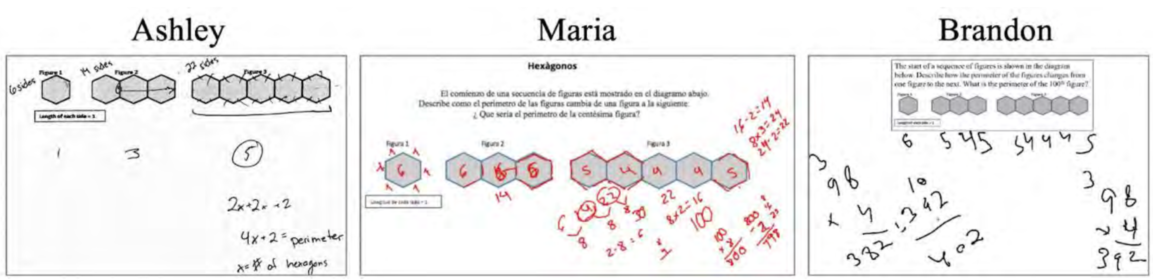

Figure 2: Student evidence featured in The Hexagon Task module

Design Principle (DP3): Engage in Cycles of Inference and Prediction. Effective teachers do more than analyze and interpret student strategies; the inferences they make about students' understanding and their predictions about students' next steps help inform their responses to students (Hill et al., 2005; Jacobs et al., 2010; Lesseig & Hine, 2019). We target DP3 by sequencing video clips from the three featured students throughout the module.

Design Principle (DP4): Describe the Mathematical Challenges for Students. Teachers who understand the specific mathematical challenges students may encounter with a non-routine task are better positioned to respond to students in ways that support students' productive struggle (Stein et al., 1996). We target DP4 by including multiple opportunities for TCs to recognize the challenges students face when focusing on how the configuration of hexagons in each figure contributes to the perimeter.

Methods

We report on this design-based research project following its third revision cycle and after implementation at six public US university Uteach replication sites (Uteach, n.d.). Participants (n=73) were undergraduate students enrolled in the study's partner instructors' mathematics courses. Six partner instructors either taught at the host university or were recruited through the Uteach listservs and annual conferences.

Data were collected via the project's digital platform. TCs submitted responses to open-ended, single- or multiple-selected response, ranking, and upload prompts. Prior to analyzing module responses, three researchers from the design team discussed the expected range of responses. Though we anticipated a range of quality across responses with respect to descriptions and inferences about student work (Jacobs et al., 2010; van Es, 2011), we assumed TCs would have sufficient mathematical knowledge to make sense of the task and the students' work. With each design cycle, the research team applied a variation of magnitude coding (Saldaña, 2016) to review the quality of alignment between expected response and the actual response data. Three researchers independently reviewed all TC responses for each module prompt and compared them to what the prompt had been intended to elicit. Then, researchers reviewed the sets of individual TCs' responses to each sequence of prompts focused on a featured secondary student in the module. Researchers met weekly to discuss emergent areas of concern and to reach consensus on module content warranting revision. The areas of concern connected to TCs' own mathematical knowledge which emerged from these cycles of analysis and discussion are the focus of this paper's results and inform our conclusion and its implications for future research.

Results

Due to limited space, we focus on DPs 3 and 4. An area of concern related to DP3 arose when we were unable to determine whether TCs' superficial predictions stemmed from weaknesses in mathematical knowledge. For example, in Maria's first clip, she narrates how the two outer hexagons in the third figure contribute five units each, while the interior three hexagons contribute four units each, to yield a perimeter of 22. When predicting Maria's next steps, some TCs predicted that Maria will try to solve the task. In the next video segment, Maria creates a sequence of the first three figures' perimeters and computes the common difference of 8 between them. When asked how Maria's next steps compared to TCs' predictions, some TCs made a judgment about her progress with the task, seemingly ignoring the shift in her problem-solving process. In the third video clip, Maria successfully solves the task. Interestingly, some TCs appeared to either not value or not notice some aspects of Maria's productive work. Sample TC evidence elicited from this cycle illustrates the challenge we encountered (Figure 6).

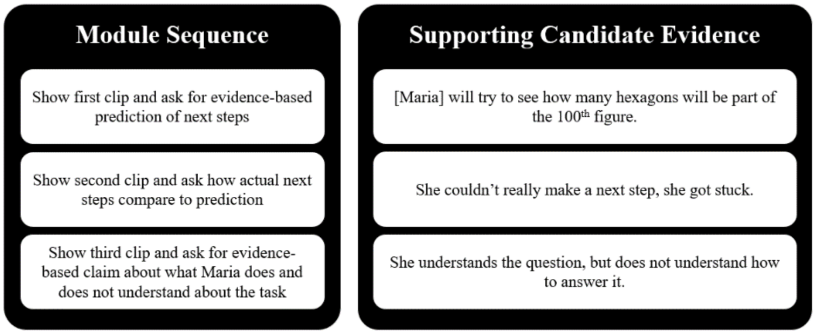

Figure 6: Sample TC Evidence Elicited During a Prediction Cycle

To address this concern, we followed the cycles of prediction in the online component with an adjusted version of the task, the Octagon Task, and provided selected-response prompts which asked TCs to apply each featured students' approach to the new task. This revision accomplishes two goals: (1) it provides scaffolded support to TCs who struggle with the task and (2) it allows us to differentiate between TCs who simply struggle to articulate the mathematical approach they might notice from those who struggle to make sense of the approach itself.

The DP4 area of concern is related to TCs' persistent inability to articulate the mathematical challenges for students. Many of these TCs also appeared to struggle with the same mathematical challenge themselves. To address this concern, we added an official Exit Ticket to the module where we explicitly direct TCs' attention to the challenge that arises when students focus first on the relationship between the number of hexagons and the perimeter and then try to connect their reasoning to the figure number. By Year 4, the Exit Ticket featured video of our third student, Brandon, who made an initial misstep by assuming there were 100 hexagons in the 100th figure. After he noticed his original perimeter of 402 was incorrect, he observed:

> It's going to be more. That's plus two, plus two. I guess it's going to be plus two again. Okay. Five, seven, nine. Okay, [the perimeter's] going to be more than 402. Because there's more than 100 hexagons in the 100th figure. [...] But I need to find how many hexagons the 100th figure has. Um, so it increases by two every time. One plus two equals three. Three plus two equals five. Seven, nine ..Ihat would take too long.

We follow this with two prompts: "Describe the challenge that Brandon encountered in this clip. Use evidence from the clip to support your answer" and "Describe how Brandon's approach to the Hexagon Task is contributing to the difficulty he is experiencing."

Conclusion

Given the importance of conceptual understanding in mathematics education (Ball, 1990), we feature non-traditional tasks that require conceptual understanding to solve in the VCAST modules. With each of our design principles, we work to position TCs in ways that support their ability to analyze, interpret, and make inferences about the range of student reasoning they might encounter in their future classrooms. Our intent is to support TCs' development of the knowledge and skills they will need to enact responsive teaching practices in the classroom.

Yet with each round of our iterative design process, we have discovered areas of ambiguity in the TC response data. We found that we cannot make assumptions about the reasons behind TCs' struggles to make sense of student reasoning evidence while engaged in those same tasks. Engaging in this revision process improves our ability to discern whether TCs' mathematical knowledge might be posing barriers to their attentiveness development and also improves our understanding of the attentiveness construct itself. By sharing some of the challenges encountered in refining the operationalization of design principles through iterative cycles of implementation data collection and analysis and illuminating implicit researcher assumptions which impact design decisions, we hope to support others in similar work.

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APPLYING SYSTEMIC FUNCTIONAL LINGUISTICS TO UNPACK THE LANGUAGE OF ADDITIVE WORD PROBLEMS

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Research has not yet examined the linguistic patterns of additive problem types nor explored how linguistic analysis might be applied to support preservice elementary teachers in making sense of the semantic and structural differences amongst them. Using a corpus of 150 word problems, the authors conducted an ideation analysis, drawing from systemic functional linguistics theory. The findings resulted in a distillation of language features key to the mathematical processes in the three types of additive problems (i.e. change, part-part-whole, and compare). Preservice elementary teachers (n=97) were introduced to metalanguage as a tool for analyzing structural differences in additive word problems. Analysis of their work demonstrated their ability to employ the functional metalanguage to identify challenges and describe specific linguistic characteristics associated with each type of additive problem.

Keywords: Preservice Teacher Education, Number Concepts and Operations, Elementary School Education, Instructional Activities and Practices

Introduction & Background

Encompassing mathematical operations within contextual word problems is an important pedagogical strategy for supporting children in developing meaning for operations and connecting computational knowledge with an understanding of how such calculations are applicable in everyday life (e.g., Briars & Larkin, 1984; Carpenter & Moser, 1982; Nesher et al., 1982). The Common Core State Standards in Mathematics (CCSS-M; NGA Center & CCSSO, 2010), adopted by the majority of the United States, identify the need for children to be able to apply the four basic operations to all possible types of contextual situations in which they naturally occur. If students' exposure to different types of situational word problems is limited, it may cause them to develop misconceptions about the meanings for the operations (Van de Walle et al., 2019) and/or limited solution strategies (Carpenter et al., 2015).

However, research shows that the language of word problems can pose greater obstacles than the mathematical concepts involved in solving for solutions (Kintsch, 1987; Lager, 2006; Wyndham & Säljö, 1997). One strategy for overcoming such challenges encourages learners to identify "keywords" (such as "altogether") to determine operations, but this practice has been shown to be highly problematic (Huang & Normandia, 2008). Taking a different approach, the authors of this work aimed to help preservice elementary teachers (PTs) develop disciplinary linguistic knowledge (Turkan et al., 2014) that could enable them to unpack the language of different types of word problems. Systemic Functional Linguistics (SFL) is a social semiotic language theory that offers such a meaning-based metalanguage: a language for helping students and teachers talk about the functions of language and how it shapes meaning in the subject areas (Fang & Schleppegrell, 2010; Halliday, 1975; Halliday & Matthiessen, 2013). Our goals were to describe the salient linguistic features of the multiple types of additive word problems, as well as explore PTs' uptake and application of the linguistic metalanguage.

Van de Walle et al. (2019) categorize addition and subtraction problems into four disjoint situations: Change-Join, Change-Separate, Part-part-whole, and Compare. For the purposes of this paper, we use the singular term "change" to refer to problems involving an action that causes a set to undergo an additive increase (Join) or decrease (Separate) collectively. Part-part-whole (PPW) problems involve the relationship amount a set and its two disjoint subsets, whereas Compare problems involve the additive comparison of two sets.

Research to date has not yet examined the linguistic patterns of these particular problem types, nor explored how linguistic analysis might be applied to support PTs in making sense of the semantic and structural differences amongst them. Therefore, our research questions were: 1) What linguistic patterns exist in one-step additive word problems? 2) How can PTs use SFL metalanguage to identify the semantic and structural features of additive word problems?

Methods

The sample for this study included undergraduate PTs in a College of Education at a southern public university in the United States. All PTs were enrolled in a mathematics problem-solving course designed for education majors (n=97). The intervention developed through this research project was implemented by two instructors across three sections of this course. Data collection and analysis were conducted in two phases: 1) analysis of word problems to develop an intervention and 2) implementation of intervention and analysis of resulting student work.

Development of Intervention

Word problem analysis. First, using a corpus of 150 one-step additive word problems, our research team conducted an ideation analysis, examining taxonomic relations of participants, process types, and connectors (Martin & Rose, 2003). *Participants* refer to the nouns or noun phrases in the word problem. To distinguish between significant and contextual participants, we used the term *referents* to denote the specific participants that are quantified and tracked through the word problem. The term *processes* refers to the verbs and verb phrases, and *connectors* comprise of conjunctions and words or phrases that convey relationships between clauses. These initial analyses resulted in a distillation of language features that are key to the mathematical processes of the three types of additive word problems: *time markers* and *active processes* in change problems; *taxonomic relations* among participants in PPW problems; and *connectors of comparison* in compare problems.

Linguistic patterns in one-step additive word problems. Findings from the word problem analysis indicated distinct linguistic patterns across the three types of additive word problems. Change problems involve one referent being tracked as it is acted upon over time, in which there occurs either an additive increase or decrease in the quantity of the referent. In change problems, the process moves from stasis into action and back to stasis as the referent is being acted upon. Furthermore, because actions happen over time, change problems utilize internal conjunctions of time (i.e. *time markers*) that often serve to differentiate distinct stages in the problem with indicating language such as "then/now, before/after, yesterday/today," etc.

Compare problems are characterized by situations where two referents are being compared at one point in time. Therefore, is it highly unlikely for time markers to appear in compare problems. Instead, these problems include the quantification of two unique referents and the assessment of how these two quantities compare additively. Thus, these types of problems include *comparative connectors*, such as "more than/less than, greater than/fewer than." Also, since the quantities of the referents are only being compared, there tends not to be any significant active process occurring as a key feature, thus processes are typically static in nature.

PPW problems include three referents that are situated in a specific hierarchical fashion. Two of the referents are subsets of the third overarching referent. These "sub-referents" illustrate one way in which the items in the overarching referent can be broken into two inclusive and disjoint subsets according to some differentiating characteristic, at one point in time. Thus, time markers are typically not used and the processes in these problems also tend to be static.

The results of this analysis informed our design for an instructional intervention. Instructors implemented a series of lessons that introduced the metalanguage of *participants, referents, processes*, and *time markers* during PTs' initial exploration of additive word problem types. Our goal was to support PTs in developing the taxonomic categories of additive word problems through inductive reasoning instead of providing them with a taxonomy upfront.

Implementation of Intervention

The intervention occurred after PTs completed an introductory task in which they analyzed and sorted 22 one-step additive word problems into categories based on what they initially noticed about the problems' similarities and differences. Approximately 70% of the PTs sorted the 22 problems into two categories: addition and subtraction, only considering the operation required to solve the problem.

Following this initial sorting activity, instructors introduced the metalanguage of *participants*, *referents*, and *processes* and as a way for PTs to analyze the word problems based on the meaning of the language used in them. After these terms were introduced to the PTs, they were asked to consider three word problems (one change, one PPW, and one compare). The purpose was to highlight the fact that each of these problems can be solved using addition, yet the operation is used in different ways to address a variety of situations (though no categorical language was explicitly revealed to PTs at this time). After working together in class to identify the *participants, referents,* and *processes* of these problems, the PTs were then asked to apply this SFL metalanguage to find new ways of sorting the original set of 22 word problems, again.

During discussions around iterative cycles of this work, the idea of *time markers* was introduced to help distinguish between situations that are happening at one point in time (i.e., PPW and compare problems) and those involving changes occurring over time (i.e., change problems). By the end of the intervention, PTs had *inductively* arrived at the distinguishing characteristics underlying the three taxonomic categories of additive word problems based on their attention to the linguistic patterns, at which time the instructors introduced the associated vocabulary related to each category (i.e., change, PPW, and compare).

Analysis of Student Work

We collected PTs' reflections on their challenges in classifying word problems, as well as their articulations of the linguistic differences they found among the problems. The first source of data came from a reflection question where PTs were asked to identify the word problems they found most challenging to classify. The initial, iterative analysis (Strauss & Corbin, 1997) of PTs' responses identified the types of word problems PTs struggled with, as well as their use of SFL metalanguage. Five weeks after the intervention, PTs completed a midterm video reflection in which they were asked to explain the difference between additive words problems that have a change structure and those that have a PPW structure. Transcripts from these video reflections (n=40) were analyzed for metalanguage articulated by the PTs.

Findings

Analysis of PTs' work demonstrated an ability to employ the functional metalanguage to describe specific linguistic characteristics associated with different types of additive word problem types. PTs were able to use metalanguage to: 1) Distinguish between contextual participants and referents; 2) Identify the number of referents and use this knowledge to distinguish amongst change, compare, and PPW problems; 3) Identify active versus static processes and apply this knowledge to distinguish between change and PPW problems; and 4) Identify time markers in change problems. In what follows, we showcase PTs' responses to illustrate how they used the SFL metalanguage to identify challenges, and in turn key features, of the additive word problems.

PTs initially found it challenging to identify the referent(s) amongst several participants in a given word problem. However, the metalanguage of referent versus participant eventually helped PTs identify patterns in the word problems based on the number and characteristics of the *participants* and *referents*. This led to an inductive categorization scheme separating the problems with one referent (i.e., change) from those with two referents (i.e., compare) and three referents where two sub-referents are encapsulated in one major referent (i.e., PPW). In turn, the identification of the number (and structure) of referents became a key feature PTs analyzed when categorizing additive problems.

PTs initially articulated challenges in identifying processes as being static or active, but this also developed into a tool they later used to help distinguish between problem types. During the sorting activity, the metalanguage of static versus active helped PTs identify patterns in the word problems based on their *processes*, inductively categorizing the problems with active processes (i.e., change) separately from those with static processes (i.e., compare and PPW). Later in midterm reflections, they showed a deeper understanding of the role of active and static processes, particularly their role in change problems, articulating several key features of the *processes* within change problems: 1) change problems indicate a change in a referent's quantity, 2) a relationship exists between time markers and the type of process, i.e., that time markers indicate a change over time in which the *process* moves from stasis to active and back to stasis, and 3) the active process denotes either an additive increase or decrease in the resulting referent. Last, attending to the metalanguage of time markers helped PTs identify change problems. Most PTs were able to initially identify time markers, but some struggled to determine the timing of events in a word problem. By the midterm reflection, however, most PTs did note that time markers were a significant key feature in change problems.

Discussion

The language of mathematical word problems can greatly influence how students make sense of situations posed and identify appropriate solution strategies, which vary according to the type of word problem they are being asked to solve (Carpenter & Moser, 1982; Carpenter et al., 1988; Sarama & Clements, 2009; Verschaffel et al., 2006). PTs need tools for unpacking the language in word problems so that they (and their future students) can access their mathematical meanings. The findings presented here demonstrate how a targeted application of SFL metalanguage can succinctly describe the linguistic features that articulate the key mathematical functions in additive word problems. This work likewise demonstrates that such a targeted approach can support PTs' abilities to apply this metalanguage in accurate and purposeful ways. The functional metalanguage enabled PTs to pay attention to the key *functions* in additive word problems, rather than identifying "keywords" to identify operations, a strategy researchers are encouraging teachers to remove from work with students (e.g., Van de Walle & Lovin, 2006).

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VIDEO ANALYSIS OF PRESERVICE ELEMENTARY TEACHERS: SUPPORTING PRODUCTIVE STRUGGLE IN A MATHEMATICS METHODS COURSE

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In this study, we explored how elementary preservice teachers (PTs) demonstrate supporting productive struggle in a peer teaching activity, after they were engaged in supporting productive struggle practice through a series of activities in a math methods course. While PTs demonstrated strategies that support productive struggle in giving time, asking questions, task, and discussion/feedback in several different ways; the results indicated that they were not able to support productive struggle in use of tools, scaffolding, praising, and mistakes categories. Implications for teacher educators are discussed through connections between and among the strategies.

Keywords: Productive Struggle, Pre-service Elementary Teachers, Mathematics Method Course, Peer Teaching, Effective Mathematics Teaching Practices

Supporting productive struggle in learning math is identified as one of the eight effective math teaching practices by the National Council of Teachers of Mathematics' ’NCTM, 2014) *Principles to Actions (PtA)*. Math methods courses that focus on how productive struggle fosters learning math with understanding may help pre-service teachers (PTs) use supporting productive struggle practice in their future teaching, as methods courses affect PTs’ use of effective instructional practices (Clift & Brady, 2005). There is, however, little evidence in the literature that indicates the introduction and study of supporting productive struggle teaching practice at the teacher preparation programs (Warshauer et al., 2019). In this study, we explored how elementary PTs demonstrate supporting productive struggle in a peer teaching activity, after they were taught a variety of strategies to support productive struggle, and engaged in supporting productive struggle practice through a series of activities in a math methods course. To this end, the research questions guiding the study were: (1) How and in what ways do PTs demonstrate supporting productive struggle in learning math in a peer teaching activity in the math methods course? (2) How and in what ways do PTs not demonstrate supporting productive struggle in learning math in a peer teaching activity in the math methods course?

Theoretical Framework

Struggle is often attributed to something negative as opposed to an opportunity for learning in math education (Borasi, 1996; Hiebert & Wearne, 2003). In the U.S., learning is considered as an activity that is supposed to be fun; however, students do not learn deeply if learning is only fun, without any struggle (Stigler, 2016). Although, struggling is considered a critical component of learning math deeply (Hiebert & Grouws, 2007), due to the adverse effects associated with struggle, such as frustration and overwhelm, teachers often look for ways to remove the cause of students' struggle (Adams & Hamm, 2008; Borasi, 1996). Therefore, teachers, particularly in the U.S., tend not to use high cognitive demanding tasks in their lessons, or they often transform the tasks into procedural exercises during instruction (Stigler & Hiebert, 2004), perhaps because they do not know how to support struggle productively.

Hiebert and Grouws (2007) described productive struggle as "students expend effort in order to make sense of math, to figure out something that is not immediately apparent" ”p.387). Warshauer (2015a) elaborated on Hiebert and Grouws's’notion of struggle in three parts: the purpose of struggle to make sense of math; the characteristics of math tasks and their connection to students' struggle; and teacher's role in supporting the productive struggle. Teaching practices that maintain productive struggle with high cognitive level tasks promote a high degree of engagement and persistence throughout the task, deepen students' understanding (Dweck, 1986; Santagata, 2005; Stein et al., 1996; Sun, 2018), and hence help students make sense of the math concepts. Introducing productive struggle to PTs in their math methods courses may not only expand their understanding of math concepts for teaching, but also equip them with strategies to support struggle productively. This could potentially help PTs embrace struggle as a learning opportunity instead of avoiding it, and use high cognitive demanding tasks in their teaching.

Research on supporting productive struggle is still in its infancy, and little is known in the context of PT education. Prior research on productive struggle focused on students' struggle (Warshauer, 2011; 2014a), effective math teaching practices to support productive struggle with inservice teachers (Smith et al., 2017), and PTs' understanding of productive struggle with professional teacher noticing of student thinking (Warshauer et al., 2019). PTs’ development and implementation of strategies to support productive struggle remains undiscovered.

Methods

This study presented preliminary results that were part of an extensive study and was exploratory in nature. The participants were 25 PTs in an elementary/middle education program at a Mid-Western university in the U.S. They were enrolled in a math methods course, which they typically take in the semester before their student teaching experience. The PTs studied and practiced supporting productive struggle throughout the methods course through a variety of readings, discussions, and activities within eight effective math teaching practices stipulated in PtA, as well as in relation to differentiation, inclusion, and equitable pedagogies. The data analyzed in this study consisted of video-recorded instances of peer teaching as well as lesson supplements or relevant materials that were used to assist the research team's analysis of the videos. While the videos were focused mainly on the PT as the teacher, it also included recordings of the peer PTs communicating with the presenting teacher. To analyze the videos and strategize a consistent coding criterion, we developed a rubric that helped us identify and describe the supporting productive struggle practice across a variety of criteria and sub-criteria. The criteria were derived from the literature, as well as from the content of the math methods course. An expert validation and a coding agreement were established through discussion and consensus among five researchers. The coding rubric consisted of eight main criteria including task, tools, giving time to struggle, asking questions, scaffolding for access to productive struggle, mistakes and confusions, praising and encouraging students, and discussion/feedback. Beneath eight criteria, there were a total of 37 sub-criteria that describe specific strategies. For instance, mistakes and confusions included five sub-criteria: (1) helping students realize that confusion and errors are a natural part of learning by facilitating discussions on mistakes, misconceptions, and struggles; (2) calling on students who may not have the correct answer, and then guiding students in the process of questioning their thinking; (3) acknowledging student contributions whether it is a misstep, misconception, or inappropriate approach; (4) publicly valuing mistakes; and (5) creating disequilibrium. Praising and encouraging students comprised of two sub-criteria: (1) praising for their efforts and struggle in making sense of mathematical ideas and perseverance in reasoning through problems; not for being smart and/or fast; and (2) for generating multiple representations and methods even if these do not lead to a successful solution. It is worth noting that during the video analyses, the research team was cautious in providing credit to the type of praise that explicitly values students' efforts and struggles, as well as varied strategies and representations, not their intelligence, success, speed, or correct answers. After the rubric preparation, the entire research team received training and gained experience in how to code the video clips using the newly developed coding rubric. To ensure inter-rater reliability, the research team individually coded the same video footage collaboratively, discussed coding differences and similarities among people, and finally reached a consensus. Later on, 25 videos, 30-35 minutes each, were coded by the research team independently using qualitative thematic coding.

Results

A total of 482 codes were generated for the instances that PTs demonstrated supporting productive struggle strategies across 25 videos. Percentage frequencies were computed for each criterion and sub-criterion, for three phases of the lesson. The PTs' ’se of supporting productive struggle strategies were similarly distributed among the three parts of the lesson; launch (29.6%), explore (37.6%), and reflect (32.8%). As shown in Figure 1, the PTs, most frequently, demonstrated supporting productive struggle in the discussion/feedback, task, giving time, and asking questions categories. In the discussion and feedback category, PTs most frequently (10.4%) encouraged students to make their ideas public; (6.0%) engaged students in purposeful sharing of mathematical ideas and reasoning; and or (2.7%) honored and built on student thinking. In the task category, PTs supported productive struggle by implementing tasks that allow for/encourage multiple solutions, strategies, and representation (4.4%); have real life context (7.1%); or require written explanations for reasoning (4.1%). The PTs often provided time for struggle by working in pairs and groups and engaging in whole group discussion. PTs supported productive struggle by asking assessing (8.7%) and advancing (4.1%) questions more than any other questioning strategies in the same category such as questions that help students identify the source of their struggle and or surface an error (0.2%).

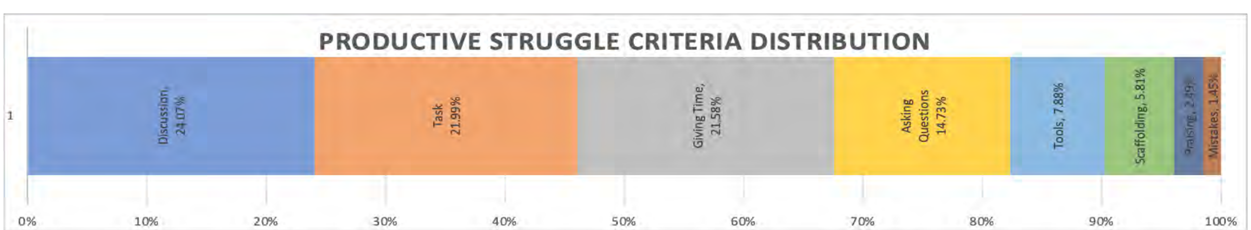

Figure 1: Distribution of Supporting Productive Struggle Strategies

On the other hand, the PTs rarely demonstrated supporting productive struggle strategies in tools, scaffolding, praising, and mistakes/confusions categories. As displayed in Figure 1, the

PTs low frequently gave access and or required using a variety of tools, strategies, and representations in order to support thinking processes. Scaffolding for access to productive struggle was observed only when PTs elicited prior knowledge by purposefully planned questions or activities that illuminated key mathematical ideas that would likely be useful when students faced the subsequent task (3.7%). Praising, in the context of productive struggle, was defined as praising for effort and struggle, and or for generating multiple representations and methods. The PTs seldom praised for struggle and effort (0.8%) and multiple representations (1.7%). Although, mistakes and errors were emphasized as an essential component of supporting productive struggle, the PTs rarely (0.3%) helped students realize that mistakes and errors were a natural part of learning, or (0.4%) publicly valued mistakes; and they never created a disequilibrium.

Implications and Conclusions

While PTs demonstrated strategies that support productive struggle in giving time, asking questions, task, and discussion/feedback in several different ways; the results indicated that they were not able to support productive struggle in tools, scaffolding, praising, and mistakes categories. Giving access and requiring students to use tools and representations supports students' thinking processes (NCTM, 2014), and enforces the idea that it is not about how fast you work, it is about knowing a lot of strategies and tools that can help you (Huinker & Bill, 2018), and hence fosters productive struggle (Warshauer, 2011). Requiring use of multiple tools, representations, and strategies is often linked to implementing high-cognitive demand tasks (NCTM, 2014). Therefore, PTs' infrequent use of the strategy could be attributed to their infrequent use of high-cognitive demand tasks. Providing scaffolding without removing the demands or doing the thinking for students has been often discussed in relation to productive struggle (Huinker & Bill, 2018; Kapur, 2014), and possible techniques have been suggested in Barlow et al. (2018) and Lynch et al. (2018). Although PTs engaged in a variety of such techniques during the methods course, they were unable to utilize them in peer teaching, except for eliciting prior knowledge scaffolding technique (Barlow et al., 2018). This can be explained by PTs' infrequent use of high cognitive demand tasks, as scaffolding for access to productive struggle is applicable when the tasks require complex and non-algorithmic thinking, and considerable cognitive effort. The results revealed that the PTs did not utilize praising, as defined in the context of productive struggle. Praising students for their efforts in making sense of mathematical ideas and perseverance in reasoning through problems, rather than for correct answers, is also linked to productive struggle (Baker et al., 2020; Bray, 2018; NCTM, 2014). PTs' infrequent use of the praising strategies possibly arose from their failure to require a variety of tools, representations, and strategies. Although, publicly valuing mistakes, creating disequilibrium (Boaler, 2016), and viewing mistakes and struggle as inevitable parts of learning (Bray, 2014; Kapur, 2014; NCTM, 2014; Warshauer, 2015) are considered essential components of supporting productive struggle, the PTs were not able to utilize mistakes and errors as a portal for supporting productive struggle. The PTs', as well as teachers', difficulty in helping students question or leverage an incorrect idea toward a more productive strategy, and in treating student mistakes and flawed approaches as objects of inquiry was noted in the literature (Hallman-Thrasher, 2017), and asking questions that encourage reflection and justification was suggested as an avenue to help PTs facilitate learning around mistakes and struggle (Morissey et al., 2019). The PTs' difficulty with using mistakes as learning opportunities can be linked to their low-none frequency of asking questions that help students identify the source of their struggle and or surface an error. Our goal is to take the positive findings of this study and further extend them with a focus on the strategies that PTs had difficulty with, so that future K-12 students will have teachers who provide them opportunity to productively struggle through the important mathematical concepts and relationships.

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EXPANDING CONNECTION IN PRACTICE: ATTENTION TO MULTIPLE MATHEMATICAL KNOWLEDGE BASES IN INTEGRATED STEAM LESSONS

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As integrated STEM/STEAM education gains prominence in PreK-5 classrooms, prospective elementary teachers face new challenges in learning to focus on children's mathematical thinking and community funds of knowledge (i.e., multiple mathematical knowledge bases) in instruction. We explored prospective teachers' attention to multiple mathematical knowledge bases in integrated STEAM lesson plans, co-planned and co-facilitated at an informal STEM event for preschool children and families. Analyzing three lesson plans, we asked how prospective elementary teachers connected children's mathematical thinking, funds of knowledge, and STEAM. We found transitional connections – explicit attempts that were underdeveloped in one or two areas – in two lesson plans and meaningful connections in one. We discuss implications for elementary teacher learning and integrated STEAM practice.

Keywords: Integrated STEM/STEAM; Learning Trajectories and Progressions; Preservice Teacher Education; Equity, Inclusion, and Diversity

Increasingly, educational initiatives are pushing for the development of teachers' capacity for STEM integration in K-12 classrooms (e.g., Division of College Career & Technical Education, 2018). However, learning to teach integrated STEM education presents new challenges for prospective teachers and teacher educators (e.g., Ryu et al., 2019; Shernoff et al., 2017; Stohlmann et al., 2012). Among these challenges lies the concern that situating mathematics within integrated STEM will negatively impact the quality of mathematics curriculum and instruction (Weinstein et al., 2016). In other words, efforts to center children's mathematical thinking (e.g., Carpenter et al., 1996) and funds of knowledge – the cultural, community, linguistic, and cognitive resources from home and community settings (e.g., González et al., 2001) – in mathematics may fall to the side as science and engineering take center stage.

We share preliminary findings from efforts to extend the work of the Teachers Empowered to Advance CHange in Mathematics (TEACH Math) group, which explored how prospective elementary teachers learn to incorporate multiple mathematical knowledge bases (MMKB), namely children's mathematical thinking and funds of knowledge, into mathematics instruction (e.g., Turner et al., 2012). We asked: *In what ways do prospective elementary teachers connect children's MMKB and STEAM in integrated lessons they design?*

Theoretical Framework

We build on the learning trajectory proposed by the TEACH Math group (Aguirre et al., 2012; Turner et al., 2012) for prospective teacher learning of critical practices for incorporating MMKB into instruction (Figure 1). Accordingly, we share the perspective that teacher learning along this trajectory is "dynamic and assume that [prospective teachers'] understanding and practices will reflect different points on the trajectory at different times" (Aguirre et al., 2012, p. 180). We also assume that teacher learning requires multiple entry points and collaboration.

Initially, teachers engage in three interrelated practices: (1) attention – what teachers notice; (2) awareness – how teachers interpret what they notice; and (3) eliciting – how teachers interact with children and families to elicit MMKB (Turner et al., 2012). The second phase reflects teachers' initial attempts to make connections to children's MMKB, and the third phase involves ongoing and purposeful incorporation of MMKB in instruction (Aguirre et al., 2012). We sought to extend this framework by considering prospective teachers' efforts to leverage children's MMKB with and through connections to other academic disciplines incorporated within integrated STEAM (Figure 1). By integrated STEAM, we mean that mathematics learning occurs alongside and/or through science, computer science, engineering, and/or arts learning. We take a broad stance on what counts as "art" and include fine arts, music, and humanities (e.g., literacy, history). Our study focused on the second phase within the learning trajectory, making connections. We explored the extent to which integrated STEAM provided additional access points for prospective teachers leveraging MMKB.

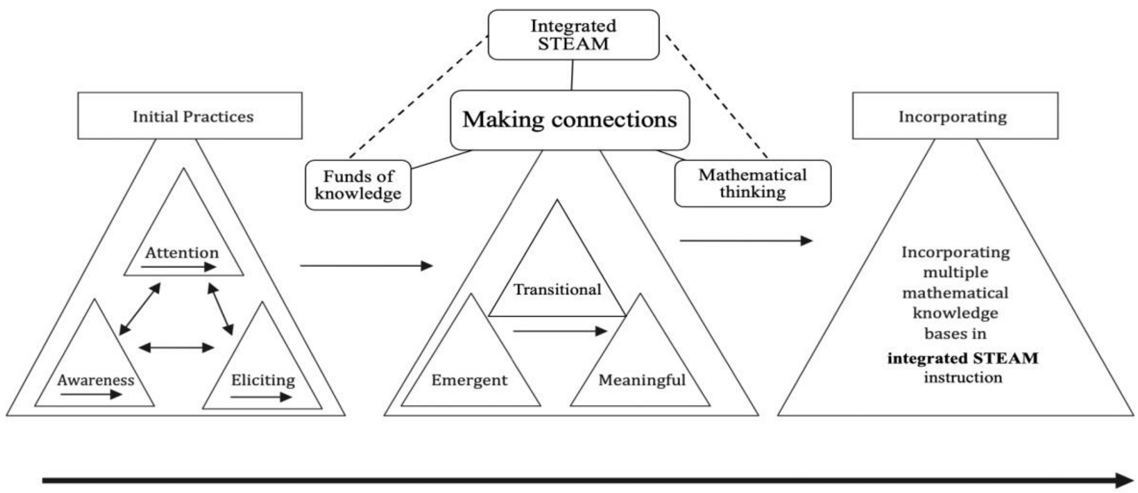

Figure 1: Prospective teacher learning trajectory of key practices for engaging with children's multiple mathematical knowledge bases in transdisciplinary instruction

Methods

Data sources for this analysis came from 124 prospective elementary teachers collaboratively producing 44 lesson plans across three years. Data was generated from 2018-2020. Harper taught all sections of the courses and used a modified version of the TEACH Math Community Engagement Module (Turner et al., 2015) to support prospective teachers' collaborative lesson planning and facilitation. Enrolled prospective teachers designed, planned for, and facilitated integrated STEAM lessons using MMKB at informal STEM events. These lessons and events were hosted by a nearby public title I preschool in a predominately Black community, and children and their families attended. In this brief report, we share findings from our preliminary analysis of three lesson plans. The following groups of prospective teachers from the PreK-3 licensure program co-planned and co-facilitated the lessons: (1) *Vet Clinic* by Georgia, Khloe, and Helen in 2018 (All names are pseudonyms); (2) *Sink or Float?* by Devon, Zara, and Chelsea in 2019; (3) *Codename: Mouse* by Dallas and Hope in 2019.

We coded the lessons by section as our unit of analysis; this allowed for the greatest consistency across all coders and lesson plans. Lesson plans included four sections: (1) the task; (2) learning intentions; (3) plan for enactment (i.e., anticipated student strategies; teacher responses); and (4) wrap up. We began analysis with iterative rounds of inductive coding (Sa , 2013) of the "Vet Clinic" and "Sink or Float?" lessons, meeting to reach consensus and develop a codebook. Our emergent codes were compared to the codebook from previous analyses (Harper et al., 2021) and pre-existing codes were adopted as appropriate. Parent codes included: Science; Technology; Engineering; Art; Literacy; Social Studies; Mathematics; and Funds of Knowledge. We elaborate on subcodes, as themes within these broader domains, in the findings section. Harper deductively applied codes to the "Codename: Mouse" lesson and then categorized (i.e., theoretical coding; Saldaña, 2013) the connections among mathematical thinking, funds of knowledge (i.e., MMKB), and STEAM in all three lessons as *emergent* (superficial attempts to make connections), *transitional* (explicit attempts to make connections, but connections remain underdeveloped), or *meaningful* (connections support rich, problem-solving) (Aguirre et al., 2012). Finally, Harper generated data displays (e.g., code co-occurrence) in Dedoose (8.3.45) to identify the themes described in the next section.

Findings

We found that prospective teachers made meaningful connections in the "Codename: Mouse" lesson. Prospective teachers made transitional connections among children's MMKB and STEAM in the "Vet Clinic" and "Sink or Float?" lessons. The transitional connections included explicit attempts to connect MMKB and integrated STEAM, but attention to funds of knowledge and STEAM integration were both underdeveloped in the "Vet Clinic" lesson. The focus on children's mathematical thinking was underdeveloped, and explicit attempts to connect MMKB and integrated STEAM were consistent but problematic in the "Sink or Float?" lesson.

Meaningful Connections

Prospective teachers Dallas and Hope visited the preschool community and, with attention, identified several important community sites nearby to include in their lesson (e.g., local businesses, parks). In "Codename: Mouse," Dallas and Hope planned for students to choose two familiar sites (pre-identified ones or ones named by students), students placed markers on a grid and navigated a robot mouse from one site to another (for more details, see Harper et al., 2021). The lesson was authentic to community mathematics in that it elicited children to draw on their experiences navigating between familiar places and encouraged children and family members to become experts by "engage in conversations with their children about directions and distances between places in their daily lives." We categorized these MMKB connections as meaningful from the attention Dallas and Hope paid to the local community sites and their awareness that families and students are the "experts" of the site locations. Navigating the robot mouse was consistently connected to children's mathematical thinking about counting principles (e.g., one-to-one correspondence) related to the concept of unit measurements of distance and computational thinking, namely putting steps in a sequence to perform actions. We categorized these STEAM integrations as meaningful because the mathematics engagement supported the development of computational thinking and vice versa.

Transitional Connections

Georgia, Khloe, and Helen visited an animal hospital near the preschool and learned about the authentic ways in which a veterinarian calculates dosages for pet medication. In their "Vet Clinic" lesson, students chose a "sick" animal and determined the correct medication dosage based on the animal's weight (half a pill per five pounds). To support the necessary proportional reasoning, prospective teachers consistently elicited children's mathematical thinking across various strategies, such as direct modeling with fraction bars, drawings, and paper "pills" and counting strategies (e.g., skip counting). Funds of knowledge remained underdeveloped, focusing on children and families understanding "the intricacy of a veterinarian's job." Here we found the prospective teachers had an awareness of funds of knowledge but ultimately missed the opportunity to leverage children and families' knowledge of taking medicine or giving it to pets. Further, although the task was set within the context of veterinary/medical science, with the goal of children "understanding a specific weight coincides with a specific dosage," the specific science concepts remained vague and underdeveloped.

Devon, Zara, and Chelsea visited local businesses near the preschool but decided to use a commercially available "sink or float" STEM set as the main inspiration for planning. In their lesson, students chose two objects, judged their weight and shape, predicted whether they would sink or float, and then tested the hypothesis. Attention to funds of knowledge was consistent throughout the lesson with encouragement for children to "think back to experiences they have already encountered (swimming pool, bathtub) and connect or compare those items they have observed to the new items introduced during the lesson." Further, prospective teachers asked children to weigh objects using an informal, home-based method of holding one object in each hand to judge which was heavier. Attention to children's mathematical thinking about comparisons of weight and properties of shapes (e.g., size) were consistent but remained vague. Although the scientific method and physical science concepts, namely density, were integrated throughout the lesson, prospective teachers never made explicit the inversely proportional relationship between mass and volume; therefore, we categorized the STEAM integration as transitional. Some parts of the lesson seemed to reinforce a misconception that heavy objects sink while light objects float (i.e., weight alone determines density), which may be problematic.

Discussion and Conclusion

Elementary teachers are not new to the daunting task of instructing across disciplines, and exemplars exist that show the potential for integrated STEAM to foster deep mathematics learning in K-5 classrooms (e.g., Bush & Cook, 2019). We found that integrating computer science education into mathematics lessons fostered productive struggle for our prospective elementary teachers, which facilitated meaningful connections to both children's mathematical thinking and funds of knowledge (this analysis; Harper et al., 2021). Sense of place, or place attachment and identity that is contextually dependent, offers one such entry point for integrating MMKB and computer science (Harper et al., 2021; Leonard et al., 2016; Rubel et al., 2017). This analysis suggests that integrating science into mathematics lessons can provide different entry points for prospective teachers to make connections among children's MMKB. In the "Sink or Float?" lesson, attention to children's scientific thinking and funds of knowledge were more developed than the focus on mathematical thinking. Nevertheless, the focus on the science and mathematics practices observed in everyday activities (e.g., bath) provided an additional entry point for leveraging MMKB and highlighting weight, shape, and comparison during the scientific processes of predicting and observing, which also engaged children in practices authentic to STEM communities and professions (Civil, 2016). Leveraging funds of knowledge, in addition to children's mathematical thinking, in integrated STEAM lessons is necessary within broader efforts to diversify STEM. Such attention to funds of knowledge can disrupt the expectation that those from marginalized groups will disregard their unique lived experiences and perspectives and conform to (white, masculine, middle class) professional norms upon entering STEM fields (Verdin et al., 2016).

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PRE-SERVICE MATHEMATICS TEACHERS IDENTIFY CRITICAL EVENTS: WHAT CHANGES IN THEIR ATTENTION?

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This study explores the changes in pre-service mathematics teachers' 'ttention to critical events within the context of a year-long teachers' 'reparation program in which noticing critical events was a key preparation tool. We asked 20 pre-service teachers to identify and describe critical events they witnessed during school observations/teaching. We then used an empirical and theoretical-based model developed to explore the ways in which pre-service teachers changed their attention to these critical events. Our findings reveal that the pre-service teachers' 'oci of attention were broadened; they were attentive to more details, especially to students' 'ffect. We demonstrate these findings using the case of Nasim.

Keywords: Preservice Teacher Education, Teacher Noticing, Critical Events, Professional development

Introduction and Theoretical Background

Critical events have been defined in many ways. Here, we consider critical events to be classroom situations in which students' mathematical thinking is apparent and can serve as an opportunity to delve into the mathematics that was presented (Leatham et al., 2015). We use critical events as a tool to prepare pre-service teachers (PSTs) to teach high-level secondary school mathematics, in the context of a larger research project. In particular, we use critical events to teach PSTs to notice instances where the teacher can develop students' mathematical thinking. Noticing students' thinking is considered a core teaching practice (Grossman et al., 2009), leading researchers to study ways to support PSTs' learning of these skills (Jacobs et al., 2010; Simpson & Haltiwanger, 2017; Sun & van Es, 2015).

We conceptualize noticing according to Jacobs et al.'s (2010) framework of *teachers' professional noticing of students' thinking*, which consists of three components: (1) attending to critical events that involve student thinking that the teacher could build on, (2) interpretation of a student's thinking, and (3) offering alternative teaching responses. Some scholars argue that current conceptualizations do not fully reflect the complexity inherent in noticing (Jacobs, 2017; Scheiner, 2016). The complexity of noticing is connected to the complexity of classroom situations, which stems from the different dimensions of learning and teaching mathematics – cognitive, affective, and social (Op't Eynde, et al., 2006; Skott, 2019). Being attentive to multiple dimensions embedded in a critical event means being attentive to its details. This is important as it allows a more appropriate interpretation of students' thinking, which, in turn, may lead to informed and effective responses (Barnhart & van Es, 2015; Mason, 2011).

In this paper, we focus on investigating the changes in PSTs' attention to critical events within the context of a year-long teachers' preparation program in which noticing critical events was a key preparation tool. To explore changes in PSTs' attention to details over the year, we defined the term of *degree of foci of attention*, which is the number of combinations of the different dimensions (cognitive, affective, and social) embedded in the PSTs' descriptions of the events. We ask: *What is the change in the PSTs' foci of attention when identifying critical events, they have witnessed while observing/teaching mathematics classrooms?*

Theoretical Framework

To characterize the different foci of PSTs' attention, we draw on two theoretical perspectives, which together with our previous empirical work (Rotem & Ayalon, in preparation) constitute a three-axis model: (1) *the participants of the event*, (2) *the content of the event*, and (3) *the dimensions of learning and teaching mathematics* (see Figure 1). To characterize the participants and the content of critical events, we build on van Es and Sherin's (2008) *Content and Stance framework*, drawing on two components: the *actor* in the excerpt that the teachers commented on (student, teacher, or other) and the *topic* of teachers' noticing, i.e., the mathematical thinking and/or the pedagogy (e.g., teaching strategies, classroom climate and management). To characterize the dimensions in learning and teaching mathematics, we draw on Op't Eynde et al.'s (2006) perspective, which addresses the interplay between different participants in different dimensions – cognitive, affective, and social. All in all, the three-axis model consists of 27 possible triples that we refer to as combinations of attention.

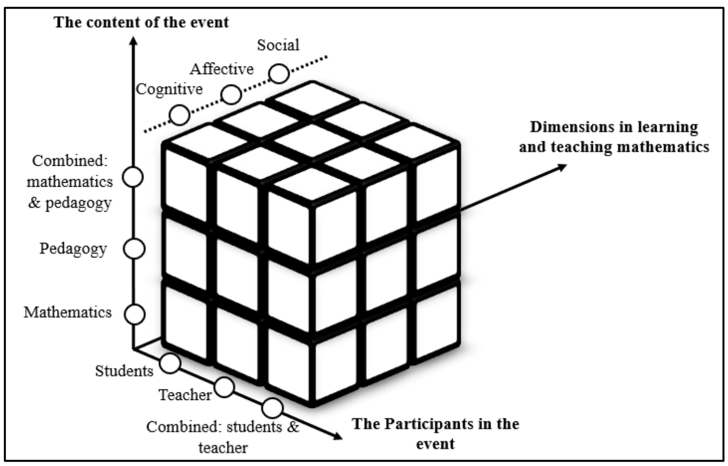

Figure 1. The three-axis model for characterizing critical events

Methodology

Data Collection

This study is framed within a larger research project, ACLIM-5. The project aims to examine the longitudinal change PSTs undergo as they identify, interpret, and respond to critical events in mathematics classrooms. The participants here were 20 PSTs who participated in the program in 2017-2019. They were asked to identify critical events while observing and teaching lessons as part of the program's practical experience. Additionally, they were required to submit 4-5 written reports in which they described and interpreted these events. The reports were built on Jacobs et al. (2010) conceptualization of noticing (see above). Overall, the 20 PSTs submitted 80 reports. To explore the change in foci of attention, we consider the first and last reports, 40 in total.

Data Analysis

We consider each report a unit of analysis and distinguish between the PSTs' first reports (20 units) and last reports (20 units). In the data analysis process, we first used our model (Figure 1) to code each of the analysis units. We identified 11 combinations (out of a possible 27). For example, the combination *student-mathematics-cognitive* [SMC] is the combination of the component 'students' from the *x* axis: the participant of the event, the component 'mathematics' from the *y* axis: the content of the event, and the component 'cognitive' from the *z* axis:

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dimensions of teaching and learning mathematics. Due to the lack of space, we will not share all 11 combinations here, but rather use them according to our needs; we will describe and demonstrate all the categories in our presentation.

We coded each unit of analysis with a specific combination if that combination was present within the analysis unit at least once. For example, when a critical event was coded as SMC, it means that this code appeared at least once. Then, to reveal a change in the PSTs' degree of foci of attention, we counted the number of combinations for each PST's first and last critical event. We then compared the degree of foci between the first and last critical event.

Preliminary Findings

The analysis reveals that for most PSTs, the degree of foci of attention was broader in the last critical event; 12 PSTs were attentive to more combinations in the last critical event. For four PSTs, the degree of foci was unchanged, and four PSTs attended to fewer combinations in the last critical event. Here we focus on the 12 PSTs whose degree of foci of attention was broadened. We illustrate these preliminary findings using Nasim's case as an example. Below is Nasim's first critical event, followed by our analysis (Table 1). Nasim's first critical event occurred during a lesson on trigonometric function analysis; he did not give details regarding the lesson, however, in Israel this would normally be taught in a high-level 11th-grade class.

> A student was asked to find the extreme point of the function. The derivative obtained was: $\cos\left(2x+\frac{\pi}{3}\right)=0$. He suggested solving by making $2x+\frac{\pi}{3}=0$ and [then] finding the x. The teacher turned to the class [CPS]: "remember what is $\cos(0)$?" and everyone answered: "1" [SMC]. Then the teacher said: "but we obtained that $\cos\left(2x+\frac{\pi}{3}\right)=0$ and not $\cos\left(2x+\frac{\pi}{3}\right)=1$. [Also,] what about the unit circle?" [TCC].

Table 1: Analysis of Nasim's first critical event using the model

Combination [Acronym]: Name	Definition	Brief analysis of Nasim's event
[SMC]: Students-Math-Cognitive	The students suggested various mathematical ideas, concepts, representations, solution strategies, or considerations for solution strategies.	In the event, the students raised two solution-related considerations: the student's suggestion that the cosine's argument equals to zero, and the students' mathematical statement that $\cos(0)$ is 1.
[TCC]: Teacher-Combined math & pedagogy-Cognitive	The teacher used various mathematical ideas, concepts, representations, or solution strategies in their teaching.	The teacher used various mathematical ideas and concepts. He simplified the students' statement by asking them about $\cos(0)$ instead of asking them about the original expression. Additionally, he also mentioned the unit circle.
[CPS]: Combined Teacher & Students-Pedagogy-Social	The teacher used strategies that make students' statements public to prompt the students' participation.	The teacher made the student's statements public. The teacher turned to the class and asked questions to prompt their participation.

Nasim's last critical event occurred in a 10th-grade high-level mathematics lesson, focusing on the circle theorems (for analysis see Table 2):

> In the past, they studied the theorem that, in front of equal arcs, there are equal chords. [In this lesson], the teacher taught the theorem that if two angles are inscribed on the same arc, they are equal. Then, coming to the question where there were two angles - each from the other side of the chord - one student said the two angles are equal because they rest on the

same chord. Some students said that it did not look right because the first angle seemed larger than the second angle. Other students started shouting [SMA] and saying that the two angles are not on the same side [SMC]. The teacher pointed toward the direction of the students' statement that the two angles were not from the same side until the students were convinced that the side of the chord is important [CCS]. [He] reminded them that before they were talking about arches and not chords. [he] draw them a drawing on the board [TCC].

Table 2: Nasim's last critical event's analysis using the model

Combination [Acronym]: Name	Definition	Brief analysis of Nasim's event
[SMC]: Students-Math-Cognitive	The students suggested various mathematical ideas, concepts, representations, solution strategies, or considerations for solution strategies.	The students offered some considerations for refuting the claim that the two angles are equal; some judged the claim by the angle's appearance while others used the theorem that the angles should be on the same side of the chord.
[TCC]: Teacher-Combined math & pedagogy-Cognitive	The teacher used various mathematical ideas, concepts, representations, or solution strategies in their teaching.	The teacher used two representations in his teaching: he went back to the exact words of the theorem and then used a drawing to illustrate why the theorem does not hold in this case.
[SMA]: Students-Math- Affective	The students indicated feelings, emotions, beliefs, attitudes, motivation, values, and moods towards mathematics.	Nasim articulates the students' behavior to indicate their motivation and interest toward the mathematics that was taught. The students were "shouting"; it seems that they were engaged in the debate in a way that got them involved emotionally.
[CCS]: Combined Teacher & Students-Combined math & pedagogy-Social	The teacher prompted and connected the students' ideas in the discussion, so one idea is built on another idea.	The teacher built on the students' statements and connected their ideas in the discussion.

In Nasim's case, the degree of foci of attention changed from 3 in the first critical event to 4 in the last. The broadening of his attention was reflected not only in his attention to the cognitive aspect of the students' mathematics, but also in attention to affective and social dimensions.

Discussion and Implications

In this paper, we started to explore PSTs' change in attention to critical events within the context of a preparation program in which critical events were a key preparation tool. We found that for most PSTs, the foci of attention were broadened from the first to the last critical event. These findings add to the existing research by indicating that learning to notice includes broadening the degree of foci of attention, expressed in attention to the details of different dimensions of teaching and learning. In particular, our findings demonstrate that within the moment in which noticing occurs, the different dimensions – cognitive, social and affective – play a part. A possible explanation for the change is the PSTs' experience within classrooms, both observing and teaching, an experience which has been demonstrated to be productive for learning to notice students' thinking (Simpson & Haltiwanger, 2017).

Currently we are examining if and how this change is reflected in the PSTs' interpretations of critical events. We investigate Jacobs's (2017; p. 278) question: "when teachers do not report noticing something of interest, did they fail to notice it or simply fail to report noticing it?" We believe that examining PSTs' interpretations of critical events will shed light on the change (or lack thereof) that could not be captured solely by analyzing the descriptions of the critical events.

Additionally, we propose that noticing could be seen more broadly, and suggest the three-axes model as a starting point for further conceptualization of noticing of critical events. Had we analyzed our data using existing tools, usually comprised of rubrics for levels of quality of attention (Scheiner, 2016), the changes we found here would have been overlooked.

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PST LEARNING TO FACILITATE ARGUMENTATION VIA SIMULATION: EXPLORING THE ROLE OF UNDERSTANDING AND EMOTION

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The present study focuses on examining transitions in elementary pre-service teachers (PSTs)' understanding of, and skills in, leading argumentation-focused discussions in mathematics during participation in a sequence of three different practice-based activities, collectively referred to as the Online Practice Suite (OPS). We will examine 14 PSTs' responses to post-activity surveys targeting their understanding of argumentation-focused discussions and emotional experiences, over the course of a single semester. From this initial coding, we will select three to four cases that represent a range of understandings and emotional experiences and conduct in-depth analyses on the patterns of engagement in the OPS, drawing on records of practice from their experiences in the OPS. We discuss ways that teacher educators can scaffold PSTs' experiences as they develop the skills to facilitate argumentation-focused discussions.

Keywords: Instructional Activities and Practices, Preservice Teacher Education, Classroom Discourse, Technology, Approximations of Practice, Simulation

Purpose of the Study

This study is situated within a larger project focused on examining how participation in a sequence of three practice-based activities, collectively called the *Online Practice Suite* (OPS), supports PSTs in facilitating argumentation-focused discussions in elementary mathematics. The larger project aims to address two challenges simultaneously, one acute and the other longer-term. The first challenge is COVID's impact on teacher preparation, which has pushed methods courses online and limited preservice teachers' (PSTs) access to field work (e.g., Reich et al., 2020; Saenz-Armstrong, 2020). The second is the endemic challenge of providing PSTs authentic and appropriately scaffolded opportunities to engage in the work of teaching—ideally across different contexts and diverse student populations (Grossman, 2018; Lampert et al., 2013; Sleeter, 2001; Whitaker & Valtierra, 2018). The OPS is designed to allow PSTs to experience a carefully scaffolded set of approximations of practice (Grossman, Compton, et al., 2009), all via an online environment that is resilient to COVID-induced constraints and provides teacher educators with an alternative resource to access field placements for PSTs. It is also designed to be appropriate as a future tool for teacher development across multiple contexts, including to complement field placement, and with a goal of supporting PST learning in ways that are easily adaptable to what teacher preparation comes to be in a post-pandemic world.

In the larger project, we will study teacher educators' use and adaptation of the suite across elementary and secondary mathematics and science teacher preparation and measure PST learning via a pre/post design. The study described here draws on pilot data from implementation of the elementary mathematics OPS to explore the research question: *how do PSTs'* *understandings of, and skills in, leading argumentation-focused discussions emerge over the course of an OPS engagement semester*?

Perspective(s)

The Importance of Argumentation-Focused Discussion

Communication is fundamental to mathematics teaching and learning. In mathematics education, there is a tradition of focusing on classroom discussions as a space to examine the teacher's role in facilitating communication among students (National Council of Teachers of Mathematics [NCTM], 2014; Stein & Smith, 2011). This line of research considers whose mathematical contributions the teacher takes up, how students are encouraged to talk about their ideas with one another and how the teacher moves the class toward collective meaning-making. With recent education standards, argumentation has taken the spotlight as a preferred discursive practice for students to learn and take up (National Governors Association & Council of Chief State School Officers [NGA & CCSO], 2010). Argumentation in mathematics involves students comparing, analyzing and critiquing one another's approaches to solving mathematics problems (NCTM, 2014; Smith et al., 2008). Facilitating these kinds of interactions among students in such a way that the outcome is, in fact, productive for students' sense making is difficult (Ball, 1993; Lampert, 2001). The combination of a focus on argumentation as a goal for students and the difficulty teachers have facilitating it has drawn recent attention to how pre-service and in-service teachers learn this practice (Gosek et al., 2018; Hallman-Thrasher, 2017).

Approximations of Practice as A Site for Teacher Learning

During the last decade, there have been increased calls for a focus on practice-based teacher education to address the widespread challenge of providing opportunities for PSTs to rehearse components of complex practice, like facilitating argumentation (Ball & Forzani, 2009; Francis et al., 2018; Grossman, Hammerness, et al., 2009; Lampert, 2010). Research has shown that teacher candidates are more effective when their preparation is directly linked to practice (Association of Mathematics Teacher Educators, 2017; Goodson et al., 2019). Approximations of practice entail immersing novices in the activities in which they routinely engage during instruction (Grossman, Compton, et al., 2009). More recently, technological advances have enabled the development of digital practice spaces and virtual classrooms (such as those used in this work) to engage teachers in rehearsals. These digital practice and classroom spaces, while simplified, contain core elements and behaviors typical of real classrooms (Brown, 1999; Dieker et al., 2014), thereby providing sheltered environments for PSTs to engage in repeated practice, (Badiee & Kauffman, 2015; Garland et al., 2016; Grossman, 2010; Straub et al., 2014), and can be customized to meet a teacher educator's instructional purposes (Herbst & Kosko, 2014).

Emotions are closely connected to cognition and action, and are considered influential on how teachers make, and enact, instructional decisions (Hargreaves, 2000; Oatley, 1991). Research around teachers' emotions has increased significantly over the last decade, with studies focused on understanding the relationship among emotions and other constructs more emergent, and very little focus on how these relationships unfold in virtual learning environment. Recent work (Cross Francis et al., 2020) has shown that elementary mathematics teachers experience a complex array of emotions that vary in fluctuation patterns as teaching unfolds. These patterns served to stimulate productive teaching practices (e.g., professional noticing; Cross et al., 2017) for some teachers, while for others, they exacerbated adverse physiological and cognitive responses non-conducive to effective teaching. Virtual classroom spaces afford opportunities to systematically investigate the relationships between teachers' emotions and instructional practices and provide a supportive space for PSTs to hone their emotional regulatory skills.

Methods

Study Context: The Online Practice Suite

The practice-based activities that make up the OPS include: (1) Focused-Practice Spaces (FPS): interactive, online digital games that create targeted practice spaces to engage PSTs in considering and responding to students' content-focused ideas; (2) Avatar-Based Simulations (ABS): performance tasks that provide opportunities for PSTs to practice facilitating discussions with a group of five upper elementary avatars; and (3) Virtual Teaching Simulator (VTS): a virtual reality environment that allows for verbal, textual, and non-verbal interactions between a teacher avatar (played by the PST) and 24 student avatars in an immersive whole classroom.

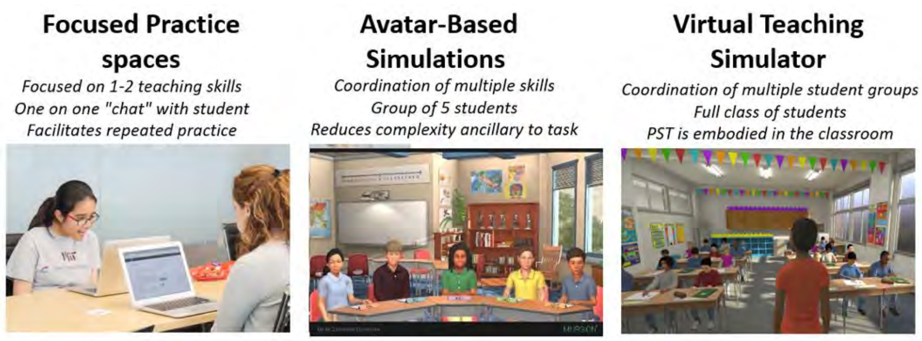

Figure 1: Activities that make up the Online Practice Suite

Participants

For the purpose of this paper, we focus on data from a pilot study involving PSTs enrolled in elementary mathematics methods courses at two different university sites in the Northeastern United States. All enrolled PSTs (24) engaged in three practice-based activities sequentially over the course of a single semester. A subset (14) consented to have their records of practice retained for research purposes; among those eight completed all required research activities.

Data Sources

Selected responses to post-activity surveys. Each PST completed a post-task survey after each activity (three time points), including a combination of Likert and open-ended responses. We analyzed items focused on PSTs' perceptions of discussions and argumentation (e.g., how they define discussion and argumentation) and items focused on PST's emotional experiences during engagement (e.g., what emotions they report feeling).

Records of practice. Each activity generates records of the PST's practice. For FPS this includes transcripts of PSTs' responses that are typed or spoken into the system; for ABS and VTS, the records are video-recordings of the PSTs facilitating small and whole-group discussions with student avatars.

Analyses

There are two phases of analyses, the first of which is complete as of the submission of this brief report. In phase one, PSTs' responses to the post-activity surveys across the three time points were coded to capture their ideas about argumentation-focused discussion and their

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emotions as they engaged in the activities. We looked across these responses for patterns present in the data set to select two cases for further investigation in phase two.

During phase two, we examined the records of practice for the selected PSTs, attending in particular the degree to which their stated understandings of what teachers should to do support argumentation-focused discussion are observable in the actual teaching moves they make during the simulations, and the degree to which we can observe evidence of their reported emotional state hindering or supporting their engagement.

Results

Phase 1 Results

A total of eight PSTs consented to participate in research and completed all three surveys. PSTs described high quality discussions and argumentation as centered around “student-to-student interaction”, and involved “critical thinking around math ideas” with a “focus on understanding”. With respect to argumentation, PSTs additionally emphasized debate grounded in defense of claims, reasoning and justification of ideas. We observed different trends in emotional experiences across the activities. Half of the PSTs (four) experienced a consistent emotion across the three activities, with two experiencing anxiety (negative) and two excitement (positive). The other four PSTs experienced a mix of positive and negative emotions.

We selected two PSTs for deeper analysis, whose responses with respect to argumentation-focused discussion and what teachers should do to support it were both robust and similar, but whose emotional experiences were quite different. Their descriptions of the teaching moves that support argumentation-focused discussion included “asking effective questions”, “promoting respectful student interaction”, “guide discussion”, “creating opportunities to hear and learn from other ideas”. One PST consistently experienced excitement, making statements such as, “I was excited to take part of an innovative experiences like this one!” whereas the other expressed anxiety in comments such as “It was new and scary. I don't know who I am talking to”.

Phase 2 Anticipated Results

We anticipate, based on initial observations of the OPS implementation, that understanding argumentation-focused discussion may be a necessary but not sufficient condition for success in leading such discussions, and that the case study PSTs may or may not consistently make the teaching moves they describe as important, although we expect to see more such moves across the time points. The role of emotion in PST learning is complex, and we hope, from this analysis, to develop provisional hypotheses about whether and how the nature of the observed emotional pattern may hinder or support learning that can then be tested in later analysis of the broader set of PST data.

Discussion

As the world recovers from COVID-19, teacher education will inevitably re-normalize, but will likely never be completely the same. It is critical that we capitalize as a field on the innovation sparked by necessity, incorporating the best inventions borne out of crisis into our future work. Technology-based interventions, such as the OPS, designed to focus on complex, content-intensive teaching practices, not only afford us a bridge to the end of the pandemic, but also a way of thinking about deliberately scaffolding PST learning around the very practices that are most difficult to learn to do well, and that PSTs are least likely to encounter by chance in field experiences. Understanding how PSTs learn from such experiences and how that learning intersects with emotion is critical in informing the design of such experiences and in helping teacher educators to make sense of and use such innovative tools effectively.

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MIDDLE SCHOOL MATH PRE- SERVICE TEACHERS AND ABILITY-BASED COURSE ASSIGNMENT: EXPERIENCES, BELIEFS, AND DISPOSITIONS

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This study aims to understand the beliefs that middle school math PSTs hold about tracking practices in the middle grades and potential influential experiences that have lead to the development of these beliefs. Through a survey and semi-structured interviews, I examine what experiences middle school math PSTs have had with ability-based course assignment as well as what affordances and constraints they perceive for teaching and learning within tracked math classrooms. I find a lack of awareness in how course placements were determined in middle school. The lack of mobility in more highly tracked systems was seen as an important constraint for PSTs. Most PSTs did not explicitly attend to the equity aspects of detracking when presented with the idea. I propose some potential implications for math teacher educators.

Keywords: Preservice Teacher Education, Teacher Beliefs, Equity, Inclusion, and Diversity, Middle School Education

Although tracking in middle school mathematics is widespread in the United States, NCTM (2019, 2020) has called for an end to tracking due to the inequities the practice perpetuates, particularly for students of color or from families living in poverty. One potential reason that tracking persists despite these inequities is that teachers feel underprepared to teach in more diverse classrooms (Jackson et al., 2017; Loveless, 2013). To support new teachers in this effort, pre-service teachers need to be adequately prepared to teach in and promote these heterogeneous classroom environments. Since entering beliefs and dispositions often interfere with efforts to pre-service teachers to promote more diverse classrooms (Major & Brock, 2003), an initial step in this effort is to better understand the beliefs and dispositions these pre-service teachers hold as well as the experiences that have contributed to their development.

In this study, I focus on understanding middle school mathematics pre-service teachers' (PSTs) experiences with ability-based course assignment, beliefs about how course assignment should be structured, their dispositions toward more diverse classrooms, and how these three components relate to each other during teacher preparation. I build upon Garmon's (2004, 2005) framework for promoting positive change in PST's beliefs in regard to cultural diversity. This framework, posits that three types of experiences (intercultural, educational, and support group) along with three dispositions (self-awareness/reflectiveness, openness, and commitment to social justice) are key factors associated with changing PST's attitudes toward and beliefs about diversity. To address this, I ask: In what ways have middle school math PSTs experienced ability-based course assignment? What beliefs about and dispositions toward teaching and learning within tracked and non-tracked learning environments do middle school math PSTs hold? What experiences do these PSTs credit as influential in the development of these beliefs?

Background

Many teachers and administrators, even those who espouse ideals of equity and diversity, support tracking in practice (Linchevski & Kutscher, 1998). Some believe that tracking allows teachers to better match content and pedagogy to students' ability level (Garmon, 2004; Jackson et al., 2017; LeTendre et al., 2003). Some find it challenging to adequately support the full range of students in mixed-ability classes (Loveless, 2013). Despite these beliefs, heterogeneous classrooms have consistently shown more equitable outcomes than homogenous ones (Boaler & Staples, 2008; Cohen & Lotan, 1997; Linchevski & Kutscher, 1998). Although tracking may be intended as a way to meet the needs of all students, in practice students of color and from families living in poverty are more likely to be placed in lower-level courses (Battey, 2013; Morton & Riegle-Crumb, 2019). Furthermore, there is a lack of transparency in how course placement decisions are made and how early placements may restrict future course options (Akos et al., 2007; Kilgore, 1991; Rice, 1997). Historically marginalized families often do not share in the cultural capital needed to navigate this inconsistent system in ways that would advantage their children (Antony-Newman, 2019; Kilgore, 1991; McDonough, 1997).

Teachers have an important role to play in potentially disrupting the status quo of tracking, starting in middle school where most formal tracking begins. However, if teachers are to do so, they must develop equity-oriented beliefs about how middle school classrooms should be organized. One way to address this is by focusing on teacher preparation, supporting PSTs to understand the issues associated with tracking and preparing PSTs to teach in more heterogeneous settings. Teachers often enter the workforce underprepared to teach mathematics effectively in more diverse classrooms (Howard, 1999; Kitchen, 2005; Sleeter, 2001; Wiggins & Follo, 1999) and to advocate for more equitable instruction, especially when it involves disrupting the status quo (Herbel-Eisenmann et al., 2013). PSTs enter into this tracked system and must navigate the cognitive dissonance between what they may have been taught about equity in their education programs and what practices they perceive as successful in classrooms (Neumayer-Depiper, 2013; Webel & Dwiggins, 2019).

To support PSTs in developing an equity-focused stance toward tracking, we must first understand what beliefs they hold and how they developed. Causey et al. (2000) called for more research on how students' prior experiences and dispositions may influence their responses to more diverse classrooms. Major and Brock (2003) highlight that the interference of PSTs entering beliefs and dispositions create persistent problems in fostering positive dispositions toward diversity. PSTs are more likely to embrace information consistent with their existing beliefs and prior experiences and so, it is important for teacher educators to learn about these beliefs, experiences, and dispositions in order to incorporate educational experiences that will effectively challenge them (Garmon, 2005). If we wish to increase teacher advocacy for more heterogeneous classrooms for the sake of equity, PSTs must believe that more diverse classrooms are more equitable, feel confident in their ability to teach in heterogeneous classrooms, and be able to navigate the cognitive dissonance they may face between their beliefs and the practices in their schools. To achieve this, we need to better understand the beliefs PSTs hold about tracking and how these beliefs have developed through their preparation and transition from student to teacher.

Methods

To address these research questions, I first surveyed a sample of PSTs from a mid-Atlantic university who are majoring in elementary education and have chosen a second certification in middle school math. I do this to provide a broader overview of the types of experiences that PSTs in this concentration have had with tracked math environments. I received responses from 20 PSTs. From this group, I selected a sub-sample of 9 students (1 freshman, 3 sophomores, 5 juniors) to participate in semi-structured interviews. Interview participants were chosen due to varied experiences with course assignment (as indicated on the survey) and to reflect different points within their undergraduate course of study. This sample consisted of eight females and one male. Seven participants identified as White (two of whom also identified as Hispanic), one participant identified as Asian, and one participant identified as both White and Asian. All interview participants attended public middle schools in the Mid-Atlantic or Northeast regions.

The interview consisted of three components. First, the PSTs were asked to elaborate on their survey responses about their course placements in middle and high school and to reflect on how their course placements were determined. They were also asked about their own feelings toward their course placements and their level of preparedness for future math classes. Finally, they were asked about how, if at all, their university coursework has addressed components of ability-grouping or ability-based course assignment. Next, the PSTs were provided two descriptions of fictional middle school math teaching positions, one school more highly tracked than the other. They were asked to indicate in which school they would prefer to teach and to explain the affordances and constraints of each option. Finally, the PSTs were presented with an excerpt from a recent article about one district's detracking efforts (Yoder, 2020), asked to annotate the article, share their reactions, and reflect on if and how their own experiences relate to their responses.

I analyzed surveys for common themes in mathematics placement (all on-level, all advanced, mixed levels), grade of first ability-based division, and the level of awareness students had of how placement decisions were made. The interview analysis was guided by the framework of Garmon (2004, 2005) and Mills and Ballantyne (2010). I coded interview transcripts for dispositions and types of influential experiences (as student, classmate, or PST), also using open coding for beliefs related to ability-based course assignment and PST's feelings toward their own course placement.

Results

All 20 PSTs experienced ability-based course assignment, beginning between late elementary school and ninth grade. The majority of PSTs were placed in an advanced class for all or most of their math placements. None of the participants were ever enrolled in a remedial math class. Almost all respondents were aware of how (n=19) and by whom (n=18) course assignment decisions were made in high school. In contrast, about half the PSTs were aware of how (n=12) or by whom (n=8) course placement decisions were made in middle school. In interviews, students described or hypothesized what factors determined their math course placements in high school and middle school. For high school, PSTs cited that student input and grades were key factors (see Table 1 for the hypothesized factors in middle school). Furthermore, eight of the nine indicated that they felt their course placement was appropriate at least some of the time.

Table 1: Middle School Course Placement Factors by Placement Trajectory

	All Advanced	Mostly Advanced	Mostly On-Level	All On-Level	Total
Grades	3	-	2	2	7
Teacher	3	1	1	1	6
Student input	1	-	-	2	3
Test Scores	1	1	-	1	3
Guidance Counselor	1	-	-	1	2

When asked to choose where they would prefer to teach, PSTs were given two options: a more highly tracked school (Jupiter) that divided math classes beginning in 6th grade based on standardized test scores and a less highly tracked school (Zeus) that divided math beginning in 8th grade based on student choice. Four PSTs chose Jupiter and five PSTs chose Zeus. See Table 2 for the affordances and constraints cited for each school.

Table 2: Affordances & Constraints for Each School Choice

	Jupiter		Zeus	
	Affordance	Constraint	Affordance	Constraint
Match student ability	7	2	-	3
Standardized testing	3	4	2	-
Student choice	-	2	4	1
Difficulty of mixed-ability	1	-	-	5
Mobility	-	4	2	-
Student emotional wellbeing	1	1	3	-
Mixed ability classroom	-	-	3	1
Dividing on ability	2	-	-	-

Finally, when asked to respond to the article about detracking (Yoder, 2020), most PSTs were open to the idea of detracking at least somewhat, but they varied widely in their reasoning. Some interviewees focused only on the academic statistics prompting the change. Only four PSTs mentioned the equity aspects of detracking highlighted in the article and it was only the focus of the response for one PST.

Discussion and Conclusion

This study focused on examining the beliefs, experiences, and dispositions middle school math PSTs hold in regard to tracking in the middle grades. Results indicate that, while PSTs were aware of how course assignment decisions were made in high school, they experienced a lack of transparency in how their middle school course decisions were made. Lack of mobility between tracks and the use of standardized test scores were important factors that PSTs in this study considered a constraint of a more highly tracked system. The beliefs PSTs hold about course mobility and standardized testing could be potential entry points for developing equity-oriented beliefs around detracking. Notably, none of the PSTs included the practice of dividing students into classes based on ability as a constraint within a tracked system. Only two PSTs mentioned it as a benefit of the tracked school. When reflecting on the article, a number of PSTs indicated this was the first time they had heard of middle schools without this division despite being several semesters into their education program. One possible implication for math teacher educators is that they need to introduce their PSTs to such possibilities if they wish to see the PSTs become advocates for detracking. PSTs cannot question the status quo of tracking if they are unaware of alternatives.

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HOW MATH ANXIETY AFFECTED MY TEACHING PHILOSOPHY: PERSPECTIVES FROM PRE-SERVICE SECONDARY MATHEMATICS TEACHERS

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Teachers with math anxiety can pass their anxiety on to students and even affect student achievement. In this project, we attempt to analyze pre-service secondary mathematics teachers' (PSMTs) feelings about how their math anxiety affects the way they view teaching as a profession. Preliminary findings indicate that math anxiety could affect multiple areas of a teacher's outlook on teaching, including preparing for class, engaging with students, and their overall philosophy about teaching. This case study can serve as a launching point for further research into how math anxiety affects both pre-service and in-service secondary mathematics teachers.

Keywords: Affect, Emotion, Beliefs, and Attitudes; Teacher Beliefs; Preservice Teacher Education

Math anxiety, defined as "an adverse emotional reaction to math or the prospect of doing math" (Maloney & Beilock, 2012), is a condition that is estimated to affect 50% of students (Beilock & Willingham, 2014) and has been shown to affect elementary education majors to a large degree (e.g., Hembree, 1990; Sloan, 2010). Unfortunately, the literature is lacking when it comes to exploring math anxiety in secondary mathematics teachers. In the research that does exist, any discussion of math anxiety in teachers is often included as a subset of discussions of identity or teacher beliefs (e.g., Ertekin, 2010; Ren & Smith, 2018; Unlu et al., 2017) A meta-analysis showed that only 28% of research articles focusing on math identity were focused on teachers, and only 17% of similar articles focused on pre-service teachers (Darragh, 2016). As such, we felt it was pertinent to dedicate this study to mathematics anxiety in pre-service secondary mathematics teachers (PSMTs).

Background

Research has shown that teacher math anxiety negatively affects student performance, particularly for female elementary teachers of female elementary students (Beilock et al., 2010). Another study demonstrated that 9th-grade students' math achievement was negatively affected by teacher math anxiety (Ramirez et al., 2018). This effect of math anxiety seems to be largely due to negative emotions expressed by teachers as they address mathematical content in the classroom, which leads to the students adopting similar attitudes that may encourage the development of math anxiety (Ramirez et al., 2018; Rozgonjuk et al., 2020; Szczygiel, 2020). This relationship presents a myriad of possible research topics, such as the prevalence of math anxiety in secondary math teachers, common experiences among secondary math teachers with math anxiety, and why someone with math anxiety would decide to become a secondary math teacher. For this paper, we decided to investigate the following research question: What areas of teaching do those with math anxiety identify as connected to their anxiety?

Methodology

Data were gathered from four PSMTs (Caroline, Rachelle, Shelby, and Willow) at a large, research-focused university in the southeastern United States. The participants initially responded to an online survey that was developed by the research team and distributed to all PSMTs at the university. After answering questions about their feelings towards math and self-reporting whether or not they had math anxiety, the potential participants either agreed or disagreed to be contacted for further research. If they agreed, they were sent a link to another survey, where they could schedule a time for a one-on-one interview with the first author.

The semi-structured interviews were conducted via online conferencing tools and lasted anywhere from 20 minutes to one hour and were composed of several questions about the participant's mathematical background and feelings about math in general. Interviews were recorded for the purpose of transcribing. The interview transcripts were analyzed individually by both authors for large themes and commonalities between participants. The interviews were coded using descriptive coding, defined as coding that "summarizes in a word or short phrase…the basic topic of a passage" and in vivo coding, which is described as "a word or short phrase from the actual language found in the qualitative data record" (Salda , 2013). These codes emerged as the transcripts were reviewed. After each author had reviewed the transcripts, we met to discuss our codes and worked together to develop our final codebook and coding scheme.

Findings

The four PSMTs included in this study shared stories about how they came to identify as mathematically anxious, and to situate the findings, we outline some of the key points from each of their narratives in Table 1.

Table 1: Math Anxiety Narratives by Participant

Participant	Math Anxiety Narrative
Caroline	Math anxiety tends to manifest in avoidance of mathematical tasks, especially homework. Grew up being told she was gifted in math, which is why she believes she was good at math; her success was not correlated to effort.
Rachelle	Remembers answering an order-of-operations question incorrectly in class, and the class laughed at her. Considered mathematically gifted growing up and felt pressure to perform perfectly.
Shelby	Intimidated by the formal language of mathematics. Might fall behind and then worries that she will stay behind and never fully understand that material.
Willow	Has had many unsupportive and disparaging math teachers. Does not believe herself to be a quick mathematician. Tends to be anxious in other STEM classes, as well.

Each of the participants in this study shared connections between their math anxiety and teaching practices. Several commonalities existed between participants, including ways in which math anxiety affected their lesson planning, experiences with students and as a student, and their overall teaching philosophies.

Lesson Planning

Each of the participants discussed how math anxiety had affected their lesson planning process in some capacity. Three of the PSMTs, Caroline, Shelby, and Rachelle, explicitly stated that their lesson planning processes were hindered by their math anxiety. These hindrances manifested in the several ways, including that participants:

- Believed they spent longer planning lessons than they should.
- Felt the need to overprepare for class or tutoring.
- Became distracted by tangential topics in their planning.
- Found themselves including too many topics in their lesson, leaving little time for practice or questions in class.
- Doubted whether they had thought of enough possible student responses, methods, or questions to be adequately prepared for class.

These PSMTs worried that they would not be effective teachers due to their anxieties about the content. Specifically they worried that they would spend too much time preparing to understand the content that they would have to sacrifice time planning how best to explore the topics with their students. Even so, Caroline looked at these hindrances through an optimistic lens, stating, “I think [math anxiety] hinders my side of the planning, but I’d like to think that it benefits the end result. The end of it is justified by the means.”

Experiences with Students

Several of the participants described how their math anxiety manifested in physical symptoms while working with students. Willow described visibly shaking and playing with her nails; Caroline also recalled times when she was so nervous to be in front of students that she would shake, stutter, and notice her palms were sweaty. Shelby noted that she would often play with her hair or twiddle her fingers, and she described her experiences with students in the following way: “When I'm actually up at the front, I am, start to finish, kind of nervous, like ‘stomach in knots’ kind of thing.” Fortunately, Shelby was able to identify events that lessened her anxiety, primarily when students would ask questions that she was able to answer confidently.

Shelby also noted that she was more at ease with the students when she did not have to be up at the front of the room. If she could meander through the desks and work with students on a more individualized basis, she noticed that her anxiety was lower than when she was standing at the board. Willow also attributed her ease in the classroom to primarily having had experiences working one-on-one with students. These interactions led her to get to know her students better, which also lessened her anxiety in the classrooms. As such, these PSMTs came to realize the importance of developing meaningful relationships with their students.

Caroline described her experiences leading class as “diving off the diving board into the deep end of the pool.” At the beginning of a lesson, she might be nervous, but as she received positive feedback from her students that they were understanding the material, her anxiety went away. She also noted that if she received negative feedback from the students, her anxiety increased and caused her to doubt her knowledge of the content and abilities as an instructor. Similarly, Rachelle remembered a time when she was tutoring a 4th-grade student and felt helpless upon encountering a method for multiplying two-digit numbers that she had not seen before. “I felt clueless and incompetent. I thought ‘Am I smarter than a 4th grader? No.’ I was there to help her, and I couldn’t do anything about it. That was stressful.” Rachelle used this opportunity to revise her thinking by researching the method that the student wanted to use. In the end, she found that she really enjoyed the method, so much so that she "used it on the SAT." Because of their experiences, these participants developed an appreciation for the feedback they received from students and the effect this feedback could have on their anxiety in the math classroom.

Teaching Philosophies

In accord with the lessons learned through their experiences with students, each of the PSMTs' teaching philosophies centered on the relationship between the teacher and the student. Each of our participants expressed a unique point of view when it came to their philosophy, but when taken together, they created a cohesive picture of a thoughtful classroom.

Willow set the scene by envisioning a classroom where students feel welcome and open to "contribute without being shamed or [made to] feel like they can't ask questions." She believes that math is a broad subject to be explored by all students, not just a select few. Caroline echoed this sentiment in her musings, stating that "every student is great and has potential, and every student can do what they put their mind to." She believes that no student is unable to succeed in math or any other academic subject, for that matter.

Shelby seemed to agree with Caroline, but she issued a word of warning in her philosophy. She cautioned against "hyping someone up too much," citing this as something that happened to her in school and saying that it was detrimental to her in the long run. She encouraged teachers to meet students where they are, academically and developmentally, without making the students feel like they are better or worse than they really are. Her final thoughts hearkened back to Willow's sentiments about "creating a very hospitable environment where students can speak up about concerns."

While Rachelle shared many of the sentiments expressed by her peers, particularly about creating wholesome, safe environments for students, she shared that she was leaving the College of Education at the end of the semester because of her math anxiety. She came to realize that she "was probably drawn to teaching more for connecting with people than the actual teaching itself." Rachelle had struggled to complete her Calculus sequence, and she developed the mindset that she might not have the content skills necessary to be a truly excellent math teacher. For her, being good wasn't good enough, and for the sake of her future students, she decided to leave the profession before her career even started.

Discussion and Implications

The four PSMTs in this study each developed math anxiety in unique ways, but they shared many similarities in their teaching philosophies. Each of the participants recognized that their anxiety might cause them to spend additional time preparing for class when compared with their non-mathematically anxious peers. They also believe that a positive teacher-student relationship is a critical component of any successful math classroom.

The findings presented in this paper indicate that there is a viable research agenda in the realm of math anxiety in secondary mathematics teachers. Future research might address any of a litany of questions, including:

- How prevalent is math anxiety in secondary math teachers?
- Are there common experiences among secondary math teachers with math anxiety?
- What other facets of a secondary math teacher's career are affected by math anxiety, and how so?
- What causes people with math anxiety to decide to become secondary math teachers?

The answers to these questions, and many more, will help us to better serve our students in math classes across the country and around the world. By determining the causes and effects of math anxiety in secondary math teachers, we can work toward reducing the prevalence of math anxiety, thereby opening the doors to untold mathematical endeavors for students, and teachers, everywhere.

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'MY HOST TEACHER'S MATH TIME': POWER DYNAMICS DURING ENACTMENTS OF STUDENT-CENTERED ROUTINES IN FIELD PLACEMENT

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In this paper we share data regarding preservice teachers' (PSTs') experiences enacting a number talk routine within a clinical experience, focusing on the ways that the PSTs' described their mentor teachers' influence on the routine. In particular, we describe the case of Ms. Brooks, a PST who lamented several instructional decisions made by her mentor teacher, including interrupting the routine with questions and comments that countered the goals of the number talk routine. The case of Ms. Brooks' challenges can help teacher educators consider the ways they might support PSTs' learning in situations where routines like number talks might come into conflict with status quo practices and instructional norms within clinical experiences.

Keywords: Instructional activities and practices, Preservice teacher education

Effective teacher preparation features coordination between methods courses and hosting schools where preservice teachers (PSTs) have opportunities to observe and practice teaching prior to certification (Darling-Hammond & Bransford, 2005). At the same time, mathematics teacher educators are developing programs, courses, and assignments to explicitly "help beginning teachers challenge deficit views about learning by questioning the status quo at a systemic level" (Association of Mathematics Teacher Educators, 2017, p. 35; see also White, Crespo, & Civil, 2016). This includes interrogating instruction that privileges only certain kinds of mathematical competence or certain kinds of students. Importantly, this type of instruction is likely occurring at sites where PSTs are engaged in clinical experiences.

Within most traditional models, clinical experiences are usually wide-reaching, obligatory activities where preservice teachers are socialized into a teaching space and develop a teaching identity (Bolton, 1997). The novice-to-expert frame generates an inherent power imbalance between the positioned novice and expert, which can lead to professional conflict, novice complicity in reinforcing the status quo, and/or missed opportunities for shared learning (Graham, 1993; Graham, 1997; Valencia et al., 2010). Further research is needed in the development of positive, productive student teaching placements which supports productive interaction between preservice teacher and mentor teacher (Darling-Hammond, 2006; Tang, 2003; Graham, 1999) and also enables novices to develop visions of equitable instruction.

Within this paper, we explore tensions inherent in clinical experiences during PSTs' enactment of a student-centered, number talk routine. Number talks feature class discussions focused on building students' number sense through exploring multiple strategies, shared student-teacher authority, and opportunities for public praise of student ideas (Parrish, 2010). In particular, we ask, "In what ways do host teachers' impact the enactment of the instructional routine, as noted by PSTs in their reflections on the enactment?"

Methods

Context. This study took place within a traditional undergraduate teacher education program at a large state university in the Midwestern United States, during the year prior to student teaching. As part of the program, PSTs were required to spend 60 hours over the course of the semester in an elementary classroom that was assigned by the university. Several of their methods course assignments required them to engage in and/or reflect on their experiences in their field placement classes. In their elementary mathematics methods course, taught by the second author and assisted by the first, PSTs were asked to create and rehearse number talk routines before enacting one with students in their respective host classrooms.

Data Collection. PSTs captured audio recordings of their enactment, which were later reviewed by a classmate and a course instructor. All three parties utilized a video annotation tool (VoiceThread) to provide time-stamped comments on their enactment.

Data Analysis: In reading PSTs' annotations on their recorded number talk routines, the authorship team noticed several PSTs mentioning their host teacher's influence on the enactment. We categorized these in terms of whether the influence was supportive (e.g., aligned with the goals of a number talk) or detractive (opposed to the goals of a number talk), and then attended to the ways that PSTs characterized the host teachers' influence. In particular, we focused on tensions that PSTs sought to navigate between the expectations embedded in the number talk assignment, the expectations of their host teachers, and their own ideas about mathematics teaching and learning. In this paper, we feature an episode involving a PST, Ms. Brooks, and her mentor teacher, Ms. Smith, in order to exemplify some of these tensions.

Results

Patterns in PSTs' mentions of mentor teacher's influence. Of the 17 PSTs in the elementary methods course, seven (including Ms. Brooks) added video annotations that mentioned actions or expectations of the host teacher in their implementation of the number talk routine. Two of these seven PSTs described ways in which host teachers supported the routine, such as one PST who shared they were "glad Ms. Donaldson (pseudonym) caught me & told me to change the [marker] color," a consideration when representing student thinking to provide easier access to various students' strategies.

Five of the seven PSTs, including Ms. Brooks, made comments about ways their host teachers detracted from their ideal enactment of the number talk routine. Three of these 5 PSTs only shared ways in which teachers influenced the enactment in non-math specific ways, such as being pressed for time, as illustrated in the following anecdote:

> "My teacher basically only allotted 10 minutes just because they had a really busy math schedule and the time was running out. So after the turn and talk she kind of signaled to me to wrap it up, and that kind of made me a little flustered, so I wish that I had had them discuss what they had just talked to her neighbors about."

The case of Ms. Brooks. Ms. Brooks' discussion of Ms. Smith's influence on her number talk enactment was particularly revealing. In the recording, Ms. Smith introduced Ms. Brooks' number talk routine with the following message:

> Alright, Ms. Brooks is gonna kind of do a little warmup with some number stuff. So your attention needs to be on her and I expect you to have all the same rules with her as you do with me. You sit on your bottom, you have learner look, you face forward, you raise your hand if you have something to say. Ok? They're all yours.

After providing students with the procedures of a number talk, Ms. Brooks asked her 3rd grade students to mentally solve 33+ 58. Later, when addressing the allocated time, Ms. Brooks shared her desire for more time, but also sought to comply with Ms. Smith's expectations.

In my own classroom, I may have waited a little longer, but I wanted to make sure we could get through plenty of strategies without taking up too much of my host teacher's math time.

In the episode, Ms. Brooks collected student answers and then asked individual students to share their ideas, representing their strategies in various colored markers on the whiteboard. The first two students shared solutions based on number decomposition, which featured a “turn and talk” opportunity, verbal questioning and praise from Ms. Brooks, and no engagement from Ms. Smith. A third student, Josie, then shared that she “did the numbers up and down.” As Ms. Brooks asked Josie to clarify her words by asking “Up and down?”, Ms. Smith interjected by saying “So you’re doing the algorithm?” Josie agreed, proceeding to explain the required steps of the multi-digit addition algorithm (see Figure 1). Ms. Smith praised Josie for her explanation, stating “Very good. We haven’t had anybody explain that yet. That was excellent. Very good.”

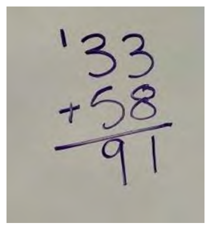

Figure 1: Ms. Brooks’ representation of Josie’s solution

Next, Ms. Brooks praised the clarity of Josie’s explanation, and then called upon Milly to share her solution strategy. While sharing, Ms. Brooks asked Milly “Can I pause you for one second? So you said five plus three…,” to which Milly responded, “equals 11.” While Ms. Brooks prepared to continue questioning, Ms. Smith interjected and had the following exchange with Milly:

Ms. Smith: Five plus three got you 11?
Milly: Wait, no nine.
Ms. Smith: I think, I think you were right on the first one where you said 3 plus 5 is 8 but it really should be 9 because what do you want to do with your answer?

The dialogue continued, concluding with Ms. Smith asking Milly, “so you want to change it, right? OK.”

After a final student shared his strategy, Ms. Brooks concluded the routine by praising the students’ mathematical thinking and identifying them as “good math thinkers,” before asking them to give themselves a round of applause. Ms. Smith agreed, and concluded the experience with the following words of her own:

> “And I also liked to see you work with another teacher because it shows what you’ve learned from the beginning of the year until now. Because I would say most of you probably couldn’t have explained all of this at the beginning of the year and now you can. And we’re talking about open number lines, tens and ones, the algorithm, you know, regrouping, all those really good things.”

Tensions revealed in the episode and Ms. Brooks’ reflections. In this episode, while Ms. Smith and Ms. Brooks were both heard praising and questioning students throughout the recording, the ways in which they did so differed. Ms. Brooks provided verbal praise to each student who shared a strategy, projecting equal value for each strategy. Ms. Smith, on the other hand, interjected to praise Josie's algorithmic solution, and then again did so for Milly's use of the standard algorithm. These interjections, in our view, signaled a high value placed on procedural competence, undermining the number talk's emphasis on multiple strategies and the employment of place value reasoning.

When questioning students, Ms. Brooks primarily asked students to re-voice, further explain and/or interact with other students' ideas, or utilized strategies such as having turn to a partner and talk about a specific idea that had been raised. Ms. Smith's questions primarily required recall and led students to a desired response, as seen in her exchange with Milly. Ms. Brooks noted these differences in her annotations, stating:

> I wish my host teacher hadn't broken in here. I would have liked to have gotten through the rest of the thinking and asked her a few more questions about it to see if [Milly] would have come to this conclusion and changed her mind on her own… I would have continued letting her talk through the strategy, then I may have asked the other students if anyone had any questions for her…or had them turn and talk about how two people got two different answers.

Ms. Brooks' peer reviewer, a classmate, agreed, noting:

> Yes! I wish she wouldn't have said anything. Allowing the student to continue to talk through it would have been a great teaching experience for you & the students. A turn & talk would have been great here as well.

Ms. Brooks also noted that she had wanted to close the number talk with an emphasis on the importance of "being able to change our minds [as] a great way to be smart at math," and suggested that "it probably would have helped if [Milly] had come to the conclusion on her own instead of with the teacher's help." Instead, the routine closed with Ms. Smith's emphasis on demonstrating the various ways in which they had "learned" new ways of doing mathematics throughout the school year, including the algorithm and re-grouping. Despite the challenges in its enactment, Ms. Brooks expressed a desire to use number talks in her future classroom:

> I am really looking forward to incorporating number talks into my future classroom. I think they are a really powerful way to build the big ideas behind math concepts, keep older content fresh in our minds, teach students how to respectfully engage with others' ideas, and build students' confidence and comfort with math.

Conclusion

Though Ms. Brooks was able to acknowledge and reflect upon the conflicting pedagogical goals apparent throughout this enactment, we are struck by Ms. Smith's use of her position and power to reinforce traditional emphases on computation in the midst of a PST's efforts to enact a student-centered talk routine. We also acknowledge that Ms. Smith has been placed into a difficult position with the number talk assignment. Ostensibly, the practices being used by Ms. Brooks were uncommon in her classroom, and there are many reasons why their use could cause discomfort, including concerns for students about the use of unfamiliar routines and possible even perceived implications about her own instructional approaches. Thinking about this episode from the perspective of both Ms. Brooks and Ms. Smith raises questions about the use of such assignments in methods classes and the need for further consideration of potential conflicts with status quo practices and instructional norms in host classrooms. In particular, we are interested in considering, on the one hand, how to better support host teachers to learn about and develop comfort with such routines in their classrooms, and, on the other hand, how we might be more intentional about the assignment of PSTs to host classrooms.

Further research could investigate each of these possibilities, to better understand how the power invested in experienced teachers might be utilized to challenge rather than reinforce the status quo, and to explore the implications this might have for novice teachers' future classroom practices.

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FEAR OF FAILURE OR LACK OF MOTIVATION? UNDERSTANDING THE MATHEMATICS ANXIETY OF ELEMENTARY PRESERVICE TEACHERS

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Mathematics educators agree that elementary teachers should possess confidence and competence in teaching mathematics. Yet many preservice elementary teachers pursue careers in elementary teaching despite repeated experiences of mathematics anxiety. Previous studies have defined and documented a range of different experiences of mathematics anxiety related both to learning mathematical content and to pedagogical skill to teach mathematics. This study analyzes the reported experiences of forty-eight preservice elementary teachers in relation to the range of different experiences documented in the literature.

Keywords: Pre-Service Teacher Education; Affect, Emotion, Beliefs, and Attitudes; Teacher Beliefs

Purpose of the Study

High proportions of elementary preservice teachers (PSTs) have been shown to struggle with mathematics anxiety, both as students (Beilock et al., 2010; Olson & Stoehr, 2019; Stoehr & Olson, 2017) and as developing teachers (Brady & Bowd, 2005; Brown et al., 2011; Bursal & Paznokas, 2006; Gresham, 2007; Marbán et al., 2020; McGlynn-Stewart, 2010; Peker, 2009; Sloan et al., 2002). The negative impact of generalized mathematics anxiety on the education and development of elementary mathematics teachers has been well studied (Ball, 1988; Bursal & Paznokas, 2006; Ma, 1999; Mizala et al., 2015; Roberts & Maiorca, 2020; Sanders et al., 2019). However, in order for teacher educators to support the development of confident and competent mathematics teachers, they must first understand the specific experiences PSTs have with mathematics anxiety (Beilock et al., 2010; Bursal & Paznokas, 2006; Johnson & vanderSandt, 2011; Mizala et al., 2015; Sloan et al., 2002; Stoehr & Olson, 2017; Swars et al., 2006).

The Role of Elementary Methods in Supporting PSTS to Cope with Mathematics Anxiety

Mathematics methods courses offer an appropriate space for teacher educators to help support PSTs to process and begin to cope with mathematics anxiety (Ganley et al., 2019). A number of studies have focused on how to construct appropriate learning opportunities in methods courses where PSTs can acknowledge and cope with mathematics anxiety while they are supported to (re)learn elementary mathematics content and pedagogy (Gresham, 2007; Harper & Daane, 1998; McGlynn-Stewart, 2010; Sloan et al., 2002). For example, Harper and Daane (1998) demonstrated that more than 80% of PSTs in their course experienced reduction in mathematics anxiety when the instructors incorporated a student-centered discussion style rather than lecture-based teaching. Gresham (2007), McGlynn-Stewart (2010), and Sloan (2010) added to the understanding of student-centered possibilities using a variety of teaching strategies, heavy use of manipulatives, problem-solving opportunities, and cooperative learning experiences. In all these studies, the kinds of mathematics classrooms we hope to see at the elementary level were modeled for PSTs, with the result of improving PST confidence and competence with the content and pedagogy.

Theoretical Framework: Mathematics Anxiety

The literature demonstrates five major ways mathematics anxiety is defined and studied. First, some researchers frame mathematics anxiety in affective or emotional ways. From these perspectives, mathematics anxiety is a "state of discomfort" triggered by mathematical stimuli (Trujillo & Hadfield, 1999, p. 173). For some people, this discomfort may be mild, but anxious populations often report more intense emotional experiences including fear, panic, shame, and hatred (Jensen et al., 2020; Tobias, 1978; Swars et al., 2006).

Another way researchers study mathematics anxiety is through a physiological rather than emotional lens. For example, Luo et al. (2009) describe mathematics anxiety as a type of fight or flight response, complete with physical reactions such as increased heart rate, sweaty palms, and shortness of breath. Researchers taking this perspective examine how the body engages in a hormonal stress response to a threat assessment triggered by mathematical stimuli.

Some researchers define mathematics anxiety as a type of performance anxiety. This perspective argues that anxious feelings are elicited by the possibility of failure (Trujillo & Hadfield, 1999). A number of researchers taking this approach have looked at the associations between test anxiety and mathematics anxiety (e.g., Brady & Bowd, 2005; Bursal & Paznokas, 2006; Olson & Stoehr, 2019), particularly as it relates to fear of failure. However, individuals may experience mathematics anxiety even when contemplating mathematics, with no plans to engage in a mathematical performance (Hembree, 1990).

The fourth approach examines the relationship between mathematics anxiety and self-beliefs, such as self-esteem, self-efficacy for mathematics, and identity as a "math person" (Lee, 2009; McGlynn-Stewart, 2010; Stoehr & Olson, 2015, 2017). As a way to explore and potentially treat mathematics anxiety from this approach, researchers have used retrospective and autobiographical approaches to explore the potential causes of mathematics anxiety (Hadfield & McNeil, 1994; Usimaki & Nason, 2004). Such studies have demonstrated that humiliating experiences in K-12 mathematics lessons are associated with later experiences of mathematics anxiety (Brady & Bowd, 2005; Dowker et al., 2016, Stoehr & Olson, 2017; Usimaki & Nason, 2004). However, it is also clear from this body of work that the third and fourth approach are not discrete: whether poor performance causes negative self-beliefs about mathematics or whether negative self-beliefs cause poor performance remains unclear.

The last approach examines motivational effects. Correlational work demonstrates that the experience of mathematics anxiety results in decreased motivation and interest engaging in mathematical contexts (Hembree, 1990; Luo et al, 2009). Mathematically anxious individuals are also more likely to opt out of optional and advanced mathematics courses (Beilock et al., 2010; Brady & Bowd, 2005), which is believed to contribute to the negative relationship between mathematics anxiety and mathematics achievement (Hembree, 1990; Ma, 1999).

The study of mathematics anxiety is not new, but the varied affective, physical, performance-oriented, self-oriented, and motivational constructions and approaches may make it more challenging for teacher educators to predict what sorts of experiences of mathematical anxiety PSTs bring to their classroom. In an attempt to begin to explore this issue, the research question that guided this study was: What attributes do elementary PSTs use to describe their experiences of mathematics anxiety?

Methods

Participants and Context

The participants included 48 PSTs (n = 45 identified as women, n = 3 identified as men) enrolled in a 20-week elementary methods course as part of their Master's level initial certification program at a small, private university, located in the western United States. The participants included twenty-four White women, eight Latinas, nine Asian women, two African American women, one Indian woman, and one Persian woman. There were also two men who identified as Latino and one man who identified as Asian. The PSTs were primarily in their early to mid-twenties.

Data Sources

Each PST wrote a mathematics autobiography as part of the methods course. Additionally, the PSTs were asked to reflect on individual experiences that occurred when they were students of mathematics and how their experiences impacted their attitude towards understanding mathematics. They were also asked to think about how their own mathematics experiences might impact their role as teachers. These narrative sources provided a view of their experiences with mathematics and mathematics anxiety.

Data Analysis

We began our analysis by separately reading through each of the 48 PSTs' written prompts. We engaged in an iterative analysis (Bogdan & Biklen, 2006) by demarcating sections that pertained to the ways in which the PSTs experienced episodes of anxiety in learning and teaching mathematics. We identified key words in these sections and then compared the demarcated words and phrases to the five attributes of mathematics anxiety described in the literature (affect, physical, performance, self-beliefs, and motivation). Individually, we coded each word or phrase and then met to compare our coding. We then reconciled our differences in coding and added and/or redefined any codes until we reached a 100% inter-rater reliability.

Findings

Table 1 shows the keywords and phrases each PST associated with mathematics anxiety. The table is sorted from most prevalent to least prevalent. In total, 87 unique words and phrases were coded. Of these, 33 were coded as performance, 27 were coded as affect/emotion, 14 were coded as self-beliefs, 7 were coded as physical, and 6 were coded as motivation.

Table 1: Key Words PSTs Used to Describe Experiences of Mathematics Anxiety

Attribute	Key Words and Phrases
Performance	Completely lost, Difficult, Struggled with understanding concepts, Did not make sense to me, Just plain hard, Struggling so much, Dreaded math class, Slow learner, Struggled, Trying to live up to others' perceptions of me, Extremely afraid to make mistakes, Tedious chore, Cloud of failure looming over me, Repeated 4th grade because of low scores, Abandoned and forgotten by teachers, Not important to teachers, Merely surviving, Hours of work to try and keep up, Placed in remedial math classes, My grades lowered, Left behind, Experienced real struggles with math, Hard for me to keep up, Hard to search for help, Too confusing, Too many steps, Missing key concepts, Falling behind, Felt rushed, Always a few steps behind, Punished for being slow, Failed miserably, Could not "get" math

Affect/Emotion	Nervous & scared, Devastating, Defeated, Despised math, Fear, Despair, Sadness, Embarrassed, Discouraged, Intimidated, Feeling hopeless, Frustrated, Burn out, Stopped enjoying math, A scary shadowy figure, Afraid to take risks, Pressure, Negative attitude, No sense of enjoyment, Isolated, Confused, Stressed me out, Painful & torturous, Shamed, Insulted, Helpless, Humiliated
Self-Beliefs	Lacked confidence, Felt stupid, Not smart enough to do math, Not quick enough, Might not be good at math, Not able to ever understand, Never be good enough, Unsure, Extremely dumb, Confidence in math plummeted, Hurt my self-esteem, Felt inferior, math is not my thing, math is not my strong suit
Physical	Felt sick to my stomach, Tears, Heart racing a million times a minute, Panic attack, Made my stomach turn, Want to cringe and hide, Negative gut reaction
Motivation	Closed the door on math, Gave up, Journey with math ended, Avoid math altogether, Turned off from math, Felt apathetic towards math

Discussion

Previous research has clearly documented that many elementary PSTs have experienced mathematics anxiety while learning mathematics and learning to teach mathematics (Beilock et al, 2010; Brady & Bowd, 2005; Brown et al., 2011; Bursal & Paznokas, 2006; Gresham, 2007; McGlynn-Stewart, 2010; Olson & Stoehr, 2019; Peker, 2009; Sloan et al., 2002; Stoehr & Olson, 2017). Our study confirms that PSTs continue to experience mathematics anxiety while learning mathematics content and learning to teach mathematics and supports previous research that has established that mathematics anxiety cannot be approached from a one size fits all perspective (Bursal & Paznokas, 2006; Hembree, 1990; Luo et al., 2009; McGlynn-Stewart, 2010; Trujillo & Hadfield, 1999). However, our study adds to the literature by presenting the different ways that anxiety manifests in this population. As Table 1 demonstrates, the most common reflections include PSTs recalling performance deficits and emotional responses to mathematical contexts. Research with this same group of PSTs is currently ongoing with an examination of how their intended coping strategies align with their individual experiences of mathematics anxiety. Future research is also needed to confirm these findings across different populations of elementary PSTs and to relate their autobiographical and retrospective findings to the professional experiences of mathematics anxiety teachers may have once they enter the classroom.

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A CRITICAL PARTICIPATORY ACTION RESEARCH OF SOCIAL JUSTICE MATHEMATICS

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Current events have underscored the need for mathematics teachers to facilitate and engage in critical conversations of social justice in their secondary classrooms. After completing a social justice mathematics course, three prospective mathematics teachers (PMTs) and one prospective teacher educator engaged in a critical participatory action research study to explore: (a) how to support PMTs as they engage in social justice mathematics; and (b) how PMTs engage secondary students with social justice mathematics. This study informs ongoing efforts to support PMTs' development of critical mathematics literacy. Furthermore, this study provides an example of the possibilities of engaging PMTs in collaborative research that serves to (a) reemphasize and amplify teachers' voices; (b) support PMTs to engage their students in social justice mathematics; and (c) encourage PMTs to connect education research into practice.

Keywords: Equity, Inclusion, and Diversity, Social Justice, Preservice Teacher Education

Current events, including the climate crisis, racial capitalism, white supremacy, and the global coronavirus pandemic of 2019, have underscored the need for teachers, including those of mathematics, to facilitate and engage in conversations of social justice with their students (Gutstein, 2020). As such, prospective mathematics teachers (PMTs) can benefit from an introduction to critical mathematics (Frankenstein, 1983; Skovsmose, 1994), or social justice mathematics (Gutstein, 2007; Kokka, 2015), which similarly utilize mathematics to understand current social realities (i.e., *read the world*) in order to change (i.e., *write*) the world (Gutstein, 2006, 2020). Learning to read and write the world with mathematics promotes the use of mathematics to model and shape a more socially just world (Gutstein & Peterson, 2013). Social justice mathematical tasks have shown promise for highlighting the voices and experiences of students of color (Harper, 2019) and supporting students' development of socio-political understandings (Rubel, 2017). Recently, Berry et al., 2020 summarized the benefits of engaging secondary students in social justice mathematics tasks: (a) "build an informed society; (b) connect mathematics with students' cultural and community histories; (c) empower students to confront and solve real-world challenges they face; and (d) help students learn to value mathematics as a tool for social change" (p. 23). Our study supports PMTs' development of social justice mathematics pedagogies after completing *Knowing the World Through Mathematics (KWM;* Lolkus & Newton, 2020), a social justice mathematics course Lolkus designed and taught in Fall 2020.

Lolkus continued collaborating with three of the 11 PMTs post-*KWM*. The three PMTs (i.e., Grimes, Adkison, & Miller) and Lolkus (hereafter "the research team", or "we") engaged in a critical participatory action research (CPAR; Kemmis et al., 2014) study in which we explored how to support PMTs' as they developed and interrogated social justice and mathematics pedagogies and engage secondary students with social justice mathematics. In this study, each member of the research team was positioned a as co-researcher; a process that allowed Lolkus to engage in research *with*, not *on*, PMTs (Vithal, 2004) as they developed social justice mathematics practices in secondary classrooms. Engaging in this CPAR provided opportunities for the PMTs to reflect on their social justice mathematics instruction while also further developing their expertise and professional status in the field as full participants in the research and teaching communities (Wenger, 1998). In this CPAR study, we investigated two research questions: (a) how do PMTs engage with social justice mathematics? and (b) how do PMTs engage their students with social justice mathematics?

Relevant Literature & Theoretical Perspectives

We designed our study based on the findings of social justice mathematics education research, a social theory of learning, and the qualities of critical research. Gutstein and Peterson (2013) encouraged teachers and teacher educators to connect liberatory education to the lived experiences of students and to trends in society beyond the classroom through engagement with "critical mathematics literacy" (Frankenstein, 1983, 1990; Skovsmose, 1994). Drawing on Freire's (1970/2018) theory of liberatory education, scholars have defined critical mathematics literacy as the specific understandings about how mathematics can be used to determine whose knowledge is valued (Frankenstein, 1983, 1990; Skovsmose, 1994). Mathematics teacher educators, like Lolkus, have a responsibility to ensure that PMTs are prepared to support their secondary students in learning to *read* and *write* the world with mathematics (Gutstein, 2006). Our research builds on these perspectives to prepare teachers to facilitate and engage in critical conversations about social justice issues in their secondary mathematics classrooms.

Our research is informed by Wenger's (1988) social theory of learning in that our processes of learning and knowing were characterized by participation in our situated and social communities of practice. From this perspective, the research team collaboratively constructed (e.g., Cobb et al., 1990; Vygotsky, 1978) our understandings of social justice mathematics in *KWM,* this CPAR study, and in secondary classrooms, or our shared communities of practice (Lave & Wenger, 1991; Wenger, 1998). Throughout our collaboration, we integrated all three of Wenger's modes of belonging (i.e., *engagement*, *imagination*, *alignment*) as we negotiated our participation and developed common discourse, aims, and practices.

We draw on Skovsmose and Borba's (2004) qualities of critical research (i.e., *current situation, imagined situation, arranged situation*). Analysis of the *current situation* refers to our development of an understanding and recognition of the historical and social contexts that limit our engagement with liberatory and social justice-oriented instruction. As we develop ideas for things that could be (i.e., *imagined situation*), we work toward implementation of the *imagined situation* with consideration of the structural and practical limitations of our contexts (i.e., *arranged situation*). Whereas CPAR is similar to other research methods (e.g., reflective teaching [Schön, 1983; Zeichner & Liston, 2014]), it also incorporates the imagination of, and action toward, a better future (Avci, 2020). Thus, we engaged in CPAR, not through the focus on *what is*, but the focus on *what could be* (Skovsmose & Borba, 2004). Through our research design centered around these concepts, we investigated alternative constructions (Lincoln, 2002) of mathematics pedagogies through social justice mathematics in secondary classrooms.

Methods

Drawing from the CPAR process: (a) planning, (b) acting and observing, (c) reflecting, (d) revisiting the plan, (eIcting and observing, and (f) reflecting (Kemmis et al., 2014), we worked to extend participatory action research by adding a critical approach and utilizing a dialectical perspective on practice (Kemmis, 2011). Beginning in Spring 2021, the research team conducted analyses and reflected on the connections between our ideas for teaching realities that promote social justice mathematics (i.e., *imagined situation*), and our work toward that implementation with consideration of the structural and practical limitations of our contexts (i.e., *arranged situation*; Skovsmose & Borba, 2004). Our CPAR design "invites [PMTs] to engage in education as a double process of helping students to live well, and helping to form a world worth living" (Kemmis, et al., 2014, p. 70). This approach is aligned with Gutstein's (2006) notion of promoting critical mathematics literacy development for students to *read* and *write* the world with mathematics.

Participants and Setting

Three of the 11 PMTs (i.e., Grimes, Adkison, & Miller) in *KWM* volunteered to participate in the CPAR study. All three PMTs identified as white women, a relatively homogenous population that is also representative of the teaching force (Marx & Moss, 2011). Adkison and Miller engaged their secondary students in social justice mathematics during their student teaching field placements in two suburban midwestern cities. In the same city as Adkison, Grimes collaborated with her students in the semester prior to student teaching. The research team became familiar with each other through interactions in class (i.e., *Methods II*, *KWM*), as well as weekly CPAR meetings. While Grimes, Adkison, & Miller knew each other from other classes and contexts, our ongoing meetings allowed Lolkus and the PMTs to get to know one another better, which is necessary for establishing partnerships amongst the research team (Pitts & Miller-Day, 2007) and decentering traditional researcher-participant power relations (Råheim et al., 2016).

Sources of Evidence

As we engaged in the CPAR process, we heeded Kemmis' (2011) recommendations to engage in critical self-reflections of our prejudices and perspectives own understandings of how students learn and why we adhered to those beliefs. From our insider-outsider perspectives (i.e., participant, PMT, researcher), our reflections related to the *arranged situation* provided us with opportunities to modify the research process (i.e., reflexivity). The research team collected multiple sources of evidence and documentation, including written reflections from the PMTs and secondary students, as well as written documentation of social justice mathematics implementation (e.g., lesson plans, mathematical tasks, class handouts).

Nature of Analysis and Interpretation

Essential for CPAR methods (Kemmis et al., 2014; Vithal, 2004), PMTs negotiated their continued participation in the project after completion of *KWM* through (a) active engagement in the coding and writing processes, (b) member checking and revising written drafts of all deliverables, and (c) delivering professional presentations. The research team engaged in thematic analysis (Braun & Clarke, 2012) of all documentation (i.e., written reflections, field notes, enacted mathematics tasks). As an investigation and organization of common themes across evidence sources, Braun and Clarke suggest that thematic analysis is beneficial for novice researchers because it teaches the mechanics of analyzing evidence while encouraging connections to broader theoretical and conceptual ideas.

We attended to the trustworthiness of our evidence through an 18-month-long collaboration and development of shared discourse to ensure our findings were credible, dependable, and transferable (Lincoln & Guba, 1985). Each source of evidence was evaluated by at least two members of the research team to support the triangulation of our findings (Flick, 2018) through open coding. To further ensure ensure we established trustworthy results, we maintained detailed documentation of the coding and debriefing processes, engaged in peer debriefing, and kept reflexive journals throughout each phases of our thematic analysis process (Nowell et al., 2017).

Results

We identified three preliminary themes in the three PMTs' weekly unstructured teaching reflections, weekly CPAR meeting semi-structured reflections, secondary student questionnaires and exit tickets, and curricular documents.

Classroom Environment

In order to effectively engage in social justice mathematics, all three PMTs shared an affinity for first developing a safe and supportive classroom environment that would encourage their secondary students to fully engage in what could be challenging conversations about issues of social justice. Each of the PMTs prioritized developing relationships and subsequently, trust, with students while also working to keep students at the center of activities with group-worthy tasks. Over the course of the semester, each of the three PMTs engaged their secondary students in at least one social justice mathematics activity, covering topics pertaining to climate change and gender inequities in STEM disciplines.

Rigorous Mathematics

The PMTs named that ensuring their students had access to rigorous mathematics was, in and of itself a social justice issue. As such, the PMTs prioritized engaging their students in rigorous mathematical tasks throughout their teaching experiences. While the PMTs worked to simultaneously promote rigorous mathematics and social justice issues, they reflected that developing their own social justice-focused mathematical tasks was challenging. Despite having pre-existing resources available (e.g., Berry et al., 2020), the PMTs either modified or developed their own lessons to promote social justice issues relevant to them and their students (e.g., climate change, women in science).

Discomfort

Beginning in *KWM*, the PMTs shared an underlying discomfort with engaging in social justice conversations they were not experts on, nor having experienced many of the social justice issues first-hand. This perceived tension between the PMTs' identities as white women and their racially diverse students resulted in a mantra from *KWM,* that as white teachers, we need to be comfortable with being uncomfortable. Meaning, as we engage in conversations about race, or racialized policies (e.g., housing, policing, schooling), we need to recognize the discomfort we feel as people who have benefited, implicitly or explicitly, from the systems that have marginalized many of our students. As the PMTs continue on their journeys to better understand issues of social justice, like racialized policies and practices, they seek to draw upon their students' experiences and understandings of current and historical social justice issues.

Discussion

Our study provides insight into PMTs' critical mathematics experiences through a field placement under the co-development and co-direction of the PMTs. As these PMTs began their journeys toward teaching mathematics for social justice, they benefited from unpacking their own understandings of social injustices and developing relationships with their secondary students prior to engaging in social justice mathematics tasks. Our work can inform ongoing efforts to support PMTs' development of critical mathematics literacy and their implementation of social justice mathematics in secondary classrooms. Furthermore, our research showcases the potential for engaging in CPAR with PMTs while also for providing leadership and intellectual opportunities for PMTs through active participation in the research community.

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TEACHER CANDIDATES' CONSTRUCTION OF CHARACTERS IN SCRIPTS

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Scripting tasks are a commonly used pedagogy in which teacher candidates (TCs) are presented with a scenario and asked to write a dialogue continuing the discussion. Little attention has been paid to the students that TCs imagine as part of the script. We describe our initial efforts to attend to the characters constructed in scripts in which we observed multiple character traits and ways such traits can coalesce into recognizable characters. This work suggests the importance of intentionally considering various aspects of the scripts as part of efforts to reveal TCs' perceptions of students and the resources they bring to the work of teaching.

Keywords: Preservice Teacher Education; Instructional Activities and Practices

Scripting tasks are a commonly used pedagogy in which teacher candidates (TCs) are asked to write a dialogue that shows how they might continue a class discussion (Crespo et al., 2011; Zazkis, 2017). The scripts TCs write, along with TCs' reasons for writing the scripts as they did, have been used to make visible how TCs' engage in aspects of the work of teaching and their emerging resources, such as mathematical knowledge and instructional practice (Campbell et al., 2019; Zazkis & Herbst, 2018). Little attention has been paid to the students that are "imagined" by TCs in such scripts (Zazkis, 2018). Through viewing scripts as a form of storytelling, we describe our initial efforts to attend to the characters in scripts written by TCs, as part of broader work considering what can be revealed about TCs' resources through scripting tasks.

Background and Theoretical Lens

Representations of practice serve as one category of teacher education pedagogy that structures TCs' investigations into and enactments of the work of teaching (Grossman et al., 2009). Scripts of classroom interactions are commonly used representations of practice in mathematics teacher education (Zazkis & Herbst, 2018) and have been found to support teachers' learning of mathematical content (e.g., Koichu & Zazkis, 2018) as well as pedagogical practices (e.g., González, 2018). While scripts are often developed by teacher educators or researchers, opportunities for TCs to engage in script writing as an approximation of practice (Grossman et al., 2009) have been increasingly used to support and assess TCs' mathematical and pedagogical learning (e.g., Buchbinder & Cook, 2018; Campbell et al., 2019; Crespo, 2018).

Across the literature focused on the use of scripting tasks in mathematics teacher education, there has been little focus on the students that are "imagined" as part of the writing of a script (Zazkis, 2018). Research on scripts tends to focus on the teacher (e.g., Rougée & Herbst, 2018) or acknowledges TCs' potential difficulties in producing student responses (e.g., Lim et al., 2018). Furthermore, the imagining of students in scripts involves not just the nature of the students' turns of talk, but the construction of characters who interact with others in the script and, together, enable the script to tell a story. This is consistent with Herbst's (2018) framing of script writing as "storytelling," as well as Zazkis's (2018) recognition that scripts reveal characters that are created for particular mathematical or pedagogical purposes. Investigation around these representation of students in scripts have the potential to reveal novel aspects of how TCs envision students and classroom interactions. We highlight our initial investigation of student characters in scripts written by TCs by pursuing the following research questions: (1) What character traits are evident in the students created by TCs when writing scripts in response to provided scenarios? (2) In what ways do sets of traits coalesce into characters in these scripts?

Methods

Our work occurs in the context of a multi-year collaboration situated in secondary mathematics methods courses at two large, public research universities, centered on the design and use of practice-based pedagogies, including coached rehearsals and scripting tasks (Baldinger et al., 2020; Baldinger & Campbell, 2019; Campbell et al., 2020, 2019). We consider the scripts TCs across sites (n = 27) wrote during the 2019-2020 school year in response to four different scenarios, each centered on a different classroom activity (Graph Interpretation Task, Number Talk, Sorting Task, and Representation Talk). Each scenario included contextual information (including the mathematical task and goal) and a few lines of script that included a student error to open the discussion. TCs were prompted: "Imagine that you are the teacher. Write the next 5-8 lines of script continuing this discussion." TCs completed the four distinct scripting tasks using Qualtrics. The graphing scripting task was administered multiple times as part of the course design, resulting in a set of 162 scripts for analysis—six from each TC.

The initial analyses we report on here included the writing of analytic memos for each script focusing on the question of what character traits were included among the "students" in the script, defined by how the students were represented as participating and their apparent role in the script. Multiple researchers read each script individually, and then compared observations about the characters and traits being observed. This report reflects our findings from this initial round of open analysis, with a focus on four salient character traits present in the scripts written by TCs, as well as two examples of how those traits formed more complete characters.

Findings

Student Character Traits Emerging from Scripts

One emergent trait was students' ability to readily provide contributions of their own reasoning with little to no prompting. For example, in one TC's script, the teacher asked the class for another description of a provided graph, first commented on by the student in the provided scenario. A new student, Trina, introduced in the script responded,

> Tom walks at a consistent rate of 2 meters/sec for the first 100 meters from school. Then he had to walk back towards home for 20 seconds. Finally he turns back and continues to the bus stop at 4 meters/sec until he reaches the bus stop.

This contribution was mathematically correct and precise and came without any additional prompting from the teacher. At times, these contributions were used to move the story forward in ways that made the student seem almost telepathic. In these cases, this character trait was similar to teacher telling, though with mathematical ideas presented in the voice of a student.

Another character trait was students' ability to readily restate or reason about other students' ideas when prompted by the teacher. For example, in one script, the teacher asked, "Does anyone want to add onto Priya or Amir's claims?" A new student, Joyce, responded, "I agree with Priya that the function is not linear but I think it is because the change in x is 3, 5, 7, and 9." Joyce was positioned as understanding Priya's idea sufficiently to agree with it and extend it. In other cases, this trait took the form of disagreement. In a different script, the teacher asked, "How do you feel about what Priya is saying, Amir?" The TC had Amir respond, "I don't agree. When a linear function has a negative slope the x can increase while the y will decrease so that doesn't mean it isn't still linear?" The TC positioned Amir as able to defend his own thinking and willing to publicly disagree with another student's contribution. This trait allowed for additional students to be added to the conversation in the script and as another way to insert mathematical ideas.

A third trait that emerged was students' explicit willingness to be unsure or to change their thinking. For example, in a script where a student, Jessie, was sharing thoughts on a shape's classification as a polygon, she asked, "It's still a square though right, it just has a line added?" This trait demonstrated the TC's acknowledgement that a student could be unsure about something and also be willing to express that uncertainty. Students in some scripts were also characterized as being willing to change their minds. This took the form of explicit statements that they made an error and now would like to change their thoughts. In one script, in response to another student's reasoning, Foster stated, "Oh that makes sense. So mine was wrong." Often, a student's change in thinking was used to move the discussion towards a resolution.

A final trait we highlight was students' demonstration of actions consistent with actively not listening to the discussion. While most TCs wrote scripts where all represented students were listening in a way that enabled them to directly respond to other students and the teacher, some created students who stood in stark contrast to this. For example, one TC had the teacher ask, "Can anyone restate what Eli just said?" In response to this open call, a newly introduced student, Amira, responded, "Nope, I wasn't paying attention." Amira was represented as not paying attention and brazen in her admittance of not doing so. The creation of Amira through this trait does not necessarily seem to move the story along in the same way as the other traits we observed. Instead, it suggests an attempt to construct "authentic" students, perhaps for the TC to show the teacher educator that they recognize students will not always pay attention.

Compilations of Traits into Characters

As we saw traits emerging from student talk, we also found that they could coalesce into identifiable characters. This notion of character entails considering the role the student plays in service of the overall script—whether mathematically or pedagogically.

The "savior" character. A recurring character across scripts was one we called "the savior." Given that all four scripting task scenarios included the contribution of a student error, what was to be "saved" was both the need for a more correct or complete answer as well as the ability for the discussion to move past the error. The savior student readily provided an alternative (correct) answer to what had been presented and was able to communicate that idea completely and precisely. Key to defining the savior character was not just the contribution from the student, but that the response from the teacher or other students leveraged the contribution from the savior student as something that could resolve and move on from the error. The following script shows an example of a savior character.

Teacher: Who sees things differently?
Trina: I see that Tom starts walking to school, comes back towards home, and then turns around and continues to the bus stop.
Teacher: Trina, can you explain why you think that?

Trina: I noticed that the y-axis of the graph is distance from home so Tom is walking further from home at first.
Teacher: Who agrees with Trina? Does anyone else see it differently?
Foster: I would like to change my answer. I didn't see what was on the y-axis at first but now I see that it is distance. I now agree with Trina.
Teacher: Does everyone now agree with Trina?
Students: Yes.

Trina stepped in to provide a very detailed description of the graph that contrasted Foster's. Her contributions convinced Foster to change his answer, and the teacher took the opportunity to move on with Trina's idea, in part because of apparent agreement among all students.

The "easily convinced" character. A second character we saw across scripts was one who would change their response with little to no prompting—they were "easily convinced," relative to the information provided to them, that their initial idea was incorrect. These characters were willing to change their thinking, and they often were able to readily and robustly restate what another student said. Most importantly, such characters typically functioned to bring the discussion of the error to a close. The following script shows an example of this character in action. The student, Foster, had misinterpreted a graph.

Teacher: Okay Foster, why do you think that?
Foster: Well the graph goes up first and then it goes down and then up again like he is walking up and down a hill.
Teacher: Okay thank you. (to class) What does it mean when the graph goes up?
Stacy: It is increasing.
Teacher: Yes, so the values are increasing over time. What value is increasing?
Foster: Oh the distance from home. I think I thought that that was the height.
Teacher: So now we can see that when the graph goes up, the distance from home increases. So what happens when the graph goes down?

Foster changed his opinion following a very small amount of information from Stacy and a question from the teacher. Moreover, he clearly connected his new thinking to the previous error. This effectively closed the conversation about Foster's initial interpretation and the discussion progressed with the teacher confirming Foster's new opinion and adding in new information.

Discussion and Conclusion

In recognizing the range of character traits and characters in scripts, we noticed that the students made possible certain kinds of teacher moves, which are typically the focus of analyses of scripts. As a result, the practice TCs are approximating through scripting is contingent on the student characters they create, making these student characters vital to understanding scripts and what they make visible about the TCs who write them. This preliminary work highlights the potential in responding to the call from Zazkis (2018) to consider this feature of scripts.

As we continue to pursue the intentional consideration of students that are imagined in scripts, we also recognize a number of open questions. For instance, what do TCs see themselves representing and what is informing that representation? These scripts are telling a story, intended to be consumed by the teacher educator. Are TCs representing actual or idealized classrooms in their storytelling? What is the motivation of the storytelling? TCs might be motivated to "impress" teacher educators with their pedagogical skill, or they might be motivated to develop characters who help them learn to navigate complex classroom situations. TCs might interpret that the script needs to depict closure or resolution, which would influence what they represent. Continued efforts to understand what TCs' scripts reveal about the resources they bring to the work of teaching would necessitate consideration of these types of questions.

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EXPLORING RELATIONSHIPS BETWEEN PROSPECTIVE TEACHERS' DECIMAL MODELS AND THEIR PROCEDURAL AND CONCEPTUAL DECIMAL KNOWLEDGE

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Robust knowledge of the mathematics that one teaches plays an essential role in quality teaching and is therefore important for prospective teachers (PTs). For elementary PTs, this must include both conceptual and procedural decimal knowledge. Research reveals that mastery in this domain is elusive for children and adults. However, limited rich descriptions of PTs' knowledge of decimals exist. Even less research explores the role of models, though modeling is an integral part of doing mathematics. In this study, I examine 225 elementary PTs' responses, when asked to create a model for comparing 0.4 and 0.32 and explain the mathematical ideas addressed. Comparing responses by PTs who created area versus linear models reveals that procedural fluency is similar, but more users of area models demonstrate conceptual understanding.

Keywords: Preservice Teacher Education, Rational Numbers, Modeling, Mathematical Knowledge for Teaching

There is growing consensus that mathematical knowledge for teaching (Ball et al., 2008) is an essential component of quality mathematics instruction and student learning (Charalambous, 2010; Hill et al., 2005, 2008; Wilhelm, 2014). This includes deep understanding of the mathematical topics to be taught, such as decimals, a prominent topic in elementary school (NGACBP, 2010). Research shows that decimals can be difficult for students (Graeber et al., 1989; Mehmetlioğlu, 2014; Moody, 2010; Sackur-Grisvard & Léonard, 1985; Steinle & Stacey, 1998) and prospective elementary teachers (PTs) (Burroughs & Yopp, 2010; Depaepe et al., 2015; Muir & Livy, 2012; Putt, 1995; Stacey et al., 2001). Further, PTs' knowledge of decimals is under-researched compared to other content areas (Kastberg & Morton, 2014). There is also little scholarship on PTs' decimal modeling, which is problematic since modeling is an essential component of learners' mathematical activity (Fosnot & Dolk, 2002).

This study begins to address this gap by exploring relationships between PTs' choice of model and their procedural and conceptual decimal knowledge. Improved understanding of these relationships and of PTs' decimal knowledge and abilities generally may better equip mathematics teacher educators to prepare PTs for the classroom. In this study, I pose the following research question: when comparing decimals, what relationships exist between PTs' choice of models and the way they describe decimal place value concepts and procedures?

Background and Theory

Here, I present some of the literature surrounding learners' understanding of and skill with decimals. I then characterize *model* as well as *procedural fluency* and *conceptual understanding*.

Learning of Elementary Decimal Concepts

The base ten place value system is a system of different-sized units, added together, each related to adjacent places by a factor of ten (Kastberg & D'Ambrosio, 2011). Decimal place value concepts are prominent in upper elementary school, where students represent quantities as decimals, plot decimals on number lines, compare decimals, and operate on decimals (*Meeting Standards with EM - everyday Mathematics*, n.d.; *The Curriculum*, n.d.; NGACBP, 2010).

Despite the prominence of decimals in school mathematics, research has shown that mastery can be challenging. Elementary students may apply false rules when comparing decimals, such as assuming the number of digits impacts decimal magnitude (e.g., Mehmetlioğlu, 2014; Sackur-Grisvard & Léonard, 1985; Steinle & Stacey, 1998). Some students believe longer decimals are greater in magnitude (like for whole numbers), others, that shorter decimals are greater (at times due to a misapplication of the fact that further left decimal places represent larger "pieces").

Studies which attend to PTs' knowledge of decimal magnitude and ability to compare have found that *longer-is-larger* thinking is less prominent, while other incomplete understandings persist from childhood to adulthood (Burroughs & Yopp, 2010; Depaepe et al., 2015; Graeber et al., 1989; Muir & Livy, 2012; Stacey et al., 2001). A small number of studies of PTs' decimal knowledge have also attended to modeling, generally finding that PTs may not be able to effectively use models for making sense of decimals (D'Ambrosio & Kastberg, 2012; Starks & Feldman, 2020; Thipkong & Davis, 1991). That decimal mastery remains elusive for many learners suggests that we must learn more about teaching and learning this content.

Models

Models are "mental maps of relationships that can be used as tools when solving problems" (Fosnot & Dolk, 2002, p. 90). Decimal squares, one common model, are squares systematically divided into ten, one hundred, or one thousand sections, with some shaded in to represent a specific quantity. Number lines are also common, and often include sections split further into groups of ten, to represent tenths, hundredths, or thousandths, using a dot plotted at the appropriate location to represent a quantity. Modeling decimals can support processes (e.g., comparing) as well as concept development (such as sensemaking around decimal place values).

Procedural and Conceptual Knowledge

The National Research Council (NRC, 2001) includes *procedural fluency* and *conceptual understanding* as two of five main strands of mathematical proficiency (alongside *strategic competence*, *adaptive reasoning*, and *productive disposition*). *Procedural fluency* is the ability to apply algorithms and solution methods flexibly and effectively. This knowledge of mathematical processes may also be grouped with knowledge of mathematical symbols, vocabulary, and conventions (Hiebert & Lefevre, 1986). *Conceptual understanding* is demonstrated when a learner has a robust internal web of interconnected knowledge of new and previously understood mathematical ideas, as opposed to knowing only isolated mathematical facts (NRC, 2001).

While learners may develop procedural fluency more quickly than conceptual understanding (perhaps due overemphasis on procedures by curricula and instructors), it is flawed to view these strands as hierarchical levels. Rather, scholars have identified both conceptual understanding and procedural fluency as essential components of proficiency, neither more important than the other (NRC, 2001). In the current study, differentiating between demonstrations of procedural and conceptual knowledge allows me to characterize the content of PTs' writing about decimals.

Methods

Data for this study were collected in the context of the NSF-funded Elementary Mathematics Project (EMP), which has developed curriculum for use in mathematics content courses for elementary PTs (Chapin et al., 2021; Gibbons et al., 2018). Decimal topics within *Number Concepts*, one of the EMP units, include places, naming, magnitude, comparing, and modeling. In the 2017-2018 academic year, EMP collected *Number Concepts* posttest data from 302 PTs, taught by 11 instructors at nine institutions of diverse sizes and selectivity, both public and private, in eight U.S. states. Following engagement with the unit, participants completed a five-item open-ended test, including this two-part item.

> As a future teacher, you may encounter a student who is having difficulty determining which of two decimal values is greater. For example, 0.4 and 0.32. (a) Provide a model that would help a student to think about the sizes of 0.4 and 0.32. (b) Explain how your model would help a student compare these quantities and which important mathematical ideas it addresses.

The current analytic sample includes PTs who provided decimal squares or a number line (not both) in response to this prompt; 225 participants met these criteria. The other 77 participants' work will be analyzed in a later phase of this study. I used qualitative content analysis (Hsieh & Shannon, 2005) to make claims about PTs' understanding and hypothesize about relationships with models. I used open coding in a pilot study (Starks & Feldman, 2020), creating a codebook describing the type and features of the PTs' models, and capturing the content of PTs' writing. More recently, I refined the coding scheme, in part by differentiating between knowledge of decimal concepts (11 codes) such as *decimals are parts of wholes*, *tenths are greater than hundredths,* or *decimals may be decomposed additively by place value*, and decimal comparison procedures, conventions, and vocabulary (11 codes) such as "*the tenths place", 0s may be annexed to the end of a decimal,* or *comparing by place value.* My adopted definitions of *conceptual* and *procedural* allowed to do this differentiation; codes were classified as *conceptual* generally if they had to do with mathematical connections, and *procedural* if they had to do with processes or language. My coding was completed using NVivo 12 software. I continued my analysis by exploring to what extent users of each model addressed place value (concepts *and* procedures), as well as exploring the extent to which indicators of procedural fluency and conceptual understanding were present among users of each model.

Preliminary Findings

Of the 225 participants, 72% correctly stated that 0.4 is larger, less than 2% selected 0.32, and the remaining participants did not make an explicit statement. A notable difference was not observed between users of decimal squares (n = 132) and users of number lines (n = 93) in terms of correct identification of the larger decimal.

The concepts and conventions of the base ten place value system are *the* key mathematical ideas which underly models of decimal magnitude and strategies for comparison. When prompted to identify important mathematical ideas addressed in their models and strategies, nearly nine out of ten decimal square users (DSUs) addressed place value, while only two thirds of number line users (NLUs) did so. Some participants accomplished this superficially, simply using the words *place value* (e.g. NLU 0699: "My model would help a student … see which number is longer than the other one. This helps students to see the different place values and how they compare to one another.") Others addressed place value in more robust ways. For instance, over half of DSUs wrote about "tenths" and "hundredths" as mathematical or concrete objects (as opposed to labels for positions), and described their role in modeling and comparing quantities (e.g. DSU 0704: "This would help show students better that there are ten smaller hundredths inside of a tenth so the 0.02 after the .3 doesn't have as much value as the 3 or 4 in the tenths places. This mathematical idea is the idea of place value and that when using decimals the further right you go the smaller the number gets.") This sort of response was far less common among NLUs, only roughly one in nine spoke about tenths or hundredths as pieces of the whole.

Participants demonstrated reasonable procedural knowledge, without substantial differences between DSUs and NLUs. Beyond successfully identifying the larger value, over 90% of both groups received at least one code pertaining to place value procedures, vocabulary, and conventions. In both groups, most described how to interpret their model to compare the quantities, nearly half used the term *place value*, over one quarter described that a zero may be annexed to facilitate comparison, and roughly one in five described comparing by place value.

Conversely, conceptual understanding appeared to be very different between groups. While responses from a large majority of DSUs (eight out of ten) addressed at least one relevant concept, responses from fewer than three out of ten NLUs did so. For instance, most DSUs addressed the basic concept that a decimal quantity represents a part of a whole (e.g., DSU 0746: "The model helps explain this by showing the whole number *one* broken down into 100 smaller parts …"; DSU 0581: "… think of 0.4 as 40/100 and 0.32 as 32/100; DSU 2292: "… [students] can count which has more squares filled … 1 filled column = 1 tenth, 1 box = 1 hundredth …"). However, less than one in six NLUs included similar explanations. Other relevant concepts present in some responses (across users of both models) included the equivalence of 0.4 or four tenths and 0.40 or forty hundredths, the decomposition of 0.32 into 0.3 + 0.02, and the fact that tenths are greater than hundredths. While well under half the responses in each group addressed these concepts, and while statistical significance has not been verified, these ideas consistently seemed to surface more often among the sample of DSUs than they did among NLUs.

Discussion

Since procedural fluency is an essential component of mathematical proficiency (NRC, 2001), it is encouraging that PTs successfully identified the larger decimal and showed facility with decimal place value conventions, vocabulary, and processes. However, it is concerning that not all PTs displayed conceptual understanding, and interesting that it seemed less developed among NLUs than DSUs. We cannot conclude that use of decimal squares necessarily supports development of richer conceptual understanding, but we may conjecture as to why this association emerged. Perhaps certain features of decimal squares support PTs' reasoning about decimal quantities; for instance, they intrinsically capture the relationship between 0 (empty square) and 1 (full square) in a way that number lines may not, if truncated. Alternatively, as decimal squares are introduced before number lines and other linear models in the EMP curriculum, perhaps they are overemphasized, leading to tighter mental connections between them and important concepts in this sample. Or, perhaps learners in general are more comfortable with area models – consistent with a finding by Thipkong and Davis (1991) – and therefore develop or demonstrate deeper understanding when using them versus linear models.

The findings reported here are preliminary findings of a larger study which is currently in process. Next steps in this study are to explore associations between features of the models and the same procedural and conceptual indicators. Further, I intend to re-examine PTs' writing, focusing specifically on when and how they explicitly connect concepts and procedures back to their models. Additional future research in this area is needed to explore potential causation; are there ways in which use of certain models or pedagogies around modeling can or do improve understanding of particular decimal concepts? Additional scholarship in this area will continue to support mathematics teacher educators in providing essential, quality instruction to future elementary school teachers.

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ESTABLISHING COMMON GROUND THROUGH GESTURAL SCAFFOLDING: A FIRST-GRADE PRESERVICE TEACHER'S USE OF PROBES

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Many studies have focused on students' gesture use and learning; however, little attention has been paid to preservice teachers' gesture use in their instruction. Therefore, this study aims to expand existing research by investigating a preservice teacher's gestural scaffolding in her mathematics probes. Sources of data included a video of the preservice teacher's teaching, her identification of probes in a stimulated recall interview, and researchers' identification of probes. Results showed that the preservice teacher packaged spatio-motoric information in her iconic and metaphoric gestures to establish common ground with her students. The research findings provide further implications about how teacher educators teach probing practices in preservice teacher education.

Keywords: Preservice Teacher Education, Classroom Discourse, Elementary School Education, Embodied Cognition, Gesture Use

Many studies (e.g., Alibali et al., 2019; Nathan et al., 2017) have focused on students' gesture use and learning; however, little attention has been paid to preservice teachers (PSTs), who are still learning to teach mathematics and manage their own expressions and displays of meaning. Prior studies on the use of probes have been dominated by the powered lenses of researchers, leaving little space for the teachers' voices of their own probing experiences. Therefore, this study aims to 1) address how gesture scaffolds communicative meaning-making through establishing common ground between teachers and students, and 2) delve into similarities and differences between PSTs' and researchers' identification of probes.

Theoretical Framework

Gesture accompanies speech and contributes to meaning-making by conveying similar or additional information (McNeil, 2017). Gesture in action (e.g., deictic gesture) interacts with concrete materials in context; gesture in verbal communication (e.g., representational gesture) functions as an imitation of action and thus takes on certain selective features of movements so that listeners can make sense of what is being represented (Clark & Gerrig, 1990; Gerwing & Bavelas, 2004). An example of selective movements is the gesture of counting by ones (i.e., pointing to imaginary dots one by one in the air). Its sequenced points and regular stops between two adjacent imaginary points are symbolic of counting concrete objects one by one, therefore signaling the messages speakers intend to convey.

McNeill (1992) categorized gestures into four types: deictic gestures, iconic gestures, metaphoric gestures, and beat gestures. *Deictic gestures,* often called pointing gestures, refer to pointing movements toward concrete objects such as PSTs pointing at numbers when talking about them in a problem. *Iconic gestures* refer to the gestures that directly describe the motion included in the semantic content of speech. Consider a teacher who moves her hands in a circular motion while talking about circles in geometry; the circular hand movement is an iconic gesture. The pace of the movement, which does not exist in speech, sometimes expresses additional information. For example, slow finger movement could imply that the teacher has noticed her students struggling and is slowing down her instruction as a form of scaffolding. *Metaphoric gestures*, which are similar to iconic gestures in terms of referring to imagery representation, describe abstract concepts expressed in speech. An example is a grasping gesture: A student says he wants to understand place value with accompanying gestures, i.e., opening a hand and closing it into a fist. The place value concept is compared to something that the student can grasp. Therefore, this gesture acts as a metaphorical base to hold the concept of knowledge or ideas (McNeill, 1992). *Beat gestures*, which do not usually bear semantic content, are rhythmic movements that reside in the speaker's comfortable or habitual movements, like quick flicks of fingers or rapid pats on the lap.

Gestures, when grounded in physical contexts such as objects and actions, facilitate students' access to the meaning embedded in speech (Alibali & Nathan, 2007). *Common ground* refers to the shared information between speakers and listeners (Alibali et al., 2013; Gerwing & Bavelas, 2004; Holler, 2009). In classroom instruction, when common ground between teachers and students is lost, students become confused. In response, teachers tend to slow down their instruction and apply more verbal and non-verbal signs to assist students in learning. Prior research has confirmed that teachers are likely to increase their use of grounding gestures in trouble spots (Alibali & Nathan, 2007) and in the introduction of new information (Gerwing & Bavelas, 2004). Deictic gestures are a frequently used means of establishing common ground in mathematics classrooms when teachers connect speech with visual representations (e.g., drawings, diagrams). Speakers also change their gesture use based on the common ground that they share with their listeners. Gerwing and Bavelas (2004) asked participants to describe their actions while playing with the toys to those who had played with the toys and those who had not. They found that when describing the action to those with playing experience, participants used less informative and less complex gestures; by contrast, participants' descriptions tended to include more informative and complex gestures in interactions with those with little playing experience. In other words, the more knowledgeable other used more precise and informative gestures to facilitate the learner's comprehension. Meanwhile, the learner developed a better understanding thanks to the more knowledgeable other's use of gestures.

Methods

Participants and Data Collection

This study examines a PST's— Clara— use of probes in a first-grade classroom. Clara was taking an elementary mathematics course as required by the teacher certification program at a Midwest university. After doing lesson planning in the methods course, Clara implemented her lesson plan about two-digit number comparison at an urban elementary school. Her 39-minute lesson targeted the facilitation of meaningful discussions and was videorecorded. Afterwards, a stimulated recall interview was conducted with Clara to record her understanding of her own probe use. The interviewer (the author) asked only pre-planned facilitating questions to minimize influence on the PST. Finally, two researchers (the author and a colleague) independently coded Clara's teaching video and discussed their codes of probes until 100% agreement was achieved.

Analysis

To investigate the issue of gestural scaffolding, I built on existing literature on probing and following-up questions (e.g., Boaler & Humphreys, 2005) and categorized the PST's probes into four types: concept probe, strategy probe, reasoning probe, and clarification probe (see Table 1). Using these codes, my second coder and I coded the PST's teaching video, identified speech and gesture use when the PST probed students' mathematical thinking, and finally formed a list of researcher-identified probes. I provided descriptive explanations for how the PST used speech and gestures to establish common ground with students and particularly captured the scaffolding role that gesture played in the expressions of probes. To explore to what extent the PST-identified probes aligned with the researcher-identified probes, I elaborated on the commonalities and differences between the researchers' and the PST's identification of probes.

Table 1: Categorizations of Mathematics Probes

Probe Type	Description	Example
Concept probe	Points to underlying mathematical relationships and meanings between mathematical ideas (Boaler & Humphreys, 2005)	What does 91 have more of?
Strategy probe	Wants to get students' descriptions of their strategies (Chen et al., 2020)	How do you get 12?
Reasoning probe	Wants students to justify their problem-solving strategies (Franke et al., 2009)	How do you know they were equal?
Clarification probe	Asks students to clarify their explanations; repeat students' responses to resolve uncertainty about what students said (Chen et al., 2020; Franke et al., 2009)	What do you mean? Just two?

Findings and Discussion

Overall, Clara's variation of probes grounded in gesture helped students practice using the mathematical language, helped them dig into their perception of place value, and provided them with opportunities to sharpening their adaptive reasoning. A case in point is when Clara probed a student's conceptual understanding of place value after the student said 59 is bigger than 58. She first probed, "What's bigger?" with a deictic gesture (pointing to the problem on the worksheet) and found out that the student understood her probe as "Which is bigger, 58 or 59?" Then she revised her concept probe as a combination of the question "What does it have more of?" with the same deictic gesture. The student incorrectly answered that "It has more of ten." Going a step further, Clara probed by asking "What does it have more of? Is it ten or one?" and a metaphoric gesture (putting her right hand up vertically, then her left hand up vertically, see Figure 1A), thus making her probe objectively accessible to the student by restricting her probe to two choices. The metaphoric gesture with two hands up vertically abstractly represents ten and one, with the right hand standing for a side for ten and the left hand for the other side for one. Clara confined her probes by means of asking the student to choose a side to stand by; she grounded these two choices in her two hands and made them accessible to the student. One the one hand, the metaphoric gesture bears a deictic aspect, referring to ten and one [gesture is image-based (McNeill & Duncan, 2000)]; on the other hand, the metaphoric gesture represents two choices, i.e., 59 has more of tens or more of ones than 58 (place value). Despite the student's failure to answer what 59 has more of, Clara herself constantly refined her speech and increasingly used grounding gestures to better probe student thinking.

A B

Figure 1: Concept Probe and Clarification Probe

Clara's clarification probes functioned with the purpose of prompting or pressing students to say more (Chapin et al., 2009; Ghousseini, 2015). She either repeated previous talk moves or directly asked students to clarify their thoughts. There were also times when Clara repeated her own questions as clarification probes. For instance, Clara initiated a talk turn with a student by asking "Which one is bigger?" (problem: 60 vs. 88) without using any gesture; the student's response was 100. Realizing that this student might not have understood her initial question, Clara intentionally enriched her probes by using multiple gestures and revised speech. First, Clara probed by asking, "Which one, 60 or 88? Which one is bigger?" accompanied by her hands moving up and down (Figure 1B-a: lifting her right hand up while her left hand was ready to go up; Figure 1B-b: leveling up the palm of her right hand, which was ready to go down while her left hand was going up; Figure 1B-c: both palms leveled up and almost at the same level but in an opposite tendency of movement). Clara moved both her hands up and down three times. She used the up-and-down gesture to denote her conception of a big number and a small number, which may originate from her learning experience of a balance scale or a vertical number line. Thus, Clara packaged the spatio-motoric information in her up-and-down gesture (Kita et al., 2017). She transferred her probe from a conceptual domain (which is bigger?) to a physical space (which is higher or heavier?). Further, she did not reveal the answer by raising a hand along with the verbalization of 88. Instead, she encoded the signal of her waiting for an answer in the motion of moving her hands up and down three times. The series of gesture and speech grounded the teacher's probe tightly in the physical context. Clara, however, did not stop her clarification probe after using the metaphoric gesture. She continued by alternating her way of probing: "Which one is the alligator"— (making an alligator mouth with hands, see Figure 1B-d)— "going to eat?" Connecting with the alligator mouth, which the student was already familiar with, Clara clarified her probe further with an iconic gesture (two hands forming an acute angle, a metaphoric shape of an alligator mouth).

Unsurprisingly, the researchers identified more probes than the PST did, but both identified that concept probes and reasoning probes were used most often. There was a high rate of consistency in speech use between the researcher-identified probes and the PST-identified probes. The PST did not identify gesture use in her strategy probes except the instance of using only gesture in a strategy probe. She tended not to identify the probes followed by students' brief input or silence whereas the researchers did. Clara sometimes did not notice a research-identified probe if the probe followed tightly behind another probe or if she thought she was checklisting students' responses.

Conclusions

To sum up, Clara varied her gestures when she noticed students struggling to understand her questions. The scaffolding role that Clara's gestures played helped establish common ground with students. Varied gestures and questioning imparted flexibility and richness in meaning-making to her probes. With respect to the alignment between the PST-identified probes and the researcher-identified probes, much alignment was found on the speech level rather than the gesture level. These findings highlight needed changes in how the field interprets and investigates probing and further suggest the inclusion of gestures in the probing practices in preservice teacher education.

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PRESERVICE TEACHERS' LANGUAGE USE FOR FRACTION PROBLEMS IN CONTEXT

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Authors asked preservice elementary teachers to write word problems for fraction equations, then use manipulatives and draw diagrams to model the solutions to the word problems. Through analysis of problems and a follow-up group interview, authors discovered that a focus on contexts using partitive division, as well as the lack of precise mathematical language, impeded further understanding of fraction concepts.

Keywords: Preservice Teacher Education, Rational Numbers

Research on fraction understanding and effective teaching methods is extensive. Teaching fraction concepts using real-world contexts has been seen as a way to improve understanding (Roesslein & Codding, 2019). According to an analysis of fraction problems in three textbooks undertaken by Cady et al. (2015), the most common contexts for word problems with fractions were cooking, money, sharing, and shopping. While fraction multiplication is straight-forward, fraction division problems can be solved by the use of either partitioning or measurement (Lo & Luo, 2012). Typically, researchers seek to determine preservice teachers' (PTs) understanding of fraction multiplication and division in context by providing PTs word problems with real life contexts to solve, asking them to write word problems for equations or expressions, or asking them to analyze given solutions for word problems (e.g., Adu-Gyamfi et al., 2019; Ball, 1990; Jansen & Hohensee, 2016; Mack, 2001; Nillas, 2003; Stohlmann et al., 2020).

For this study, we sought to determine PTs' understandings of fraction multiplication and division by asking them to write word problems for given equations. We provided solved equations so that PTs would focus on the meaning of the equation rather than the solution. PTs wrote word problems, then used manipulatives and drew diagrams to model the solutions to the word problems. To investigate PTs understandings, we asked the questions: 1. Which contexts do PTs relate fractions to? 2. How does the context of a word problem facilitate or impede the understanding of fraction multiplication and division?

Review of the Literature

Fraction multiplication and division have been shown to be difficult concepts to understand, with most PTs, teachers and students resorting to algorithms without knowing the underlying reasoning (Ball, 1990; Unlu & Ertekin, 2012; Zembat, 2004). Language use can be misleading, as seen in the problem provided by Adu-Gyamfi et al. (2019) which was worded as "There is 3/¾f a pie left. Christina and Candace want to divide what is left in half so they each get an equal part. How much does each of them get?" (p. 516). This resulted in half (18) of the PTs rating as correct solutions that showed an answer of 1 1/½which would result from (3/4) ÷ (1/2). This problem was intended to be an example of partition division, where a quantity is partitioned into smaller pieces, with the expression (3/4) ÷ 2 used to generate a unit rate. The wording of the problem is problematic, in that it implies (3/4) ÷ (1/2). Therefore, the misunderstanding could be expected, although it was surprising that 18 of 24 PTs who initially solved the problem correctly accepted an incorrect answer. Based on the wording of the above problem, we can see why many PTs wrote story problems involving division by two in Ball's (1990) study of 19 PTs who struggled to write a story problem for (1 3/¾÷ (1/2). Nillas (2003) also commented that PTs often interpret division by 1/½s division by 2, as was seen in her study of four elementary PTs who were asked to write a story problem for (5/7) ÷ (1/2).

The problem described above from Adu-Gyamfi et al. (2019) could also be solved by taking 1/½f the 3/¾f pie, which is a multiplication problem of (3/4) x (1/2). This method of finding a unit rate was described by Mack (2001) in a study of six fifth graders who built on their informal knowledge of partitioning to solve problems using fraction multiplication. Mack used problems in context such as "Find 2/3 of 3/¾f one whole pizza" (p. 279). Mack's use of partitioning to build up to multiplication skills was recommended by Lamon (1996), who studied students' partitioning skills in grades four through eight. Lamon found partitioning to be foundational for multiplicative reasoning and recommended its continued use beyond third grade.

Jansen and Hohensee (2016) also tasked 17 elementary PTs with looking at partitioning, in this case with the use of fraction division. They found that PTs were mostly successful at writing a correct word problem and discussing their solution in terms of unit rate for the problem "24 DVDs ÷ 4 Hours = ?" (p. 508). However, most wrote a word problem that was represented by multiplication of 24 by 1/¼hen given the problem "24 oz of water ÷1/4 hours = ?" (p. 508) and did not interpret the solution in terms of unit rate. Failure to attend to referent units is a common issue, and largely ignored by most PTs, teachers, and students (Lee, 2017; Stohlmann et al., 2020). Looking at fractions from a measurement perspective, as was the focus for Stohlmann, et al. (2020), presents a model that aligns with the definition of division, in that problems look at how many of one quantity are present in another quantity. Stohlmann, et al. focused on teachers' ability to write a word problem for and solve (2 1/½÷ (3/4) after receiving professional development. Teachers showed significant improvement in their ability to create both a correct word problem and diagram. However, about 17% of teachers struggled with referent units, in that they were unclear on whether the remainder referred to 1/3 of a whole or to 1/3 of 3/¾Similarly, Lee (2017) asked PTs to use a measurement perspective when solving the question, "How many 1/20 sticks can you make from the 3/5 stick" (p. 335). Fifty-two of 111 PTs solved the problem correctly, but only 13 of those showed an understanding of referent units.

Language use and referent units are just some of the issues that make the study of fraction multiplication and division in context difficult. Wording can suggest an improper procedure, whereas failure to attend to referent units leaves PTs unclear on the meaning of the result.

Methodology

To investigate how context facilitated or impeded fraction multiplication and division, we implemented an assignment, followed by a group interview, in math methods courses. The sample was composed of PTs from two universities taught by two of the researchers. Ninety-three elementary/middle school PTs came from a Mid-Western university in the U.S. They were enrolled in a math methods course, which they typically take in the semester before their student teaching experience. Ten elementary PTs came from a South-Eastern university in the U.S., who were in a math methods course in the second semester of their junior year. All PTs completed an assignment that asked them to create word problems for the given fraction equations involving operations of fractions (multiplication and division). The problems instructed students to, (1) Create a real word problem for the given equation; (2) Solve the problem you have created, without using the standard algorithm/rule, using drawings and manipulatives; and explain your solution; (3) Solve the problem you have created using the standard algorithm/rule; and (4) Explain how the standard algorithm is connected to your drawing and your moves with the manipulatives. A variety of equations were given, e.g., $3 \div (1/2) = 6$, $(2/3) \div (1/10) = 6\ 2/3$. The problems used were adapted from Principles to Actions (NCTM, 2014) and the Praxis study guide (Educational Testing Service, 2018, p. 13). Using open coding (Strauss & Corbin, 1990), one researcher analyzed 297 word problems to determine the frequent contexts that PTs relate fractions with. Individually each of the other two authors reviewed the categories and agreed 100%. Next, the researcher, who was not the instructor, conducted a group interview with nine of the PTs from the South-Eastern university. The aim was to gain a better understanding of the PTs' reasoning while solving the given problems. The PTs were first asked to explain how their drawing represented the corresponding equation as well as how it related to the standard algorithm. The transcript of the group interview was analyzed using open coding to identify how a context supports or interferes with understanding of fraction multiplication and division.

Results

First, seven categories were identified in 297 fraction word problems: (1) pizza/cookie/cake/ pie (47%); (2) recipe (13%); (3) distance/height/length (14%); (4) time (5%); (5) weight/volume (8%); (6) population (5%); and (7) other (8%). The PTs most frequently related fractions to the pizza/cookie/cake/pie context. Then, the group interview focused on PTs' reasoning in their non-algorithmic representational solution for the pizza/cookie/cake/pie problems that they created, and connections between their representation and the standard algorithm (e.g., how keep-change-flip rule is connected to the representation). The quotes from the interview, which took an hour and 15 minutes, were coded for the instances where pizza/cookie/cake/pie context was a basis for their reasoning and explanation. Five categories emerged from the interview (see Table 1).

Table 1: Description of the Categories

Categories (frequency)	Description
Split 1 (20)	The word "split" emerged from the context (pizza/cookie/cake/pie) and resulted in a misleading conversation as in "split in half" which indicates dividing by 2 instead of dividing by 1/½
Whole and Parts (9)	Both whole (pizza/cookie/cake/pie) and parts (slices) are referred to as pizzas, cookies, cakes, or pies.
Split 2 (4)	The action of splitting (pizza/cookie/cake/pie) resulted in a misconception that once you divide (i.e., split) then there are no wholes anymore.
Pieces (10)	The word "pieces" emerged from the context and used as in "piece of pizza/cake/cookie/pie)" with ambiguity in size as in daily life language.
How Many (7)	"how many" is used, referring to the number of slices, instead of "how much" "what fraction" which lead to an understanding of a fraction as two separate numbers at the top and bottom instead of "fraction as a number"

In the majority of the instances, the PTs used the word "split" in relation to splitting the pizza or cake, which led to an imprecise and inaccurate interpretation of fraction division (see Split 1 category in Table 1). For example, for the equation 3 ÷ (1/2), one PT explained, “when you split [cake] in half, you are gonna end up in smaller number,” and another PT said, “You have three cakes and then you split those in half, so now you have six cakes.” Both quotes show that splitting the cake in halves idea created confusion. On one hand, the language used, “split in half” suggested that they interpreted the equation as “division by 2.” Their solution, on the other hand, was consistent with the equation, “divide by 1/½ and hence there was a mismatch between the equation, language, and the reasoning. PTs often referred to both whole and parts with the same word, or used “piece(s)” with ambiguity, which impeded an accurate understanding of parts of a whole and unit whole concepts. For example, “..Ike the fact that you end up with six pieces [of cake] versus three” shows how PTs referred to both whole cakes and half cakes as pieces of cakes with no distinction. Another way that PTs were misled with the pizza and cake context was the misconception that once you divide then there are no wholes left. This misconception also arose from the daily life meaning of splitting as evidenced in, “..It now you have six halves [cake] like you don't have any wholes left anymore.” Lastly, PTs often used “how many” referring to the number of slices of a pizza or a cake, which led to an incorrect understanding of a fraction as two separate numbers (top and bottom), instead of a fraction as a single number representing a single quantity. For example, for (2/3) ÷ (1/10), one PT explained, “I did if Katie has two thirds of a pizza that had 10 slices originally how many slices does she have left.” The quote shows how the pizza context led the PT to focus on the number of slices per se, and hence deviated from the fractional content. In summary, the results showed that PTs most frequently interpreted fractions in the context of pizza/cake/pie/cookie; and these contexts impeded their understanding of fractions, instead of facilitating, in many different ways.

Conclusions and Implications

Problems with real-world contexts are commonly used in teaching fractions because such an approach has been seen to support student understanding of fractions. Our study, however, suggests that this might not always be the case. The majority of the word problems PTs created for given fraction multiplication and division equations used pizza, cookie, cake, or pie context. However, their explanations about how their visual representations related to the context of the problem and to the standard algorithm revealed that the context might have affected their understanding of fractions. The use of pizza, cookie, cake or pie context led to an understanding of partitive division but impeded their understanding of measurement division. Similar to past research, they used “split in half,” dividing by 2, to interpret the equation that involves division by 1/½Additionally, PTs also used the word “split” to describe an action of “splitting a pizza/ cookie/cake/pie,” which led them to the conclusion that there were no wholes anymore. The PTs used the terms “piece(s)” or “slice(s)” to refer to both the whole unit and part of the whole, with ambiguity in size, as used in everyday life. Consequently, this may interfere with an appropriate understanding of the part of a whole and unit whole relationship. Finally, PTs focused on “how many” parts were left out of the whole, suggesting their understanding of fractions as two numbers separated by a fraction bar rather than understanding fractions as numbers.

Results of our study suggest that using the context of pizza/cookie/cake/pie might impede an appropriate understanding of fractions. It is essential to pay close attention to the language used by consistently requiring PTs to use mathematical terms when explaining their solutions and reasoning. PTs’ use of words such as slice or piece to refer to both a whole and a part of the whole should be explicitly addressed by asking them to clearly define the whole and the part of the whole in the context of the problem using mathematical terms. The use of words such as slice and piece also seem to present obstacles to understanding fractions as numbers. All word problems with a pizza, cookie, cake, or pie context participating PTs created were partitioning problems that required answers to the question, "how many parts are left out of a whole." Such questions further support an understanding of a fraction as two numbers separated by a fraction bar. In contrast, by requiring PTs to use mathematical terms, the question "how much of an amount is contained in another amount" may be introduced logically, thus beginning to develop an understanding of a fraction as a number.

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USING A CRITICAL FRAMEWORK TO MAKE TEACHER RESOURCES VISIBLE

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While practice-based teacher education (PBTE) commonly draws upon multidimensional frameworks for teacher learning, I argue for the expansion of currently utilized frameworks to include dimensions related to justice. I discuss how this expanded framework was developed and draw upon examples from a larger project, focusing on the design and use of practice-based pedagogies centered on discretionary spaces, to illustrate the need for and benefits of this framework. I offer implications for future use of the framework and practice-based pedagogies.

Keywords: Preservice Teacher Education, Teacher Educators

The current wave of PBTE has resulted in the reconceptualization of *what* is taught in teacher education and *how* those things are taught. Alongside these changes has been the formulation of frameworks of teacher learning (Hammerness et al., 2005) that align with, and attempt to understand, the multiple dimensions of learning to teach that support the aims of PBTE. Despite this progress, some have argued the focus of "practice-based" has marginalized justice (Philip et al., 2019). I look to contribute to this body PBTE of scholarship by articulating an expanded framework for teacher learning that sits at the nexus of practice and justice-based teacher education – and then illustrate its benefits for making teacher candidate (TC) learning visible. I use the data from two TCs' engagement with a representation of practice (Grossman et al., 2009) centered in a discretionary space (Ball, 2018) to illustrate the necessity of this framework – showing how "practice-based" can become more contextualized and justice-based. My findings suggest implications for future use of the framework and practice-based pedagogies.

Background and Theoretical Framework

PBTE scholars have sought to build research, pedagogy, and curriculum around teaching practice. This has resulted in a multidimensional approach to teacher education which encompasses necessary knowledge and understandings, the development of tools and practices, and the cultivation of productive dispositions and vision for education. Despite the strides made by PBTE scholars, some critique that by breaking teaching into its constituent parts PBTE fails to maintain the complex, and situational nature of teaching (Horn & Kane, 2019) and situates justice as peripheral to the 'core' (Philip et al., 2019; Dutro & Cartun, 2016).

In response to these critiques, some PBTE researchers have drawn on the "Framework of Teacher Learning" (FLT; Hammerness et al., 2005) – attempting to robustly capture multiple dimensions of learning to teach. The FTL is multifaceted - –t works to capture not only different areas of teacher development, but also seeks relationships between teacher development and the surrounding context. While the FTL has been used by PBTE researchers to demonstrate how the current movement of "practice-based" is not just focused on the technical skills of teaching (e.g., Ghousseini & Herbst, 2016), much of its use has only used the initial definitions set forth, with researchers focusing analysis on a specific core-practice (e.g., orchestrating a whole class discussion). While this has helped some practice-based scholars to contextualize teacher learning, there exists a gap in understanding how TCs learn about teaching for justice through practice-based means. Thus, I argue here for an expanded framework of teacher learning that aims the foci of learning to teach within PBTE onto the enactments of justice-based resources.

This expanded framework entails using the FTL dimensions – dispositions, understandings, practices/tools, and vision – to capture dimensions of teacher learning as has been done previously (e.g., Baldinger et al., 2020) – but then adding four accompanying dimensions – *noticing*, *naming*, *confronting*, and *transforming* – connecting the existing framework to notions of teaching for justice, thus arriving at a Critical Framework for Teacher Learning (CFTL).

To accompany the dimension of dispositions I propose the addition of noticing. While noticing relates to the habits of thinking and action described by Hammerness and colleagues (2009), here the development of critical consciousness (Bell, 2016; Freire, 1998; Schiera, 2019) grounds these habits in the broader sociopolitical sphere. Thus, it is not enough for TCs to develop dispositions related to a belief that 'all students can learn,' but they must additionally notice when broad structures and inequities infringe upon a student's opportunity to learn.

Complimenting the conceptualization of understandings is the dimension of naming. Extending from learning to notice oppressive features in education, TCs need to have understandings that will allow them to name the features of oppression (Bell, 2016) happening at multiple levels (Adams & Zúñiga, 2016). Additionally, teacher learning in this dimension includes naming oppressive features and functions of content and/or pedagogy as well as those working within the intersecting social identities of students – further demonstrating robust understandings in relation to content, pedagogies, students, and social contexts.

Extending the conceptualization of practices/tools is the area of confronting. Much like the way Hammerness and colleagues (2005) outline how the development of practices and tools can help TCs understand complex conceptual tasks and translate them into approaches and rationales, the dimension of confronting is aimed at capturing the complex work that goes into a teacher addressing inequities within practice. Some scholars have already begun to articulate examples of some skills that might exist within this dimension. Examples include posing alternatives, empowering students (Kavanagh, 2017), as well as practices/tools that can be (re)envisioned as justice-based such as, asking clarifying questions (Adams & Zúñiga, 2016), offering feedback, and orchestrating whole class discussions (Hammerness et al., 2005).

To expand Hammerness and colleagues' (2009) dimension of vision, I propose a dimension of transforming. Opportunities must be provided for TCs to not only think about how education stands to be transformed, but to actually begin to enact that work. Directly building from the previous dimensions, transforming encompasses the need to develop a not only a curricular vision (Hammerness et al., 2005), but one that has justice centralized. Additionally, notions of 'good' practice, what is possible, as well as desirable in teaching are shaped by what can transform education and move toward justice. Particularly in mathematics, images of practice are clouded by intersecting layers of oppression. Thus, what is seemingly possible is limited.

Methods

This illustration of two TCs' work is a portion of a larger design-based research project working to design and facilitate cycles of pedagogies of practice (Grossman et al., 2009) centered in discretionary spaces (Ball, 2018) related to authority in secondary mathematics classroom discussions. In this paper, the focus is specifically on how Tacy and Evan's engagement with a single representation of practice (Grossman et al., 2009) illustrates the utility and need for the CFTL in practice-based scholarship.

Data was collected from TCs enrolled in a seminar course that accompanied their student teaching experience (Spring 2021). The set of data included their annotations on a written case of classroom activity. This written case was created to highlight authority relationships that may exist within secondary mathematics classroom (Langer-Osuna, 2017) and how the struggle for authority intersects with social identities such as gender. Within the class session of interest, despite the teacher's use of discourse moves often positioned as productive during a whole group discussion, there exists many discretionary spaces where voices of (white) men are elevated, while those of women are stifled. TCs were given a scenario and asked to annotate the document for things they find interesting, questions they had, as well as anything they noticed or wondered.

A priori codes based on the eight dimensions of CFLT (dispositions, understandings, practices/tools, vision, noticing, naming, confronting, and transforming) were used to conduct an initial round of coding on the annotations. This round sought to identify the dimensions that were made possible by the representation of practice. Following this identification, a round of open, pattern of coding (Saldaña, 2016) was conducted to add contextualization for *how* each dimension of teacher learning was made visible. At this stage, the data from Tacy and Evan was chosen because of their illustration of the use of the CFTL.

Illustrative Examples

Upon reading the written case, Tacy made 14 annotations. During the first round of a priori coding, these 14 annotations translated to 18 instances of Tacy demonstrating a dimension of the CFTL. The complete breakdown of these instances can be found in Table 1.

Table 1: A Priori Coding – Tacy and Evan

	Tacy	Evan
Dispositions	1	1
Understandings	3	3
Practices/Tools	5	2
Vision	1	0
Noticing	4	1
Naming	3	1
Confronting	1	0
Transforming	0	0

Throughout her annotations, Tacy focused on analyzing student thinking (Understandings) as well as identifying teaching moves and lesson planning routines that support students in having a whole class discussion (Practices/Tools). For example, Tacy made the following annotation:

> I like that she consistently goes to the board with the students' thoughts. She highlights, underlines, draws those ideas on the board, which can help students see where their peers' thoughts are coming from. Also, I like that she kept control of how ideas are presented on the board.

This annotation serves as evidence of Tacy making connections of the teacher's gestures, writing on the board, and the organization of that writing to the ways she sees this supporting students who are "visual learners" to engage in the whole class discussion.

Tacy was also able to notice (four occurrences) and name (three occurrences) inequities throughout the written case. Often in Tacy's annotations the coding for "noticing" or "naming" was accompanied by another dimension such as "Understandings" or "Practices/Tools". Within the written case, the teacher gives credence to the ideas of white boys, even when those ideas are very similar to another idea already shared, or less mathematically complete than others. In one of her annotations, Tacy was able to recognize the similarity of two mathematical ideas shared (one by a white boy, Jaron, and another by a white girl, Clara), demonstrating an understanding related to analyzing student thinking. In that same annotation, she went on to say,

> To spend the rest of the lesson discussing "Jaron's idea" when the idea belongs to not only Jaron but Clara and Jess and anyone in their groups who came up with a similar method. Again, it establishes a white boy as the owner and director of the ideas in the classroom when in reality there are multiple students who deserve empowerment in this situation.

Here, Tacy recognizes, despite a very similar idea already being presented by a group of girls, how the teacher named the idea as belonging to Jaron (Noticing). Additionally, she names the implications of these actions (Naming) as positioning white boys as the "owner and director of the ideas in the classroom." This annotation also served as evidence of Tacy contextualizing the teaching moment and seeing it holistically. Attaching a student's name to their idea is a "move" typically noted as being valuable when orchestrating a whole group discussion. But as Tacy notes, within context, this move is harmful particularly for non-white, non-female students.

As seen in Table 1, Evan had demonstrated dimensions of the CFTL eight times across his six annotations. Like Tacy, Evan drew on various aspects of the mathematics and orchestrating a whole class discussion. In one instance, Evan wrote, "I like how she [teacher] restates Clara's comment, but adds more precise language so the class can fully understand what Clara is sharing". Here, Evan has identified where he believes the discussion needs to go mathematically (Understandings). Thus, he identifies this move of "teacher restating" as an appropriate insertion of mathematical language (Practices/Tools) because it allows for students to work on the content.

Similar to Tacy, Evan worked on multiple dimensions simultaneously. At the end of the discussion, Evan felt that the teacher "took over". He recognized an unproductive insertion of ideas by the teacher because it took the mathematical work from the students (Dispositions), suggests a different course of action based on tools of classroom talk (Practices/Tools), and illustrates his ability to analyze student thinking (Understandings) by saying, "students were slowly making connections toward making an equation to model the figures."

Despite his success in demonstrating the four previously established dimensions of the CFTL, Evan was only able to demonstrate his ability to notice and name inequities once through his annotations. Evan wrote, "This is the 2nd time she ignored Jess. The teacher was trying to use different talk moves, but Jess could easily see this as bias against her, especially given her race in a predominantly white setting." Here, Evan noticed Jess, a Black girl, being ignored on two occasions. Attempting to name this moment, Evan assigned the name "bias" and the outcome as being ignored in math class, but simultaneously projected this onto Jess, rather than the teacher. This is representative of Evan's annotations in that he failed to contextualize the written case, and the actions of the teacher, on multiple levels. Other examples of this come from the two instances previously mentioned. Evan failed to notice the teacher's insertion of "more precise" mathematical language into Clara's idea as a possible act of undermining her. Additionally, when Evan felt the teacher "took over," he was focused on the students who were "slowly making connections" – which were all white boys. He did not recognize the mathematically sound, and advanced, contributions of the group of girls represented in the written scenario.

Discussion and Conclusions

Through these two illustrative examples using the CFTL, teacher resources across practice and justice-oriented dimensions were able to be illuminated – an important takeaway for both practice-based teacher educator/researchers and for TCs. By utilizing the CFTL to make teacher resources visible, dimensions needing to be worked on can be identified and opportunities that attend to these dimensions of teacher learning can now be designed, facilitated, and reflected upon to support TCs in learning to teach for justice through practice-based pedagogies.

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ELEMENTARY TEACHER CANDIDATES' CONNECTIONS BETWEEN MATHEMATICS AND LITERACY TEACHING

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One persistent challenge in elementary teacher education is supporting teachers in developing confidence in teaching mathematics in ways that develop children's conceptual understanding. This challenge is intensified by the math anxiety that many elementary teachers experience. I argue that elementary teachers' position as subject area generalists could allow them to draw on their strengths in teaching other subject areas. This paper explores the connections that elementary teacher candidates (TCs) make between mathematics and literacy teaching. TCs noted similarities in content and in instructional practices. Creating spaces for TCs to reflect on similarities and differences between subject areas could support them in drawing on strengths in other subject areas to increase their confidence in mathematics teaching.

Keywords: elementary school education, preservice teacher education, instructional activities and practices, teacher knowledge

Elementary teachers tend to be less confident in teaching mathematics in ways that support children's conceptual understanding than they are in teaching other subject areas (Buss, 2010; Wilkins, 2010). Much of this discomfort can be attributed to elementary teachers' feelings of anxiety towards mathematics (Hadley & Dorward, 2011; Wood, 1988). In contrast, elementary teachers report reading and language arts as the subjects they feel most confident teaching (Wilkins, 2010), which makes teaching literacy an area of strength for many elementary teachers.

Researchers have frequently taken a deficit view of elementary teachers' subject-area generalist status, arguing that it limits the depth of content knowledge (e.g., Ding, Li, & Capraro, 2013; Kajander, 2010; Lovin, Stevens, Siegfried, Wilkins, & Norton, 2016; Ma, 1999). I argue that being a subject-area generalist can be a strength for elementary teachers. Their familiarity with pedagogy in multiple subjects gives them a unique opportunity to reflect on teaching practices across subject areas. However, the way that teacher education is structured is unlikely to promote connections across subject areas. Most learning opportunities for teachers are divided by subject areas, including university courses, professional development programs, and journals for teachers, with little communication across those subject area divides. What connections might elementary teacher candidates (TCs), make if they were encouraged to consider the similarities and differences in teaching mathematics and teaching literacy?

Literature Review

One way of connecting mathematics and literacy is content-area literacy. In mathematics, this work tends to be about ways to use reading comprehension skills to help students read mathematics textbooks or to understand word problems (e.g., Adams, Pegg, & Case, 2015; Armstrong, Ming, & Helf, 2018; Beaudine, 2018; Caputo, 2015; Halladay & Neumann, 2012) or strategies for learning new vocabulary words (Altieri, 2009). Writing is also used as a way for students to make sense of mathematics and to explain their thinking. This includes strategies such as keeping mathematics journals to record their thoughts (Armstrong et al., 2018) and bringing writing composition skills into mathematics writing (Carter, 2009).

Much of the research that has been done on connections between mathematics and literacy have focused on connections chosen by the researchers or teacher educators (e.g., Gilles et al., 2016; Lemley, Hart, & King, 2019; Phillips, Bardsley, Bach, & Gibb-Brown, 2009). Although the teachers were doing the inquiry in these studies, the focus on literacy practices in their discipline was chosen by the researchers. In contrast, Matthews and Rainer (2001) specified the subject areas to connect, but left the types of connections to the teachers as they created learning frameworks for mathematics and literacy. As the teachers created separate lists of mathematics and literacy skills, they began to notice more abstract connections between the two subject areas, such as concern with both conceptual understanding and skills, and constructing understanding of texts and mathematical concepts. The study described in this article adds to our current understanding of how teachers and prospective teachers make connections between subject areas, by focusing on the kinds of connections that teacher candidates (TCs) make, rather than those suggested by the researchers.

Methods

This study draws on sociocultural understandings of language and knowledge (Lave, 1996; Rogoff, 2008; Vygotsky, 1978; Wenger, 1998) In groups with peers and experts, we use language to make sense of what we were learning and build a shared understanding of what is "true" (Gee & Handford, 2012).

The data for this paper is drawn from focus group interviews that are part of a larger case study of elementary TCs learning to teach multiple subject areas. Focus groups use group interaction as part of the interview (e.g., adding to others' ideas, responding to others' experiences, etc.; Kitzinger, 1995), allowing an in-depth exploration of the TCs ideas about teaching literacy and mathematics. Eight TCs, who were taking concurrent mathematics and literacy methods courses participated in the focus groups. All participants, including myself, self-identified as White women.

I used an open-coding approach to identify themes in the video transcript of each focus group (Charmaz, 2014). For each group, I collected the TCs' statements about teaching mathematics, organized them by the themes identified in the coding phases, and arranged them into a *collective portrait* (Anderson, 2005), a narrative that expressed their shared understanding of teaching mathematics. I repeated this process with the TCs' comments about teaching literacy, and the connections they noticed across the two subjects. In the findings, I will share excerpts from these narratives that illuminate the themes from data analysis and the connections TCs made.

Findings

Content Connections

The first connections that occurred to the TCs were reading comprehension of word problems and mathematics vocabulary. "Students need to know how to read and distinguish what the problem's asking." They also emphasized the importance of the specialized meaning of words in mathematics. "When you're reading a math text, this is what [this word] means and when you're reading a story about friends, this is what it means." Learning mathematics vocabulary is important to students' understanding of mathematics, and those terms which exist in everyday language (e.g., difference), but with a different meaning may be the most difficult for children to learn (Schleppegrell, 2007).

Another content connection the TCs made is that background knowledge is important in both mathematics and literacy, but it takes on different aspects in the two subject areas. In both subject areas, background knowledge included skills and ideas learned in previous lessons, but only literacy was concerned with contextual background knowledge.

> [In literacy] you teach more explicitly, like "This is what - – don't know baseball terms - –his is whatever this means," so that they can understand it a little better. Whereas, with math, I just don't think there's that same context. Math can be a little more, I guess, straightforward, because you know you're working with numbers, and you already understand that.

Although it is true that a good deal of mathematics in school is without context, mathematics in the real world is often embedded in a context that must be understood to make sense of the mathematics.

Pedagogical Connections

As our conversations turned to the similarities and differences in teaching mathematics and literacy, one group noted that you can use "manipulatives for understanding things in math or blocks for making letters and sounds go together for literacy instruction," as well as using visuals such as diagrams or graphic organizers in both subjects. This focus group tended to take an early elementary perspective (i.e., letters, sounds, early mathematics concepts), which may be why they thought about using manipulatives in literacy.

Another similarity the TCs noticed was using small groups for differentiation and learning from peers. The TCs saw small group instruction as a possible way to meet a large range of learning needs but had some hesitations about managing small groups, as well as about grouping students by "ability."

> And definitely allow kids to work at their ability levels, but also have them work with students who aren't necessarily at their levels, too, so if one's really strong in this area, and this student's not, let them be able to share their strengths, because I do think students learn from each other.

This sort of ambivalence towards small group instruction – that it could help meet students' learning needs and allow them to share strengths, but that it could also reinforce hierarchies of who is perceived as "smart" or "dumb" - –as a big concern for two of the TCs, based on their own experiences with instruction, particularly in mathematics.

A third similarity was the need for assessments to "know where students are at" and to "change up your strategy of how you're teaching this to the student." Although they agreed that the purpose of assessment was the same, they described assessments in the two subjects in very different ways. In literacy, they referred several times to a variety of reading assessments they had learned in a previous course. In contrast, they mentioned very few mathematics assessments, and typically in a negative light.

> I really don't like timed tests. They gave me, as a student, a ton of anxiety. I could probably look at [my students' thinking] with worksheets they do. I know eventually you're going to have to have a math test, to see where the kid is at, and see if they've learned.

Their perception of mathematics assessment was limited to timed tests, worksheets, and written math tests, which were likely the kinds of assessments they had experienced as students, and they may not have been able to imagine mathematics assessments that were different from what they experienced.

A final similarity that was noted by the TCs was the role of process in both subjects, and the way that rough draft thinking could be beneficial in mathematics teaching and learning.

> I think the idea of writing being a recursive process should be transferred over to math as well. We do so many rough drafts and we do outlines and so many steps before you get to the final version of what you're producing, and the growth that happens in between all those revisions. If you sort of approach math in that way, that can help make it seem more like a process where it's okay to trip up and go back and edit it and learn from that.

Discussion

The TCs made many connections between mathematics and literacy when given the opportunity to reflect on their ideas about how to teach the two subject areas. The immediate connections they made were similar to those in the content-area literacy research - reading comprehension of mathematics problems and mathematics vocabulary (e.g., Adams, Pegg, & Case, 2015; Armstrong, Ming, & Helf, 2018; Beaudine, 2018). Noticing these connections would allow these TCs to draw on this literature to support their students in learning mathematics. Continued reflection and discussion led these TCs to make further connections, including pedagogical similarities. This may allow them to draw on their strengths in one area to teach another. For example, the TCs were familiar with a variety of literacy assessments, including both assessments of specific skills (e.g., letter-sound correspondence) and reading comprehension assessments. The mathematics assessments they mentioned (e.g., timed tests, worksheets, math tests) are mainly skills-based, rather than conceptual. Perhaps, by reflecting on how reading comprehension assessments surface students' understanding of text, they could consider mathematics assessments that elicit students' mathematical thinking. A connection these TCs made that I believe holds potential for supporting more conceptually based teaching is that of mathematics as a process similar to writing. By thinking of initial work in class as "rough draft" mathematics, the focus becomes more about revising mathematical understanding rather than getting the correct answer quickly the first time.

Unfortunately, teacher preparation programs, with subject areas siloed in their own courses, with few or no connections between them, are not set up to promote this type of cross-curricular thinking. The focus groups, as a space set aside for reflecting on the similarities and differences in teaching mathematics and literacy, allowed TCs to make connections that did not happen in their separate methods courses. This raises several questions and challenges for elementary teacher education. How might elementary teacher preparation programs reimagine their structure in a way that more closely reflects the day-to-day realities of an elementary school, where a single teacher works with the same group of children in multiple subject areas? In the absence of program-wide structural changes, how can mathematics teacher educators take advantage of knowledge in other subject areas? Collaboration with teacher educators in other subject areas has been explored as one possibility (e.g., Draper & Siebert, 2004; Wohlhuter & Quintero, 2003). Other teacher educators have examined the challenges and benefits of integrated coursework (e.g., DeLuca, Ogden, & Pero, 2015). Other researchers documented the challenges in teaching an integrated course (e.g., Kalchman & Kozoll, 2012), or collaborating with peers in other disciplines in institutions set up to separate disciplines (e.g., Miller & Stayton, 2006).

Despite these challenges, reimagining elementary education in ways that celebrate the strengths of knowing multiple subject areas can result in increased confidence in teaching various subject areas. Researchers in science education have found that integrating science and literacy methods increases elementary TCs' confidence in teaching science (e.g., Akerson & Flanigan, 2000; Brand & Triplett, 2012; Ledoux & McHenry, 2004). In mathematics education, perhaps we can draw on these examples to help TCs develop confidence in teaching mathematics by leveraging their strengths from other subject areas.

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PRE-SERVICE TEACHERS' INTERPRETATIONS OF REAL-LIFE EXAMPLES RELATED TO PERCENT

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Theoretical Perspectives and Research Question

Although the understanding and use of percent is vital in mathematics and our daily life, studies report that it is often misinterpreted (Lo & Ko, 2013; Parker & Leinhardt, 1995; Watson & English, 2013). This difficulty exists among students as well as among in-service and pre-service teachers (White & Mitchelmore, 2005). Studies point out that instruction often relies too much on teaching the formalized algorithms rather than exploring various contexts and intuitive strategies, which causes more difficulties (Freudenthal, 1983; Parker & Leinhardt, 1995). This study investigates the following research question: What types of understanding do pre-service teachers (PSTs) demonstrate while engaging in the tasks of interpreting real-life examples related to percent?

Data Collection and Analysis

Forty-six PSTs who enrolled in a mathematics methods course in a Midwestern University in the U.S participated in the study. Individual PSTs were asked to select one real-life example that uses percent (e.g., commercials, news reports, etc.). Each PST was asked to facilitate an online discussion forum on the example of his/her choice for four weeks using various questions/prompts and to report the conclusive interpretation of the initial problem upon completion. The entire discussion forum entries were analyzed by following the inductive content analysis approach (Grbich, 2013).

Summary of Findings and Implications

This presentation will report the detailed progress of the discussion and results of the study. A few aspects noted include the following: (a) Many PSTs focused on applying typical computation procedures they knew, but those efforts were often unsuccessful when using real-life examples where essential information was provided in a vague form compared to the typical textbook problems. (b) One significant difficulty was identifying referent units to interpret relative sizes, which led to incorrect fractional relationships. This study shows that PSTs were familiar with computations procedures for typical percent-related problems but lacked flexibility in applying their knowledge to real-life examples. Designing tasks that promote PSTs' flexible knowledge application in various mathematical content domains is necessary.

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PRESERVICE TEACHERS AS RESEARCHERS: A MENTORSHIP MODEL

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Participating in research activities can enhance undergraduate students' learning experience during their college education and provide them with opportunities for professional growth (Jahan & Aly, 2018; Madan & Teitge, 2013). For pre-service teachers (PSTs), these research experiences support their professional development as both action researchers and effective classroom teachers. By engaging in undergraduate research focusing on model eliciting activities (MEAs), PSTs can better understand mathematical concepts related to mathematical modeling, practice teaching mathematics with real-world contexts (COMAP & SIAM, 2016), and better conceptualize the research designs in the mathematics education field.

Our goals for the PSTs engaged in undergraduate research are: 1) advance their mathematical content knowledge by designing and developing the modeling activities, 2) improve their pedagogical content knowledge by teaching mathematics in informal settings, and 3) strengthen their understanding and analysis of student thinking and learning (Ball, Thames, & Phelps, 2008). To attend to these goals, we designed a six-phase mentorship model for developing and implementing MEAs.

Table 1: Mentorship Model Phases

Phase	Description
Development	Research and develop the modeling task
Rehearsing	Consider how the task would be implemented in a classroom setting
Implementation	Implement task with intended audience (e.g., local summer camp, classroom setting) and collect data of students' thinking and learning
Reflecting	Reflect on what went well and how could the task be improved post-implementation
Analysis	Analyze student data collected during task implementation
Dissemination	Prepare conference proposals and presentations to share research findings

Thus far, we have mentored or co-mentored six PSTs (two PSTs completed the mentoring cycle and the other four are still in progress). The PSTs' written reflections during the mentorship cycle were collected to assess the effectiveness of the program and to report PSTs' professional growth. These written reflections were often related to PSTs' beliefs about MEAs (e.g., "By including real-life situations, current events, and scenarios relevant to each individual's lives, I will facilitate a learning environment that builds on how the world truly works"), teaching mathematics (e.g., "These implementations really helped me grow as a teacher…I can see my questioning start to get stronger every time I get into the 'classroom'"), and conducting research (e.g., "Students' varying thought processes can create the window of opportunity for many ideas and theories to be explored, discussed, questioned, and potentially solved").

This experience not only provides rich teaching and learning experiences for the PSTs, but it also benefits the faculty mentors. This mentorship model provides faculty mentors the opportunity to work with PSTs in a non-graded setting. This experience also helps faculty mentors to integrate their teaching and their research expectations.

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LEARNING TO ASK QUESTIONS ABOUT INSTRUCTION FROM VIDEO

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Practicum placements, where preservice teachers (PSTs) observe in a classroom over an extended period of time, are a typical feature of teacher education programs. PSTs' opportunities to learn from observations are influenced by discussions with their cooperating teachers (CTs), as discussions can reveal CTs' reasoning about instructional decision-making and student thinking. How can PSTs learn to elicit such discussions? We propose that learning to ask meaningful questions after an observation is a skill that can be learned using video. Video allows teachers the opportunity to reflect on the richness of real classroom interactions while providing needed time for examination (Sherin, 2003, 2007). In turn, video-based opportunities to learn support teachers in nurturing an inquiry stance (Sherin, 2003; van Es et al., 2014), or the ability to consider teaching and learning in new ways (Putnam & Borko, 2000). In this exploratory study, we examine whether using video in a mathematics methods course to practice question-posing influenced the kinds of questions PSTs reported having posed or imagined posing to their CTs following observations of mathematics instruction.

Participants were PSTs in an elementary mathematics methods course (n= 12) at a university in the Midwest. Data were collected from participants at three timepoints: Surveys 1 and 2, given at the start and end of the course, respectively, and a class discussion about video mid-course. Here, we focus on PSTs' responses to one prompt given across these sources, "What questions have you asked your CT (or could you have asked, in the case of the video activity) to learn from what you observed?" Using a combination of top-down and bottom-up coding (Miles et al., 2018) we examined three salient dimensions of PSTs' questions: topic, specificity (e.g., van Es & Sherin, 2008), and function. We ask: How did the topic, specificity, and function of PSTs' observation questions change from Survey 1 to the video-based class discussion, and from Survey 1 to Survey 2? The topic of PSTs' questions were: curriculum, pedagogy, student thinking or needs, or assessment. Questions either referred to a *specific* moment, decision, student, or question or something *general.* Function refers to what the questioner was trying to understand: the observed lesson, the CT's pedagogical approach or belief(s), or logistics. We then created memos detailing changes from Survey 1 to the classroom discussion, and Survey 1 to Survey 2, and compared memos to derive three preliminary findings.

Topic: Between Survey 1 and the class discussion, PSTs shifted from primarily focusing on pedagogy to considering students' thinking and needs; by Survey 2, PSTs' questions focused on students' thinking and needs the majority of the time. *Specificity:* Between Survey 1 and the class discussion, PSTs' shifted from asking nearly all general questions to asking a majority specific questions; by Survey 2, PSTs' questions were nearly all specific. *Function:* Between Survey 1 and the class discussion, PSTs' questions shifted from focusing primarily on logistics to rooting nearly all questions in moments from the video; this held true in their Survey 2 responses. These findings suggest that PSTs can learn to pose substantive questions about classroom observations through video in their methods courses, and point toward the need for larger studies to explore the efficacy of using video to practice posing questions to enrich PSTs' practicum placement experiences.

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LEADING GROUP DISCUSSIONS ENACTMENTS IN A METHODS COURSE

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The teacher education field has shifted from emphasizing knowledge for teaching into core teaching practices (McDonald, Kazemi, & Kavanagh, 2013). Focusing on core teaching practices helps teacher educators deconstruct teaching into learnable parts and helps pre-service teachers (PSTs) manage the complexity and uncertainty of teaching (Forzani, 2014). Programs focusing deliberately on core practices have the potential to narrow the gap between university courses and clinical experiences.

Leading group discussions (LGD) is a fundamental practice in teaching mathematics. PSTs are often not sure how to orchestrate a discussion when there are multiple different solutions, when they want to draw other students into the conversation, or when they want to facilitate the conversation towards a particular goal. This practice is still complex but can be deconstructed into learnable components such as using discussion enabling prompts (TeachingWorks, 2019).

Following the rationale described above, I systematically integrated LGD core practice in a methods course and investigated PSTs' reflections on their own implementations. The participants of the study consisted of 11 senior undergraduate students enrolled in a mathematics methods course offered to middle school majors during the Fall 2019 semester (five PSTs, in-person instruction) and the Fall 2020 semester (six PSTs, virtual, synchronous).

During the first half of the course, the PSTs were introduced to the components of the LGD framework (TeachingWorks, 2019), student thinking monitoring tools (Smith & Stein, 2018) and assessing and advancing questioning techniques (National Council of Teachers of Mathematics, 2014). PSTs solved mathematical tasks, anticipated possible solutions, created questions to assess and advance student thinking in the solutions, and analyzed classroom videos. During the second half of the course, the PSTs created their own tasks, prepared a monitoring tool with possible student strategies as well as assessing and advancing questions for each strategy. Next, they decided on follow-up questions for orchestrating the discussion which would unpack 1) key mathematical points, 2) underlying logical necessities of those points, 3) questions to push student thinking further and advance the learning by making connections, and 4) connections between different strategies. Each PST implemented their LGD task during the class sessions in a "peer-teaching" format by utilizing the LGD framework. Additionally, the Fall 2020 group conducted their second implementations with middle school avatar students in a simulated classroom environment (Mursion, 2020). Finally, PSTs wrote reflections on their implementations. These reflections were analyzed using comparative analysis (Merriam, 1998).

Overall, all 11 PSTs indicated that this instructional effort provided them with various learning opportunities related to orchestrating group discussions, eliciting and orienting student thinking, and/or responding with appropriate questions. Almost all PSTs stated that they need to improve their questioning techniques. For example, one PST stated, "*I learned to target my questions so that they aren't yes or no answers but ones that require students to think deeper.*" Another PST stated, *"'Does that make sense?' is not a very productive question because most students are just going to say yes… A better way … is to ask them to explain in their own words* *what you said/what their peer said."* The sophistication of reflections varied based on the cognitive demand of the LGD tasks as well as each PST's mathematical content knowledge.

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EXAMINING PRESERVICE TEACHERS' RESPONSES TO AREA CONSERVATION AND VOLUME TASKS

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Building procedural fluency from conceptual understanding is one of the effective teaching practices recommended by the National Council of Teachers of Mathematics (National Council of Teachers of Mathematics [NCTM], 2014). Among many topics in mathematics, with a significant number of formulas, area and volume measurements have a high likelihood of students relying on procedural understanding to solve problems (Vasilyeva et al., 2013). As we worked with our PSTs, we were interested in exploring their existing knowledge about area conservation and volume measurement. Our results showed that PSTs in the present study experienced difficulty determining the bases and heights of multiple figures (especially non-prototypical ones), relied on visual comparison of shapes as opposed to geometric and referring to irrelevant information such as the length of the sides or the largest number. Some of the conceptual limitations found in the present work may lead future teachers to lean more on procedurally-focused area and volume measurement lessons, resulting in continued students' learning challenges. In addition to limitations in curriculum materials, PSTs' limited knowledge in area and volume measurements showed an alarming pattern in the teaching and learning of area and volume measurement, and suggest that area and volume measurement may continue to be treated as a primarily formulaic, procedure-drive topic (Hong, Choi, Runnalls, & Hwang, 2018, 2019; Runnalls & Hong, 2019). With a small number of participating PSTs, we can't generalize from our results; however, we suggest that being exposed to the activities specifically address PSTs' challenges to be included in mathematics content classes and/or their teacher education programs. In turn, PSTs can have the potential to develop lessons that can reflect conceptual ideas of area measurement.

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NAVIGATING THE IMPLEMENTATION OF BEST PRACTICES IN MATH INSTRUCTION WHILE STUDENT TEACHING DURING COVID-19

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With the shift to hybrid and online learning in the spring of 2020, due to COVID-19, preservice teachers' internships in classrooms were drastically changed. While research has shown the importance of field experiences (e.g., Darling-Hammond et al., 2002; Philipp et al., 2007; Zeichner, 2010), little is known about the impact of virtual field experiences and how COVID-19 has impacted preservice teachers' preparedness to teach. While Kennedy & Archambault (2012) argue *for* the use of virtual field experiences for preservice teachers, they acknowledge that online learning does not take the place of traditional classrooms.

Understanding the impact of the restrictions and policies placed on preservice teachers due to COVID-19 is important as they move into their student teaching placements and into their first years of teaching. Specifically, understanding how COVID-19 is impacting preservice teachers' implementation of best practices in their mathematics classrooms is crucial for teacher education programs. The two main questions guiding this portion of the research study were:

1. How is learning to teach impacted by restrictions and policies due to COVID-19?
2. What does mathematics teaching look like during COVID-19?

Participants and Data Collection

Twenty-four preservice elementary teachers volunteered to participate in a study exploring what student teaching looked like during the COVID-19 pandemic. Of those, ten agreed to take part in more in-depth sharing of their experiences during student teaching. Participants were asked to engage in three interviews and complete three journal entries over the course of the semester. Interviews for this larger study included questions specific to their knowledge of best practices in teaching mathematics and ways that implementation of those practices was enhanced or limited during their student teaching experience due to restrictions from COVID-19 policies.

Results and Implications

This study is relevant and timely as it helps teacher educators understand the implications of learning to teach during COVID-19. Preliminary results of this study show the impact of moving field experiences online during the 2020 school year. Teacher candidates adjusted to this transition to online teaching during their field experiences in the fall and many gained confidence in implementing best practices in that virtual environment. However, when they entered the classroom in a face-to-face setting during their student teaching, many of them struggled with planning for instruction that incorporated the best practices they studied in their mathematics methods courses. Teacher candidates commented on how they struggled to incorporate shared manipulatives, group discussions, and high-level tasks, while meeting the limitations placed on instruction due to COVID-19 policies. While this study is ongoing, the initial findings show the struggle these teacher candidates experienced and how they are persevering through the challenges of learning to teach in the midst of a pandemic.

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PROSPECTIVE TEACHERS' QUESTIONING PATTERNS IN A SIMULATED STUDENT INTERVIEW

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Mathematics teacher questioning is an important and complex instructional practice which is critical to enacting student-centered teaching (Franke et al., 2009). Questioning is essential to successful teaching, adaptable to a variety of teaching contexts and circumstances, and is a manageable and discrete skill which can be developed during pre-service teacher education; it is thus considered a "high-leverage practice" (Grossman, Hammerness & McDonald, 2009). Conducting structured interviews with students can improve prospective teachers' (PTs') questioning skills (Jenkins, 2010; Moyer & Millewicz, 2002), but logistical considerations of field placements often limit these opportunities.

Research Question, Design, and Analysis

This poster presents preliminary results from a study in which PTs interacted with a *simulated student* (programmed using an AI chatbot and the authentic responses of 5 children) to conduct a diagnostic interview about geometric thinking. This work adds to existing research about simulated interviews (Shaughnessy, Boerst, & Ball, 2015) by introducing and investigating a novel technological tool. Participants (20 PTs enrolled in a geometry course) conducted a simulated interview with the chatbot, which we dubbed "Matilda." Each PT independently "interviewed" Matilda, using a given set of example shapes to frame their questions, with the goal of revealing and diagnosing her understanding of geometry. We analyzed the interview transcripts to classify the questioning types and patterns that PTs used. Future work will examine associations between PTs' questioning patterns and their resulting diagnoses of student thinking.

Findings and Implications

Results suggest that PTs asked a variety of question types during their simulated interview. The most common type of question asked was Specific Classification questions (e.g., "Is Shape 5 a rectangle?"), asked 178 times by 16 different PTs (89% of participants), followed by General Classification questions (e.g., "What kind of shape is Shape 7?"), which asked 118 times by 17 different PTs (94% of participants). Nearly all PTs (94%) asked at least one follow-up question to probe Matilda's response to a previous question, with the majority of these follow-up questions (a total of 100 questions) asking *why* Matilda had given a particular response. This suggests that PTs were attending to the simulated student's answers and attempting to probe more deeply. However, the simulated student had been programmed with a variety of children's invented vocabulary for describing the shapes (e.g., "it's too spinny" for a square oriented with no horizontal sides), and only 6 PTs (33%) asked a follow-up question about such vocabulary. This suggests that their questioning patterns could be more specifically targeted to gain richer understanding. Overall, these results, and other data we will share in the poster, suggest that PTs may need specific interventions to improve their ability to ask rich and meaningful questions to uncover student thinking in order to guide their instruction.

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EXAMINING THE INSTRUCTIONAL READINESS OF PRE-SERVICE TEACHERS VIA THEIR EPISTEMIC AND MATHEMATICS INSTRUCTION BELIEFS

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Following *teaching-in-context theory* (Schoenfeld, 1998, 2011), this study investigated pre-service teachers' instructional readiness in mathematics by answering the following questions: (1) How familiar are pre-service teachers with reform-based standards for mathematical practice? (2) What type of epistemic and mathematics instruction beliefs do pre-service teachers have? and (3) What role do these beliefs, prior experience, and program of study play in pre-service teachers' instructional readiness?

Method

Design

This poster presents results from a mixed-methods study that combined online survey responses with follow-up, in-person interviews (QUANT → Qual; Creswell, 2009).

Participants. The sample ($N = 104$) included pre-service elementary teachers, pre-service secondary math teachers, and students minoring in education enrolled at a preeminent research university in the southern United States during the 2017-18 academic year. All participants declared they planned to teach after graduation. A subset of participants ($n = 6$) who were towards the end of their teacher training was also interviewed.

Instruments. In addition to general information, two instruments were used to collect data on participants' beliefs: Mathematics Instruction Beliefs Survey (MIBS), which was designed to assess alignment with reform-based standards for mathematical practice, and Hofer's (2000) Discipline-focused Epistemological Beliefs Questionnaire (DEBQ), which assessed participants' mathematics epistemic beliefs. In-person interviews included questions about participants' prior school experiences, specific questions regarding the MIBS and DEBQ scales, and questions about their program of study and future teaching practice, including using vignettes that aligned with one or more standards for mathematical practice to unpack instructional readiness.

Results

Overall, less than 15% of pre-service teachers in the sample were familiar with reform-based standards for mathematical practice. Participants' epistemic and mathematics instruction beliefs seemed positively correlated; a categorical regression showed participants' MIBS overall scores and program of study as the strongest predictors of their DEBQ scores, $F(4, 102) = 10.187$, $p < .001$. Follow-up interviews showed subtle differences between pre-service elementary teachers and education minors, who leaned more toward traditional mathematical practices and believed mathematics to be about correct answers, and pre-service secondary teachers, who favored reform-based mathematical practices and believed mathematics entails a process of discovery.

Conclusion and Implications

Meaningful mathematics reform requires teachers to be familiar with the principles behind reform-based mathematical practices (Opfer et al., 2017; Schoenfeld, 2020). More needs to be done to ensure students entering the teaching profession are not only familiar with reform-based standards for mathematical practice but aware of their role in implementing them as intended.

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A PRELIMINARY INVESTIGATION INTO PROSPECTIVE TEACHERS' PRODUCTIVE STRUGGLES FOR MAKING SENSE OF MATHEMATICAL PRACTICES

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For prospective teachers to become teachers who can bring the mathematical practices (NGACBP & CCSSO, 2010) into their classrooms, first, they need to experience those practices themselves. We used the frog problem (see Dixon & Watkinson, 1998; Andrews, 2000 for details) as the first activity of a middle school methods course to provide a chance for the 13 prospective teachers enrolled to *make sense of problems and persevere in solving them* (MP1; NGACBP & CCSSO, 2010). After engaging in making sense of the problem across multiple class sessions, the prospective teachers completed an activity where they analyzed their experiences in relation to the mathematical practices (MPs).

Since productive struggle is an opportunity for students to make sense of mathematics within their *zone of proximal development* (Vygotsky, 1978) and deepen their mathematical ideas and the relationship among those ideas (NCTM 2014; Warshauer, 2015; Zeybek, 2016; Peterson & Viramontes, 2017), productive struggle related to the MPs may also support prospective teachers in making sense of MPs. Our observations of the prospective teachers' struggles during the frog activity led us to the following research question: *What struggles of prospective teachers when engaging with the frog problem support them in making sense of mathematical practices?*

Data collection included videos of all classroom sessions and electronic copies of all prospective teacher work completed in connection with the frog activity. We used Warshauer's (2015) kinds of student struggles for identifying instances of struggle. Then, we applied Kelemanik et al. 's'(2016) diagram of three avenues (*Quantities and Relationship*, *Structure*, and *Repetition*) leading to MP1 as a framework for identifying which avenues or specific MPs the prospective teachers struggled with. After that, we used *expectations for students* from Smith (2000, as cited in NCTM, 2014) as formative indicators to help us decide whether the identified struggles were productive. We used the prospective teachers' MP analysis document as a final indicator of whether the struggles identified as productive supported the prospective teachers in making sense of MPs.

The initial results show that the prospective teachers struggled with looking for a pattern to generate a rule that works for all cases (e.g., "I have a pattern but I don't know how to put it in the equation.") and with providing a meaning behind their expressions or equations in relation to the frogs' movements (e.g., "Even if we know the answer, we don't know why we need to move like that."). Both types of struggles generated opportunities for the prospective teachers to make sense of MPs, especially, MP2, MP4, MP7, and MP8. Those struggles seemed productive as they provided the prospective teachers with an object lesson in the meaning and purpose of the MPs. For instance, most of them mentioned the same key statement from CCSS about MP1 that related to their experience of engaging with the frog problem— "[Mathematically proficient students] consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution" (NGACBP & CCSSO, 2010, p. 6).

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LEARNING FROM PEDAGOGICAL MISTAKES IN TEACHER EDUCATION

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Learning from mistakes has a strong and deep foundation in theory and research. Although there have been recent calls to foreground the roles of mistakes in learning how to teach (Wieman & Hiebert, 2018), little has been written about what this might look like in teacher education. In this poster, we share what we are learning in response to this call.

The questions guiding our research are: (1) *What happens when mathematics teacher educators (MTEs) treat pedagogical mistakes as valuable sites for learning?* (2) *To what extent do pre-service teachers (PSTs) self-report pedagogical mistakes as sites for learning?* and (3) *In what ways do PSTs discuss pedagogical mistakes as sites for learning?* We explore these questions in the context of a two-semester secondary mathematics methods course sequence. As part of this course, PSTs attended a weekly practicum with a partner in a local secondary math classroom and engaged in at least one iterative cycle of collaborative planning, rehearsal, revision, enactment, and reflection loosely modeled on structures and protocols from Lesson Study (e.g., Lewis, 2016, Lampert, et al., 2013, Wieman, 2019). Rehearsals were purposefully designed to position pedagogical mistakes as normal, invaluable opportunities for professional learning and growth, rather than as evidence of professional inadequacy.

Our data set comprised 2 cohorts of PSTs' (C1, N=22; C2, N=12) end-of semester responses to these questions: (1) *What is the most significant way in which you have contributed to your own learning this semester?* (2) *What is the most significant way in which you have contributed to your classmates' learning this semester?* (3) *What is the most significant way that you have changed as a result of this class?* Taking an inductive content analysis approach (Schreier, 2012), we worked within the *Dedoose* data analysis platform. We engaged in initial open coding of C1's data to create our codebook, and then coded data independently. We discussed and came to agreement for every instance when text was coded differently. Using iterative cycles, we continued to code and analyze data from C2.

Our emergent findings indicate that these PSTs experienced learning from mistakes as powerful, valuable, and normal. They saw pedagogical mistakes as opportunities to learn about the effectiveness of specific moves, to make improvements to lesson plans, and to develop pedagogical knowledge and skill. PSTs also saw mistakes as an important part of their own developing identity as teachers. Pedagogical mistakes helped them see themselves and their peers as valuable resources in a professional community dedicated to learning and improvement, and PSTs also described mistakes as connected to their own developing confidence. We hope that this preliminary work will help the field become more thoughtful and purposeful in positioning pedagogical mistakes as normal and valuable opportunities to learn.

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"EDUCATION FISH IN A WORLD FULL OF SHARKS": PRESERVICE TEACHERS' EXPERIENCES IN MATHEMATICS CONTENT COURSES

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University-level mathematics content courses often serve as the primary context in which secondary mathematics preservice teachers (PSTs) develop mathematical content knowledge which is crucial for effective instruction in mathematics (Ball et al., 2008). These courses allow PSTs to explore and make connections across a breadth of mathematics domains, and foster self-confidence in their own mathematical abilities (Hodges et al., 2010). Mathematics content courses are typically taught by mathematicians who do not have a formal background in education (Oleson & Hora, 2014), and are often markedly different in their delivery compared to traditional teacher education courses taught in faculties of education (Leikin et al., 2018). Given the limited research investigating the experiences of PSTs engaging in these courses during their teacher education programs, this study explored the question: What are the experiences of secondary mathematics PSTs taking university-level mathematics content courses?

The data for this study were collected from a series of semi-structured interviews with one cohort of secondary mathematics PSTs' (*n*=7) enrolled in a four-year undergraduate teacher education program at a Canadian university. Interviews were transcribed verbatim and coded for emerging themes using the constant comparative method (Maykut & Morehouse, 2002).

Findings suggest that secondary mathematics PSTs experienced a range of challenges, including difficulties connecting with and understanding course content, and being ignored and dismissed by mathematics professors. Importantly, challenges with mathematical learning did not simply result in roadblocks to completing a course, but also became significant barriers for PSTs trying to complete their teacher education program altogether. When PSTs spoke about specific experiences with mathematics professors, most of their recollections were negative. PSTs shared that when mathematics professors knew they were education students, they were viewed and treated differently. Specifically, PSTs indicated that the mathematics professors spoke down to them and did not think that they were capable of doing advanced mathematics. PSTs also critiqued the pedagogical choices of their mathematics professors.

Yet, the secondary mathematics PSTs leveraged their negative experiences into opportunities to be reflective practitioners. First, after seeing what they felt was evidence of poor teaching, PSTs reflected that they learned what they "don't want to do…as a teacher" and considered the ways that they would be different as future teachers (Zazkis & Leikin, 2010). Additionally, the PSTs' first-time encounters with conceptual difficulties in mathematics (Goulding et al., 2003) helped them develop empathy for the way that some of their future students might feel in secondary mathematics. The PSTs also developed a community by leaning on each other for both mathematical and emotional support throughout their time in the content courses. This *de facto* community helped them successfully complete their mathematics content courses while alleviating feelings of isolation (Grossman, et al., 2001). While the PSTs in this study were able to find some positive features of largely negative experiences, it is worth considering what adjustments need to be made to provide all PSTs with the positive learning environment that we hope they will provide to their own students in the future.

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SECONDARY STEM PRESERVICE TEACHERS' CONTINUOUS IMPROVEMENT: FOCUSING ON STUDENT ENGAGEMENT DURING A PANDEMIC

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Supporting preservice teachers (PSTs) to practice, in the field, what they have learned in their teacher preparation courses is an ongoing challenge for STEM education (e.g., Roehrig & Luft, 2006; Tatto & Senk, 2011). Some of these challenges are influenced by the structure of teacher preparation programs. Thus, education policy has suggested that programs adopt "holistic practice-based" methods to support PSTs (Hollins, 2011). One such promising method is the use of continuous improvement. Preservice teachers who completed elementary mathematics content courses developed through continuous improvement applied what they learned, when planning lessons, during their first years in the field (Morris & Hiebert, 2017). Thus, applying similar improvement science methods within a teaching and learning course may support PSTs to learn from teaching (see Hiebert & Morris, 2012).

To support this type of learning, continuous improvement methods can be implemented within teaching methods courses (e.g., Bryk et al. 2015; Lewis, 2015). Continuous improvement cycles involve selecting a goal, planning and implementing a strategy to achieve that goal, and improving the strategy through reflection. The cycle is repeated as needed. When seeking to improve practice, small changes over time can influence greater outcomes than one-time attempts to overhaul a system or process. However, few studies discuss improvement science in the context of teacher preparation. Hence, this study focused on how to support PSTs to utilize improvement science through lab assignments implemented in field placements concurrent with methods coursework. The results from this study answer: In what ways do lab assignments support teacher candidates to engage in continuous improvement during their field experiences?

The participants in this study were ten PSTs enrolled in a secondary STEM teaching methods course while completing their teaching field experience. In alignment with continuous improvement cycles, the students planned an activity, implemented the activity, and reflected on their practice. The data consisted of completed lab assignments. This study focused specifically on an assignment where the PSTs worked to improve student engagement in their virtual classes while teaching during a pandemic. The data were analyzed using constant comparative analysis (see Strauss, 1987) to discover themes and trends within the completed lab assignments.

Results showed that through the completion of the student engagement lab assignment, the PSTs reflected on their teaching and learned how to improve the design and implementation of instructional tools. For example, the teachers saw an improvement in student engagement by using online tools such as collaborative presentations slides. They also acknowledged that classroom environment influenced student participation. Lastly, the PSTs reflected on how to improve their implementation of these tools. These results are important as they show that the use of lab assignments can support PSTs to learn from their teaching and show continuous improvement in their instruction.

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I thank my students who survived the pandemic with me. Excellent work on your assignments!

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FENDING FOR THEMSELVES: SECONDARY MATHEMATICS PRE-SERVICE TEACHERS' EXPERIENCES AS OUTSIDERS IN THEIR PROGRAM

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For preservice teachers (PSTs), the majority of their teacher education (TE) program is comprised of university-based coursework. In our context, secondary mathematics PSTs engage in general education courses centred on subject-non-specific topics (e.g., assessment, technology, etc.), mathematics content courses to support their mathematics content knowledge (Ball et al., 2008), and mathematics methods courses for teaching secondary mathematics (Albayrak & Unal, 2011). While the content of these courses is important, a successful teacher education program is not only reliant on PSTs' academic prowess, but it is also dependent on PSTs' subjective experiences. Indeed, post-secondary students' experiences in their programs of study impact their academic success (Crisp et al., 2015; van Rhijn et al., 2016), feelings of self-efficacy, and development of future goals (Carpi et al., 2017; Pearlman-Avnion & Aloni, 2016). With this in mind, the overarching research question guiding this paper is: What are the experiences of secondary mathematics PSTs in the university-based coursework of their TE program?

The study took place within the context of a four-year, undergraduate-level TE program at a large Canadian university. Participants in this study were the secondary mathematics PSTs in one graduating class of the TE program (*n*=7). General education and mathematics methods courses were offered by the Faculty of Education, and mathematics content courses were offered by the Department of Mathematics and Statistics. Participants were interviewed throughout their program and all interviews were transcribed verbatim and coded for emerging themes using the constant comparative method (Maykut & Morehouse, 2002).

Findings suggest that secondary mathematics PSTs broadly felt like outsiders in their TE program. In general education courses, they were outside of the typical PST experience as *mathematics* PSTs, and in mathematics content courses, they were outside of the typical mathematics- or STEM-major experience as mathematics *education* students. Consequently, PSTs banded together and developed a community of secondary mathematics PSTs. This community was formalized in the secondary mathematics methods courses where the PSTs finally felt like insiders. We argue that this PST-developed community is akin to a cohort, because it was "a group of students who beg[an] and complete[d] a program of studies together, engaging in a common set of courses, activities, and/or learning experiences" (Barnett & Muse, 1993, p.401). In this cohort, the secondary mathematics PSTs supported one another in many ways (Bullough et al., 2001), developed a strong sense of community, and increased professional growth (Beck & Kosnik, 2001; Govender & Dhunpath, 2011; Mandzuk et al, 2005). Because of this self-developed cohort, the PSTs were empowered where they once felt like outsiders in their program.

We hope that these insights propel TE programs to think critically about the ways that they do and do not support all of their students. We encourage teacher educators to consider ways of making courses more inclusive and TE programs, more broadly, to facilitate the formal creation of spaces earlier in programs for students who are not well-represented in the program.

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AUTHENTICITY IN ELEMENTARY PRESERVICE TEACHERS' MATHEMATICAL TASK DESIGN

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This poster session described the results of 30 elementary preservice teachers' (PSTs) task design in a mathematics methods course. The context and task authenticity of the mathematical task were examined. When the task could be set up in a decontextualized, simulation, or real-life setting, more than half of the participants used a simulation (54%) inspired by the contextual features of a storybook in their task design. In terms of task authenticity, PSTs were inclined to include pure computation problems (47%) and modeling/manipulative activities (40%) in their task design. The results contribute to the literature on the quality of task and teaching designs in mathematics education (Watson & Ohtani, 2015).

The conceptual framework involved context authenticity, including an academic setting (e.g., classroom or laboratory), a simulation setting (e.g., a learning environment that either resembles real-world complexity or is established on a virtual platform), and a real-life setting (e.g., field experience or workplace). Another dimension is task authenticity, which consisted of computational, modeling, and realistic categories (Herrington et al., 2009; Strobel et al., 2013) while describing the type of tasks.

The research questions that guided this study include:

1. How do the mathematical tasks designed by elementary PSTs demonstrate context and task authenticity?
2. What are the characteristics and relationship between the context and task authenticity of the designed mathematical tasks?

In this study, PSTs were exposed to children's mathematical literature in the mathematics methods course and then required to design a task that could be employed to facilitate particular mathematical concepts. The collected 30 mathematical tasks were analyzed using a constant comparative method based on the framework addressing context and task authenticities. Axial coding was applied to describe the similarities and differences among the tasks, as well as to relate the identified concepts across various categories (Corbin & Strauss, 2008).

The results showed that 40% of PSTs' tasks focused on memorization and procedure, highlighting the popularity of computation problems when they were not likely to be set in a real-life setting. Simulations also seemed favored as a task setting after PSTs were exposed to various story contexts. Among the tasks set in a simulation and real-life setting, half of the tasks required students to demonstrate their conceptual understanding of the procedure. It is encouraging to observe that relatively more tasks required students to express their understanding through multiple mathematical representations and were situated in a simulation/real-life context. This research sheds light on the correlation between context and task authenticity in mathematical task design, which has pedagogical implications for the coordination of the task type and the assignment of its scenario, as well as how these components interact when designing mathematical tasks at the elementary level.

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PRODUCTIVE STRUGGLE IN LEARNING MATHEMATICS: PRESERVICE ELEMENTARY AND MIDDLE SCHOOL TEACHERS' 'ESSON PLANS

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Participants of this study were 80 preservice teachers (PTs) enrolled in an elementary and middle grades math methods course at a Mid-Western university teacher education program in the U.S. The PTs engaged in a series of activities regarding the value and importance of productive struggle in learning and learned and practiced various strategies that support the productive struggle. The strategies included selecting tasks that are conducive to productive struggle and anticipating student thinking (Smith et al., 2018); utilizing multiple tools, representations, and strategies (NCTM, 2014); giving time to struggle (Warshauer, 2015); asking questions (Freeburn & Arbaugh, 2017); scaffolding for access to productive struggle (Barlow et al, 2018; Huinker & Bill, 2018); using mistakes, errors, and confusion as learning portals (Boaler, 2015; Carter, 2008; Catmull, 2014); and mindfully facilitating discussions around activities and giving feedback (Boston et al., 2017; Huinker & Bill, 2017). At the end of the course, they produced a fully documented lesson plan that employs eight effective teaching practices, including productive struggle. Through a literature search, we developed a coding rubric composed of a comprehensive collection of strategies that indicates supporting the productive struggle. Using thematic coding (Gibbs, 2007) and the rubric, we analyzed PTs' lesson plans to determine how and to what extent they supported productive struggle in their lesson plans.

Our results indicated that both scaffolding and high cognitive demand tasks were paid little attention by the PTs. The low frequency of use of scaffolding techniques can be explained by the low frequency of planning for high cognitive demand tasks. NCTM (2014) draws attention to the importance of anticipating solutions in lesson plans to support productive struggle. However, the PTs rarely anticipated solutions in their plans which echoes Kartal et al. (2020)'s findings regarding the deficiencies of PTs in anticipating student thinking. PTs did not plan to emphasize that making mistakes is a natural part of learning (NCTM, 2014), even though publicly valuing mistakes (Boaler, 2015), and facilitating discussions on mistakes, misconceptions, and errors have been considered as important components of productive struggle. Anticipating mistakes and incorrect solutions to mathematical tasks is emphasized as a focal planning practice in the context of effective math teaching and in relation to facilitating discussions (Huinker & Bill, 2017; NCTM, 2014). Therefore, absence of discussion plans for mistakes and struggles can be explained by absence of anticipated solutions and difficulties from the lesson plans.

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KNOWING WHAT, WHEN, HOW AND WHY: PROSPECTIVE MATHEMATICS TEACHERS TALKING ABOUT MATHEMATICAL REASONING

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Keywords: Reasoning and Proof, Prospective teacher education, High school education, Mathematical Knowledge for Teaching

Mathematical reasoning [MR] has been at the core of school mathematics in several countries for more than 30 years. However, the concept of MR remains vague both in research and in practice. It is associated with a plethora of terms such as thought, deduction, generalization, justification, and proof (Jeannotte, 2015). Very little is known about the different ways in which teachers use this concept. Moreover, within their daily practices, teachers have to create, choose, or even discriminate means of developing and evaluating MR of their students. Davidson et al. (2019) point to the complexity underlying the professional judgment teachers place on student MR. In addition, the discourse of teachers about MR contingents several pedagogical choices they make (Hill et al., 2005; Stylianides and Ball, 2008). This poster aims at exploring how prospective mathematics teachers [PMT] conceptualize MR.

This project is based on a commognitive perspective which is cultural and discursive. From this perspective, cognition and communication are two aspects of the same ontology, i.e., discourse. Discourses are constituted of keywords, visual mediators, rules of discourse, routines, and generally endorsed utterances (Sfard, 2008). They are specific to a particular community. Thereby, PMTs are viewed as a particular community that shares a certain discourse. Three PMTs each participated in one 60 minutes individual interview and one 120 minutes collective interview. All interviews were video recorded. They were finishing their last semester of a four-year Bachelor's degree in mathematics education for secondary teachers. During those four years, they did four teaching internships and some substitute teaching.

By analyzing their discourse, we were able to highlight what MR is for them. First, the participants generally talked strongly about what MR is not. For them, it is clear that MR is not applying techniques per se, rather it is applying in a controlled manner. They endorsed that MR is knowing what concepts or techniques to apply, when it is relevant to apply them, how they are applied, and why it is valid to apply them. This vision of MR can be linked to the concept of control developed by Saboya (2010). Second, exemplifying, as a particular MR, serves a particular purpose and is very close to, but contrasted with trial and error: *She thought, there must be something, I'm going to try some cases… It's not even trial and error. It's more like giving yourself examples.* Compared to primary school teachers' discourse (Jeannotte, Dufour, & Sampson, 2020), PMTs refer less to the aspect of being able to explain (to someone) what, how, and why. The difference, between explaining (primary school teachers) and knowing (PMTs), might reside in the different experiences of those two communities. Indeed, explaining can be related to the traces needed to evaluate students' MR. And, evaluation occupies a major place in teachers' everyday practice but it is an aspect that PMTs have not yet fully experienced.

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IDENTITY JOURNEY MAPS: A GEOSPATIAL ROAD TRIP OF MATH CONTENT IDENTITY

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Keywords: Teacher Educators, Preservice Teacher Education, Technology

Introduction

The goal of the Identity Journey Map project (IJM) is to explore the impact of student teachers creating IJMs is to allow for the opportunity to reflect and set the stage for all the (possibly different) ways student teachers will experience their mathematics content identity in the classroom. Through geospatial story-telling software (StoryMaps - https://storymaps.arcgis.com/en/), students will communicate their journey and experiences to this point that have brought them to be connected math educators in the hopes of exploring ways to connect their student's content identities in math.

As a teacher educator, many math classrooms I observe maintain to incorporate a one-size-fits-all approach to supporting students through their learning. There is no wonder why many students are turned off by math and begin to believe, "I'm just not a math person." They lack an identity or connection to the content that many of our teachers possess - – love for mathematics. We are referring to this as their *mathematical content identity*. This lack of identity can affect performance in classrooms and lead to a sinking feeling that they do not belong in that math class (Huvard, H., et al., 2020). A growing body of research has revealed the diversity of student strengths and challenges and a wide variety of pathways there are to effective learning (Vincent-Ruz, P., & Schunn, C. D., 2018). How can educators transform their approach and teach math in a way that effectively addresses learner variability? Productive struggle focuses on the math journey instead of simply the destination. It recognizes that there are multiple ways to tackle a problem and views mistakes as opportunities for learning rather than failures. It is this journey that is the focus of this proposal. We are proposing to explore Identity Journey Maps (IJMs). In this project we have students build web-maps using the geospatial technology StoryMaps to explore their own mathematical identities and consider ways to use these journals as entry points to engage students in believing they too have an identity as a mathematician.

Identities are developed and shaped by different experiences influencing how we perceive and relate to the world around us. Identity theory states that our identities are filled with meaning based on how we perceive our roles as an individual and within the larger cultural groups we are part of within society (Burke & Stets, 2009). People hold multiple identities, and each identity is attributed a specific set of meanings. This paper extends this line of inquiry to investigate how participation in the creation of these geospatial stories impacts student teacher's mathematical identities and in turn their perceptions of their student's identities as mathematicians.

In the Summer of 2019 we started to use IJMs in an introductory Planning for Learning in STEM Education course (for Math and Science student teachers) as a way to explore the contexts of the student teacher's that led them to their path as a math educator. They created a narrative using StoryMaps that explained both their connections to content, important opportunities, people and places that led them to their appreciation of mathematics as well as how they believed that this could bring their student's lived experiences to math as well. This become even more important throughout the following 2 years as we explored how content identity connected to student's connections to math.

The aim of this study is to explore how the IJMs impacted their perceptions of their content identity and how this connects to their student's identities in math. Using case study methodology, we will study graduate-level courses at our institution to provide an in-depth examination of students' sense of content identity. We use identity theory to explore the nature of content identity development in preservice teachers who engaged in creating identity journey maps at a university in the western USA. Constructs of content identity development are seen as critical outcomes of experiential math inreach and outreach programs.

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CONNECTING PROSPECTIVE TEACHERS' MATHEMATICS UNDERSTANDING

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Theoretical Perspective

This study is framed by the Pirie-Kieren (P-K) theory for the Dynamical Growth of Mathematical Understanding (Pirie & Kieren, 1994). P-K theory characterizes learning by non-linear movements through eight *embedded* layers of understanding. *Embedding* layers of understanding means formal layers of understanding (e.g., formulas/rules) are connected to informal layers of understanding (e.g., pictorial images). *Disjointed* understanding is used to describe how understanding can be fragmented (e.g., algorithm is separated from concept).

Research Question and Design

This case study investigates how prospective teachers (PTs) used their existing understanding to connect elementary mathematical concepts. The setting for the study was a ten-day professional development course and 15 PTs agreed to participate. The course was designed for PTs to elicit their existing understanding by working on elementary mathematics tasks, generating multiple solution strategies and making connections between the solution strategies. This poster will share Deb's (pseudonym) work on the subtraction question: 91 -70.

Data Collection and Analysis

Video recordings and participants' journals provided both real-time and daily-summary data for the study. I used this data to identify and analyze PTs existing images, disjointed understandings, and how they noticed properties and made connections.

Summary of Findings and Implications

To make subtraction easier, Deb used a friendly number strategy she knew worked in addition. Deb recorded: 91 – 79 = 90 – 80 (see Figure 1a). Prompted to use a number line, Deb constructed an image (see Figure 1b) and noticed the difference of 10 between 80 and 90, the +1 on each end, and the total difference of 12. In her journal (see Figure 1c), Deb drew another number line using different numbers, 21 - –4 = 20 – 13 and said how shifting maintained "the difference between the two numbers," whereas her original image "had changed the difference." She referred to this as an "aha moment" that "explained the idea of constant difference."

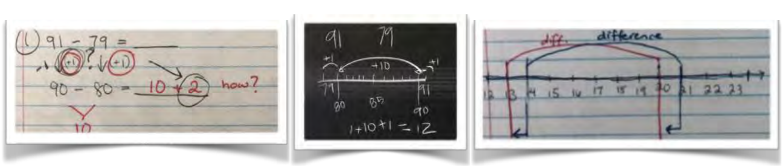

Figure 1a: 91-79=90 - –0, Figure 1b: Number line, Figure 1c: Constant difference

Prior to working with her friendlier number strategy, Deb's formal understanding was disjointed from an image of difference for subtraction. By translating her use of the number line into a new equation and seeing the numerical relationships, Deb's Formalising layer of understanding for subtraction now embeds an image of constant difference. Deb's work suggests a process for how prospective teachers can make connections within their existing understanding of elementary mathematics.

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PROSPECTIVE ELEMENTARY TEACHERS' EVALUATIONS OF STUDENT SOLUTIONS TO DIVISION STORY PROBLEMS

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Keywords: Preservice Teacher Education, Teacher Beliefs, Teacher Noticing

The AMTE (2017) *Standards for Preparing Teachers of Mathematics* state, "Well-prepared beginning teachers of mathematics analyze both written and oral mathematical productions related to key mathematical ideas and look for and identify sensible mathematical reasoning, even when that reasoning may be atypical or different from their own" (p. 10). Understanding and evaluating students' written solutions (e.g., on formative assessments to determine what students understand (Joyner & Muri, 2011)) is a critical, everyday skill for the job of teaching.

Various factors can affect teachers' evaluations of student work. For example, teachers tend to rate solutions that match their own preferred methods for solving a problem as better than other correct solutions (Van Dooren et al., 2002). Bartell et al. (2013) noted that prospective teachers' (PTs') evaluations of student understanding were swayed positively by features that superficially seemed to be conceptual (e.g., pictorial representations), even when the solution did not reflect clear evidence of conceptual understanding. Shaughnessy and Boerst (2018) suggest that beginning PTs will often dismiss student work and reasoning when the numerical answer is incorrect. Conversely, Jansen and Spitzer (2009) found that PTs often base their assessments of student understanding solely on whether the answers students obtained were correct. Teacher education must help PTs develop more appropriate bases upon which to evaluate student work.

The data I report came from a pilot test of an interview protocol for a study in progress. Participants were 24 prospective elementary teachers enrolled in their first mathematics content course. Each participant was shown one of four division story problems involving decimals (n=6 for each problem). After seeing the problem, participants were shown seven (simulated) student solutions to the problem, one at a time. Participants were prompted by the interviewer to first reconstruct each student's solution (Philipp, 2018), priming them to attend to the mathematical details in the solution and potentially interpret what the student understood (Jacobs et al., 2010) rather than make evaluative snap judgements. After seeing and reconstructing all seven solutions, participants were asked to rank the solutions in any way they wanted and to select which solutions they would want to see their future students produce. I analyzed and coded the responses qualitatively to answer the research question: *What criteria do prospective elementary teachers say they use when evaluating student solutions to a division story problem?*

When asked which solutions PTs would want to see their future students produce, some PTs cited factors mentioned above (e.g., obtained the correct answer, used a method the PT would have taught) as criteria. However, several PTs preferred solutions in which the student's process was clearly communicated, even if it was incorrect. A similar criterion cited was whether the PT felt the solution served as a productive base from which to build student understanding. These two criteria align with a belief that mathematics instruction should center student thinking (NCTM, 2014). On the ranking task, PTs often used multiple criteria and considered factors on a spectrum. Some PTs employed more objective criteria (e.g., validity of the solution strategy, use of pictures) whereas others cited subjectively relative criteria (e.g., how easily the PT understood the solution). Future research will examine how these criteria evolve during teacher preparation.

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ADVANCING PRESERVICE MATHEMATICS TEACHERS' CULTURAL AWARENESS THROUGH AN EMBEDDED METHODS COURSE EXPERIENCE

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Mathematics teacher education is in a time of necessary focus on issues of equity and access. This goal is urgent, in part because while populations in the U.S. have grown more diverse, the demographics of teachers have remained mostly the same, with the vast majority being White, monolingual in English, middle-class, and culturally different from many of their students (Marx & Moss, 2011). Thus, teachers' responses toward cultural difference must be a focus of mathematics teacher preparation to meet the needs of the nation's diverse student population.

The Developmental Model of Intercultural Sensitivity (DMIS; Bennett, 1986) is a framework that establishes a continuum of ways to respond to cultural difference ranging from monocultural to intercultural. Monocultural is "the experience of one's own culture as 'central to reality'" (Bennet, 2004, p. 62). In contrast, intercultural is "the experience of one's own beliefs and behaviors as just one organization of reality among many viable possibilities" (Bennett, 2004, p. 62). The framework includes two categories related to monocultural (Denial and Polarization), two categories within intercultural (Acceptance and Adaptation), and a category (Minimization) which is thought of as a transition between monocultural and intercultural. The Intercultural Development Inventory (IDI; Hammer & Bennett, 1998) is a validated instrument that places respondents along the DMIS continuum.

During the spring 2021 semester, a university in the Midwestern United States began offering a program that embedded preservice teachers (PTs) methods classes at a local elementary school. PTs in this program spent every morning throughout the spring semester at this school, participating in their university classes and engaging with students in classrooms. This allowed the opportunity for PTs to immerse themselves in a school where about 80% of the students in this school identify as Black or Hispanic and 68% are eligible for free and reduced lunch (Public School Review, 2018). These are different demographics from PTs participating in this program, many of which identify as white, middle-class, and from small, rural towns in the Midwest.

Twenty-three PTs participated in this program during the spring 2021 semester. Throughout the semester, we explored how participation in this program influenced PTs' sense of cultural awareness and their ability to work across cultures. Data collection included pre- and post-IDI results and journal reflections. At the beginning of the program, 9 students fell in monocultural categories of the DMIS, 2 fell in intercultural categories, and 12 fell in the transitional category of minimization. Throughout the semester, PTs participated in discussions and activities related to culture (e.g. culturally relevant math pedagogy). By the end of the program, many had advanced along the DMIS continuum, becoming more aware of culture and the importance of including their students' cultures in their classrooms and curriculum. One student wrote: "*While my culture (norms, beliefs, practices) is important and makes me who I am, I need to dig deeper into cultures around me, especially my future students*." We argue that this program allowed PTs to develop their sense of cultural awareness while building the skills to work with a diverse group of students. The implications of this study will support other teacher educators in developing their PTs' cultural awareness in embedded methods courses.

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LEARNING TO CODE FOR MATHEMATICS TEACHING: THE CASE OF MARIA

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Many schooling jurisdictions have begun to include coding and computational thinking in K-12 curricula. Our own school jurisdiction of Ontario, Canada has just revised our mathematics curriculum for grades 1 through 8 (2020) to include coding expectations connected to algebraic reasoning in every grade. Notably, much of our elementary teaching workforce are women. Yet, since the 1990's in most countries women have been dissociating from actively engaging in many forms of ICT including computational reasoning (Patitsas et al., 2014; West et al., 2019). This poses a new problem for our K-12 system especially, like in Ontario, when computational reasoning curricular expectations are intertwined with mathematics expectations. We need to understand how to support K-12 educators, and especially women, with learning to code for teaching mathematics.

While there is scant research on the support of women learning to code in order to teach coding, we do know from studies in ICT and women that fostering a sense of belonging (e.g., Esquinca et al., 2021) and establishing an identity (e.g., Ulriksen et al., 2010) as a person who codes are important to the persistence of women in coding. Numerous studies have shown that feeling a sense of community (e.g., Goos, 2004) is vital for the success of women and equity. Women associate perceived identity compatibility and perceived social support with a greater sense of belonging in STEM (Rosenthal et al., 2011).

In our poster we report on a case study with one secondary mathematics preservice teacher, Maria, a confident woman, and high achiever with an interest in, but no prior experience with, coding. Maria was in her third semester of a 16-month teacher education program that focused on pedagogies which made use of various educational technologies. Maria was introduced to Scratch ©, a block-based and user-friendly coding environment. A member of the research team met with Maria weekly to support her learning and progress. We analyse Maria's experiences through the lens of Hannula's (2002) framework for analysing attitude and changes in attitude.

Acknowledgements

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BLOGS AND LEARNING JOURNALS: MATHEMATICS TEACHER CANDIDATES' REFLECTIONS ON LEARNING TO TEACH

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This poster describes our shared exploration of teacher candidates' (TC) engagement in two kinds of *personal writing space* across two courses in our secondary mathematics education program. In our courses, *learning journals* are semi-private student-instructor writing spaces where TCs relate to the subject of mathematics, while *blog entries* prompt candidates to "go public" with their ideas about teaching. In support of our program's balance between reflection and enactment, we explore the potential of these two formats, extending Cohen's (2016) ideas on writing in mathematics by analyzing writing as a space for mathematics TC learning. We use writing to facilitate opportunities for TCs to develop critical, antiracist perspectives that center students' humanity (Gutiérrez's 2002; 2018; Love, 2019), and to respond to Felton-Koestler's (2020) call for mathematics teacher educators to attend to sociopolitical issues in TC learning.

We sought to understand how secondary mathematics TCs in our Noyce program (NSF, n.d.) used writing to learn about teaching toward equity and antiracism, asking: *When teacher candidates engage in writing as a reflective practice, how do they write about mathematics, equitable mathematics teaching, and/or teaching toward antiracist classrooms?* We selected three Noyce scholars, Rhianna, Valerie, and Yolanda, who took our two courses simultaneously in Fall 2020, and analyzed the ways they used these different writing spaces for their learning.

Journals enabled these TCs to capture ideas about the beauty of mathematics, express empathy with students, and consider parallels between their own needs and those of students. Yolanda wrote about "explor[ing] content through an intrinsic perspective," noting "In K-12, math was always about numbers and solutions." She valued "see[ing] mathematics as what it is – beautiful worlds that we can visit and engage with." Rhianna used her own experiences to express empathy with students. For instance, she used her feeling that topological equivalence was "foreign" to imagine that her students might find *fraction* equivalence similarly foreign. Finally, Valerie mused that a reflective journal could be as valuable for her students as it was for her, saying "I really like this format of thought collection" and that she wanted her students, too "to have something personal they made and look back on how they constructed knowledge…."

Blogs allowed TCs to reflect on emerging ideas through the lens of classroom practices. Valerie asked, "how do we develop students' disciplinary agency?" reflecting that "it starts with knowing… Know the strengths of your students. Know the ways they like to play and work. Then, put it all together." She returned to this idea in processing a teaching moment when a student made and revised a conjecture as he articulated his way of thinking. Rhianna described how videos, chats, and nearpod assignments gave her knowledge of her students and "the work, emotions, and explanations my students had about their own identity, what it meant to be smart, … and differences correlated to real world events mathematically and socio emotionally." Lastly, Yolanda aimed to use her understanding of students to help them connect to the power and beauty of math: "I want my students to recognize their power as mathematicians and participate in math that is personally and socially meaningful to them. It is because of this that I must also know where my students come from, their backgrounds, and how they are positioned in society."

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REIMAGINANDO EL APOYO A LOS FUTUROS MAESTROS UTILIZANDO UN MODELO DE ESTUDIO DE LECCIONES

RE-VISIONING SUPPORT TO PROSPECTIVE TEACHERS USING A LESSON STUDY MODEL

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Palabras clave: Maestros mentores, formación de maestros, estudio de lecciones, experiencias de campo

Esta investigación se contextualiza en una innovación que utiliza el estudio de lecciones (Lewis et al., 2019) para establecer conexiones entre los cursos de métodos y las experiencias clínicas. A pesar de que varias investigaciones han reconocido la importancia de las experiencias de campo en la formación de los candidatos a maestros de matemáticas (Bieda et al., 2015), son pocas las investigaciones que se focalizan en determinar el conocimiento de los mentores de las experiencias de campo. Utilizamos cuatro situaciones hipotéticas para explorar cómo los mentores utilizarían su conocimiento para lidiar con posibles problemas que se presentan en el estudio de lecciones. Los datos se analizaron utilizando las estructuras discursivas descritas por Horn (2010) y el modelo de Toulmin (1958).

El diseño utilizado fue el estudio de caso (Yin, 2003). Participaron tres mentores de candidatos a maestros de matemáticas a nivel secundario (grados 7 a 12). El proceso de recopilación de la información se llevó a cabo mediante un grupo focal en el que los participantes reaccionaron a cuatro situaciones hipotéticas de posibles actuaciones en el proceso de estudio de lecciones. Específicamente, preguntamos ¿Cómo los maestros mentores fomentarían la participación de todos los participantes del equipo? ¿Cómo llamarían la atención a un candidato a maestro que rompe el protocolo en el momento en que se está implementando la lección de investigación? ¿Cómo fomentarían discusiones fundamentadas en las observaciones en la fase de reflexión? ¿Cómo incorporarían las preocupaciones de los candidatos a maestros que desean cambiar completamente la lección de investigación?

Lo resultados demuestran que, para motivar la participación de los candidatos a maestros en las lecciones de estudio, los mentores crearían un ambiente adecuado, estimularían la comunicación por medio de preguntas y el involucramiento en la clase desde el inicio de la experiencia de campo, pero también dejarían espacios para la reflexión. En casos inesperados en los que los candidatos a maestros rompan los protocolos convenidos, tratarían de establecer un balance entre explorar las razones por las cuales no se respetan y permitir las actuaciones inesperadas. Establecerían de antemano los criterios para realizar las observaciones y en la etapa de reflexión no permitirían datos que no provengan de observaciones. Los cambios a la lección de investigación estarían limitados a cambios en la forma de enseñar ya que los contenidos matemáticos no son negociables.

Encontramos que los mentores de futuros maestros utilizan principalmente el conocimiento que han adquirido en su práctica para dar respuesta a las situaciones hipotéticas. Ese mismo conocimiento lo quieren transmitir a los candidatos a maestros que supervisan a través del ejemplo y de crear un ambiente participativo que permita la discusión y la reflexión. Se necesita más investigación para establecer los aspectos que permiten que estas comunidades de estudio de lecciones sean verdaderos espacios para crecimiento profesional cuando hay personas con diferentes experiencias y conocimientos previos.

Reconocimientos

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RE-VISIONING SUPPORT TO PROSPECTIVE TEACHERS USING A LESSON STUDY MODEL

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Keywords: Mentor Teacher, teacher education, Lesson Study, field experiences.

The context of this study is an innovation using Lesson Study (LS) (Lewis et al., 2019) to establish connections between methods courses and clinical experiences. Although several researchers have recognized the importance of field experiences in mathematics teacher education (Bieda et al., 2015) there are few that focuses on determining mentors´ knowledge. We use four hypothetical situations to explore how mentor teachers would use their knowledge to manage problems that could surface during Lesson Study. Data was analyzed using the discursive patterns described by Horn (2010) and the Toulmin´s (1958) model.

We used a case study design (Yin, 2003). Three mentors of high school math teacher candidates (grades 7-12) participated. We collected data through a focus group in which participants reacted to four hypothetical situations of possible actions in the lesson study process. Specifically, we asked: How would mentor teachers encourage the participation of all team members participating in LS? How would they draw attention to a teacher candidate who breaks the LS protocol when the research lesson is implemented? How would they encourage discussions based on observations during the reflection phase? How would they incorporate the teacher candidates' concerns who want to change the research lesson completely?

The results show that to motivate the participation of prospective teachers in LS, the mentors would create a suitable environment, foster communication through questions and class involvement from the beginning of the field experience, but also leave room for reflection. In unexpected cases where the prospective teachers break agreed LS protocols, they would try to find a balance between exploring the reasons why the protocols are not followed and allowing unexpected actions. They would establish in advance the criteria for making the observations and at the reflection stage they would not allow data that did not come from observations. Changes to the research lesson would be limited to changes in the way of teaching since the mathematical contents are non-negotiable.

We found that mentor teachers primarily use the knowledge that they have acquired in their practice to respond to the hypothetical situations. They want to share that knowledge to the prospective teachers whom they supervise through example and create a participatory environment that allows for discussion and reflection. More research is needed to establish the aspects that allow these communities of LS to be true spaces for professional growth when there are participants with different experiences and knowledge.

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UNDERSTANDING THE EVOLUTION OF A TEACHER'S PHILOSOPHY OF MATHEMATICS: PART I, HIS INITIAL CONCEPTIONS

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Teachers' beliefs and conceptions about the nature of mathematics affect their instructional practices (Thompson, 1992). Thus, considerable research has been done to understand the development of teachers' beliefs and conceptions using mostly qualitative methodologies (Cooney, 1985; Eggleton, 1995; Fernandes & Vale, 1994).

The goal of this research is to understand the evolution of a teacher's philosophy of mathematics and the impact of his philosophy on his classroom practice. In this poster, I will report on the teacher's conceptions about mathematics as he entered the teacher education program (thereafter called his initial conceptions) and the critical experiences that, from his own perspective, influenced those conceptions. As a main research method, I'm using the biographical approach, a particular method of narrative inquiry.

To better understand teachers' knowledge, thoughts, and conceptions, researchers have used narrative, biographical research, and self-study methodologies (e.g., Butt, Raymond, McCue, and Yamagishi, 1992; Elbaz, 1990; Fernandes 1995; Frost, 2010; (Hamilton et al., 2008; Kelchtermans, 1993, 1994; Knowles, 1992; Suazo-Flores et al. 2019).

The biographical approach involves the use of heuristic tools such as critical events and formative experiences. Damian Torres, the participant of the study, is writing a journal of reflections in which he reflects on his evolving philosophy of mathematics and provides answers to questions or discuss issues related to the nature of mathematics.

Damian Torres's initial conceptions about the nature of mathematics included a utilitarian view, a platonic view, a logical view, and a problem-solving view. He saw mathematics as a service subject because it is used to solve daily-life problems faced by ordinary people, and advanced practical problems faced by some people having a professional degree such as engineers, architects, etc. In addition, he conceived of mathematics as a tool to describe the physical world and that the goal of scientists was to discover the mathematical laws that govern nature and the universe. He also held a platonic view about the nature of mathematics. He thought that mathematics was a finished, immutable, and absolute body of knowledge. He said that mathematical formulas and theorems were true yesterday, are true today, and will be true tomorrow. The work of past mathematicians consisted of discovering the theorems and procedures. Damian also viewed mathematics as a logical, objective, and rational discipline. He stated that the solutions to mathematical problems, including the proofs of mathematical theorems, can always be checked because the solutions and the proofs are based on logical reasoning rather than on personal opinion or physical instruments subject to error. Finally, Damian Torres also conceived of mathematics as problem solving. He mentioned that mathematics provides problems to practice and the solutions to these problems involve applying appropriate formulas and theorems. In addition, he stated that mathematics helps develop our thinking and reasoning skills and that also provides the tools to solve problems in the real world. Damian Torres said that that some factors that shaped his initial conceptions include his experiences in solving daily-life problems and the problems and theories posed and described in mathematics and science textbooks.

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WHAT AN ANGLE MEASURE MEASURES FOR PROSPECTIVE TEACHERS

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Keywords: Geometry and Spatial Reasoning, Measurement, Angularity

Angle measure is a central topic in mathematics curricula at a variety of levels (Barabash, 2017). Yet, how individuals reason about angles and their measures has been the focus of fewer research studies in comparison to other geometric attributes like length and area (Smith & Barrett, 2017). Some researchers have argued that productive ways of reasoning about angularity can be engendered through principles of quantitative reasoning (e.g., Hardison & Lee, 2019; Hardison, 2020; Moore, 2013). More specifically, by working to foster a conception of angularity as a quantity—a measurable attribute of an object (Thompson, 2011). Attributes must be conceptualized in order to be quantified. Therefore, it is important to attend to the attribute(s) an individual holds in mind when considering an angle's measure. Openness and amount of rotation are two common characterizations of the angular attribute appearing in the literature, both of which can be conceived in relation to a circle. In this poster, I report on 19 prospective teachers' (PTs) conceptions of the attribute characterized by an angle's measure. At the time of the study, the participants were enrolled in an undergraduate geometry content course for middle and secondary PTs at a large public university in the United States. Data for the study included PTs responses to two prompts. In the first prompt, PTs provided a typed response to the question: "When you measure a line segment in inches, you're measuring its length; when you measure an angle in degrees, what are you measuring?" In the second prompt, PTs were provided with multiple copies of an angle model and asked to indicate (by drawing, shading, etc.) what exactly they would be measuring; multiple copies of the angle model were provided in the event that PTs wished to indicate more than one option for what they were measuring. Responses to the first prompt were coded based on the objects and attributes referenced. Responses to the second prompt were coded based on the characteristics of the drawings produced. For both prompts, responses could receive multiple codes.

PTs responses to the first prompt indicate relatively little consistency in the verbal characterization of the attribute that an angle's measure references. The most frequent responses included statements of uncertainty (21% of responses; e.g., "I don't know" or "I am not sure") or references to an angle as an object without nominalization of an attribute (21% of responses; e.g., "you are measuring the angle two lines make with each other"). Attributes mentioned included width (16%), opening (11%), rotation (11%), tilt (5%), distance (5%), area (5%), and position (5%). Other non-attribute responses included radius (5%) and radians (5%). Circles were mentioned in 16% of responses. For the second prompt, the 19 PTs produced a total of 44 drawings indicating what they thought they were measuring (an average of 2.3 drawings per PT). In these drawings, curved inscriptions suggesting circles were prominently featured with 48% of responses including a shaded circular sector and 39% of responses indicating a circular arc without a shaded region. Few responses indicated attention to linear attributes like side length (2%) or distance between the sides of the angle model (2%). The differences in PTs responses to these two tasks suggest that prospective teachers should be supported through opportunities to develop verbal characterizations of angle measure that are consistent with their mental imagery of angle measure. Further results and implications will be shared via the poster.

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TOPICS CHOSEN BY PRE-SERVICE ELEMENTRY TEACHERS ENGAGING IN SELF-DIRECTED LEARNING IN A GEOMETRY CLASS

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Keywords: Pre Service Teacher Education, Geometry and Spatial Reasoning, Measurement, Culturally Relevant Pedagogy

Self-Directed Learning (SDL) is a pedagogical approach described as "a process in which individuals take the initiative, with or without the help of others, in diagnosing their learning needs, formulating learning goals, identifying human and material resources for learning, choosing and implementing appropriate learning strategies, and evaluating learning outcomes" (Malcolm Shepherd Knowles, 1975, p. 18). Participating in SDL allows learners to draw from their own *mathematical funds of knowledge* (Velez-Ibanez & Greenberg, 2005) while also developing informal learning skills similar to those used by teachers throughout their careers (Kyndt et al., 2016; Wagner, 2011; McNally et. al, 2009).

This study explores themes among the types of self-directed learning projects completed by elementary education majors as part of a content course for pre-service teachers (PSTs). PSTs were told to learn something related to geometry and measurement and to then present a project as an "artifact" of that learning. Projects were assessed according to six criteria: clear communication of the mathematics, accuracy of the mathematics, a statement of why they chose the project, a statement of what they learned that they did not know before, inclusion of contextual information and references, and overall organization and neatness. Students were provided with a list of 100+ starter ideas but no structured steps. We assume, given the option to explore anything they like, that PSTs chose projects that felt relevant to them.

Data were collected at a large university outside of a major city in the Southeastern U.S. which attracts a diverse student population. The projects analyzed for this study were generated by 113 PSTs over the course of three semesters and seven course sections. Each PST completed a series of three mini projects each which could be related or unrelated to one another. PSTs were permitted to work with a partner.

The project set was reviewed for themes, and inductive codes were generated. Some themes occurred more than others, and some projects merited multiple codes. We will not report the frequency of each theme in order to not marginalize project types which were less common but equally important. Themes included:

- Companions – involved their child, pet, friends, or other relatives
- Art – analysis of existing art or generation of new mathematical art
- Work – unpacking or exploration of how concepts apply in their job
- Sports / Hobbies – technical components of soccer, dance, track, cheer, etc.
- Crafts / Baking / DIY projects – how mathematics plays a role in creating
- Scavenger Hunt – identifying concepts in everyday life
- Travel – Planning travel or observing concepts while away
- Wonderment – extending a topic via history, related topics, or a challenge problem
- Practice – exploring or practicing a concept from class to better understand it

While the specific applications of these themes varied from student to student, these general trends help identify areas of mathematics that are relevant to PST populations.

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INFUSING PROOF AND JUSTIFICATION, MATHEMATICAL MODELING AND TECHNOLOGY: THE CASE OF MATHEMATICAL SERIES

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Keywords: Reasoning and Proof, Modeling, Technology, Preservice Teacher Education

Study has shown that teachers believe proof is a subject that not all students could learn (Knuth 2002). However, this is not the case for mathematical modeling or technology. This study aims to focus on enriching preservice teachers' knowledge about proof and justification by infusing other mathematical practices. To do this, we intentionally integrate frameworks from different subjects to build a structure in which the goal of learning and teaching proof and justification could be achieved. Since there are different definitions and frameworks for each mathematical practice that is mentioned, this is indeed a very challenging task. The content that we chose for this purpose is arithmetic and geometric series, which helps the purpose of the study since it is connected to a very rich context.

While it seems clear why mathematics is essential in any society, it is not clear how educators should prepare students to use mathematics when using it to make sense of everyday life's challenges in solving real-life issues (Abassian et al., 2020). Mathematical modeling represents how students use mathematics and apply it to solve those real-world and everyday life problems. The definition that we chose for this study is the models and modeling perspective (MMP)/contextual modeling. According to Lesh & Doerr (2003), MMP extends the goal of problem-solving to develop the applications of mathematics in real-world situations in order to make a deeper understanding of mathematical concepts.

Although there are different definitions for mathematical proofs, researchers consistently indicate that proof and proving have a central role in mathematical practice (Zazkis, Weber & Mejia-Ramos, 2014). Mathematics educators also agree that in K-12 school mathematics, curriculum proof needs to have a central place (Stylianides & Stylianides, 2006) because, in mathematics itself, the proof has a central role, and K-12 mathematics should maintain that structure for students. Since the focus of tasks in this study is on mathematical sequences and series, we chose to work with Stylianides' (2008) analytical framework of reasoning and proving. The mentioned framework has three components: mathematical, psychological, and pedagogical, which suits the purpose of our study and working with preservice teachers.

Common Core State Standards for Mathematics (CCSSM, 2010) states that technology is a powerful tool to help students understand and solve real-world problems and situations that are modeled mathematically. The role of technology is emphasized in Principles and Standards for School Mathematics (NCTM 2000). The technology principle says, "Technology is essential in teaching and learning mathematics; it influences the mathematics taught and enhances students' learning." In this study, we use Mishra Koehler's (2006) framework that was introduced for research in preparing teachers to integrate technology in classrooms.

Using the different frameworks, this study will lead to a chain of tasks in investigating mathematics sequences and series. Those designed tasks will help preservice teachers by providing them opportunities to explore how proofs are constructed through a productive challenge.

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Chapter 11:

Student Learning

THE EVOLUTION FROM LINEAR TO EXPONENTIAL MODELS WHEN SOLVING A MODEL DEVELOPMENT SEQUENCE

EVOLUCIÓN DE MODELOS LINEALES A EXPONENCIALES AL RESOLVER UNA SECUENCIA DE DESARROLLO DE MODELOS

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This article describes the results of an investigation based on a Models and Modeling Perspective [MMP]. We present the evolution of the models built by university students when solving a model development sequence designed to promote their learning of the exponential function. As a result, we observed that students' thinking was modified, expanded, and refined, as they developed different iterations of their models. Students' models evolved by creating, first, linear models that required direction; second, models where there was no dissociation between linear and exponential behavior; then, situated exponential models; and finally, sharable, and reusable exponential models.

Keywords: Modeling, College-level mathematics, Exponential function

In the research literature (Ärlebäck, Doerr & O'Neil, 2013; Ärlebäck & Doerr, 2018) it is mentioned that high school and undergraduate students have difficulties with the learning of the exponential function because it is a mathematical object whose learning requires a high cognitive transfer capacity. In addition, its understanding implies also understanding other concepts. For example, learning the exponential function requires students to develop covariational reasoning (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Thompson & Carlson, 2017). Research by Ärlebäck and Doerr (2018) and Ärlebäck, Doerr and O'Neil (2013) shows the importance of designing Model Eliciting Activities [MEAs] for students to develop knowledge related to the exponential function. Technology can support the learning of the exponential function because it allows the use of different representations and the connection between them to interpret, describe and predict phenomena, in addition to simplifying calculations. Due to its dynamic nature, technology can also support students to delve into the concepts such as variation (Stillman, Blum & Kaiser, 2013) and therefore to develop their covariational reasoning.

The problem addressed in this research was to better understand how a model development sequence can contribute to the expansion and refinement of the knowledge of exponential function developed by undergraduate students in business administration and accounting [LAEC]. The research question posed was: How did LAEC students' models and covariational reasoning - related to the exponential function- evolve when solving a model development sequence based on real-life problems, with the support of technology?

Conceptual framework

The conceptual framework of this research was based on the MMP proposed by Lesh and Doerr (2003) and the covariational reasoning framework proposed by Carlson et al. (2002).

Models and Modeling Perspective

According to the MMP, learning mathematics is a process of developing conceptual systems, which are continually modified, extended and refined based on the student's interactions with their environment (e.g., teachers and peers) and by solving problems (Lesh, 2010). Solving problems implies “differentiating, integrating, reorganizing, adapting or extending interpretation systems that are in intermediate stages of development” (Lesh, 2010, p. 27). Cycles of Modeling are interpretations that students exhibit when solving MEAs, in which the ways of thinking are repeatedly expressed, tested, and revised (Lesh, 2010; Sevinc & Lesh, 2018). From the MMP, models are defined as:

> Conceptual systems (consisting of elements, relations, operations, and rules governing interactions) that are expressed using external notation systems, and that are used to construct, describe, or explain the behaviors of other system(s)—perhaps so that the other system can be manipulated or predicted intelligently.
>
> A mathematical model focuses on the structural characteristics (rather than, for example, physical or musical characteristics) of the relevant systems. (Lesh and Doerr, 2003, p. 10)

In this way, the MMP proposes structuring experiences for the student, in which they express, test and refine their ways of thinking during the process of building a mathematical model to solve a situation that is presented to them. These situations, intentionally designed for students to generate models using specific mathematical ideas, are called Model Eliciting Activities [MEAs], and they are situated in everyday contexts (Doerr, 2016; Lesh & Doerr, 2003; Aliprantis & Carmona, 2003). As in everyday life, these situations are open and can be solved in many ways. Therefore, students generate various approaches and levels of sophistication of mathematical thinking that are explicitly expressed in the models they build.

Lesh, Cramer, Doerr, Post and Zawojewski (2003) propose a standard organizational scheme for model development curricular sequences composed of a MEA, a Model Exploration Activity [MXA] and a Model Adaptation Activity [MAA].

A proposal to analyze the types of models that students build when solving a MEA is the “MEA Quality Assessment Guide” (Lesh, 2010, p. 33) designed to help teachers and students evaluate the quality and level of sophistication of the models they develop in response to MEAs. The evaluation is proposed in terms of the relevance of the model developed by the students in solving the situation that has been presented to them in the MEA, identifying five levels. The solution: a) requires redirection, b) requires major extensions or refinements, c) requires only minor editing, d) is useful for these specific data given. and e) is sharable and reusable.

Conceptual framework for covariational reasoning

Learning exponential functions implies that students develop a covariational reasoning. That is, "the cognitive activities involved in the coordination two varying quantitative while attending to the ways in which they change in relation to each other" (Carlson et al., 2002, p. 124). The conceptual framework proposed by Carlson et al. (2002) is oriented to the study of covariational reasoning that students develop when solving problems that contain situations that involve the use of two quantities that change simultaneously. Four of the five levels of covariational reasoning identified by Carlson are shown in Table 1.

Table 1: Four levels of covariational reasoning from Carlson et al. (2002, p. 358)

Level 1 (L1). Coordination
At the coordination level, the images of covariation can support the mental action of coordinating the change of one variable with changes in the other variable (MA1).
Level 2 (L2). *Direction*
At the direction level, the images of covariation can support the mental action of coordinating the direction of change of one variable with changes in the other variable. The mental actions identified as MA1 and MA2 are *both* supported by L2 images.
Level 3 (L3). *Quantitative Coordination*
At the quantitative coordination level, the images of covariation can support the mental actions of coordinating the amount of change in one variable with changes in the other variable. The mental actions identified as MA1, MA2 and MA3 are supported by L3 images.
Level 4 (L4). *Average Rate*
At the average rate level, the images of covariation can support the mental actions of coordinating the average rate of change of the function with uniform changes in the input variable. The average rate of change can be unpacked to coordinate the amount of change of the output variable with changes in the input variable. The mental actions identified as MA1 through MA4 are supported by L4 images.

Based on these two theories, Montero-Moguel & Vargas-Alejo (2021) proposed a classification of models called "Guide for the evaluation of models related to the concept of function" [GEMF] that allows describing the evolution of the models and the covariational reasoning developed by the students when solving MEAs where the concept of exponential function underlies.

Methodology

The research was qualitative because the interest was to study the development process of students' concept of exponential function in order to identify and describe the evolution of the models developed during the model development sequence. The research participants were 10 first-semester university-level students (women and men). The students were in a course focusing on mathematics applied to business. Before they were exposed to the model development sequence designed for this study, the students had not covered the topic of exponential function as part of in this course.

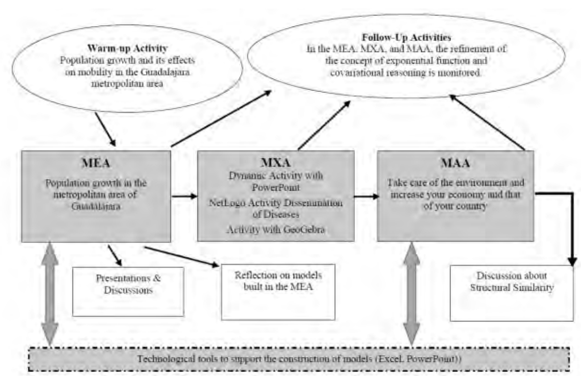
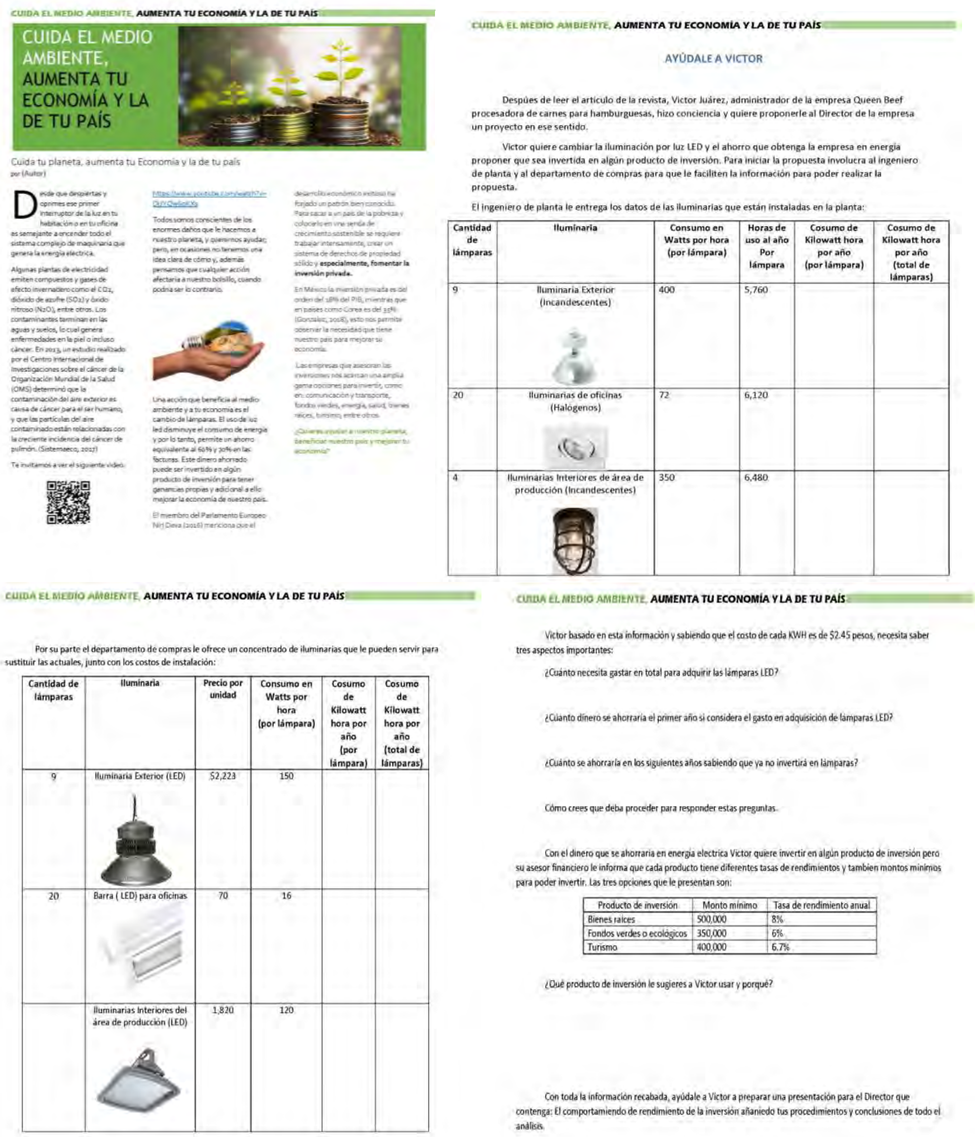


Figure 1: Didactic sequence diagram

The model development sequence was designed based on the proposal of Lesh et al. (2003). It was made up of three activities (Figure 1): a) MEA in the context of population growth

(Montero-Moguel &Vargas Alejo, 2021), b) MXA divided into three parts, PowerPoint activity, NetLogo activity, and GeoGebra activity, c) MAA in the context of taking care for the environment and investments.

The MEA and MAA were designed with the same structure to elicit students' conceptions of exponential function, including three parts: a newspaper designed for this MEA, context questions, and a situation. The newspaper and the situation of the MAA are shown in Figure 2.



CUIDA EL MEDIO AMBIENTE, AUMENTA TU ECONOMÍA Y LA DE TU PAÍS

CUIDA EL MEDIO AMBIENTE, AUMENTA TU ECONOMÍA Y LA DE TU PAÍS

Cuida tu planeta, aumenta tu Economía y la de tu país
por (Autor)

Desde que despiertas y oprimes ese primer interruptor de la luz en tu habitación o en tu oficina es semejante a encender todo el sistema complejo de maquinaria que genera la energía electrica.

Algunas plantas de electricidad emiten compuestos y gases de efecto invernadero como el CO_2, dióxido de azufre (SO_2) y óxido nitroso (N_2O), entre otros. Los contaminantes terminan en las aguas y suelos, lo cual genera enfermedades en la piel o incluso cáncer. En 2013, un estudio realizado por el Centro Internacional de Investigaciones sobre el cáncer de la Organización Mundial de la Salud (OMS) determinó que la contaminación del aire exterior es causa de cáncer para el ser humano, y que las partículas del aire contaminado están relacionadas con la creciente incidencia del cáncer de pulmón. (Sistemaeco, 2017)

Te invitamos a ver el siguiente video.

Todos somos conscientes de los enormes daños que le hacemos a nuestro planeta, y queremos ayudar, pero, en ocasiones no tenemos una idea clara de cómo y, además pensamos que cualquier acción afectaria a nuestro bolsillo, cuando podría ser lo contrario.

Una acción que beneficia al medio ambiente y a tu economía es el cambio de lámparas. El uso de luz led disminuye el consumo de energía y por lo tanto, permite un ahorro equivalente al 60% y 70% en las facturas. Este dinero ahorrado puede ser invertido en algún producto de inversión para tener ganancias propias y adicional a ello mejorar la economía de nuestro país.

El miembro del Parlamento Europeo Nirj Deva (2016) menciona que el desarrollo económico exitoso ha forjado un patrón bien conocido. Para sacar a un país de la pobreza y colocarlo en una senda de crecimiento sostenible se requiere trabajar intensamente, crear un sistema de derechos de propiedad sólido y **especialmente, fomentar la inversión privada.**

En México la inversión privada es del orden del 18% del PIB, mientras que en países como Corea es del 35% (González, 2018), esto nos permite observar la necesidad que tiene nuestro país para mejorar su economía.

Las empresas que asesoran las inversiones nos acercan una amplia gama opciones para invertir, como en: comunicación y transporte, fondos verdes, energía, salud, bienes raíces, turismo, entre otros.

¿Quieres ayudar a nuestro planeta, beneficiar nuestro país y mejorar tu economía?

CUIDA EL MEDIO AMBIENTE, AUMENTA TU ECONOMÍA Y LA DE TU PAÍS

AYÚDALE A VICTOR

Despúes de leer el articulo de la revista, Victor Juárez, administrador de la empresa Queen Beef procesadora de carnes para hamburguesas, hizo conciencia y quiere proponerle al Director de la empresa un proyecto en ese sentido.

Victor quiere cambiar la iluminación por luz LED y el ahorro que obtenga la empresa en energía proponer que sea invertida en algún producto de inversión. Para iniciar la propuesta involucra al ingeniero de planta y al departamento de compras para que le faciliten la información para poder realizar la propuesta.

El ingeniero de planta le entrega los datos de las iluminarias que están instaladas en la planta:

Cantidad de lámparas	Iluminaria	Consumo en Watts por hora (por lámpara)	Horas de uso al año Por lámpara	Cosumo de Kilowatt hora por año (por lámpara)	Cosumo de Kilowatt hora por año (total de lámparas)
9	Iluminaria Exterior (Incandescentes)	400	5,760		
20	Iluminarias de oficinas (Halógenos)	72	6,120		
4	Iluminarias Interiores de área de producción (Incandescentes)	350	6,480		

CUIDA EL MEDIO AMBIENTE, AUMENTA TU ECONOMÍA Y LA DE TU PAÍS

Por su parte el departamento de compras le ofrece un concentrado de iluminarias que le pueden servir para sustituir las actuales, junto con los costos de instalación:

Cantidad de lámparas	Iluminaria	Precio por unidad	Consumo en Watts por hora (por lámpara)	Cosumo de Kilowatt hora por año (por lámpara)	Cosumo de Kilowatt hora por año (total de lámparas)
9	Iluminaria Exterior (LED)	$2,223	150		
20	Barra (LED) para oficinas	70	16		
	Iluminarias Interiores del área de producción (LED)	1,820	120		

CUIDA EL MEDIO AMBIENTE, AUMENTA TU ECONOMÍA Y LA DE TU PAÍS

Victor basado en esta información y sabiendo que el costo de cada KWH es de $2.45 pesos, necesita saber tres aspectos importantes:

¿Cuánto necesita gastar en total para adquirir las lámparas LED?

¿Cuánto dinero se ahorraría el primer año si considera el gasto en adquisición de lámparas LED?

¿Cuánto se ahorraría en los siguientes años sabiendo que ya no invertirá en lámparas?

Cómo crees que deba proceder para responder estas preguntas.

Con el dinero que se ahorraria en energia electrica Victor quiere invertir en algún producto de inversión pero su asesor financiero le informa que cada producto tiene diferentes tasas de rendimientos y tambien montos minimos para poder invertir. Las tres opciones que le presentan son:

Producto de inversión	Monto mínimo	Tasa de rendimiento anual
Bienes raices	500,000	8%
Fondos verdes o ecológicos	350,000	6%
Turismo	400,000	6.7%

¿Qué producto de inversión le sugieres a Victor usar y porqué?

Con toda la información recabada, ayúdale a Victor a preparar una presentación para el Director que contenga: El comportamiendo de rendimiento de la inversión añaniedo tus procedimientos y conclusiones de todo el análisis.

Figure 2: Newspaper and situation of the MAA

The GEMF was used to analyze the data, it allowed to analyze the models built by the students when they solved the model development sequence (Table 2).
The experimentation lasted three sessions of three hours each. It was important to collect data from different sources to support the phenomenon studied, including: student worksheets, audio recordings, video recordings, and teacher's log.

Table 2: Classification of models

Model T1. The model requires direction
The model is not associated with the function (exponential, in this case) that allows to better describe, interpret, predict and control the situation. Students associate a linear behavior to the situation. Students need additional comments from their classmates or questions that encourage reflection by the teacher, that allow them to redirect their way of thinking. In relation to covariational reasoning, students show level 1 of Carlson et al. (2002, p. 358): "the images of covariation can support the mental action of coordinating the change of one variable with changes in the other variable (MA1)".
Model T2. The model requires major extensions or refinements
The model is associated with the (exponential) function that best describes the situation; however, students are unable to dissociate linear behavior. The student needs to work further to obtain greater extensions or refinements. Regarding covariational reasoning associated with the function that best describes the situation, the student shows coordination and direction of the variables. Students' reasoning relates to level 2 of Carlson et al. (2002, p. 358): " the images of covariation can support the mental action of coordinating the direction of change of one variable with changes in the other variable ".
Model T3. The model is situated
The model is associated with the (exponential) function that best describes the situation. It is only useful for the context of the situation presented. The student's conceptual system is extended and refined by differentiating between exponential and linear behavior. In relation to covariational reasoning associated with the function that best describes the situation, students exhibit coordination, direction, and quantification of the variables. Students' reasoning relates to level 3 of Carlson et al. (2002, p. 358): "the images of covariation can support the mental actions of coordinating the amount of change in one variable with changes in the other variable".
Model T4. The model is sharable and reusable
The tool not only works for the proposed problem, but it would also be easy for others to modify and use it in similar situations outside the context of the situation posed. Regarding covariational reasoning associated with the function that best describes the situation, students exhibit coordination, direction, quantification, and average rate of change of the variables. Students' reasoning relates to level 4 of Carlson et al. (2002). The images of covariation can support the mental actions of coordinating the average rate of change of the function with uniform changes in the input variable. The average rate of change can be unpacked to coordinate the amount of change of the output variable with changes in the input variable. (Carlson et al., 2002, p. 358)

Results Analysis and Discussion

A qualitative data analysis was conducted based on the following modeling cycles.

First Modeling Cycle

Models were developed during teamwork as students solved the MEA. The four teams built model T1 (Require Direction) and included only tabular representations. The students did not recognize an exponential pattern, they focused on solving the situation using linear models. Their level of covariational reasoning was level 1 (coordination) based on Carlson et al. (2002).

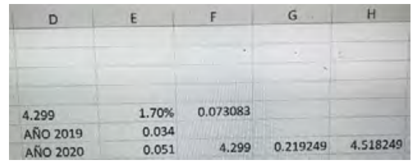
Models T1. Teams A and D multiplied the data included in the MEA (the initial population of 4.299 million times the growth rate of 1.7%) and obtained the value of 0.073803 (million people) that they assumed constant (Figure 3a). Team C detected that the growth for the years 2019, 2020 and 2021 was 0.073, 0.074 and 0.075 million inhabitants, respectively; they thought that the population increased 0.001 million people per year, that is, they believed that the growth was constant (Figure 3c). Team B model was characterized by the use of the "rule of three". A member of the team commented the following.

S4: Let's see, then we need to get the 1.7, so… would it be like a rule of three?

AÑO	2018	2019	2020	2021	2022
POBLACIÓN INICIAL	4.299	4.373	4.447	4.520	4.594
CRECIMIENTO ANUAL	0.073803	0.073803	0.073803	0.073803	0.073803
POBLACIÓN FINAL [POBLACIÓN INICIAL + CRECIMIENTO]	4.373	4.447	4.520	4.594	4.668

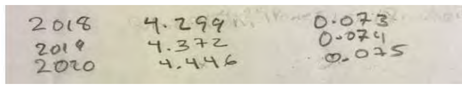
Nota: Datos en millones de habitantes

a) Teams A and D model



D	E	F	G	H
4.299	1.70%	0.073083		
AÑO 2019	0.034			
AÑO 2020	0.051	4.299	0.219249	4.518249

b) Team B model



2018	4.299	0.073
2019	4.372	0.074
2020	4.446	0.075

c) Team C model

Figure 3: Models first modeling cycle of the equipment

Second Modeling Cycle

These models were developed by students after they self-evaluated their first model and interacted with the teacher. Three types of models emerged.

Models T2. Teams B and D built tabular and graphical representations. The teams did not dissociate the exponential growth from the linear. Covariational reasoning level 2 from Carlson et. al (2002)

Models T3. Team A built an exponential and situated model (Figure 4a). The representations were tabular. Covariational Reasoning Level 3 from Carlson et. al (2002).

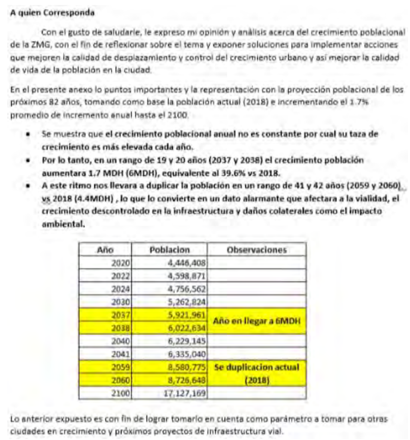
Models T4. Team C built an exponential model, integrated algebraic representations; the model was sharable and reusable (Figure 4b). Covariational reasoning level 4 from Carlson et. al (2002).

Third Modeling Cycle

These models were developed after each team of students engaged in a whole class discussion. Prior to this, students wrote their letters individually as a homework assignment.

Models T3. Students S4 and S5 included graphical and tabular representations in their models; they expressed that growth was not constant and it depended on the rate of 1.7%. The model was situated.

Models T4. Eight students included tabular, graphical, verbal, and algebraic representations in their models, which are modifiable and reusable for similar situations outside the context of the population growth situation.



A quien Corresponda

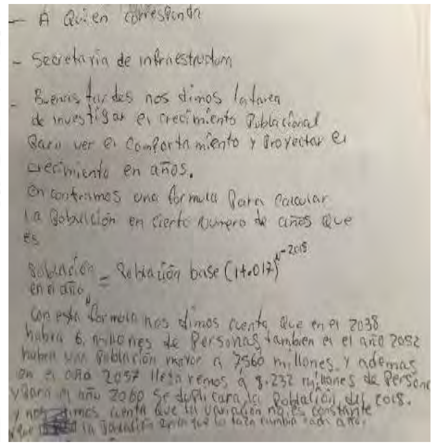
Con el gusto de saludarle, le expreso mi opinión y análisis acerca del crecimiento poblacional de la ZMG, con el fin de reflexionar sobre el tema y exponer soluciones para implementar acciones que mejoren la calidad de desplazamiento y control del crecimiento urbano y así mejorar la calidad de vida de la población en la ciudad.

En el presente anexo lo puntos importantes y la representación con la proyección poblacional de los próximos 82 años, tomando como base la población actual (2018) e incrementando el 1.7% promedio de incremento anual hasta el 2100.

- Se muestra que **el crecimiento poblacional anual no es constante por cual su taza de crecimiento es más elevada cada año.**
- **Por lo tanto, en un rango de 19 y 20 años (2037 y 2038) el crecimiento población aumentara 1.7 MDH (6MDH), equivalente al 39.6% vs 2018.**
- **A este ritmo nos llevara a duplicar la población en un rango de 41 y 42 años (2059 y 2060) vs 2018 (4.4MDH) , lo que lo convierte en un dato alarmante que afectara a la vialidad, el crecimiento descontrolado en la infraestructura y daños colaterales como el impacto ambiental.**

Año	Poblacion	Observaciones
2020	4,446,408	
2022	4,598,871	
2024	4,756,562	
2030	5,262,824	
2037	5,921,961	Año en llegar a 6MDH
2038	6,022,634	
2040	6,229,145	
2041	6,335,040	
2059	8,580,775	Se duplicacion actual (2018)
2060	8,726,648	
2100	17,127,169	

Lo anterior expuesto es con fin de lograr tomarlo en cuenta como parámetro a tomar para otras ciudades en crecimiento y próximos proyectos de infraestructura vial.



- A Quien corresponda
- Secretaria de infraestructura
- Buenas tardes nos dimos la tarea de investigar el crecimiento Poblacional para ver el comportamiento y proyectar el crecimiento en años.
En contramos una formula para calcular la Población en cierto numero de años que es
Población en el año n = Población base $(1+.017)^{n-2018}$
Con esta formula nos dimos cuenta que en el 2038 habra 6 millones de personas tambien en el año 2052 habra una población mayor a 7.560 millones y ademas en el año 2057 llegaremos a 8.232 millones de personas y para el año 2060 se duplicara la población del 2018. y nos dimos cuenta que la variación no es constante [illegible] la población ya que la taza cambia cada año.

a) Model team A b) Model team C

Figure 4: Example of models built in the second modeling cycle

Fourth Modeling Cycle

Models were developed when the students solved the MAA. The students exhibited model T4. The characteristics of the fourth modeling cycle were the following.

5. Two students (S4 and S5) improved their models from Model T3 to T4. They included a diversity of representations in their models.
6. Eight students (S1, S2, S3, S6, S7, S8, S9, and S10) affirmed their model T4.
    - a. Four students (S3, S8, S9, and S10) mathematized based on only one investment rate to explain the situation.
    - b. Four students (S1, S2, S6, and S7) mathematized based on the three investment rates.
        - i. Three students (S1, S2, and S6) proposed the choice of only one investment product (green funds) (Figure 5).
        - ii. One student (S7) proposed a combination of different investment products.

CAMBIO DE ILUMINARIA INCANDESCENTE A LED – CUIDA EL MEDIO AMBIENTE E INVERSION

Con el ahorro en 10 años ($602,825), podemos invertir en 3 opciones de inversión que a continuación represento de forma tabular y grafica .
T = Inversión Inicial (1 + % de Rendimiento)
En conclusión, por sentido común la inversión en bienes raíces genera mejores rendimientos, sin embargo, el fondo del proyecto además de ahorrar recursos económicos también están los recursos naturales, por lo que en mi punto de vista sugiero reinvertir lo ahorrado en este concepto para seguir desarrollando tecnologías y acciones que promuevan la sustentabilidad ambiental.

	Año	Bienes Raices	Ambiental	Turismo
AHORRO POR CONSUMO	1	$ 34,464	$ 34,464	$ 34,464
	2	$ 97,615	$ 97,615	$ 97,615
	3	$ 160,767	$ 160,767	$ 160,767
	4	$ 223,918	$ 223,918	$ 223,918
	5	$ 287,069	$ 287,069	$ 287,069
	6	$ 350,220	$ 350,220	$ 350,220
	7	$ 413,371	$ 413,371	$ 413,371
	8	$ 476,523	$ 476,523	$ 476,523
	9	$ 539,674	$ 539,674	$ 539,674
	10	$ 602,825	$ 602,825	$ 602,825
RENDIMIENTOS DE INVERSION	11	$ 651,051	$ 638,995	$ 643,214
	12	$ 703,135	$ 677,334	$ 686,310
	13	$ 759,386	$ 717,974	$ 732,292
	14	$ 820,137	$ 761,053	$ 781,356
	15	$ 885,748	$ 806,716	$ 833,707
	16	$ 956,608	$ 855,119	$ 889,565
	17	$ 1,033,136	$ 906,426	$ 949,166
	18	$ 1,115,787	$ 960,811	$ 1,012,760
	19	$ 1,205,050	$ 1,018,460	$ 1,080,615
	20	$ 1,301,454	$ 1,079,568	$ 1,153,016

Rendimiento de Inversion	
Bienes Raices	8.0%
Turismo	6.7%
Ambiental	6.0%

Figure 5: Example of models from the fourth modeling cycle

Figure 5 is an example of the model T4 built by the students in the fourth cycle, during which students dissociated the linear and exponential behavior and included different representations (verbal, tabular, graphical, and algebraic). Regarding the linear function, students used it to describe energy savings and identified a constant growth. Regarding the exponential function, students included an analysis of three types of investment instrument at ten years, which allowed them to propose a form of investment portfolio. A summary of the evolution of the linear to exponential models built by the students can be observed in Table 3.

Table 3: Scheme showing the evolution of the models

Team	Student	Type of Model built by the students			
		First cycle MEA (Team)	Second cycle MEA (Team)	Third cycle MEA (Individual)	Fourth cycle MAA (Individual)
A	S1	T1	T3	T4	T4
	S2	T1	T3	T4	T4
B	S3	T1	T2	T4	T4
	S4	T1	T2	T3	T4
	S5	T1	T2	T3	T4
C	S6	T1	T4	T4	T4
	S7	T1	T4	T4	T4
	S8	T1	T4	T4	T4
D	S9	T1	T2	T4	T4
	S10	T1	T2	T4	T4

Conclusions

This analysis allowed us to address the research question, how did LAEC students' models and covariational reasoning - related to the exponential function- evolve when solving a model development sequence based on real-life problems, with the support of technology?

The evolution of the models developed by students was observed in each modeling cycle. In the first cycle, all the teams built T1 models. They identified variables, but did not understand the type of relationship between them. The ideas and procedures associated with the situation were linear. The teams gave more importance to the answers obtained for the situation than to the construction of models.

In the second cycle, three types of models were built (T2, T3, and T4) characterized by various attributes, including: a) Model T2: Teams B and D used language associated with the linear function to describe the exponential function. They failed to dissociate linear behavior from exponential. b) Model T3: Team A exhibited coordination, direction, and quantification of the variables. This team dissociated linear behavior from exponential; the model was situated. c) Model T4: Team C exhibited coordination, direction, quantification, and average rate of change between the variables. This team not only dissociated linear behavior from the exponential, but they also built useful models for a specific client (sharable) who was interested in solving the situation, and any similar situation with different initial conditions (reusable).

In the third modeling cycle, the students individually reconstructed the models, based on the group discussions generated in class. They all participated in the evaluation and self-evaluation of their models. Students' progress in developing their knowledge and skills to mathematize evolved to situated (T3), and shareable and reusable models (T4). In the fourth modeling cycle, the students, individually, transferred their knowledge obtained by performing the MEA and MXA, which allowed them to deepen their knowledge regarding concepts such as: variation, exponential function, variables and use of different representations. When solving the MAA, the refinement of ideas was noted, all the students built sharable and reusable models (T4).

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EVOLUCIÓN DE MODELOS LINEALES A EXPONENCIALES AL RESOLVER UNA SECUENCIA DE DESARROLLO DE MODELOS

THE EVOLUTION FROM LINEAR TO EXPONENTIAL MODELS WHEN SOLVING A MODEL DEVELOPMENT SEQUENCE

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En este artículo se describen resultados de una investigación basada en la Perspectiva de Modelos y Modelación [PMM]. Se presenta la evolución de los modelos construidos por estudiantes universitarios al resolver una secuencia de desarrollo de modelos creada para propiciar el aprendizaje de la función exponencial. Como resultado, se observó que el *pensamiento de los estudiantes se modificó, amplió y refinó, ya que los modelos evolucionaron. Primero, se construyeron modelos que requerían dirección por ser lineales; después, modelos donde no se exhibía disociación entre comportamiento lineal y exponencial; enseguida, modelos exponenciales situados; y finalmente, modelos exponenciales compartibles y reutilizables.*

Palabras clave: Modelación, Matemáticas de nivel universitario, Función exponencial

En la literatura de estudios de investigación (Ärlebäck, Doerr y O'Neil, 2013; Ärlebäck y Doerr, 2018) se menciona que los estudiantes de nivel superior tienen dificultades con el aprendizaje de la función exponencial porque es un objeto matemático cuyo aprendizaje requiere una alta capacidad cognoscitiva de transferencia, ya que su comprensión implica entender otros conceptos. El aprendizaje de la función exponencial requiere que los estudiantes desarrollen un razonamiento covariacional (Carlson, Jacobs, Coe, Larsen y Hsu, 2002; Thompson y Carlson, 2017). Investigaciones realizadas por Ärlebäck y Doerr (2018) y Ärlebäck, Doerr y O'Neil (2013) muestran la importancia del diseño de Actividades Provocadoras de Modelos [MEAs, por sus siglas en inglés] para que los estudiantes desarrollen conocimiento relacionado con la función exponencial. La tecnología podría apoyar el aprendizaje de la función exponencial debido a que posibilita el uso de distintas representaciones y la conexión entre ellas para interpretar, describir y predecir fenómenos, además de simplificar cálculos. Debido a su carácter dinámico la tecnología puede apoyar a los estudiantes a profundizar en conceptos como variación (Stillman, Blum y Kaiser, 2013) y por lo tanto a desarrollar su razonamiento covariacional.

El problema que interesó abordar en esta investigación fue conocer cómo una secuencia de desarrollo de modelos puede contribuir a la ampliación y refinamiento del conocimiento sobre la función exponencial de estudiantes de licenciatura en administración de empresas y en contaduría [LAEC]. Por lo tanto, la pregunta de investigación que se planteó fue ¿Cómo evolucionaron los modelos y el razonamiento covariacional−relacionados con la func exponencial− de estudiantes de LAEC al resolver, con el apoyo de tecnología, una secuencia de desarrollo de modelos compuesta por problemas cercanos a la vida real?

Marco Conceptual

El Marco conceptual de esta investigación se estructuró con base en la PMM propuesta por Lesh y Doerr (2003) y el razonamiento covariacional propuesto por Carlson et al. (2002).

La Perspectiva de Modelos y Modelación

De acuerdo con la PMM, aprender matemáticas es un proceso de desarrollo de sistemas conceptuales, que se modifican de manera continua; se extienden y refinan a partir de las interacciones del estudiante con su entorno (e.g., los profesores y compañeros) y al resolver problemas (Lesh, 2010). Resolver problemas implica "diferenciar, integrar, reorganizar, adaptar o extender sistemas de interpretación que se encuentran en etapas intermedias de desarrollo." (Lesh, 2010, p. 27) Los ciclos de modelación son interpretaciones que los estudiantes exhiben al resolver las MEAs, en las cuales las formas de pensamiento se expresan, prueban y revisan repetidamente (Lesh, 2010; Sevinc y Lesh, 2018). Desde la PMM, los modelos se definen como:

> Sistemas conceptuales (que consisten en elementos, relaciones, operaciones y reglas que gobiernan las interacciones) que se expresan mediante sistemas de notación externa, y se usan para construir, describir o explicar los comportamientos de otros sistemas –Quizás de tal forma que otro sistema pueda ser manipulado o predicho de manera inteligente.

> Un modelo matemático se enfoca en las características estructurales (más que, por ejemplo, en características musicales o físicas) de los sistemas relevantes. (Lesh y Doerr, 2003, p. 10)

De esta manera, la PMM propone estructurar experiencias para el alumno, en las cuales exprese, pruebe y refine sus formas de pensamiento durante el proceso que desarrolla al generar un modelo matemático para resolver una situación problemática que le es presentada. Estas situaciones diseñadas intencionalmente para que los alumnos generen modelos utilizando ideas matemáticas específicas se llaman Actividades Provocadoras de Modelos (MEAs), y están situadas en contextos cotidianos (Doerr, 2016; Aliprantis y Carmona, 2003). Tal como sucede en la vida cotidiana, estas situaciones son abiertas y se pueden resolver de muchas maneras. Por tanto, los estudiantes generan varias aproximaciones y niveles de sofisticación de pensamiento matemático que quedan expresados de manera explícita en los modelos que generan en sus soluciones.

Lesh, Cramer, Doerr, Post y Zawojewski (2003) proponen un esquema organizacional estándar para secuencias curriculares de desarrollo de modelos compuesto por una MEA, una Actividad de Exploración de Modelos [MXA] y una Actividad de Adaptación de Modelos [MAA].

Una propuesta para analizar los tipos de modelos que los alumnos construyen al resolver una MEA es la “guía de evaluac alidad MEA” (Lesh, 2010, p. 33) dise ada para ayudar a los maestros y estudiantes a evaluar la calidad y nivel de sofisticación de los modelos que desarrollan en sus respuestas a las MEAs. La evaluación se propone en términos de la pertinencia del modelo desarrollado por los estudiantes al resolver la situación problemática que se le ha presentado en la MEA, identificando cinco niveles: a) Requiere redirección, b) Requiere mayores extensiones o refinamientos, c) Sólo requiere ediciones menores, d) Útil para estos datos específicos dados y e) Compartible y reutilizable.

Marco conceptual de razonamiento covariacional

El estudio de las funciones exponenciales implica que los alumnos desarrollen un razonamiento covariacional, es decir “actividades cognitivas implicadas en la coordinación de dos cantidades que varían mientras se atiende a las formas en que cada una de ellas cambia con respecto a la otra” (Carlson et al., 2002, p. 124). El marco conceptual propuesto por Carlson et al. (2002) se orienta al estudio del razonamiento covariacional que desarrollan los estudiantes al resolver situaciones problema que implican el uso de dos cantidades que cambian simultáneamente. Cuatro de los cinco niveles de razonamiento covariacional identificados por Carlson se observan en la Tabla 1.

Tabla 1: Cuatro niveles de razonamiento covariacional de Carlson et al. (2002, p. 358)

Nivel 1 (N1). ***Coordinación***
En el nivel de coordinación, las imágenes de la covariación pueden sustentar a la acción mental de coordinar el cambio de una variable con cambios en la otra variable (AM1).
Nivel 2 (N2). ***Dirección***
En el nivel de dirección, las imágenes de la covariación pueden sustentar a las acciones mentales de coordinar la dirección del cambio de una de las variables con cambios en la otra. Las acciones mentales identificadas como AM1 y AM2 son sustentas por imágenes de N2.
Nivel 3 (N3). ***Coordinación cuantitativa***
En el nivel de la coordinación cuantitativa, las imágenes de la covariación pueden sustentar a las acciones mentales de coordinar la cantidad de cambio en una variable con cambios en la otra. Las acciones mentales identificadas como AM1, AM2 y AM3 son sustentadas por las imágenes de N3.

Nivel 4 (N4). ***Razón promedio***
En el nivel de la razón promedio, las imágenes de covariación pueden sustentar a las acciones mentales de coordinar la razón de cambio promedio de una función con cambios uniformes en los valores de entrada de la variable. La razón de cambio promedio se puede descomponer para coordinar la cantidad de cambio de la variable resultante con los cambios en la variable de entrada. Las acciones mentales identificadas como AM1 hasta AM4 son sustentadas por imágenes de N4.

A partir de estas dos teorías se propuso en Montero-Moguel y Vargas Alejo (2021) una clasificac los denominada "Guía de evaluac los relacionados con el concepto de func [GEMF] que permite describir la e los modelos y el razonamiento covariacional desarrollado por los estudiantes al resolver MEAs donde subyace el concepto de función exponencial.

Metodología

La investigación fue de tipo cualitativa, porque interesaba analizar el proceso de desarrollo de conocimiento de los estudiantes para lograr identificar y describir la evolución de los modelos que ellos desarrollan al resolver la secuencia de desarrollo de modelos para elucidar el concepto de función exponencial. Los participantes de la investigación estaban conformados por 10 alumnos (mujeres y hombres) de primer semestre de nivel universitario. Los alumnos estaban cursando la materia de matemáticas aplicadas a los negocios. Previo a la experimentación, los alumnos no habían visto el tema de función exponencial en el curso.

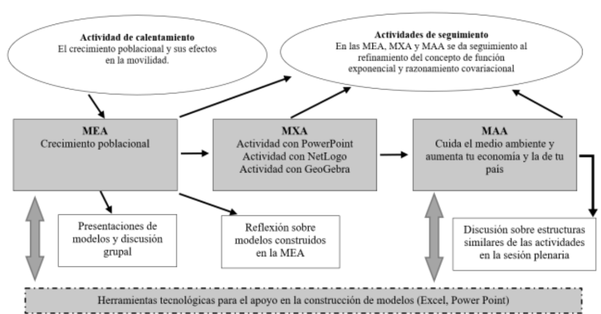
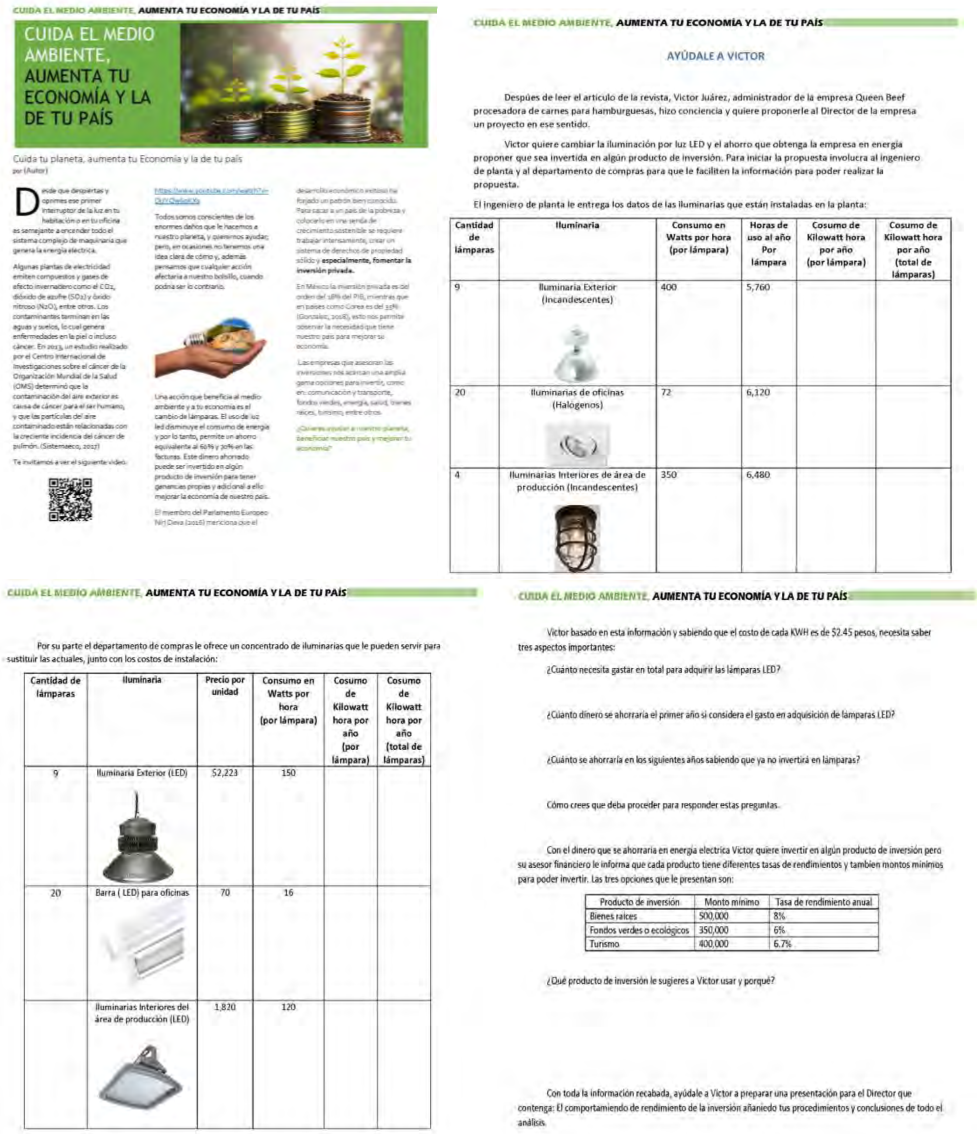


Figura 1: Esquema de secuencia didáctica

La secuencia de desarrollo de modelos se diseñó con base en la propuesta de Lesh et al. (2003). Se conformó por tres actividades (Figura 1): a) MEA en el contexto del crecimiento poblacional (Montero-Moguel y Vargas Alejo, 2021), b) MXA dividida en tres partes, actividad con PowerPoint, actividad con NetLogo y actividad con GeoGebra y c) MAA en el contexto del cuidado del medio ambiente e inversiones.

La MEA y la MAA fueron diseñadas con la misma estructura para elucidar las concepciones de los estudiantes de función exponencial, incluyendo tres partes: nota periodística diseñada exprofeso, preguntas de contexto y situación problema. La nota periodística y la situación problema de la MAA se presentan en la Figura 2.



CUIDA EL MEDIO AMBIENTE, AUMENTA TU ECONOMÍA Y LA DE TU PAÍS

CUIDA EL MEDIO AMBIENTE, AUMENTA TU ECONOMÍA Y LA DE TU PAÍS

Cuida tu planeta, aumenta tu Economía y la de tu país
por (Autor)

Desde que despiertas y oprimes ese primer interruptor de la luz en tu habitación o en tu oficina es semejante a encender todo el sistema complejo de maquinaria que genera la energía eléctrica.

Algunas plantas de electricidad emiten compuestos y gases de efecto invernadero como el CO_2, dióxido de azufre (SO_2) y óxido nitroso (N_2O), entre otros. Los contaminantes terminan en las aguas y suelos, lo cual genera enfermedades en la piel o incluso cáncer. En 2013, un estudio realizado por el Centro Internacional de Investigaciones sobre el cáncer de la Organización Mundial de la Salud (OMS) determinó que la contaminación del aire exterior es causa de cáncer para el ser humano, y que las partículas del aire contaminado están relacionadas con la creciente incidencia del cáncer de pulmón. (Sistemaeco, 2017)

Te invitamos a ver el siguiente video:

[illegible]

Todos somos conscientes de los enormes daños que le hacemos a nuestro planeta, y queremos ayudar; pero, en ocasiones no tenemos una idea clara de cómo y, además pensamos que cualquier acción afectaría a nuestro bolsillo, cuando podría ser lo contrario.

Una acción que beneficia al medio ambiente y a tu economía es el cambio de lámparas. El uso de luz led disminuye el consumo de energía y por lo tanto, permite un ahorro equivalente al 60% y 70% en las facturas. Este dinero ahorrado puede ser invertido en algún producto de inversión para tener ganancias propias y adicional a ello mejorar la economía de nuestro país.

El miembro del Parlamento Europeo Nirj Deva (2016) menciona que el desarrollo económico exitoso ha forjado un patrón bien conocido. Para sacar a un país de la pobreza y colocarlo en una senda de crecimiento sostenible se requiere trabajar intensamente, crear un sistema de derechos de propiedad sólido y **especialmente, fomentar la inversión privada.**

En México la inversión privada es del orden del 18% del PIB, mientras que en países como Corea es del 35% (González, 2018), esto nos permite observar la necesidad que tiene nuestro país para mejorar su economía.

Las empresas que asesoran las inversiones nos acercan una amplia gama opciones para invertir, como en: comunicación y transporte, fondos verdes, energía, salud, bienes raíces, turismo, entre otros.

¿Quieres ayudar a nuestro planeta, beneficiar nuestro país y mejorar tu economía?

CUIDA EL MEDIO AMBIENTE, AUMENTA TU ECONOMÍA Y LA DE TU PAÍS

AYÚDALE A VICTOR

Despúes de leer el articulo de la revista, Victor Juárez, administrador de la empresa Queen Beef procesadora de carnes para hamburguesas, hizo conciencia y quiere proponerle al Director de la empresa un proyecto en ese sentido.

Victor quiere cambiar la iluminación por luz LED y el ahorro que obtenga la empresa en energia proponer que sea invertida en algún producto de inversión. Para iniciar la propuesta involucra al ingeniero de planta y al departamento de compras para que le faciliten la información para poder realizar la propuesta.

El ingeniero de planta le entrega los datos de las iluminarias que están instaladas en la planta:

Cantidad de lámparas	Iluminaria	Consumo en Watts por hora (por lámpara)	Horas de uso al año Por lámpara	Cosumo de Kilowatt hora por año (por lámpara)	Cosumo de Kilowatt hora por año (total de lámparas)
9	Iluminaria Exterior (Incandescentes)	400	5,760		
20	Iluminarias de oficinas (Halógenos)	72	6,120		
4	Iluminarias Interiores de área de producción (Incandescentes)	350	6,480		

CUIDA EL MEDIO AMBIENTE, AUMENTA TU ECONOMÍA Y LA DE TU PAÍS

Por su parte el departamento de compras le ofrece un concentrado de iluminarias que le pueden servir para sustituir las actuales, junto con los costos de instalación:

Cantidad de lámparas	Iluminaria	Precio por unidad	Consumo en Watts por hora (por lámpara)	Cosumo de Kilowatt hora por año (por lámpara)	Cosumo de Kilowatt hora por año (total de lámparas)
9	Iluminaria Exterior (LED)	$2,223	150		
20	Barra (LED) para oficinas	70	16		
	Iluminarias Interiores del área de producción (LED)	1,820	120		

CUIDA EL MEDIO AMBIENTE, AUMENTA TU ECONOMÍA Y LA DE TU PAÍS

Victor basado en esta información y sabiendo que el costo de cada KWH es de $2.45 pesos, necesita saber tres aspectos importantes:

¿Cuánto necesita gastar en total para adquirir las lámparas LED?

¿Cuanto dinero se ahorraría el primer año si considera el gasto en adquisición de lámparas LED?

¿Cuánto se ahorraría en los siguientes años sabiendo que ya no invertirá en lámparas?

Cómo crees que deba proceder para responder estas preguntas.

Con el dinero que se ahorraria en energia electrica Victor quiere invertir en algún producto de inversión pero su asesor financiero le informa que cada producto tiene diferentes tasas de rendimientos y tambien montos minimos para poder invertir. Las tres opciones que le presentan son:

Producto de inversión	Monto mínimo	Tasa de rendimiento anual
Bienes raices	500,000	8%
Fondos verdes o ecológicos	350,000	6%
Turismo	400,000	6.7%

¿Qué producto de inversión le sugieres a Victor usar y porqué?

Con toda la información recabada, ayúdale a Victor a preparar una presentación para el Director que contenga: El comportamiendo de rendimiento de la inversión añaniedo tus procedimientos y conclusiones de todo el análisis.

Figura 2: Nota periodística y situación problema de la MAA

Para el análisis de los datos se utilizó la GEMF que permite analizar los modelos construidos por los estudiantes al resolver la secuencia de desarrollo de modelos (Tabla 2).

La experimentación tuvo una duración de tres sesiones de tres horas cada una. El investigador fungió como profesor-investigador. Fue importante colectar datos de distintas fuentes que ayudaran a describir el fenómeno estudiado: hojas de trabajo de los estudiantes, audios, videos y bitácora del docente.

Tabla 2: Clasificación de modelos

***Modelo T1.* El modelo requiere dirección**
El modelo no está asociado a la función (exponencial, en este caso) que permite describir, interpretar, predecir y controlar mejor la situación problema. Los estudiantes asocian un comportamiento lineal a la situación; necesitan comentarios adicionales de sus compañeros o preguntas que propicien la reflexión por el profesor, que les posibiliten redireccionar su manera de pensar. En relación con el razonamiento covariacional, los estudiantes exhiben el nivel 1 de Carlson et al. (2002, p. 358): "las im enes de la covariación pueden sustentar a la acción mental de coordinar el cambio de una variable con cambios en la otra variable".
***Modelo T2.* El modelo requiere mayor extensión o refinamiento**
El modelo está asociado a la función (exponencial) que describe mejor la situación problema. Sin embargo, los estudiantes no logran disociar el comportamiento lineal de su sistema conceptual. El estudiante necesita trabajar más en la resolución del problema que le permita mayor extensión o refinamiento. Respecto al razonamiento covariacional asociado a la función que describe mejor la situación problema, los estudiantes exhiben coordinación y dirección de las variables. Se puede considerar que alcanzaron el nivel 2 de Carlson et al. (2002, p. 358): "las im nes de la covariación pueden sustentar a las acciones mentales de coordinar la dirección del cambio de una de las variables con cambios en la otra".
***Modelo T3.* El modelo es situado**
Está asociado a la función (exponencial) que describe mejor la situación problema es útil únicamente para el contexto de la situación problemática presentada. El sistema conceptual de los estudiantes se amplía y refina al diferenciar entre un comportamiento exponencial y lineal. En relación con el razonamiento covariacional asociado a la función que describe mejor la situación problema, los estudiantes exhiben coordinación, dirección y cuantificación de las variables. Se puede considerar que alcanzaron el nivel 3 de Carlson et al. (2002, p. 358): "las im nes de la covariaci pueden sustentar a las acciones mentales de coordinar la cantidad de cambio en una variable con cambios en la otra".
***Modelo T4.* El modelo es compartible y reutilizable**
La herramienta no sólo funciona para el problema propuesto, sino que también sería fácil para otros modificarla y utilizarla en situaciones similares fuera del contexto de la situación problemática planteada. Respecto al razonamiento covariacional asociado a la función que describe mejor la situación problema, los estudiantes exhiben coordinación, dirección, cuantificación y razón de cambio promedio de las variables. Se puede considerar que alcanzaron el nivel 4 de Carlson et al. (2002). Las imágenes de covariación pueden sustentar a las acciones mentales de coordinar la razón de cambio promedio de una función con cambios uniformes en los valores de entrada de la variable. La razón de cambio promedio se puede descomponer para coordinar la cantidad de cambio de la variable resultante con los cambios en la variable de entrada (Carlson et al., 2002, p. 358).

Análisis de resultados y Discusión

El análisis de los datos se hizo con base en los ciclos de modelación siguientes.

Primer Ciclo de Modelación

Se desarrolló durante el trabajo en equipo para resolver la MEA. Los cuatro equipos construyeron *modelos T1* (Requieren dirección) e incluyeron únicamente representaciones tabulares. Los alumnos no reconocieron un comportamiento exponencial. Se centraron en resolver la situación problema mediante modelos lineales. Su nivel de razonamiento covariacional fue del nivel 1 (coordinación) de acuerdo con Carlson et al. (2002).

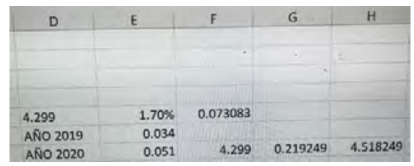
Modelos T1. Los Equipos A y D multiplicaron los datos contenidos en la MEA (la población inicial de 4.299 millones por la tasa de crecimiento de 1.7%) y obtuvieron el valor de 0.073803 (millones de personas) que supusieron constante (Figura 3a). El equipo C detectó que el crecimiento para los años 2019, 2020 y 2021 era de 0.073, 0.074 y 0.075 millones de habitantes, respectivamente; pensaron que la población aumentaba 0.001 millones de personas por año. Es decir, los estudiantes creyeron que el crecimiento era constante (Figura 3c). El modelo del equipo B se caracteriz or el uso de "la regla de tres". Un integrante del equipo come o siguiente.

S4: A ver, entonces tenemos que sacar el 1.7, ¿entonces… sí sería como una regla de tres no?

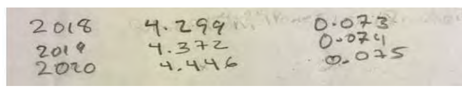
AÑO	2018	2019	2020	2021	2022
POBLACIÓN INICIAL	4.299	4.373	4.447	4.520	4.594
CRECIMIENTO ANUAL	0.073803	0.073803	0.073803	0.073803	0.073803
POBLACIÓN FINAL [POBLACIÓN INICIAL + CRECIMIENTO]	4.373	4.447	4.520	4.594	4.668

Nota: Datos en millones de habitantes

a) Modelo equipos A y D



b) Modelo equipo B



c) Modelo equipo C

Figura 3: Modelos primer ciclo de modelación de los equipos

Segundo Ciclo de Modelación

Se desarrolló posterior a la autoevaluación del primer modelo e interacción con el profesor. Emergieron tres tipos de modelos.

Modelos T2. Los Equipos B y D construyeron representaciones tabulares y gráficas. Los equipos no disociaron el crecimiento exponencial del lineal. Nivel de razonamiento covariacional 2 de Carlson et. al (2002)

Modelos T3. El equipo A construyó un modelo exponencial y situado (Figura 4a). Sus representaciones fueron tabulares. Nivel de razonamiento covariacional 3 de Carlson et. al (2002).

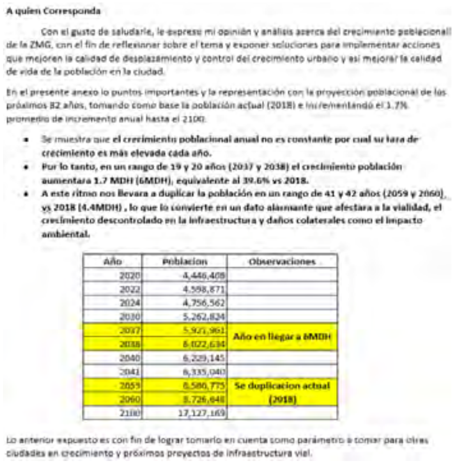
Modelos T4. El equipo C construyó un modelo exponencial, integró representaciones algebraicas. Su modelo es compartible y reutilizable (Figura 4b). Nivel de razonamiento covariacional 4 de Carlson et. al (2002).

Tercer Ciclo de Modelación

Se desarrolló después de la sesión plenaria. Los alumnos realizaron sus cartas de forma individual como tarea extra a los trabajos de clase.

Modelos T3. Los alumnos S4 y S5 incluyeron representaciones gráficas y tabulares en sus modelos; expresaron que el crecimiento no era constante y que dependía de la tasa del 1.7%. El modelo de los alumnos fue situado.

Modelos T4. Ocho alumnos incluyeron representaciones tabulares, gráficas, verbales y algebraicas en sus modelos, los cuales son modificables y reutilizables para situaciones similares fuera del contexto de la situación problemática de crecimiento poblacional.



A quien Corresponda

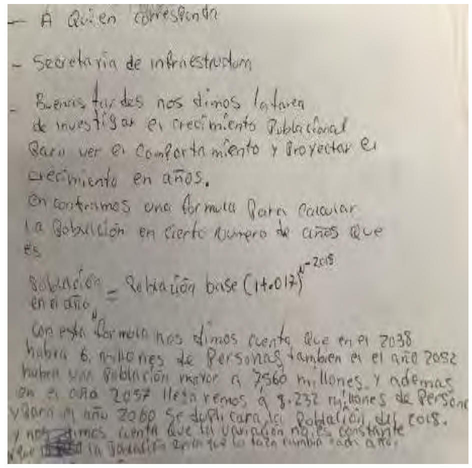
Con el gusto de saludarle, le expreso mi opinión y análisis acerca del crecimiento poblacional de la ZMG, con el fin de reflexionar sobre el tema y exponer soluciones para implementar acciones que mejoren la calidad de desplazamiento y control del crecimiento urbano y así mejorar la calidad de vida de la población en la ciudad.

En el presente anexo lo puntos importantes y la representación con la proyección poblacional de los próximos 82 años, tomando como base la población actual (2018) e incrementando el 1.7% promedio de incremento anual hasta el 2100.

- Se muestra que **el crecimiento poblacional anual no es constante por cual su tasa de crecimiento es más elevada cada año.**
- **Por lo tanto, en un rango de 19 y 20 años (2037 y 2038) el crecimiento población aumentara 1.7 MDH (6MDH), equivalente al 39.6% vs 2018.**
- **A este ritmo nos llevara a duplicar la población en un rango de 41 y 42 años (2059 y 2060) vs 2018 (4.4MDH), lo que lo convierte en un dato alarmante que afectara a la vialidad, el crecimiento descontrolado en la infraestructura y daños colaterales como el impacto ambiental.**

Año	Poblacion	Observaciones
2020	4,446,408	
2022	4,598,871	
2024	4,756,562	
2030	5,262,824	
2037	5,921,961	Año en llegar a 6MDH
2038	6,022,634	
2040	6,229,145	
2041	6,335,040	
2059	8,586,775	Se duplicacion actual (2018)
2060	8,726,648	
2100	17,127,169	

Lo anterior expuesto es con fin de lograr tomarlo en cuenta como parámetro a tomar para otras ciudades en crecimiento y próximos proyectos de infraestructura vial.



- A Quien corresponda
- Secretaria de infraestructura
- Buenas tardes nos dimos la tarea de investigar el crecimiento poblacional para ver el comportamiento y proyectar el crecimiento en años.

Encontramos una formula para calcular la población en cierto numero de años que es

$$\text{Población en el año } n = \text{Población base}(1+.017)^{n-2018}$$

Con esta formula nos dimos cuenta que en el 2038 habra 6 millones de personas también en el año 2052 habra una población mayor a 7.560 millones y ademas en el año 2057 llegaremos a 8.232 millones de personas y para el año 2060 se duplicara la población del 2018. Y nos dimos cuenta que la variación no es constante por que la tasa cambia cada año.

a) Modelo equipo A b) Modelo equipo C

Figura 4: Ejemplo de modelos construidos en el segundo ciclo de modelación

Cuarto Ciclo de Modelación

Se desarrolló cuando los estudiantes resolvieron la MAA. Los alumnos exhibieron modelos T4. Las características del cuarto ciclo de modelación fueron las siguientes.

1. Dos estudiantes (S4 y S5) pasaron de Modelo T3 a T4. Incluyeron en sus modelos una diversidad de representaciones.
2. Ocho estudiantes (S1, S2, S3, S6, S7, S8, S9 y S10) se mantuvieron en el modelo T4.
    - a. Cuatro estudiantes (S3, S8, S9 y S10) matematizaron con base en sólo una tasa de inversión para explicar la situación.
    - b. Cuatro estudiantes (S1, S2, S6 y S7) matematizaron con base en las tres tasas de inversión.
        - i. Tres estudiantes (S1, S2 y S6) propusieron la elección de sólo un producto de inversión (fondos verdes) (Figura 5).
        - ii. Un estudiante (S7) propuso una combinación de diferentes productos de inversión.

Año	Bienes Raices	Ambiental	Turismo
1	$ 34,464	$ 34,464	$ 34,464
2	$ 97,615	$ 97,615	$ 97,615
3	$ 160,767	$ 160,767	$ 160,767
4	$ 223,918	$ 223,918	$ 223,918
5	$ 287,069	$ 287,069	$ 287,069
6	$ 350,220	$ 350,220	$ 350,220
7	$ 413,371	$ 413,371	$ 413,371
8	$ 476,523	$ 476,523	$ 476,523
9	$ 539,674	$ 539,674	$ 539,674
10	$ 602,825	$ 602,825	$ 602,825
11	$ 651,051	$ 638,995	$ 643,214
12	$ 703,135	$ 677,334	$ 686,310
13	$ 759,386	$ 717,974	$ 732,292
14	$ 820,137	$ 761,053	$ 781,356
15	$ 885,748	$ 806,716	$ 833,707
16	$ 956,608	$ 855,119	$ 889,565
17	$ 1,033,136	$ 906,426	$ 949,166
18	$ 1,115,787	$ 960,811	$ 1,012,760
19	$ 1,205,050	$ 1,018,460	$ 1,080,615
20	$ 1,301,454	$ 1,079,568	$ 1,153,016

Rendimiento de Inversion	
Bienes Raices	8.0%
Turismo	6.7%
Ambiental	6.0%

Figura 5: Ejemplo de modelos del cuarto ciclo de modelación

La Figura 5 es un ejemplo del modelo T4 construido por los estudiantes en el cuarto ciclo, en la cual se observa que los estudiantes disociaron el comportamiento lineal y exponencial e incluyeron diferentes representaciones (verbal, tabular, gráfica y algebraica). Respecto a la función lineal, la utilizaron para describir el ahorro de la energía e identificaron un crecimiento constante. Respecto a la función exponencial, los estudiantes incluyeron el análisis de los tres tipos de instrumentos de inversión a diez años lo que les permitió proponer un plan de inversión. En resumen, la evolución de los modelos lineales a exponenciales construidos por los estudiantes se puede observar en la Tabla 3.

Tabla 3: Esquema que muestra la evolución de los modelos

Equipo	Alumno	Tipo de Modelo construido			
		Primer ciclo MEA (equipo)	Segundo Ciclo MEA (Equipo)	Tercer Ciclo MEA (Individual)	Cuarto Ciclo MAA (Individual)
A	S1	T1	T3	T4	T4
	S2	T1	T3	T4	T4
B	S3	T1	T2	T4	T4
	S4	T1	T2	T3	T4
	S5	T1	T2	T3	T4
C	S6	T1	T4	T4	T4
	S7	T1	T4	T4	T4
	S8	T1	T4	T4	T4
D	S9	T1	T2	T4	T4
	S10	T1	T2	T4	T4

Conclusiones

Respecto a la pregunta de investigación ¿Cómo evolucionaron los modelos y el razonamiento covariacional−relacionados con la func encial− de estudiantes de LAEC al resolver, con el apoyo de tecnología, una secuencia de desarrollo de modelos compuesta por problemas cercanos a la vida real? La evolución de los modelos construidos se pudo observar en cada ciclo de modelación. En el primer ciclo todos los equipos de estudiantes construyeron modelos T1. Identificaron variables, pero no entendieron el tipo de relación que había entre las mismas. Las ideas y procedimientos asociados a la situación fueron lineales. Los equipos dieron más importancia a las respuestas obtenidas para la situación que a la construcción de modelos.

En el segundo ciclo se construyeron tres tipos de modelos (T2, T3 y T4) caracterizados por varios atributos, entre los que se pueden mencionar los siguientes: a) Modelos T2: los equipos B y D usaron lenguaje asociado a la función lineal para describir la función exponencial. No lograron disociar el comportamiento lineal del exponencial en su sistema conceptual. b) Modelo T3: El equipo A exhibió coordinación, dirección y cuantificación de las variables. Disoció el comportamiento lineal del exponencial, el modelo fue situado. c) Modelo T4: El equipo C exhibió coordinación, dirección, cuantificación y razón de cambio promedio entre las variables. No sólo disoció el comportamiento lineal del exponencial, sino que, además, construyó modelos útiles para un cliente (compartibles) interesado en resolver la situación y cualquier situación parecida (reutilizable), con condiciones iniciales distintas.

En el tercer ciclo de modelación los estudiantes, de manera individual, reconstruyeron los modelos con base en las discusiones grupales generadas en clase y participaron en la evaluación y autoevaluación de los modelos. Su progreso en cuanto al desarrollo de su conocimiento y habilidades para matematizar evolucionó a los modelos situados (T3) y compartibles y reutilizables (T4). En el cuarto ciclo de modelación los estudiantes, de manera individual, transfirieron su conocimiento obtenido al realizar la MEA y la MXA, las cuales les permitieron profundizar en su conocimiento respecto a conceptos como: variación, función exponencial, variables y uso de diferentes representaciones. Al resolver la MAA se notó el refinamiento de ideas, ya que todos los estudiantes construyeron modelos, compartibles y reutilizables (T4).

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PRODUCTIVE STRUGGLE LEADING TO COLLECTIVE MATHEMATICAL CREATIVITY

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In this article we show how students' productive struggle on a mathematical task can lead to collective mathematical creativity. We use observable (co)actions and interactions from a video record that features three Grade 6 students in a problem-solving session to document the emergence of collective creativity leading to a solution. We discuss some key features of the task and the learning environment and present implications for classroom practices aimed at helping students to capitalize on their mathematical struggles.

Keywords: Elementary school education, Problem solving, Number concepts and operations

In this paper we show how elementary students' productive struggle on a mathematics task can lead to collective mathematical creativity and what that process might look like in practice. We discuss some key features of the task and the learning environment and present implications for classroom practices aimed at helping students to capitalize on their mathematical struggles.

Literature Review

Productive Struggle

In the field of mathematics education, Boaler (2016) described a vision of mathematics learning where students are offered opportunities to engage in productive struggle, to thrive, and to become mathematical problem solvers. Lesh & Zawojewski (2007) noted that such a productive way of thinking involves iterative cycles of "expressing, testing, and revising mathematical interpretation—and of sorting out, integrating, modifying, revising or refining clusters of mathematical concepts from various topics within and beyond mathematics" (p. 782). There is an extensive literature discussing ways to support students in this kind of productive struggle in the mathematics classroom. The NCTM (2014) noted that effective teaching values productive struggle as a means to deepen conceptual understanding and "embraces a view of students' struggles as opportunities for delving more deeply into understanding the mathematical structure of problems and relationships among mathematical ideas" (p. 48). In recent years, many authors (e.g., Townsend et al., 2018; Warshauer, 2015) have emphasized the socioemotional dimension of learning and have focused on the importance of building supports for, and valuing, struggle in the classroom. It is widely recognized that without appropriate supports students can spend a lot of time in unproductive struggle and that, for those students, timely intervention is key in nudging them forward from unproductive to productive struggle (Jonsson et al., 2014). Some studies, though, report that students are able to sustain productive struggle, given supports such as an appropriate task, successful strategy choice, and relevant tools. For example, in a study using GeoGebra, Granberg (2016) reported that the majority of the students were able to engage in productive struggle that enabled them to solve problems together. Successful students did this by observing knowledge gaps between their prior knowledge and the target knowledge, correcting incorrectly recalled information, and reconstructing partly forgotten knowledge.

Creativity—Individual and Collective

While some see creativity as confined to special people, particular arts-based activities, or undisciplined play, scholars generally agree that creativity involves the combination of originality and task appropriateness or effectiveness (Beghetto & Kaufman, 2013; Runco & Jaeger, 2012). The word *creativity*, both in its origins and in most of its different uses, reflects a kind of newness, originality, or novelty; it indicates bringing something new and fruitful into being. Craft (2001) claimed that creativity in learning environments enables learners to generate and expand ideas, suggest hypotheses, apply imagination, and look for alternative, not-yet imagined approaches. In the field of mathematics education, Levenson (2011) characterized collective mathematical creativity using characteristics of individual creativity—namely, fluency, flexibility, and originality—and concluded that working as a collective may encourage students to persevere and try new ideas and that teachers can promote the emergence of creativity in their classrooms by encouraging diversity, supporting interactions, and allowing for a certain amount of instability (Levenson, 2014).

Theoretical Framing

Building on this scholarship, herein we draw on the first author's work on collective creativity in mathematics learning environments (Aljarrah, 2017, 2018, 2020; Aljarrah & Towers, 2019), and the work of the second author on the emergence of collective mathematical understanding (Martin & Towers, 2009, 2011; Martin et al., 2006). We bring these theoretical frameworks together to document and analyze the trajectory from productive struggle to the emergence of collective creativity.

Collectivity and Emergence

The second author and colleagues (Martin et al., 2006) laid the groundwork for the present study of collective creative acts in mathematics learning environments. They argued that doing and understanding mathematics are creative processes that should be considered at both the individual and the collective levels. Drawing on improvisational theory, Martin and Towers (2009) suggested that, when students are working together, acts of mathematical understanding "[can] not simply be located in the minds or actions of any one individual, but instead [emerge] from the *interplay* of the ideas of individuals, as these [become] woven together in shared action, as in an improvisational performance" (p. 2, emphasis in original). Martin et al. (2006) used the notion of coaction "to describe a particular kind of mathematical action, one that whilst obviously in execution is still being carried out by an individual, is also dependent and contingent upon the actions of the others in the group" (p. 156).

One of the most important ideas in the study of collectivity in learning settings is the notion of emergence. In our analysis of data later in this paper, we concentrate on three key features of collective emergence adopted from improvisational theory and already articulated in the mathematics education literature (e.g., Martin & Towers, 2009, 2011; Martin et al., 2006): (1) potential pathways, (2) collective structure and striking a groove, and (3) etiquette and the group mind. Noteworthy here is that the actions and interactions of a group working as a collective are usually prompted and constrained by a common purpose that guides the development of a collective structure. In referring to the development of such a collective structure, Martin and Towers (2011) adopted Berliner's (1994, 1997) expression *striking a groove*.

> Striking a groove involves 'the negotiation of a shared sense of the beat,' and is a subtle and fundamental process to allow the performance to develop to its fullest…. The 'groove' is the

> underlying element of the structure that allows the improvisation to proceed in a coherent and productive way, and it is the responsibility of all the players to collectively maintain the groove. (Martin & Towers, 2011, p. 257)

Martin and Towers (2011) also borrowed the expression "etiquette" from Becker (2000) to refer to a number of conventions (group norms) that "govern the ways in which an improvisational performance develops and group flow[1] emerges" (Martin & Towers, 2011, p. 258). Based on the study of improvisational theater, Sawyer (2001) noticed that actors use guidelines (principles) to create better conversations. Three simple, yet overarching, principles were proposed by Sawyer (2001) as rules of improv: (1) Yes, and…, (2) Don't write the script in your head, and (3) Listen to the group mind. According to Sawyer (2001), the "Yes, and …" rule implies that every student should accept the material introduced by preceding student(s) and add something new to it. The second rule, "Don't write the script in your head," is intended to keep all improvisers, moment by moment, within the scene. It means do not plan in advance by foreshadowing or pre-determining where the problem-solving is going, for to do so shines the spotlight on oneself and results in "a lack of the necessary outward focus, toward the group creativity" (Sawyer, 2001, p. 17). Hence, an outward focus requires adherence to the third rule—listening to the group mind—being willing to abandon personal motivations to further the emerging collective structure.

Collective Creativity

Sawyer (2003) also asserted the improvised and the collective nature of group creativity. According to him, group creativity is: (1) unpredictable, in that each moment emerges from preceding flow of the performance, (2) collective, in that members of the group influence each other from moment to moment, and (3) emergent, in that the group demonstrates properties greater than the sum of its individuals. Based on the above ideas, and the first author's study of the nature of collective creativity in mathematics learning settings (Aljarrah, 2018), we define collective creative acts as particular kinds of "(co)actions and interactions of a group of curious learners while they are working collaboratively on an engaging problematic situation. Such acts, which may include (1) summing forces, (2) expanding possibilities, (3) divergent thinking, and (4) assembling things in new ways, trigger the new and the crucial to emerge and evolve" (p. 136). Below, we elaborate on the four metaphors for creativity, first proposed by Aljarrah (2018), that form core of our definition of collective creative acts:

Summing forces: This metaphor encompasses the ways in which learners coordinate their efforts to enable productive steering (Aljarrah, 2019) towards a mathematical understanding "that is not simply located in the actions of any one individual but in the collective engagement with the task posed" (Martin et al., 2006, p. 157).

Expanding possibilities: Expanding might be understood as broadening the learners' horizon by gaining new insights based on previous insights. It is a kind of stretching of the space of the possible as a result of the evolving and the growth of the learners' basic insights.

Divergent Thinking: Divergent thinking requires students to consider many potential pathways, look in many directions, journey outside a known content universe, go beyond the problem's clearly given conditions and information, and think outside-the-box (Aljarrah, 2019).

Assembling (things in new ways): This metaphor implies looking for associations and making connections. It is a vision of creativity based on an assumption that many educative things are with(in) the reach of learners in their learning environment.

In our analysis, we show how collective creativity emerges from productive struggle by detailing the students' pathways to collective creativity in pursuit of a solution to a mathematical problem.

Methods

The data described below are part of a broader, design-based research study exploring collective creativity in elementary mathematics learning environments (Aljarrah, 2018). Two mathematics teachers and 25 of their sixth-grade students in a Canadian school setting participated in the study. Students participated in problem-solving sessions in their regular mathematics classroom and in small groups under task-based interview conditions with the first author. Video-recordings of these group activities formed the core of the data.

The processes of analysis followed Pirie's (1996) advice to "sit, look, think, look again" (p. 556) supported by Powell et al.'s (2003) analytical model for studying the development of mathematical thinking, which consists of seven interacting, non-linear phases: (1) viewing the video data, (2) describing the video data, (3) identifying critical events, (4) transcribing, (5) coding, (6) constructing a storyline, and (7) composing a narrative (p. 413). Following Flanagan (1954), an event was considered to be critical if it was helpful in triggering and/or explaining the emergence of collective creativity in elementary mathematics learning environments. These events were transcribed and the key features of collaborative emergence (Martin & Towers, 2011) together with the first author's definition of collective creativity and metaphors for creativity as outlined in the previous section, were used to code the students' collaborative practices that were effective in the emergence of new and crucial ideas. For the purpose of this article, we selected one video excerpt that best displayed the way that productive struggle led to the emergence of collective mathematical creativity.

Findings

In order to explain how students' productive struggle on a mathematical task can lead to collective mathematical creativity, we use a video excerpt that features a group of sixth grade students, who were assigned the pseudonyms Maddie, Adam, and Frank, engaged in a problem-solving session with the first author. The first author introduced the following task to the group and asked them to work on it together: *What are the possible combinations to obtain a sum of one dollar using pennies, nickels, dimes, and quarters such that the four different types of coins are included in each combination?* Due to space limitations, we focus on describing three collective creative acts, namely, summing forces, expanding possibilities, and divergent thinking, which resulted from the group's productive struggle on the assigned mathematical task. (Note: In the transcript we use dashes to show an interruption of one speaker by another).

Productive Struggle Leading to Summing Forces

The presence of multiple potential pathways was evident at the beginning of the scene. The students started by negotiating the task, and a variety of ideas and suggestions were put forward as possible approaches to find all combinations to obtain a sum of one dollar. Quite quickly, one potential pathway garnered attention. Adam suggested getting "the basic ones [i.e., one penny, one nickel, one dime, and one quarter]." Maddie gave the sum of those basic ones: "Okay, there is forty-one—" and Frank suggested that they could "use all pennies" to make up the rest of the dollar (i.e., fifty-nine cents). He also started to pool the group's thoughts and ideas on their shared document. For example, he wrote down the expression 41¢ = 1 penny + 1 nickel + 1 dime + 1 quarter and labelled it as a fixed amount. He also wrote down 59¢ and under it he wrote 59 pennies as a first suggestion to make 59¢. Maddie noted that they "need at least one of each, though still." Frank responded by pointing to their shared document and explaining, "Yes, those are forty-one—" (he was trying to remind her that they already had one of each coin in their basic combination to a total of forty-one). Maddie agreed that they could "have all pennies," so Frank continued the discussion by wondering, "So, forty-one, um, that means there is, um, how many?" Adam responded "fifty-nine." Maddie was still doing the calculation in her head while she was whispering "fiftyyyyyy, um—" so Frank stressed Adam's answer by completing Maddie's whispering, "Nine, yes, fifty-nine." Frank summarized and rearticulated their initial thoughts by stating, "Okay, so fifty-nine left. Out of fifty-nine, how many can we make? So, one of them is fifty-nine pennies, um— [while he was looking to Adam and Maddie]." Our interpretation of Frank's pause and questioning look towards Adam and Maddie is that the space was open equally to all suggestions. As such, it was impossible to predict the direction of the group's unfolding interaction. None of the students seemed to be trying to force his/her ideas on the group, and none of them tried to convince the others to follow a specific strategy. To this point, what we found of particular noteworthiness was the group's collective engagement in "summing forces." They tried to understand the problem and to consider the conditions of it. Thus, decisions about where to start and how to proceed emerged from their interactions as a group. They listened respectfully to each other and responded thoughtfully to the wonderings and suggestions that emerged through the conversation. The respectful collaboration between the students set them on a pathway towards the mathematics that emerged.

Productive Struggle Leading to Expanding Possibilities

As the interaction continued, the task the students set for themselves shifted from finding all possible combinations to obtain a sum of one dollar to finding all possible combinations to obtain a sum of 59 cents. From here on, a collective structure started to evolve. This conceptual structure was located in, and stemmed from, the actions and doings of the group as a collective. Those acting and doings "determine[d] both the nature of the potential that [was] created, and also how the potential [was] then developed into a coherent performance" (Martin et al., 2006, pp. 159–160). Take as an example the occasion just mentioned above, where Frank initiated a space for a conversation to navigate potential pathways to proceed: "Okay, so fifty-nine left. Out of fifty-nine, how many can we make? So, one of them is fifty-nine pennies, um— [while he was looking to Adam and Maddie]." This opening prompted Adam to suggest making a table within which to arrange the group's choices, and, on their shared piece of paper, he drew an initial table with four columns and a few rows. Maddie pulled the paper toward her side of the table, labeled the columns of Adam's table (1¢, 5¢, 10¢, & 25¢), and started to suggest, with effective participation from Frank, some possible combinations to total fifty-nine cents (see Table 1). At this moment we see the students striking a groove (Berliner, 1994, 1997). Maddie and Frank needed no explanation of Adam's table, nor did Adam attempt to offer an explanation. Maddie didn't seek Adam's permission (and nor did he show any sign that such seeking was expected) to take control of the shared document containing Adam's blank table. Maddie added column headings, and these were not contested in any way. Maddie and Frank then began suggesting possible combinations of coins that would sum to 59. This kind of synchronous participation is characteristic of the coactions that are needed to sustain a collective structure. The metaphor of growth—of expanding possibilities—seems to characterize the students' participation in this episode as they built on and expanded the ideas, concepts, and approaches already developed.

Table 1: The Group's Initial Table Filled with Twelve Suggestions

1¢	5¢	10¢	25¢
59	0		
54	1		
49	2		
44	3		
39	4		
34	5		
29	6		
24	7		
19	8		
14	9		
9	10		
4	11		

Productive Struggle Leading to Divergent Thinking

While trying to lay out all possible combinations to a total of fifty-nine cents in their shared table, the students engaged in an interactional conversation to find an effective way to do this. Their interaction and conversation supported them in considering many potential pathways, looking in many directions. For example, Frank started to fill the table with some possible combinations while whispering, "Um, fifty-nine pennies—." But suddenly, a different potential pathway seemed to present itself to Adam who suggested trying to "get the total amounts [of combinations]; we can get the total amount for, like, if we change this (the fifty-nine) to fifty, and then we had some sort, like, the two combinations of nine (i.e., nine pennies; and four pennies and one nickel)—." Maddie, still wedded for the moment to the idea of finding combinations that made 59 not 50, tried to make sense of Adam's suggestion. She asked him to "wait, wait, wait," and then to "continue." Adam explained his suggestion by stating, "You could change the number to fifty [instead of fifty-nine], and then go from fifty, because it is easier to go from fifty and then multiply the answer by two." While Adam was explaining to Maddie "why [he would] multiply the answer (i.e., the number of combinations to a total of 50 cents) by two," Frank continued filling their existing shared table while whispering words like "forty-seven, um, forty-nine."

Two possible pathways were now in play and the group faced a choice about which pathway to follow. As the students negotiated their varied suggested strategies to proceed, Maddie pondered the options. The "Yes, and" rule, sometimes called the "Do not deny" rule, does not mean that you must agree with everything that comes from fellow learners, but it does mean that you have to listen to them thoughtfully, and fully respect, embrace, and respond to their contributions, which is what Maddie did when faced with the two potential pathways. Maddie made a commitment to their existing strategy, saying to Adam, "Okay, let us actually listen to him (i.e., to Frank)." Following Maddie's suggestion, the group suspended Adam's suggestion (to begin with 50 instead of 59) in favor of trying the strategy that Frank was still pursuing—to lay out all potential options to combine two or more types of coins, and then to find all possible combinations to obtain a sum of 59 cents under each option. They inferred that there were eleven options that were the basis of all possible combinations totaling 59 cents: pennies and nickels;

nickels and dimes; dimes and quarters; pennies and dimes; pennies and quarters; nickels and quarters; nickels, dimes, and pennies; nickels, dimes, and quarters; pennies, dimes, and quarters; quarters and dimes; and all (i.e., pennies, nickels, dimes, and quarters). For a while, all subsequent actions of the group were about developing a fast (or an effective) strategy to find all possible combinations that met these criteria (a collective goal). All the three students' contributions were critical in keeping the mathematics moving forward. Most speaking turns followed the "Yes, and…" rule, and listening to the group mind was evident throughout the whole problem-solving session with this group. Though Adam had interjected a new suggestion (to find combinations to sum to 50, then multiply by two), which could have destabilized the group process, the group collectively agreed to shelve Adam's idea for now and continue working on their present strategy. Although the group elected to listen to the group mind, they later returned to test Adam's suggestion but at the end of their second consideration of his mathematical idea, Adam was willing to abandon his personal motivations and to defer to the group mind (Martin & Towers, 2011) as the group returned once again to using 59 cents as their focus for generating combinations.

The metaphor of divergent thinking characterizes the students' collective process during this part of their problem-solving journey. Two competing solution paths emerged and were given consideration and one was agreed upon and pursued by the group. The group showed that it valued divergent thinking by re-considering the rejected proposal a second time, before ultimately letting it go.

Discussion

In the above extracts we can see that engaging in productive struggle, when viewed through the lens of improvisational concepts such as emergence of multiple pathways, collective structure and striking a groove, and etiquette and group mind, is an iterative process. The students began by considering multiple potential pathways and establishing an etiquette of working together and listening to group mind. At this stage of their problem solving the metaphor of summing forces can be used to describe their actions. At each point during the scene when the group faced or was confronted by a challenge, all the members of the group were eager to contribute their ideas and thoughts and to listen responsively to the others' contributions. The *momentum* that helped students to overcome such challenges and make remarkable progress should be attributed to the whole group as a result of the interaction between their ideas, thoughts, representations, metaphors, gestures, and words.

They gradually refined their problem-solving through striking a groove resulting in a collective structure of focus. Here, the metaphor of growth and of expanding possibilities characterizes their creative process. Students' creative acts were not just about finding their route around/through the problem. Even though they settled on an initial strategy, they still continued to generate alternative possible pathways. By continuing to explore (play with) ideas and thoughts, new spaces of possibility were opened. Learning was not just about zeroing in on a final end product or conclusion but about participating in a continuous process of growing (coming to understand). Later, although the collective structure could have been disrupted as they once again considered competing pathways to a solution, the metaphor of divergent thinking, which characterizes their creative process during this part of their collaboration, helps us to recognize the value of continually seeking out divergent views while still retaining the capacity as a group to defer to group mind to keep the collective moving towards a creative solution. In this data extract, we see students iteratively scope out multiple potential pathways to a solution, 'agree' (without ever discussing rules of engagement) on a way of working together (an etiquette) that allows them to defer to the group mind, develop a collective structure of engagement that affords insight into a credible route to solving the problem, create and reject further potential solution pathways, and again defer to the group mind to coalesce on a solution. This iterative process, we believe, is characteristic of the creative process, and we anticipate that it would be evident in other data extracts featuring collaborating groups who are able to sustain productive struggle in the pursuit of mathematically sound problem-solving.

Implications for Classroom Practice

The iterative process leading from productive struggle to collective creativity suggests a number of implications for classroom mathematics learning. We note that the task offered to this group of students was rich enough to allow for the possibility of multiple potential solution pathways to emerge. According to Martin and Towers (2011), although there exists the potential for many different directions for the 'performance' to take at any point of the scene, it is at the start that "the potential is unlimited…[and] it is here that the widest range of choices are open to the actors" (p. 256). However, for students to sustain productive struggle, the task also needs to afford the possibility of multiple potential pathways to emerge at many points in the solution so that the possibility of better alternative pathways can emerge during problem-solving.

In addition, the learning environment (and this includes structures such as resources offered to students) needs to afford the emergence of collective structure and striking a groove. As we have noted elsewhere (Martin et al., 2006), offering single piece of paper for students to share has proved fruitful in promoting the growth of collective mathematical understanding in that it becomes a place to 'pool' thinking. As we saw in the data presented here, the shared document enabled the emergence of the initial solution idea by providing a single focus for striking a groove based on which "a collectively created structure start[ed] to emerge" (Martin & Towers, 2011, p. 269). Finally, our data suggests that the kind of teaching that supports productive struggle is teaching that models and encourages the kind of etiquette and valuing of group mind that generate good improvisational performances. These students had learned such etiquette in a classroom that valued genuine collaboration, mathematical argumentation, and problem solving.

Conclusion

The students in this problem-solving session are good examples of *attentive and responsive listeners*. Their conversation was fundamentally creative; it required "trust among the group; the ability to listen and to respond to each other; the ability to work without a script or a director" (Sawyer, 2001, p. 196). Thus, they were able to struggle productively by listening to and watching what others were saying and doing and responding accordingly. No comment or gesture was ignored, i.e., mathematical ideas and actions stemming from any one of them became "taken up, built on, developed, reworked, and elaborated by others and thus emerge[d] as shared [structures] for and across the group, rather than remaining located within any one individual" (Martin et al., 2006, p. 157).

As VanLehn et al. (2019) concluded, though, it is not easy to create environments in which this kind of collaborative productive struggle can be sustained and in which there are opportunities for students to "work hard together to solve challenging, open-ended problems that afford many mathematical insights and discussions" (p. 8) and in which successful pedagogy "engages the students in mathematically meaningful, productive, collaborative behavior" (p. 8). Jardine et al. (2003) reminded us though that "children like to work hard—if that work is meaningful, engaging, and powerful" (p. 102). They used the expression "hard fun" to describe this kind of learning, which is rich in productive struggle, recognizing that it is the kind of learning that is called for to thrive in this rapidly changing and challenging world.

Note

[1] A property of the collective, where "everything seems to come naturally; the performers are in interactional synchrony" (Sawyer, 2003, p. 44). Sawyer (2003) suggested this expression based on Csikszentmihalyi's (1990) conception of flow. According to Sawyer (2003), "Csikszentmihalyi intended flow to represent a state of consciousness within the individual performer, whereas group flow is a property of the entire group as a collective unit" (p. 43).

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CIRCULAR REASONING: SHIFTING EPISTEMOLOGICAL FRAMES ACROSS MATHEMATICS AND CODING ACTIVITIES

RAZONAMIENTO CIRCULAR: CAMBIOS DEL MARCO EPISTEMOLÓGICO EN ACTIVIDADES EN LA INTERSECCIÓN DE MATEMÁTICAS Y CODIFICACIÓN

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STEM integration holds significant promise for supporting students in making connections among ideas and ways of thinking that might otherwise remain "siloed." Nevertheless, activities that integrate disciplines can present challenges to learners. In particular, they can require students to shift epistemological framing, demands that can be overlooked by designers and facilitators. We analyze how students in an 8th grade mathematics classroom reasoned about circles, across math and coding activities. One student showed evidence of shifting fluently between different frames as facilitators had expected. The dramatic change in his contributions gauge the demands of the activities, as do the contributions of other students, who appeared to work within different frames. Our findings have relevance for the design and facilitation of integrated STEM learning environments to support students in navigating such frame-shifts.

Keywords: Integrated STEM / STEAM, Computational Thinking, Geometry and Spatial Reasoning, Middle School Education

Introduction

The "STEM" and "STEAM" labels in education (Takeuchi et al, 2020) signal possibilities for integrative experiences involving multiple disciplines. These experiences can be valued as *workforce preparation,* recognizing that interdisciplinarity is increasingly vital in professional STEM fields (National Science Foundation, 2020; Nersessian, 2017). Or, they can reflect the observation that problems in the world of work are seldom confined to a single school subject area (Lesh, Hamilton, & Kaput, 2007). Alternatively, a case for integrative STEAM activities can be based in goals such as enhancing students' motivation and engagement, and increasing the sense of relevance of STEM subjects (National Science and Technology Council, 2018). Recognizing the motivation for engaging in them, the *value* of such integrative STEM activities hinges on learners' successfully constructing productive relations among the integrated disciplines. Lehrer & Schauble (2020) warn that this can be a challenging proposition indeed, showing how activities that promise to connect mathematics with other STEM disciplines can unfold in ways that diverge from teachers' intended learning goals, or can raise thorny questions that participants may not be equipped to navigate. Connecting with the PME-NA conference theme of *productive struggle*, Lehrer and Schauble's (2020) work highlights the challenges (and opportunities) involved in making struggles over mismatches between disciplinary ways of knowing in integrative STEAM activities into productive inter- and meta-disciplinary experiences for learners.

Charting a course for this line of work calls for rich descriptions of the classroom experience of integrative STEAM activities that engage learners at the intersection of epistemic practices fundamental to different disciplines. Integrative STEAM activities of this kind position learners as *boundary crossers* (Akkerman & Bakker, 2011) with epistemic agency to connect mathematical practices of representation and inquiry with those of other disciplines. The construct of boundary crossing is widely studied in the context of professional and organizational learning. To conceptualize what kinds of learning might be possible by positioning students as boundary crossers, and to calibrate the challenges involved, we draw (with caveats) on that literature of professional boundary crossing and interdisciplinarity. A useful review by Akkerman and Bakker (2011) outlines essential themes that are foundational to our analysis. Boundary crossing research typically studies professional practices in which individuals and groups find themselves at the intersection of communities that are concretely embodied in disciplinary and institutional practices that play critical roles in their work lives. In such settings, boundary crossers can pioneer new directions of organizational and professional growth.
In classroom settings, institutional and disciplinary forces are present in very different ways from how they appear in professional settings. Nevertheless, research from the professional context offers us models for how learners might be supported to negotiate tensions at the intersections of disciplines, models that can offer guidance through target stances and forms of interaction. Table 1, below, describes analogies that we leverage between professional STEM and classroom STEM education contexts.

Table 1: Tracing the key concepts of boundary crossing and epistemic cultures and frames—between professional STEM and STEM education contexts

Key Concept	Manifestation in Science & Technology Studies and organizational research	Manifestation in educational activity designs and analyses
Boundary Crossing	Shared problems and enterprises create the need for transdisciplinarity. Stable procedures and institutional structures emerge that reflect the interface between distinct disciplinary cultures (Osbeck & Nersessian, 2017).	Activity designs create the need for students to construct connections across subject areas. Diverse participation reflects the interface between distinct ways of thinking. New hybrids prove their *viability* by being useful in practice (Brady, Eames, & Lesh, 2015).
Epistemic Cultures and Epistemic Frames	Epistemic cultures (Knorr Cetina, 1999) have characteristic discourses and representations for concepts relevant to their shared enterprise. Shifts appear in boundary crossing, facilitated by boundary objects and by "creoles" (Galison, 1997) to mediate the boundary.	In talk and interaction, different participants interpret activity settings using interpretive frameworks built out of disciplinary and everyday knowledge resources. Breakdowns in activity can reflect clashes between these epistemic frames (Hall & Stevens, 2015) and provoke repair, negotiations, and shifts.

In this paper, we analyze a classroom episode, in which we describe the distinct *epistemological frames* (Scherr & Hammer, 2009; Thoma, Deitrick, & Wilkerson, 2018) and the *shifts* between such frames, which the facilitators assumed students would navigate. Understanding the demands we are making of students as designers and facilitators of integrative

STEAM activities, and learning how to support students are two critical issues of research and praxis for making such activities scenes for productive struggle in mathematics education.

Theoretical and methodological approaches

The *framing* of a situation or interaction reflects participants' determination of "what is going on" there (Goffman, 1974). Faced with a barrage of information that is overwhelming and often conflicting, humans have to make snap decisions about what "kind" of situation they are in, in order to determine what is relevant, what the rules are, and how they should act. Framing is both interactional and individual; contexts can invite particular frames, but frame signaling can be ambiguous (Wisittanawat & Gresalfi, 2020) or can suggest different frames to different people (Hand, Penuel, & Gutierrez, 2012). It is remarkable, then, that this can be mostly done unthinkingly and without uncertainty rising to conscious experience, especially since framing is a matter of shared agreement and coordination (Goffman, 1974).

In designed or otherwise exceptional environments, however, questions and even disputes about framing can come to occupy the foreground (DeLiema, Enyedy, & Danish, 2019). Novel settings make it possible for multiple candidate framings to emerge, as people look for contextual clues about the tools, participation structures, language, and interactions that are appropriate. Such settings can offer different frames for different people (Hand, Penuel, & Gutierrez, 2012), or make it ambiguous to both participants and outside observers what is actually going on (Gresalfi, Brady, Knowe, & Steinberg, 2020).

Within learning environments, such indeterminacy in framing can be seen as a liability, making it more difficult for individual students to participate or more challenging for teachers to facilitate a student group in activities that require coordination. On the other hand, moments that provoke frame indeterminacy can also offer the potential to bring together different interpretations of shared experience, and thus could also offer powerful learning opportunities. Goffman's (1974) extended analysis of frames and their transformations shows how frame breaks and frame disputes surface fundamental assumptions about the "primary frameworks" that underlie social interactions in various contexts. They thus offer an opportunity to see and discuss the consequences of these underlying frameworks. In the context of frames governed by disciplinary ways of seeing and acting, frame breaks and frame disputes offer a setting where the nature and consequences of epistemic frameworks that are fundamental to the philosophy of a discipline can be made palpable and experiential.

Setting and Participants

We focus our analysis on one session from an 8th grade mathematics classroom at an urban public-school in the southeastern United States. The teacher, Ms. T, has been a co-design partner with the authors in an NSF-funded project (CAMPS, NSF#1742257), to design and study learning environments that integrate mathematics, computer science, and art. To this point in the project, Ms. T and the authors had collaborated in an informal-learning setting, a summer "Code Your Art" camp. In the 2018-19 school year, at Ms. T's initiative, the project team worked to adapt activities and ideas from "Code Your Art" camp to Ms. T's math class on "Code Fridays." Throughout the school year, the research team worked with her to co-design and co-facilitate coding activities on many Fridays, using the NetLogo (Wilensky, 1999) modeling environment. Ms T's school is a community middle school serving a racially and economically diverse population, and the class that experienced Code Fridays sessions comprised 34 students.

On the day in question, Ms. T reviewed practice problems for a high-stakes state assessment before moving on to coding. Facilitators' in-the-moment decisions about how to transition from this phase to the Code Fridays activity created an opportunity for integration across math and computation around *circles* in a sequence of two conversations. Students reasoned very differently about circles across these two instructional moments, leading to the appearance that they did not make connections between the same set of ideas as they switched activities. One possible interpretation is that students demonstrated a failure to "transfer," in that resources and ideas leveraged in one activity were not leveraged in the second. Instead, we argue that the different resources students brought to bear on questions about circles suggested differences in their epistemological framing (Scherr & Hammer, 2009; Thoma, Deitrick, & Wilkerson, 2018) of the two activities, and revealed mismatches between some students' framing and facilitators' expectations. Recognizing the roles of framing and frame-expectations focuses our attention on features (and shortcomings) of our design and facilitation, rather than on shortcomings in students' thinking.

Methods of Analysis

We apply epistemological frame analysis to our focal episodes, to understand how students experienced and responded to signals for framing of two successive activities about circles. Data analyzed in this paper include video from two sources, a camera in back of the room positioned to capture the teacher's projected computer, and a second camera set up in the front to capture students' talk, gestures, and interaction at their tables. Through multiple viewings of the record, we narrowed our focus to two brief episodes involving circles—one from the math exam practice and the other from the coding session. We used discourse analysis, including an analysis of gesture, to investigate how different epistemological framings were recruited with respect to expected framings across the two activities.

Findings

We found that across the two focal activities, distinct epistemological framings of circles emerged. We identified one student, Mateo (a pseudonym), who navigated the shift between these two activities successfully (i.e., as the teacher and researchers had expected). We studied the forms of expression and argumentation that he exhibited, as a measure of the difference in framing. We also identified other students in the class, whose contributions appeared to come from frames less well aligned with the expectations of the facilitators. These students did not appear to lack conceptual sophistication or resources; rather, their framing prevented them from participating in the discussion as the facilitator intended. Our goal in the analysis was to gauge the nature of the epistemological discontinuity between these mathematics and coding activities: both the success that Mateo had in constructing compelling accounts across the two settings, and the challenges other students faced, help to characterize this discontinuity between activities.

Mathematics activity: Mapping given numbers to elements of the area formula

The class session began with a review of practice problems for the state exam. Problem 37 asked for the area of a semicircle, given that its diameter measured 6 units. Mateo volunteered to share his work:

Mateo: So, uh, since we know the formula for the area of a circle is, pi times uh radius squared, so for half the circle, we just need to do, uh one-half of pi times radius squared. So, I did…so I did for the radius I found that it was 3 because the radius is half of that, the the diameter. So I did uh uh 3? Uh, squared? which is 3 times 3 which is 9, and I did 9 times, uh, 3.14, divided by two, so I got 14.13.

Mateo's contribution suggested he framed the problem as a challenge of *mapping* given elements of the figure to their meanings in a memorized formula, and enacting the operations called for by that formula. His explanation took care to unpack each element in the memorized area formula (the relation of a semicircle's area to that of a circle; the value of p; the meaning of r and its relation to the diameter; and the meaning of squaring), which was sensitive to classmates who might have missed any of these elements. Moreover, his use of pronouns (e.g., "we know," "we just need," "I did") suggested that Mateo was positioning this mapping against a backdrop of communal and normative mathematical knowledge, which authorize his procedure. Finally, Mateo's manner of pointing and gesture-writing in the air with his pencil as he provided his explanation (Figure 1) is an instance of what McNeill (1992) calls an *observer-viewpoint* gesture. Together these features suggest he is visualizing a figure and that his reasoning was occurring in a mapping between recognized inscriptions and arithmetic calculations.

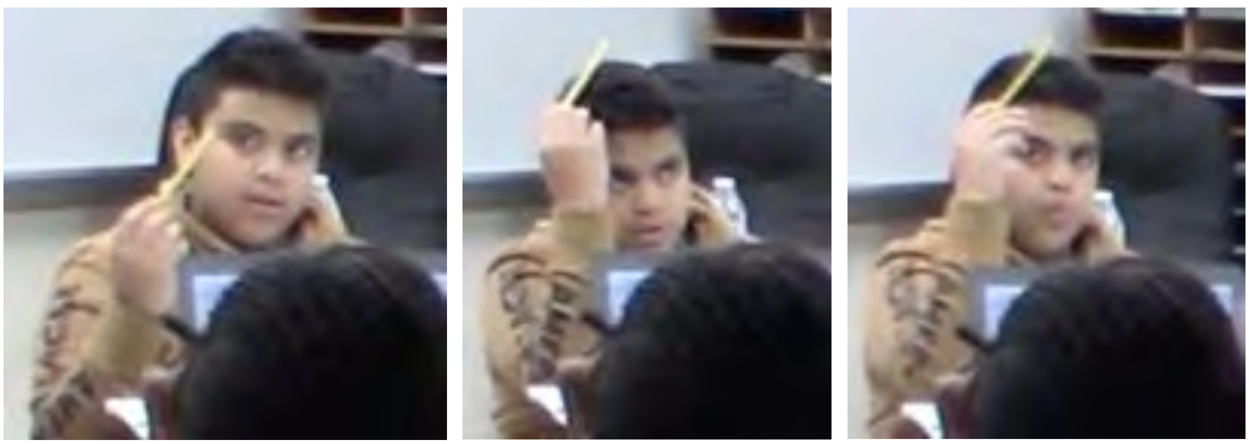

Figure 1. Mateo wrote in the air with his pencil as he described steps to calculate the area of the semicircle.

Mateo's contribution expressed a coherent framing, but his was not the only framing possible. In volunteering an alternative solution, ("I have another way") Edgar made a contribution that framed the activity in terms of voicing diverse strategies for sense-making, a framing valued in Ms. T's classroom at other times:

Edgar: Well, what I do is I multiply uh 6 times the 3.14, and from that I think I got the uh, it was like, yeah, 18.84, then then I multiply it, multiply that number by 6, again, and I get, like, 113.04, and then I divide it by 4 and then half.

Edgar's solution was both correct and well-reasoned. As he later explained, squaring the diameter (twice the radius) and then dividing by a factor of 4 ("since I multiplied twice...times two times two") accommodated the givens of the problem. And with a calculator, his method was no more computationally cumbersome than Mateo's. Yet Edgar's approach appeared to be out of sync with the framing of the activity assumed by the teacher (and the students who followed her lead). Edgar's reasoning was questioned ("Where'd you get the four from?") and critiqued ("you added a bunch of unnecessary steps") by other students. Moreover, Ms. T reinforced these responses, saying "[Edgar], don't confuse yourself. On the test, you don't have that much time to go through all those steps, ok? Stick to the formula…."

Edgar's status in this classroom was quite high; indeed, he had been celebrated minutes earlier for using "a process of elimination" to reason about multiple-choice responses. Yet Edgar's own first-person pronoun use positioned his work as an *idiosyncratic* approach ("What I do," "Then I multiply"), in contrast with Mateo's normative "we." Finally, faced with the responses of classmates and Ms. T, Edgar explained "that's how I've been doing it…because I have no idea…." Overall, the differential rhetorical success of Mateo and Edgar suggest that Edgar's solution was received as less responsive to the "epistemic game" (Shaffer, 2006) of efficiently filling the "epistemic form" (Collins & Ferguson, 1993) of the formula.

Coding activity: Reasoning from the intrinsic perspective of the turtles.

On turning to Coding, the class was introduced to NetLogo *turtles* (agents that can move). The researcher leading the activity, CB, set the stage by creating 100 turtles on the projected computer, noting they were "piled up" at the center of the screen. With students following along, he typed forward 5, to be run by all turtles.

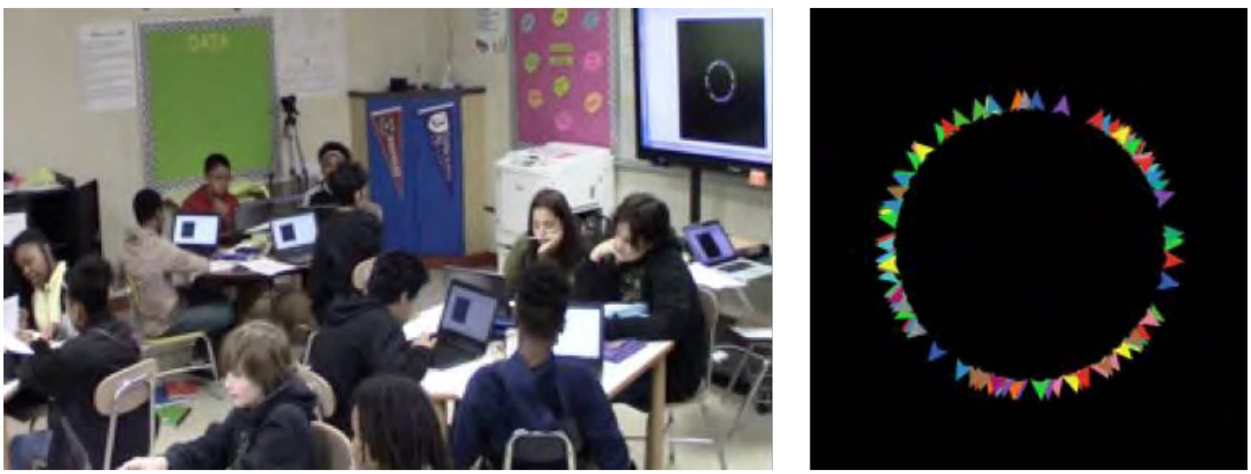

Figure 2. When student created 100 randomly-oriented turtles and executed forward 5**, a circle was formed.**

The turtles each moved forward 5 steps from the center of the screen and created a circle (Figure 2), which surprised the class. Making a connection to the first half of the session, CB asked, "By the way, since you guys were just talking about circles, what's the RADIUS of this circle?" Students shouted out three answers: "Three-sixty!" "Five!" and "Two point five!" CB then asked the class to discuss their reasoning in groups and different groups came to different conclusions. Marissa and Elena shared first and then Mateo joined the discussion:

Marissa: 2.5
CB: 2.5? And why?
Elena: The the the diameter is 5, all the way across. And the radius is half of that.
CB: Ok, so IF the diameter was 5, then the radius would be 2.5, for sure. How…what is the diameter of this guy?
Mateo: 10.
CB: 10. Why?
Mateo: Because if all the patches are going forward five, all facing in different directions/

CB: /turtles//
Mateo: //Ah turtles. So, they're all going 5 in every direction. The diameter's going to be 10.

<u>Figure 3</u>. Mateo gestured to show the movement of two oppositely-oriented turtles and the diameter they made.

The class's discourse about the circle of turtles illustrates the discontinuity between the two activities and the CB's expectations about how students would shift their framing to participate. On one hand, Marissa and Elena's group reasoned within the epistemic frame of mapping given values to formulas, as provided by the prior math activity. They interpreted the number 5 as mapping to the turtle-circle's *diameter*, as had happened earlier in Problem 37. Thus they argued for a radius of 2.5. In contrast, Mateo's explanation revealed a different epistemic frame and a new form of reasoning, distinctive to the context of agent-based programming. From the group of 100 turtles, he selected an imagined pair facing in opposite directions. With two hands, he gestured to simulate their movement, and then gestured (Figure 3A-B) to show how they would produce a line segment passing through the center of the circle, 10 steps long. Finally, (Figure 3C) he interpreted this to be the diameter of the circle formed by all the turtles.

Each of Mateo's moves arises from successfully constructing and operating a mathematization of the computational agent-based environment. First, the selection of two turtles from the *agentset* of 100 relies on a characteristic feature of computational simulations, which use randomization to present a finite sample of an infinite outcome space. The "circle" is only *suggested* by the turtles' bodies, and yet the "professional vision" (Goodwin, 1994) of a mathematically-attuned user of this representation can reason from the particular sample of turtles to imagine two of them oriented with precisely opposite headings. Next, Mateo uses a peculiar species of communicative character-viewpoint gesture (McNeill, 1992; Ochs et al. 1994), in which he embodies the *pair* of turtles with his hands, positioning his own head as the invisible center of the constructed circle. This gestural achievement stabilizes the mathematical objects (center, radial points, radius, diameter) that are necessary to align the situation and enable a link with the forms of reasoning about circles used Problem 37 can be applied effectively.

Conclusions and implications for future work

Mateo was successful in reasoning across the two activity contexts. But to do so, he had to make a substantial leap between epistemic frames. The differences in reasoning showed in the "embodied modeling" approach he employed (cf, Wilensky & Riesman, 2006) and in the different gestural resources that he recruited. The connection that CB assumed would be straightforward, in fact required a significant conceptual reorganization. Many students in this class exhibited strong and flexible resources for reasoning about circles, across each of the two activity contexts, as shown in Edgar's example. Nevertheless, differences and shifts in the discourse and forms of reasoning demanded by the two activities suggested that the "circle" in the math problem and the "circle" formed by the NetLogo turtles were substantially different kinds of objects. We take this example as indicating a challenge for the design and facilitation of activities that aim to provide STEM integration. Specifically, we must recognize that in moving across disciplinary contexts, we may be unknowingly asking students to bridge between epistemic frames, to carry ideas and resources from one domain to the other.

STEM integration has high potential. Indeed, treating the circle from an agent-based perspective offered Mateo a significant resource for mathematically conceptualizing it. Mateo's gestures suggest that he was able to use the turtles to imagine a circle as a *set* or a *locus* of *points* (turtles), and to infer that relations among those turtles/points gave rise to a property of the circle (its diameter). Utilizing a "point-set perspective" is typically viewed as a notable achievement in mathematics. Reasoning in this style, (Figure 3) Mateo leveraged the relation between two turtles and the vacated "center" (his head), as an embodied support for describing the emergent circle's diameter. His description was compelling in the classroom discourse, but it is not clear that his turtle-based reasoning and means of bridging computational and mathematical worlds were fully shared. Providing disciplinarily hybridized learning environments where students can reap the benefits of bridging the disciplinary divide between mathematics and computer science is a challenge for both research and praxis.

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CHARACTERIZING OPPORTUNITIES FOR MATHEMATICAL AND SOCIAL PARTICIPATION: A MICRO-ANALYSIS OF EQUITY IN SMALL-GROUP ZOOM INTERACTIONS

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This study focused on issues of equity related to small-group participation in a distance learning calculus class. Equity is defined as the fair distribution of opportunities for students to participate and learn. I examined how opportunities for mathematical and social participation were constructed through acts of positioning for four students. Findings suggest that creating fair opportunities requires: 1) conceptualizing opportunities for participation as connected to students' positionings and developing identities, 2) acknowledging that what counts as an opportunity for one student does not necessarily count as an opportunity for another student, and 3) leveraging both mathematical and social contributions in creating equitable, supportive, and intellectually rich learning communities.

Keywords: Classroom discourse; Equity, inclusion, and diversity; Online and distance education

Educational inequities constrain the opportunities students have to participate and learn in math classes (Cobb & Hodge, 2002; Esmonde, 2009). Often rooted in societal-level biases, inequities are promulgated by patterns of marginalization (e.g., racialization, sexism) that distribute power unfairly through classroom interactions (Esmonde & Langer-Osuna, 2013; Gutiérrez, 2012; Martin, 2009). Inequities occur when some students are positioned as having more to contribute than others (Herbel-Eisenmann, Wagner, Johnson, Suh, & Figueras, 2015) and/or face additional barriers to participation (Leyva, Quea, Weber, Battey & López, 2020). Pursuing equity requires studying classroom interactions at a micro-scale to understand how inequities play out for individual students in specific classroom contexts (McDermott & Roth, 1978). Equity is conceptualized as the fair distribution of opportunities for students to engage in meaningful ways, supporting the development of rich content knowledge and positive identities (Esmonde, 2009; Schoenfeld, 2014). This study focuses on micro-level issues of equity related to small-group participation in math. I examine how opportunities for mathematical and social participation were constructed through acts of positioning in a distance learning calculus class.

Small-group learning tasks hold potential to address issues of equity by engaging all students in meaningful content while also supporting students in building positive identities as thinkers, learners, and community members (Boaler & Staples, 2008; Cohen & Lotan, 2014). While potential benefits are substantial, implementing successful small-group learning is not easy (Barron, 2003), and working with virtual learning constraints makes it even harder (Wong, 2020). In video-conferencing platforms like Zoom, students face additional physical and social barriers to interaction (e.g., background distractions, ambiguous body language, and limited visibility of student work). For some students, Zoom breakout rooms have been the only opportunity to interact with people outside of their immediate families, exacerbating the necessity for small-group interactions to support both students' academic and social needs.

This study explored the interactions of four students working on a small-group task, focusing on the construction of opportunities for mathematical and social participation. Specific research questions are: *1) How did each student contribute during the task? 2) How were students invited to contribute? 3) How did group participants respond to students' contributions?*

Theoretical Framework

This study is informed by sociocultural and situated theories that consider learning as occurring through participation in cultural activities (Lave & Wenger, 1991; Vygotsky, 1978). Classroom participation is defined broadly and includes more than content-related talk. Non-verbal forms of communication are believed to be valuable for learning (Esmonde, 2009), and "off-task" or social participation is deemed relevant and potentially productive (Gholson & Martin, 2014; Langer-Osuna, Gargroetzi, Munson, & Chavez, 2020). Learning is defined as changes in students' participation in collective classroom practices (Lave & Wenger, 1991). Students' participation in learning activities is a function of the opportunities they are given to participate (Gresalfi, Martin, Hand, & Greeno, 2009); if opportunities to participate are unfairly distributed, then learning will be inequitable as well. Opportunities to participate are shaped by the roles and responsibilities students are assigned through acts of positioning (van Langenhove & Harré, 1999). Through positioning, racialized and gendered narratives (i.e., storylines) come into play (Esmonde & Langer-Osuna, 2013; Reinholz & Shah, 2018). As students interact, expectations are negotiated for what each student can and should do, distributing power among students (Herbel-Eisenmann et. al., 2015). Equitable learning processes require that each student be positioned as a valuable contributor to their own and their peers' learning. Students positioned with competence and authority have more opportunities to participate in consequential and influential ways, and therefore have better access to rich mathematical learning and identity development (Cohen & Lotan, 2014; Gresalfi et al., 2009; Langer-Osuna, 2011).

Methods

Data Collection

Participants. Classes used Zoom for fully distanced learning at the time of observation (Feb. 2021). The focal group consisted of four 12th grade students in a Calculus AB course at an urban public high school: Yonas, Guadalupe, Hosein, and Elijah (pseudonyms). Guadalupe is the only student in the group who identifies as female, and Elijah is the only student who identifies as White. Ms. B was the calculus teacher. Mr. K was a student teacher. Ms. F was the researcher (began daily observations Sept. 2020). Participants are shown in Figure 1 (with permission).

Figure 1: Participants in the Focal Group on Zoom

Task. Students joined Zoom breakout rooms to work on a Related Rates problem. They had not yet received formal instruction on this topic. The teacher wanted students to think about the underlying ideas before formalizing solving strategies. Students were instructed to work with their teams. The problem read: *1. A ladder leans against a wall. It begins to slide down the wall.*

Olanoff, D., Johnson, K., & Spitzer, S. (2021). *Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Philadelphia, PA.

Does the top of the ladder move at the same rate as the bottom of the ladder? 2. Suppose the bottom slides away from the wall at a rate of 1 ft/sec. How fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall? Assume the ladder is 10 ft.

Video. Video of the focal group was recorded using Zoom functionality. The video was approximately 10 minutes long, the amount of time the students spent working on the task.

Data Analysis

Video was transcribed for speech and salient expressions/gestures, then divided into *contributions*. A contribution was uninterrupted speech by one person of a single *type* (defined below). Sometimes a single talk-turn contained two contributions, such as when Yonas began reading the problem aloud (contribution 1), then shared mathematical reasoning (contribution 2). All talk was coded as either a mathematical or social contribution. Codes were assigned to every contribution based on *acts of positioning* (van Langenhove & Harré, 1999), operationalized into three categories: contribution types, contribution invitations, and contribution responses.

Contribution types. Each contribution is an act of positioning (Gholson & Martin, 2014). The contributing student positions themself through the type and content of their contribution. Did the student contribute sound mathematical reasoning? Or did they contribute a comment that made everyone smile? Contribution *types* were coded as: Asks a question, Makes a comment, Shares mathematical reasoning, Shares solution with reasoning, Shares solution without reasoning, Expresses agreement, Expresses disagreement, or Reads the problem aloud.

Contribution invitations. Each contribution invitation is an act of positioning (Langer-Osuna, 2011; Radinsky, 2008), including explicit and implicit prompts. Students position themselves and each other depending on how contributions are prompted. Was a student called on by name to contribute a math idea? Or did they interrupt another student to share an idea with seemingly no invitation at all? Contribution invitations were coded as one of the following: Participant actions, Silence, or Interruptions. Invitations were coded as *Participant action* when the words or actions of someone in the group prompted a contribution from someone else, either explicitly or implicitly. The person whose actions invited the contribution was also coded. Invitations were coded as *Silence* if a contribution was made when no one was speaking and was not connected to previous contributions. *Interruptions* were coded when someone cut off another person's contribution, indicating a lack of invitation.

Contribution responses. Participants' responses to contributions are acts of positioning as well (Anderson, 2009; Hand, 2010). Students are positioned by their peers and teachers through the reactions they get to the contributions they make. Is the contribution met with explicit affirmation? Is the validity of the contribution challenged? Or is the contribution ignored? Contribution responses were coded as: Positive (verbal agreement or smile), Negative (verbal disagreement or interruption), or Neutral (silence or no change in facial expression).

Findings

Collectively, findings address the central question of how opportunities for mathematical and social participation were constructed for each student. Findings are organized by the specific research question: *1) How did each student contribute during the task? 2) How were students invited to contribute? 3) How did group participants respond to students' contributions?*

Contribution Types

Findings in this section address the question: *How did each student contribute during the task?* Contributions were quantified by totaling the number of words spoken and the number of contributions made by each student, shown in Table 1. Words and contributions were categorized as either mathematical or social. The number of words metric provides insight into the amount of airtime occupied by each student without any indication of contribution quality.

Table 1: Number of Mathematical & Social Words and Contributions by Student

	Yonas	Guadalupe	Hosein	Elijah
# of Mathematical Words Spoken	454	55	227	213
# of Social Words Spoken	9	130	1	0
TOTAL # of Words Spoken	**463**	**185**	**228**	**213**
# of Mathematical Contributions	23	4	10	10
# of Social Contributions	2	9	1	0
TOTAL # of Contributions	**25**	**13**	**11**	**10**

Yonas spoke more than twice as many words as his peers, resulting in a total of 25 contributions, 23 of which were mathematical. Hosein and Elijah both spoke over 200 words, resulting in 10 mathematical contributions each. Hosein had a one-word social contribution as well. Unlike her peers, Guadalupe's social words and corresponding social contributions more than doubled her math words and math contributions. Table 2 shows the types of contributions for each student.

Table 2: Number of Mathematical & Social Contributions by Type by Student

# of Contributions by Type			Yonas	Guadalupe	Hosein	Elijah
	Mathematical	Asks a question		1	3	
		Makes a comment	7		2	1
		Shares mathematical reasoning	7		3	1
		Shares solution with reasoning	2	1	1	2
		Shares solution without reasoning		1		
		Expresses agreement	3	1	1	4
		Expresses disagreement	1			2
		Reads the problem aloud	3			
	Social	Asks a question		2		
		Makes a comment	2	7	1	

All students shared at least one solution with mathematically valid reasoning and each expressed verbal agreement with a peer at least once. Yonas and Elijah were the only students who expressed disagreement, and Guadalupe and Hosein were the only students who asked questions. Based on this data, it appears Yonas contributed the most in terms of the quantity of math interactions, and Guadalupe contributed the most in terms of social interactions. Hosein and Elijah both made considerable math contributions, while Guadalupe shared some math ideas too.

Contribution Invitations

Findings in this section address the question: *How was each student invited to contribute?* Invitations were categorized as either mathematical or social. From there, invitations were determined to be connected to participants' actions, silence, or interruptions. The number of invitations by type and by student are presented in Table 3.

Table 3: Invitations to Contribute by Type and by Student

	Mathematical				Social			
	Participant Actions	Silence	Interruptions	Total	Participant Actions	Silence	Interruptions	Total
Yonas	12	10	1	23	2	0	0	2
Guadalupe	4	0	0	4	8	1	0	9
Hosein	4	5	1	10	1	0	0	1
Elijah	6	2	2	10	0	0	0	0

Table 3 shows that invitations for Yonas's and Hosein's mathematical contributions were relatively balanced between participant actions and silence (Yonas: 12 participant actions vs. 10 silence; Hosein: 4 participant actions vs. 5 silence). The relatively high number of silence invitations suggest that Yonas and Hosein were comfortable initiating math contributions without an explicit prompt. Yonas and Hosein also had one interruption each, but both students apologized. When Guadalupe was speaking, Yonas said, "Oh my god. Wait. Sorry, sorry. I just had a theory." When Yonas was speaking, Hosein said, "Um, Yonas? I'm sorry to interrupt. I have an idea." Elijah's math contributions were most often connected to other participants' actions (6 out of 10 invitations). Elijah also made the same number of contributions when the room was silent as when someone else was speaking, suggesting that if Elijah had a thought to share, he shared it regardless of what other people were doing. Unlike Yonas and Hosein, Elijah offered interruption apologies. All of Guadalupe's math contributions were prompted by other people's actions; she made no math contributions while the group was silent, nor interrupted anyone. These data suggest that Guadalupe was not as comfortable as her peers initiating math contributions on her own. Participants' actions dominated Guadalupe's invitations for social contributions as well. However, she did make one social contribution that occurred during silence, suggesting she was more comfortable initiating social contributions than mathematical.

The majority of all contributions were prompted by participants' actions; participants included Ms. F (researcher), Ms. B (teacher) and Mr. K (student teacher) in addition to the four students. Ms. F was there the entire time. Ms. B visited the group once for 28 seconds, and Mr. K visited the group once for 33 seconds. Ms. B's and Mr. K's visits overlapped by 8 seconds. Figure 1 provides a closer look at the invitations that were attributed to *participant actions*. The figure contains two rectangles per student, one for math invitations (purple) and one for social invitations (blue). The top two rectangles show data for Yonas (bold outlined name). Orange arrows point away from Yonas representing the number of times Yonas's actions prompted a contribution from someone else. For example, the orange arrow from Yonas to Hosein in the top left rectangle shows that Yonas's actions prompted two math contributions from Hosein. Blue arrows point toward Yonas representing the number of times someone else's actions prompted a contribution from Yonas. For example, the blue arrow pointing from Hosein to Yonas shows that Hosein's actions prompted three of Yonas's math contributions. The thickness of arrows corresponds to the number of invitations, also shown as a number next to each arrow.

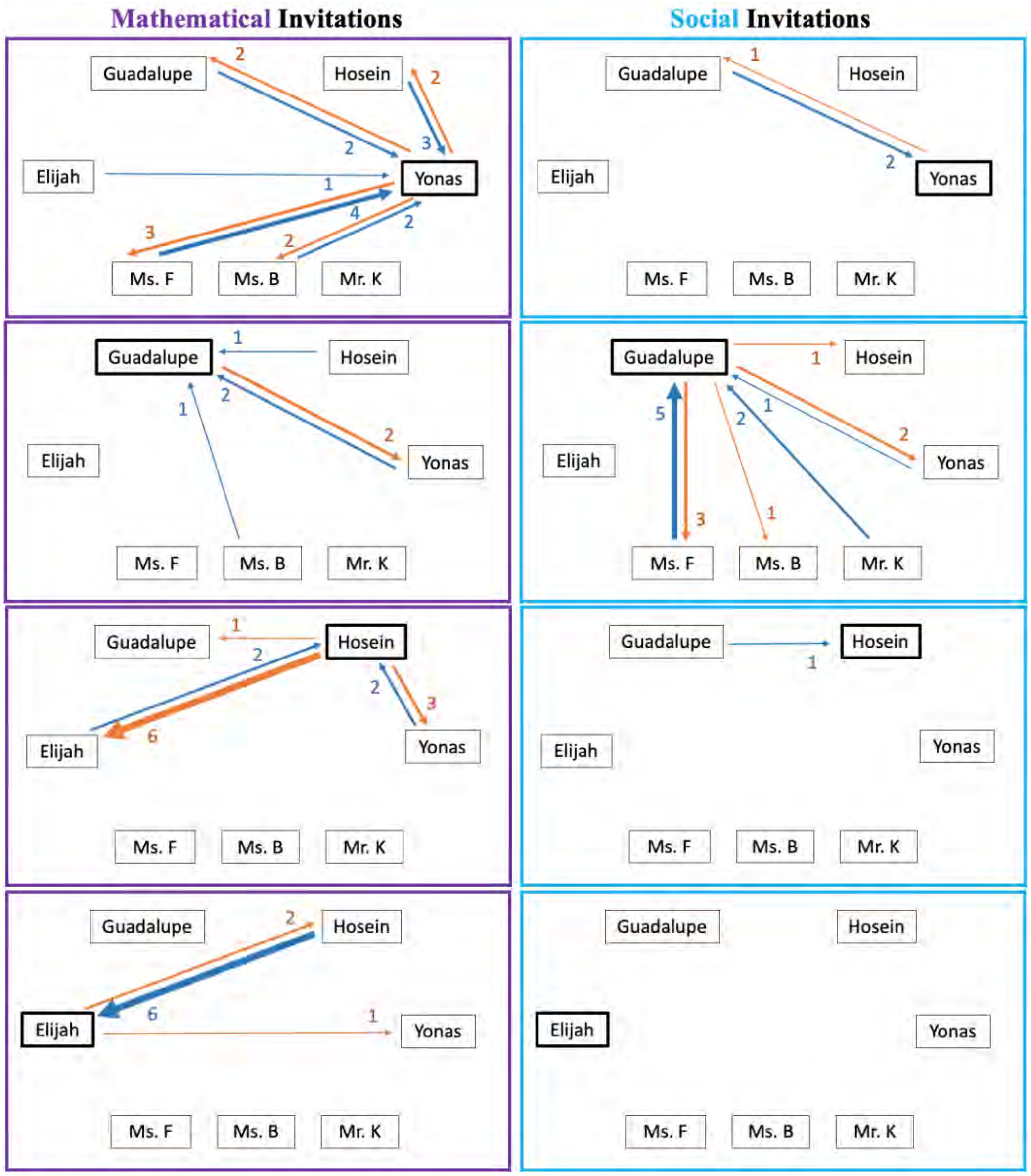

Figure 1: Participant Action Invitations *by* and *to* each Student

The top two rectangles show that Yonas connected through participant action invitations with everyone except Mr. K. Interactions between Yonas and others were relatively balanced and reciprocal; he interacted roughly the same amount with each person, and invitations *by* and *to* each person were relatively even. Elijah was an exception, with just one interaction with Yonas.

The next two rectangles, highlighting Guadalupe's interactions, show that she was connected to everyone except Elijah, and most of her interactions were social. In fact, Guadalupe was involved in *all* of the social contributions that took place during this task; she either made the contribution or her actions invited someone else to make a social contribution. Most of Guadalupe's social interactions involved adults, Ms. F in particular, and can be characterized as friendly, casual, and often humorous. For example, when Ms. F first entered the breakout room, Guadalupe greeted her with, "[Ms. F], oh my God! I get so excited!" *Ms. F's appearance in the group invited Guadalupe's contribution*. In response to Mr. K's sleepy appearance, Guadalupe teased, "[Mr. K], you look hecka bored." *Mr. K's appearance invited Guadalupe's contribution.* And, in response to Yonas's virtual whiteboard drawing, Guadalupe commented with sarcasm, "very sturdy looking ladder!" *Yonas's drawing invited Guadalupe's contribution.* Guadalupe made only four mathematical contributions, the fewest in the group. Her first math contribution was invited by Hosein's direct question, "[Guadalupe], what are you thinking about [the problem]?" The second contribution was an expression of agreement (i.e., "I agree with you,") in response to an explanation shared by Yonas just after Ms. B joined the room. The third contribution was invited by Ms. B's question to the group, "Are you guys saying yes or no?" Guadalupe's final mathematical contribution was a question she asked Yonas about what he was doing, invited by Yonas's virtual white board drawing.

The next row of rectangles shows that most of Hosein's interactions occurred with Yonas and Elijah. Two of Hosein's math contributions were prompted by Elijah's actions and two by Yonas's. The bottom two rectangles highlight the very limited scope of Elijah's interactions. Elijah interacted almost exclusively with Hosein with six of his contributions prompted by Hosein's actions. There were several back-and-forth math conversations between Hosein and Elijah which sometimes included Yonas peripherally, but never Guadalupe. One example occurred toward the end of the discussion when Hosein asked the group, "So, does that mean that the top falls twice as fast as the bottom?" *Silence invited Hosein's contribution.* Elijah responded right away, "Um. I don't think it's twice as fast because it's six verses eight." *Hosein's question invited Elijah's contribution.* Hosein explained further, "No, but it has to move an additional four on the bottom compared to the eight that it has to move at the top." *Elijah's comment invited Hosein's contribution.* Elijah contemplated Hosein's response, saying, "Hmm… True. Hmm… Interesting." *Hosein's comment invited Elijah's contribution.* This 2-person exchange illustrates the type of back-and-forth conversation Elijah engaged in with only Hosein.

Contribution Responses

Findings in this section address the question: *How did participants respond to each student's contributions?* Every contribution received a response from each person who was in the room at the time of the contribution, coded as positive, negative, or neutral based on participants' words and actions. Each contribution received 4-6 responses, depending on how many people were there. The four students and Ms. F were in the room the entire time. Ms. B and Mr. K were there for less than a minute each. Table 4 shows responses for each student's contributions.

Table 4: Responses to each Student's Contributions

	# of Mathematical		Positive		Negative		Neutral		# of Social		Positive		Negative		Neutral	
	Contributions	Responses	#	%	#	%	#	%	Contributions	Responses	#	%	#	%	#	%
Yonas	23	95	9	9%	3	3%	83	87%	2	9	5	56%	0	0%	4	44%
Guadalupe	4	18	0	0%	1	6%	17	94%	9	39	17	44%	0	0%	22	56%
Hosein	10	40	8	20%	2	5%	30	75%	1	5	4	80%	0	0%	1	20%
Elijah	10	40	0	0%	2	5%	38	95%	0	0	n/a	n/a	n/a	n/a	n/a	n/a

The majority of responses to math contributions were neutral for all four students. However, Yonas and Hosein received at least some positive responses to their math contributions (Yonas: 9 positive; Hosein: 8 positive), mostly in the form of verbal agreement (e.g., "yeah"). Neither Guadalupe nor Elijah received positive responses for any math contributions. On the other hand, all students received at least one negative response. When Yonas interrupted Guadalupe to share an idea, this counted as a negative response for Guadalupe. Negative responses also occurred when someone disagreed with an idea that was shared. For example, in the Elijah-Hosein conversation shared previously, Elijah responded negatively to Hosein's suggestion that the top of the ladder fell twice as fast as the bottom by disagreeing with Hosein. Then Hosein responded negatively to Elijah by further supporting his initial claim. Elijah was eventually convinced by Hosein's explanation and, consequently, responded positively to Hosein's final contribution.

Social contributions received more positive responses than math contributions, most coming in the form of smiles and laughs and many involving the adults. For example, shortly after Mr. K joined the room, Guadalupe accused him of looking “hecka bored.” (See Figure 1 in Methods.) Guadalupe then admitted that she sometimes turns off her camera in class so she can lie in bed. She clarified by saying, "but not in this class. Never in this class." Smiling, Hosein responded, "Never." (This was Hosein's one and only social contribution.) Laughing, Yonas responded, “Jeez, does that actually happen?” Guadalupe, Hosein and Yonas all received positive responses to these social contributions from everyone except Elijah. There were big smiles and chuckles from Ms. F and Mr. K, but Elijah's expression did not change. In fact, Elijah did not smile once during the task. Even though Yonas and Hosein did not speak many social words, they indicated their support of Guadalupe's numerous social contributions through their frequent smiles.

Discussion

The goal of this study was to examine how opportunities for mathematical and social participation were constructed through acts of positioning during a small-group task. Students in the same classroom were positioned differently through their contributions. Yonas was positioned as a collaborative, talkative math contributor who shared his thinking freely, had easy access to the conversational floor (Erickson, 2004), and engaged in social interactions either directly with Guadalupe or indirectly by listening and smiling. Guadalupe was positioned as a jovial, caring, social contributor who brought smiles to participants' faces and shared math ideas when asked explicitly. Hosein was positioned as an inquisitive and polite math contributor who offered his own ideas and asked other people for theirs. Elijah was positioned as a deep-thinking math contributor who shared ideas and opinions freely and was oblivious to social norms.

These various positionings had implications for the opportunities students had to contribute to the group's collective learning experiences. Opportunities for mathematical and social participation looked different for different students. What counted as an *opportunity* for one student to participate did not count as a genuine *opportunity* for another. For example, silence in the breakout room constituted a clear opportunity for Yonas, Hosein, and Elijah to offer mathematical contributions, but not Guadalupe. If someone else was talking, that too counted as an opportunity for everyone except Guadalupe, though Yonas's and Hosein's interruption apologies suggest the opportunity was not as clear as it was for Elijah. Guadalupe's threshold for math contributions was much higher than it was for her peers; she needed an explicit invitation to share her mathematical ideas. However, Guadalupe's threshold for social participation was low; the appearance of an adult was enough to prompt a greeting or a light-hearted joke from her. The opposite was true for Elijah. Elijah's threshold for math contributions was the lowest of all four students, yet the threshold for social contributions was the highest. In fact, the threshold was so high that it was never reached in this episode. It is unclear what an opportunity for social participation might look like for Elijah since he did not participate in any social interactions.

Prior research shows that classroom participation is a function of the opportunities students have to participate, and opportunities are shaped by classroom contexts (e.g., how competence is constructed and how tasks are designed) (Gresalfi et al., 2009). However, to understand how opportunities for participation are differentially constructed within a single classroom, looking beyond classroom-level contextual factors is needed. This study suggests that constructing fair opportunities to participate requires: 1) conceptualizing opportunities for participation as deeply connected to students' positionings and developing identities in classroom communities, 2) acknowledging that what counts as an *opportunity* for one student to participate does not necessarily count as an *opportunity* for another, and 3) leveraging both mathematical and social contributions in creating equitable, supportive, and intellectually rich learning communities.

Working toward participatory equity – cultivating classrooms with *fair* (not necessarily the *same*) opportunities to participate (Esmonde, 2009; Secada, 1989) – requires taking into account that calculus classes are historically White, male-dominated spaces, in which females and racially minoritized students face additional barriers to participation (Leyva et. al., 2020). Opportunities to participate for Guadalupe, a woman of color, were undoubtedly different from those of her White and/or male peers. Exploring how racialized and gendered discourses shape students' opportunities to participate is an important direction for future research.

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GESTURE INDICATES PRODUCTIVE STRUGGLE IN PROOF WRITING: CASE STUDIES FROM A BASIC TOPOLOGY COURSE

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Many students struggle with proof writing. However, struggle is not universally bad: researchers have distinguished between productive and unproductive forms of struggle and have identified productive struggle as essential for learning mathematics. Yet, in practice, recognizing when learners are engaged in productive struggle or unproductive struggle can be challenging. In this report, I argue that students' gesture production may indicate engagement in productive struggle. I observed three undergraduate students from an introductory point-set topology course, collaborating in pairs to complete proof tasks. I present evidence from the students' work on two proof tasks that undergraduate students' gesture frequently when they are engaged in productive struggle and that gesture is rare during engagement in unproductive struggle.

Keywords: Cognition, Communication, Reasoning and Proof, Undergraduate Education

Writing proofs is known to be challenging for mathematics students (Alcock & Weber, 2010; Azrou & Khelladi, 2019; Harel & Sowder, 1998; Iannone & Inglis, 2010; Leron, 1983, 1985; Moore, 1994). Hiebert and Grouws (2007) identified that allowing students to struggle with mathematics was an important feature of effective mathematics teaching; still, not all struggle is beneficial to students' learning. In this paper, I present evidence that undergraduate students' uses of gestures when working on proof tasks can be used as an indicator of engagement in productive struggle.

Vygotsky (1978) defined the *zone of proximal development*: "the distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem-solving under adult guidance, or in collaboration with more capable peers" (Vygotsky, 1978, p. 86). To support students' learning of new material, educational tasks should be designed so that the concepts involved fall into the students' zone of proximal development: they should be challenging to students, but achievable through appropriate scaffolding and support from peers or a teacher. Hiebert and Grouws (2007), in a meta-analysis of effective teaching practices for conceptual understanding, echoed this idea, referring to the notion of *struggle*, defined as "effort to make sense of mathematics, to figure something out that is not immediately apparent" (ibid., p. 387). In this paper, I will refer to this kind of struggle as *productive struggle*, and distinguish it from *unproductive struggle*, or "needless frustration or extreme levels of challenge created by nonsensical or overly difficult problems... [or] the feelings of despair that some students can experience when little of the material makes sense" (Hiebert & Grouws, 2007, p. 387).

Gesture use is known to be directly connected to cognition and perception (Alibali et al., 2014; Bernard et al., 2015; Goldinger et al., 2016; Hostetter & Alibali, 2008; Lakoff, 2012; Lakoff & Núñez, 2000; McNeill, 1992, 2005; Straube et al., 2011; Varela et al., 1993; Wilson, 2002). Research on undergraduate students' gesture use has shown that the use (or lack of use) of gestures influences strategy choices in problem solving (Alibali et al., 2011) and that gesture use can support recognition of important ideas in the construction of proofs (Gallagher, 2020; Pier et al., 2019; Williams-Pierce et al., 2017) and communication about ideas related to proof (Kokushkin, 2020).

In this paper, I present evidence that undergraduate students' use of gestures when working on tasks related to proof may be indicative of engagement in productive struggle.

Theoretical Framework

To frame this work, I utilize Sfard's theory of commognition as well as the notion of productive struggle.

Commognition is a portmanteau of the words *communication* and *cognition*; Sfard described it in the following way:

> Once we adopt the claim that thinking may be usefully defined as the individualized form of the activity of communicating, thinking stops being a self-sustained process separate from and, in a sense, primary to any act of communication and becomes an act of communication in itself, although not necessarily interpersonal. This self-communication does not have to be in any way audible or visible and does not have to be in words. In the proposed discourse on thinking, cognitive processes and interpersonal communication processes are thus but different manifestations of basically the same phenomenon. (Sfard, 2008, pp. 82-83)

The crux of the theory of commognition is that thinking and communicating are intrinsically linked. Rather than thinking of cognition as *preempting* communication or communication *following from* cognition, commognition adopts the perspective that these two actions are indeed one and the same. Furthermore, thinking can be conceptualized as *self-communication*; thus, commognition encompasses the practices of internal thought and "thinking out loud" as acts of communicating ideas with oneself.

In line with Sfard's assertion above, I assume that self-communication does not need to take the form of speech, and I include the production of gestures during self-communication as a form of commognition. Gestures are known to be produced spontaneously during thought, particularly when students are initially orienting to a problem or trying to communicate complex information (Alibali et al., 2014; Bernard et al., 2015; Hostetter & Alibali, 2008; Lakoff, 2012; Lakoff & Núñez, 2000; Straube et al., 2011).

With this in mind, in this work I associate gesture use with the concept of *productive struggle*, using the definition from Hiebert and Grouws (2007) given in the introduction to this paper. In the results that follow, I will show that gesture does not always occur spontaneously. Rather, I will argue that spontaneous gesture use occurs concurrently with productive struggle and can be used to distinguish productive struggle from unproductive struggle in undergraduates working on proof-related tasks.

Methods

Four undergraduates were recruited from a general topology course for a teaching experiment to gain insight into the ways undergraduates leverage examples, diagrams, and gestures when writing proofs in general topology. The author served as the researcher leading the teaching experiment. A total of 9 one-hour sessions comprised the teaching experiment, during each of which the students were asked to prove a true statement and disprove a false statement, although they only engaged with only one of these tasks during some sessions due to limitations on time. Each session was video recorded, and each video was transcribed. Videos and transcripts were then coded for instances of students engaging in productive struggle and unproductive struggle (operationalized definitions provided later in this section) and instances of students producing gestures.

A descriptive case study methodology (Cohen et al., 2013; Yin, 2003) was used to analyze the behaviors of Stacey, Tom, and Rachel, specifically focusing on when and how they used gestures while reasoning about proof tasks. The fourth student was excluded from this analysis, as he attended only one session, and participated very minimally in that session. As the students worked collaboratively to complete the proof tasks in a given session, and different groups of students were present during each session, I consider each session to constitute one "case" in this case study.

In this paper, I assume that the meaning of *struggle* is self-evident, but I distinguish between unproductive struggle and productive struggle. For the purposes of this paper, I claim that students are engaged in *unproductive struggle* any time they give visual or audible signs of focusing on the task under consideration but are not performing an action (such as drawing a diagram, considering an example, or writing notation) or proposing ideas or making conjectures (statements or questions like "I think I need to take the union of these sets" or "What happens if I take the intersection here?"). In other words, students are engaged in unproductive struggle when they appear to be thinking about a problem but seem to be unable to interact with the ideas involved in its statement or its solution. Most often, this is evidenced by students staring at the board in silence or expressing sentiments like "I'm not sure what to do here." In contrast, students are said to be engaged in *productive struggle* any time they are performing an action or proposing an idea or conjecture related to the task under consideration but seem to be uncertain about the usefulness or consequences of those actions, ideas, or conjectures. Examples of productive struggle include consideration of examples, drawing diagrams, attempting to write logical statements to move forward in a proof, and thinking aloud about the meaning of notation.

For this paper, I use the definition of *gesture* given by Rasmussen, Stephan, and Allen (2004) as "movement made by the hand with a specific form: the hand(s) begin at rest, moves away from the position to create a movement, and then return to rest" (p. 303). Gestures may be further divided into deictic gestures (pointing) and representational gestures (movements made to depict an idea, object, or action), though in this paper I do not consider these kinds of gestures separately.

Results

Throughout all nine sessions, instances of struggle were evident from all participants. I present results from two tasks: the prove task from Session 1 and the disprove task from Session 2. Stacey was present for all three (indeed, all nine) sessions; she was joined in Session 1 by Tom, and in Session 2 by Rachel.

Session 1

The students engaged in unproductive struggle when they were faced with notation they had used before but were unaccustomed to working with. In Session 1, Stacey and Tom struggled to get started on the following task, which was written on a chalkboard: *Let $f: S \to T$ be a function, and let $\{U_i\}_{i\in I}$ be a family of subsets of T. Prove that $f^{-1}(\bigcap_{i\in I} U_i) = \bigcap_{i\in I} f^{-1}(U_i)$.* After some initial thought, Stacey expressed the general proof strategy: "First, we have to prove that the first one is a subset of that [pointing from $f^{-1}(\bigcap_{i\in I} U_i)$ to $\bigcap_{i\in I} f^{-1}(U_i)$], and then we have to prove that this one is a subset of that one [pointing from $\bigcap_{i\in I} f^{-1}(U_i)$ to $f^{-1}(\bigcap_{i\in I} U_i)$." Tom suggested to start by proving the inclusion $f^{-1}(\bigcap_{i\in I} U_i) \subseteq \bigcap_{i\in I} f^{-1}(U_i)$, which he indicated by drawing the relation "⊆" in the air with his finger. Stacey wrote "Let $x \in f^{-1}(\bigcap_{i \in I} U_i)$" on the board.

The students then spent the next full minute in silence, both staring at the problem on the board, motionless. At the end of that minute, Stacey wrote "$\{1,2,3\}$" on the board and stated that "the intersection of all of those subsets would be the null set, 'cause there's nothing that would be common to every single one of them," a statement Tom agreed with. I interpreted this as Stacey's attempt to consider an example in which $T = \{1,2,3\}$, and that she has taken the family of subsets $\{U_i\}_{i \in I}$ to be the power set of T. She indicated that she knew the family of subsets did not need to contain *all* subsets of T, but she clarified that she was "just trying to figure out something to think about, I'm a little bit lost."

After another minute of silent consideration, Stacey turned to Tom and asked, "Do you understand... what an inverse of an intersection would even look like?" She proposed, as another attempted example, that if the intersection was the set $(2,3]$, then the "inverse" of that intersection might be "all of the other elements other than this?" Both students continued to stare at the board in silence.

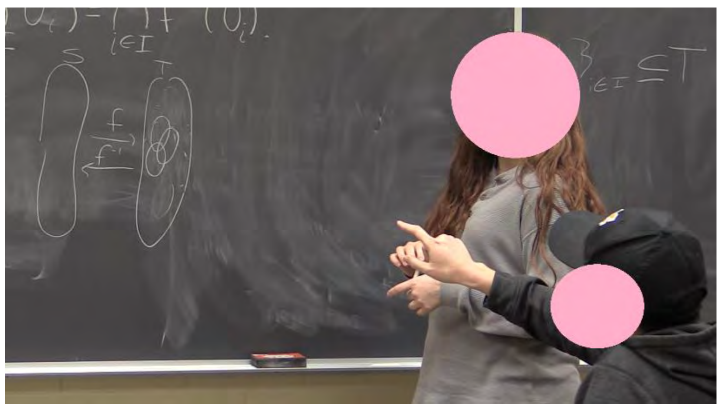

Figure 3: Tom pointing to Stacey's diagram.

I then prompted the students to draw a picture to represent the situation, and Stacey drew a standard set-theoretic diagram. Almost immediately, Tom pointed at the subsets of T in the diagram (Figure 1), claiming that "x is gonna be a point inside all three, in the intersection." Stacey considered this suggestion for a moment, then replied "... Is it?" In response, Tom began to explain his reasoning, but after pointing to the notation for $f^{-1}(\bigcap_{i \in I} U_i)$, he paused and second guessed his suggestion, pointing to the set S and saying "It's gonna be in this"; Stacey agreed, elaborating, "It's not in this [pointing to the text $\{U_i\}_{i \in I}$ in the problem statement], it's in the inverse of the intersection of that." Tom continued, explaining that for each U_i, $f^{-1}(U_i)$ represented a subset of S, first pointing to the notation $f^{-1}(U_i)$ and then tracing the outline of the corresponding subset of S with his finger in the diagram, and he noted that taking the intersection of those subsets would result in "only one area," tracing out a smaller region in the overlap of those sets. "Yeah, and x is *in* that area," Stacey concluded.

The remainder of the students' time spent on this task continued in a similar fashion, with Tom and Stacey pointing to notations from the problem statement and to regions of their diagram and using dynamic representational gestures to indicate elements being mapped between the sets S and T (Figure 2). Although they did not write a formal proof due to time constraints on the session, Tom and Stacey were able to articulate the key ideas of this proof and construct an oral and visual argument that appeared to convince both of them why this statement was true.

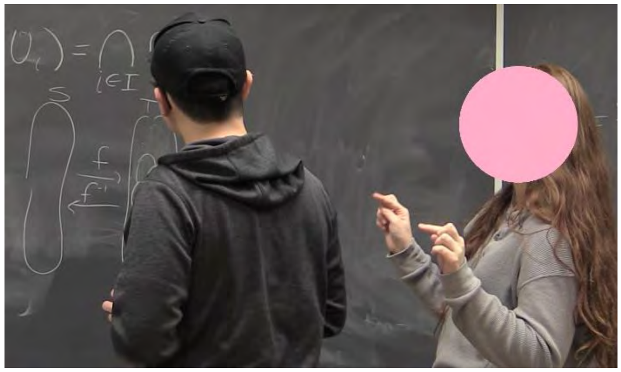

Figure 4: Stacey's dynamic representational gesture indicating a point mapping from T to S via f^{-1}.

Session 2

Stacey and Rachel worked together in Session 2 and did not experience the same immediate struggle that Stacey and Tom experienced in Session 1. Rather, they were able to discuss, relatively comfortably, the notions of symmetry, transitivity, and reflexivity that were necessary to work on the following task: *Disprove: Every relation* C *that is both symmetric and transitive must be reflexive.*

Upon reading the problem, Stacey immediately began by writing "$\{(1,2),(2,1),$'' at which point she paused and pointed with her index finger to the corresponding components of her writing as she read aloud, "So we have one-two... two-one... we have *a* related to *b*... it'd be one-one, if that was symmetric," and Rachel suggested adding $(2,2)$: "[pointing to where Stacey had written $(1,2)$ and $(2,1)$] I think you have to have both anyway, because it's 'for all.'" Stacey continued, "[pointing sequentially to each digit in $(1,2)$] We could do one to two, and then [writing] two to three, and then one-three, and that'd be transitive" (Figure 3).

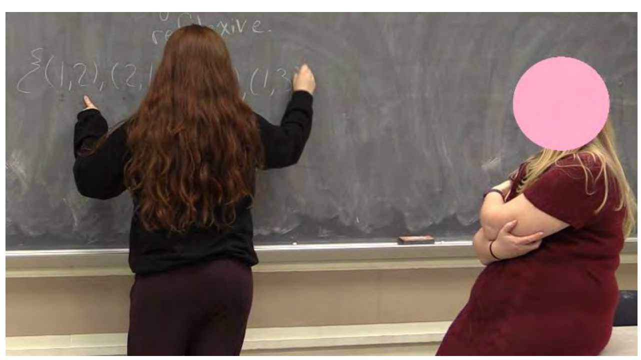

Figure 5: Stacey referencing element $(1,2)$ while adding element $(1,3)$.

After two minutes of work, Stacey and Rachel presented the relation $\{(1,2),(2,1),(2,3),(1,3),(3,1),(3,2)\}$ as their counterexample (the reader will note, however, that they did not specify a set on which to define this relation), at which point I informed them that, although their relation was symmetric on a certain set, it was not transitive, and thus could not serve as a counterexample to the given statement. Inspecting their work, Rachel pointed to the pair $(3,2)$, and Stacey pointed to the pair $(2,3)$ as she announced disappointedly "Then we'd need three-three, and we can't" throwing her hands into the air. Rachel replied, "No, we can have three-three," reminding Stacey that the reflexive property would not be satisfied unless their relation contained *all* of $(1,1), (2,2)$, and $(3,3)$ – although they would later realize that all three of these pairs must be present for their relation to possess the transitive property (Figure 4).

Figure 6: Rachel realizing that $(1, 1)$ must be included in the relation.

After adding $(1,1), (2,2)$, and $(3,3)$ to their relation, both students backed away from the board and stared at their work, both silent and standing still. Rachel explained "We need to have one that's not like, one-one, two-two, or three-three, but it still satisfies symmetric and transitive, which I don't think that we can." Reading over the definition of the reflexive property, Rachel noticed that they had not specified a set for their relation, and she wrote $X = \{1,2,3\}$ under their relation. "We need something to not be in there, like one-one, two-two, or three-three... exactly where I'm stuck." Both students continued staring at the board, no longer writing nor gesturing. Near the end of this session, Rachel suggested a viable solution to their problem – but both students rejected it. She proposed, "I mean, if you threw a four into X... but then you'd just have to make more elements," referring to a misconception expressed by both students during this session that if $4 \in X$, then 4 would have to be related to the other elements in X, and thus would need to appear as a component of some ordered pairs in their relation. Stacey agreed, and they continued to stare at the board in silence (Figure 5).

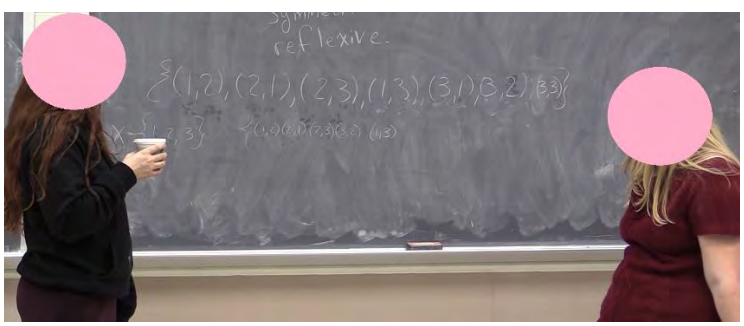

Figure 7: Stacey and Rachel near the end of Session 2, confused about how to prevent this relation from satisfying the reflexive property.

Olanoff, D., Johnson, K., & Spitzer, S. (2021). *Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Philadelphia, PA.

Discussion

Throughout the data presented, and indeed, throughout the data corpus, the study participants produced significantly more gestures during times of productive struggle than during times of unproductive struggle. In fact, during times of unproductive struggle, students seldom produced gestures, which is in sharp contrast to periods of productive struggle, during which gestures were commonplace.

Sfard's theory of commognition treats thought and communication as two sides of the same coin. Taking this perspective and treating unproductive struggle and productive struggle as two distinct forms of cognitive activity, I note that, in these data, these corresponded to two distinct forms of communication. In addition to distinct modes of verbal communication – silence versus speech – these data also show distinct forms of nonverbal communication: stillness versus gesture. In this comparison, stillness may be thought of as a form of nonverbal silence.

Consider Stacey's and Tom's behaviors from Session 1. Stacey and Tom initially seemed unable to make any progress on the task, as evidenced by them taking little action and seeming to be confused by the terminology, notation, and concepts involved in the task. Although Stacey attempted to generate examples, those examples were inappropriate to model the conjecture, and they did not seem to provide Stacey or Tom with any advantages. However, when I suggested that Tom and Stacey draw a picture to represent the situation, they began to negotiate meaning for the various pieces of notation used in the statement of the conjecture and to develop intuition for the scenario it described, eventually gaining personal insights into why the statement was true.

Stacey and Rachel were not immediately hindered in Session 2. In fact, they were able to produce an equivalence relation on the set $\{1,2,3\}$ and competently discuss the concepts of symmetry and transitivity that were necessary to produce an appropriate counterexample. However, both students seemed to lack a complete understanding of the definition of reflexivity (or perhaps of relations more generally), which caused them to struggle as they tried to violate this property. Throughout their discussion, however, Stacey and Rachel produced numerous pointing gestures as they negotiated how to make their relation satisfy the symmetric and transitive properties and as they discussed why the relation they had chosen also satisfied the reflexive property. When they tried to identify a way to violate the reflexive property, they became "stuck," and their gestures ceased.

Conceptualizing gestures as a component of cognition gives a window into students' mental activities. These results show that the students in my sample produced gestures when they were engaging in a meaningful way with the content of a given proof task, and, conversely, that they did not gesture when they were not participating in such engagement. To be clear, I do not mean to imply that productive struggle will *always* be accompanied by gestures, but rather that *when* a student produces gestures, these may act as an indication that the student is engaged in productive struggle.

Conclusion

Struggle is essential in the process of learning mathematics. Unproductive struggle, however, prohibits learners from making learning gains and increases their frustration, leading to a decrease in motivation. Educators should strive to engage their students in productive struggle, as this is the part of the problem-solving process during which students grow their understanding, make connections, and feel like their efforts might be rewarded with success.

In this paper, I framed students' gesture use as a way for teachers to discern whether students are engaged in productive struggle or unproductive struggle. With this tool, teachers can determine, at a glance, whether a task that has been set may be beyond the zone of proximal development for their students, and whether they may need to intervene to prevent students from losing motivation or simply let their students continue to work and develop their ideas.

However, as online instruction becomes more prevalent, researchers should attend to other means for distinguishing productive struggle from unproductive struggle, as gestures are not only more difficult to notice in the online environment, but may also be less frequent due to the inefficiency of pointing in such settings. Indeed, as reports from teachers and students indicate some students struggling to learn in the online environment, and as some classrooms transition back to in-person instruction, educators must be hypervigilant to notice signs of students struggling, and gestures serve this purpose well.

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IMAGE OF MATHEMATICS IN- AND OUT-OF-SCHOOL: A CASE STUDY OF TWO ORIGINAL PARTICIPANTS IN AN AFTERSCHOOL STEM CLUB– GIRLS EXCELLING IN MATH AND SCIENCE (GEMS)

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People often view mathematics as abstract, cold, and irrelevant to real-life, and their school experiences influence such views. In this case study, we investigated the mathematics learning experiences of two women who participated in an afterschool girls STEM club 26 years ago. We explored their experiences in and out of school and how such experiences informed their images of mathematics. Data were collected from a survey, focus group interviews, and individual interviews. Using qualitative analysis, we learned that their school mathematics experiences influenced the participants' images of mathematics. The findings also revealed the participants' continuous and discontinuous learning experiences between school and out-of-school mathematics. This study suggests creating spaces to develop curricula that bridge the gap between school and out-of-school learning experiences.

Keywords: Attitudes, Belief, Informal education, Integrated STEM

Introduction

People's experiences with learning mathematics in schools inform their images of mathematics (Sam & Ernest, 2000). The public often describes mathematics as difficult, cold, abstract, and primarily masculine (e.g., Darragh, 2018, Epstein et al., 2010, Ernest, 1996). In a *Journal of Research in Mathematics Education* commentary, Stephan et al. (2015) reported "grand challenges" for mathematics education, including "changing the public's perception about the role of mathematics in society," "achieving equity in mathematics education," and "changing perceptions about what it means to do mathematics" (p. 139). These challenges necessitate altering the public's image of mathematics. Sam and Ernest (2000) conceptualized the image of mathematics as "a mental representation or view of mathematics, presumably constructed as a result of social experiences, mediated through school, parents, peers or mass media" (p. 195). Researchers suggested that the widespread and narrow public image of mathematics may have resulted from instruction in mathematics education that portrays the subject as isolated from out-of-school experiences (e.g., Darragh, 2018; Sam & Ernest, 2000).

Reconstructing the public image of mathematics requires widespread support from many stakeholders. Specifically, the mathematics education community is responsible for supporting the public to see mathematics as "normal and ordinary but at the same time important and useful" (Darragh, 2018, p. 203). Researchers have found that in contrast to the image of school mathematics; out-of-school mathematics, including everyday mathematics and mathematics learning in designed informal environments, is often viewed as useful and real (Civil, 2007; Cooper, 2011; Nunes, 1999; Pattison et al., 2017). Exploring out-of-school mathematics could be a way to disseminate alternative images of mathematics (Nemirovsky et al., 2017). Therefore, informal mathematics learning has the potential to address the described grand challenges to change the public image of mathematics. However, limited extant literature has reported continuities and discontinuities between people's mathematical experiences in in- and out-of-school contexts. Conducting such a study has the potential to inform school mathematics teaching and broaden the public image of mathematics.

We investigated two women's in-school and out-of-school mathematics learning experiences to: (a) examine how these experiences informed their images of mathematics, and (b) identify continuities and discontinuities across their experiences. The women had participated in an afterschool STEM program, *Girls Excelling in Math and Science* (*GEMS*), in fifth and sixth grades. The two participants had different mathematics learning experiences as well as different views of mathematics. The following research questions guided this study:

- How did the participants' mathematics experiences inform their images of mathematics?
- How did the participants' descriptions of mathematics reflect continuities and discontinuities between their in-school and out-of-school experiences?

Literature Review and Theoretical Perspectives

School contexts typically have highly regulated learning environments. Below, Bronkhorst and Akkerman (2016) summarized key characteristics at school:

> (a) learning is intended; (b) students and teachers as main actors, with teachers as knowledgeable others; (c) what and how one learns is formalized in a curriculum; (d) validation of learning by assessment; (e) cumulative qualification; (f) school building; (g) mandatory attendance. (p. 22)

Unlike in-school learning, out-of-school learning often has more flexibility regarding time and space and is not constrained by the school schedule, national or state standards, and standardized tests. Informal learning is usually voluntary and allows students to bring in their cultural knowledge and personal experiences (Copper, 2011). In this section, we synthesize literature on in- and out-of-school mathematics learning as well as continuities and discontinuities between school and out-of-school contexts.

In- and Out-of-School Mathematics Learning

Out-of-school learning refers to curricular and non-curricular learning experiences that are provided for students outside of the school environment (Resnick, 1987). Resnick mentioned several discontinuities between in- and out-of-school learning. For example, school environments often focus on individual performance, independent thinking, symbolic representations, and generalized skills and knowledge. In contrast, learning out of school typically involves collaborative engagement, tools, and is situated in authentic environments. School mathematics learning frequently focuses on preparing students for standardized tests, and providing isolated instruction with limited opportunities to make connections between mathematics and daily life experiences (Copper, 2011). Students often view school learning as completing assignments required by teachers, which at times diminishes their motivation and interest in learning mathematics (Nunes et al., 1993).

Learning in school is not necessarily disconnected from out-of-school learning; in particular, some intended continuity efforts strengthen school learning by bridging learning between school and out-of-school spaces (Bronkhorst & Akkerman, 2016). The out-of-school contexts provide a rich environment for authentic and experiential learning (Nielsen et al., 2009). Authentic mathematics is introduced in the classroom to remedy the common disengagement in school mathematics, wherein students are expected to develop formal mathematics by mathematicizing their informal mathematical activities (Bonotto, 2005). In doing so, school activities further engage students and provide opportunities for them to consolidate knowledge and develop deeper understandings (Nielsen, 2009). Some researchers have suggested fostering informal learning as a supplemental formal learning method (e.g., Xiao & Carroll, 2007). Other researchers have proposed that the schooling system should incorporate informal curriculum to bridge learning in formal and informal contexts (e.g., Hung et al., 2012).

The term "informal learning environment" is often used as a general reference for a learning setting which is different from school. Though there is not an agreed-upon definition of informal learning environment in the literature, it often refers to everyday activities such as family discussions, pursuing one's hobbies, or daily conversations and designed environments such as museums, science centers, or afterschool programs (e.g., Civil, 2007, National Research Council, 2009). Nemirovsky et al. (2017) stated that designed informal mathematics learning environments are "intentionally designed to support mathematics learning, whether because they are structured through programs with regular schedules and assigned educators or because they host technologies, tools, or exhibits designed to engage the user with mathematics" (p. 970). In this study, we use the terms *informal mathematics* and *out-of-school mathematics* interchangeably to refer to mathematics practices in everyday life, in professions, and in designed informal learning environments.

Continuities and Discontinuities between In-School and Out-of-School Settings

Researchers have found that people use flexible strategies in diverse settings to solve mathematical problems outside of school, which are significantly different from those taught in school (Lave, 1988; Nunes et al., 1993; Saxe, 1988). Nunes (1999) suggested that school mathematics, concepts, methods, and procedures are the goals for instruction, solving problems for the teacher's sake. As such, "informal mathematics education is an emerging field of learning with a unique potential to disseminate alternative images about the nature of mathematics and to realize the potential for everyone to engage with mathematics in creative and diverse ways" (Nemirovsky et al., 2017, p. 975).

Bronkhorst and Akkerman (2016) synthesized 186 empirical studies to investigate continuity and discontinuity in students' learning across school and out-of-school contexts. They found that both continuity and discontinuity can result from different educational intentions, but it also occurs as a given. Due to the fundamental role in students' learning, the school is responsible for establishing the continuity for students' learning across contexts. There are possibilities and challenges for schools to develop this continuity. First, introducing out-of-school practices in school might be a challenge for teachers who usually have limited expertise in out-of-school teaching practices, while on the other hand this can be an opportunity for teachers to engage in a new practice. Second, a challenge for teachers adopting informal practices is to ensure that all required content and standards are met. Third, in order to extend schools' influence, some informal learning contexts provide opportunities to supplement school education, such as a tutoring center, that intentional continuity limits students' experiences differing from school.

Akkerman and Bakker (2011) claimed that discontinuities result from boundaries, which can be seen as socio-cultural differences between different contexts. As learners engage in different practices, learning is not necessarily bounded in a particular stable domain. Rather, learning involves crossing boundaries between multiple practices, in which a learner should be approached as a whole person who participates in school and many other places. Our theoretical perspective is informed by learning as crossing boundaries between in-school and out-of-school contexts (e.g., Akkerman & Bakker, 2011; Bronkhorst & Akkerman, 2016), which provides a lens to explore each participant as a whole person with interconnected identities when they engage in different practices.

Methods

In this study, the designed informal learning environment particularly refers to an afterschool STEM program–*GEMS*. *GEMS* was initiated in 1994 by a mother who aimed to help her daughter and other girls develop positive dispositions towards mathematics and science. The first *GEMS* club was started at an elementary school in Virginia; at that time, most participants were white and from working class families. The mother and her daughter's fifth-grade classroom teacher co-led the club; fifth and sixth grade girls were enrolled by teacher's recommendation, parental request, and/or voluntarily.

This study draws its data from a larger study of the original *GEMS* participants in 1994-1995. In the current study, we used a collective case study approach (Yin, 2017) to understand two participants' descriptions of their mathematics learning experiences in formal and informal settings and to explore their images of mathematics. We aimed to explore across the two cases to draw case-specific characteristics. As such, the goal of this case study was not to generalize the image of mathematics from the two cases. Instead, our goal was to identify the possible impact of mathematics experiences on the participants' images of mathematics (Simons, 2009).

Data Collection

Based on information-oriented sampling (Yin, 2017), Kate and Stella (pseudonyms) were selected from the larger study because of their different experiences and views on mathematics. Kate expressed her positive experiences with school mathematics; she identified herself as "naturally good in mathematics." In contrast, Stella, consistently reported struggling with mathematics in and out of school, saying, "I had always been awful at math."

Reflecting on Yin's (2017) emphasis on the role of theory in guiding case study research, we developed data collection protocols based on relevant literature. For instance, building on the literature on mathematical identity (Boaler, 2002), we designed an interview question, which asked the participants to describe themselves as math learners. We also built from Sam and Ernest's (2000) work that proposed that adults' images of mathematics are influenced by their mathematics teachers. The data sources included surveys, focus group interviews, and individual interviews.

Data Analysis

Data analysis involved a review of the three data sources to derive relevant themes across cases regarding mathematics experiences and conceptions of mathematics in- and out-of-school. Drawing on Akkerman and Bakker (2011) and Bronkhorst and Akkerman's (2006) foundational work on continuities and discontinuities crossing boundaries, we conceptualized continuities when individuals make connections between their participation in various contexts(Bronkhorst & Akkerman, 2006) in which individuals might change roles across contexts. Discontinuities refer to an individual's experiences, interests, and perspective in one context that conflict with his/her experiences in another context. We followed a qualitative content analysis protocol, which allowed coding to be both data driven and theory driven (e.g., Schreier, 2012).

The first author reviewed all data and developed the primary codes. Using the primary codes, the two members of the research team then independently coded each participant's survey, focus group interview, and individual interviews. The researchers also wrote research memos to record the emerging new codes and suggestions for revising the codebook. The third researcher oversaw the coding process and tracked the discrepancies between the first two researchers' codes. Then the authors discussed and resolved these divergences. Any unresolved divergences, confusions, ambiguities were brought to the whole research team and discussed in the team meeting until a consensus was reached.

Findings

In this section, we report our findings to highlight Kate and Stella's mathematics learning experiences, how they influenced their views of mathematics, and continuities and discontinuities within their mathematical learning experiences across settings.

Kate's Description of Mathematics

Math is binary. In Kate's descriptions, mathematics is objective and a natural aptitude. She explicitly mentioned her positive learning experiences in elementary and secondary mathematics classrooms, saying "when I was growing up, I knew I could do the math and science as well as the boys....I was interested in math and science at an early age. It just came naturally to me." Kate perceived mathematics as a neutral and objective subject (i.e., right and wrong answers). The objective aspect of math gave her autonomy when she interacted with math. She stated:

> I liked knowing there was an answer, and that you just had to go through the steps to get it right or wrong. And you know why you got the grade you did in math and science. I knew that if I studied, I would get it right. I could and can control it.

Kate compared math and science with other subjects in which there is no right or wrong answer; the learners then need to rely on evaluation from authorities, which from her perspective is subjective. She stated that "in English, instead [of a right or wrong answer], you could write an essay and the teacher may or may not like it depending on what day she reads it."

Even though Kate enjoyed the binary right and wrong aspect of mathematics and considered herself a natural mathematics learner, she was also hindered by this aspect of mathematics. She was hesitant to share her answers in the class as she did not want to present wrong answers. She said, "but you completely second guess about yourself. I do not want to raise my hands just because if I am wrong that's gonna feel really bad in front of everyone."

Discontinuities between school and GEMS broaden the image of mathematics. Kate described the nature of mathematics learned in a formal space as "rigid of structure" and objective ("there is a right answer") while the mathematics learned in the informal learning space was flexible and more subjective. The discontinuities of mathematics within GEMS and the classroom broadened her image of mathematics.

Kate connected her mathematics learning experiences with her daily life activities and experiences. Meanwhile, she acknowledged that she was not able to perceive the continuity in formal classroom mathematics and her daily-life activities in her childhood, "I remember we got our first computer. We're going to have fun games. They're so much fun. It was always learning without making it feel like you're learning. It was always the problem-solving games. This is super fun!"

Kate acknowledged that she engaged in those games without realizing she was learning mathematics (i.e., logic and reasoning). She distinguished between her mathematics learning experiences in the mathematics classroom and in *GEMS*. She mentioned:

> [in a mathematics class], it's gonna look really bad, if I get this wrong in front of everybody. I think one thing that helped in GEMS [that] it was okay if you were not super confident, like yes, I know this. And then they [leaders] were like you're close, but you're not quite there.

This indicated that her learning experiences in these two spaces were discontinuities in terms of being open to share her answers for two reasons. First, the mathematics content presented in the *GEMS* was different from the school mathematics content; it was more subjective, and it did not necessarily depend on "Yes or No type of answers." She stated,

> [Math] was just fun and group kind of thing and, what if add this in and how does that affect? It was not obviously a rigid structure. It's not like a question/answer. So, that definitely helped in a way that builds confidence.

Second, Kate described that the learning environment in the *GEMS* was safer ("it's okay not to get the right answer") and less judgmental ("everybody here they like the same thing as you do") than the classroom learning environment. Kate mentioned that *GEMS* made her realize that mathematics can be learned in a fun way, which aligned with her childhood experiences of learning mathematics with computer games. She stated that she never felt she was learning "rigid structure[s]" and "just question/ answer" in *GEMS*, but it was like building something as a group and having a product at the end. Specifically, she mentioned:

> There was not that big of a, I guess, a fear of that as much as you have in class, it was just a smaller group of people and everybody kind of understood we all like this and we all love doing this, and it's okay to not get it right.

Kate enjoyed the discontinuities of mathematics between school and *GEMS*; in fact, she intentionally sought out these discontinuities. When she was in high school, she joined an after-school math club and expected to do activities similar to *GEMS*. But when she got there, she found that "it was just sitting there doing mathematics problems. I don't need another math class after school, no thanks."

Stella's Description of Mathematics

Math is multifaceted. Stella struggled a lot in basic mathematics, in particular, numbers, but she perceived mathematics as multifaceted, including logic, process, language, statistics, and spatial design. She described her early school years, saying *"*I always sucked with numbers. I was awful at the times table. I was a straight A student except for any math class that I had; I was constantly getting Cs [in math]. *"* Stella said she was traumatized by mathematics:

> I had always been awful at math, especially after third grade where I always failed the multiplication table timed quizzes and tests because I didn't understand the concepts. I don't think I passed a single one of those tests besides the 5s, maybe the 9s because of logic. I was traumatized by math.

In elementary and middle school, people surrounding her did not understand her difficulties, which exacerbated her struggles in mathematics. She said,

> I really struggled in elementary school. And my mother did not get it at all because I was a straight A student in any of the language arts and I really did well in writing. But I just could not understand math and they didn't have, I guess, it wasn't something that was really talked about or knew. So, nobody really thought to be like, hey, this girl has a learning disability because I was doing so well everywhere else. I really grew to hate math because of it.

In high school, Stella was often quiet in math classes. In spite of her hard work on algebra, Stella mentioned that she performed poorly. She said that it was incomprehensible for her at the time to get "such a bad grade." However, she performed well in geometry and got an A. Because she did not want to take extra mathematics classes, she opted to take a programming course instead of trigonometry or calculus. For Stella, programming was not related to mathematics, just logic and languages, "even if it is like math in that it is a logical process, I did not have to do any of the calculation and numbers, that was really hard for me."

Supports discontinuities in mathematics learning. Stella portrayed herself early on as a person who struggled with mathematics. To avoid being called on to do numerical problems in the class, she would make sure to raise her hand and be overly participating in any process- and logic-related questions. The discontinuities between mathematical results and process provided opportunities for her to enhance her strength (i.e., logical thinking and conceptual understanding). Stella mentioned that in GEMS, her teacher helped her address these discontinuities.

Unlike in the mathematics classroom, Stella was not nervous in *GEMS*, "[The teacher] was so good and motivated everyone involved. You can ask questions, there was definitely no stupid question." *GEMS* did not focus on memorizing rather promotes application of math which released Stella's struggles, she said "*GEMS* made me think that that math wasn't as scary. I learned that I understood applied math in a way that I didn't get from memorizing times tables." Even though she was traumatized by numbers and did not like math, she developed confidence in math, "[in *GEMS*] I learned that I even though I was bad at writing down equations on paper, I was decent at applied math. I ended up doing computer programming in high school because I felt comfortable with applied math." At home, Stella's father also supported her conceptual understanding by encouraging Stella to use tools, she recalled "he would even let me use my fingers and he gave me an Abacus so I could use that. It really helped because it took the numbers out of the math and I could understand."

The discontinuities between learning mathematics that focus on getting correct results and understanding concepts supported Stella to appreciate different foci of mathematics. Rather than hindering her development, the discontinuities redeemed her struggles in mathematics and provided room for her to develop her confidence and interests, which impacted her course selection and later career decision. Looking back, she laughs, "even with all my math issues, I ended up going into STEM."

Kate and Stella's Explorations of Continuities In and Out-of-School

Even though Kate experienced discontinuities in her mathematics learning between formal and informal learning environments, Kate's learning experiences in informal settings and everyday life indicated that her descriptions of the nature of mathematics were somewhat continuous as she described mathematics presented in informal classrooms and everyday life as "fun-type" learning. Kate wants her daughter to engage with more mathematical toys because she wants her daughter to choose a STEM-related career: "She has blocks we are building, for her it's just let's stack them down, and right now she accesses the construction all that comes in and knocks it over, but you do not realize this in a way you are engineering but something you are building blocks." Kate also mentioned that her experiences in *GEMS* are parallel with this because in *GEMS* she "never thought [they] were learning, [they] tried fun experiments" without realizing they were "actually learning bigger concepts."

Stella consistently experiences mathematics struggles across different settings. She said, "I still really struggle with the [math], when I'm on the phone with somebody and I need to read off a credit card number, I always mess it up" she added, "I still cannot do basic math like a *normal* person." However, she accepts these continuities and uses resources to overcome the challenges. She depends on Excel and other software when she is doing finances in her personal and professional life. In *GEMS*, continuities of mathematics struggles did not hinder her development. Being with friends and gaining support from her teacher, Stella felt a sense of belonging and being involved, "it was no stupid questions for me [in *GEMS*]. If she [the teacher] could see that you were not understanding it, she would explain it a different way. She had like a million different ways to explain something until you understood it."

Discussion

In this study, we sought to answer two research questions: How did the participants' descriptions of mathematics experiences inform their images of mathematics? How did the participants' descriptions of mathematics reflect continuities and discontinuities between their in-school and out-of-school experiences? For the first research question, we learned that participants' school experiences influenced Kate and Stella's images of mathematics. Kate reported liking mathematics and referred to happy memories of doing mathematics across her experiences at school, home, and her professional life. Stella, who explicitly reported not liking mathematics, referred to school experiences wherein she remembers struggling or feeling defeated. These findings are aligned with existing literature that reports that schools and mathematics teachers greatly influence people's image of mathematics (Sam & Ernest, 2000). Though Kate and Stella hold different attitudes toward mathematics, their views of mathematics are aligned with the public image of mathematics. School mathematics for the two participants was perceived as difficult, cold, abstract, and masculine (e.g., Darragh, 2018, Ernest, 1996, Epstein et al., 2010). Kate and Stella perceived that people either can do school mathematics naturally or not. Kate said that she liked math since she was young while Stella described how she "was awful" in math. Kate's mathematics experiences were pleasant, she enjoyed doing mathematics at school, with her family, and at *GEMS*. Yet, for Stella, school mathematics, especially elementary mathematics, was difficult and unpleasant. In contrast, Stella enjoyed doing mathematics that was related to logic and application, such as her experiences doing mathematics at *GEMS* or with her family.

Concerning findings related to the second research question, we identified continuities and discontinuities in both participants' in-school and out-of-school experiences. A continuity we identified is that when participants reported having uplifting experiences with mathematics in school, the confidence continues in other learning contexts. We identified a discontinuity when Stella reported not liking school mathematics, but doing mathematics in her daily life. The two participants' descriptions of their professional mathematics activities were uplifting and reflected confidence. It appears that even though school experiences had a profound influence on the participants' images of mathematics, their experiences in other learning contexts, particular positive learning experiences, also impact on their construction on alternative images of mathematics.

Future studies could design school mathematics interventions and study their influences on students' images of mathematics. We learned from the participants that doing mathematics in their professions is doable and enjoyable, which prompted us to think that designing activities that resemble those done by professionals might be a way to bridge in-school and out-of-school mathematics, which could broaden students' images of mathematics. Participants reported using mathematical practices and concepts in their daily lives. Curriculum designers might use those practices and concepts to develop K-12 curricula.

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A HYPOTHETICAL LEARNING TRAJECTORY FOR THE UNDERSTANDING OF NUMBER DENSITY IN HIGH SCHOOL STUDENTS

UNA TRAYECTORIA HIPOTÉTICA DE APRENDIZAJE PARA LA COMPRENSIÓN DE DENSIDAD NUMÉRICA CON ESTUDIANTES DE BACHILLERATO

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During their school life, students learn mathematical topics that can be considered essential for the understanding of the property of density in the set of real numbers. Therefore, we detected a need to design and elaborate a Hypothetical Learning Path to include topics to help promote the learning of this property. This report shows results of a first stage of an educational experiment as part of an ongoing research. It describes how through the trajectory, high school students are able to recognize ways of finding numbers in an interval using various semiotic representations. We also describe some difficulties that students had to recognize the non-existence of a successor in real numbers.

Keywords: Learning Trajectories and Progressions, Number Concepts and Operations, Mathematical Representations, High School Education

Introduction and Research Questions

It has been seen that many students, from the beginning of their elementary school to the end of high school, even university school, show a deficiency in understanding the density property in the set of real numbers (Tirosh et al., 1999). In elementary school, the student only has one opportunity to learn about the density property of decimal numbers (Ávila & García, 2008). For example, in Mexico, the teacher shares with his sixth-grade students (around 12 years old) a unit related to this property using the number line (SEP, school year 2020-2021). In the project by Vamvakoussi and Vosniadou (2010), high school students between 12 and 17 years of age show difficulty in understanding the density property in real numbers. Some students in this project believe that there is no other number between 0.005 and 0.006, or between 1/3 and 2/3. That is, for them, these pairs of numbers are “consecutive” each. Students seem to believe in the existence of a successor in real numbers. And other participants in the study refer that there is a finite number of decimals between 0.005 and 0.006.

On the other hand, Vamvakoussi and Vosniadou (2010) have noted how there are students who are affected by the symbolic representation of the extremes of an interval. The authors observed that students tend to express that there are no fractions between decimals, that there can only be decimals; and vice versa. For this reason, it is important, as Duval (2004) points out, that different representations are handled throughout the students’ school journey. According to Duval, teaching and learning mathematics implies that some cognitive activities (conceptualization, reasoning, comprehension, among others) require, in addition to natural language or that of images, the use of different registers of semiotic representation. For this author, in mathematics there are different writing systems for numbers, symbolic notations for objects, relations and operations, as well as a variety of graphs; each of the above activities constitutes a different semiotic form.

In order to overcome the difficulty that students must understand the density property in real numbers, a Design Based Research is being carried out (Cobb & Gravemeijer, 2008), in which a Hypothetical Learning Trajectory (HLT) was designed as a didactic proposal using various representation registers. Simon (1995) proposes that a HLT is a sequence of activities to attend to some mathematical concept, in which these activities are built from one or more hypotheses that support the student in the construction of new knowledge.

Considering the above and the problems that are generated to understand the density property, the following research questions were elaborated:

a) How to design a HLT to promote the learning of the density property in the set of real numbers?

b) How do high school students understand the property of density in real numbers using semiotic representations that can be produced during HLT?

Theoretical framework

Vamvakoussi and Vosniadou (2004) indicate that understanding the density property of rational numbers is a gradual process: from the discrete to the dense. The authors conclude that prior knowledge about natural numbers supports the student to use the property of the discrete to solve tasks related to rational numbers, in effect restricting the understanding of the density property. Using the ideas of Ni and Zhou (2005), the act of counting by a child is his first approach to a representation of the natural number, of the discrete; this representation persists in the child to such an extent that in problems related to fractions or decimals he considers properties of the natural numbers to solve them. Vamvakoussi and Vosniadou, in 2004, carried out an investigation to find out how much ninth-grade students (around 15 years of age) know about the number density, later, the authors elaborated five categories based on the responses of the participants (see Table 1).

Table 1: Categories of thinking about the quantity number of numbers in an interval

Naive thinking about the discrete	It is thought that there is no other number between two *consecutive false* rational numbers. Vamvakoussi and Vosniadou (2004) created this expression to refer that exists a successor of a rational number.
Advanced thinking about the discrete	It is thought there is a finite quantity of numbers between two consecutive false rational numbers.
Mixed thinking between discrete and dense	In some cases, it is thought that between two rational numbers there is an infinity of numbers; and in other cases, that there is a finite number.
Naive thinking about the dense	It is understood that there is an infinity of numbers in an interval, but this situation is not justified by using the density property. The symbolic representation of the extremes of an interval influences the way of thinking; it is believed there can only be an infinite number of decimal numbers between decimals and an infinity of fractions between fractions, but not an infinity of fractions between decimals or otherwise.
Advanced thinking about the dense	There is a sophisticated understanding of the density property; that is, it is understood that there is an infinite number of numbers between two rational numbers, regardless of their symbolic representation and this is justified through the use of the density property.

To understand the property of density in real numbers, it is necessary to consider that the real number has different representations (Apóstol, 2006). However, Duval (1983) points that this task is not easy for a student, it is difficult for him to recognize the same mathematical object through various contexts of representation. As happened in the study done by Neuman (2001), seventh grade students (around 13 years old) did not accept that there could be a fraction between 0.3 and 0.6, since for them fractions and decimals were different mathematical objects that did not have any relationship. For Duval (1983), learning a mathematical concept requires the resource of several semiotic systems of representation, which implies a deeper mathematical thinking by the student.

Relevant aspects of an HLT

Complementing the definition of a HLT, Simon and Tzur (2004) describe that this expression refers to the predictions (hypotheses) that the researcher, or the teacher, think about how a student can challenge the proposed scenarios in a learning trajectory. The path becomes "real" (real trajectory) when the student's conceptions or ideas are known during the socialization of activities (Simon, 1995). For Simon, during the trajectory, the researcher can modify various aspects, including the duration and the design of the learning lessons, this as a result of the interactions that arise with the students. Thus, a HLT provides the researcher with a rational criterion to decide the design that he is considering, as well as the best hypothesis of how the learning process can advance in the student. Following the author's discourse, three essential components of a HLT are considered: a) *the objectives*, understood as the set of statements of which it is expected to carry out the fulfillment of the actions, b) *the route of the learning activities*, in which students progress, made up of increasingly complex levels with instructional activities that promote the passage from one level to another, and c) *hypotheses by the researcher*, understood as the conjectures that a researcher plans about the learning process.

Methodology

Our research methodology is situated in the context of Design Based Research (DBR), with a qualitative approach of a case study. DBR is a methodological approach in which the researcher tries to examine, in a systematic and detailed way, how students do proposed tasks, and analyzes teaching strategies and tools (Cobb & Gravemeijer, 2008). In terms of these authors, a DBR tries to experiment to support learning, through the design of an HLT. Cobb and Gravemeijer recommend testing and improving the conjectures or hypotheses outlined in the trajectory, which is why they suggest the execution of several cycles of analysis and design of activities. In the present investigation, we focus on a HLT for the learning of the number density, whose refinement contemplates two cycles. At this moment, the investigation is completing the first cycle of design and analysis.

Participants

The current population is made up of four students aged between 15 and 17, who are in high school in Colombia. Due to the COVID-19 pandemic, the activities have been carried out individually and hybrid, some at the student's home and others virtually, over a period of three months. This report will show the participation of two students, Néstor and Paola (pseudonyms), who completed the proposed HLT route.

HLT design

The educational experimentation corresponding to the first cycle has three phases, the first and third consist of the application of a pretest and posttest respectively, and the second consists of the HLT. This report will show results of the first and second phases, and on the conclusions section some details the last phase.

First and second phase. Students solve a pretest and a posttest –with four questions–, each test lasts 30 minutes and measure how much they know about the discrete property of natural numbers and the density of real numbers. Through the pretest and the posttest, it is investigated whether at the end of the implementation of the HLT there have been changes in the way of reasoning of the student regarding the property of the discrete of the natural numbers, which is why the same questionnaire is used. The questions in this questionnaire were inspired by questions asked by Suárez-Rodríguez and Figueras (2020).

Second stage. In preparing this phase there are three objectives:

1. *Establish the stages of the HLT*. These stages are specified as levels of learning lessons, exactly four (see Table 2); however, it does not mean that the student depends on a previous level to complete the next. Each level lasts approximately one hour and 30 minutes.

2. *Define the hypotheses for each level*. The learning hypotheses are raised based on topics that students have studied in their school life and that it is believed that they can guide them to the understanding of density (see Table 2). McMullen and Van Hoof (2020) mention that although this property is not studied in class, there are moments in school mathematics that can provide opportunities to talk about it.

3. *Reach a metaconceptual awareness*. The student is expected to achieve metaconceptual awareness and can identify the characteristics that make a set dense. A student assumes a metaconceptual awareness when he reflects on some of his assumptions that they are not true, and that they also limit the way he interprets the new information (Vosniadou, 1994).

Table 2: HLT Levels and Hypotheses

Levels	*Hypotheses*
Level 1. First approaches to number density.	From two situations: one related to everyday life and the other related to a hypothetical scenario, it is thought that the student can have his first approaches to the property of density.
Level 2. Approach to number density through the similarity of triangles.	It is contemplated that by using triangle similarity students can learn about the density property of rational numbers.
Level 3. Approach to number density from arithmetic progressions and geometric progressions.	It is possible that by finding arithmetic and geometric halves in an interval the student understands the density property of rational numbers in the set of real numbers.
Level 4. Approach to numerical density through the property of continuity.	Using the continuity property, students are believed to understand the density property of irrational numbers in the set of reals.

Results

Pretest results

For the analysis of the responses given to the pretest by the students, the characterization made by Vamvakoussi and Vosniadou in 2004 was considered (Table 1).

Naive thinking about the dense. Figure 1 shows Néstor's response to the first item of the pretest. The process of infinite subdivisions in an interval was the representation that this student elaborated the most. It is observed how he uses decimal writing as a semiotic representation in an arithmetic register; and how he uses colloquial language to refer that there are infinite numbers between 0 and 1, and to express that the numbers follow the "same cycle". This last sentence suggests that he does not consider the irrational numbers in the interval, since the numbers that he writes are numbers in periodic decimal writing.

1. How many numbers are there between 0 and 1? Explain your answer.

Rta: Para mi, los números que hay son infinitos. Porque sigue el mismo ciclo; y porque no tienen un final. los: Ejemplo. 0.111111....
0.111112....
0.111113....

Figure 1: Naive thinking about dense (example)

Naive thinking about the discreet. Figure 2 shows Paola's responses to the fourth item of the pretest. It is appreciated that Paola answers the questions according to her natural number knowledge. Apparently, Paola believes in the existence of "several successors" (0.2, 0.11, 0.111) in decimal numbers. The difficulty of her to perceive the equivalence between 0.1 and 0.10 is also appreciated, since she indicates that 0.2 is the successor of 0.1, but that if the number were 0.10, her successor would be 0.11.

4. Pedro wants to draw a line where he can locate decimal numbers. He starts with the number 0.1. Then he places the 0.2. At that point, Laura interrupts him and says that the number that "follows" 0.1 is 0.11.
a. Who is right, Pedro or Laura? Neither of them? Or both? Explain your answer.

Pedro tiene la razón, ya que después de 0,1 sigue 0,2, Laura tendría razón en el caso que fuera 0.10.

b. Is there a number that follows 0.1? If your answer is yes, write the number that you think follows 0.1. But if your answer is not affirmative, explain why.

Si, existe un número que le sigue a 0,1.
0,11, 0,111, 0,111...

Figure 2: Naive thinking about discrete (example)

HLT Learning Lesson Results

Some actions of Paola and Néstor during the HLT are described.

Actions related to Level 1. In this first scenario, the students solved two learning lessons whose purpose is to have a first approach to density. In one of the lessons, the student must describe the movements that a frog makes (first the frog jumps halfway, then it jumps half of what was left, and so on). In Figure 3 it is seen that Néstor answers the question by approaching a thought related to the dense; however, his expression "too many times" can mean something finite. He uses a common language representation in his response. In the question in Figure 4, Néstor performs the procedure to find the following terms of the given sequence and uses semiotic representations such as fractional and exponential writing in an arithmetic register. In the last question (see Figure 5), he claims that there is an infinity of fractions between 0 and 1, but he does not justify with the density property.

2. How many times does the frog jump between points A and B? Explain your answer.

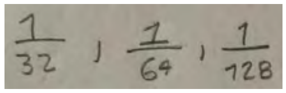
Rta: demasiadas veces; Por que la rana tarde o temprano va a llegar a su destino. [B]

4. Considering the previous question, how many fractions can you find between 0 and 1? Explain your answer.

Rta: se pueden encontrar infinitas fracciones Porque hay una infinidad de números.

Figure 3: Néstor's Record 1 (Level 1) **Figure 5: Néstor's Record 3 (Level 1)**

3. If 1 is the distance between A and B, that is, the interval from 0 to 1, the sequence of terms that the frog performs is: 1/2, 1/4, 1/8, 1/16 ...
Now find the next three terms of the sequence. Describe your procedure.



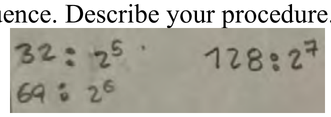
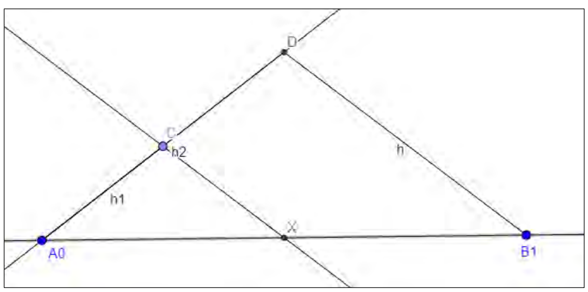


Figure 4: Néstor's Record 2 (Level 1)

Actions identified in Level 2. The learning lesson of this session is inspired by activities done by Tovar (2011). It is believed that by using similarity of triangles the student can understand the property of density to find middle terms in an interval. At first, the student reads the instructions so that he can construct similar triangles on a number line in GeoGebra. The image in Figure 6 shows the exercise carried out by Paola in which is observed that between "A0 and B1" (that is, between 0 and 1) she located the point *x*. She is asked to find this point. After thinking about it several times, Paola concludes that she must use proportions, as shown in Figure 7. She uses graphical representations of similar triangles in a geometric register, and in an algebraic register she finds the value of *x*. In the exercise instructions, the values of the segments, $\overline{AC}$ = 1 y $\overline{AD}$ = 2, are described for the resolution of this.



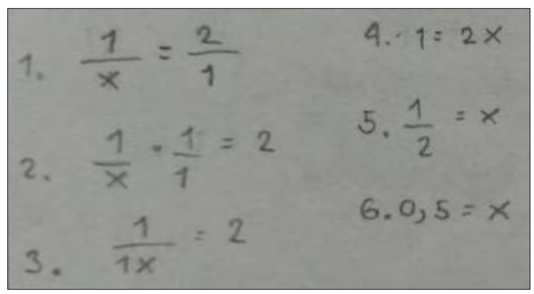
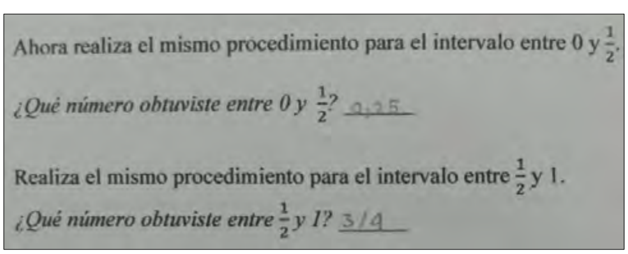


Figure 6: Paola's graph (GeoGebra) **Figure 7: Paola's Record 1 (Level 2)**

Paola continues to solve the questions (see Figure 8) that lead her to carry out the same procedure to find a number between 0 and 1/2 (see Figure 9), and another between 1/2 and 1. The numbers that she has found up to moment are: 0.25, 1/2 and 3/4; that is, the unit segment was divided into four equal parts. Finally, the students answer the question that appears in Figure 10. It is observed in this image that Paola approaches a thought related to the dense, since she contemplates that each time halves are obtained and that this has an infinite process. While Néstor, apparently, does not consider infinity but "several rational numbers", which could mean a finite quantity.


Ahora realiza el mismo procedimiento para el intervalo entre 0 y $\frac{1}{2}$.

¿Qué número obtuviste entre 0 y $\frac{1}{2}$? 0,25

Realiza el mismo procedimiento para el intervalo entre $\frac{1}{2}$ y 1.

¿Qué número obtuviste entre $\frac{1}{2}$ *y 1?* 3/4

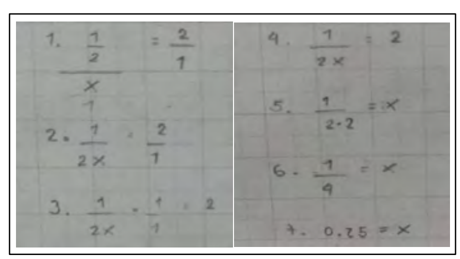
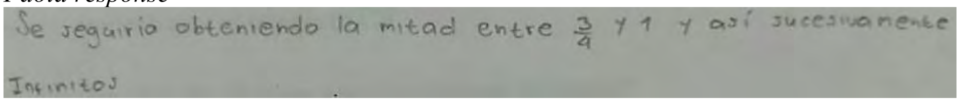


Figure 8: Paola's Record 2 (Level 2) **Figure 9: Paola's Record 3 (Level 2)**

5. What happens if you continue to perform the same procedure between 3/4 and 1? And so on, how many rational numbers would you get?

Paola response



Néstor response

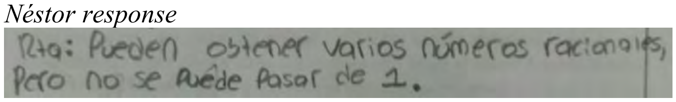


Figure 10: Record of Paola and Néstor in question 5 (Level 2)

Actions associated with Level 3. For this level, three learning lessons concerning arithmetic progressions and geometric progressions were designed to understand the density property of rational in real numbers. In one of the lessons the student reads a short paragraph about the definition of arithmetic progression. Next, the student must find five arithmetic means between 4 and 22 using the nth term of a sequence: $u = a + (n - 1)\, d$, in which a is the first term, n is the number of terms and d the difference between one term and another. In Figure 11 Paola's resolution to the exercise is shown –in an algebraic register – and she concludes that $d = 3$, and writes the terms found.

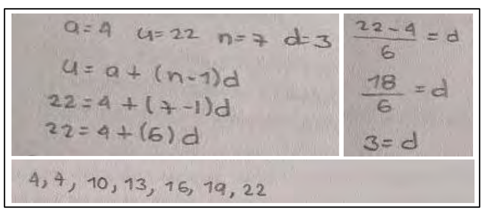


Figure 11: Paola's Record (Level 3)

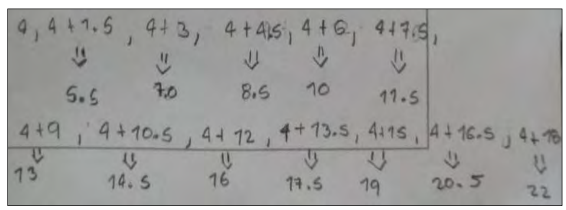


Figure 12: Néstor's Record (Level 3)

Subsequently, the student must answer the following situation: If d, from the previous question, were reduced by half, what would be the new arithmetic means between 4 and 22? Figure 12 shows that Néstor only uses the first term to find the new arithmetic means. He adds 1.5 to 4, then double 1.5 to 4, then triple 1.5 to 4, and so on until he reaches 22. Néstor makes use of representations of additions of decimal numbers in an arithmetic register. Finally, students are asked to find the arithmetic mean between a and u using the expression to find the nth term of a sequence. However, Paola and Néstor could not resolve this point of the activity because it seemed difficult to them.

Actions identified in Level 4. For this scenario, a learning lesson inspired by activities by Tovar (2011) was designed. It is expected that the student can learn the density of irrationals in reals, from the continuity of the line with respect to the non-correspondence between the rational numbers and the points of the line. Néstor explored in GeoGebra how the diagonal of a square with side 1 "translates" on the number line and observes that point x is between 1.2 and 1.6 (see Figure 13). Then he was asked to write four intervals where this point is located. To do this, he made several zooms on the GeoGebra screen and noted two intervals (see Figure 14). Both students observed that the point x does not correspond to a rational and that the intervals that enclose it are getting smaller and smaller. Finally, they found the diagonal of the square using the Pythagorean Theorem, and then concluded that between two rational numbers lies $\sqrt{2}$. However, both students were still thinking about the presence of a successor, in this case for $\sqrt{2}$, when they were questioned about it, they did not answer what it could be. This leads us to think that since $\sqrt{2}$ is irrational, it is difficult for them to find a supposed successor even though they affirm its existence.

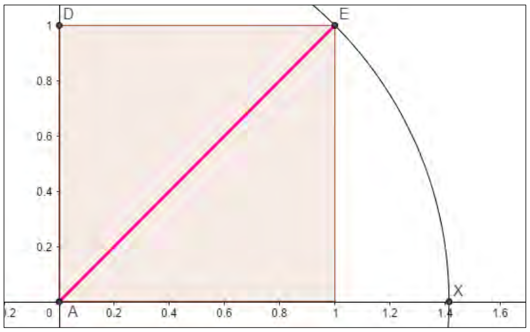


Figure 13: Néstor Record 1 (Level 4)

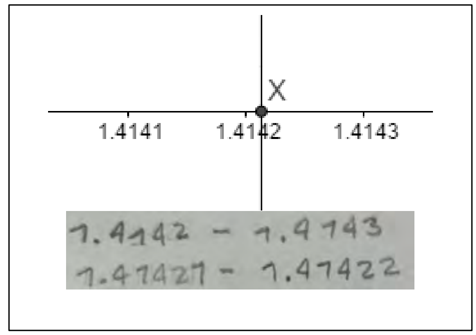


Figure 14: Néstor Record 2 (Level 4)

Conclusions

The design and development of the HLT involved the search and analysis of school mathematics topics that could guide the student to learn the number density. Topics such as similarity of triangles, arithmetic and geometric progressions, property of continuity and diagonal of a square, helped the students not only in their learning process but in their process of understanding the property of density. However, both students retained the idea of the existence of a successor in a set other than that of natural numbers. In the last phase of this first cycle, although the students improved their skills to find intermediate numbers in an interval, it was still difficult for them to make a metaconceptual awareness about the non-existence of a successor in the real numbers. This aspect will be considered for the refinement of the learning trajectory in the second cycle of activities. On the other hand, the students used various semiotic representations such as fractional and decimal writing. Colloquial language was one of the semiotic registers most used by students to express their thoughts in their own words. Finally, the hypotheses raised showed that it is feasible for the student to mitigate the difficulties on the density property. It is suggested that these hypotheses are more aimed at identifying characteristics that make a set dense as a discrete one, which would possibly lead to the understanding of a unique successor in natural numbers.

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Durante su vida escolar, el estudiante aprende temas de las matemáticas que podrían considerarse esenciales para la comprensión de la propiedad de densidad en el conjunto de los números reales. Por ello se contempló la necesidad de diseñar y elaborar una Trayectoria Hipotética de Aprendizaje para incluir temas que ayuden a propiciar el aprendizaje de esta propiedad. Este informe muestra los resultados de la primera experimentación educativa de una *investigación que aún está en curso. Se expone la delineación de la trayectoria en la que el estudiante, de bachillerato, logra reconocer formas de hallar números en un intervalo usando varias representaciones semióticas. Así mismo, se describen algunas dificultades que tiene el estudiante para reconocer la no existencia de un sucesor en los números reales.*

Palabras clave: trayectorias de aprendizaje y progresiones, conceptos de números y operaciones, representaciones matemáticas, educación media superior

Introducción y Preguntas de Investigación

Se ha visto que muchos estudiantes, desde que inician su educación primaria hasta que terminan con la educación media superior, incluso con la universitaria, muestran deficiencia para comprender la propiedad de densidad en el conjunto de los números reales (Tirosh et al., 1999). En la primaria, el estudiante solo tiene una oportunidad para aprender acerca de la propiedad de densidad de los números decimales (Ávila y García, 2008). Por ejemplo, en México, el docente comparte con sus estudiantes de sexto de primaria (alrededor de 12 años) una unidad relacionada con esta propiedad usando la recta numérica (SEP, ciclo escolar 2020-2021). En el proyecto de Vamvakoussi y Vosniadou (2010), estudiantes de secundaria (y bachillerato) entre 12 y 17 años de edad, muestran dificultad para comprender la propiedad de densidad en los números reales. Algunos estudiantes de este proyecto creen que no hay otro número entre 0.005 y 0.006, o entre 1/3 y 2/3. Es decir, para ellos, estos pares de números son "consecutivos" cada uno. Al parecer, los estudiantes creen en la existencia de un sucesor en los números reales. Y otros participantes del estudio mencionado, refieren que hay una cantidad finita de decimales entre 0.005 y 0.006.

Por otro lado, Vamvakoussi y Vosniadou (2010) han notado cómo hay estudiantes que se ven afectados por la representación simbólica de los extremos de un intervalo. Las autoras observaron que estudiantes tienden a expresar que no hay fracciones entre decimales, que solo puede haber decimales; y viceversa. Por ello, resulta importante, como señala Duval (2004), que se manejen representaciones diferentes a lo largo el trayecto escolar de los estudiantes. Según Duval, enseñar y aprender matemáticas conlleva a que algunas actividades cognitivas (conceptualización, razonamiento, comprensión, entre otras) requieran, además del lenguaje natural o el de las imágenes, la utilización de diferentes registros de representación semiótica. Para este autor, en matemáticas se encuentran distintos sistemas de escritura para los números, notaciones simbólicas para los objetos, relaciones y operaciones, así como también una variedad de gráficas; cada una de las actividades anteriores constituye una forma semiótica diferente.

Con miras a superar la dificultad que tienen los estudiantes para comprender la propiedad de densidad en los números reales, se está llevando a cabo una Investigación Basada en el Diseño (Cobb y Gravemeijer, 2008), donde se diseñó una Trayectoria Hipotética de Aprendizaje (THA) como una propuesta didáctica utilizando diversos registros de representación. Simon (1995) plantea que una THA es una secuencia de actividades para atender algún concepto matemático, donde dichas actividades se construyen a partir de una o varias hipótesis que apoyen al estudiante en la construcción de nuevos conocimientos.

Teniendo en cuenta lo anterior y la problemática que se genera para comprender la propiedad de densidad se elaboraron las siguientes preguntas de investigación:

a.) ¿cómo diseñar una THA para promover el aprendizaje de la propiedad de densidad en el conjunto de los números reales?, y
b.) ¿cómo estudiantes de bachillerato comprenden la propiedad de densidad en los números

reales usando representaciones semióticas que se pueden producir durante la THA?

Marco Teórico

Vamvakoussi y Vosniadou (2004) plantearon la hipótesis de que la comprensión de la propiedad de densidad de los números racionales es un proceso gradual: de lo discreto a lo denso. Las autoras concluyen que el conocimiento previo sobre los números naturales apoya al estudiante al uso de la propiedad de lo discreto para solucionar tareas vinculadas con los racionales, en efecto restringe la comprensión de la propiedad de densidad. Recurriendo a las ideas de Ni y Zhou (2005), el acto de contar por un niño es su primer acercamiento a una representación del número natural, de lo discreto; esta representación perdura en el niño a tal grado que en problemas vinculados con fracciones o decimales él toma en cuenta propiedades de los números naturales para solucionarlos. Vamvakoussi y Vosniadou, en el 2004, realizaron una investigación para conocer qué tanto saben los estudiantes de noveno grado (alrededor de 15 años de edad) acerca de la propiedad de densidad, posteriormente, elaboraron cinco categorías con base en las respuestas de los participantes (ver Tabla 1).

Tabla 1: Categorías del pensamiento sobre la cantidad de números en un intervalo

Pensamiento ingenuo sobre lo discreto	Se considera que no hay otro número entre dos números racionales *consecutivos falsos* (expresión instaurada por Vamvakoussi y Vosniadou, en 2004, para referir que existe un sucesor en los números racionales.)
Pensamiento avanzado sobre lo discreto	Se cree que hay un número finito de números intermedios entre dos números racionales consecutivos falsos.
Pensamiento compuesto entre lo discreto y lo denso	En algunas situaciones se piensa que entre dos números racionales hay una cantidad infinita de números, y en otros, que hay una cantidad finita.
Pensamiento ingenuo sobre lo denso	Se contempla que hay una infinidad de números en un intervalo, pero no se justifica la situación usando la propiedad de densidad. La representación simbólica de los extremos de un intervalo influye en la forma de pensar; se cree que solo puede haber una infinidad de decimales entre decimales, pero no fracciones; de igual modo sucede con fracciones entre fracciones.
Pensamiento avanzado sobre lo denso	Hay una comprensión bastante sofisticada de la propiedad de densidad, es decir, se pone de manifiesto que se entiende que entre dos números racionales hay una infinidad de números, independientemente de su representación simbólica, y se justifica con la propiedad de la densidad.

Para comprender la propiedad de densidad en los números reales es necesario tener en cuenta que el número real tiene diversas representaciones (Apóstol, 2006). No obstante, Duval (1983) recalca que esta tarea no es sencilla para un estudiante, le cuesta reconocer el mismo objeto matemático a través de varios contextos de representación. Como sucedió en el estudio hecho por Neuman (2001), estudiantes de séptimo grado (alrededor de 13 años edad) no aceptaban que podía haber una fracción entre 0.3 y 0.6, pues para ellos las fracciones y los decimales eran objetos matemáticos distintos que no tenían relación alguna. En términos de Duval, el aprendizaje de un concepto matemático requiere del recurso de varios sistemas semióticos de representación, lo que implica un pensamiento matemático más profundo por el estudiante.

Aspectos relevantes de una THA

Complementando la definición de una THA, Simon y Tzur (2004) describen que esta expresión se refiere a las predicciones (hipótesis) que tiene el investigador, o el profesor, sobre cómo un estudiante puede desafiar los escenarios propuestos en un camino de aprendizaje. El camino se vuelve "real" (trayectoria real) cuando se conoce las concepciones o las ideas del estudiante durante la socialización de las actividades (Simon, 1995). Para Simon, durante el recorrido de la trayectoria, el investigador puede ir modificando varios aspectos, entre ellos la duración y el diseño mismo de las lecciones de aprendizaje, esto como resultado de las interacciones que van surgiendo con los estudiantes. Así, una THA le proporciona al investigador un criterio racional para decidir el diseño que él está considerando, así como la mejor hipótesis de cómo puede avanzar el proceso de aprendizaje en el estudiante. Siguiendo el discurso de Simon, se consideran tres componentes esenciales de una THA: a) *los objetivos*, entendidos como el conjunto de enunciados de los que se espera llevar a cabo el cumplimiento de las acciones, b) *la ruta de las actividades de aprendizaje*, en la cual los estudiantes progresan, constituida por niveles cada vez complejos con actividades instruccionales que fomentan el paso de un nivel a otro, e c) *hipótesis planteadas por el investigador*, entendidas como las conjeturas que planea un investigador sobre el proceso de aprendizaje.

Metodología

Nuestra metodología de investigación se sitúa en el contexto de la Investigación Basada en el Diseño (IBD), con un enfoque cualitativo de un estudio de casos. La IBD es un enfoque metodológico en el que el investigador intenta examinar, de manera sistemática y minuciosa, cómo los estudiantes afrontan tareas propuestas, y también analiza estrategias y herramientas de enseñanza (Cobb y Gravemeijer, 2008). En términos de estos autores, en una IBD se intenta experimentar para apoyar el aprendizaje, a través del diseño de una THA. Cobb y Gravemeijer recomiendan ir probando y mejorando las conjeturas o las hipótesis que se esbozan en la trayectoria, razón por la cual sugieren la ejecución de varios ciclos de análisis y diseño de actividades. En la presente investigación, nos enfocamos en una THA para el aprendizaje de la propiedad de densidad, cuyo refinamiento contempla dos ciclos. En este momento la investigación está culminando el primer ciclo de diseño y análisis.

Participantes

La población de la investigación en curso está conformada por cuatro estudiantes con edades entre 15 y 17 años, quienes cursan la educación media vocacional (bachillerato) en Colombia. Debido a la pandemia de COVID-19, las actividades se han realizado de manera individual e híbrida, algunas en casa del estudiante y otras de manera virtual, durante un periodo de tres meses. En este informe se mostrará la participación de dos estudiantes, Néstor y Paola (pseudónimos), quienes completaron la ruta de la THA propuesta.

Diseño de la THA

La experimentación educativa correspondiente al primer ciclo tiene tres fases, la primera y tercera constan de la aplicación de un pretest y postest respectivamente, y la segunda se constituye de la THA. En este informe se mostrarán resultados de la primera y segunda fase, y en el apartado de las conclusiones se mencionarán detalles breves de la última fase.

Primera y segunda fase. Los estudiantes resuelven un pretest y un postest –de cuatro ítems– con una duración de 30 minutos cada uno, y miden qué tanto saben sobre lo discreto de los números naturales y lo denso de los números reales. A través del pretest y el postest se indaga si al final de la implementación de la THA ha habido cambios en forma de razonar del estudiante respecto a la propiedad de lo discreto de los números naturales, razón por la cual se utiliza el mismo cuestionario. Las preguntas de dicho cuestionario se inspiraron en preguntas hechas por Suárez-Rodríguez y Figueras (2020).

Segunda fase. En la preparación de esta fase se tienen tres objetivos:

1. *Establecer las etapas de la THA*. Se concretan estas etapas como niveles de lecciones de aprendizaje, exactamente cuatro (ver Tabla 2); sin embargo, no significa que el estudiante

dependa de un nivel anterior para completar el siguiente. Cada nivel tiene una duración de una hora y 30 minutos aproximadamente.

2. *Definir las hipótesis para cada nivel*. Se plantean las hipótesis de aprendizaje con base en temas que han cursado los estudiantes en su vida escolar y que se cree que pueden guiarles a la comprensión de la densidad (ver Tabla 2). McMullen y Van Hoof (2020) mencionan que si bien esta propiedad no se profundiza en clase, hay momentos de la matemática escolar que pueden proporcionar oportunidades para hablar de ella.
3. *Alcanzar una conciencia metaconceptual*. Se espera que el estudiante pueda lograr una conciencia metaconceptual para que pueda identificar las características que hacen que un conjunto sea denso. Un estudiante asume una conciencia metaconceptual cuando reflexiona sobre algunas de sus suposiciones y que estas no son ciertas, y que además limitan la forma en que interpreta la nueva información (Vosniadou, 1994).

Tabla 2: Niveles e hipótesis de la THA

Niveles.	*Hipótesis*
Nivel 1. Primeros acercamientos a la propiedad de densidad.	A partir de dos situaciones: una relacionada con la cotidianidad y otra vinculada con un escenario hipotético, se piensa que el estudiante puede tener sus primeros acercamientos a la propiedad de densidad.
Nivel 2. Acercamiento a la propiedad de densidad a través de la semejanza de triángulos.	Se contempla que usando semejanza de triángulos los estudiantes puedan aprender sobre la propiedad de densidad de los números racionales.
Nivel 3. Aproximación a la propiedad de densidad a partir de progresiones aritméticas y progresiones geométricas.	Es posible que hallando medios aritméticos y geométricos en un intervalo el estudiante comprenda la propiedad de densidad de los números racionales en el conjunto de los reales.
Nivel 4. Aproximación a la propiedad de densidad por medio de la propiedad de continuidad.	Se cree que usando la propiedad de continuidad los estudiantes comprenden la propiedad de densidad de los números irracionales en el conjunto de los reales.

Resultados

Resultados del pretest

Para el análisis de las respuestas dadas al pretest por los estudiantes se tuvo en cuenta la caracterización hecha por Vamvakoussi y Vosniadou en 2004 (ver Tabla 1).

Pensamiento ingenuo sobre lo denso. En la Figura 1 se muestra la respuesta de Néstor al primer ítem del pretest. El proceso de subdivisiones infinitas en un intervalo fue la representación que más elaboró este estudiante. Se observa cómo él usa la escritura decimal como representación semiótica en un registro aritmético; y cómo él usa lenguaje coloquial para referir que hay infinitos números entre 0 y 1, y para expresar que los números siguen el "mismo ciclo". Esta última frase hace pensar que él no toma en cuenta a los números irracionales en el intervalo, pues los números que él anota son números en escritura decimal periódica.

1. ¿Cuántos números hay entre 0 y 1? Justifica tu respuesta.

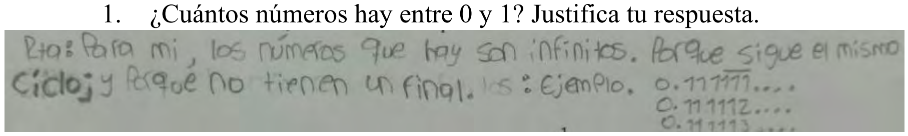


Figura 1: Pensamiento ingenuo sobre lo denso (ejemplo)

Pensamiento ingenuo sobre lo discreto. En la Figura 2 se muestran las respuestas de Paola al cuarto ítem del pretest. Se aprecia que Paola responde a las preguntas de acuerdo con sus conocimientos de número natural. Al parecer, Paola cree en la existencia de "varios sucesores" (0.2, 0.11, 0.111) en los números decimales. También se aprecia su dificultad para percibir la equivalencia entre 0.1 y 0.10, pues ella indica que 0.2 es el sucesor de 0.1, pero que si el número fuera 0.10, su sucesor sería 0.11.

4. Pedro quiere dibujar una línea donde pueda ubicar números decimales. Inicia con el número 0.1. Después coloca el 0.2. En ese momento, Laura lo interrumpe y dice que el número que "sigue" de 0.1 es 0.11. a. ¿Quién tiene la razón, Pedro o Laura?, ¿ninguno de los dos?, o ¿ambos? Justifica tu respuesta.

Pedro tiene la razón, ya que después de 0,1 sigue 0,2, Laura tendría razón en el caso que fuera 0.10.

b. ¿Existe un número que le sigue a 0.1? Si tu respuesta es afirmativa, escribe el número que crees que sigue a 0.1. Pero si tu respuesta no es afirmativa explica por qué.

Si, existe un número que le sigue a 0,1.
0,11 , 0,111 , 0,111...

Figura 2: Pensamiento ingenuo sobre lo discreto (ejemplo)

Resultados de las lecciones de aprendizaje de la THA

Se describen algunas actuaciones de Paola y Néstor durante la puesta en marcha de la THA.

Actuaciones vinculadas con el Nivel 1. En este primer escenario los estudiantes solucionaron dos lecciones de aprendizaje cuya finalidad es tener un primer acercamiento a la densidad. En una de las lecciones, el estudiante debe describir los movimientos que efectúa una rana (primero salta a la mitad, luego salta la mitad de lo que quedó, y así sucesivamente). En la Figura 3 se contempla que Néstor responde la pregunta acercándose a un pensamiento afín con lo denso; no obstante, su expre masiadas veces" puede significar algo finito. Él usa una representación de lenguaje común en su respuesta. En la pregunta de la Figura 4 se aprecia que Néstor realiza el procedimiento para hallar los siguientes términos de la sucesión dada, y usa representaciones semióticas como las escrituras fraccionaria y exponencial en un registro aritmético. En la última pregunta (ver Figura 5), aunque él afirma que hay una infinidad de fracciones entre 0 y 1, no justifica con la propiedad de densidad.

2. ¿Cuántas veces saltará la rana entre los puntos A y B? Explica tu respuesta.

Rta: demasiadas veces; Por que la rana tarde o temprano va a llegar a su destino. [B]

Figura 3: Registro 1 de Néstor (Nivel 1)

4. Teniendo en cuenta la pregunta anterior, ¿cuántas fracciones puedes encontrar entre 0 y 1? Justifica tu respuesta.

Rta: se Pueden encontrar infinitas fracciones Por que hay una infinidad de números.

Figura 5: Registro 3 de Néstor (Nivel 1)

3. Suponiendo que 1 es la distancia entre A y B, es decir, el intervalo de 0 a 1, la sucesión de términos que realiza la rana es: $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16} \ldots$
Ahora, halle los siguientes tres términos de la sucesión. Describe su procedimiento.

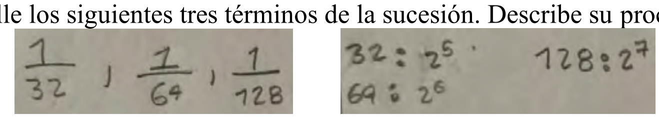


Figura 4: Registro 2 de Néstor (Nivel 1)

Actuaciones identificadas en el Nivel 2. La lección de aprendizaje de esta sesión está inspirada en actividades hechas por Tovar (2011). Se cree que al usar semejanza de triángulos el estudiante puede comprender la propiedad de densidad para hallar términos medios en un intervalo. Al principio, el estudiante lee las instrucciones para que pueda construir triángulos semejantes sobre una recta numérica en GeoGebra. La imagen de la Figura 6 muestra el ejercicio realizado por Paola en la que se observa que entre "A0 y B1" (es decir, entre 0 y 1) 1 punto *x*. Se le pide que halle este punto. Después de pensarlo varias veces, Paola concluye que debe usar proporciones, como se nota en la Figura 7. Ella emplea representaciones gráficas de triángulos semejantes en un registro geométrico, y en un registro algeraico halla el valor de *x*. Cabe mencionar que en las instrucciones del ejercicio están descritos los valores de los segmentos, $\overline{AC}$ = 1 y $\overline{AD}$ = 2, para la resolución de este.

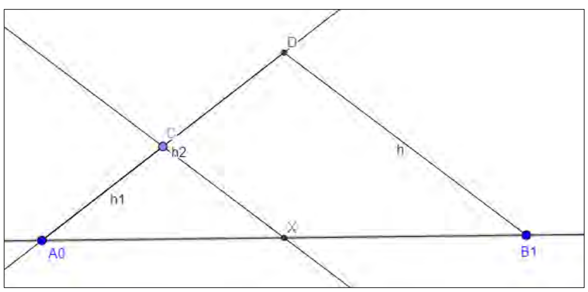


Figura 6: Gráfica de Paola (GeoGebra)

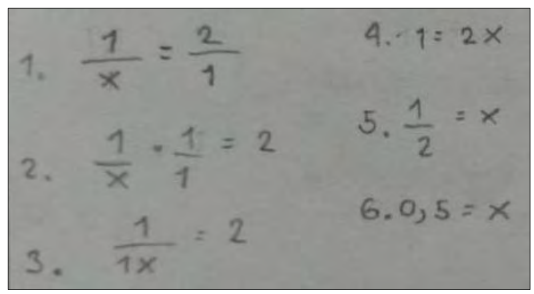
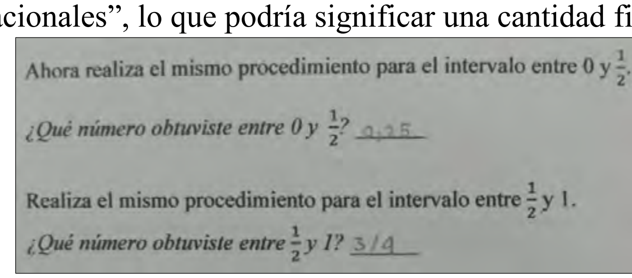


Figura 7: Registro 1 de Paola (Nivel 2)

Paola continúa solucionando las preguntas (ver Figura 8) que conllevan a efectuar el mismo procedimiento para hallar un número entre 0 y 1/2 (ver Figura 9), y otro entre 1/2 y 1. Los números que ha encontrado ella hasta el momento son: 0.25, 1/2 y 3/4; es decir, el segmento unitario fue dividido en cuatro partes iguales. Finalmente, los estudiantes responden la pregunta que aparece en la Figura 10. Se observa en esta imagen que Paola se acerca a un pensamiento afín con lo denso, pues contempla que cada vez se obtienen mitades y que esto tiene un proceso infinito. Mientras que Néstor, al parecer, no considera la infinidad sino "varios números racionales", lo que podría significar una cantidad finita.


Ahora realiza el mismo procedimiento para el intervalo entre 0 y $\frac{1}{2}$.

¿Qué número obtuviste entre 0 y $\frac{1}{2}$? 0,25

Realiza el mismo procedimiento para el intervalo entre $\frac{1}{2}$ y 1.

¿Qué número obtuviste entre $\frac{1}{2}$ y 1? 3/4

Figura 8: Registro 2 de Paola (Nivel 2)

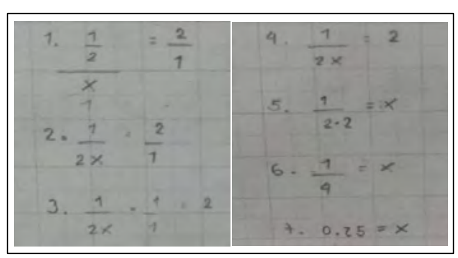
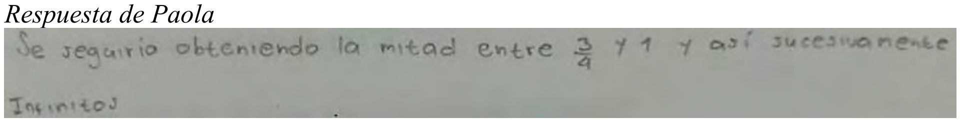


Figura 9: Registro 3 de Paola (Nivel 2)

5. ¿Qué sucede si continúas realizando el mismo procedimiento entre $\frac{3}{4}$ y 1? Y así sucesivamente, ¿cuántos números racionales se obtendrían?

Respuesta de Paola



Respuesta de Néstor

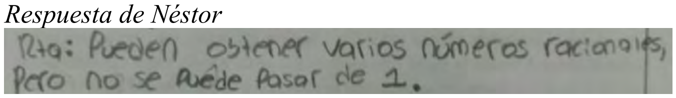


Figura 10: Registro de Paola y Néstor en la pregunta 5 (Nivel 2)

Actuaciones asociadas con el Nivel 3. Para este nivel se diseñaron tres lecciones de aprendizaje concernientes a progresiones aritméticas y progresiones geométricas para comprender la propiedad de densidad de los racionales en los números reales. En una de las lecciones el estudiante lee un breve párrafo sobre la definición de progresión aritmética. En seguida, el estudiante debe hallar cinco medios aritméticos entre 4 y 22 usando el término n-ésimo de una sucesión: $u = a + (n - 1)d$, donde a es el primer término, n el número de términos y d la diferencia entre un término y otro. En la Figura 11 se muestra la resolución de Paola al ejercicio –en un registro algebraico– y concluye que $d = 3$, y escribe los términos hallados.

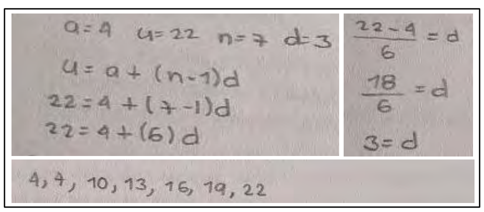


Figura 11: Registro de Paola (Nivel 3)

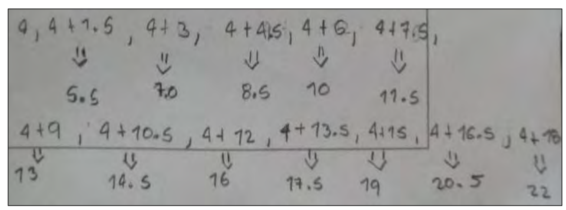


Figura 12: Registro de Néstor (Nivel 3)

Posteriormente, el estudiante debe responder la siguiente situación: *Si d, de la pregunta anterior, se redujera a la mitad, ¿cuáles serían los nuevos medios aritméticos entre 4 y 22?* En la Figura 12 se muestra que Néstor solo utiliza el primer término para encontrar los nuevos medios aritméticos. Él suma 1.5 a 4, luego suma el doble de 1.5 a 4, en seguida, el triple de 1.5 a 4, y así hasta llegar a 22. Néstor hace uso de representaciones de sumas de números decimales en un registro aritmético. Finalmente, se les solicita a los estudiantes hallar la media aritmética entre a y u usando la expresión para hallar el término n-ésimo de una sucesión. Sin embargo, Paola y Néstor no pudieron resolver este punto de la actividad porque les parecía difícil.

Actuaciones identificadas en el Nivel 4. Para este escenario se diseñó una lección de aprendizaje inspirada en actividades de Tovar (2011). Se espera que el estudiante pueda aprender la densidad de los irracionales en los reales, a partir de la continuidad de la recta con respecto a la no correspondencia entre los números racionales y los puntos de la recta. Néstor exploró en GeoGebra c a diagonal de un cuadrado de lado 1 se "traslada" sobre la recta numérica y observa que el punto x está entre 1.2 y 1.6 (ver Figura 13). Luego se le pidió escribir cuatro intervalos en donde esté ubicado este punto. Para ello, él efectuó varios zooms en la pantalla de GeoGebra y anotó dos intervalos (ver Figura 14). Ambos estudiantes observaron que el punto x no corresponde a un racional y que cada vez son más pequeños los intervalos que lo encierran.

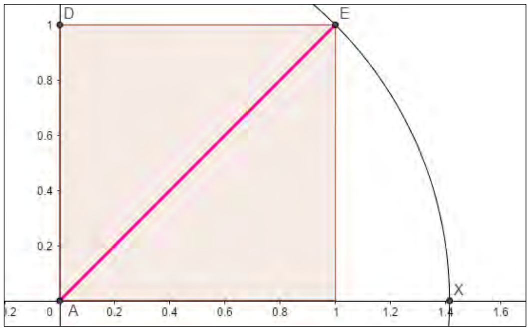


Figura 13: Registro 1 de Néstor (Nivel 4)

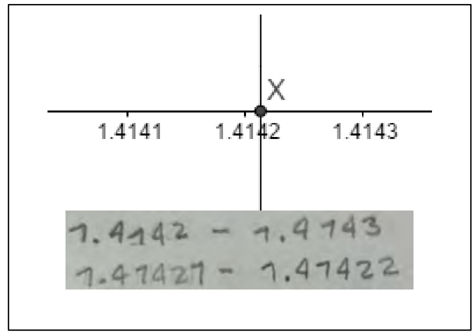


Figura 14: Registro 2 de Néstor (Nivel 4)

Por último, hallaron la diagonal del cuadrado por medio del Teorema de Pitágoras, y luego concluyen que entre dos números racionales se encuentra $\sqrt{2}$. No obstante, ambos estudiantes seguían pensando en la presencia de un sucesor, en este caso para $\sqrt{2}$ cuando se les cuestionó por ello, pero no respondieron cuál podría ser. Esto lleva a pensar que al ser $\sqrt{2}$ un irracional, les es difícil encontrar un supuesto sucesor pese a que ellos afirman su existencia.

Conclusiones

El diseño y elaboración de la THA implicó la búsqueda y análisis de temas de la matemática escolar que pudieran guiar al estudiante al aprendizaje de la propiedad de densidad. Temas como semejanza de triángulos, progresiones aritméticas y geométricas, propiedad de continuidad y diagonal de un cuadrado, ayudaron a los estudiantes no solo en su proceso de aprender sino en su proceso de comprender la propiedad de densidad. Sin embargo, ambos estudiantes conservaron la idea de la existencia de un sucesor en un conjunto que no sea el de los números naturales. En la última fase de este primer ciclo, aunque los estudiantes mejoraron sus habilidades para hallar números intermedios en un intervalo, todavía se les dificultaba hacer una conciencia metaconceptual sobre la no existencia de un sucesor en los números reales. Este aspecto se tendrá en cuenta para el refinamiento de la trayectoria de aprendizaje en el segundo ciclo de actividades. Por otro lado, los estudiantes emplearon varias representaciones semióticas como las escrituras fraccionaria y decimal. El lenguaje coloquial fue uno de los registros semióticos más usados por los estudiantes para expresar con sus propias palabras sus pensamientos. Finalmente, las hipótesis planteadas mostraron que es viable que el estudiante pueda mitigar las dificultades sobre la propiedad de densidad. Se sugiere que estas hipótesis se encuentren más encaminadas en la identificación de características que hace que un conjunto sea denso como de uno discreto, lo que conllevaría, posiblemente, a la comprensión de un sucesor único en los números naturales.

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DIFFERENTIAL BACKWARD TRANSFER EFFECTS FOR STUDENTS WITH DIFFERENT LEVELS OF LINEAR FUNCTION REASONING ABILITIES

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Backward transfer is when learning about a new topic influences an individual's prior ways of reasoning about a topic they previously learned about. This study looked at how quadratic functions instruction differentially influenced students' prior ways of reasoning about linear functions. Specifically, we compared students at three levels of reasoning about linear functions, low-, mid-, and high-level, using a pre/posttest design that bracketed a two-week quadratic functions math program. Results showed that students at different reasoning levels experienced different backward transfer effects, that particular mathematical reasoning processes were most involved in the effects, and that the effects spanned two dimensions of productiveness of mathematical reasoning. Results from this study are significant for better understanding the construct of backward transfer, and have implications for teaching quadratic functions.

Keywords: algebra and algebraic thinking; cognition; learning theory

The study reported in this article integrates two ideas that thus far have not yet been intentionally studied together. The first idea is that when individuals learn about a new concept (C_2), that learning may have the unintended side-effect of influencing the individuals' ways of reasoning about a previously-encountered concept (C_1) (i.e., a concept they previously learned about and already developed ways of reasoning about). We call this effect *backward transfer* (BT) (Hohensee, 2014). A number of studies have reported a variety of BT effects (e.g., Bagley et al., 2015; Hohensee, 2014; Melhuish & Fagan, 2018; Van Dooren, 2004). Importantly, these studies have also shown there can be different BT effects for different students.

The second idea is that students develop ways of reasoning that are more or less productive. Greeno (1989) characterized *productive ways of reasoning* about problem situations as when reasoning is deeply embedded in a problem situation and when that reasoning accounts for the essential properties and relations of that problem situation.

Our study examined the interplay between these two ideas. To explain what we mean, imagine two students with different pre-established ways of reasoning about C_1, who participate in the same learning experiences about C_2. How the BT effects on those students' ways of reasoning about C_1 might compare is an open question. Our study set out to make these kinds of comparisons.

Insights these comparisons reveal would be consequential for the development of BT theory because research thus far has only looked at *what* effects are produced (e.g., Hohensee, Gartland, Willoughby, & Melville, 2021), without trying to account for *how* those effects are different across students. Insights would also be consequential for future research on BT because, once more is known about differences in BT effects across students, future research can examine how to address the differences (e.g., find ways to promote backward transfer that enhances productiveness of ways reasoning about C_1).

Theoretical Perspective

Our theoretical perspective has three parts: part one is about mathematical reasoning in general, part two is about what productive mathematical reasoning is (and is not), and part three is about BT effects on mathematical reasoning. It is the relationship between productiveness of mathematical reasoning and BT effects on reasoning that is the topic of our study.

Mathematical Reasoning

Our theoretical perspective on mathematical reasoning aligns with Jeannotte and Kieran's (2017) view that all mathematical reasoning is made up of thought and communicational elements that are organized on two interrelated dimensions, a process dimension and a structural dimension.[1] The process dimension refers to the steps taken by thought or communicational elements to reach an intended mathematical goal. Jeannotte and Kieran specify nine such processes: generalizing, classifying, comparing, identifying a pattern, validating, justifying, proving, formal proving, and exemplifying. Of the nine processes, classifying, comparing, and identifying a pattern were most central to our study.

One type of reasoning important for our study was *quantitative reasoning*. As we conceive it, quantitative reasoning requires the processes of *classifying* and *comparing*. Classifying is defined as the process of inferring "a class of objects based on mathematical properties and definitions" (Jeannotte & Kieran, 2017, p. 11). For an example of a falling rock, two quantities that could be classified are the distance and time the rock falls. Additionally to measure a particular quantity, comparisons must be made between amounts of a quantity and a standard of measure for that quantity (e.g., compare a meter stick to the distance a rock falls).

A second type of reasoning important for our study was *covariational reasoning*. As we conceive it, covariational reasoning requires the process of *comparing*. Comparing is defined as "the search for similarities and differences [to infer a] narrative about mathematical objects or relations" (Jeannotte & Kieran, 2017, p. 11; parenthetical added). During covariational reasoning, what is being compared are the ways "two varying quantities . . . change in relation to each other" (Carlson et al., 2002). For the falling rock example, corresponding differences in distance and time could be compared. Note that while quantitative and covariational reasoning are tied to classifying and/or comparing, other process likely also play a role (e.g., making *generalizations* during quantitative reasoning, justifying one's covariational reasoning, etc.).

Productiveness of Mathematical Reasoning

Our theoretical perspective on productiveness of mathematical reasoning aligns with Greeno (1989), who characterizes productiveness on four dimensions, two of which are the following: (a) the degree to which reasoning is deeply embedded in the problem situation, and (b) the degree to which reasoning accounts for essential properties and relations in a problem situation.[2] According to our interpretation of these dimensions, when comparing students, those who engage in *more* of a particular kind of reasoning (e.g., more of the same kind of classifying or comparing) in ways that are relevant to a particular problem situation, are more deeply embedded in the problem situation. Similarly, we interpret those students who engage in the kinds of classifying and/or comparing that is more relevant to the problem situation, as better accounting for the essential properties and relations of the problem situation. Thus, these are two ways students' reasoning can be categorized in terms of its productiveness.

Backward Transfer Influences on Mathematical Reasoning

Finally, our theoretical perspective on how BT influences mathematical reasoning is based on Lobato's (2012) perspective on forward transfer, which is that transfer is "the influence of a learner's prior activities on her activity in novel situations" (p. 233). BT, which is "the influence that constructing and subsequently generalizing new knowledge has on one's ways of reasoning about related mathematical concepts that one has encountered previously" (Hohensee, 2014, p. 136), aligns with Lobato's perspective on forward transfer because we too were interested in studying *all* influences. However, our definition of BT also departs from Lobato's definition of forward transfer because we were interested in influences on reasoning about previously-encountered concepts by new knowledge, rather than in the opposite direction.

A number of other mathematics education researchers have observed this phenomenon (e.g., Bagley et al., 2015; Lima & Tall, 2008; Melhuish & Fagan, 2018; Van Dooren et al., 2004). However, to our knowledge no studies have specifically examined the relationship between BT effects and productivity of mathematical reasoning. In this study, we were driven by the following research question: *In what ways are BT effects similar and/or different for students whose prior ways of reasoning are more productive (e.g., deeply embedded in a problem situation) compared to students whose prior ways of reasoning are less productive (e.g., not grasping as essential properties or relations of a problem situation)?*

Methods

Setting and Participants

Our study took place during a summer mathematics program in the Mid-Atlantic region of the United States. Participants were recruited from an organization that helps students of color enhance their college readiness. The students' grade-level ranged from 9th to 11th grade. Our study was centered around a two-week summer math program on quadratic functions. The program took place at a local university and was taught by the primary investigator. Students had two 60-minute lessons per day. This study focused on data from four students whose reasoning about linear functions represented varying levels of productiveness.

Procedure

The study began on the first day of the math program with a linear functions paper-and-pencil pre-assessment. The students had previously learned about linear functions, and as the assessment showed, came in with varying levels of productiveness in reasoning about linear functions. Students were also interviewed about their solution methods on the assessment. Next, students participated in 16 lessons about quadratic functions that focused on covariational reasoning (i.e., the math program). At the end of the program, students took a linear functions paper-and pencil post-assessment and were interviewed again about their solution methods.

Assessments. The assessments assessed the students' abilities to reason about various linear function problems. There were three main problems on the assessment each containing several sub-questions. The first problem made use of graphical representations, the second made use of tabular representations, and the third made use of pictorial representations. Two versions, A and B, of the linear functions assessment were developed. The versions varied in context and in numerical values, but not in structure or in mathematical intent. Students were randomly assigned to one version for their pre-assessment and the other version for their post-assessment.

Math program instructional pattern. The math program was designed as a two-week course on quadratic relationships. The principle investigator, a university professor who was previously a high school mathematics teacher, was the instructor for the course. The focus of the program was to develop students' abilities to reason covariationally with quadratic functions. An inquiry-oriented instructional approach was utilized and quadratic functions were represented with tables and with SimCalc dynamic software.

Data Set

Our data set consisted written responses to the pre- and post-assessments, and video/audio recordings and transcripts from semi-structured interviews.

Data Analysis

We began by reviewing the assessments and the interviews in order to identify four students, of varying levels of productiveness in reasoning about linear functions. We looked for at least one student from each of the following categories: higher-, mid- and lower-level linear function reasoners. We also looked for students who appeared to exhibit changes in ways of reasoning from pre- to post-assessment. We ended up choosing one high-level, one low-level, and two mid-level linear function reasoners.

During analysis, each member of the research team analyzed one student's data, taking a grounded theory approach (Strauss & Corbin, 1998). During open coding, each research team member went sub-question by sub-question through the written and interview responses for their student, looking for changes in ways of reasoning from pre- and post-assessment. Each new change in reasoning became a new code. When each student had been analyzed, we compared the codes and consolidated those that were similar. For each student, a second member of the research team reviewed the coded changes in ways of reasoning f to triangulate the data. During axial and selective coding, each team member identified associations between categories of changes in ways of reasoning and organized and integrated the categories into a story for each student and presented the story to the group for feedback. Finally, the team interpreted each change in reasoning in terms of Jeannotte and Kieran's (2017) mathematical reasoning processes and Greeno's (1989) dimensions of productive reasoning.

Results

In this section, we present each student's changes in ways of reasoning, starting with Rashana, the higher-level linear function reasoner, followed by Layla, the lower-level linear function reasoner, followed by Yolanda and Damien, the mid-level linear function reasoners. For each student, we state the core category and several subcategories of how their reasoning changed from pre- to post-assessment. Then, we illustrate the core category and one subcategory.

Rashana

Rashana, the highest-level linear function reasoner of the four students, changed some of her ways of reasoning linear functions from the pre- to post-assessment. There was a core category we called *improved quality of the responses*, and four subcategories we called (a) *expansion of covariational reasoning*, (b) *more quantities notice*, (c) *exploration of relationship between quantities*, and (d) *different representations used*. Each subcategory represents a dimension on which Rashana's reasoning became more productive from pre- to post-assessment.

Core category: Improved quality of responses. From pre- to post-assessment, Rashana improved the quality of several responses. Interestingly, however, on the six problems that we coded her response as having improved from pre- to post-assessment, the correctness of her answers did not change. For example on problem 3(a) of the pre-assessment, Rashana correctly solved a problem about a plant growing at a constant rate by first finding the equation $y = 1.6(x+1) + (-.2)x$. This equation was technically correct. However, Rashana indicated she was uncertain about why the $(-.2)x$ was needed, other than that the equation did not work without adding that expression. In contrast, on problem 3(a) of the post-assessment, Rashana correctly solved a similar problem by representing the data set with a table, without any uncertainties.

Subcategory: More quantities noticed. We subcategorized some changes in Rashana's ways of reasoning as *more quantities noticed*. For example, on 3(d) of the pre-assessment Rashana noticed the following quantities for a plant growing at a constant rate: the changes in the day, and changes in the height. In contrast, on problem 3(d) of the post-assessment, Rashana again noticed the changes in the day and the changes in height, but also changes in the changes in the day and the changes in the changes in height.

Applying Jeannotte and Kieran's (2017) conceptualization of mathematical reasoning to this subcategory, we interpreted the change to notice more quantities as an increase in the process of *classifying* (i.e., classifying more quantities). Also, applying Greeno's (1989) conceptualization of productivity of reasoning to this subcategory, we interpreted this a productive change in favor of becoming more deeply embedded in the problem on the post-assessment problem.

In sum, Rashana, who represented a high-level linear function reasoner, nevertheless exhibited BT changes in her ways of reasoning that considering correctness alone did not reveal.

Layla

Layla, the lowest-level linear function reasoner of the four students, also changed some of her ways of reasoning about linear functions from the pre- to post-assessment. There was a core category we called *mixed changes in quantitative reasoning*, and three subcategories we called (a) *new ways of reasoning with quantities* (b) *new ways of reasoning with changes in quantities*, and (c) *new ways of finding and reasoning about rates of change.* In contrast to Rashana, Layla's responses in several instances became less productive from pre- to post-assessment, although at times there were also more productive aspects.

Core category: Mixed changes of quantitative reasoning. From pre- to post-assessment, Layla's ways of reasoning changed on six responses. Moreover, four of the six were less correct from pre- to post-assessment, and the other two stayed at similar levels of correctness. However, we did observe some productive development in her ability to reason quantitatively. For example, on problem 1(a) of the pre-assessment, Layla *correctly* applied the slope formula to a linear graph representing gas left in a car's gas tank vs the distance the car traveled. However, her explanation lacked evidence of quantitative reasoning: "So basically, I started out by finding the total amount. So I did the slope equation for these two first and then I found out that that was the total number of gas use between point A and point C."

On problem 1(a) of the post-assessment, Layla *incorrectly* divided corresponding values of gallons used by distance driven. However, her explanation had more reasoning with quantities:

> So, I said the gas in Car 1 is decreasing as the miles driven increases. The gas in Car 1 has decreased drastically by point C. So, basically, I did the distance driven over the gallons left in the tank . . . Those were the changes. The changes in the - oh my gosh - the changes in the um, we were just talking about this! The changes in, um, I would say the changes in gallons.

We interpreted this excerpt as evidence of Layla trying to reason with several quantities, distance driven, gallons of gas in the tank and changes in the gallons. Most changes in Layla's ways of reasoning similarly reflected increased attempts at quantitative reasoning.

Subcategory: New ways of reasoning with changes in quantities. We subcategorized some changes in Layla's ways of reasoning as *new ways of reasoning with changes in quantities*. For example, on problem 2(a) of the pre-assessment, Layla used the slope formula to correctly determine that for a table displaying the additional cost for extra megabytes of data used on a cell phone plan "it increased by .75 cents, each time you used a megabyte." In this response, reasoning with the changes in quantities involved multiplicatively comparing changes in one quantity and changes in the other quantity (i.e., by dividing).

On problem 2(a) of the post-assessment, Layla found changes in additional megabytes used and changes in additional cost between rows of the table to "find the constant rate of additional, that used. But that wasn't really working for me." Thus, Layla went from multiplicatively comparing changes in quantities on the pre-assessment, to looking for additive patterns in the changes in each quantity separately. Altogether, changes in how Layla reasoned with changes in quantities were observed on five problems.

Applying Jeannotte and Kieran's (2017) conceptualization of mathematical reasoning to this subcategory, we interpreted changes in reasoning about changes in quantities as primarily a change in the process of *comparing* (i.e., going from multiplicatively comparing to additively comparing). Also, applying Greeno's (1989) conceptualization of productivity of reasoning to this subcategory, we interpreted this an unproductive change toward grasping *less* of the essential properties and relations for the post-assessment problem than the pre-assessment problem.

In sum, Layla, who represented a low-level linear function reasoner, exhibited BT changes in her ways of reasoning that were mostly not productive but did reflect increased attempts at quantitative reasoning..

Yolanda

Yolanda, one of the mid-level linear function reasoners, also changed her ways of reasoning from pre- to post-assessment. There was a core category we called *greater focus on changes in quantities*, and three subcategories we called (a) *more changes in quantities found*, (b) *more changes in quantities represented*, and (c) *changes in reasoning about changes in quantities.* Like Rashana, each subcategory represents a dimension on which Yolanda's reasoning became more productive from pre- to post-assessment.

Core category: Greater focus on changes in quantities. From pre- to post-assessment, Yolanda's reasoning changed in favor of a *greater focus on changes in quantities*. For example, on problem 3(d) of the pre-assessment, Yolanda focused only on the changes in the height for the growing plant, recording magnitudes of each change in height, and adding brackets to indicate where each change in height applied. On problem 3(d) of the post-assessment, Yolanda again focused on the plant's height, adding brackets to indicate where the changes in height applied. She also focused on the changes in days and the changes in changes in the days. We found evidence of this increased focus on changes in quantities on five problems.

Subcategory: Changes in reasoning about changes in quantities. We subcategorized some changes in Yolanda's ways of reasoning as *changes in reasoning about changes in quantities.* For example, on problem 1(b) of the pre-assessment, Yolanda compared changes in the gallons left in the tank from points D to E and from points E to F (see Figure 6), saying:

> Car 2 does not use the same gas at the same rate between D and E as it does between E and F due to the reason that D to E takes up 1.50 gallons while E to F took up only .75 gallons.

In contrast, on 1(b) of the post-assessment, Yolanda tried to iterate a difference in one quantity to go from one data point to the other:

> So one way I found out, well made me confident, was I just did the pattern again and again on the whiteboard I had. And since I just did 42 times like 42 times 9, 42 times 8 to try to get to 408 but I didn't come to that number.

Thus, Yolanda went from comparing two changes for the same quantity (i.e., between points D and E and points E and F), to iterating a difference in one quantity to go from one value of the quantity to another (i.e., iterating 42 miles to go from 42 to 408 miles). Changes in how Yolanda reasoned about changes in quantities were observed on three problems.

Applying Jeannotte and Kieran's (2017) conceptualization of mathematical reasoning to this subcategory, we interpreted changes in reasoning about changes in quantities as a change in the process of *comparing* (i.e., how Yolanda compared changes in quantities). Also, applying Greeno's (1989) conceptualization of productivity of reasoning to this subcategory, we interpreted this a productive change in favor of grasping more of the essential properties and relations for the post-assessment problem. We claim this because iterating a change in one quantity repeatedly is more consistent with a constant rate of change than comparing static changes in a particular quantity. In sum, Yolanda, who represented a mid-level linear function reasoner, exhibited productive changes in her ways from pre- to post-assessment but that, like Rashana, did not impact correctness.

Damien

Finally, Damien, one of the mid-level linear function reasoners, also changed his ways of reasoning from pre- to post-assessment. There was a core category we called *improved covariational reasoning* and three subcategories we called (a) a *change in reasoning about different quantities*, (b) *better understanding of rates of change*, and (c) *change in the stability of correct application of the slope formula*. Overall, Damien's reasoning appeared to change in favor of an increased ability to reason covariationally in a more productive manner.

Core category: Improved covariational reasoning. From pre- to post-assessment, Damien improved his ability to reason covariationally. In particular, on each of the five problems we coded as having changed responses from pre- to post-assessment, despite not all responses becoming more correct, Damien provided evidence of improved covariational reasoning. For example, on problem 1(a) of the pre-assessment, which was about the graph of the gas remaining in the tank of the car and the distance driven, Damien wrote down the correct slope formula, but incorrectly calculated the slope by dividing Δx by Δy instead of vice versa. Trying again, he subtracted Δx from Δy rather than dividing Δy by Δx. When asked what his calculation meant, Damien struggled to reason covariationally, replying, "for each, um, mile driven, 30 gallons are wasted." This incorrect response suggested he did not have a clear understanding of the meaning of slope. Understanding slope is an important aspect of reasoning covariationally.

On problem 1(a) of the post-assessment, Damien used the slope formula to correctly calculate that the slope between points A and B and between points B and C was -0.031, and correctly wrote "per mile driven 0.031 gallons of gas are used." In the interview, Damien confirmed, by interpreting the slope, that there had been somewhat of a productive change in his covariational reasoning, saying "It's negative 0.031 because that's how much is decreasing by." This response suggests Damien was reasoning more covariationally.

Subcategory: Better understanding of rates of change. We subcategorized some changes in Damien's ways of reasoning as indicating a *better understanding of rates of change*. For example, on problem 2(a) of the pre-assessment, Damien was asked to consider the cell phone data table. Damien correctly applied the slope formula but was unclear about why that worked: "I don't know how to describe it, but, um . . . when I was in slope intercept in eighth grade and I just remember doing this for every question that I would get that would be like this." In contrast, on problem 2(a) of the post-assessment, Damien correctly found and correctly interpreted the rate of change:

> I found out that the one megabyte of data costs 0.75 cents. So they said that they wanted to know how much, um, an additional 51 MB of data would cost. So I've taken 0.75 and multiply . . . Since one megabit of data is 75, well .75, I want it to multiply that by 51 times.

This excerpt suggested Damien had a better understanding of the rate of change.

Applying Jeannotte and Kieran's (2017) conceptualization of mathematical reasoning to this subcategory, we interpreted changes in reasoning in favor of better understandings of rates of change as a change in the process of *comparing* (i.e., rates of change are multiplicative comparisons between changes in one quantity and the corresponding changes for a related quantity). Also, applying Greeno's (1989) conceptualization of productivity of reasoning to this subcategory, we interpreted this a productive change in favor of better grasping the essential properties and relations of a problem situation.

In sum, Damien, who like Yolanda was a mid-level linear function reasoner, exhibited productive changes in his ways of reasoning from pre- to post-assessment. However, in contrast to Yolanda, whose reasoning changed primarily in favor of a greater focus on changes in quantities, Damien's reasoning changed primarily in favor of improved covariational reasoning.

Discussion

The results from this study can be summed up with the following five points. First, three of four students' level of correctness remained stable from pre- to post-assessment, while one student's level of correctness dropped. Second, all students, including the student whose level of correctness dropped, showed at least some productive changes in reasoning from pre to post (i.e., most BT effects were productive). Third, productiveness was impacted on both of Greeno's (1989) productiveness dimensions. Fourth, BT effects largely involved changes in quantitative reasoning and somewhat involved covariational reasoning. Fifth, the reasoning process that appeared most involved in BT was the process of comparing.

Significance. This study is significant because it provides new insights into how BT influences mathematical reasoning processes and productiveness, as well as into how the reasoning of students at different levels is influenced by BT. With respect to reasoning processes, this study is significant because it showed that particular reasoning processes (i.e., *classifying* and *comparing*) can be influenced by BT. Moreover, it showed that BT can influence the amount that reasoning processes are used (e.g., classifying more quantities) and the ways reasoning processes are used (e.g., comparing different quantities).

With respect to mathematical reasoning productiveness, this study is significant because it showed how productiveness can be influenced by BT. Although other studies have reported productive and unproductive BT effects on mathematical reasoning (e.g., Hohensee, 2014), this study was the first to show that these effects can manifest themselves on two of Greeno's (1989) dimensions of productiveness.

Finally, with respect to students who represent different reasoning levels, this study is significant because it showed that BT can influence the reasoning of students at all levels. This finding challenges our previous theory about BT (Hohensee, 2014), that BT primarily affects mid-level reasoners, and that high-level reasoners know too much and low-level reasoners too little to be influenced by BT. It is also significant that our study unpacks ways that students at different levels are influenced by BT. To our knowledge, our study is the first to do so.

Implications. We mention two implications for practice. An implication from our finding that our lower-level linear function reasoner, whose reasoning became less correct from pre to post but who also showed some new quantitative reasoning, is that perhaps it could be useful for this level of reasoner, if teachers revisited an old topic after covering the new topic. By reasoning more quantitatively, these learners may be more ready to further their thinking of the old topic.

A final implication is that emphasizing quantitative and covariational reasoning during quadratic functions instruction should be promoted. Our results suggest that this emphasis productively influences most students' ways of reasoning about linear functions.

Notes

[1] The structural dimension, which is about whether the mathematical reasoning is deductive, inductive, or abductive, was not examined.

[2] Creativity and flexibility, the other two dimensions of productiveness of reasoning, were not examined.

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QUANTITATIVE REASONING AND COVARIATIONAL REASONING AS THE BASIS FOR MATHEMATICAL STRUCTURE FOR REAL-WORLD SITUATIONS

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In this paper we address the question, how do quantitative reasoning and covariational reasoning present as students build structural conceptions of real-world situations. We use data from an exploratory teaching experiment with an undergraduate STEM major to illustrate the explanatory roles quantitative reasoning and covariational reasoning play in, (a) coordinating more than two interdependent quantities, (b) conceiving of real-world situations in more than one way, (c) constructing networks of quantitative relationships, and (d) creating a mathematical expression. We make the case that looking at mathematical model construction through the lens of quantitative reasoning and covariational reasoning may provide insights into students' mathematical decisions as they structure complex real-world scenarios.

Keywords: Mathematical Modeling, Quantitative Reasoning, Covariational Reasoning

Mathematical Modeling plays a central role in supporting students' views of relevance of mathematics to the real world. Many scholars as well as curricular materials have advocated for the importance of including mathematical modeling into the mathematics curriculum because it would motivate the use of mathematics in the world outside of a classroom, for students (Blum & Niss, 1991, Zibek & Connor, 2006; CCSSI, 2010). In addition, empirical studies have also shown that incorporating mathematical modeling in the teaching of mathematics can positively impact both students' learning of mathematics and affective factors which influence students' learning such as interest, motivation, and self-efficacy (e.g., Czocher, 2017; Rasmussen & Blumenfeld, 2007; Schukajlow et al., 2012; Zbiek & Conner, 2006). Despite the field's desire to motivate the learning of mathematical modeling and its inclusion to curricula, mathematical modeling remains highly challenging for students (Stillman, Brown, & Galbraith, 2010; Jankvist & Niss, 2017; Blum, 2011; Blum & Leiss, 2007).

The challenges students experience while engaging in mathematical modeling are multifaceted. Mathematical modeling involves translating between real-world and mathematics in both directions simultaneously. This translation requires the appropriate mathematical and real-world knowledge (Blum, 2011), so that the modeler can associate the appropriate mathematics with the real-world situation. Existing research on mathematical modeling investigates students' mathematical modeling activities, informing the field about challenges students face in given content areas or real-world scenarios. These studies focus on how students simplify the real-world situation, identify important parameters and variables from the simplified situation, transform these identifications into a mathematical representation, and check the validity of the mathematical representation created against real world constraints. However, these studies have given little attention to describe a modeler's model evolution in terms of quantities and relations among those quantities.

The literature on mathematical modeling is clear that mathematical knowledge alone is insufficient for choosing viable mathematics to represent a real-world situation. The modeler, would also need an understanding of the entities present in the real-world situation, how these entities contribute to the aspects that needs to be modeled, and relations among these entities.

That is, the modeler may have to conceive the real-world situation through quantities and relations among those conceived quantities. Before the field of mathematical modeling can promote this kind of understanding among modelers, the field would first need an idea of how quantitative relations are established during model construction. Conceiving of a real-world situation through quantities and establishing relations among those quantities involves quantitative reasoning and covariational reasoning. Therefore, the purpose of this paper is to discuss how covariational reasoning and quantitative reasoning is present in the construction of mathematical models of real-world situations. We first present research that bridges mathematical modeling and quantitative reasoning, Next, we provide our theoretical orientation that was used towards the analysis of data. Next, we present four examples of how quantitative reasoning covariational reasoning is present in model construction. Finally, we discuss implications.

Mathematical Modeling and Quantitative Reasoning

Thompson (2011) claimed that "mathematical modeling is simply mathematics in the context of quantitative reasoning" (p. 52). By this Thompson means, in mathematical modeling a modeler uses mathematical notation and methods to express a relationship among quantities that were constructed by the modeler. Larson (2013) in her study with linear algebra college students explored the role of quantities and quantitative reasoning in mathematical model construction. Larsen operationalized a mathematical model as a system that consists of elements, relationship among elements, and operations that describe how these elements interact. Larsen made the case that quantities act as "elements" in students' mathematical model. Further, she stated that quantitative reasoning provides a language to describe (i) how students consider quantities that are relevant, (ii) how students express the relationship among these quantities, (iii) use these relationships to operate on these quantities, and finally (iv) how these operations would give rise to new quantities that are also "elements" of the students' model. Larsen claims that quantitative reasoning is a central mechanism in model development because *products* (derivation of new quantities by operating on identified quantities) of one stage at model development become the *objects* at the next stage.

Czocher & Hardison (2020) presented methodological approaches for understanding the quantities that modelers identify as situationally relevant in a given modeling tasks and how the conception of these quantities are manifested as observations through external inscriptions and utterances the students' make. They formulated eight observable criteria that can be used as indications that the modeler engaged in the process of quantification. Further, they defined the construct *modeling space* (Czocher & Hardison, 2019) as the set of mathematical models the modeler constructs within a given modeling task to conceptualize students' mathematical model. Collectively, contributions lead the mathematical modeling field in a new path to trace the genesis of students' quantities and to understand how the meanings students attribute to these quantities may change over time in the context of mathematical modeling. In this paper we contribute to the existing conversation that bridges mathematical modeling and quantitative reasoning. In particular, we address the following question: Given that quantitative and covariational reasoning are foundational to mathematical modeling, how do they present in students' conceiving of mathematical structure within a real-world situation?

Theoretical Orientation

We take on the cognitive perspective on mathematical modeling (Kaiser, 2017). In this perspective, mathematical modeling is considered to be the cognitive processes involved in constructing a mathematical model of real-world scenarios. We view *mathematical model* an individual constructs as an external representation of the relationship between the quantities the individual conceived as relevant to the real-world situation she is given to model. We take on Thompson's (2011) view on quantity, where quantity is a mental construction of a measurable attribute of an object. Thompson (2011) defines *quantification*, the mental construction of quantities, as "the process of conceptualizing an object and an attribute of it so that the attribute has a unit of measure, and the attribute's measure entails a proportional relationship with its unit" (p.37). *Quantitative reasoning* (QR) refers to conceiving and reasoning about quantities and the relations among the conceived quantities. A *quantitative operation* is a "mental operation by which one conceives a new quantity in relation to one or more already-conceived quantities" (Thompson, 1994, p.10). Examples of quantitative operations include combining two quantities additively or multiplicatively and comparing two quantities additively or multiplicatively. Each quantitative operation creates a relationship among the quantities operated upon with the quantitative operation and the result of operating. As Thompson (1994) noted, "Conception of complex situations are built by constructing networks of quantitative relationships" (p.11).

Carlson and colleagues (2002) define *covariational reasoning* (CR), a form of quantitative reasoning, to be "the cognitive activities involved in coordinating varying quantities while attending to the ways in which they change in relation each other" (p. 354). Carlson et al. (2002) identified five mental actions students exhibited when engaging in covariational reasoning. These five mental actions include, *coordination* of quantities, coordinating the *direction of change* of quantities, coordinating the *amounts of change* of quantities, coordinating *average rate of change* of one quantity with respect to the other quantity, and coordinating the *instantaneous rate of change* of one quantity over the interval of the domain. Similarly, Thompson and Carlson (2017) proposed six major levels of covariational reasoning. These levels are smooth continuous variation, chunky continuous variation, coordination of values, gross coordination of values, pre coordination of values, and no coordination of values. Moore et al (2020) extend the work of QR and CR by introducing the construct *abstracted quantitative structure*: "a system of quantitative relationships a person has interiorized to the extent they can operate as if it is independent of specific figurative material" (p.752, Moore et al., 2020) as means to explain the construction of a concept. Borrowing ideas from these constructs, we define a *structure for real-world situation* to be the network of quantitative relations among the quantities one constructed as relevant to model a real-world situation. By network of quantitative relations, we mean the set of quantitative relations that was created as a result of operating on conceived quantities. We call this way of understanding about the real-world situation as *a structural conception for the real-world situation.* Students' structural conception of a real-world situation will be analyzed by seeking instances of quantitative and covariational reasoning while they engage in mathematical modeling activities.

Methods

We present data from an exploratory teaching experiment (Steffe & Thompson, 2000) conducted with an undergraduate STEM major at a large university in the United States of America. We used a teaching experiment methodology because it afforded us means to build explanatory accounts of students' mathematical reasoning while they conceive of real-world situations through quantities and relations among those quantities. The overall goal of the exploratory teaching experiment was to investigate how students use quantitative reasoning and covariational reasoning to construct mathematical models of complex real-world scenarios.

Our participant Baxil, a non-native English speaker, participated in a total of ten clinical interview sessions comprising the teaching experiment. Baxil was an undergraduate mathematics major who at the time of the experiment was enrolled in a differential equations course. Each session was approximately an hour long. In this report, we draw data from three teaching sessions where Baxil engaged in the *Baker's Yeast Task*, *The population Dynamics Task*, and *The Fruit Ripening Task*. We focus on these tasks because they provide illustrations of how conceiving of quantities, operating on conceived quantities, reasoning about conceived quantities that change in tandem are present in the structuring for real-world scenarios and because they offer insights into networks of quantitative relations that can be built up into models.

> *The Baker's Yeast Task*: Baker's yeast is a type of fungus that reproduces through budding. Each cell reproduces once every 30 minutes. To grow yeast for baking bread, you have to proof it first (allow it to form a colony) in a bowl of warm water. Suppose that in a particular bowl, after six hours, the surface of the water is covered in yeast cells. Can you come up with an expression that gives the number of cells present after 6 hours if we start with n cells initially?
>
> *The Population Dynamics Task*: Suppose in a laboratory setting, we are looking at large populations of breeding stock in which species give birth to new offspring but also die after some time. Suppose that the given population has a birth rate of α% and, the death rate of the population due to natural causes is β%. If P is the population of species at any given time, write a mathematical expression for the rate at which the population changes with time.
>
> *The Fruit Ripening Task*: There is a surprising effect in nature where a tree or bush will suddenly ripen all of its fruit or vegetables, without any visible signal. If we look at an apple tree, with many apples, seemingly overnight they all go from unripe to ripe to overripe. This will begin with the first apple to ripen. Once ripe, it gives off a gas known as ethylene (C_2H_4) through its skin. When exposed to this gas, the apples near to it also ripen. Once ripe, they too produce ethylene, which continues to ripen the rest of the tree in an effect much like a wave. This feedback loop is often used in fruit production, with apples being exposed to manufactured ethylene gas to make them ripen faster. Develop a mathematical model that captures the dynamics of the ethylene gas produced.

The primary goal of the exploratory teaching experiment was to build accounts of Baxil's mental activities as he reasoned quantitatively and covariationally to construct mathematical models of the task scenarios. Since we did not have direct access to Baxil's mental activities, we created second-order accounts (Steffe & Thompson, 2000) of inferences we made from Baxil's observable activities including his language, verbal descriptions and discourse, written work, and his mathematically salient gestures. Each episode was video recorded, and his written work was digitized.

We conducted both ongoing analysis and retrospective analysis (Steffe & Thompson, 2000). The ongoing analysis involved testing and formulating hypothesis during the teaching experiment based on ways Baxil was reasoning with the quantities he conceived as relevant to model the situations. After the completion of the teaching experiment, we revisited the data to perform an in-depth retrospective analysis. Our retrospective analysis consisted of two phases: observing and describing Baxil's mathematical modeling activities and constructing and refining accounts of Baxil's use of quantitative reasoning and covariational reasoning to mathematically structure the task scenarios. The two phases of the retrospective analysis of Baxil's engagement in the three tasks, comprised of five rounds of data analysis to arrive at examples that could serve for theory-building. First, we watched the videos or subsets of videos without interruption to observe patterns in Baxil's activities. Second, we paid closed attention to Baxil's utterances, gestures, and written work and described his mathematical modeling activities for the three tasks. Third, we identified instances where Baxil was reasoning with conceived quantities (Thompson, 2011; Czocher & Hardison, 2020), operating on the conceived quantities (Thompson, 1994), and engaging in covariational reasoning (Carlson et al, 2002; Thompson and Carlson, 2016). Fourth, we constructed annotated transcripts of such instances that provided rich descriptions of Baxil's mathematical modeling activities. Finally, we constructed and refined explanatory models of Baxil's structuring of the three task scenarios.

Findings

Coordination of three interdependent quantities

In the fruit ripening task, Baxil conceived ripeness as "readiness to eat" the fruit. When the teacher-researcher (TR) asked to draw an ethylene gas production-time graph, Baxil sketched Figure 1. He reasoned "I would say increasing slowly at the beginning, then increasing faster as they are ready to eat because after you're ready to eat, it will produce more instead if it didn't ripe yet." Here, Baxil conceived of a relation among gas production and time where, as time goes on, the rate at which ethylene gas produced increases because as fruits are ripening, they produce more gas. The TR probed his rationale for why the ethelyne gas production would be faster as the fruit ripens. Baxil explained "When you're not ready to eat, it's just like a little bit amount of the gas, I would think, but after it's ready, it goes faster because everywhere have the gas". Baxil engaged in coordination of three interdependent quantities (amount of ethelyne gas produced, gas production, and time), while maintaining pairwise coordination between amount of gas vs. time and gas production vs. time, and production of gas vs. amount of gas.

Figure 1: Baxil's graph for ethylene gas produced vs time.

There can be more than one way of conceiving a real-world scenario.

In the fruit ripening task, The TR asked Baxil to construct a mathematical expression for the amount of ethylene gas produced. He presented two expressions to represent the same scenario and discussed the merits of each:

i. Amount of gas produced by the apple which is ready to eat = $e^{rate\ of\ gas*\ time}$

Olanoff, D., Johnson, K., & Spitzer, S. (2021). *Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Philadelphia, PA.

ii. Amount of gas produced by the apple which is ready to eat = $rate^{time}$

In expression (i), Baxil conceived of rate of gas to be the "percentage of gas inside the apple". By that, Baxil meant the ripeness to ethylene conversion rate. Whereas in the second expression, he indicated that rate would be "the rate of gas that affect the (ripeness of the apple)". Baxil further indicated that the amount of gas, as represented in the first expression, would be increasing slowly. Whereas in the second expression, the amount of gas would increase quickly. This interpretation was evident in his following explanation:

> May I make an example like the raw apple there is a little bit of gas like I say 10% of them I guess, so it might be a 20% of them and the next there is something like that and there is a 40% then a 60% it doesn't add to 100% that's the second equation thinking and for the first equation I was thinking if it is 10% the rate won't be changing... I mean not the rate the like the amount then I say like its 10% it might be and depend on the tense it will be increasing by one-tenth, two-tenth, third-tenth, four-tenth... something like that.

Here Baxil conceived two distinct measurable attributes of the same object, apple. One was by how much the apple produces the gas and the other being by how much the gas affects the apple. As a result, he constructed two expressions that, despite being mathematically equivalent, behaved different to him in terms of quantities and quantitative operations.

Constructing a network of quantitative operations

Baxil initially conceived of the population dynamics scenario in terms of a birth rate and a death rate. Baxil was thinking about the population changing continuously along 1-second chunks of time, indicating chunky continuous variational reasoning. This was evident when he reasoned "every second have some people die and every second have people born". While he was reasoning about how the population changes, he was also coordinating the quantities population, people born, and people dead simultaneously. This was evident when he reasoned "Because if someone is born, so the population is growing as well. That means the principle is changing too. But when people die…the population, it's also going down." By principle, Baxil meant initial population during any time chunk. He also indicated that the quantities birth rate (α) and death rate (β) will be non-varying.

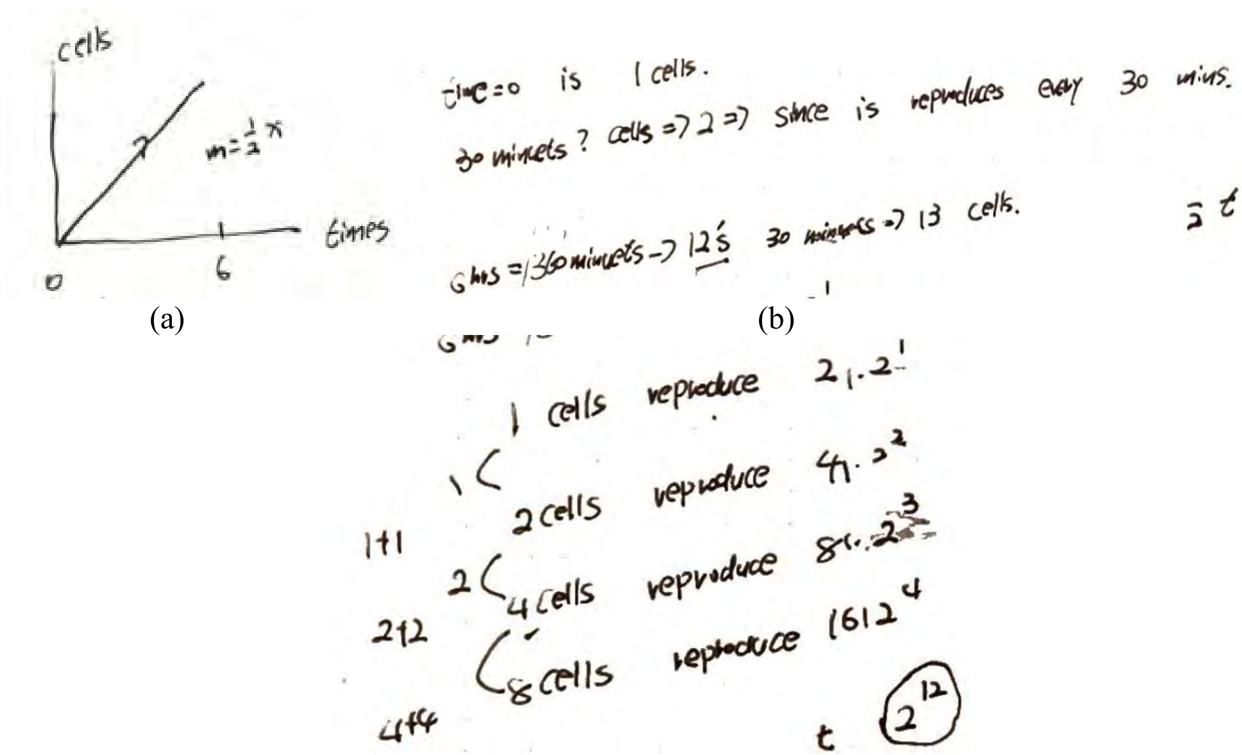
Baxil's final model for this real-world scenario was $P'(t) = P(t)\alpha - P(t)\beta,$ where $P(t)$ is the population over time, $P(t)\alpha$ is the "percentage of people to be born" and $P(t)\beta$ is the "percentage of people that will be dead." Here, he constructed the quantities "percentage of people be born" and "percentage of people be dead" via the multiplicative combination of the quantities (α, $P(t)$) and (β, $P(t)$) respectively. Baxil also gave evidence of thinking of rate of change of population as the net rate of change by meaningfully adding the percentage of people that can be born and subtracting the percentage of people that would die. By additive combination of two quantities, he created an expression for $P'(t)$, which itself is a multiplicative combination of population and time. Baxil was also aware that all of the quantities he constructed, were implicitly dependent on time itself. Therefore, Baxil expressed a network of quantitative operations as a network of arithmetic operations, by operating on the quantities $P(t), t, \alpha, \beta, \alpha P(t), \beta P(t),$ and $P'(t)$.

QR and CR supports in constructing a mathematical expression for a real-world situation.

Baxil's initial conception of the Baker's Yeast scenario was that the "number of cells" and "time" share a linear relationship. He drew the graph in Figure 2(a) reasoning as "because times is increasing…then the cells is also increasing". At this instance, there is evidence to claim that

Baxil was reasoning covariationally, where not only did he coordinate the quantities “number of cells” and “time”, but he also established a directional relationship between them.

The TR then asked him what happens to the number of cells at each 30-minute mark, to which Baxil constructed the representation in Figure 2(b). Although he reasoned that at the end of the first 30 minutes there would be 2 cells “because it reproduces every thirty minutes”, at the end of the 6 hours, he reasoned as “six hours has 360 minutes, which is 12 of thirty minutes, there for 12+1=13 cells”, attempting quantitatively coordinate how the number of cells would change at the end of every 30 minutes. The TR intervened, asking what happens to the 2 cells after at the end of an additional 30 minutes, Baxil answered that 2 cells become 4 cells “because both [cells] reproduce”. As a result, Baxil produced the representation in Figure 2(c) and deduced that at the end of six hours there would be 2^{12} cells. He then wrote down 2^{2t} as the number of cells produced after t hours of time, given that he starts with 1 cell. He attained this structure through coordinating the amounts of change of time and number of cells. Baxil established a quantitative relationship between the number of cells and time through coordinating the direction of change and amounts of change of those quantities. Through his quantitative and covariational reasoning, Baxil was able to structure the Baker’s yeast scenario.



(a) (b)

(c)

Figure 2: (a) Baxil’s graph for how number of cells varies with time from $t = 0$ to $t = 6$, (b) Baxil’s explanation for the number of cells present at the end of 6 hours and (c) Baxil’s final model when starting with 1 cell.

Conclusion and Discussion

In the fruit ripening task Baxil coordinated three interdependent quantities to reason how production of ethylene gas changes with time and brought in two distinct structures, in his perspective, to build a model for the scenario. In the Population Dynamics task, Baxil was able to establish a structure for the situation through reasoning with the quantities: rate of change of prey population, the population of prey, the birth rate, and death rate of prey, through coordinating the direction of change of the quantities population of prey and time and engaging in quantitative operation to produce other quantities. And finally, In the Baker's Yeast task, Baxil was able to establish a structure for the situation through coordinating the value and direction of change of quantities time and number of cells. Although initially his coordination of the amounts of change of these two quantities of these two wasn't representative of the situation (from the TR's perspective), through TR's intervention, he was able to conceive how the number of cells would increase every thirty minutes. Being able to internalize this quantitative coordination, helped him to create a mathematical expression that would predict the number of cells at time t. In conclusion, Baxil structured these three scenarios through conceiving quantities, operating on those conceived quantities, and through engaging in covariational reasoning to discuss how those conceived quantities are related.

A primary goal of the work in QR and CR is describing students' learning of specific mathematical ideas and their attendant reasoning processes. We make the case that QR and CR influences a modelers' structuring for real-world situations. This is because, the manner in which one choses to operate on two or more already conceived quantities, establishes a relationship among the old quantities and the newly created quantity (that resulted by operating on the old ones). This newly constructed quantitative relation influences the construction of other new quantities and operations on those quantities. This way, multiple conceptions of real-world situations can originate through engaging in quantification and covariational reasoning. Therefore, having a structural conception of the real-world situation provides students the opportunity to realize how seemingly different real-world situations can be mathematically modeled using the same mathematics and how the same situation can be modeled using different mathematics, depending on the conceived quantities and the operations performed on them. Looking at QR and CR in model construction allows researchers to pay close attention to get an understanding of the quantities the modeler conceived and the modeler's reasoning about those quantities. This understanding will provide a better picture about the mathematical decisions the modeler makes to mathematize complex situations, particularly during the simplifying and specifying phases of modeling (Zbiek & Conner, 2006; Blum, 2011) that precede the formal mathematical expression of a model. We believe attending to the quantitative and covariational rationales for these decisions will open opportunities for appropriate intervention when necessary.

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STUDENT STRUGGLE DURING COLLABORATIVE PROBLEM-SOLVING IN ONE MATHEMATICS CLASSROOM

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In solving mathematics problems in collaboration, students encounter a range of mathematical and social struggles. As teachers cannot possibly respond to every such struggle, they may need to respond to those with which students require most support. Yet, little is known about students' success in overcoming the various types of struggle they encounter. In this study, we examined the types of struggle students experienced as they worked together in solving a cognitively-demanding problem. We analyzed the relative proportions of the various types of struggle they encountered, their success in overcoming each type, and the resources they leveraged in doing so. While students overcame many mathematical struggles, they had less success overcoming struggles related to reaching consensus or having their questions answered by peers. We argue that teachers may merit from support in learning to attend to these latter, more social struggles.

Keywords: Problem Solving, Middle School Education, Teacher Educators

Conceptual Framework

Struggle is of crucial importance to student learning (Hiebert & Grouws, 2007; NCTM, 2014). Indeed, if students already know all they need to know to solve a problem, the problem is unlikely to result in much struggle or any new learning. Struggle is a sign that one's prior knowledge is inadequate for solving a problem and that something new needs to be learned. When struggling, students may identify gaps in their understanding, which can result in new learning if addressed (Loibl & Rummel, 2014). Research even shows that, when given the chance to struggle in solving a problem before a lesson on the underlying concepts, students develop richer understandings of these concepts than when given the lesson without first having had opportunities to struggle (Kapur, 2010).

Recent research has described the mathematical, or cognitive, struggles students encounter when solving challenging mathematics problems, as well as teachers' responses to such struggles (Warshauer, 2015). In today's mathematics classroom, however, students are likely to encounter not only individual, cognitive struggles, but also a host of other struggles related to the collaborative context in which they are increasingly being asked to solve problems (NCTM, 2014). When solving problems in collaboration with others, students encounter various social struggles, such as the struggle to have their questions answered or ideas taken up by their peers (Langer-Osuna, 2011). If not overcome, these social struggles may limit students' opportunities to engage substantively with the mathematics under consideration and to benefit in their learning as a result. In solving problems collaboratively, students may also encounter struggles that reside at the intersection of the mathematical and the social. For example, they may struggle to explain a solution strategy of theirs to a peer or to understand a peer's explanation (Franke et al., 2015). If granted the opportunity to work through such struggles, both the student providing an explanation and the student listening to one may develop important mathematical understandings (Ing et al., 2018; Webb et al., 2009).

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Prior research has examined teachers' responses to these various cognitive struggles (Warshauer, 2015), social struggles (Dunleavy, 2015), and struggles to engage with one another's mathematical thinking (Franke et al., 2015). Given the many demands on their time and attention when teaching, teachers cannot possibly attend and respond to every such struggle that students encounter. They may thus need to respond to those struggles students have the hardest time overcoming. And yet, knowledge of the relative success students have in overcoming the various types of struggle they encounter remains underspecified. Moreover, little is known about the resources students call upon when struggling or which of these resources prove most helpful in overcoming their struggles.

As a first step to addressing these perceived research needs, the current study involved an analysis of video portraying students' collaborative mathematical work. The study is part of a broader project seeking to unpack the nature of students' struggles, both productive and unproductive, in learning mathematics. We examined the types of struggle a group of four students encountered as they worked together to solve a cognitively-demanding problem (Stein et al., 1996) over the course of a lesson. We analyzed the number and proportion of each type of struggle encountered by the group as a whole and by each individual student in the group. We also examined the extent to which students overcame each type of struggle they encountered and the resources they drew upon in doing so. Specifically, we examined the following research questions:

1. What types of struggle do students in one classroom encounter when solving a cognitively-demanding mathematics problem in collaboration?
2. How successful are the students in overcoming the various types of struggle they encounter?
3. What resources do the students leverage to overcome these struggles?

Methods

Study Context

The data we examined in this study consisted of one classroom video portraying a group of four 7th-grade students solving a cognitively-demanding mathematics problem collaboratively. We chose this video from a large collection of videos collected from the same school district. This particular district is a large, metropolitan district serving students from diverse racial, linguistic, and cultural backgrounds. The district uses a task-based curriculum comprised of challenging tasks designed to be solved using multiple solution strategies and representations, and for which an existing solution strategy should not be immediately apparent (Stein et al., 1996). Moreover, the district has a strong focus on and commitment to the tenets of Complex Instruction (Cohen & Lotan, 2004). For example, in each of the district's middle-school mathematics classrooms, teachers strive to delegate authority to students. They also assign students group roles to ensure that they all contribute substantively to the group's problem-solving efforts and to disrupt a pattern whereby only the voices of high-status students are heard.

Clip Selection

We chose to analyze video as the analysis we describe below would not have been feasible to conduct in a classroom in real time. The video we analyzed for this study was chosen through an iterative process. To begin, the first author viewed 83 videos in their entirety, each about 45-minutes in length. These videos portrayed groups of 3-4 students solving challenging mathematics tasks and were collected as part of a separate research study. The videos were collected with an iPad placed on a tripod, while audio was collected using a table mic. Following data collection, audio from the table mic was synced with the video from the iPad. In viewing these 83 videos, the first author flagged videos in which students appeared to be struggling often, for sustained periods of time, and in varying ways. This struggle was evidenced by students' disagreements, questions, and expressions of confusion. Altogether, this process yielded a collection of 30 classroom videos. Next, the first author revisited notes he had written describing the audio- and video-quality of the videos, dropping those that were of inadequate quality (e.g., in which a student was out of view or students were hard to hear). This left 19 videos. Finally, he read through descriptions he had written for each of these 19 videos, choosing eight that portrayed a variety of different types and magnitudes of struggle. This collection of eight videos became the focus of a larger research program examining struggle.

Together, the two authors of the present study narrowed down this collection of videos from eight to three for detailed analysis. Our selections were guided by a set of criteria rooted in literature on productive struggle and the use of classroom video (Sherin et al., 2009; Warshauer, 2015). The criteria specified that the videos portray substantial student discussion of the mathematics, a range of resources being called upon, and various types of struggle.

In applying these criteria, we identified three videos, one that we analyzed in depth for this paper. We chose to analyze this video first as we had developed some familiarity with it through a separate analysis and believed this familiarity would facilitate the process of applying codes as part of the current analysis. This video portrays a group of four students solving the Mathematics Assessment Resource Service, or MARS, task referred to as Design a Garden (Figure 1).

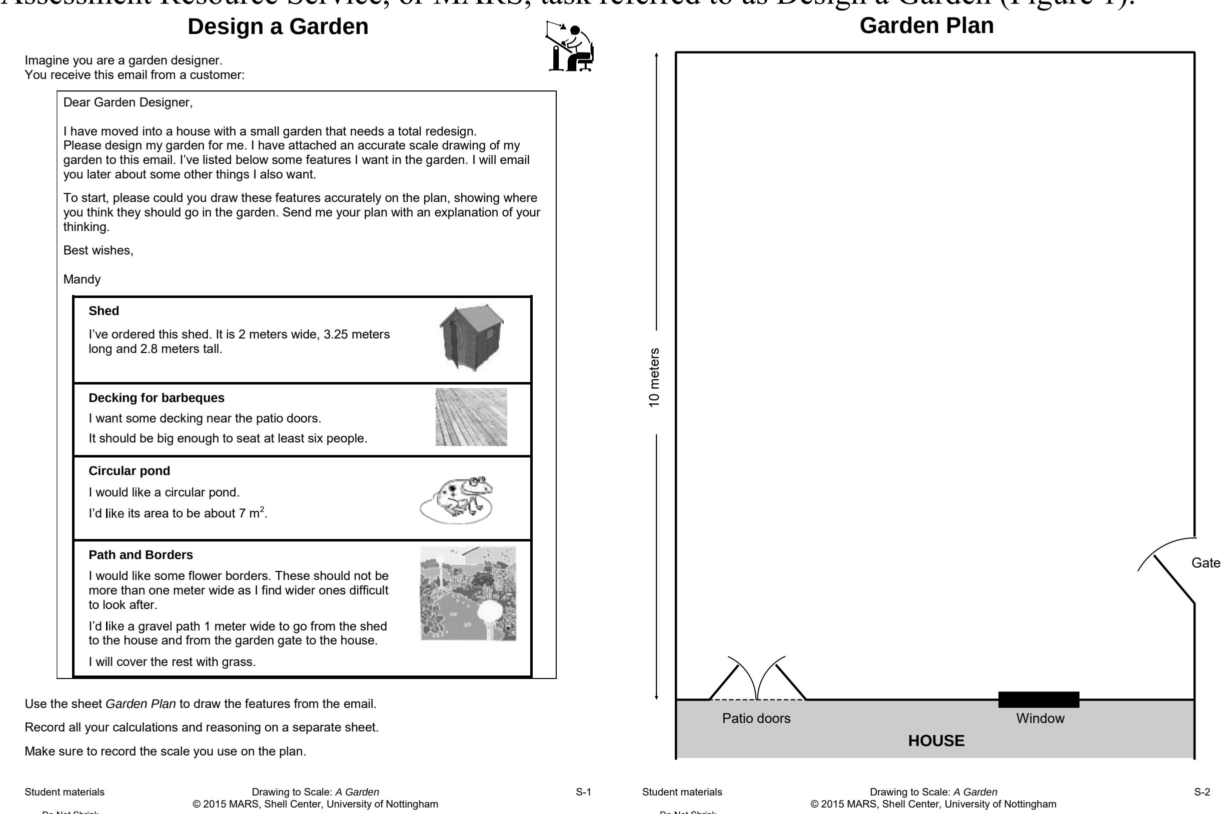

Figure 1. The Design a Garden Task

For this task, students were asked to create a scale drawing of a garden containing the following items: a) a shed, b) a deck (i.e., a patio), c) a circular pond, and d) a path and some borders. We analyzed the 45 minutes and 14 seconds of video portraying students solving this particular problem, excluding a brief warm-up activity at the start of the lesson. This 45:14 of video begins with the teacher launching the Design a Garden task, during which time she asked students to identify all the different items they might find in a garden. Students then used the remaining time to work together to design their garden, ensuring that they included each of the items the task asked them to include. Although each student had their own copy of the task in front of them, they worked together throughout, cognizant that they needed to arrive at consensus regarding where they placed the various items in their design, as well as the dimensions of each item.

Data Analysis

We began our analysis of this video by applying codes to a 12-minute sample, which represented about 10% of the total duration of all three videos we ultimately selected. We did this to refine our coding procedure and establish inter-rater agreement before then splitting off to each code part of the video. Ultimately, however, we decided that we would both code the entire video together.

We coded the video in three phases. First, the two authors independently parsed the 12-minute sample of video into segments that captured any instance in which a student in the video encountered a “roadblock,” which we defined as an impediment or obstacle that slowed students’ progress. Although it was common for more than one student to be involved in particular roadblocks, we created roadblock segments one student at-a-time, as it was too challenging to create segments for multiple students simultaneously. We did not distinguish smaller roadblocks from more substantive ones, instead creating segments for any impediment students encountered, regardless of its magnitude. Next, we met to compare our segments and to arrive at consensus regarding the start and end times for each one. This resulted in a collection of segments for each of the four students in the 12-minute video-clip. In the second phase of coding, we independently applied the following codes to each segment: 1) type of struggle encountered (cognitive, socio-cognitive, social, or materials) and 2) struggle overcome (yes or no). *Cognitive* struggles consisted of individual, mathematical struggles like the struggle to understand the problem or implement a procedure for solving it. *Socio-cognitive* struggles consisted of students struggling to explain a strategy to a peer or to reach consensus regarding a particular approach for solving the problem. *Social* struggles consisted of struggles related to group dynamics, including the struggle to have one’s questions or ideas taken up. Finally, the *materials* code captured students’ struggles to access or use a material (e.g., a ruler, a calculator). We applied a series of rules for determining whether or not a struggle was overcome, which varied somewhat depending on the type of struggle under consideration. As an example, if a student repeatedly asked a question that was not answered, we determined that this struggle, a social struggle, was not overcome. As another example, the socio-cognitive struggle to reach consensus was overcome if students ultimately reached agreement regarding the idea over which they were in disagreement. If the conversation shifted to a different topic before such agreement was reached, we determined that the struggle was not overcome. In cases were a student acknowledged understanding something mathematical that had previously puzzled them, we determined that they had overcome a cognitive roadblock. After applying these codes in this second phase of coding, we came together to compare our codes and discuss, then resolve, any disagreements. For the vast majority of segments, we had applied the same codes independently. In the third phase of coding, we independently applied codes for the various resources students asked for or were offered related to each roadblock we identified previously. These resources consisted of: a) a peer, b) the teacher, c) a tool, d) the problem itself, e) students' multiple mathematical knowledge bases or MMKBs (e.g., linguistic resources) (Turner et al., 2012), and f) notes/the board. We then met to discuss and resolve any disagreements in these code applications.

We then repeated the steps described here with the remainder of the video.

Examining the coded data. After coding all the data, we each independently examined the coded data, then wrote and shared analytic memos documenting our observations. Our analysis of the coded data was guided by several conjectures. First, we conjectured that the students would encounter different types of roadblocks and that different students would encounter these roadblock types to varying degrees. To evaluate this conjecture, we examined the relative proportions of each type of roadblock encountered by the group, as well as the relative proportions of each type of roadblock each student encountered. Second, we conjectured that students would have more success overcoming certain types of roadblocks than others. To evaluate this conjecture, we examined the proportion of each type of roadblock that students overcame. Third, we anticipated that calling upon certain resources might prove more helpful to students in overcoming the roadblocks they encountered. To evaluate this final conjecture, we identified which resources were called upon most for those roadblocks that students overcame.

Findings

Overall, students encountered 107 roadblocks in the 45:14 of video we analyzed. Of these 107 roadblocks, 30 (28.0%) were *cognitive*, 48 (44.9%) were *socio-cognitive*, 16 (15.0%) were *social*, and 13 (12.1%) were related to students accessing or using *materials*. Hence, a full 59.9% of the struggles students encountered (i.e., socio-cognitive and social) were related to the collaborative nature of their work.

In terms of the number of roadblocks each individual student encountered, and for which we had evidence, Student 1 encountered 25, Student 2 encountered 32, Student 3 encountered 13, and Student 4 encountered 37. Table 1 portrays the number and proportion of each of the four types of roadblock each individual student encountered. The proportion of cognitive roadblocks each student encountered was fairly similar, although for Student 3, the total *number* of cognitive roadblocks he encountered was smaller than was the case for the other students. Moreover, while Students 1, 2, and 4 encountered a similar number and proportion of socio-cognitive roadblocks, Student 3 grappled with far fewer socio-cognitive roadblocks. Lastly, the number of social roadblocks Student 3 encountered was the same as the number encountered by Students 1, 2, and 4 combined.

Table 1: Number & Proportion of Each Type of Roadblock Encountered by Each Student

	Student 1	Student 2	Student 3	Student 4
Cognitive	10 (40.0%)	8 (25.0%)	3 (23.0%)	9 (24.3%)
Socio-cognitive	13 (52.0%)	17 (53.1%)	1 (7.7%)	17 (45.9%)
Social	1 (4.0%)	3 (9.4%)	8 (61.5%)	4 (10.8%)
Materials	1 (4.0%)	4 (12.5%)	1 (7.7%)	7 (18.9%)
Total	25	32	13	37

Note. Percentages show the proportion of the total number of roadblocks each student encountered for each roadblock type (e.g., 10/25 or 40% of Student 1's roadblocks were cognitive).

With regard to students' success in overcoming roadblocks, we found that students overcame more cognitive and materials roadblocks than either socio-cognitive or social roadblocks (Table 2).

Table 2: Number & Proportion of Each Type of Roadblock Overcome or Not Overcome

Type of Roadblock	Overcome	Not Overcome	Total
Cognitive	15 (50.0%)	15 (50.0%)	30
Socio-cognitive	2 (4.2%)	46 (95.8%)	48
Social	4 (25.0%)	12 (75.0%)	16
Materials	11 (84.6%)	2 (15.4%)	13

Specifically, students overcame half of the *cognitive* roadblocks they encountered and a full 84.6% of the *materials* roadblocks they encountered. This latter number is likely as high as it is in part because it includes certain materials roadblocks that were fairly easy to overcome (e.g., accessing an eraser). One such roadblock, however, involved students figuring out how to use a SAFE-T compass to draw the circular pond in their garden design. This roadblock was, in our view, harder to overcome than the roadblock of finding an eraser, yet was ultimately overcome. Unlike most of the materials roadblocks, the cognitive roadblocks were more challenging. One such roadblock involved Student 2 trying to find the radius of a circle with the area being given, something the student ultimately overcame with support from Student 1, who located a website that calculated the circle's radius when the area was entered. Regarding the *socio-cognitive* roadblocks, the vast majority of these involved three students – Students 1, 2, and 4 – trying to reach consensus regarding the inclusion, dimensions, and placement of various objects in the garden. For one such roadblock, Student 1 suggested placing the patio in a particular location in the garden, which Student 4 disagreed with. To convince Student 1 to place the patio in a different location, Student 4 pointed out that the existing garden plan included a pair of patio doors, and that the patio should be placed by these doors, something Student 1 immediately agreed with. This was one of only two socio-cognitive roadblocks that was overcome. Another such struggle regarded whether or not to add stairs to the patio. While Student 4 wanted to include stairs, the other students, especially Students 1 and 2, disagreed, pointing out that the problem did not say to include stairs and that they had seen patios before that did not have stairs. This particular struggle re-surfaced multiple times, yet consensus was never reached and the struggle remained unresolved. In terms of *social* roadblocks, students overcame one-fourth of this type of roadblock. However, further analysis revealed that Student 3, who encountered most of the social roadblocks, overcame only one-eighth of these. For Student 3, the social roadblock repeatedly encountered involved having their ideas heard and questions responded to.

Lastly, we examined the types of resources students called upon both when successful in overcoming a roadblock and when unsuccessful in doing so. As shown in Table 3, when students overcame a roadblock, the resource they most often called upon was one another (i.e., peers). At times, a peer's support was asked for, while at other times, a peer offered support that was unsolicited. As an example, for the socio-cognitive struggle mentioned above, in which Students 1 and 4 sought to reach consensus regarding the placement of the patio, Student 4 pointed out the patio doors to Student 1. In this example, Student 1 had asked Student 4 if they agreed with Student 1's idea, and as such, the peer resource was asked for. Although the peer resource was often called upon when roadblocks were overcome, it was also called upon often when students did not manage to overcome a roadblock.

Table 3: Resources Leveraged in Overcoming Each Roadblock

	Peer	Teacher	Tool	Problem	MMKBs	Notes/board
Overcome	18 (56.3%)	2 (6.3%)	15 (46.9%)	3 (9.4%)	0 (0.0%)	3 (9.4%)
Not overcome	63 (84.0%)	9 (12.0%)	11 (14.7%)	23 (30.7%)	9 (12.0%)	3 (4.0%)

Note. There were 32 roadblocks overcome and 75 not overcome; totals here exceed these numbers, as multiple resources were called upon for certain roadblocks. Percentages represent the proportion of roadblocks overcome, or not overcome, when a given resource was called upon (e.g., for 18/32 or 56.3% of the roadblocks overcome, students called upon the peer resource).

Noteworthy is that one resource – students' multiple mathematical knowledge bases (MMKBs) – was called upon only when students were engaged in a disagreement and striving to reach consensus. As an example, when discussing whether or not to add stairs to the patio, Student 2 referenced knowledge of their home patio, specifically, that this patio had no stairs and was so low from the ground that stairs were not needed. Lastly, we think it is worth noting that, for the 32 roadblocks students overcame, the teacher was accessed as a resource only twice.

Discussion

In this study, we examined three research questions: 1) What types of struggle do students in one classroom encounter when solving a cognitively-demanding mathematics problem in collaboration? 2) How successful are the students in overcoming the various types of struggle they encounter? 3) What resources do the students leverage to overcome these struggles? We anticipated that students would encounter a variety of different types of struggle, experience greater success in overcoming certain types of struggle, and call upon certain resources more than others when striving to overcome their struggles.

We found that the majority of the struggles, or "roadblocks," students encountered were related to the collaborative context in which they solved the Design a Garden problem. Specifically, 59.9% of their struggles were either socio-cognitive or social in nature. This suggests that, if teachers are asked to train their attention primarily on students' cognitive (i.e., individual, mathematical) struggles, as prior work has sought to do (Warshauer et al., 2021), teachers may miss a significant part of the struggle picture. These findings also suggest that teacher educators may find it beneficial to support teachers in attending to a greater range of different types of struggle that students encounter.

This seems particularly important given the potential association between social, socio-cognitive, and cognitive struggles. Of the roadblocks that Student 3 encountered, 61.5% were social in nature. Moreover, this student overcame only one of the eight social struggles they encountered. Unlike Students 1, 2, and 4, few of Student 3's struggles were mathematical in nature. While the majority of the roadblocks Students 1, 2, and 4 grappled with were cognitive or socio-cognitive, this was not the case for Student 3. This suggests that, unless a student overcomes the social struggles they encounter during collaborative problem-solving, they may lack opportunities to grapple with the mathematical struggles that seem likely to result in them arriving at important mathematical insights (Webb, 1991). Given that so few of Student 3's social struggles were overcome, it may be important for teachers to attend and respond to (Jacobs et al., 2010) social struggles like a student struggling to have their questions heard and ideas taken up, perhaps more so than the other types of struggle we describe here. Existing research provides guidance in this regard, describing practices (e.g., Shuffle Quizzes) teachers may enact to ensure all students are included in a group's mathematical conversations (Dunleavy, 2015).

Teachers may also wish to attend and respond to students' socio-cognitive struggles given how few of the struggles of this type students overcame. Prior work provides guidance regarding the sorts of moves teachers might make to help students engage with each other's strategies (Franke et al., 2015). Why so few socio-cognitive roadblocks were ultimately overcome is not immediately apparent. However, we think this may be related to the particular nature of the socio-cognitive roadblock with which these students engaged most: deciding whether or not to include stairs in the garden design. Although this was related to the problem, it was less mathematical than, say, the struggle to reach consensus regarding the solution to a problem, something that would likely be less open to debate. Moreover, we imagine that a discussion of the solution to a problem may do more to merit students' mathematical understandings than a discussion of whether or not to include stairs in the design of a garden. As such, some socio-cognitive roadblocks appear more worthy of students' time than others. Finally, we find it noteworthy that, despite referencing the problem itself and calling upon their out-of-school knowledge, students did not overcome many of their socio-cognitive struggles related to reaching consensus. This suggests that such roadblocks may be difficult to overcome even if students call upon the sorts of resources teachers might hope that they call upon.

Of the various types of roadblocks students encountered, they had most success overcoming cognitive and materials roadblocks. Indeed, students overcame half of their cognitive roadblocks, often with the support of their peers and rarely with the support of the teacher. In our view, this suggests that, rather than intervening right away when students appear to be grappling with a mathematical struggle, it may be best to leave the students to continue grappling, as there is a good chance that they will overcome the struggle on their own, without the teacher's support.

Limitations and Future Directions

The purpose of this analysis was to make visible the complexity of students' struggles in the course of one lesson. The patterns discernable in this video, however, cannot broadly predict what students' struggles may look like in other contexts. For instance, in the classroom observed in this video, like many others in this district, norms appeared to have been established whereby students understood they were to rely on each other for support. This may explain, in part, the degree to which students turned to each other as a resource and suggests that students in classrooms where such norms are not yet present may call upon their peers with lesser frequency.

Additionally, while certain resources (e.g., students' multiple mathematical knowledge bases) did not appear to be called upon much when students overcame a particular roadblock, we do not believe this suggests that these resources are not helpful with regards to overcoming struggle. It is possible that such resources would prove more helpful with problems involving contexts other than designing a garden. We think this is an area worthy of further examination.

Lastly, there are some limitations to our coding procedure. For instance, when students ultimately reached agreement, we determined that a socio-cognitive roadblock to reach consensus had been overcome. However, such agreement could have been reached as a result of one student leveraging their status or overpowering a peer. It is important to distinguish such instances from instances when agreement is reached as a result of a more equitable exchange, yet our coding procedure does not capture such distinctions. Future work could examine the role of students' status, power, and positioning in overcoming struggles like the struggle to reach consensus.

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ENACTED TASK CHARACTERISTICS: SETTING AN INFRASTRUCTURE FOR STUDENTS' QUANTITATIVE REASONING

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In this study, we explored enacted task characteristics (ETCs) that supported students' quantitative reasoning (QR). We employed a design-based methodology; we conducted a teaching experiment with eight secondary school students. Through ongoing and retrospective analyses, we identified ETCs which supported students' quantitative reasoning. The ETCs can set the infrastructure for students' QR when students are: (a) identifying changing attributes of the tasks or situations, (b) coordinating the change among quantities, and (c) making generalizations about quantitative relationships. ETCs play an important role in development of students' meaningful understanding when tasks are designed with focus on quantitative reasoning and representational fluency.

Keywords: Enacted Task Characteristics, Quadratic Functions, Representational Fluency, Functional Thinking, Quantitative Reasoning

Rational and Research Aim

This research aims to identify sets of enacted task characteristics that support students' co-development of representational fluency and functional thinking in learning about quadratic functions within a quantitative context. Historically, quadratic functions have been identified as one function family students develop less sophisticated reasoning. Scholars reported that students often develop an unsophisticated understanding of quadratic functions, such as (a) conceiving a graph as an object (a pictorial entailment) (Ellis & Grinstead, 2008; Zaslavsky, 1997); (b) only articulating the parameters of quadratic functions with an unsophisticated understanding (Ellis & Grinstead, 2008; Even, 1998); (c) providing inappropriate generalization (Ellis & Grinstead, 2008); (d) conceiving of quadratic growth as exponential (Altindis & Fonger, 2018; Altindis & Fonger; 2019; Fonger & Altindis, 2019); and (e) depending heavily on algebraic representations, which limits the development of a robust understanding of quadratic functions (Ellis & Grinstead, 2008).

Developing greater sophisticated reasoning in learning about quadratic functions requires co-development of representational fluency and functional thinking. *Representational fluency* (RF) is defined as "the ability to create, interpret, translate between, and connect multiple representations—is a key to a meaningful understanding of mathematics" (Fonger, 2019, p. 1). *Functional thinking* (FT) is a creative thinking style about functions, creating patterns, and generalizing the functional relationships within concrete representations of functions (Blanton & Kaput, 2011; Stephens et al., 2017). In this study, FT included two types of reasoning about functions: correspondence and covariational reasoning. *Correspondence reasoning* is understanding the relationship between the x and y values by looking at the x and the y as corresponding dependent and independent values or quantities (Confrey & Smith, 1991; 1994; 1995). According to Thompson and his colleagues, covariational reasoning is being able to think about "two quantities' values varying" and the two quantities "varying simultaneously" (Thompson & Carlson, 2017, p. 425).

Although we learn about the meaningful understanding grounds within the co-development of RF and FT from the literature (Even, 1998, Altindis & Fonger, 2019), we are still left with an inquiry on what type of enacted task characteristics may support the co-development of RF and FT. This study is guided by the research question: How can secondary school students be supported to develop a sophisticated understanding of quadratic functions?

Theoretical Framework and Background Literature

We networked the theory of quantitative reasoning (QR) (Thompson, 1994) with a theory of representations (Kaput, 1987a; 1987b) to support students' sophisticated understanding of quadratic functions. Quantitative reasoning sets a foundation for students' algebraic and covariational reasoning. Thompson's quantitative reasoning theory is based on Piaget's work on the mental images that learners create or *mental constructions* (Thompson, 1994). The creation of mental constructions is a demanding process for students to conceptualize quantities, quantification, and relationships among quantities (Thompson, 2011). According to Piaget (1967), *images* are conceptualizations that people must create, not something that already exists in their understanding of functions or the world. Piaget (1967) theorizes that a given subject's mental operation of a function and their mental image are connected and that the subject makes sense of an object by interacting with it. Following this logic, students might form an image of a function through reasoning about quantities that covary (Thompson, 1994). According to Thompson, when students try to grasp the concept of functions as equations that vary, they often focus on one variable as the source of the variation, usually the dependent variable. According to Thompson (1994), students' ability to build an image of changing quantities involves several layers: first, perceiving a change in one quantity; second, shifting into conceiving the two quantities as coordinated; and, finally, constructing an image of the two changing quantities as they covary simultaneously. These categories are based on Piaget's constructivist theory of learning.

Representations have been a focus of the mathematics education research community for decades. Scholars have explored students' understanding of mathematics regarding their representational activity, particularly their translations between and among representations—creating, interpreting, and transforming representations (e.g., Adu-Gyamfi & Bosse, 2014; Janvier, 1987a;1987b). In general, the relationship between mathematics and representations is understood as cause and effect—as long as teaching and learning mathematics exists, representations and their role will exist within it. In this study, we will focus on external (concrete) representations. Throughout this study, the word representation refers to the concrete representations of functions: graphs, tables, symbolic equations, and diagrams.

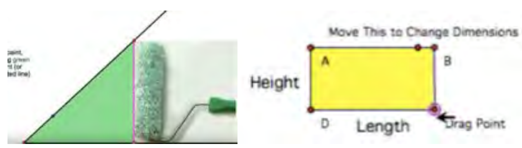
In the current study, we intend to network the theory of QR and the theory of representations to support students' sophisticated understanding of quadratic functions. We set the design principles and instructional supports, by the affordances of QR and representations, as follows: (a) creating opportunities for students to construct mental images of covarying quantities; (b) getting students to focus on quantitative operations rather than numerical operations; (c) emphasizing the role of concrete representations in quantitative processes; (d) grounding students' RF within the meaning of quantities; and (e) getting students to present the models of quantities in their minds via concrete representations.

Methodology

In the present study, we employed a design-based research methodology (Cobb et al., 2017). We conducted a teaching experiment with eight Turkish American middle and high school students in the 8^{th}, 9^{th}, and 10^{th} grades from urban and suburban school districts. The teaching experiment consisted of eight instructional lessons for two weeks lasting 45 to 60 minutes. We networked theories of quantitative reasoning (Thompson, 1994; 2011) and representations (Kaput, 1987a;1987b; Dreyfus, 2002) in designing a well-crafted learning ecology framework: enacted task characteristics, small and whole group dynamics, and teacher' pedagogical moves. In this research report, we will be focusing on enacted task characteristics. The data sources are enhanced transcription of small and whole group interactions. We are both the teacher researcher (TR) in this teaching experiment.

Tasks

In the present study, we used two tasks: the paint roller task and the growing rectangle task (see Figure 1). Both these tasks were created by Amy Ellis and her colleagues (2011; 2015). The context of these tasks may help students construct a much more profound understanding of quadratic functions as "a conception of two quantities varying simultaneously" (Thompson & Carlson, 2017, p. 444). With these tasks, students may notice attributes of the situation, such as seeing the paint roller's length and the size of the area being painted. Students may conceive of the triangle's height increasing and note that the area is also growing continually. These tasks include dynamic situations, diagrams, and videos that can help students see how a change in length affects a change in the area using color-coding that might help make the change in variables more visible to students (Johnson et al., 2018).



(a) (b)

Figure 1: (a) The Paint Roller Task (b) Growing Rectangle

Analyses

We used Cobb and Whitenack's (1996) techniques, which drew from Corbin and Strauss's (2008) constant comparison method. In the initial analysis, using phase one, we identified regularities in participants' interactions in small- and whole-group settings by creating enhanced transcriptions, structured and extended memos, and researcher journals. In the episode-by-episode analysis, we created the initial coding schema by coding the enhanced transcriptions of day 1 to day 8 using phase two. Then we re-coded to refute or agree with the codes or form the top-level codes—an emergent coding schema—using phase three. We then formed a developed coding schema—a learning-ecology framework—using phase four. In the analysis of analyses, we coded using the predetermined analytical frameworks of RF and FT—using phase five. Then we identified shifts in students' understanding of quadratic functions concerning the supports students received during the teaching experiment and verified the learning-ecology framework by coding 25% of the data using phase six.

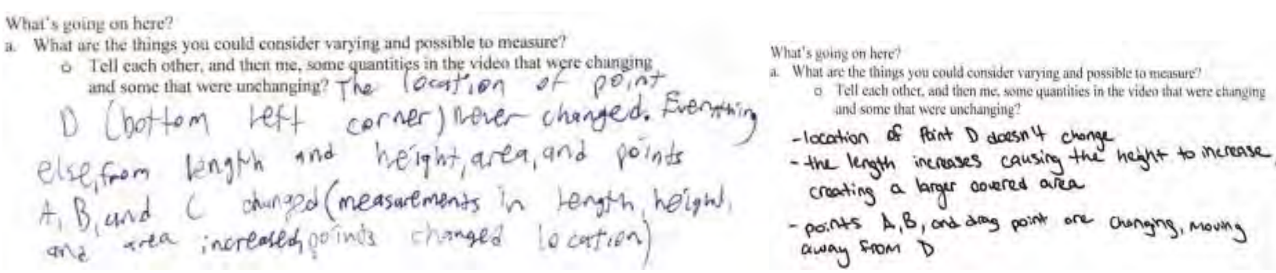
Result

Enacted Task Characteristics

We define ETCs as the instances in which students are given opportunities to articulate, talk about, answer, and discuss quantitative relationships within tables, graphs, and symbolic equations during small- and whole-group interactions (King, 2011; Stein et al., 2007). In other words, ETCs are statements and questions about a problem or a set of problems that encourage students to articulate, talk about, discuss, and create representations to present quantitative relationships. ETCs are a form of instructional support; the characteristics cluster around promoting students' QR and RF. ETCs can set the infrastructure for students' QR when students are: (a) identifying changing attributes of the tasks or situations, (b) coordinating the change among quantities, and (c) making generalizations about quantitative relationships.

Identifying changing attributes of tasks. One of the enacted task characteristics is asking students to identify attributes of a situation or their tasks—identifying relevant quantities and units to measure the quantities. Students were asked or prompted to identify quantities by looking at the task's attributes and identifying relevant quantities. After tracing appropriate quantities within the task context, they were prompted to think about a unit to measure the quantities.

Asli and Yener watched a video in the following vignette —featuring a growing rectangle being sketched via dynamic geometry software. Student handouts were structured so that students were asked to think and talk to each other about varying quantities and possible ways to measure those quantities. The task was structured to ask students to identify varying quantities; for example, the question in Figure 2: "What are the things you could consider varying and possible to measure?"



(a) (b)

Figure 2: (a) Yener's and(b) Asli's Varying Quantities of the Growing Rectangle

See the vignette below, which is the conversation students had in responding to the question on the task: "What are the things you could consider varying and possible to measure?"

6 Asli: Location of point D does not change.
7 Yener: Yeah. [Figure 2 (a) shows Yener's written answer: The location of point D (bottom left corner) never changed. Everything else, from the length and the height, area and the points A, B, and C changed (measurements in length, height, and area increased, points changed location)]
8 Researcher: Can you talk to each other?
9 Asli: We just wrote down when we talked about before we got the paper. [Figure 2 (b).]

Asli and Yener identified the rectangle's corner; D (D is a point on the rectangle) was not changing (line 6–7). Asli referred to it as point D's location; Yener stated that D is at the "bottom left corner," not changing (Figure 2). They agreed that everything else is changing on the task. Asli noticed that "the length increases causing the height to increase, creating a larger covered area" (see Figure 2 (b)). Asli also recognized that the corners of the rectangle are changing, so she wrote, "Points A, B, and drag points are changing, moving away from D." Yener agreed with Asli that A, B, and C changed. Length, height, and area changed as well. Yener recognized that the change in height, length, and area increases when the locations of A, B, and C (corners of the rectangle) change (line 7). Hence, we concluded that creating a foundation for students' QR might involve getting students to determine what is changing or varying in a dynamic task context. The tasks' structure, along with necessary tools, supports students in identifying varying relevant quantities. Students begin to recognize constant and variable quantities and how to measure them. This is evidence to suggest that enacted task characteristics should include questions or prompts that direct students' attention toward identifying relevant varying quantities on a task and noticing that the quantities are changing together.

Coordinating change among quantities. Another ETC is the coordination of change among quantities: probing, asking, or reinforcing students to coordinate changes among quantities. The tasks were structured to ask students how a change in one quantity affects the change in another in order to get students to coordinate the change between quantities. For example, one of the ETCs asks students: "How does the change in height affect change in area?" In the following vignette, Asli and Yener investigated the relationship between the height, length, and area of the growing rectangle task.

10 Yener: How does change in height is affect the change in area? If the height changes, the length changes.
11 Asli: The change in height increases the area covered. Because it contributes to the formula to get the area.
12 Yener: When the height changes, the area changes. Here is the area changes too.
13 Researcher: Can you be more specific? About how the height changes, the length changes. This also be an area.
14 Asli: When the length is increasing, the heights increase.
15 Yener: Increase Uhm. I think they might increase by the same amount. Yeah, they probably started over different, and then they increased amount each time the height and length.
16 Yener: Oh, I found this when height changes by 2, length changes by 3. That means that is constant.
17 Asli: Okay. So, what I wrote is the change in height increases the area covered because it contributes to the formula necessary to calculate the area [Figure 3 (a)].
18 Yener: Mine is same thing with height is affecting the change. [Figure 3 (b); he wrote: "The change in height is affecting the change in area by contributing to the formula for area therefore affecting the area."]

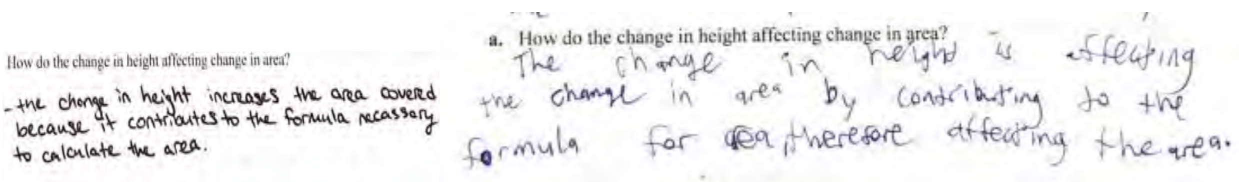


Figure 3: (a) Asli and Yener's (b) Response to "How Do the Change in Height Affecting Change in Area?

For this type of ETC, students are asked to see how the change in one quantity affects the change in another quantity (Figure 3). These questions (e.g., how does change in height affect the change in area?) form a foundation upon which students can coordinate change in quantities. For instance, Yener read the question (line 10): "How does change in height is affect the change in area?" Then he coordinated height with the length such that if the height changes (line 12), the length changes. Asli built on Yener's reasoning by stating (line 11), "The change in height increases the area covered."

Yener and Asli engaged in the task jointly; Yener agreed with Asli's statement, which encouraged Asli to justify her statement (line 11). She said, "Because it contributes to the formula to get the area." Asli's justification is about the corresponding reasoning. Yener said: "Increase, Uhmm. I think they might increase by the same amount, Yeah, they probably started over different, and then they increased amount each time the height and length." Yener noticed that the growing rectangle's height and length started with a different amount that changed in magnitude or amount each time (line 15). Then Yener said: "Oh, I found this when height changes by 2, length changes by 3. That means that is constant." Asli read her written responses: "Okay. So, what I wrote is the change in height increases the area covered because it contributes to the formula necessary to calculate the area" (line 17).

In responding to the task characteristics, students not only respond to questions on the tasks, but they also attempt to justify their responses. As we saw from Asli, she read her answer and even explained it (line 17). Furthermore, Yener read his response by comparing and contrasting his answer for the same question with Asli's (line 18).

Observing the results of this student exchange, we can infer that this student's ability to reason about relevant quantities and coordinate changes in quantities develops when prompted to consider how a change in one quantity affected change in another quantity. In other words, asking students about how a change in one quantity may affect the change in another can be an effective way to support healthy peer deliberation and the development of more advanced reasoning.

Generalization. Lastly, ETC involved structuring tasks to ask students to generalize the relationship between quantities. In terms of this study, a generalization is a form of support that pushes students to think about a pattern representing the relationship between quantities (e.g., the length of the paint-roller and its area). With ETC, students were asked to answer the same focus questions[1] in small- and whole-group settings in their handouts and had individual writing time for answering the same problem in their journal. The below vignette is taken from a whole-group interaction when students explored the relationship between the paint roller's length and the area covered by the paint roller. ETCs were structured with a focus question to allow the students to look for a pattern about the quantitative relationships.

And in the vignette below, the students were exploring the focus question: "What is the relationship between the length of the paint roller and the amount of the area being covered?"

The focus question is designed to prompt students to coordinate a change in the paint roller's length and a change in the area it is covered. In other words, the question itself states that there is a relationship between the length of the paint roller and the area covered, which pushes students to generalize about the relationship.

Consider the vignette below:

31 Researcher: So, we will present the focus question ["What is the relationship between the length of the paint roller and the amount of the area being covered?"]. I will ask this group to present first. Yener. Ready.

32 Yener: I did not finish everything. But I have my answer.

33 Researcher: Okay. So, when someone is presenting, we want to ask questions, and we want to compare their thinking with ours—what they have on there. All right?

34 Yener: Wait. So, I just answer the focus question?

35 Researcher: Okay. Yeah. We are just answering the focus questions. But we are providing some evidence for our thinking.

36 Asli: Do you want to start first?

37 Yener: Okay, I'll do it first.

38 Yener: So, the focus question is, what's the relation between the length of the paint roller and the amount of area covered? And my answer is that every time the length increases by one centimeter, the amount the area changes by or the change in the change of area, it increases by 1 centimeter.

The focus Question: What is the relationship between the length of the paint roller and amount of the area covered?

Figure 4: A Focus Question for the Paint Roller Task

As we see with the above vignette, the TR stated that as a classroom community, students were trying to answer the focus question, which was about generalizing the relationship between quantities (line 31). Subsequently, the student's attention was directed to the relationship between the growing triangles' length and area (line 34). The paint roller task creates a growing triangle; the students' attention is directed to how the growing area is related to its length. As we see, the TR asked Asli and Yener if they could present, and when they agreed to present, she restated that as a community, they were trying to answer the focus question (line 31–33). Yener confirms that they were just answering the focus question by saying, "Wait. So, I just answer the focused question" (line 34). The TR oriented Yener toward answering the focus question and providing evidence to the claim they made in answering the focus question (line 33). Yener read the question (Figure 4): "What is the relationship between the length of the paint roller and amount of the area being covered?" and answered it by saying, "And my answer is that every time the length increases by one centimeter, the amount the area changes by or the change in the change of area, it increases by 1 centimeter" (line 38).

We concluded that having students answer the same focus questions about covarying quantities in social (small- and whole-group settings) and individual contexts (journals and individual handouts during writing time) might provide students with opportunities to articulate their thinking a more sophisticated understanding of their reasoning.

To use this ETC, the students' handouts and journals center on a focus question. For example, "What is the relationship between the length of the paint roller and the amount of the area being covered?" Students' handouts are designed to aid students in answering the focus question. Additionally, the TR's prompts in whole- and small-group settings, along with students' journals, center on answering the same focus questions. ETCs are a form of support in small- and whole-group settings where students are encouraged to generalize quantitative relationships.

In this example, we see that ETCs ask students to generalize the relationship by getting students to answer the focus question in small- and whole-group settings centered around identifying a pattern between quantities. Thus, ETCs are pushing students to generalize a relationship between quantities. The study suggested that enacted task characteristics that help support students' learning can include setting a focus question about quantitative relationships, which provides opportunities for students to generalize the quantitative relationships. Setting a focus question (see Figure 4) that asks students to explore the relationship among quantities is a form of support that may reinforce advanced reasoning about quantitative relationships. The findings suggest that the focus question provides students opportunities to articulate quantitative relationships in individual, small, and whole-group settings. Students benefited from the focus questions about quantitative relationships because students could answer the questions on their own, then discuss the same quantitative relationship in small- and whole-group settings where everyone articulated their thinking about the situation.

Discussion and Conclusion

We analyzed ETCs in the context of setting infrastructure for students' QR. Enacted task characteristics are purposefully designed elements that contribute to students' meaningful understanding of quadratic functions. Such characteristics allow students to talk, articulate and discuss quantitatively rich tasks while learning about quadratic functions. This study's findings parallel prior literature that posits that enacted tasks' design characteristics are a form of instructional support in learning and teaching about mathematics (King et al., 2011; Stein et al., 2007). In particular, the findings indicated that enacted task characteristics could effectively support student learning by setting infrastructure for students' QR. Thus, the present study's significance is in showing that the task characteristics should be designed with an emphasis on QR and RF and that these kinds of task characteristics can support students in co-developing RF and FT.

The findings showed that enacted task characteristics supported students' learning when ETCs enabled students to notice changing quantities and identify these quantities' attributes when learning about quadratic functions. The task characteristics made quantities and quantitative relationships visible to students. They provided opportunities for students to measure the magnitude of the quantities in the tasks, which effectively aligns with the prior research (e.g., Johnson et al., 2018). As the findings corroborate previous research on making quantities and their relationships visible to students, they further the literature by showing how task characteristics should be emphasized when focusing on RF and FT. Specifically, we found three salient task characteristics that enabled students to form foundations for QR. These characteristics include, typically, stating, probing, or asking students about (a) identifying changing attributes of the tasks, (b) coordinating change among quantities, and (c) generalizing the quantitative relationships.

First, we found that task design characteristics that direct students' attention to covarying quantities support students' meaningful understanding. Furthermore, such features support students' development of robust quantitative reasoning. Second, the current findings also focus on purposefully designing tasks to allow students to coordinate the change in one quantity with the change in another quantity. Lastly, the present results demonstrated that creating tasks with features, such as focus questions, that allow students to explore quantitative relationships is an effective form of support for students that further helps them form generalizations about quantitative relationships. While prior literature focused on making quantities visible to students (e.g., Johnson et al., 2018), this study builds on previous literature by suggesting that designing tasks with prompts, statements, or questions that redirect students' attention towards recognizing coordination among quantities can provide effective support for students' meaningful learning. This study also suggests that designing tasks with focus questions that require students to articulate or seek a generalized pattern about a quantitative relationship is beneficial to students' to develop quantitative reasoning skills.

Note

[1] For example, “What is the relationship between the length of the paint roller and the amount of the area being covered?”

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TEACHING PRODUCTIVE STRUGGLE IN CALCULUS

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Students often lack the cognitive and metacognitive strategies to maximize their learning. However, instruction may help students use these strategies. We redesigned a calculus course to teach students metacognitive strategies, with three components: frequent in-class discussions, corrections on exams, and a student essay on metacognitive strategies. We used a mixed method study design to qualitatively analyze the students' essays and quantitatively measured changes in students' attitudes towards mathematics using a pre-post assessment. We found the students attitudes improved at a practical and statistically significant level ($p<0.0001$) over the course.

Keywords: Affect, Emotion, Beliefs, and Attitudes. Calculus. Metacognition.

Introduction

To learn new ideas, a learner needs two types of knowledge: knowledge of the subject, and knowledge about learning itself (Alexander, 2005). In our experience, most college courses have curricula filled with subject matter knowledge, but do not provide our students with enough guidance about learning. We attempted to remedy this in a first-semester calculus course.

Calculus is difficult for many students, and often serves as a gatekeeper to other STEM courses (Bressoud, et al. 2013). Many students leave calculus with less confidence in their mathematical abilities than before they started, and less desire to continue taking mathematics.

We redesigned a first-semester calculus course by adding three components: (i) frequent class discussions on metacognition; (ii) allowing students to correct their midterm exams and improve their scores; and (iii) assigning students to read articles and watch videos on metacognitive topics, and write a reflective essay on these materials.

Literature Review

Calculus Attitudes and Instruction

Calculus 1 is a required course for many programs of study in STEM fields. For the last 40 years, educators, politicians and students have called for calculus to move from "a filter to a pump" (Steen, 1988; Bressoud, Mesa & Rasmussen, 2015). Unfortunately, according to the *Characteristics of Successful Programs in College Calculus* (CSPCC) study (Bressoud, et al., 2015), little progress in improving retention rates has occurred over this time.

The CSPCC study found that many students lose interest in mathematics during their first calculus course (Sonnert & Sadler, 2015), and the students reported decreased confidence in their mathematical abilities and decreased enjoyment from mathematics. This finding was confirmed with the *Mathematics Attitudes and Perceptions Survey*, where Code, et al. (2016) found that during a full year calculus sequence, students' attitudes moved away from expert-like orientations. However, pedagogical factors including "classroom interactions that acknowledge students" can lead to improvements in students' attitudes during calculus (Mesa, et al., 2015).

Metacognitive Instruction

To effectively learn new material in any domain, students should have sufficient knowledge of cognitive and metacognitive strategies for learning (Alexander, 2005; Dunlosky, et al., 2013; Winne & Hadwin, 1998), which is comparable to a teacher having both subject matter knowledge and pedagogical knowledge (Shulman, 1987). Unfortunately, students vary widely in their knowledge of appropriate learning strategies (Askell-Williams & Lawson, 2005ab; Askell-Williams, Lawson & Murray-Harvey, 2007; Askell-Williams, Lawson & Skrzypiec, 2012).

However, students can improve their knowledge of cognitive and metacognitive strategies via instruction (Hattie, 2009). The instruction in these strategies can be integrated into other instruction for just-in-time intervention, and result in meaningful increases to student knowledge of cognitive and metacognitive strategies (Askell-Williams, Lawson & Skrzypiec, 2012).

Methodology

The redesign was implemented in two sections of a calculus 1 course, taught by the second author. Six sections of this course were offered, but only two participated in the redesign. The students were not aware of this redesign or this study when they enrolled in the course.

In the two redesign sections, 56 students were enrolled. However, only 46 students are included in our analysis. The other students either elected not to participate, did not correctly fill out the consent form, or did not complete both the pre- and post-surveys.

The goal of the redesign was to alter the students' attitudes about learning mathematics, especially the mindset of the students, and their attitudes toward failure (Dweck, 2006). The redesign had three components. The first component was in-class talks and discussions on productive failure and mindset. Some talks were planned, specifically those given on the first and last day of class, and on the days when exams were returned. An instance of this occurred prior to the instructor handing back the first exams, when the instructor said:

> Before you get your exam back, I want you to think about how you're going to frame your score in your mind, and how you're going to put into practice everything that we've talked about up to this point regarding mindsets and productive mistakes. If you flip your exam over and see a good score, are you going to attribute that to you being a "math person?" That your score is solely the result of your "natural talent?" Or did you score well because you put in the time, hard work, and effort? One of these framings leads to a fixed mindset, and the other a growth one.

This example is typical of the pre-planned talks given throughout the semester.

In addition, mini discussions occurred spontaneously and with greater frequency both in class and office hours. For example, mistakes that the instructor made at the board, and mistakes that the students made while working on in-class problem sets provided opportunities to talk about the importance of mistake-making and struggle for learning.

To encourage the students to return to their mistakes and make them productive, the students could correct mistakes on their exams and return them for partial credit. These exam corrections made up the second component of the redesign.

The third component of the redesign consisted of assigned YouTube videos and online readings on metacognition, mindsets, and productive failure. At the end of the semester, the students wrote a reflective essay that discussed the videos, readings, and class discussions. This assignment was inspired by blog posts from Matt Boelkins (2017) and Stan Yoshinobu (2016).

Data collection

To assess the redesign, we utilized a mixed methods design. We qualitatively analysed the students' essays using a constant comparative method to develop themes (Creswell, 2007).

Also, we gave the students the Mathematics Attitudes and Perceptions Survey (MAPS) twice (Code, et al., 2016). The pre-test was administered during the first week, and the post-test was administered during the penultimate week. This survey assesses the attitudes of students towards mathematics along seven dimensions: beliefs about growth mindset, the applicability of mathematics to the real world, confidence, interest in studying mathematics, persistence in problem solving, making sense of mathematics, and the nature of answers in mathematics. The MAPS assessment was validated with both undergraduate and mathematician groups, and expert consensus was achieved on all items, except two growth mindset indicators.

We processed the data by matching pre-test to post-test and eliminating the participants who did not have both. The survey included a filter question to ensure the participants were reading the questions carefully, and we removed one participant who answered that question incorrectly. Then for each survey item we recorded a 1 for a response that indicated a positive attitude towards mathematics, and 0 for neutral or negative responses, as specified by Code, et al. (2016).

Results

Quantitative Results

The MAPS survey data was analysed with paired t-tests. The difference between the pre-test and post-test was calculated for each student for the overall survey, and within all seven subcategories. The mean of these differences was computed, and we ran a paired t-test. The difference between pre-test and post-test was highly significant (p=0.0000001), with a mean score increase of 3.478 points on a 31-point survey. Furthermore, (see Table 1), each of the

Table 1: Means of differences and p-values for each paired t-test.

Category (points)	Mean of Differences	Standard Deviation	p-value
Overall (31)	3.478	3.897	0.0000001*
Growth Mindset (4)	0.565	1.003	0.000203*
Real World (3)	0.652	0.924	0.000009*
Confidence (4)	0.500	1.457	0.012238
Interest (3)	0.217	0.841	0.043187
Persistence (4)	0.543	1.005	0.000321*
Sense Making (5)	0.435	1.377	0.018831
Answers (6)	0.304	1.171	0.042408

* indicates significance at the 0.01 level

seven subcategories showed a significant improvement in attitudes at the 0.05 level. The categories of growth mindset, real world applicability and persistence in problem solving were also significant at the 0.01 level. Thus, this course resulted in a significant positive change in attitude for the students involved. Furthermore, previous results have shown a significant negative change on this instrument after taking a full year calculus sequence (Code, et al., 2016).

Qualitative Results

The constant comparative analysis resulted in two major themes: the transition from fixed to growth mindsets and adopting the belief that struggle can be productive. Note that all student names are changed to pseudonyms to protect their privacy.

The transition from a fixed mindset to a growth mindset was a topic in many of the essays. Several students acknowledged that they previously held a fixed mindset about mathematics, such as Taylor who said

> Coming into this class this semester I was suffering from a fixed mindset. I believed that I lacked the ability to be good at math no matter what I did. Since I already made up my mind that I was never going to understand math, I didn't put in the work needed to get better at it. I now understand that this is not the case. I now understand that I can be successful in calculus if I work hard and apply a growth mindset. … The mental switch from a fixed mindset to more of a growth mindset is the reason for that success.

Taylor's quote is representative of several students' beliefs that they previously felt they were "not a math person", that they were "naturally worse at math than other people" or that they "had hit [their] ceiling in mathematical ability." Yet all these students acknowledged that these beliefs started to change due to the redesign.

The other major theme from the analysis was the students' growing belief that struggles and failures can be productive. Abby expresses this belief writing "This course has truly taught me how to learn from my mistakes and almost laugh at the failures because it's usually easy for me to get it correct on my next try." Another student, Bradley said "If we embrace our failures and learn from them rather than dwelling on them and pitying ourselves, we can learn faster and improve our success." Several students echoed the sentiments from these quotes that they now find failure to be a normal learning experience.

For many students, normalizing failure led to a focus on learning rather than grades. For instance, Maddie said "I know that all I focus on is getting good grades and I have this fear of failing. I should look at it in a way of learning and understanding why it is that I failed." Similarly, Carrie said "there we many times throughout the class that I didn't do so well on my exams, but through these productive failures I feel I have learned new things."

One concern is that several students seemed to conflate productive struggles with test corrections. We observe this in Andrew's statement "the circumstances that have been given to me during this calculus class have been amazing. In other math classes I get the test back look it over once and put it away but with the corrections I look through it thoroughly and figure out my mistakes." We find it concerning that some of the students seem to associate the concept of making struggles productive to this single task, but we also understand that it was a valuable opportunity for the students to experience productive failure.

Conclusions and Implications

Although the primary goal of calculus is for students to learn the concepts and applications of differential and integral calculus, it is a crucial course for recruiting and retaining students in STEM majors (Bressoud, et al., 2013). Furthermore, we believe that changing attitudes about mathematics is a mechanism for recruitment. As such, the redesign of teaching metacognition alongside calculus concepts was successful in changing attitudes about mathematics.

While the redesign was successful, it may not be possible to scale this to larger sections. In particular, the exam corrections were time consuming, and would not be easily scalable.

Furthermore, we acknowledge that we cannot separate the redesign from possible conflating variables, such as the charisma of the instructor, the previous knowledge of the students, and the scheduling of the course. It was not possible to have an additional section of the course to serve as a control group, due to logistical constraints. As such, we cannot claim that the redesign alone caused the change in attitudes. However, due to the overwhelming changes in attitudes recorded by surveys, we are confident that the redesign contributed to this result.

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A CASE OF UNITS COORDINATION STAGE CHANGE IN MIDDLE SCHOOL

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A case study of a 7th grade student, Emily, was conducted to understand her change from units coordination stage 1 to 2 over the fall semester of 7th grade. Emily participated in a unit-long classroom design experiment conducted that semester by two of the authors, where units coordination stage change was not an explicit goal. Units coordination refers to how students conceive of relationships between discrete units, like 1s, or measurement units, like lengths, as they solve problems. Students at stage 1 can coordinate two different types of units as they solve a problem, such as the number of packages in a crate containing 6 boxes with 8 packages in each box. However, these students are essentially in a "one's world" and do not create structures like six 8s. In this paper we carefully document Emily's stage change. We also consider tasks in the unit that may have supported this significant change.

Keywords: Number Concepts and Operations, Rational Numbers, Learning Theory

Students enter middle school operating at three different stages of units coordination (Steffe, 2017). Units coordination refers to how students conceive of relationships between discrete units, like 1s, or measurement units, like lengths. Students who are operating at stage 1 can learn to coordinate two different types of units as they solve a problem. Consider the problem of how many packages are in a crate if it contains 6 boxes with 8 packages in each box. Students at stage 1 can track a box of 8 packages, and then another, and another, until they accumulate six 8s (Steffe, 1992). However, the result, 48 packages, is only 48 ones for them—they do not create the structure that observers can see, six 8s. Furthermore, if students are to build on this result to solve another problem, they have to keep re-establishing the 48 as six 8s. In contrast, students operating at stage 2 create the 48 packages as six 8s (Ulrich, 2016a). Yet like stage 1 students, stage 2 students do not maintain the 48 as six 8s: 48 becomes 48 ones. In contrast, students operating at stage 3 maintain the 48 as six 8s as they work on solving more problems.

Units coordination influences students' reasoning in many mathematical domains that are the focus of middle school, such as proportional reasoning (Shin, et al., 2020) and equation writing (Hackenberg & Lee, 2015). Students at stage 1 are basically pre-fractional (Hackenberg, 2013), which puts them at a great disadvantage in working on these domains. Steffe (2017) estimates about 30% of incoming 6th grade students are at stage 1, while a recent study found 61% of 6th grade students were at stage 1 (Zwanch & Wilkins, 2021). Therefore, the field needs to understand how changes in units coordination stage occur, especially from stage 1 to stage 2.

To our knowledge, only one study has addressed stage change (Norton & Boyce, 2015). These researchers supported a 6th grade student to shift from stage 1 to 2 by engaging the student in fairly rapid questions about the number of different types of units in a "embedded units" problem similar to our Crate Problem (shown in the Method section). We did not engage in this approach because we did not intend, necessarily, to promote a shift in units coordination stage. However, our study implies another possible avenue for promoting stage change.

The purpose of this paper is to show evidence of a student, Emily, who entered 7th grade at stage 1 of units coordination. During fall semester she transitioned to stage 2. That semester she participated in a unit on proportional reasoning co-taught by the first and second authors; the second author was Emily's classroom teacher at the time of the study. In this paper we document how we assessed Emily's units coordination stage at the start and end of the semester to demonstrate a case of change in stage. We do not claim that the classroom unit caused this stage change. However, we consider tasks in the unit that may have supported this significant change.

Our research question is: What was Emily's stage of units coordination at the start and end of the first semester of her 7th grade?

Theoretical Frame: Units Coordination and Fractions Knowledge

Students' units coordination stages influence their fractions knowledge (Steffe & Olive, 2010). We view students' fractions knowledge in terms of schemes, where a scheme is a goal-directed organization of mental actions that includes an assimilatory mechanism, activity, and a result (Steffe, 2010b). For example, students at stage 1 often construct parts-within-wholes fraction schemes because they have not yet constructed a disembedding operation where they can take a part out of a whole and mentally keep the whole intact (Hackenberg, 2013).

When students are at stage 2 of units coordination, they can construct disembedding operations, and so they have the tools to construct a *partitive unit fraction scheme* (Steffe, 2010a). For students who have constructed this scheme, one-fifth of a unit means a part that can be iterated five times to make the unit. Norton and Wilkins (2013) found that 6th and 7th grade students who constructed a partitive unit fraction scheme went on to construct more advanced fractions operations and schemes in a relatively short period of time. So, evidence of a partitive unit fraction scheme implies that a student is operating at stage 2.

Method

This paper is part of a larger study, a design experiment to investigate how to differentiate mathematics instruction with a regular-level 7th grade class with 18 students. Before the unit began, we sought to develop initial understanding of students' units coordination stages and fractions knowledge and to select six focus students, two operating at each stage. Toward this end we administered two written assessments of students' units coordination stages (e.g., Norton, et al., 2015), and a written assessment of students' fractions knowledge (Wilkins, et al., 2013). We used results of these assessments to select 16 students for 40-minute interviews prior to the start of the unit. Following the interviews, we had 5 students at stage 1, 9 at stage 2, and 4 at stage 3. Emily was one of the focus students at stage 1.

For this paper we have developed a second-order model of Emily. A second-order model is a researcher's constellation of constructs to describe and account for another person's mathematical activity (Steffe & Olive, 2010). To make the model, we repeatedly reviewed video of Emily's interviews; video of her activity during the unit; and her written work. We wrote summaries, interpretations, and conjectures. We debated interpretations at bi-weekly research meetings with a 6-member research team, coming to consensus through discussion. We also compared Emily to other middle school students at stage 1 (e.g., Ulrich, 2016b). Our model of Emily is a description of her schemes and operations, with accounts of accommodations (re-organizations) that occurred and interactions that were involved in the accommodations.

Two of the interview tasks we used to assess students' units coordination stages are:

Tiles Problem. Cara is putting down square paving tiles for a walkway with 4 tiles in each row. Her bag holds 60 tiles. She puts down 6 rows of tiles. How many rows with 4 tiles in each row can she make with the rest of the tiles?

Crate Problem. There are 4 cans of juice in a package and 8 packages in a box. A crate contains 6 boxes. How many cans of juice are in a crate and can you draw a picture to show how you know?

With these problems we assessed whether students had constructed 4 tiles or cans as a composite unit (unit of units) by iterating 4 multiple times, tracking the numbers of items and the number of rows or packages. We also assessed whether students could use that coordination in further problem solving. If a student did not, that was evidence of coordinating two types of units in activity only, a characteristic of stage 1.

Findings

Now we provide evidence of Emily's units coordination stage from her initial interview at the start of fall semester, and then from her follow up interview at the end of the semester.

Emily's Initial Interview: September 11

Tiles Problem. In her initial interview, Emily worked on the Tiles and Crate Problems. On the Tiles Problem, she initially divided 60 by 6. Although in our view this gave her 10 rows of 6 tiles, she then subtracted 4 (tiles) from the 10 (rows). So, she did not appear to view the 10 as a number of rows, and she was certainly not thinking of the rows as consisting of 4 tiles each.

When prompted to draw a picture, Emily drew 4 tiles individually in a row. When asked if she could put down 6 of those rows, Emily drew only the first two tiles in the next row for a total of 6 tiles, rather than 6 rows. When explaining, she said, "Oh, 6 rows!" She continued her drawing to make 6 rows of 4, and she said she could have "done 6 times 4" to get 24.

When Emily drew 2 tiles onto her initial 4 tiles to make 6 tiles, rather than 6 rows, she again showed that she was not organizing the tiles into groups of 4. This response is characteristic of students who have not constructed composite units (Steffe, 1992). However, when explaining she seemed to realize that 6 rows meant 6 groups of 4 tiles. Her subsequent activity showed that she could conceive of 4 as a composite unit and coordinate two types of units (tiles and rows).

The interviewer asked what the 24 meant. Emily said, "how many she's put down so far" and subtracted 24 from 60. She was "not really sure" about the result, 36: She thought it was too much for rows, so then it must be tiles. The interviewer asked how she would get the number of rows, stating again there were 4 tiles in each row. Emily said, "36 divided by 6 maybe?"

This response is striking because it shows Emily was not structuring the remaining tiles into composite units of 4. Even though she had drawn rows of 4 tiles and she and the interviewer had repeatedly talked about rows of 4, in this moment Emily suggested dividing by 6. This response is evidence that Emily's coordination of tiles and rows of 4 tiles was transient, and therefore that she was making the coordination of the two types of units in activity. It also shows the lack of a feedback system between multiplication and division, a characteristic of stage 1 (Steffe, 1992).

Crate Problem. Emily initially drew separate pictures of pairs of adjacent units: a rectangle with 4 circles for cans of juice (a package), another rectangle with 8 rectangles inside (a box), and another rectangle with 6 rectangles inside (a crate). When asked how the pictures were related, she said, "inside the box is packages and inside this crate is boxes and inside those boxes are packages and inside those packages are cans." When asked to show those relationships in a picture, she drew a second picture: a large rectangle (crate) with 6 rectangles (boxes) inside, with 8 small squares (packages) in each of those, and with 4 circular juice cans in one of the squares.

From this point on, Emily's work on the problem was characterized by conflation of all units, sometimes repeatedly and after questioning and reference to her drawings. For example, at one point she found 32 as the number of cans in all 8 packages in a box. However, she then insisted for a sustained period (1.5 min) that 32 was the number of cans in one package, not one box. When asked whether 32 was the number of cans in one package or eight, she said, "Oh yeah, eight packages." Then she tried to use the 32 to find the total number of cans in the crate by multiplying it by 4 and by 32, and she re-insisted that the 32 was the number of cans in a single package. Under questioning she remarked, "If you think about it, there *are* cans in the boxes," as though she was understanding this anew, 14.5 min into her work on the problem.

So, even though Emily had drawn cans within packages within boxes in her second picture, she was not reasoning with a box as eight 4s. Once she made that coordination, the box was 32 ones, and she had to re-establish the coordination after working on another part of the problem. Her work on the Tiles and Crate Problems is solid evidence that she was coordinating two types of units in activity, and so was at stage 1. Written assessments confirmed this conclusion.

In addition, Emily did not show evidence of a partitive unit fraction scheme. When we posed the One-Fifth Problem (below), Emily drew a copy of the bar and then another copy next to it—a bar consisting of 2 parts. When asked to draw a bar that consisted of 5 of those parts, she did so. But when asked what the answer to the problem was, she said it was the 2-part bar (Figure 1a).

One-Fifth Problem. This rectangle is a candy bar. This bar is 1/5 of another bar. Draw that bar.

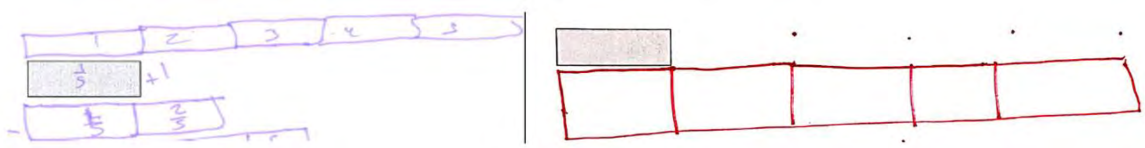

Figure 1a and b: Emily's work on the One-Fifth Problem

Emily's Final Interview: December 19

In her final interview, Emily worked mostly on problems to assess her understanding of the proportional reasoning unit. We did not pose problems to assess units coordination alone because we did not expect to see stage changes after 3 months. However, we did pose fractions problems, including the One-Fifth Problem.

This time, Emily drew a copy of the given bar. Then she spanned the given bar with her fingers and iterated that length four times to the right, drawing a 5-part bar (Figure 1b). Although the fourth part does not look equal, the dots above show she was making equal parts. When asked how she had done the problem, she said, "because it's one-fifth and that means there's 5 whole parts to it." This response was dramatically different from her initial interview, demonstrating clear evidence of a partitive unit fraction scheme. She also solved a related problem in a similar way. Her work is strong evidence that she had constructed a partitive unit fraction scheme, which implies a transition to stage 2 of units coordination.

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Conclusions

We have shown evidence that Emily experienced a change in units coordination stage during the fall semester of her 7th grade. Although we cannot claim that specific interventions caused this change, we have identified a task that may have been influential. During the proportional reasoning unit, Emily's class worked on determining how to make two cars travel at the same speed but for different distances and times. Emily's drawings of the journeys underwent a change from not showing relative size to showing it (Hackenberg, et al., 2019), and she sustained this change. It is possible that this work was one factor in her units coordination stage change. We look forward to more investigations of what promotes stage changes in units coordination.

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STRUGGLING WITH PRODUCTIVE STRUGGLE: IMPLICATIONS FOR STUDENTS WITH DIVERSE COGNITIVE RESOURCES

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The perspectives in mathematics education and special education are in tension when it comes to productive struggle. This study describes how struggle surfaced for the students and teacher/researcher in teaching experiments using learning trajectories with three students with diverse cognitive profiles. The students' activity helps to illustrate the relationships between struggle and mathematics learning. I share how students' struggle led to my own challenge in navigating tensions between mathematics education and special education. I consider how my focus on productive struggle without attending to cognitive difference reflected ableist thinking. Finally, I suggest implications of these observations for reframing productive struggle.

Keywords: Students with Disabilities; Instructional Activities and Practices; Learning Trajectories and Progressions

The perspectives of mathematics education, with a commitment to student thinking, and special education, with a commitment to explicit teaching, are often in tension—no less so when it comes to the idea of productive struggle. However, there is agreement that combinations of instructional approaches are beneficial for students (Alfieri et al., 2010; NMAP, 2008; Woodward, 2004). A number of researchers are investigating the mathematical thinking of students with learning disabilities as they engage with constructivist-based tasks (e.g., Hunt & Tzur, 2017; Xin & Tzur, 2016), but questions remain as to how best combine approaches. I conducted teaching experiments (Confrey & Lachance, 2000; Steffe et al., 2000) using a learning trajectory (LT) approach (Sarama & Clements, 2009) with three students with learning challenges, herein referred to as cognitive differences. I explored how primarily constructivist tasks and productive struggle might be supplemented with explicit support to generate learning. I drew on Hiebert and Grouws' (2007) definition of productive struggle as expending effort to make sense of mathematics.

Method

Participants

A purposeful sample of three elementary-aged girls with different learning strengths and challenges participated in this research. Table 1 provides information about these students.

Table 1: Participant information

Name	Age	Parent's description	Neuropsychological evaluation
Miranda	10	Insightful, creative. Likes to plan time to include rewards and breaks. Miranda says she wants to be interested or intrigued	Strengths: fluid reasoning, auditory memory. Difficulties: attention, inhibiting behavior; compromises accuracy for speed.
Eva	9	Great sense-of-humor, honest, loves pets, very active. Has trouble with	Strength: auditory processing. Difficulties: anxiety, speech-sound

		self-control, appropriate behavior. Very anxious about math.	disorder, mixed expressive-receptive language disorder, ADHD.
Macey	11	Compassionate, fun. Does well with hands-on and informal situations. Learns slowly and gradually, does not have a-ha moments.	Strengths: fluid and visual-spatial reasoning. Difficulties: executive functions, abstract reasoning, expressive language.

As the teacher/researcher, I am a participant in this research. I am a white, cisgender woman with over 16 years working in general education and intervention. My disciplinary commitments tend toward the mathematics education perspective, but having taught many students who struggle, I know things are not simple in practice. Therefore, in this research, I committed to supporting student progress rather than rigid use of a single instructional approach.

Teaching Experiments

The teaching experiments (Confrey & Lachance, 2000; Steffe et al., 2000) involved 45-minute individualized sessions, once or twice weekly, over 2-6 months during the COVID-19 pandemic. The conjecture guiding the experiments (Confrey & Lachance, 2000) was instruction based on a LT can center on constructivist-based tasks and support efficient progress when supplemented appropriately with reflection prompts and explicit guidance. Data sources include video and transcripts, artifacts, and planning and reflection protocols. Rigor was ensured through regular consultation with critical colleagues for their interpretations.

Retrospective Analysis

I used a three-level analysis (Simon, 2019) beginning with coding each data source for activity and strategies as indicators of student thinking. The next level of analysis involved identifying patterns and change over time, and the final level involved using the previous analyses to make inferences informing the guiding conjecture. I ensured rigor and trustworthiness through regular discussion with critical colleagues and sharing interpretations with parents as a form of member checking.

Findings

Miranda

The teaching experiment with Miranda was based on an equipartitioning LT (EPLT; Confrey et al., 2014) and reflective abstraction prompts (Simon et al., 2018). The struggle that surfaced for Miranda was sustained attention to tasks, and she resisted repeating tasks with varying number sets or contexts, a key aspect of reflective abstraction. She also wanted to figure things out without my support and did not want me to ask questions that would direct her thinking.

In a moment of insight, Miranda would enthusiastically rush through a task using sound reasoning but confuse the role of specific digits in some way and not quite "close the loop" on the idea. If she learned her solution was not right, Miranda would declare she was too confused and bored to continue. For example, when Miranda had an insight that sharing *a* objects among *b* persons results in *a/b* of an object per person, she said enthusiastically, "Mind blown!" I asked what would happen if four people shared three things. She tried to figure it out mentally but had a hard time keeping the numbers straight. When I pressed her to notate her thinking, she declared writing and drawing were boring, and only wanted to do it in her head. I tried to ask her questions to provide structure. However, Miranda then said she was too confused and bored, and she would not re-engage in the task meaningfully until the next session.

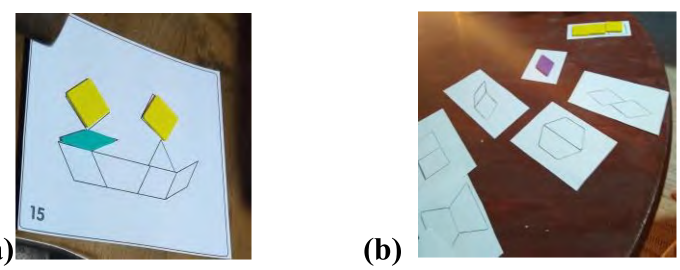
Miranda's effort to make sense of the math was productive in one sense—she would have moments of insight that were temporarily intriguing enough for her to pursue. However, it was unproductive in the sense that the process was not efficient. It was a struggle to find enough variety to sustain Miranda's motivation. Through trial-and-error, I landed on the approach of explicitly summarizing her activity from the previous session and explaining the conceptual idea at the heart of the task. I gave her feedback on where her thinking had gone astray. At this point, Miranda would solve one or two more related tasks and then express a desire to move on.

Eva

Eva had high levels of anxiety and often refused to participate during math class. Her activity in counting, arithmetic, and spatial reasoning was consistent with what is typically seen in children 4-6 years old (Clements & Sarama, 2021). I selected the shape composition LT as the focus for the teaching experiments (Clements & Sarama, 2021). The struggle that emerged for Eva was engaging with any challenge. She would look at a task and within seconds decide she could not do it. Then, she would jump up and run to the yard or play with her dogs.

Eva's first task is shown Figure 1a. Eva appeared to recognize the outlines for two squares because she quickly found these shapes and placed them at the top of the picture. Next, she used trial-and-error to find the right shape and orientation for the rhombus. Then, her attention moved to the connected shapes along the bottom, and she abandoned the task and would not return.

Because Eva would not engage with challenges, I chose to provide a very graduated increase in difficulty with extensive, explicit feedback that I viewed as eliminating struggle. Over the next few sessions, I provided outlines that gradually increased the quantity, combination, and orientation of shapes and the proportion of shared sides (see Figure 1b). I also provided Eva with extensive positive feedback. Each time she filled in a picture, I explicitly pointed out a mathematical feature of her activity: "Nice work, Eva! I noticed that … you saw that this large shape was made of two smaller shapes. Maybe you noticed this outline has three sides? Oh, you did! Great! And I saw you solve a problem—you turned this one to make it fit just right!" This approach led to fewer instances of giving up and gradual progress in the shape composition LT.



(a) (b)

Figure 1: (a) abandoned shape composition task; (b) scaffolded shape composition tasks

Macey

Macey's sessions focused on the EPLT (Confrey et al., 2014). Macey quickly took on each task, working until she felt she had achieved a satisfactory solution. However, her progress was slow, and we spent many sessions repeating variants of tasks. Macey struggled to make connections, see relationships, and construct new mathematical understanding. I used reflection prompts to guide her attention toward new ideas. However, prompts such as "What do you notice?" were typically too general. She seemed look for any feature she could describe, not one related to the mathematical ideas. For example, I asked Macey to share a whole "French fry" among an increasing number of sharers. She partitioned the whole each time, named the size of the share, and taped it to a piece of paper. When I asked her what she noticed, she said the denominator counted by ones and the numerators all stayed one. Then, I asked a more focused question, “Do you notice anything about the size of the share and the denominator?” She did not have an answer, so I became more directive: “Look at this denominator and this share. Now look at the next denominator and this share. Do you notice something? No? Can you compare this share to the one before? Which is a bigger sized share?”

I intentionally set up situations to be perturbations of her current conceptions to help her re-construct ideas. However, these situations were unproductive. Typically, Macey would look at the representations for a few seconds and then move some manipulatives or pencils or papers around, sit back in her chair, and look at something else in the area. It did not appear to be avoidance; my intuition tells me it was so inaccessible that she lost her place, maybe forgot what was expected, and was waiting for guidance. While I cannot be sure my interpretation is correct, I feel confident characterizing this as unproductive struggle. Faced with unproductive struggle, I often switched to explicit instruction to see if it supported progress. I would directly point out a relationship we had been exploring and explain the idea I wanted Macey to see. Then I would ask her to point to features of the representations I was describing and ask her to restate what I had just explained. My hope was that by guiding her attention explicitly, and with enough repetition of the idea, Macey would come to internalize the idea rather than remain lost.

Discussion

My purpose for the teaching experiments was to describe conditions under which constructivist approaches supplemented by reflection prompts and explicit guidance supported students’ learning and productive struggle. During the sessions, I frequently experienced struggle in deciding on the “right” course of action to support student learning. I wanted to provide opportunities to construct understanding through sense-making but, concerned we were not making progress, I felt I began to rely heavily on explicit guidance. However, during the retrospective analysis, I saw the increased support and explicitness was intentional and individualized, not a refutation of the conjecture or abandonment of the commitment to positioning students as active learners. In each case, the students actively engaged in the mathematical tasks without premature guidance (DeCaro & Rittle-Johnson, 2012). Explicitness was inserted within a constructivist framework to support attention to features relevant to the underlying mathematical ideas and connections.

Once I saw the pattern of purposeful, individualized explicitness, I also realized struggle was productive for all three students. Until then, my notion of productive struggle had nuances of ableist thinking. Ableism describes practices and attitudes that compare individuals to a standard of “normal” resulting in practices serving “standard” people (Stop Ableism, 2021). I had resisted “too much” explicitness because I viewed productive struggle narrowly as *expending effort to make sense of mathematics*. Viewed in this way, it established a standard reflecting cognitive strengths and needs of typically-achieving students. I had de-valued other forms of struggle such as with attention, anxiety, abstraction, or combinations of those.

These observations have two implications. First, disciplinary commitments can be re-framed as a commitment to recognizing and navigating complexity. Following on this, another implication is a more inclusive view of struggle would recognize the struggle that surfaces from cognitive difference. We can support students’ productive engagement by intentionally and purposefully planning for struggle in multiple forms—sense-making, attending, processing, remembering, reasoning, etc. From this perspective, a more inclusive definition of productive struggle is *expending effort that leads to greater levels of engagement with mathematical sense-making*. This small change in syntax and vocabulary may have large effects semantically, and those effects may better serve our goal of inclusivity.

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THAT'S CRAZY
AN EXPLORATION OF STUDENT EXCLAMATIONS IN HIGH SCHOOL MATHEMATICS LESSONS

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In this study, we explore the relationships between the types of student exclamations in an enacted lesson (e.g., "Wow!") and the varying dramatic tensions created by the unfolding content. By analyzing student exclamations in six specially-designed high school mathematics lessons, we explore how the dynamic tension between revelations of mathematical ideas at the moment and what is yet to be known connects with the aesthetic pull to react by the student. As students work through novel problems with limited information, their joys and frustrations are expressed in the form of exclamations.

Keywords: Emotions, Classroom Discourse, Mathematical Story, Exclamations, Tension

Historically, mathematics is not perceived as a popular subject among young people (COAG, 2008). National surveys show that students lose interest in mathematics as they progress in school and that by Grade 8, most report their experiences in mathematics class as unengaging and boring (Mullis et al., 2012, 2016; National Center for Education Statistics, 2015). However, students' experiences in mathematics classrooms are largely understudied (Martínez-Sierra & García González, 2014, 2015; Larkin & Jorgensen, 2015; Lewis, 2013). These studies explained students' emotions in the classroom either analyzing their interview responses (Martínez-Sierra & García González, 2014, 2015) or their responses in surveys and interviews (Dietiker, 2015; Lewis, 2013). Larking and Jorgensen (2015) allowed students to use an iPad as a video diary tool to record their experiences in the classroom. However, in this brief, we are analyzing students' experiences by observing their exclamations at different points of the lesson using the transcript, video recordings and the observation notes of each lesson.

Teachers designed lessons for Mathematically Captivating Learning Experiences (MCLE) to spark student engagement by attracting and maintaining students' attention and enhancing their curiosity and creativity. This is accomplished by withholding information from students, increasing the tension during the lesson. As tension rises and falls during the lesson, students feel compelled to shout exclamations expressing their reactions to the tension. Unlike traditional lessons, where teachers usually disclose the information too soon limiting the tension build up, MCLEs encourage students to explore relationships between mathematical ideas and make meaning. The purpose of this paper is to address: *What is the relationship between students' exclamations and the build-up of tension throughout these MCLE lessons?*

Theoretical Framework

Teachers designed lessons using the Mathematical Story Framework (Dietiker, 2015) with different aesthetic reactions. The framework considers the characters, setting, and plot within the context of a mathematics classroom. The characters are typically the students and those interacting with the lesson while the setting is the lesson materials such as, on a graph, paper, or computer screen. The plot describes how the content unfolds. We used Freytag's five-stage model (1863) to describe the development of mathematical stories and associated a degree of tension to each stage (Figure 1).

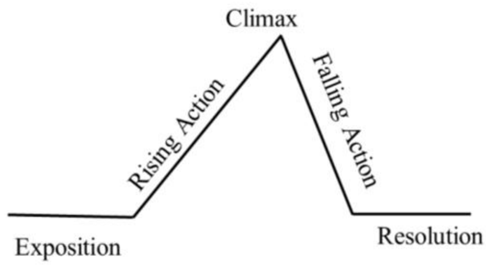

Figure 1. Freytag's Model of story development

The first stage, an *exposition,* is usually in the form of a Do Now or introductory problem that sets the narrative of the lesson. The story begins at equilibrium, that is students know the tools required to solve the given problem and there is either none or the lowest tension. The equilibrium is later disrupted as the story develops and the tension begins to rise and is propelled by a crisis during the *rising action*. At this stage, the ratio of what is unknown to known begins to increase and students wrestle with a concept and continuously make efforts to restore the equilibrium. The tension reaches its peak at the *climax* where students possibly have many questions they have yet to figure out. In the *falling action* stage*,* the level of tension quickly falls towards the *resolution*. Students use the new insights to help uncover answers and thereby creating a new equilibrium. During these stages, certain student exclamations arise. This model helps us identify the types of students' exclamations associated with the level of tension at certain points of the lesson.

Method

The students in this study were from six high school classrooms from the Northeastern region of the United States studying topics from Algebra One to AP Calculus AB. All students willingly participated in the study and the exclamations were recorded anonymously. Researchers monitored a focus group during the lesson to observe the levels of tension. The video and audio were recorded and the tone of a students' response was noted in the transcript.

We used thematic analysis to identify and analyze the patterns of meaning in a data set (Braun & Clarke, 2006). Throughout this data analysis process, the first author reviewed the lesson transcripts and noted the context and timestamp of each exclamation, and identified a pattern that relates to the levels of tension in a lesson. Four themes emerged around the tension that aligned with the Freytag model (see Table 1).

Findings

A thematic analysis of the data suggested that students' exclamations varied with levels of tension because of the information disclosed at that point in the lesson. This table shows the generalization of the exclamations found in the MCLE database organized by relative tension and their occurrence in a lesson related to the Freytag model. Areas of *low tension* occurred when students solved familiar problems, typically during the exposition of the lesson. We found students groaning or saying, "this is boring" as they applied known facts and procedures.

Growing tension occurred when teachers introduced a new problem where students could not directly apply previous knowledge. At this point, students began to question their understanding and made connections from previous mathematical ideas to make sense of this new problem. The tension grew until the point of *highest tension* (*climax* of the mathematical story) where vital information was disclosed to students. This is where students exclaimed shouts (see Table 1) in disbelief. The mathematical story ended with students saying, "wow" and *relieving tension* as they learned the applications of this new skill or topic.

Table 1: Students' Exclamations Categorized by Tension in MCLEs

Tension	Student Examples	Description	Moment in Freytag's Model
Low Tension	"This is boring" ** groans	Students are completing problems using prior knowledge in the form of a Do Now.	Exposition
Growing Tension	"I think I see a pattern" "Oh!!!! Wait"	Students are working through the investigation and a relationship or key piece of information is disclosed.	Rising Action
High Tension	"I'll bet money" "That's Trippy!"	Students react in disbelief with a new solution or tool.	Climax
Relieving Tension	"Wow"	Students are making sense of this new mathematical concept or idea.	Falling Action

The following is a breakdown of students' exclamations in an Algebra 2 class in correlation to the tension regarding the Introduction to Inverses lesson.

Low Tension "Yeah." Students worked on the Do Now problem individually and occasionally checked answers with their group members. In this scenario, the "yeah" was an affirmation to the group that the answer was correct and no follow-up discussion was needed.

Growing Tension "I'm having a brain aneurysm" In this scenario students were plugging values for *x* into the functions $h(x) = (5x - 27)/2 + 1$ and $k(x) = (2(x - 1) + 27)/5$ and $h(k(x))$. You also heard students say "Oh Jesus Christ!" and "Woah there!" Up until now, the students only had to compose functions like $f(x) = 2x + 5$. The tension built because they were using their prior knowledge on this new situation. Later in the lesson when the teacher checked the students' work and left them saying "interesting." Then students said, "We definitely did something wrong then. She does not say 'interesting' often. The tension rose as they double-checked their work and ensured everything was correct.

High Tension "Oh my god, everything cancels out!" During a full class discussion, the teacher wrote $h(k(x)) = (5((2(x - 1) + 27)/5) - 27)/2 + 1$ on the board and had students simplify it. Students said "Oh! oh! Oh!" and "I see it!" as students figured out that the solution simplified to x. Students said, "All that time it was x? So why do we need to go over this as a class?" This was a high-tension moment because the students did not yet understand and were wondering what was so special about the solution $h(k(x)) = x$. At this moment the teacher claimed that this was a special relationship called inverses. Soon after a student exclaimed, "Wait! IT'S THE SAME THING!" referring to the fact that the composition of a function of a variable (x) and its inverse is always the variable (x). This was the highest point of tension because students were starting to see the relationship between a function and its inverse.

Relieving Tension "I'm even curious" Students were working together to see if two functions were inverses using composition. They were given a list of functions and tried to match the two that were in fact inverses of one another.

There were certain instances where students become frustrated and say, "this is boring" or "I give up" but with some input and encouragement from the teacher, students were re-engaged or followed along enough until they were convinced that a process worked. For example, in one lesson, students were given the near-impossible task of finding logarithms without using a calculator! Many students initially found the process tedious and boring. However, by the end of the lesson, students said "Ohhh!" and "I get it now!" The students struggled through the tension and were rewarded with a positive outcome.

Discussion

In some instances, an exclamation was heard at multiple points of tension. For example, a student said, "Ah" to convey that they made a mistake. This was a low tension moment and was quickly resolved. However, in another lesson, a student said "Ah" in confusion and frustration. This is a high tension moment because the student struggled with the concept. These claims were grouped according to the tension in the lesson.

This study focuses on MCLE lessons because the aesthetics of the design are controlled for each lesson. So, the moment of tension when these exclamations occur aligns with parts of Freytag's Model of story development (see Fig.1). The more students engaged with the tensions in the MCLE lessons, the more they were able to voice their confusions, understanding, and emotions. These findings are useful for practitioners and researchers because it gives the insight to create future lessons focused on tension buildup and the Mathematical Story Framework. In the MCLE Research project, we have found that these lessons are significantly more interesting to students than the traditional lessons from the same teacher. A future study could focus on examining ways these exclamations differ in MCLE designed lessons than traditional mathematics lessons.

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INTERPRETACIÓN DE GRÁFICAS EN CONTEXTO POR ESTUDIANTES UNIVERSITARIOS

INTERPRETATION OF GRAPHICS IN CONTEXT BY UNDERGRADUATE STUDENTS

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En este artículo se presentan los resultados de una investigación relacionada con la interpretación de gráficas de funciones a trozos asociadas a situaciones problema. Se diseñó una secuencia de actividades basada en NetLogo. Se implementó en un ambiente online, mediante el uso de la plataforma Zoom. El marco teórico que se utilizó para analizar los resultados fue la teoría de razonamiento covariacional de Carlson. Los participantes en este estudio fueron 15 estudiantes de cuarto trimestre de nivel universitario. Como resultado se observó que los estudiantes lograron exhibir el Nivel 3 de razonamiento covariacional.

Palabras clave: Matemáticas de nivel universitario, Pre-Cálculo, Funciones a trozos, Tecnología.

Introducción

Varios investigadores (Carlson, Jacobs, Coe, Larsen y Hsu, 2002; Doerr, Ärlebäck y Staniec, 2014; Stroup, 2005) han identificado que, en distintos niveles educativos, los estudiantes exhiben dificultades para interpretar funciones asociadas a situaciones cercanas a la vida real.

El objetivo de esta investigación fue conocer, según la teoría de razonamiento covariacional de Carlson, et al. (2002), como interpretaban los estudiantes universitarios gráficas de velocidad contra tiempo en el contexto del tránsito vehicular. La investigación se llevó en el ambiente virtual Zoom, con ayuda de NetLogo y hojas de trabajo elaboradas ex profeso. La pregunta de investigación fue ¿cómo los estudiantes universitarios interpretaron y externalizaron su conocimiento sobre la función a trozos al resolver situaciones donde ésta subyace?

Marco Conceptual

El razonamiento covariacional se define como "las actividades cognitivas en coordinación de dos cantidades que varían mientras se atienden a las formas en que cada una de ellas cambia con respecto a la otra" (Carlson et al., 2002).

Para su estudio, Carlson et al. (2002) desarrolló un marco que describe cinco acciones mentales de los estudiantes asociados a cinco niveles de desarrollo del razonamiento covariacional. Tres de las acciones mentales (Carlson et al., 2002) son: a) AM1-Coordinación del valor de una variable con los cambios en la otra, b) AM2-Coordinación de la dirección del cambio de una variable con los cambios en la otra variable, c) AM3-Coordinación de la cantidad de cambio de una variable con los cambios en la otra variable. Tres de los niveles de razonamiento covariacional (Carlson et al., 2002) son: a) Nivel 1. Coordinación, b) Nivel 2. Dirección, c) Nivel 3. Coordinación Cuantitativa. Un estudiante ha obtenido un nivel, si sustenta las acciones mentales del nivel considerado y las asociadas a los niveles inferiores.

Metodología

La investigación fue de tipo cualitativa y se realizó en el ambiente virtual Zoom, debido a la pandemia del COVID-19. La población de estudio estuvo conformada por 15 estudiantes, de 19 a 22 años de edad, del cuarto trimestre de la carrera de física e ingeniería. Se organizaron en tres equipos (A, B y C). La secuencia de actividades se apoyó en a) simulaciones con el programa Traffic Basic de NetLogo (Figura 1) y b) una hoja de trabajo (Figura 2) que contiene un problema de una función a trozos en el contexto de tránsito vehicular.

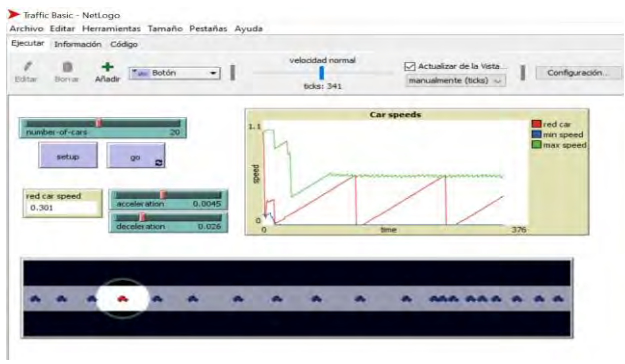

Figura 1: Simulación del Programa Traffic Basic de NetLogo

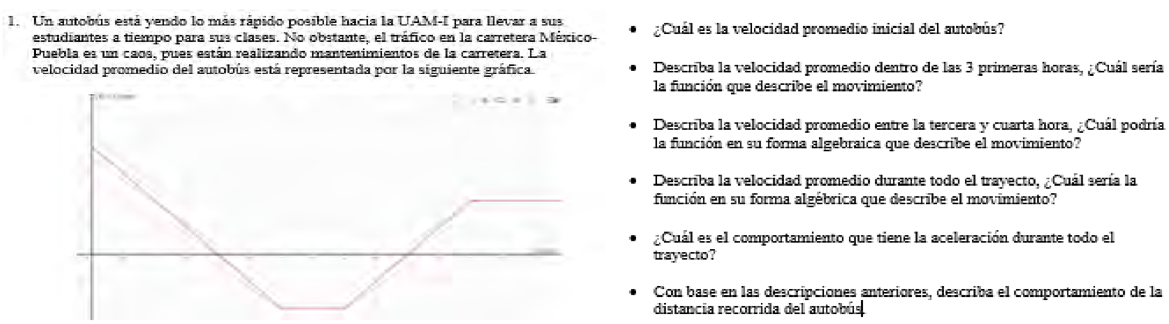

Figura 2: Hoja de Trabajo

Fases de implementación de la secuencia de actividades

Fase 1. Implementación de las simulaciones de Traffic Basic de NetLogo mediante Zoom. El objetivo fue identificar y analizar cómo el estudiante interpretaba y explicaba la gráfica (dinámica) que se desplegaba al simular el fenómeno de tránsito vehicular.

Fase 2. Resolución de la hoja de trabajo en equipos. El objetivo fue identificar y analizar cómo el estudiante interpretaba y explicaba una gráfica, sin el apoyo de la simulación.

Fase 3. Cierre de la sesión. El objetivo fue revisar si el estudiante refinó su lenguaje y comprensión para describir las gráficas.

Se recolectaron videos, respuestas de los estudiantes y notas del profesor. La duración de la actividad fue de 2 horas. Se analizó el nivel de desarrollo del razonamiento covariacional mediante los aportes de Carlson et al. (2002), donde, AMi son las acciones mentales y Ni son los niveles de razonamiento covariacional, para $i = 1,2,3$.

Resultados y Discusión

Fase 1: Implementación de las Tres Simulaciones de Traffic Basic de NetLogo

Primera Simulación. El profesor solicitó a los estudiantes describir la primera simulación (Figura 3) y la gráfica, con el objetivo de identificar sus acciones mentales.

Los estudiantes describieron, cualitativamente, la gráfica de velocidad promedio contra tiempo mediante el uso de conceptos distintos. C_2 co fico e idiendo la velocidad", "se mueve hacia arriba en un tiempo determinado" (posible AM1 y AM2). Mientras que, B_4 dijo: "Su rapidez aumenta hasta llegar a un máximo, se detiene, luego baja". Es posible que B_4 usó rapidez como sinónimo de velocidad; de acuerdo con Yildiz (2016) esto es común.

Segunda Simulación. El objetivo consistió en provocar un segundo episodio para que describieran el movimiento del auto en términos de las variables observadas en la gráfica.

El profesor modificó el parámetro de la aceleración, realizó la segunda simulación (Figura 3) y solicitó a los estudiantes describir la forma de la gráfica.

El estudiante B_2 aseveró que, en el último tramo del gráfico, la aceleración es constante. B_1discre l decir: "¡no!¡no es la acelerac elocidad, ahí la velocidad es constante y la acelerac n vale cero durante el tiempo transcurrido!" (posible AM1 y AM2).

Se pudo observar de nuevo la dificultad para describir la gráfica en términos de las variables involucradas en la gráfica. B_2 describió en términos de aceleración. B_1corrigió a B_2 y describió la gráfica en términos de la velocidad. Sus descripciones fueron cualitativas.

Tercera Simulación. El profesor realizó la tercera simulación (Figura 3), al cambiar la cantidad de automóviles, y pidió describir en términos de las variables observadas en la gráfica. C_1y B_2 dijeron: "la gr fica es de la velocidad contra el tiempo" (AM1). C_3 comentó que el auto "cambia su velocidad en cada tiempo" (AM1 y posible AM2). Podemos decir que C_1 y B_2 exhibieron N1 al identificar las variables involucradas en la simulación y C_3 pareció exhibir N2 al identificar las variables involucradas y la relación entre ellas.

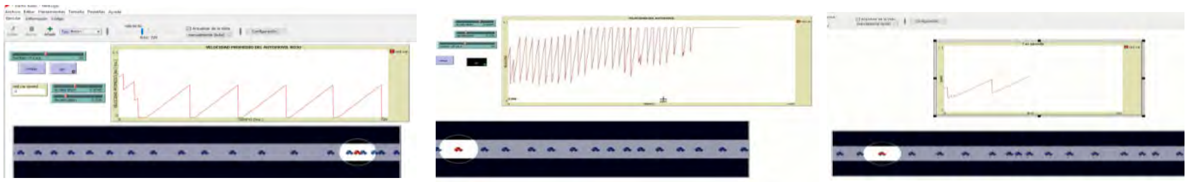

Figura 3: De izquierda a derecha, primera, segunda y tercera simulación en NetLogo

Fase 2: Implementación de la Hoja de Trabajo

Equipo A. Los estudiantes lograron identificar, verbalmente, las variables en el contexto del problema (AM1). Les fue difícil interpretar qué ocurría con el autobús, cuando su velocidad era negativa. De acuerdo con Stroup (2005) interpretar gráficas que incluyan velocidades negativas es complejo. Podemos decir que exhibieron N1.

Equipo C. Los estudiantes describieron el movimiento del autobús representado por la función a trozos, por ejemplo, mencionaron que, entre la tercera y cuarta hora, la velocidad del autobús es invariante y es igual a -1 (Figura 4). Exhibieron AM1, AM2 y AM3, pues identificaron, relacionaron y coordinaron las variables v, t. Podemos que exhibieron N3.

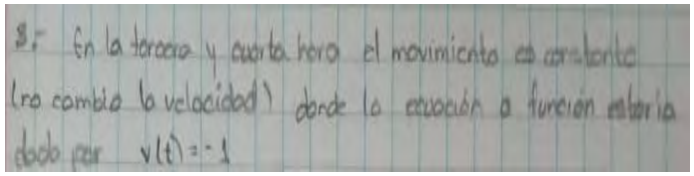

Figura 4: Respuesta del Equipo C

Equipo B. Los estudiantes interpretaron que la velocidad por debajo del eje del tiempo era negativa y, por ende, el autobús iba en reversa (Figura 5 derecha). Describieron, verbalmente, que cuando la recta intersecaba al eje del tiempo, entonces el autobús se había detenido por completo. Podemos decir exhibieron AM1, AM2 y AM3, pues identificaron, relacionaron y coordinaron las variables v, t (Figura 5 izquierda). Podemos decir que exhibieron N3.

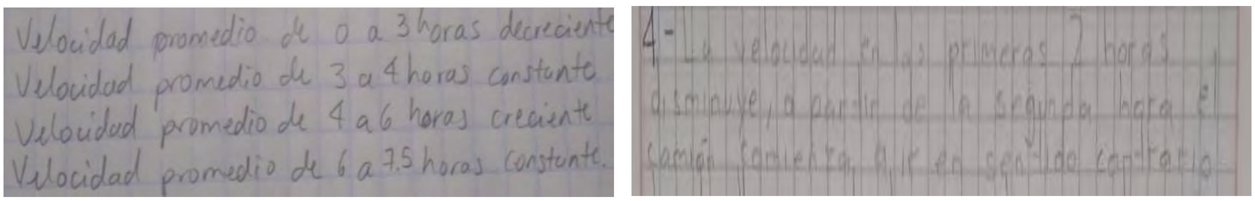

Figura 5: Respuesta del equipo B

Fase 3: Cierre de la sesión

Los estudiantes presentaron al final de la sesión sus respuestas. Su duda principal fue sobre la interpretación de las velocidades promedio negativas del autobús durante su recorrido. El profesor promovió la comprensión del problema a partir de la discusión grupal realizada con base en las distintas respuestas. En general, se observó la dificultad para describir la aceleración y sobre todo la distancia recorrida a partir de la gráfica de velocidad –lo cual coincide con lo encontrado por Stroup (2005). En esta discusión los estudiantes comentaron al profesor que no estaban acostumbrados a resolver problemas de contexto, por lo que se les había hecho difícil la actividad, pero les había gustado.

Conclusiones

Podemos concluir lo siguiente como respuesta a la pregunta de investigación planteada en este reporte: ¿Cómo los estudiantes universitarios interpretaron y externalizaron su conocimiento sobre la función a trozos al resolver situaciones donde ésta subyace? Se observó el surgimiento de diferentes tipos de descripciones al utilizar NetLogo y la hoja de trabajo. Los estudiantes denotaron mayor experiencia asociada a una descripción de gráficas estáticas en lápiz y papel fuera de contexto, que para analizar el movimiento del vehículo rojo con base en la simulación, y construcción simultánea de la gráfica. Durante la primera fase fue difícil para los estudiantes describir la gráfica en términos de las variables involucradas. Sus descripciones fueron de tipo cualitativo, asociadas a conceptos como aceleración y rapidez. No obstante, los estudiantes C_1 y B_2 exhibieron N1 y C_3 pareció exhibir N2. En la segunda y tercera fase emergieron descripciones cuantitativas, además de las cualitativas. Los estudiantes de los equipos B y C exhibieron un nivel de razonamiento covariacional más cercano al N3, al exhibir acciones mentales AM1, AM2 y AM3. Las respuestas denotan que al equipo A le fue difícil describir la gráfica e interpretarla en términos del contexto de la situación, sin embargo, exhibió N1 pues identificó las variables del problema. El equipo B por su parte exhibió más facilidad para representar de manera algebraica su interpretación de la gráfica, pero dificultad para describirla por escrito en términos del contexto de la situación. El equipo C exhibió descripciones en términos del contexto de la situación. Concluimos al igual que investigadores señalados en este documento, que aún en el nivel universitario los estudiantes requieren vivir experiencias donde razonen sobre eventos dinámicos para mejorar su razonamiento covariacional y, en particular, algo que observamos es la necesidad de brindarles experiencias donde puedan discutir conceptos como velocidad, velocidad promedio y rapidez.

Reconocimiento

La investigación tuvo apoyo de CONACYT a través de las becas de estudiantes de posgrado.

INTERPRETATION OF GRAPHICS IN CONTEXT BY UNDERGRADUATE STUDENTS

This article presents the results of an investigation related to the interpretation of graphs of piecewise functions associated with problematic situations. A sequence of activities based on NetLogo was designed. It was implemented in an online environment, using the Zoom platform. The theoretical framework used to analyze the results was Carlson's theory of covariational reasoning. The participants in this study were 15 college-level fourth-trimester students. As a result, it was realized that the students managed to exhibit Level 3 of covariational reasoning.

Keywords: College Level Math, Pre-Calculus, Piecewise Functions, Technology.

Introduction

Several researchers (Carlson, Jacobs, Coe, Larsen & Hsu, 2002; Doerr, Ärlebäck & Staniec, 2014; Stroup, 2005) have identified that, at different educational levels, students have difficulties interpreting functions associated with situations close to real life.

The objective of this research was to know, according to the covariational reasoning theory of Carlson, et al. (2002), how university students interpreted velocity versus time graphs in the context of vehicular traffic. The research was carried out in the Zoom virtual environment, with the support of NetLogo and worksheets elaborated on purpose. The research question was, how did university students interpret and externalize their knowledge about the piecewise function when solving situations where it underlies?

Conceptual Framework

Covariational reasoning is defined as "the cognitive activities involved in coordinating two varying quantities while attending to the ways in which the change in relation to each other" (Carlson et al., 2002, p. 354).

For their study, Carlson et al. (2002) developed a framework that describes five mental actions of students associated with five levels of development of covariational reasoning. Three of the mental actions (Carlson et al., 2002) are: a) AM1-Coordinating the value of one variable with changes in the other, b) AM2- Coordinating the direction of change of one variable with changes in the other variable, c) AM3-Coordinating the amount of change of one variable with changes in the other variable. Three of the levels of covariational reasoning (Carlson et al., 2002) are: a) Level 1. Coordination, b) Level 2. Direction, c) Level 3. Quantitative Coordination. A

Student has obtained a level, if he supports the mental actions of considered level and those associated with the lower levels.

Methodology

The research was qualitative and was carried out in the Zoom virtual environment, due to the COVID-19 pandemic. The study population consisted of 15 students, from 19 to 22 years old, from the fourth trimester of the physics and engineering career. They were organized into three teams (A, B and C). The sequence of activities was supported by a) simulations with NetLogo Traffic Basic program (Figure 1), and b) a worksheet (Figure 2) containing a problem of a piecewise function in the context of vehicular traffic.

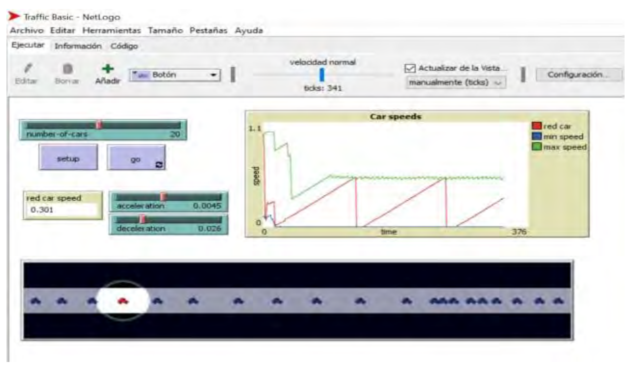

Figure 1: NetLogo Traffic Basic program simulation

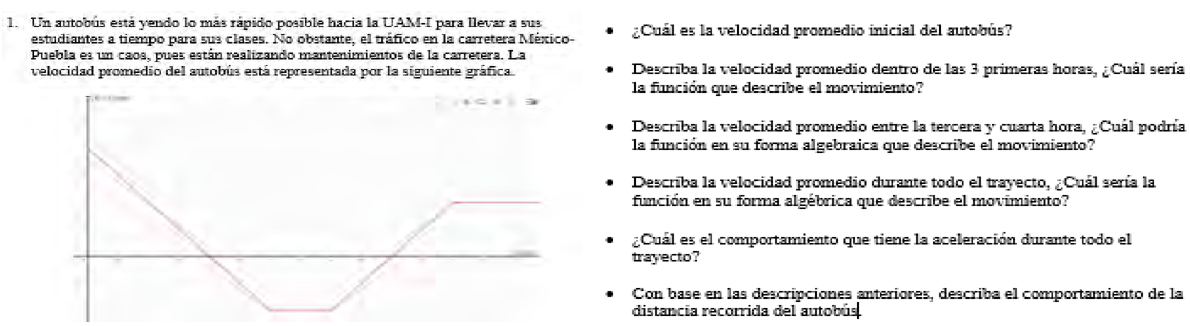

Figure 2: Worksheet

Implementation phases of the sequence of activities

Phase 1. Implementation of NetLogo Traffic Basic simulations using Zoom. The objective was to identify and analyze how the student interpreted, and explained the graph (dynamics) that was displayed when simulating the phenomenon of vehicular traffic.

Phase 2. Resolution of the worksheet in teams. The objective was to identify and analyze how the student interpreted and explained a graph, without the support of simulation.

Phase 3. Closure of the session. The objective was to check if the student refined the language and understanding of it to describe the graphs.

Videos, student responses, and teacher notes were collected. The duration of the activity was 2 hours. The level of the students' development of covariational reasoning was analyzed using the contributions of Carlson et al. (2002), where AMi are the mental actions, and Ni are the covariational reasoning levels, for $i = 1,2,3$.

Results and Discussion

Phase 1: Implementation of the three NetLogo Traffic Basic simulation

First Simulation. The teacher asked the students to describe the first simulation (Figure 3) and the graph, to identify the mental actions.

Students described qualitatively, the graph of average velocity versus time using different concepts. C_2 commented "graphic is measuring the velocity", "moves upward at a certain time" (possible AM1 and AM2). While, B_4 said: "his speed increases up to a maximum, stops, then lower". It´s possible that B_4 used speed as a synonym for velocity; according to Yildiz (2016) this is common.

Second Simulation. The aim was to cause a second episode to describe the movement of the car in terms of the observed variables in the graph.

The teacher changed the parameter of acceleration, performed the second simulation (Figure 3) and asked students to describe the shape of the graph.

Student B_2 stated that, in the last section of the graph, the acceleration is constant. B_2 disagreed, he said: "No! It´s not the acceleration, it´s the velocity, there the velocity is constant, and the acceleration is zero during the elapsed time!" (possible AM1 and AM2).

We were able to observe again the difficulty of describing the graph in terms of the variables involved in the graph. B_2 described the graph in terms of acceleration. B_1 corrected B_2 , and described the graph in terms of velocity. The descriptions of them were qualitative.

Third Simulation. The teacher made the third simulation (Figure 3) and, changing the number of cars, asked to describe in terms of the variables involved in the graph. C_1 and B_2 said: "the graph is of velocity versus time" (AM1). C_3 said the car "changes its velocity in each time frame" (AM1 and possible AM2). We can say that C_1 and B_2 exhibited N1 when identifying the variables involved in the simulation, and C_3 seemed to exhibit N2 when identifying the variables involved and the relationship between them.

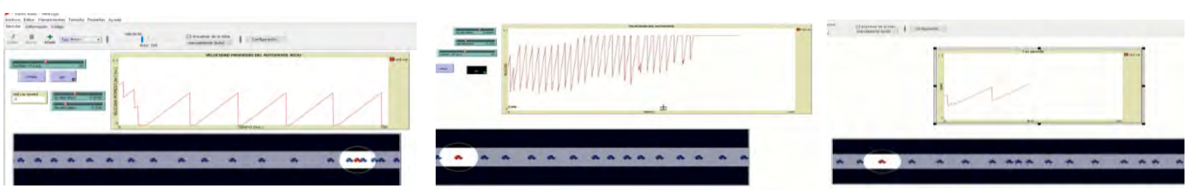

Figure 3: From left to right, first, second and third simulation in NetLogo

Phase 2: Worksheet implementation

Team A. Students were able to verbally identify the variables in the context of the problem (AM1). It was difficult for them to interpret what was happening with the bus, when its velocity was negative. According to Stroup (2005), interpreting graphs that include negative velocities is complex. We can say that the students of Team A exhibited N1.

Team C. Students described the movement of the bus represented by the piecewise function, for example, mentioned that, between the third and fourth hour, the velocity of the bus is invariant, and is equal to -1 (Figure 4). They exhibited AM1, AM2 and AM3, as they identified, related and coordinated the variables v, t. We can say they exhibited N3.

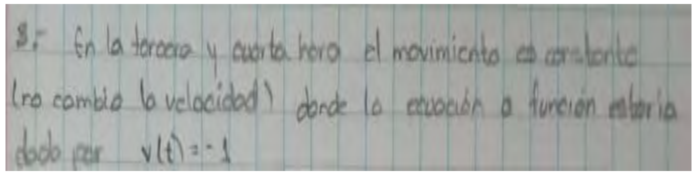

Figure 4: Team C responses

Team B. Students interpreted the velocity below the axis of the time was negative and, therefore, the bus was reversed (Figure 5, right). They verbally described that when the line intersected the time axis, then the bus had come to a complete stop. We can say they exhibited AM1, AM2 and AM3, as they identified, related and coordinated the variables v, t (Figure 5, left). We can say that they exhibited N3.

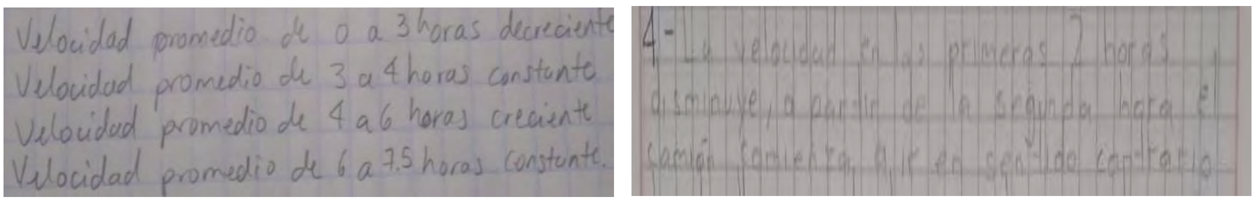

Figure 5: Team B responses

Phase 3: Closure of the session

Students presented their answers at the end of the session in a group discussion. The main question was about the interpretation of the negative average velocities of the bus during its journey. The teacher promoted the understanding of the problem based on the different students' answers. In general, it was observed the difficulty to describe the acceleration, and especially the distance traveled by the bus from the velocity graph – which coincides with what was found by Stroup (2005). At the end of the group discussion, the students commented to the teacher that they didn´t use to solve context problems, so the activity had been difficult for them, but they liked it.

Conclusions

We can conclude the following in response to the research question posed in this report: how did university students interpret and externalize their knowledge about the piecewise function when solving situations where it underlies? Different types of descriptions came up when using NetLogo and the worksheet. The students demonstrated a better experience associated to static graphic descriptions on pen and paper, out of context, than to analyze the movement of the red vehicle based on the simulation, and simultaneous construction of the graph. During the first phase it was difficult for the students to describe the graph in terms of the variables involved. Their descriptions were qualitative, associated with concepts such as acceleration and speed. However, students C_1 and B_2 exhibited N1, and C_3 appeared to exhibit N2. In the second and third phases, quantitative descriptions emerged, in addition to qualitative ones. The students of teams B and C exhibited a level of covariational reasoning closer to N3, they exhibited mental actions AM1, AM2 and AM3. The answers indicate that it was difficult for team A to describe the graph and interpret it in terms of the context of the situation, however, it exhibited N1 because it identified the variables of the problem. Team B, on the other hand, exhibited more facility to represent algebraically their interpretation of the graph, but difficulty to describe it in writing in terms of the context of the situation. Team C provided descriptions in terms of the context of the situation. We conclude, like the researchers mentioned in this study, that even at the university level, students must live experiences where they reason about dynamic events to improve their covariational reasoning and, in particular, something that we observe is the need to provide them with experiences where they can discuss concepts such as velocity, average velocity and speed.

Acknowledgement

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DEVELOPING STUDENTS' UNDERSTANDING OF PROOF THROUGH REVISING PROOFS BASED ON PEER CRITIQUES

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This study focuses on the critiquing process as an opportunity to engage high school geometry students with all aspects of the third Common Core Standard for Mathematical Practice (SMP3): construct viable arguments and critique the reasoning of others. We report on the types of critiques students provided one another and the extent to which students addressed each type of critique when revising their arguments. Nearly half of students' critiques related to the clarity of a claim or need for a mathematical justification. Students consistently revised draft arguments based on peer critiques, but at times did so in ways that decreased the generality of the argument or inserted mathematically incorrect justifications. Findings suggest that the broader instructional sequence can afford opportunities to discuss the key components of proof by drawing on students' ideas while maintaining shared authority throughout.

Keywords: Reasoning and Proof, Geometry and Spatial Reasoning, Standards, Instructional Activities and Practices

Introduction

Although research on the teaching of proof in secondary geometry has typically focused on the construction of a proof, this is but one component of the third Common Core Standard for Mathematical Practice (SMP3): Construct viable arguments and critique the reasoning of others (CCSSI, 2010, p. 6). Specifically, SMP3 calls for students to pose conjectures and investigate their validity, construct arguments that use definitions and previously established results as justifications for claims, communicate to others, and respond to peer arguments. Analysis of the types of reasoning and proof opportunities found in U.S. textbooks highlighted a difference in number of exercises that asked students to pose a conjecture versus construct a proof, as well as a lack of specific exercises asking students to critique the reasoning of others (Otten et al., 2014). These findings suggest that students may not engage in all aspects of SMP3 within a single proof task without modifications by the teacher. The present study investigated one way teachers could modify proof tasks to allow for varied reasoning-and-proving around a single diagram or context. In this paper, we focus on a subset of that practice: asking students to provide written critiques to one another and then revise their draft proofs in response to peer feedback.

Theoretical Perspective and Literature Review

Our study is based on the perspective that mathematics learning occurs through discursive practices (e.g., Pimm, 1987; van Oers, 1996). While classroom discourse is commonly studied within the context of oral communications between teachers and students, students can also engage in discourse with one another through reading and responding to each other's written work (Pimm, 1987). Within proof instruction, the classroom teacher or textbook is commonly viewed as the audience and authority for determining whether a students' constructed argument of a conjecture can be considered a proof (e.g., Herbst & Brach, 2006; McCrone & Martin, 2009; Otten et al., 2017). This view contrasts with Stylianides' (2007) definition of proof, wherein the classroom community plays a central role in determining the set of accepted statements, forms of reasoning, and modes of expression used in the proving process. Drawing on this definition of proof, we contend that it is important for students to develop their understanding of proof through engaging in varied reasoning-and-proving activity.

One way that teachers can communicate to students their role as a member of the mathematics community is by having them revise their arguments based on peer feedback. Through the process of revising their mathematical argument, students have the opportunity to make changes to their argument in ways that increase its validity, precision, amount of detail, and/or level of convincingness (Jansen, 2020). In order to be able to provide peer critiques, students must first make sense of the peer's argument, so they can pose questions or comments to clarify or improve the argument (CCSSI, 2010). Reading other students' arguments also has the potential to provide insights that students can incorporate into their own arguments and reinforce the idea that there are multiple ways to prove a mathematical conjecture. Students' evaluations of written arguments can also shed light on their understanding of proof, including aspects that may not be reflected in their own constructed arguments (Bieda & Lepak, 2014; Healy & Hoyles, 2000; Stylianides & Stylianides, 2009). While there are multiple potential benefits to this practice, peer feedback also has the potential to reinforce incorrect ideas about proof, such as the idea that examples are sufficient justifications for general claims (e.g., Knuth et al., 2009; Lee, 2016). The research questions guiding this study were 1) Which aspects of students' draft proofs did their peers attend to when providing written critiques? 2) In what ways did students take up their peer feedback, or not, when revising their argument?

Methods

This study took place across three geometry classes, taught by the first author, located in a rural high school in the Midwest region of the United States. Our analysis focuses on thirteen students' written work across four proof tasks. The first task was the Exterior Angle Theorem, presented as a diagram with specific angle measurements. Task two focused on the diagonals of parallelograms. The third task investigated the quadrilateral formed by connecting the midpoints of rectangles' sides. Task four involved classes of polygons that are similar (see Conner & Krejci, 2020 for specific prompts). For each task, students (1) formed a conjecture based on a provided geometric context; (2) constructed a draft argument; (3) exchanged papers and provided written critiques/feedback to their peers; (4) revised their argument based on the peer feedback. The task concluded with a whole class discussion of one proof based on students' ideas.

Data and Analysis

Students' critiques were open-coded (Strauss & Corbin, 1998) and then collapsed into the following categories: requests for justification (e.g., "why does this work?"), clarification of vocabulary or notation (e.g., "which angles are you referring to?"), feedback focused on the structure of the argument (e.g., "how does ____ help prove your conjecture?"), suggestions (e.g., "add how each triangle = 180") , counterexamples related to a claim, comments about the generality of a claim (e.g., "if only most angles and sides are congruent, does it always work?"), requests for examples, and compliments. After establishing the coding scheme, each researcher individually coded the remaining papers and then met to resolve any discrepancies.

We analyzed the extent students took up the peer critiques in their revised argument by identifying changes in the revised argument that appeared to be in response to each critique. When determining whether a student addressed a critique in their revised argument, we qualitatively categorized the ways in which students addressed the different types of critiques without attending to whether it improved the argument. Since students directly progressed through writing their draft argument, giving peer critiques, and writing their final argument without any additional input from the teacher, we could reasonably assume that the changes between their draft and final arguments were either a direct result of the peer critiques or an indirect result of reading their peer's draft argument. In our analysis process, we focused only on changes that appeared to be connected to the peer feedback.

Results

Critiques Provided

Table 1 reports the overall percentage of critiques within each category. Students provided clarity and justification critiques across all four tasks; in contrast, suggestion and counterexample critiques occurred most frequently in the Diagonals of a Parallelogram task while structural critiques surfaced predominantly in the Exterior Angle Theorem task. These findings show that students can attend to important aspects of proof, even when they are not provided specific directions about what to attend to during the critique/feedback process.

Table 1: Types of Critiques Students Provided Across All Tasks

	Clarity	Justify	Structure	Suggestion	Compliment	Counterex.	Example	General
Overall	23.6%	21.1%	13%	13%	8.9%	8.1%	6.5%	4.1%

Aspects of peer critiques students attended to as they revised arguments

Students addressed the majority of their peers' critiques, ranging from 60% (generality) to 83% (clarity). When revising their arguments, some changes improved the overall argument while others resulted in decreased generality of the claim or the use of incorrect justifications.

Clarity. Students addressed 24 of the 29 critiques related to the clarity of mathematical claims and diagrams when revising their arguments. For example, in the Midpoints of a Rectangle task, one group critiqued, "What triangles are you talking about?" for clarity around what Joe, Cody, and Taylor were referring to when they said "The triangles around the rhombus are all congruent because..." The students clarified this phrase in their revised argument by saying, "we know that the triangles EAF, GBF, GCH, and EDH are all congruent to each other..." in reference to their diagram. Clarity critiques typically involved a small change in wording, making them accessible for students to address during the revising process.

Justification. Of the 26 justification critiques provided, students addressed 15 of them when revising their argument. Students' new justifications fell into three categories: specific examples, mathematically incorrect justifications, and, rarely, mathematically correct justifications. For example, one group's conjecture for the Midpoints of a Rectangle task included the claim that "the inside shape [formed by connecting the midpoints of the sides of a rectangle] is a rhombus it has 4 congruent sides." Based on the critique, "how do you know shape is a rhombus? (is there a theorem?)", the students added the following claim to their revised argument: "We know the inside shape is a rhombus because rhombus opposite angle theorem." The rhombus opposite angle theorem is an incorrect justification as it can only be used *once* a rhombus has been established, and not vice versa. This example is illustrative of a pattern we encountered across multiple arguments: namely, students used an incorrect justification or did not provide a justification at all in instances where their claim required multiple steps to sufficiently justify.

Structure. Students addressed 11 of the 16 critiques students provided that related to the structure of the argument. For example, Cody addressed two structural critiques he received for the Exterior Angle Theorem task by explaining how the properties in his draft argument could be used to solve for the missing angles C and D (Figure 1). Although his revised argument does not fully explain how he could use the equation D = 180 - C to prove that B + A = D, it comes closer to connecting the equation with his conjecture. The way that Cody revised his argument to address the structural critiques was representative of the broader data: students addressed the critiques in a mathematically appropriate, but incomplete way.

<table>
<tr><th>Draft Argument (Cody)</th><th>Critiques (Kim)</th><th>Revised Argument (Cody)</th></tr>
<tr><td>A triangle = 180° and a line = 180°</td><td>Give more explanation as to why this matters</td><td rowspan="2">We know how to solve one [missing angle] because there are 180° in a triangle so you add up the two angles and subtract them from 180° that is how you get the third angle in the triangle. After that we know a line is also 180° and because the angle we just solved is on that line, we take 180° minus that number to get the outside angle.</td></tr>
<tr><td>So if D = 180 - C then B + A = D</td><td>How does “D = 180 - C” help your argument?</td></tr>
</table>

Figure 1: Draft and revised claims from Cody’s Exterior Angle task

Discussion

The lesson structure used in this study (conjecture - construct draft argument - provide written critiques - revise argument) offers a way for students to engage in multiple facets of SMP3. We found that having students provide written critiques to their peers surfaced many of the key aspects of proof, such as the need for claims to be clear and include mathematical justifications. Students’ ability to provide meaningful critiques occurred even in instances when they struggled to write arguments that adhered to all aspects of a proof. This finding is consistent with prior literature that has shown students are better at evaluating arguments than constructing them (Bieda & Lepak, 2014; Healy & Hoyles, 2000).

The process of having students revise their arguments based on peer critiques communicates the message that students play an important role as a member of the audience for whom a proof is written and furthers the sharing of authority among both students and the teacher. There was clear evidence that changes between students’ draft and final argument were due to peer critiques, demonstrating students’ ability to make sense of the peer feedback and find ways to incorporate it into their revisions. Students’ revisions in response to peer critiques ranged from those that improved the overall argument to ones that resulted in a decrease of the generality of a claim or the inclusion of a mathematically incorrect justification. This finding highlights the role of content knowledge (e.g., Cirillo & Hummer, 2021) and strategic proof understanding (Weber, 2001) needed to be able construct an argument that adheres to all aspects of a proof. For example, students’ addition of mathematically incorrect justifications may point to gaps in their content knowledge, whereas their inclusion of examples as justification could suggest incomplete understanding of the generality of the claim being proven. The differences between types of critiques that students were able to address may point to the complexity of the skills and knowledge needed to address the critique. Specifically, it may be that students are better able to attend to structural critiques once they have established the main ideas in the argument with appropriate justifications. Even though this process did not consistently result in airtight proofs, we do not discredit the value of having students revise their arguments based on peer feedback. Rather, the revision process can serve as an opportunity for the teacher to assess students’ base arguments and level of understanding in order to progress them towards a mathematically sound proof (Stylianides, 2007).

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THINKING ABOUT CONSTANT RATE OF CHANGE: A CASE STUDY OF ALEXI

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This paper presents the results of a study where a student's thinking about the idea of constant rate of change and her thinking about the foundational ideas for understanding the constant rate of change were investigated. The idea of constant rate of change involves students' quantitative reasoning, covariational reasoning, and proportional reasoning. The result of the exploratory teaching interviews also discusses how the participant thinks about quantities, the relationship between quantities, representation of quantities, changes in quantities, rate, ratio, and proportionality in relation to the idea of constant rate of change.

Keywords: constant rate of change, changes in a quantity, students' thinking, ratio of changes.

Introduction and Theoretical Background

Learning the idea of a constant rate of change associates with the concept of quantities, representation of quantities in a context, rate, ratio, proportionality, changes in quantities, and linear functions. A student engages in quantitative reasoning (Thompson, 1988, 1990, 1993, 1994 & 2011) as she conceives quantity as a measurable attribute of an object and conceives measuring it as a multiplicative comparison of two fixed quantities. When two quantities vary in relation to each other, the mental operations that support the dynamic images in students' thinking is referred to as covariational reasoning (Carlson et al., 2002; Thompson & Carlson, 2017). Proportional reasoning as a theory is based on the idea of ratio and rate and requires a conceptualization of multiplicative comparison of the measures of two quantities. A ratio is a multiplicative comparison of the measures of two non-varying quantities, and a rate is the proportional relationship between two varying quantities' measures (Thompson & Thompson, 1994). A student engages in proportional reasoning when she conceives the invariant relationship of quantities in a dynamic situation or applies her understanding of proportionality in a mathematical context. Therefore, a student conceptualizes the idea of constant rate when she envisions two quantities in a situation vary smoothly and continuously, and the changes in one quantity is a simultaneous result of changes in another quantity; and as the quantities covary the ratio of the changes in the values of two quantities remain proportional.

In this paper, I will present the results of exploratory teaching interviews (Steffe & Thompson, 2000) that was designed to address students' thinking about the idea of constant rate of change and other foundational ideas to understand the constant rate of change.

The primary research question I want to consider here is- how do students think about the idea of constant rate of change in a context in association with foundational ideas like quantities, representation of quantities, rate, ratio, and changes in the quantities?

Methods

For this study, I used an exploratory teaching interview (ETI), one of the teaching experiment elements (Steffe & Thompson, 2000). An ETI is a one-on-one interview consist of an interviewer, a student, and video/audio recording devices and was a convenient choice for this study as the spontaneous nature of proposing on-moment hypotheses would provide the groundwork to construct a model of students' thinking of the idea of constant rate of change. The participant, Alexi (pseudonym) was a precalculus student in a large south-western university in the United States, during the time of the interview. She participated in three 1.5 hours interviews. The interviews were held in zoom, and I used the zoom video recordings to analyze the data. The data analysis methods used in this study are qualitative analysis supported by grounded theory (Strauss & Corbin, 1994) and conceptual analysis (Glaserfeld, 1995; Thompson, 2008) to construct a model of Alexi's thinking. The tasks I used in this study had a dynamic nature. I presented two primary contexts- *the candle burning* context and *the circle* context (Carlson, M.P., Oehrtman, M., & Moore, K.C, 2016). The contexts used for the tasks in this study are-

- *Context 1.* The Candle Burning Problem- A 14-inches candle is lit. Using an applet, students identify the fixed and varying quantities where the applet shows how the candle's burned length and the remaining length of the candle covary as time changes. Students use variables, expressions, formulas, and graphs to represent the relationship between quantities in the context. Students think about changes in the quantities and constant rate of change as we present a situation where a candle burns at a constant rate of 1.8 inches per hour.
- *Context 2.* The Circle- Students think about the relevant ideas- quantities, the relationship between quantities, representation of the quantities, ratio and rate, and constant rate of change when a circle's circumference varies with respect to the circle's radius. Students are allowed to use an applet after they try conceptualizing the context.

Results

In Context 1, Alexi could imagine the length of the candle varying at different moments since the candle is lit and identified the candle's length before and after it was lit as quantities. She thinks she can associate numbers to the attribute of an object, and the numbers make them measurable. I probed her asking if it is possible to measure a quantity without associating numbers. Then she thought about comparing two objects with respect to one another, and she could measure attributes of one object in terms of the other without associating any numbers. In context 1 & 2, Alexi spontaneously mentioned the relationship between quantities. She noticed that the candle's remaining length decreases as the burned length of the candle increases and the circle's circumference increases or decreases as the circle's radius increases or decreases. She was careful identifying the independent and dependent quantities and how changes in the dependent quantity depend on the changes in independent quantity from the applets' visual effect. In both contexts, Alexi was natural in identifying independent and dependent varying quantities and fixed quantities and how two quantities are changing together (increasing or decreasing values of dependent quantity as there is a change in the independent quantity). She also represented the quantities and relationship between the quantities in both contexts using variables, expressions, and formulas.

In the candle burning context, the applet was set in a way that the burned length of the candle is 5 inches, and the remaining length of the candle is 9 inches. She was asked to think about the change in the values of the burned length of the candle from 5 inches to 12.40 inches is and the corresponding change in the values of the remaining length of the candle. Alexi exhibited her thinking with an analogy of the variables x_1 & x_2. She thinks x_1 is the value of the quantity from where the quantity has changed to a new value. She used hand swiping motion to say,

> "I think about it as x_1 as it (the quantity) has changed to get to x_2, so if we take x_2 and subtract x_1, it will give us the amount x_1 had to move to get to x_2. If we think about x_1 and subtracting x_2 from it, we would be going backward instead of figuring out how much x has changed from x_1 to get to x_2."

She later clarified that she thinks a change in x from x_1 to x_2 means x is moving from x_1 towards x_2. She exhibited her thinking about change in a quantity's value from one reference point to a new point as she mentioned that the candle's remaining length since it is lit decreases as the candle's burned length since it is lit increases.

In the circle context, she initially thinks about $0.56 - 0.5$ as the change in radius when it increases to 0.56 cm from 0.5 cm and $2\pi * (0.56 - 0.5)$ as the corresponding change in circumference when the radius increases to 0.56 cm from 0.5 cm. With little probing Alexi mentioned that $2\pi(0.56)$ cm and $2\pi(0.5)$ cm are values of the circumference when the radius of the circle is 0.56 cm and 0.5 cm respectively, and the difference between the values of the circumference represents the change in the values of the circumference when radius has changed from 0.5cm to 0.56 cm. However, she thinks about $2\pi * 0.06$ cm as a value of circumference when the radius is 0.06 cm not a change in the values of the radius. Alexi was asked to determine the rate of change of the circle with respect to any change in the length of the radius. She thinks that she compares the difference between two values of the circumference with respect to the difference between two values of the radii when she is thinking about the rate of change of the circumference with respect to any change in the length of the radius. She figured the numerical value of the rate of change in this situation is 2π. The answer 2π made her think that this is the resulting ratio of any circumference and corresponding radius of the circle. She exhibited thinking the formulas $C = 2\pi r$ and $\Delta C = 2\pi\Delta r$ represent the same relationship between the circumference and the radius. The following excerpt presents Alexi's response justifying why both formulas represent the equivalent relationship between the circumference of the circle and the radius of the circle-

Alexi: I think they represent the same thing because ΔC is still a circumference and Δr is still radius. We just got them differently. The 2π is always the same. No matter what C and r is C/r is 2π.

Interviewer: You mean ΔC and C represent the same quantity?

Alexi: Yes, we just got ΔC from subtracting two circumferences instead of C.

Interviewer: When you subtract a new value of the circumference value from a previous circumference value, do you think the difference is a value of the circumference? For example, if you have two values of the circumference of the circle and the difference is $c_2 - c_1$, what does the value of $c_2 - c_1$ mean to you?

Alexi: To me, this is a circumference value that we got from subtracting two circumferences. I understand how it's a change, but I also think that we got it from two circumference values so that it is still a circumference. We just got it in a different way.

Interviewer: When you have a change from quantity's two values, do you think the change represents a value of the quantity?

Alexi: Yes.

She thinks the rate of change of the circumference of the circle with respect to any change in the length of the radius is a constant rate of change because both ratios $\frac{C}{r}$ and $\frac{\Delta C}{\Delta r}$ results into a constant 2π.

Next, Alexi was presented with a modified candle burning context where a candle burns at a constant rate of 1.8 inches per hour. 5.6 hours after being lit, the candle is 9.92 inches tall. She thinks that the candle burns at a constant rate of 1.8 inches means every hour the candle burns 1.8 inches. She imagines the candle is getting shorter as time goes by, and as the unit of time gets smaller, she imagines the candle is getting less visibly shorter.

She started thinking that the candle before it is lit is 9.92 inches tall, and since it is lit the change in time is 5.6 hours, and the change in height of the candle is 9.92 inches since it is lit. However, after I insisted her on drawing a diagram of the context, and then she noticed before the candle was lit the candle would be taller than 9.92 inches after 5.6 hours being lit. She thinks as the candle burns 1.8 inches every hour, it burns 5.6 times as large as 1.8 inches after 5.6 hours being lit. She thinks 5.6*1.8 inches is the candle's length that burnt off in 5.6 hours and that is the change in length of the candle since it is lit. She thinks the changes in the time since the candle is lit and the changes in the candle's length since it is lit is proportional as 'anything happens to one side of the ratio, the same happens to the other side, that's what make them proportional.'

Reflecting on her thinking to each context, Alexi summarized what she thinks about two quantities varying at a constant rate-

> "Two things (quantities) work together or vary at a constant rate means they are moving at a constant rate, but like they are moving proportionally. If one of the quantities is moving an equal amount, the other side of the ratio will move as well proportionally."

She provided an example to support her answer as she thinks if it were 2 hours since the candle is lit the burnt length of the candle would be 2 times as large as 1.8 inches; if the time changes 1/3 of 1 hour, the candle's burned length would be 1/3 times as large as 1.8 inches.

Discussion

The findings suggest that Alexi conceives a quantity as a measurable attribute of an object and the multiplicative comparison of two fixed quantities makes a quantity measurable. Alexi showed consistency in thinking about variation of varying quantities and covariation between two covarying quantities across the contexts and her thinking aligned with 'smooth-continuous variation/covariation' discussed by Thompson & Carlson (2017). Alexi's responses about changes in a quantity or changes in quantities' values revealed her thinking that she struggles to separate any change in a quantity's value from a varying value of the quantity. She thinks the change in a quantity's value is another value of the quantity that results from taking the difference of two values of the quantity. She successfully identified desired answers in the problems with a little probing; however, her meanings for a 'change in a quantity's value' create opportunities for researchers to design tasks that will support students' thinking about changes in a quantity's values in a productive way.

Alexi's responses in the study indicate that she connects the idea of proportionality with the idea of ratio. She thinks there is a constant rate if two quantities covary proportionally. She thinks about two quantities vary at constant rate if the value of one quantity is always the same number of times as large as the other quantity's value. Alexi confuses the concept of rate with the concept of constant rate of change. When she reveals her thinking about the concept of constant rate of change, it does not involve thinking about changes in one quantity's value being proportional to the corresponding changes in the other quantity's value. Her thinking is parallel to how she thinks about changes in a quantity's value being the same as a new value of the quantity. The result of the study indicates Alexi's struggle to connect the foundational ideas like rate, ratio, changes in a quantity to construct a productive meaning of the idea of constant rate of change.

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CONFIDENCE AND PARTICIPATION IN A CALCULUS REVIEW PROGRAM: A CASE STUDY

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Staying engaged in school is especially challenging in these times. This paper presents work from a review program that is designed to help students stay connected while not attending classes and to encourage them to adopt a growth mindset. Every day during the break between semesters before students take Calculus 2 they are texted Calculus 1 review questions. Students are given opportunities such as the ability to try the problem again if they answer incorrectly, use hints, and answer challenge problems to push themselves. Each student interacts with the program in a unique way and understanding student choices can help us create a program that benefits all students. Furthermore, with specific knowledge about the student's background, the program can be improved to help similar students thus allowing for more targeted support for minority and first-generation students.

Keywords: Undergraduate Education. Calculus. Technology. Equity, Inclusion, and Diversity.

Introduction

Just like physical skills, cognitive skills grow rusty over time unless they are regularly used and practiced. This means that school breaks can have negative consequences on student learning. Indeed, by a conservative estimate, summer vacation sets K-12 students back by one month of instruction; that is, it causes them to lose one month of grade-level equivalent skills relative to national norms (Cooper, Nye, Charlton, Lindsay, & Greathouse, 1996). Although this "summer gap" effect has been documented for many school subjects, it is most pronounced for mathematics which requires a strong foundation of prior knowledge, and this phenomenon extends into higher education too. We now know that having breaks between sequential closely related mathematics courses significantly lowers performance in the second course at the university level (van de Sande & Reiser, 2018).

The Keeping in School Shape (KiSS) program was created to help students prepare for Calculus 2 by sending them a daily Calculus 1 review problem via text message or email. The program uses retrieval practice as well as push technology to engage with students in a variety of ways. (Roediger & Butler, 2011) (van de Sande & Reiser, 2020) Since the first year it was launched the program has grown to include a variety of features such as challenge problems and theme days aimed at fostering a growth mindset (Dweck, 2008) as well as furthering students review capabilities. Furthermore, the charity aspect of the program aims to motivate the students. The most recent session of the KiSS program held during the winter break of 2020 also included three surveys: an entry survey, an exit survey, and an extended feedback survey. These three surveys were used to interpret the activity of specific students throughout the program and gain insight into the students as a whole. This paper will first introduce the program and its features, then follow one student's path through the program.

The Program

Every day starting the Monday after the end of the Fall 2020 semester students who had enrolled in the program were sent a message either via email or text. When they clicked on this message, they were taken directly to the first daily problem where they viewed the problem and rated their confidence. They rated their confidence by picking one of the following five options which have been assigned a number for data analysis: not at all (1), not very (2), meh (3), somewhat (4), and super duper! (5). The student then clicked next and was presented with the daily Calculus 1 review problem along with five multiple choice options. If they answered correctly on their first attempt, they could view the solution, exit, or attempt a related challenge question. If they opted for a challenge question, they then could view its solution or exit (Figure 1).

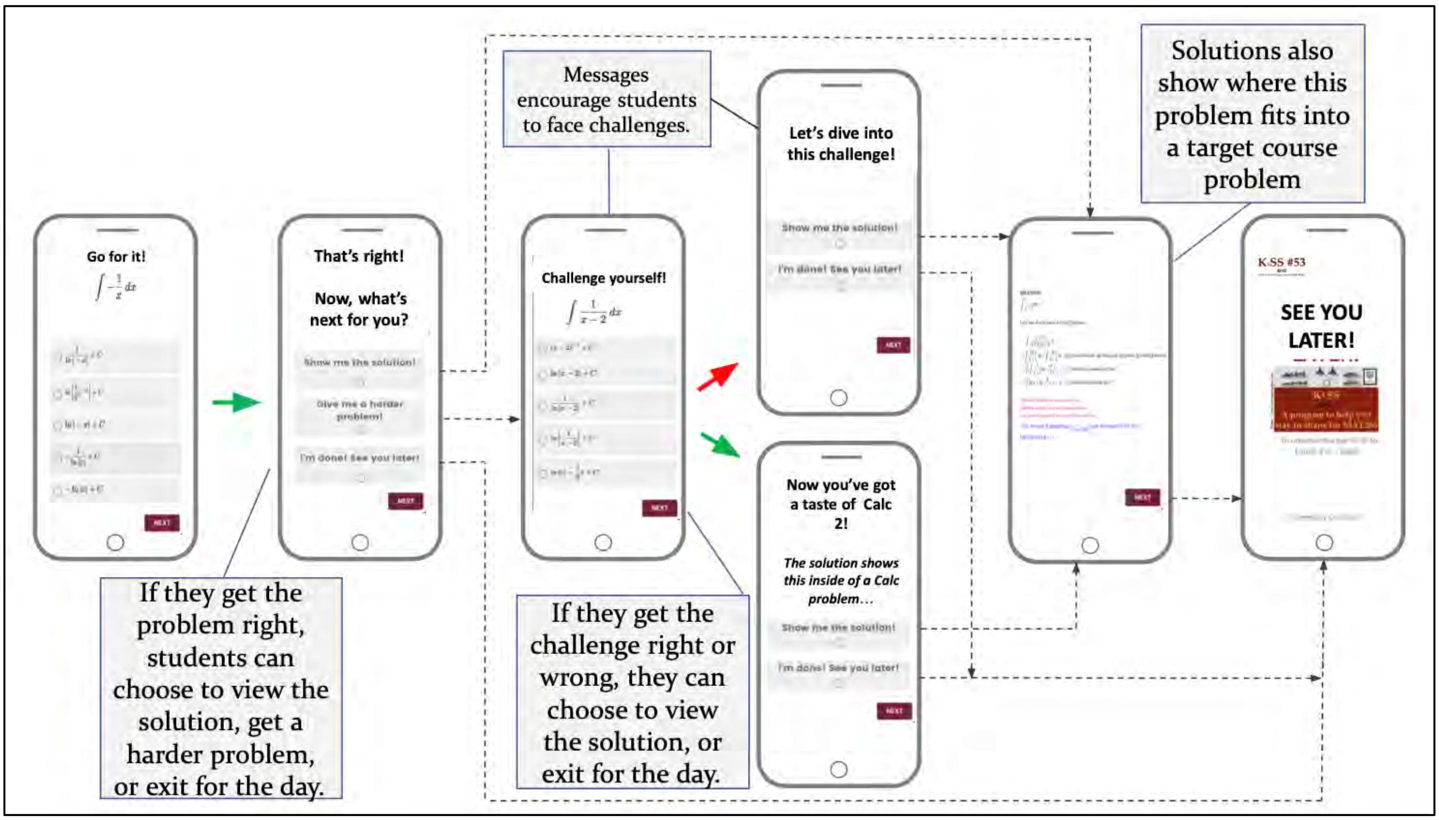

Figure 1: Path through daily review problem if student answers problem correctly

If the student answered incorrectly, they could choose to see a hint or see the solution. If they chose to see a hint, they could attempt the problem again. If they got the problem correct or incorrect on this second try, they could view the solution to the problem or exit.

There are two days of the week where the KiSS program does not follow the above explained schedule, namely 2's-days and trivia days which are held on Tuesdays and Sundays, respectively. On 2's-days, students had the option of doing an additional review problem similar to the initial problem regardless of their accuracy, or, if they answered correctly they could opt to do up to two related challenge problems. On trivia days students were given the option to only complete a math related trivia question or to complete their normal daily review followed by the math related trivia question.

Finally, there is the charity aspect of the KiSS program. Whenever a student answered a question correctly, they were allowed to select one of five good causes to receive a point. At the end of the program, the second author donated money to the charity with the most points.

Analysis

We will now follow one specific student, "John," who identified as a black and Asian male (pronoun: he/his) and a first-generation college student studying Electrical Engineering on his path through the program and responses in the extended feedback survey. John completed 19 of the 33 possible problems in the 2020 KiSS Winter Break program. Of the 19 questions he answered, he answered 14 correctly and five incorrectly and had an average confidence rating of 2.95 on a scale of one to five. So, while he answered 73.7% of the questions he attempted correctly, his confidence rating was very low, suggesting that he underestimate his capabilities.

When asked about his mindset regarding math, John stated in the extended feedback survey that that he felt he could "improve through hard work. If [he] finds the concept difficult, it means [the] will have to practice more." This mindset is evident in how John acted when he answered a question incorrectly. As shown in Figure 2, which depicts his confidence rating for each of the problems he answered and in which the problems he answered incorrectly are colored red, this happened five times. Every time John answered incorrectly, he chose to view the hint and try again, thus giving himself more practice. After viewing the hint and attempting the question again, John got the question correct three out of five times, suggesting that the hints themselves (or how he used them), were helpful to some extent. In addition, consistent with a belief that he could improve with effort, John viewed the problem solution regardless of whether he got the daily question right or wrong on the second attempt.

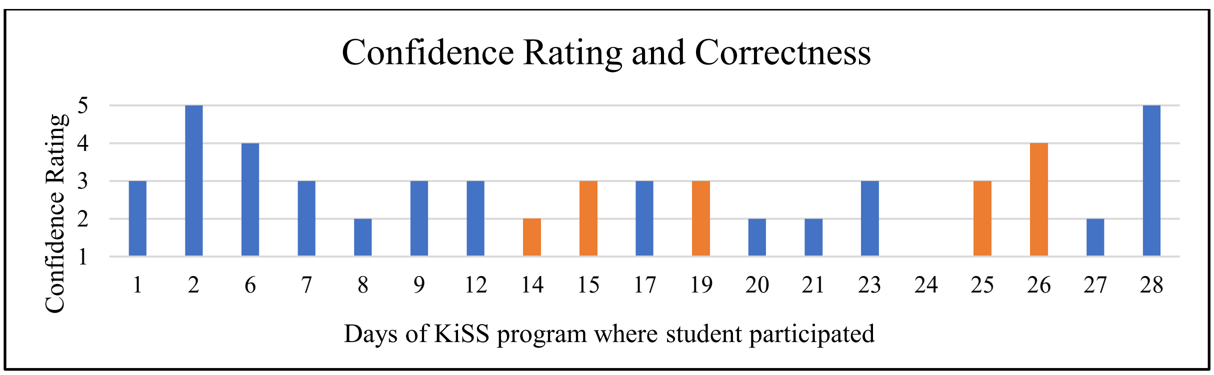

Figure 2: Student's confidence rating and correctness for each initial daily review question he answered

In the KiSS program students had the option on normal days to push themselves by answering a challenge question if he answered the first daily question correctly. John answered a total of 14 questions correctly, 12 of which were normal daily review days and two of which were 2's-days. On normal review days John only took the opportunity to attempt a challenge question three out of 12 times (25% of the time). Of those three times, he answered the challenge question correctly twice. On the other nine regular program days, John chose to view the solution eight times and exited once. This behavior is still consistent with a growth mindset at some level, since, even if he chose not to regularly push himself with a challenge problem, he still wanted to learn more from the daily problem rather than just exiting for the day. Furthermore, reviewing the solution even though he answered the question correctly suggests that he was still not confident with either himself or the material. John stated that the reason he "sometimes" did the challenge problem after getting the daily review problem correct "depend[ed] on the day/energy level". This suggests that John might be cognizant of the fact that he could push himself more but is not always motivated to for a variety of reasons. However, John may have been open to trying a question of similar difficulty.

To examine this dynamic, we can look at John's choices on the two 2's-days he participated in. On the first 2's day he participated in, John answered the initial daily review problem correctly and then chose to view the solution and do the related daily review problem which he also answered correctly. This is one of the few days on which John chose not to view the solution and instead exited, perhaps because, after getting the initial and related problem correct, he felt very confident in the way he was solving the problems. John's tendency to favor repetition over challenge, is evident in that he chose to do a second related problem of the same difficulty rather than trying the more difficult challenge problems. On the second 2's-day that John participated in, he again got the initial daily review problem correct. This time he chose to try a challenge problem, and after getting this challenge problem wrong, he clicked to see a solution. He then clicked to try the related challenge problem, but did not complete it. We infer from the fact that he viewed the second challenge problem but did not attempt it that he perhaps found it intimidating. This behavior is indicative of a fragile growth mindset since John chose to confront a second challenging problem – after not succeeding on a similar problem the first time – but then retreated (Dweck, 2008). The second 2's-day was the last day John engaged with the program mathematically, even though there were five more days with problems. He did participate in the extended feedback survey which was sent out between the last two days of the program, indicating that he was still receiving notifications for the questions. It is possible that John stopped participating at the end of the program because he got busy as the semester neared, or that his confidence was shaken after his participation in the second 2's-day on which he was unsuccessful in answering the challenge questions.

Discussion

Since every student has a unique experience within the KiSS program, there is much to be learned by studying one student's choices. The student discussed in this paper was a first-generation college student who identified as "black and Asian." Historically, first generation and minority students have more difficulty in higher education and specifically in Science, Technology, Engineering and Mathematics. Moreover, demographics have been known to be a factor in how students understand their own capabilities (Leslie et al., 2015; Litzler et al., 2014). John exhibited low confidence in his mathematical abilities and did not always choose to challenge himself. He also always generally chose to view the solution, suggesting that, even though he got the question correct, he was either still not confident in his ability or with the concepts and skills used in the question. Since confidence is often tied to success in STEM fields (Litzler et al., 2014), it is important that a program such as the KiSS program helps students build confidence.

To help students with a similar background and confidence level as John, the KiSS program could include the option to do a related daily problem that is not more challenging every day. This would allow these students to practice more and build their confidence which could lead to them attempting more challenging problems once they are more confident. We could also continue to foster a growth mindset by including hints for the challenge problems so that students have more support when they attempt a challenge problem (Dweck, 2008).

It is important to note the extended feedback surveys conducted with the option to answer the survey online, or via a Zoom interview or phone call. Had John chosen to speak via Zoom or phone, there could have been more chance for dialogue. Despite this limitation, case studies such as this help us understand how to expand the KiSS program to best serve our students.

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TRANSFER UNDERGRADUATE MATHEMATICS STUDENTS CREATING ONLINE COMMUNITY DURING COVID-19

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This study examined how transfer mathematics students reflected on their experiences with remote, online-instruction caused by the COVID-19 pandemic. Using a model of knowledge-sharing communities, we sought to better understand any challenges they faced and adjustments they made to support their learning. Through qualitative data analysis of semi-structured individual interviews with two transfer mathematics undergraduates, we found that students were aware of the difficulties they faced with a new class structure and with building community. To attend to these challenges, students exhibited a high level of agency in generating virtual communities to simulate the level of connectivity that in-person instruction affords; they perceived these communities as salient to their learning. Our findings can potentially inform instructors on practices that better support community building.

Keywords: Undergraduate Education, Online and Distance Education, Technology

Introduction

The landscape of higher education has shifted to an online and remote setting as a result of the COVID-19 pandemic. This drastic pivot in instruction became the impetus for challenges and adversities students had likely never encountered in their learning. One such challenge was the increased difficulty in community building and fostering interpersonal relationships. Integration into a community, both academically and socially, has been shown to strengthen individuals' commitment to education and their respective institutions; conversely, the lack thereof increases the likelihood of departure from a student's institution (Tinto, 2019). This poses a problem for university retention, particularly for transfer students during COVID-19. Transfer students tend to come from low-income households, identify with marginalized communities, and follow non-traditional paths to universities (Berger & Malaney, 2003). They arrive at universities already facing social challenges related to community building (Rhine et al., 2000), and this adversity is likely exacerbated by the remote learning environment brought upon by COVID-19. This study attempted to understand how transfer mathematics students perceived online instruction and engaged in community building to potentially inform mathematics instructors on practices that fostered connectivity and community. The research question of this study was: How did transfer mathematics students adjust to challenges at their new university during online remote learning?

Framing

This study used Yoon and Rolland's (2012) model of knowledge sharing in virtual communities as the conceptual framework to understand the processes in which the mathematics transfer students created and engaged with communities during this period of online remote instruction. This model is founded on self-determination theory, which concerns how people determine their choices motivated by psychological needs in the absence of external influence (Deci & Ryan, 1985). Yoon and Rolland's model investigated the effect of "psychological needs – perceived competence, perceived autonomy, and perceived relatedness – on knowledge-sharing behaviors in virtual communities," while also looking at the role of familiarity and anonymity with these needs (2012, p. 1133). Specifically pertinent to this study, perceived relatedness (the feeling of connection to others) and competence influenced knowledge-sharing behaviors. Our study involved virtual communities in which knowledge-sharing and online interaction were a means of learning, so this work helped us understand how students recognized their needs and were motivated to create a virtual community.

Methods

This study was conducted as part of a larger project exploring students' experiences in a set of mathematics courses supporting their transition to the university. Eleven undergraduate students in a Minority-Serving Institution in California participated in the study, and we used purposeful sampling (Patton, 2002) to select the two transfer students. One self-identified as male (Nathan – a pseudonym, used for all proper nouns), and one self-identified as female (Ava). In Fall 2020, both students took three mathematics courses: Course A (a transition to higher mathematics course, which introduced students to mathematical proof writing), Course B (an academic and career advising course), and Course C (an elective course that provided a space to further develop proof-writing and learn the expectations for higher level mathematics courses). The latter two courses were taken by the same set of transfer mathematics students and were specifically created in an attempt to ease the transition to the university.

In the subsequent quarter, we conducted and video/audio-recorded semi-structured interviews via Zoom (Rubin & Rubin, 2011). Students were asked to reflect on their experiences related to their enrollment in the set of courses, with special interest to how COVID-19 affected their learning experiences. The interview protocol included questions pertaining to academic and social challenges that students faced as a result of the remote-online format.

We qualitatively analyzed students' responses by watching the videos of the interviews, writing memos, and coding. Using descriptive, open coding (Strauss & Corbin, 1990), we identified themes around students' experiences with community during remote instruction brought upon by COVID-19 and generated these initial codes: Anticipation of Worsened Instruction and Connections, Instructor Challenges, Instructor Adjustments, Student Adjustments, and Impact. We then regrouped and categorized the coded data into two larger themes: *Challenges with Online Instruction* and *Student-Generated Community.*

Findings

We found that collectively, in the era of COVID-19, incoming mathematics transfer students transitioning into a four-year university yearned for and sought to develop a mathematics community. The findings are organized on the final two themes related to students' experiences during online instruction: the challenges of online instruction and the subsequent response of the student-generated online community.

Challenges with Online Instruction

Class Structure. Instructors' structure and effort varied between courses. For instance, the Course C instructor posted recorded lectures online and used designated class time as office hours, while Nathan's Course B instructor held optional synchronous lectures and posted recordings online. Regarding synchronous lectures, Nathan explained:

> [Online lectures] go faster than they would if it was an-in person class, so ...you're still writing things down, and they ask if there's any questions, but you're not even finished writing down...so your brain hasn't caught up to see if you can ask the question.

The online setting largely impeded the (often nonverbal) feedback between instructor and students, so feedback had to come and be elicited actively and intentionally. Additionally, some instructors seemed uninterested in mitigating these challenges. Nathan said positive things about his instructors' effort, whereas Ava mentioned that one of the professors engaged students as minimally as possible in the online format. When asked to elaborate, she said, "it was just [my Course A professor's] screen [in lectures] so, I actually don't know [what] my professor looks like. Actually, even during office hours, it was his name, it wasn't his face." Even though this professor hosted office hours, as mandated by the university, he did not appear to demonstrate much effort into building connections or increasing his approachability with his students, according to Ava. COVID-19 challenges may have impeded further interaction with students.

Community. Though both students felt that class sessions had worse instruction compared to in-person, they expressed concerns about their social needs too. Ava stated, "I didn't think I would meet as many people...and...was really nervous definitely about the social aspect because friends...and making those kinds of connections are really important to me." Ava emphasized the importance of making connections and anticipated that developing friendships with her peers would be a challenge. Similarly, Nathan acknowledged that the online format posed difficulties for both students and professors within the class sessions. He shared, "It is also hard on teachers, I think, to create a sense of community...because the majority of students aren't going to have their cameras on and they're not going to ask questions verbally. They'll just put it on the chat." Given the difficulties posed by online instruction, class activities were not a significant contributor to students' sense of community.

Student-Generated Community

Emergence. In-class structures did not foster the same level of community that may have naturally emerged from sharing the same physical space, so students actively invested time and effort outside of designated class time to engage with the course content. In particular, students used social media to create a learning community that would not have existed otherwise. Both Nathan and Ava noted Discord – an online platform with text and video chat, often used by gamers – as an integral piece to their experience. Discord was first used in Course B class, when a student created a channel for the class. Nathan explained, they "were kind of able to bond, and so [they] created some other channels." This online community emerged out of the desire for community and to have a more accessible space to engage with their coursework with other students. The students even invited their Course B instructor, Delilah, to join the channel, Nathan shared that this created a "simulated classroom environment in a way." In other words, the Discord channels provided aspects of an in-person classroom that the actual class sessions did not. Students were aware of the primary role they had in creating community in this online setting. Nathan said, "my classmates, I think, have done a pretty good job of creating a sense of community, even if the teachers aren't necessarily facilitating that – …it's been kind of more student-driven just because of the Discord server."

Impact. The transfer students perceived the presence of an online student community as beneficial. Both students spoke at length about its academic and social benefits; Ava explained it connected a "network of people…who were all going through the same thing and [struggling] to find the answers.". Students' mutual struggle served as a catalyst for meaningful interactions, and, as a result, Nathan considered his classmates "genuine friends, which is cool because...that is hard to do strictly online." Ava expressed even more pronounced sentiments, describing someone she met through Course B as being "her best friend in the whole world," who she stayed in contact with through Discord despite no longer having a class together. Moreover, the use of Discord led to transfer students' inclusion into larger online communities associated with the university. Ava shared that, after being exposed to Discord and integrating into the Course B created community, she was now in a larger student-run server that has existed for a few years. She described it as: "nothing official but has like a couple thousand or a couple hundreds of people in there and that's just everybody, just any student [at the university]." Both students felt that a student-generated online community in which mutual-participating was required had a positive impact in their remote, academic transition to their receiving four-year institution.

Discussion & Conclusions

During COVID-19, schools have transitioned to remote learning, often leaving students challenged to find their alternative means for making connections with other students. Using a qualitative analysis approach allowed us to examine how transfer students in these mathematics courses developed an online knowledge-sharing community. It is crucial that educators focus on this population of transfer students, as nearly 84% of those who utilize the community college pathway are underrepresented minoritized students and 50% of U.S. of transfer students intend on pursuing a STEM degree after community college (Starobin & Lanaan, 2005; Zhang 2019).

Moreover, examining the transfer mathematics student experience is of importance because there is currently very little research on supports that may assist them in their academic and social transition to a four-year university. Yoon and Rolland (2012) provided a theoretical lens to examine the relationship between relatedness and knowledge-sharing. The exacerbated social need imposed by COVID-19 prompted students to adapt and build an online community that fostered relatedness amongst the students and was conducive of knowledge-sharing interactions. Transfer students enrolled in the cohort-style courses supported each other by creating servers on Discord for themselves. On Discord, these students participated in a virtual community, where they shared important mathematics concepts pertinent to their coursework. We believe that these knowledge-sharing interactions contributed to the perceived competence of all members of the community. Future research can attend to the ways that knowledge-sharing activities influence perceived competence and relatedness.

Acknowledgements

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NURTURING STUDENTS' UNDERSTANDING OF AREA THROUGH VARIATION: A CASE STUDY

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The theory of variation (Marton & Booth, 1997) has been mostly used as a task/instructional design principle in classroom settings. In other words, studies tend to look at what is made possible to discern rather than how the discernment may happen in the interaction with the students (Kullberg et al., 2017). The purpose of this paper is to shed a light on how a 7th grade student's understanding of area progressed when prompts guided by the theory of variation were provided within a one-on-one interview setting. The concept development framework (Zhang & Manouchehri, 2016) was used to identify the student's understanding for the concept of area. The results suggested that the student's understanding significantly shifted through a series of prompts.

Keywords: Learning Theory, Measurement, Middle School Education

Various methods have been suggested by the mathematics education community to develop children's understanding of area, such as using multiple representations and real-world situations, connecting between the formulas for calculating area and the visual relationships (Barrett, Clements, & Sarama, 2017), and scaffolding along the concept developmental stages (Battista, 2012; Zhang & Manouchehri, 2016). This paper will focus on achieving such goal through the theory of variation (Marton & Booth, 1997) and unpack Jazzy's case where she worked on an area problem.

The Theory of Variation

Marton's theory of variation (Marton & Booth, 1997) posits that learning is the process of developing an awareness of critical aspects and features of a target knowledge. Such a process can benefit from a change in the presentation, or a variation, of the critical aspects and features, which could promote effective discernments towards these aspects and features. Marton and Pang (2006) identified four patterns of variation, including *contrast* (looking at different values of the same aspect), *generalization* (looking at different appearance of the same value), *separation* (changing the values of some aspects while keeping the other aspects the same), and *fusion* (looking at all the critical aspects at the same time). These four patterns provide a general idea on how to construct activities to promote learning of the targeted knowledge.

The theory of variation has been mostly used as a task/instructional design principle in classroom settings. In other words, studies tend to look at *what* is made possible to discern rather than *how* the discernment may happen in the interaction with the students (Kullberget al., 2017). Hence, the purpose of this paper is to shed a light on how a student's understanding of area progressed when prompts guided by the theory of variation were provided within a one-on-one interview setting.

Framework for the Concept of Area

The concept development framework used to identify the student's understanding for the concept of area in this paper was developed by Zhang and Manouchehri (2016). According to

Zhang and Manouchehri, the framework breaks the development into three phases: heap, complex, and concept. In the *heap* phase, the learner associates an object with another because of physical context instead of any mathematical property of the objects. In the *complex* phase, objects are related in one's mind not only by their impressions, but also by concrete relationships between them. In the *concept* phase, the relationships between objects are abstract and logical. Each phase further contains various stages; see the full framework in Zhang and Manouchehri (2016).

The concept formation framework is composed of three major components: non-measurement reasoning, unit area, and formula. This paper mainly reveals the shift of the formula component under the prompts guided by the theory of variation.

Methods

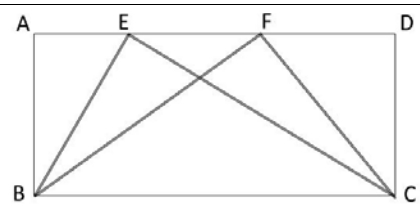
Participant and Data Collection

Jazzy was enrolled in an extracurricular program and routinely participated in one-on-one interviews with the authors from grade 6 to 8. In these interviews, she was asked to solve various mathematical problems while giving detailed explanation of her thinking process. There was no time limit on each problem and each interview was video recorded and there was no time limit on each problem. She had access to markers, poster paper, and a calculator during the interviews. In general, the interview protocol restricted the interviewers from giving any explicit instructions and only allowed them to ask clarification questions. Jazzy was in 7th grade when she completed the interview episode in this paper.

Contexts

Jazzy was asked to solve the problem below. This was one of the problems where she concluded that she couldn't solve within a short period of time.

Table 1. Compare Triangle problem

	Consider the graph on the left: Which triangle has a bigger area, triangles BEC or triangle BFC?

Jazzy spent a total of 2.5 minutes solving the problem. She first asked for numbers before attempting to solve it, then stated that triangle ABE and triangle CDF (part of the outer area of each target triangle) looked very different so she couldn't tell. As a follow up of her initial request, the interviewer provided her two hypothetical numbers, where BC was 10 units long and AB was 5 units long, and asked her whether she could solve the problem with the given information. Jazzy explained that she needed to further figure out the lengths of AE and DF, hence there was not enough information to answer the question. At this point, the interviewer acknowledged her conclusion and started the second part of the interview, which last 8 minutes. Jazzy attempted to solve the problem in terms of both non-measurement (comparing the outer areas) and measurement (finding the lengths of sides) reasonings before the interviewer stepped in, which revealed an understanding at a lower Complex stage for the concept of area, i.e., comparing parts randomly and (attempt at) using incorrect formula.

Results

Since little of what Jazzy knew was revealed in the first part of the interview, the interviewer started by asking her how to find the area of a triangle. Jazzy almost immediately wrote down the formula, $\frac{1}{2}bh$. The interviewer continued to ask what the formula meant for her and asked her to draw a picture to show how the formula worked (visually). Jazzy drew a right triangle and labeled the two legs as height and base (Figure 2.1) and explained that you multiplied them together and then multiplied by a half. The interviewer asked Jazzy why we multiplied them together and multiplied it by one half. She immediately drew a triangle to form a rectangle (Figure 2.2) and explained that the area of a rectangle was height multiplied base and the triangle was only half of the rectangle, hence you multiplied it by one half.

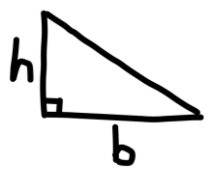


Figure 2.1: Jazzy's Illustration of the Triangle Formula

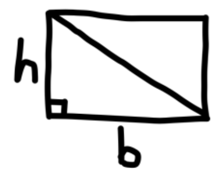


Figure 2.2: Jazzy's Explanation of the Triangle Formula

The interviewer followed up her response by asking whether the formula was only true for right triangles. Jazzy hesitantly said no. The interviewer asked her to draw a scalene triangle and show the formula still works. This time she drew an isosceles triangle; she acknowledged that it was not scalene but the interviewer encouraged her to continue with it. She identified the height of the triangle and marked the height and the base correctly, then shaded the left half of the triangle, drew it on the top-right side to complete the rectangle, and explained that "this side can be flipped here to make a rectangle" (Figure 3.1).

Since the second example Jazzy chose was plausible for her argument, the interviewer drew another example (Figure 3.2) and asked whether this case would be true according to her reasoning. Jazzy hesitantly said no, "but the formula is supposed to work, I don't know why."

Figure 3.1: Triangle Formula for a Non-right Triangle

Figure 3.2: Example Provided by the Interviewer

At this point the interviewer acknowledged her responses and redirected her to the original question (Table 1) by asking whether we could use the formula here. Jazzy thought for a few seconds and said "oh, I see." She drew the heights of both triangles and marked them as 5, then stated that the base for both was 10, hence they had the same area.

The interviewer continued asking whether they would still have the same area if the given numbers were different, such as 30 units and 40 units instead of 5 units and 10 units. Jazzy proceeded to conclude that the areas would be the same because the base would be 40 and the heights would be both 30. The interviewer finally asked what if we didn't have the measurements. She promptly answered that it wouldn't matter because the base and height would always be the same between the two triangles.

The interviewer moved on to provide a follow-up problem, where the two triangles were imbedded in a trapezoid instead of a rectangle (Figure 4.1). Being asked to compare the areas between triangle ABC and triangle BCD, Jazzy answered that they would be the same after thinking for 4 seconds. She further drew the heights of the two triangles and explained that they were the same since the base was also the same.

At last, the interviewer drew a different trapezoid (Figure 4.2) with two triangles inside and asked the same question. Jazzy was able to quickly conclude it would be the same because of the same base and the same height, which marks the end of the interview.

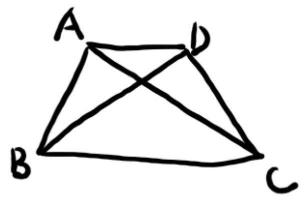


Figure 4.1: A Varied Problem Provided by the Interviewer

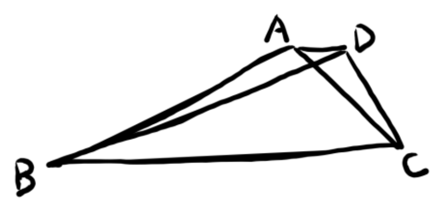


Figure 4.2: A Different Trapezoid Provided by the Interviewer

Discussion and Conclusion

Jazzy's initial understanding of the area formula was at a lower Complex stage, i.e. only used under specific occasions, which was revealed by her explanations on a right triangle and an isosceles triangle. The lower stage of understanding is possibly why she knew the formula but couldn't use it to solve the problem even with the given measurements.

The first example provided by the interviewer (Figure 3.2) was a generalized version of the two triangles in the original problem (*generalization*). Although Jazzy didn't have enough time to figure out how to adjust her reasoning to fit the new situation, she was able to make the connection between the generalized triangle to the two triangles in the problem. When the interviewer provided different measurements or even omitted the measurements (*contrast*), Jazzy focused on the fact that the base and the height were the same. Such focus was extended to the last two examples provided by the interviewer (Figure 4.1 and Figure 4.2), where the rectangle was replaced by a trapezoid (*separation*). This is considered as an understanding at the highest Concept stage for the concept of area. A potential *fusion* question could be to ask her to look at the three forms of reasoning, non-measurement, measurement, and formula, and make sense of her final answer with the two initial (failed) approaches.

In conclusion, Jazzy's stage of understanding for the concept of area significantly shifted through a series of prompts guided by the theory of variation. When the interviewer provided various measurements, she forced Jazzy to contrast her current understanding of the problem. In doing so, Jazzy was able to notice the target aspect (same base and same height) and retain the observation when one condition was changed (triangles inside of a trapezoid instead of a rectangle). For future study, it would be beneficial to test whether the improved understanding can transit under a different context in a problem-solving setting.

However, it is worth noticing that although Jazzy didn't figure out how the formula works for a scalene triangle, it did not restrain her from using the formula in a generalized way. It is plausible to argue that understanding how the formula (visually) works is not necessary to solve this problem, hence the interviewer did not focus on that part. Problems requiring such understanding to solve can be used to study those discernments.

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CALCULUS INSTRUCTION AND FEMALE SENSE OF BELONGING

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Women continue to be underrepresented in undergraduate science, technology, engineering, and mathematics (STEM) majors. Low sense of belonging and poor instruction in introductory STEM courses, especially Calculus, have been identified as key contributors to women leaving STEM. Incorporating active learning has potential to support female students in developing a stronger sense of belonging and persisting in STEM. This study investigates the instructional experiences and sense of belonging of female students enrolled in two versions of introductory Calculus – a standard course and a non-standard course infused with active learning. Females in the active learning course reported significantly higher sense of belonging and greater impact of instruction on their sense of belonging than female students in the standard course.

Keywords: Affect, Emotion, Beliefs and Attitudes; Calculus; Gender; Undergraduate Education

Problem, Perspective, and Purpose

Women continue to be underrepresented in undergraduate science, technology, engineering, and mathematics (STEM) majors (e.g., Chen et al., 2013; PCAST, 2012; Seymour & Hunter, 2019). Fewer females than males enter STEM majors, and more females than males switch out of STEM majors (Pryor et al., 2009; Eagan et al., 2013). In fact, women are 1.5 times more likely to leave the STEM pipeline compared to men (Ellis et al., 2016).

Sense of belonging can play a crucial role in females' decisions to persist in STEM (Shapiro & Sax, 2011). Sense of belonging is a "sense of being accepted, valued, included, and encouraged by others (teachers and peers) in the academic classroom setting and of feeling oneself to be an important part of the life and activity of the class" (Goodenow, 1993, p. 25). Research indicates that female students typically report a lower sense of belonging than male students in STEM courses (Shapiro & Sax, 2011; Rainey et al., 2018).

Rainey et al. (2018) conducted an interview study with 201 college seniors who were categorized as either STEM majors or STEM leavers about their sense of belonging in STEM. Students, especially females, cited interpersonal relationships as a major contributor to their feelings of belonging in STEM. Here, an interpersonal relationship means "feeling socially connected or similar to those around them in their STEM major" (p. 7).

Calculus is a critical course for STEM majors and a key juncture at which students decide whether to persist in STEM (Ellis et al., 2014; Rasmussen et al., 2013; Seymour & Hunter, 2019). Students have reported poor instruction in their foundational STEM courses as a major reason for leaving STEM -- citing Calculus in particular (Rasmussen et al., 2013; Seymour & Hunter, 2019). Seymour and colleagues (Seymour & Hewitt, 1997; Seymour & Hunter, 2019) report that students prefer teaching that is engaging and interactive and are less tolerant of poor teaching (i.e., instruction that is boring and lacks engagement) now than they were 20 years ago.

One approach that offers promise in addressing students' concerns about poor instruction and developing their sense of belonging is active learning. Active learning opportunities engage students in "the process of learning through activities and/or discussion in class, as opposed to passively listening to an expert" (Bonwell & Eison, 1991, p. iii). Using group work, class discussions, and student response systems (e.g., clickers) are some ways in which instructors can provide active learning opportunities. Instruction that includes active learning offers opportunities for students to interact with each other and their instructor, thus creating potential for building interpersonal relationships and thus strengthening their sense of belonging. Moreover, research indicates that students who engage in active learning have higher levels of achievement and persistence than students without these opportunities (Freeman et al., 2014; Lahdenperä et al., 2017; Rasmussen et al., 2019) and that women learn better in an active and collaborative learning environment (Shapiro & Sax, 2011; Kogan & Laursen, 2014).

This study compares female students' sense of belonging in two versions of introductory Calculus at the same university, and investigates how instruction might impact their sense of belonging. I address the following research questions: (1) How does female students' sense of belonging in an active learning Calculus course compare to female students' sense of belonging in a standard Calculus course? and (2) How do female students in an active learning Calculus course and a standard Calculus course characterize the impact of the instruction on their sense of belonging?

Methods

Setting and Participants

This study was conducted at a mid-sized R1 research university in the mid-Atlantic region of the U.S. during the Fall 2020 semester. The university offers two pathways for Calculus I – a standard one-semester course and a two-semester Integrated Calculus course designed to incorporate active learning. Both courses are taken primarily by freshmen, are coordinated, and use the same textbook (Stewart, Clegg, & Watson, 2021).

The standard Calculus course is typically taught in a large auditorium with lecture as the primary means of instruction. Students enrolled in this course typically intend to major in STEM. The course is coordinated in the sense that there is a common textbook and common exams. In the Fall 2020 semester, a mix of permanent and temporary faculty taught 8 sections of the course, each capped at 100 students. Due to the COVID-19 pandemic, the course was taught in a synchronous virtual format over Zoom. For the remainder of this paper, this course will be referred to as C-S.

The active learning Calculus course is typically taught in a classroom in which students sit at round tables and work together on math problems during class. Students are typically freshmen who need Calculus for their intended major but have not yet mastered all of the pre-calculus prerequisites. The first semester of the course, which is the focus of this study, develops differential Calculus, weaving in necessary pre-calculus topics as they arise. The second semester (which is not the focus of this study) develops integral Calculus, again weaving in necessary pre-calculus topics. The course is highly coordinated. In addition to common textbooks and exams, instructors teach from common lesson plans specifying which problems students work on each class, and whether they will be discussed as a whole class or in small groups. In the Fall 2020 semester, the university offered 2 sections taught by permanent faculty, each capped at 50 students. However, due to the COVID-19 pandemic, this course was also taught synchronously via Zoom. To maintain opportunities for group work, instructors used Zoom's breakout room functionality. For the remainder of this paper, this course will be referred to as C-A.

Participants were students enrolled in one of two sections of C-S taught by the same instructor and students enrolled in either section of C-A. These students received an email inviting them to participate by completing a survey about their experience in the course. In C-S, 77% (N=158) of 198 students and in C-A, 88% (N=80) of 91 students completed the survey. For this study, only students who were freshmen and self-identified as female were considered, resulting in a final sample size of 44 C-S students and 37 C-A students.

Data Collection and Analysis

The survey was distributed and completed electronically using Qualtrics, a web-based survey tool, during the eleventh week of the fourteen-week semester. To collect information about students' sense of belonging in the course, Good et al.'s (2012) Mathematical Sense of Belonging (MSoB) instrument was incorporated into the survey. The MSoB portion consists of 30 Likert items asking students to indicate the extent to which they agree with statements about their feelings of belonging in the course on a scale of 1 (Strongly Disagree) to 8 (Strongly Agree). To collect information about how students experienced the instruction, students were asked to indicate how often they experienced each of eight instructional strategies, and to identify the three most frequently used strategies. The strategies were chosen based on typical active learning instructional practices (e.g., group work), as well as strategies known to be used in at least one of C-S or C-A (e.g., lecture). To investigate how students characterized the impact of the instruction on their sense of belonging, students were asked to rate the type of impact (positive, neutral, negative) each instructional strategy had on their ability to form interpersonal relationships, a key component of sense of belonging.

Mean responses were calculated for sense of belonging and tested using IBM SPSS Statistics (Version 27) predictive analytics software. Independent samples t-tests were used to determine differences in mean responses between C-S and C-A students with $p<0.05$. To capture students' perceptions of the instruction they experienced, frequencies were computed to determine the percentage of students who reported experiencing each strategy. These proportions were tested with a two-sample proportion z-test with $p<0.05$ to determine significant differences between strategies used in the two courses. Effect size was calculated using Cohen's (1988) benchmarks for d. Finally, to investigate the relationships between the types of instructional strategies and students' abilities to form relationships, frequencies were computed to determine the percentage of students who reported each type of impact.

Results

Students' responses to the MSoB portion of the survey were used to measure their sense of belonging. Results showed a statistically significant difference ($t=3.581$, $df[79]$, $p=0.001$) with females in C-A (Mean=6.95, StDev=1.07) reporting a higher average sense of belonging than females in C-S (Mean=5.97, StDev=1.34). The effect size is large (Cohen, 1988, $d=0.80$), suggesting a substantive difference between the two groups.

To understand how students perceived their Calculus instruction, they were presented with eight instructional strategies and asked to identify the three most frequently used. In each course, only two strategies were selected by more than 55% of students. The percentage of students who identified each of these strategies as most frequently used is presented in Table 1. C-S students were significantly more likely to report working on problems individually and lecture; C-A students were significantly more likely to report group work and their instructor asking for their responses. These data indicate students perceived very different forms of instruction in the two courses, with C-A students reporting more opportunities for active learning.

Students were asked to indicate the type of impact (negative, neutral, positive) that each instructional strategy had on their ability to form relationships with their classmates and with their instructor. Table 2 presents the percentage of students indicating each type of impact for the two most frequently reported instructional strategies in each course. In terms of their ability to form relationships with classmates, less than 40% of C-S students reported that the most frequent instructional strategies had a positive impact. The majority of C-S students reported a mostly neutral impact (57% for individual work and 55% for lecture). In contrast, the vast majority of C-A students reported a positive impact (98% for group work and 78% for soliciting student responses). In terms of their ability to form relationships with the instructor, C-S students reported the most frequent instructional strategies had a mix of positive and neutral impacts. In contrast, students in C-A reported a mostly positive impact. Thus, a higher percentage of C-A students reported a positive impact of the most frequent instructional strategies on their ability to form interpersonal relationships, a key component of sense of belonging.

Table 1: Percent of Female Students in C-S and C-A Selecting Each Instructional Strategy as One of the Three Most Frequently Used.

	C-S	C-A	*z*	Cohen's *d*
Students work on math in groups	20%	97%	-6.94***	2.72
Students work on math individually	93%	8%	7.65***	2.78
Instructor asks for student responses	16%	81%	-5.85***	1.72
Instructor lectures	84%	41%	4.03***	1.13

Asterisks denote the p-values (for p<0.05, ** for p<0.01, and *** for p<0.001 significance levels).*

Table 2: Percent of Female Students in C-S and C-A Indicating What Type of Impact Each Instructional Strategy Had on Their Ability to Form Relationships.

		Relationships with Classmates			Relationships with Instructor		
	Impact	Positive	Neutral	Negative	Positive	Neutral	Negative
C-S	Individual Work	37%	57%	2%	48%	45%	5%
	Lecture	34%	55%	12%	66%	32%	2%
C-A	Group Work	98%	3%	0%	73%	27%	0%
	Student Reponses	78%	22%	0%	94%	3%	3%

Note: Percentages may not add to 100%, as students were able to select "This does not happen in my class" as a response to this question.

Conclusions

Female students in the active learning Calculus course (C-A) reported a significantly higher sense of belonging, more opportunities for active learning during class instruction, and more positive impacts of instruction on their ability to form relationships (a key contributor to sense of belonging) than their female counterparts in the standard Calculus course (C-S). While this study's design does not allow for causal claims, these findings suggest a link between opportunities for active learning and female students' sense of belonging in Calculus. Perhaps increasing opportunities for active learning in introductory STEM courses like Calculus will help retain more female students in STEM. Further research should investigate these potential relationships, including the kinds and frequency of active learning opportunities needed to make an impact.

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DEVELOPMENTAL MATHEMATICS STUDENTS' PERCEPTIONS OF (RE)LEARNING OUTCOMES: THE VALUE OF ALGEBRA ALL OVER AGAIN

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We report findings from an exploratory study on developmental mathematics students' perceived experience re-learning content they had already studied in middle- or high-school. Our findings suggest that these experiences are largely shaped by students' expected and perceived learning outcomes associated with that content. We describe how six of those learning outcomes depend on students' confidence in their previous understanding of the content to be re-learned, and how those learning outcomes influence students' modes of engagement during (and emotional reactions to) their re-learning experience.

Keywords: Affect, Emotion, Beliefs, and Attitudes; Algebra and Algebraic Thinking; Developmental Education; Undergraduate Education

Developmental (or remedial) education courses are commonly offered at U.S. colleges and universities for students that are deemed underprepared for "college-level" work in mathematics, reading, or writing. Traditional developmental courses are often nestled within a sequence, meaning that a student needs to pass multiple courses before enrolling in a credit-bearing course of the same subject. The Conference Board of the Mathematical Sciences Survey (CBMSS) found that in Fall 2015 approximately 41% of all two-year college and 11% of all four-year college and university mathematics and statistics enrollment was in developmental courses. Despite this sizable enrollment, only an estimated 50% of students beginning at public two-year institutions and 58% of students beginning at public 4-year institutions pass or earn some credit for all the developmental mathematics courses (DMCs) they attempt to take (Chen, 2016). Reasons for these failure rates have been proposed at various levels, but most research has focused on entry and exit problems with the developmental course sequence such as placement and attrition (e.g. Bahr, 2008a; Bahr 2012a; Bailey, Jeong & Cho, 2010). Grubb (2001) referred to this approach as contributing to the "black box" problem of teaching and learning in DMC (Grubb, 2001). These studies give us a broad sense of the paths students take to credit-bearing courses, but leave the reasons for rates of attrition, passing, and graduation obscured. Accordingly, he called for more research that characterizes student experiences in developmental mathematics programs and identifies aspects that appear to be critical to the formation of end-of-course outcomes over time. This exploratory study began to fill this gap in the literature by describing student experiences with content through the theoretical lens of *relearning,* or the experience of trying to learn about something one has already tried to learn about before. Relearning is an interdisciplinary phenomenon that has been studied more narrowly in the fields of cognitive psychology and preservice teacher education. This study utilizes an expanded conceptualization that cuts across these fields in order to characterize student experiences with a critical and highly critiqued component of developmental mathematics courses: repeated content.

Theoretical Framework: Relearning

Student experiences in DMCs involve significant amounts of time learning about content students have seen before, either from a previous K-12 mathematics course or from a previously-attempted DMC (Ngo & Velasquez, 2020; Fay, 2020). This similarity to content seen before is a point of *concern* for developmental math educators. Stigler, Givvin and Thompson (2010) summarize student experiences in DMCs as being, "presented the same material in the same way yet again" (p. 4). Likewise, Ngo (2020) described them as "high school math all over again," after finding that approximately 2/5 of all students in DMCs had demonstrated some degree of proficiency with course content in high school. Despite this distinctive and recurring feature, there are almost no studies that focus on student perceptions of their experience learning about content seen before, or "relearning". Thus far, usage of the term "relearning" for the purposes of research has been unnecessarily restrictive. For over 100 years, relearning has been used in studies of memory in cognitive psychology to describe the experience of committing to memory what one has already memorized (Ebbinghaus, 1885; Nelson, 1985; Bahrick, 1979). Recently the concept of 'successive relearning' has been applied to college classrooms to maximize retention of course material (Dunlosky & Rawson, 2015; Janes et al., 2020; Rawson, Dunlosky & Janes, 2020). This usage focuses on content that is considered to be learned when it is memorized and is capable of being demonstrated in one experimental session. Likewise, Zazkis (2011) proposed the term 'relearning' to better describe the experiences of preservice teachers in their mathematics content courses, in which pre-service teachers "revisit and reconstruct" what was previously learned (p. 12). According to this definition, students are only relearning if they are expanding the "domain of applicability" of their content understanding, or correcting "prior misleading learning" (Zazkis & Rouleau, 2018).

We suggest that these prior conceptualizations are instances of the same underlying phenomenon which also exists in DMCs. At the most basic level, relearning requires three things: some (mathematical) content, a "time 1" (T1) representing a past occurrence when an individual tried to learn about that content, and a "time 2" (T2) representing a most recent time when that individual tried to learn about that same content again. While the content at T1 and T2 need not be identical, it does need to cross a threshold of similarity such that the learning goals with respect to that specific content at T2 are essentially the same as those at T1. Although the name re-*learn*ing suggests some degree of mastery of the content at T1, we make no such assumption. The goal of this exploratory work was to utilize the lens of relearning to describe student experiences with content seen before. Using a conceptualization that cuts across these fields will help us better understand both context-specific features of DMC that contribute to its unique and troubling outcomes, as well as suggest avenues for future research among other fields that share this phenomenon.

Methods

This was a multiple-case study in which three Intermediate Algebra students participated in one-hour, semi-structured interviews before and after learning about a topic they indicated they had seen before in a previous algebra class (Equations of a Line and Polynomials). Simon and Zarah were students of Instructor A and had never previously taken an algebra course in college. At the time of data collection (in Spring 2020), Simon was an 18-year-old, first-year student, while Zarah was a 19-year-old, second-year student. Valeria (a 20-year-old, third-year student at the time of the study) was in Instructor B's class and had previously taken three algebra courses in college. Both Instructor A and B were recommended by the head of the department for their quality of instruction, and as many participants as possible were recruited through an online survey. All course meetings in which these topics were covered were observed and recorded. In interview 1, students were asked to describe their past history learning about the topic, confidence in their current understanding, and to predict what the experience of learning about the topic again would be like. In interview 2, after the topic had been taught, students were asked to describe what it was actually like to learn about the topic again, including how predictions aligned with what actually occurred. Discussions were anchored in problems gathered from field notes or student work as often as possible. A discussion of instructor interviews and student follow-up interviews is beyond the scope of this report, so the results reported here focus student interviews from the Spring 2020 semester. Interviews were transcribed and analyzed using a modified thematic analysis. While data collection was guided by specific theoretical propositions, the novelty of the research called for a more exploratory approach. Coding proceeded in five stages beginning with in vivo coding (Creswell, 2007) and progressing to a modified thematic analysis (Braun & Clarke, 2012) as familiarity with the dataset increased.

Results

Student descriptions of their experience with relearning, both expected and perceived, were centered around multiple types of learning outcomes. This was *not* a grade or an indication of passing/failing. In asking what was learned in a relearning scenario, we mean to answer the question: what was the value of students' learning experience in terms of their understanding of content *this time around*? The answer to this question requires one to reference, in some way, the content that was learned before which a feature unique to relearning scenarios. Zazkis' notion of reconstruction is an example of such a learning outcome in the field of teacher education. Unlike Zazkis, however, we found that students described more than one learning outcome both when talking across sub-topics and comparing expected and perceived outcomes across interviews (interview 1/I1 and interview 2/I2 below). Students described six types of learning outcomes:

- Gain a Deeper Understanding: Seeing previously-learned material in a new way and using this shift in perspective to gain a "deeper" understanding than they did before. (Zarah, I2)
- Confirm my Understanding: Feeling they already know the content and seeing relearning it as verifying their understanding is adequate. (Simon, I2; Zarah, I1, I2; Valeria, I2)
- Jog my Memory: Experiencing relearning a given content as simply refreshing their memory of material they feel they understood well at some point, but have since forgotten. After the class is completed, they do not feel like they require further practice. (Simon, I1, I2; Zarah, I1, I2; Valeria, I1, I2)
- Reconstruct Memory with Guidance: Reconstructing their memory of previously-learned material with the guidance of an instructor to avoid remembering things "incorrectly". After the class is completed, they require further practice. (Zarah, I2)
- Fix Past Mistakes: Addressing what students identify as inadequate understandings of course material gained from past experiences. (Valeria, I2)
- Accept What I Don't Understand: While perceiving to hold an inadequate understanding of course material gained from past experiences, students see themselves as being unable to address this understanding and either accept or avoid it. (Valeria, I1; Simon, I2)

While the codes are not ordered hierarchically, they were each associated with a degree of confidence in the student's perception of their current understanding of the re-learned material. 'Reconstruct my Memory with Guidance', for example, was associated with a lower degree of confidence, as students felt that they understood the material enough to partially rely on memory, but not well enough to do so without significant assistance from an instructor. Crucially, discussions of their expected and perceived learning outcomes were associated with particular methods of engagement with re-learned material and affective dispositions while re-learning. For example, in Interview 2, Valeria described different anticipated learning outcomes for sub-topics within the polynomials unit. She sometimes found herself frustrated with the experience of going over content that she already understood (confirm my understanding) to the point where she saw little value in paying attention during class.

> Valeria: Polynomials is just like, a thing to me like I honestly could care less. So when I know, I know. I'm like, I'm kind of getting lazy with this, cause this is like the third time of taking this class…Like if, like when she starts doing examples and I do them on my own and get the right answers, like maybe she'll do five and I get the first three right then I'm like I don't really need to pay attention any more. So then I'll just either like daze off or like go on my phone to be honest.

However, for other sub-topics that she found more difficult, Valeria's experience was dominated by avoidance behaviors and anxiety gained from her past history with algebra. When discussing factoring techniques she remarked, "If [a polynomial] has a number in front of the x^2 one then I'll usually just sit there and just stare at it, but if it doesn't then I can usually just do it...I think it just scares me or something. It just like scares me." Valeria went on to suggest that her difficulties with this type of factoring could only be resolved if she had learned about them in a different way the first time they were introduced (Accept what I don't understand). As her case illustrates, due to their previous experience with algebra, developmental mathematics students were frequently making judgments about what they expect to get out of their re-learning experience often based only on perceived similarity of content (to what they've seen before) and their confidence in understanding the material at that time.

Discussion

Given that some learning outcomes may be more or less desirable than others, the results provided here will be useful to DMC instructors seeking to understand the potential impacts of their pedagogy on student understanding, and how this may vary significantly from other courses in which students are learning new mathematics for the first time. The fact that student perceptions of their learning outcomes shifted not only by topic within an interview, but across interviews as they gained more information casts doubt on literature that attempts to classify students as one particular type or as having one type of learning need. Although descriptors such as "brush up" experiences were noted, they were not useful as static descriptors of student experiences as has been done in previous literature on DMC (Grubb & Gabriner, 2013; Cox & Dougherty, 2019). Instead, students described multiple types of expected learning outcomes depending on their confidence in their previous understanding of material to be re-learned. Crucially, these expectations informed student methods of engagement with and attitudes towards course material in ways that may be unseen by instructors or placement instruments working with views of students as being of monolithic "types". Placement measures may need to be sophisticated enough to provide heavy advising and the ability to re-place if the shift between expected and perceived learning outcomes the student experiences is actually harmful for their future learning. Finally, the interdisciplinary nature of the conceptualization of relearning used here means that the findings of this study have implications across the field of teacher education as well. It remains an open question the realm of potential learning outcomes possible across relearning contexts and the degree to which theories of conceptual change that have been employed in teacher education could be useful in improving the pedagogy of DMC as well.

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EMBODIED TRANSMISSION OF IDEAS: MATHEMATICAL THINKING THROUGH COLLABORATIVE CONSTRUCTION OF GEOMETRY VIDEO GAME CONTENT

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In one classroom of a Title 1 high school, students (n=12) were separated into groups and participated in a three-part study in which played and designed content for a 3D motion-capture video game, The Hidden Village (THV). This paper provides case-studies from group's work provided evidence of students' intuitions, insights, and explanations (including their gestures) of how students conceptualized the geometric transformations and how students embodied their ideas about geometry and how those ideas "traveled" (via directed actions) within and between student groups.

Keywords: Embodiment, Geometry, Collaborative Construction, Transfer

Introduction

The Hidden Village (THV) is a 3D motion-capture video game that allows players to embody mathematics learning (Swart et al., 2020). In the game, players perform movements (*directed actions*, see Nathan, 2017) prior to reading and evaluating a geometric conjecture. In effect, participants, nascent of the directed actions' relevance, are primed through body movements that are representative of both the structure of a geometric object(s) and enactive of the geometric transformation(s) of the object(s). A recent re-design of THV includes new modules that allow players to generate their own content (i.e., create new conjectures and new directed actions) to be played in the game. In the current study, students, working in groups, were invited to participate in a 3-day program to play THV, then collaboratively design their own series of directed actions for a given conjecture, and then play their conjecture and those designed by their peers. Researchers hypothesized that *directed actions* would foster mathematical insights crucial for students' insights and proofs. In this study, we present case study analyses in which students' verbal explanations, gesture production and the subsequent actions they designed to simulate geometric transformations can communicate concepts to their peers in a classroom setting.

Theoretical Background

The theory of *Gesture as Simulated Action* (GSA; Hostetter & Alibali, 2019), asserts that gestures, as spontaneous co-articulations with speech or thought, serve to activate perceptual-motor processes in the brain. In mathematics learning, Abrahamson & Sánchez-García (2016) effectively demonstrated how relative positioning of one's hands helped participants better understand concepts of ratio and proportion. According to Nathan's (2017) theory of *Action-*

Cognition Transduction (ACT), sensorimotor experiences feedforward and feedback to a predictive cognitive architecture that inductively and deductively reasons about the behaviors of ideomotor systems. ACT is a part of a larger framework called *Grounded and Embodied Cognition* (GEC; Nathan & Walkington, 2017), which proposes that body movement complements learners' verbal expressions (often seen in spontaneous gesture) by grounding understanding via the physical relationships that are the origins of mathematical thinking.

THV leverages the actions, gestures, and other body-based resources in a physically interactive social settings like collaborative game play to create opportunities for players to embody their ideas. When other groups perform the directed actions designed by their peers, we can observe the embodied transfer of ideas across groups via movement (Alibali & Nathan, *in press*). In collaborative settings, we hypothesize that these socially supported actions provide a physical medium by which their ideas "travel" between players. Moreover, we hypothesize these actions become a type of physical vocabulary that students invoke in subsequent explanations as *dynamic depictive gestures* as they mentally and physically simulate transformations of mathematical objects through multiple states (Garcia & Infante, 2012). We explore these hypotheses in multiple cases to demonstrate how embodied mathematical ideas travel within and between student groups through the creation of directed actions for game play.

Methods

Materials

The Hidden Village Game Module. THV delivers an embodied geometry curriculum in which a 3D motion capture sensor detects players enacting an in-game avatar's movements and records players' reasoning about geometry conjectures (i.e., ever false or always true). Figure 1 shows the 5 main parts to each level of game play.

Figure 1: One level of THV gameplay.

The Hidden Village Conjecture Editor Module. Students add new conjectures and design what they consider to be mathematically relevant directed actions by manipulating the sequences of poses of the avatar (Figure 2). Student groups use the Pose Editor to generate 2-3 poses (starting, intermediate, and target pose) to collaboratively co-create directed actions for each conjecture and players can preview the movements as animations. User-generated content is stored in an online database and accessible for others to play in the game module.

Figure 2: The THV Conjecture (authoring conjectures) and Pose Editors (for creating directed actions) and an example of a directed action sequence (far right).

Participants and Procedure

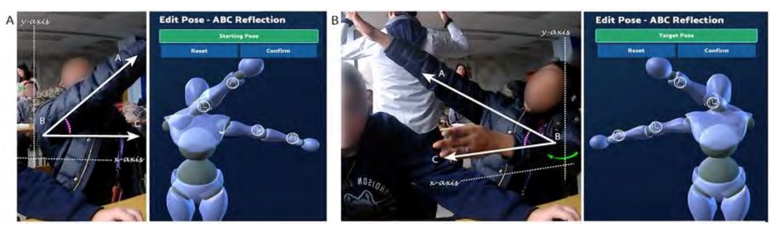
In this study, 12 students in a Title I high school in the midwestern United States were randomly assigned to groups of three or four to participate in a three-day embodied mathematics curriculum focused on geometric thinking. This paper focuses on three of the student groups.

On Day 1, group members took turns playing six conjectures in THV game module. On Day 2, student groups collaboratively constructed their own directed actions for a newly-assigned conjecture. On Day 3, student groups took turns playing the new THV curriculum (8 conjectures; 3 repeated from day 1, 3 newly designed by student groups, and 2 transfer conjectures.

Results

Within-Group Analysis

Upon noticing that the student's initial discussion of the conjecture produced a sequence of directed actions that produced the desired outcome of the geometric transformation,



Transcript 3**:** (*N.B. S1 indicates Student #1;* brackets [...] indicate gestures.)

[1] S1: Oh, wait. This is not the starting pose. Is that the starting pose? [Uses arms to make ∠ABC on the left
[2] side of the body] We are going like, this is the angle [shifted arms directly to the right side of
[3] her body by performing a reflection across the body vertical axis]... Boom! That's the angle!

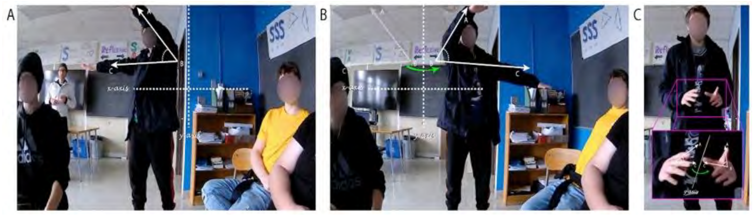
Figure 5: For *ABC Reflection* conjecture, Student 1 in Group 1 embodies the starting pose physically (also shown as designed in THV Pose Editor, panel A) and S1 performs the entire directed action, finishing on the target pose.

This transcript indicates the starting and target poses (see Figure 5) the student group used for the *ABC Reflection* conjecture. S1 embodied the idea of "using your body as the midline" through this directed action as they narrated their actions, shifting the angle from the right side over the body to the left side, which solidified their understanding of a reflective transformation.

Between-Groups Analysis

Students in Group 4 played the *ABC Reflection* conjecture as designed by Group 1. One player per group performed the directed actions while the other group members observed.

Researchers analyzed students' speech and gestures to track how Group 1's ideas traveled to peers (RQ2).



Transcript 4: (*N.B. S3 indicates Student #3;* brackets [...] indicate gestures.)

[1] S3: False. Because it can be proportionally the same, have the same angles [using hands to make an
[2] angle] while being in different locations. [S3 then, selects the correct multiple-choice answer]

Figure 6: S3 3 in Group 4 performs the directed actions (Panels A & B) for *ABC Reflection*. In Panel C, S3 provides intuition and rationale, gesturing the reflection.

In the process of proving the conjecture, S3's spontaneous gesture enacted an embodied conceptualization of the ∠ ABC that results from the transformation. In effect, this gesture is a truncated version of the authoring group's directed actions and complements S3's rationale.

Discussion

These case studies show the promises of an embodied, collaborative mathematics curriculum by demonstrating instances of how mathematical ideas "travel" through embodied actions. Students created content that explored embodied ways of reasoning those ideas were shared when their peers performed those directed actions during game play. This study provides preliminary evidence that directed actions can serve as a malleable factor that scaffolds cognition through its connections to the body, leaving historical traces that learners can feel, reinforcing their mathematical reasoning and complement their verbal explanations as conceptualization occurs. Embodied cognition offers promising ways to foster the transfer of mathematical ideas through students' collaboratively constructed movement.

Acknowledgments

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"LET'S TALK ABOUT ELECTION 2020": QUANTITATIVE REASONING AND YOUTH CIVIC COMPOSING IN AN ONLINE NETWORK

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Keywords: Communication, Social Justice, Diversity, Equity and Inclusion, Sustainability

In the months surrounding the most polarizing election in modern American history, youth produced and posted over 1,200 digital media clips on a chosen issue of concern through the educational web-based platform *Let's Talk About Election 2020.* Amongst other pressing topics, students created media related to COVID-19, Black Lives Matter, and climate change.
Given that quantitative literacy has been posed as both a civil rights priority of this millennium (Moses & Cobb, 2001) and fundamental to a contemporary democracy (Ball, Goffney & Bass, 2005; Steen, 1997), the question of how youth engage in quantitative reasoning in their civic lives has received scant attention. The body of curricular resources that examines social injustices through the lens of mathematics is growing but its use remains uncommon in math classrooms (e.g., Gutstein, 2006; Berry III, Conway IV, Lawler & Staley, 2020). Still, youth engage with quantitative data nearly every moment of every day. This study contributes insight into the ways that young people use quantitative reasoning as they engage in myriad forms of civic activity beyond the mathematics classroom. The study asked, in digital media created for *Let's Talk About Election 2020* around Black Lives Matter, COVID-19, and climate change, *what forms of quantitative reasoning did youth use in their civic argumentation, and how were these shaped by topic?*

To answer this question, transcripts from digital media segments were coded using an *a priori* set of codes defining three forms of quantitative reasoning – reasoning around a) quantity, b) relationships and change, and c) uncertainty (de Lange, 2006). Emergent coding was used to examine patterns in the use of these forms of reasoning across the three topics.

Findings show that all three forms of quantitative reasoning were present across topics. Widely evidenced mathematical practices included a) reasoning around relations of change over time, b) conceptions of scale, magnitude and normalcy, and c) set building activity surrounding notions of "we" in evoking empathy and compelling community action. Furthermore, across media segments, youth composers consistently mixed quantification and humanization. In doing so, segments cycled between mathematical practices of pattern recognition, abstraction and generalization that can be valuable for social problem solving, and narrative forms that invoked the particularities of diverse lived experiences and perspectives that can otherwise be lost in abstraction. Whereas single stories and mathematical models can each mislead in isolation, pairing them lent sophistication and texture to these media segments, suggesting the importance of quantitative civic reasoning for both mathematics learning and civic life.

Insight from this work can support mathematics educators working to bring together youth civic learning and mathematics. Specifically, we spotlight forms of quantitative civic reasoning that youth already employ outside the math classroom that can provide curricular inroads to bridge mathematics classrooms and students' civic lives.

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STUDENTS' REASONING IN A MODELING TASK: MODELING DANCE WITH MATHEMATICS

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Keywords: Modeling, Mathematical Representations, Geometry and Spatial Reasoning

There is a growing emphasis on the inclusion of mathematical modeling tasks in school mathematics where students are engaged in analytical thinking, reasoning, critical thinking, and problem-solving skills (National Council of Teachers of Mathematics, 2000). Modeling tasks are helpful in revealing student thinking and they enable students of differing performance levels to interpret, invent, and find solutions (e.g., Batista, 2017; Carmona & Greenstein, 2007; Koellner-Clark & Lesh, 2003; Mousoulides, Pittalis, Christou, & Sriraman, 2010). Mathematical modeling helps students connect mathematics to the real-world. Furthermore, modeling tasks can support student interest in the learning of mathematics, make mathematics relevant and meaningful to students, provide opportunities to improve students' problem-solving skills, and improve student achievement in mathematics (Asempapa & Foley, 2018). In our modeling task, students analyze various dance moves that have different geometric concepts embedded in each move (e.g., angles, lines, shapes, symmetries) and then generalize their findings to develop a model that could be used for future dance analysis. Students are engaged in mathematical modeling throughout this task because they are using mathematics to represent, analyze, make predictions or otherwise provide insight into the dance world.

We implemented the *Modeling Dance with Mathematics* task with two different groups of students. The first group consisted of 40 university students who are prospective teachers. The implementation with the first group was one 50-minute virtual session that took place in October 2020. The second group consisted of six high school students enrolled in an honors geometry course. The implementation with the second group was two 45-minute virtual sessions that took place in January 2021. Students analyzed five specific dance positions to better understand geometry as represented in human motion. They then created models using their knowledge of geometric concepts to support their analysis of additional dance moves. The goal of the task was to help students deepen their understanding of geometric shapes and describe the relationship between different figures and shapes. The activity sequence for this task consisted of three parts: (1) recording notices and wonders for three dance moves, (2) analyzing the five dance moves provided in picture form, and (3) developing an analysis guide (model) for future dance moves.

The common themes in students' analysis across both implementations included: angles formed in the dance moves and shapes formed in the dance moves. These themes also occurred as the most common themes in students' models that were created to be used as analysis guide for future dance moves. When completing the task, students made mathematical assumptions of angles, lines, and shapes that were formed by the body parts of the dancer and analyzed different dance moves. The results indicated that all of the students were able to create models to analyze dance moves and notice the mathematics of a real-life situation by applying their current knowledge of geometric concepts. The results of this study could help teachers in planning and implementing similar modeling tasks, especially when anticipating patterns of student solutions and preparing questions to assess and advance student thinking.

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THE INFLUENCE OF A VALUES AFFIRMATION INTERVENTION ON STUDENTS' MATHEMATICAL, SOCIAL, AND EPISTEMOLOGICAL EMPOWERMENT

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Keywords: Affect, Emotion, Beliefs, and Attitudes; Culturally Relevant Pedagogy; Equity, Inclusion, and Diversity; High School Education

Purpose of the Study

The purpose of this study is to better understand the varying impacts of educators' attention to identity construction on the mathematical, social, and epistemological empowerment of tenth grade Emergent Bilinguals and native English-speaking students. To attend to this purpose, I will address the following research questions.

1. When controlling for language identity, to what degree does a values affirmation intervention predict students' mathematical, social, and epistemological empowerment?
2. Following a values affirmation intervention, how do the perceptions of students with differing language identities relate to their mathematical, social, and epistemological empowerment?
3. Following a values affirmation intervention, how do students' mathematical, social, and epistemological empowerment profiles differ between outcome measures and personal survey responses?

Proposed Conceptual Relationships

In the context of the mathematics classroom, students' intersecting mathematical, cultural, racial, gender, and academic identities develop through critical reflection on personal strengths, values, and social positions. Critical consciousness development is the mechanism through which identity construction expands to motivation, action, and empowerment. This conceptual framework, displayed in Figure 1, details a flow of influence from identity construction to critical consciousness development to empowerment.

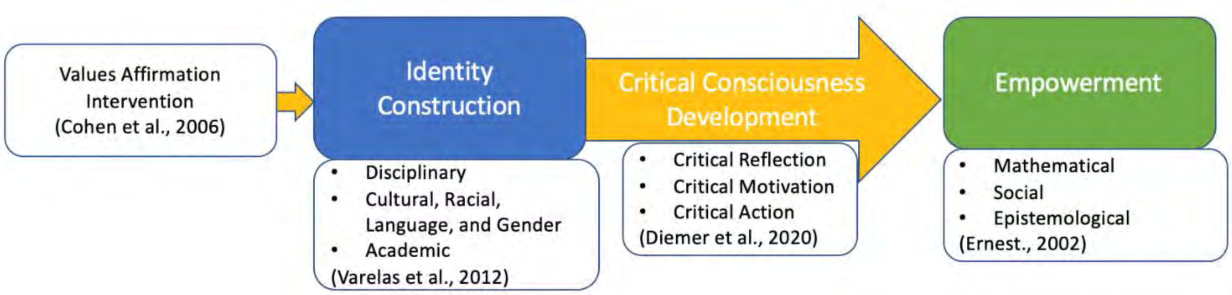

Figure 1: The Critical Identity Construction and Empowerment Framework (Crit-IC-E)

This framework informs the mixed methods transformative study design, which incorporates a values affirmation intervention intended to bolster positive student identity construction and measures to assess critical consciousness and empowerment. I will utilize quantitative, qualitative, and integrated methods to analyze students' empowerment. Approximately 150 tenth-grade Emergent Bilingual and native English-speaking students will be recruited from a western high school to participate in the proposed study.

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PERSIST AND THRIVE!: A REVIEW OF LITERATURE ABOUT BLACK GIRLS IN MATHEMATICS

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The academic performance of Black girls is often absent or scarcely represented in scholarly literature, thus creating a false or incomplete impression of their ability to achieve in mathematics (Young et al., 2018). This deficit narrative about the academic capabilities of black students, especially black girls, invades classrooms, colleges and universities. Deficit narratives could influence how Black girls position themselves in relation to mathematics, placing them at risk of disidentification (Brand et al., 2016; Ellis et al., 2016; Pringle et al., 2012). The purpose of this literature review is to determine the factors that mar the mathematical experiences of Black girls as well as the strategies and supports dedicated to supporting their persistence.

This literature review begins with an examination of the dominant views of Black learning that positions Black students as deficient and inferior to their White and Asian counterparts (e.g., Boaler, 2002; Gholson & Martin, 2019). While many ascribe these views to all black students, there are some researchers who have been able to identify perspectives that vary across gendered lines (e.g., Beekman & Ober, 2015; Marra et al., 2009). I address the complexity of the achievement (or rather, opportunity) gap, highlighting the research of scholars who look beyond the existence of a gap to the sources and contributing factors (e.g., Davis & Martin, 2018; Ellington et al., 2010; Malloy & Jones, 1998,). Next, I present literature on the dehumanization of Black girls in academic spaces (e.g., Joseph et al., 2019; Chambers et al., 2016; Young et al., 2018). The humanity of Black girls is yet to be realized in the United States as Black women and girls continue to be positioned as lacking in intelligence (Joseph et al., 2019), as disruptive in classrooms (Chambers et al., 2016), and as possessing a limited range of emotions (Gholson & Martin, 2019). Turning attention specifically to the mathematics classroom, I illuminate the various ways in which mathematics' learning spaces have been shown to be unwelcoming to Black girls (e.g., Felton-Koestler, 2015; Sleeter, 1997; Darragh, 2014). Some believe that mathematics is sterile, objective and neutral, yet a growing body of work shows that neutrality in mathematics is a myth as cultural values and expectations are expressed in the teaching thereof (Nortvedt & Buchholtz, 2018; Gutiérrez, 2013). Black girls have had to enact coping strategies to stave off feelings of isolation and exclusion (Alexander & Hermann, 2016; Gholson & Martin, 2019).

Key findings from this review of the literature include that Black girls are collaborative learners and should be afforded opportunities to work as a collective unit (e.g., Gresalfi &Hand, 2019; Kang et al., 2018), teachers play an important part in how Black girls experience mathematics instruction (Meaney & Evans, 2012; Pringle et al., 2012), and that positive mathematics identities could improved learning experiences (e.g., Fellus, 2019; Grayven & Heyd-Metzuyanim, 2019; Tao & Gloria, 2018). Although there is a growing body of asset-based studies, more research is needed that attends to high-achieving Black girls and women who are thriving in mathematics, highlighting the attributes, characteristics, and structural or institutional supports that made this possible, as well as the role that mathematics educators play.

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DO VISUAL FRACTION MODELS HELP STUDENTS COMPARE FRACTIONS? EVIDENCE FROM ELEMENTARY STUDENTS' WRITTEN WORK

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Both learning and teaching fractions can be challenging. To build students' conceptual understanding of fractions, visual models (e.g., circle model, fraction bar) are broadly promoted (Van de Walle et al., 2019) because they are theorized to communicate students' mental models (Cramer & Whitney, 2010) and make abstract mathematical concepts more accessible (McKendree et al., 2002). The phrase "visual model" or "visual fraction model" is used 19 times in the U.S. Common Core State Standards for Mathematics (CCSSM), indicating the broad consensus regarding the importance of visual models for meaningful mathematical learning. However, while the effectiveness of visual models for fractions has been well documented in small-sample studies in contexts with high involvement from university mathematics educators (e.g., Empson & Levi, 2011; Petit et al., 2015), to date, little is known about how students use visual models to justify the sizes of fractions at scale. Even less has been published after the wide adoption of the CCSSM. Accordingly, we aimed to answer RQ1: To what extent do 4th graders choose visual models to explain fraction comparison? RQ2: How effectively are such models used? RQ3: What constraints are encountered?

We designed an open-ended task that provided students with choices to explain their solutions: *Write the fractions 2/3, 3/4, and 3/8 in order from smallest to largest. Use pictures, words, or symbols to explain.* This task was administered to 214 fourth graders from seven schools in five counties of a U.S. Midwestern state. We coded whether students' justification included a visual model and categorized the models by type (e.g., fraction bar, set model, number line, etc.). For each type, we calculated the corresponding accuracy rate (correct here means order the fractions in a correct order) to determine how well it supported fraction reasoning. We analyzed incorrect responses to identify error patterns.

We found that among the 214 students, 93 (43%) used visual justification, and their accuracy rate (AR) was 28%. This rate was the same as that of those who used non-visual justification, so we found no correlation between accuracy and the choice of visual justifications for fraction comparison. However, sixteen of the 48 students (AR 33%) who used a rectangle or fraction bar model and five of the ten students (AR 50%) who employed a number line only produced the correct fraction order, suggesting that the rectangle and number line models may support students' fraction comparison more effectively than other visual models. Analysis of students' incorrect responses revealed that many students experienced constraints (1) with equal partitioning (e.g., by partitioning the circle horizontally instead of radially); (2) with referring to the same whole (e.g., by using non-congruent shapes modeling fractional area); and (3) with sector area in circle models (e.g., by drawing radial shares so inaccurately the resulting conclusion was incorrect).

Overall, we found that although visual fraction models are supposed to support students' fractional reasoning and are emphasized in the CCSSM, less than half of fourth graders in this study elect to use such models and their accuracy is not higher than students who do not. We conjure that students' units coordination (Wilkins et al., 2020) and teachers' fraction instruction

(Watanabe, 2002) may impact students' use of fraction visual models. We call for more research on fraction visual models in the CCSSM era. (c.f., Lee & Lee, 2020).

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SIXTH-GRADE STUDENTS' PERCEPTIONS OF MATHEMATICS DISCUSSIONS USING GROUP ROLES AND GROUPWORTHY TASKS

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Keywords: Classroom Discourse; Middle School Education; Equity, Inclusion, & Diversity; Problem Solving

Classroom environments that incorporate the use of groupwork as part of a broader, reform-oriented approach to mathematics teaching and learning have resulted in learning gains, improved relationships among students, and positive mathematics identities (Boaler, 2006, 2008; Boaler & Staples, 2008; Staples, 2008). However, the students who most often have opportunities to participate in groupwork—in particular—and classroom discourse more broadly are those whose voices align with the dominant discourse of math classrooms (Lubienski, 2002; Nasir & Vakil, 2017; Zevenbergen, 2000). To invite more students into doing mathematics requires better alignment of the practices of mathematics to a wider range of students.

On this poster we will share initial data and analysis of African American students' perceptions of groupwork in a 6th-grade classroom where a white teacher implemented Complex Instruction (CI), a pedagogical approach designed to minimize status differences and establish more equitable classrooms (Cohen & Lotan, 1997; Lotan, 2003). Over four weeks the teacher used "groupworthy" tasks related ratio reasoning (Lappan et al., 2014) as a context to introduce students to the use of group roles to scaffold students' interactions in small groups. After the CI intervention, we interviewed 12 students to learn how their perceptions of productive and equitable group processes aligned with the interactions we observed among groups. We created hypothetical sample dialogues based on common occurrences we observed in within-group interactions and asked students to react to the dialogues.

Our poster will share our analysis of the student interviews alongside excerpts of group discussions that we used as a basis for our interview protocol. This project prioritizes the experiences of students, and the types of interactions that they perceive as equitable and collaborative, to determine how students learn collaboration and learn mathematics through collaboration. This work can shift power for determining what constitutes mathematical activity towards students of color, better aligning the discipline with those who have been historically marginalized.

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EXPLORING STUDENTS' STATISTICAL THINKING DURING AN ENTREPRENEURIAL DESIGN CHALLENGE

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Keywords: Data Analysis and Statistics, Instructional Activities and Practices, Middle School Education, Integrated STEM/STEAM

The Design & Pitch Challenges in STEM (D&P; Confrey et al., 2019) is a novel curricular framework that situates mathematics learning within entrepreneurial pitch competitions. Leveraging features of project-based learning (PBL), design-based learning (DBL), and entrepreneurial-based learning (EBL), the framework was built to generate excitement and interest math by engaging students in authentic, math-focused entrepreneurial design challenges. In this poster, we report preliminary results from a study of students' experiences participating in the Keep It Real challenge, one of the nine D&P Challenges in STEM. In this challenge, students were tasked with inventing an app that uses data representations to help people manage their "phubbing" behavior, the act of ignoring (or snubbing) someone while using your phone ("phubbing" = "phone" + "snubbing"). Using a case study methodology (Yin, 1994), we explored the following research question: *How and to what extent does the Keep it Real challenge create opportunities for students to engage with the process of building and inventing statistical representations?* We collected data from a variety of sources, including daily written work samples and video data. We then analyzed these data using a grounded theory (Glasser & Strauss, 1967) approach.

During the competition, students drew on their interests and experiences to invent apps that would help users monitor their phubbing and incentivize improvement. As students considered how to incorporate data representations in their app designs, they defined statistical questions (e.g., how much is a user phubbing? Is a user's phubbing improving over time?); identified and considered how to quantify relevant data (e.g., phubbing as the number of minutes someone uses their phone after hearing the user's name or the number of times they access their phone during designated social events); and built prototypes of data representations using hypothetical data (e.g., line graphs to show changes in number of phubs by day, or bar graphs to show minutes spent phubbing by app or by day/week). Overall, the design challenge created a purpose for students' statistical reasoning and provided a lens through which they could make informed decisions about what questions their data representations should answer, what data were relevant to answering those questions, and how to best represent those data to effectively inform users' behaviors. Despite opening the space for students to invent novel data representations, students exclusively relied on traditional middle grades representational forms, such as circle graphs, line graphs, and bar graphs. Given the ages of the participating students and based on our conversations with them during the study, it is likely that students drew on the statistical representations with which they were familiar. This lack of variation or novelty in students' representations could limit the potential of the challenge for introducing and teaching new forms of data representation, though the findings indicate viability for motivating mathematics learning through entrepreneurial pitch competitions. More work is needed to better understand how to support students' engagement with a broader range of data representations without limiting their freedom to innovate within the entrepreneurial environment of the D&P Challenges in STEM.

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PARTICIPANT REASONING IN A MULTIVARIATE SOCIAL JUSTICE CONTEXT RELATED TO PARTICIPANT IDENTITY AND LIVED EXPERIENCES

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Keywords: Data Analysis and Statistics; Social Justice; Affect, Emotion, Beliefs, and Attitudes

Over the past twenty years, Teaching Mathematics for Social Justice (TMSJ) has emerged as a well-documented phenomenon of practice in mathematics education. Thinking about the knowledge and experience of the participant becomes a crucial element of planning a productive experience in the TMSJ context, with one important consideration being the role of the lived experiences in how participants work with social justice tasks. Findings from the literature of statistics education can be helpful in understanding how lived experiences can influence student interaction in a social justice context. Investigations by Wroughton et al. (2013) and Queiroz et al. (2017) have demonstrated that statistical content and student experience cannot be separated. Another critical linkage that has emerged is the dependence of TMSJ on data (Frankenstein, 2012; Skovsmose, 2012). The use of data sets in TMSJ can be fixed, meaning that activities direct attention in a single direction to focus student attention. However, an interesting finding from early childhood data literacy offered that questions about data can limit the ability of students to process information (Schwartz & Whitin, 2006). This limitation may also be happening in TMSJ tasks as well. One teaching strategy from statistics education that could remove the stress of a fixed point in a social justice lesson would be to facilitate a social justice lesson guided as an investigation. MacGillivray and Pereira-Mendoza (2011) indicated that when students design authentic investigations, there are significant implications for both the written and verbal communication of statistical ideas. These statistical ideas are interesting when combined with ideas like lived experiences in social justice tasks, which becomes the focus of this research brief. The question that guides the current investigation is: ***In what ways do student identities and lived experiences influence the interpretation of an open statistical task in a TMSJ context?***

To investigate this question, a task-based interview (Goldin, 1997) was designed to examine the ways in which students interacted with a multivariate social justice data set. Initial design work showed that participant background was influential in the interpretation of the task. To continue and refine the research task, participants from a mid-sized public university were used. The task was posed using CODAP (https://codap.concord.org/) to explore how participants interacted with a dataset refined from the GSS survey (https://gssdataexplorer.norc.org/variables/vfilter) concerning the prompt: *Use the data provided to find a noticeable difference or demonstrate there is no noticeable difference between groups represented in the data*. The interviews were initially analyzed based on self-identification data provided by participants to identify relevant incidents from each interview. These relevant incidents were combined to generate an aggregate set of data that was analyzed for themes. This poster highlights the preliminary themes from this aggregate set. These themes offer insight into how students use experience and identity as an entry-point for data-based tasks in social justice contexts.

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THE DEVELOPMENT AND VALIDATION OF MATHEMATICS FLOW STATE SCALE

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Introduction

Flow is an optimal psychological state first described by Csikszentmihalyi (1975, 1990, 2000). According to Csikszentmihalyi, when in flow, a person is completely engaged in activity and experiences various psychological characteristics, such as out of self-consciousness and enjoyment of the progress. Csikszentmihalyi (1991) explained that there are nine factors that consist of flow experience: challenge-skill balance, concentration, clear goals, feedback, sense of control, action-awareness merging, loss of self-consciousness, time distortion and autotelic experience. Studies found that the flow experience had a significant influence on efficiency and performance while engaging in different activities (Joo et al., 2011; Kiili & Ketamo, 2017). There are two goals for this research. First, we develop the Mathematics Flow State Scale (MFSS) to measure students' flow experiences in mathematics classrooms, adopted from the original Flow State Scale (FSS), and, second, we examine the validity and reliability of MFSS.

Methods

Two samples were collected for this study. For the first sample, we collected data from 374 fourth grade students. The second sample includes 503 fourth, fifth, and sixth grade students. MFSS was developed from the original Flow State Scale (Jackson & Marsh, 1996) to assess the degree of flow state students experience in mathematics classroom. To refine the original 36 items, we conducted Exploratory Factor Analysis (EFA) using the first sample. After the modification with the result of EFA, we ran Confirmatory Factor Analysis (CFA) to test the nine factors describe elementary school students' flow experience when learning mathematics using AMOS 20 (Arbuckle, 2011).

Results

After applying the EFA 12 times, 10 items were deleted for comprehensive reasons. All remaining 26 items had acceptable factor loading (0.4 to 0.85). To examine the factor structure of MFSS, we conducted the CFA three times with the one-factor model, the nine-factor model, and the eight-factor model to compare and select the best fitting model. Results showed that the eight-factor model with 24 items was the most suitable form of MFSS.

Discussion

The goal of this study was to develop and validate the psychometric properties of the MFSS. Results showed that the psychometric properties including reliability and validity of MFSS were acceptable, which suggests that the MFSS can be used in mathematics classroom to test elementary school students' positive experience (i.e., flow) when learning mathematics. Measuring students' positive affect would contribute valuable information to mathematics education research and practice as it can advance current knowledge about students' emotions and motivation in learning mathematics in a positive perspective.

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IMPACTS OF A COMPUTER GAME-BASED EARLY ALGEBRA INTERVENTION

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Keywords: Algebra and Algebraic Thinking; Technology; Elementary School Education

The field of *early algebra* (Stephens et al., 2017) has established itself as a promising developmental pathway to support students' later learning of formal algebra. Early algebra is not pre-algebra or algebra earlier, it is not a natural extension of number (Blanton et al., 2015), and there is evidence that it can begin alongside development of number, including prior to it (Venenciano et al., 2020). Blanton and colleagues (2015) describe 5 big ideas of early algebra: "(a) equivalence, expressions, equations, and inequalities; (b) generalized arithmetic; (c) functional thinking; (d) variable; and (e) proportional reasoning" (p. 43). Moreover, there have been important advances in the use of games to engage students in inquiry-oriented learning contexts with success (e.g., Trujillo et al., 2016). This study investigates the impact of a game-based early algebra intervention on 4th and 5th grade students' early algebra knowledge.

Participants included 457 fourth and fifth grade students across 10 schools and 28 teachers' classrooms in a mid-sized school district in the Southwestern US. Around 64% of the students self-reported as Hispanic/Latinx, 18% as White, 13% as Multi-racial and the rest as Asian, Black/African American or Native American. About 55% reported as Female and 45% Male. Data is from a pre/post-assessments of students' early algebra knowledge (Engledowl, 2020), pre/post-surveys of mathematics anxiety, total gameplay time, grade level, gender, and race.

The intervention focused on two key ideas of early algebra that overlap with Blanton et al.'s (2015) big ideas: write and interpret expressions, and express patterns and relationships between quantities. Teachers implemented 3 inquiry-oriented lessons that each involved three phases: 1) gameplay, 2) a supplemental activity, and 3) another gameplay session. Each lesson was associated with one of two games and an interactive tool. Data was analyzed using a 2-level hierarchical linear model, with students (Level 1) nested within teachers (Level 2), using the R package *lmerTest* (Kuznetsova et al., 2017). Assessment scores were converted to latent person-ability logit scores using Rasch analysis (Bond & Fox, 2015) and the eRm package in R (Mair & Hatzinger, 2007). Students' mathematics anxiety was measured as a composite score with the same items as OECD (2014), and demographics were dichotomous.

Preliminary findings show that the null model's intraclass correlation (ICC) indicated about 35.2% of the variance in post-test scores could be explained by classroom membership—teachers played a major role in students' knowledge. Full model analysis revealed that all independent variables except for *Gender* and *Time* were statistically significant (all $p < .10$). Although we found expected results, such as no differences on gender, and that grade level, pre-test, and anxiety (e.g., Ashcraft & Krause, 2007) are significant predictors, a shocking result was that total time students played *Agrinautica* was not a significant predictor. Analysis is ongoing to further explore contextual factors at both student and teacher levels.

Acknowledgements

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PANDEMIC INSTRUCTION: A TALE OF SPATIAL COLLAPSE AND PLATIAL COLLISIONS

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Keywords: Online and Distance Education, Undergraduate Education

> I heard a thing about: "Chew mint gum when you study, and then chew the same gum, when you take the test. And all those things will come back." … It's like that but in reverse for me. …When I'm sitting [at home] at my desk with my computer, I'm in lay-back-have-fun-play-a-video-game mode, not sit-forward-and-take-notes-for-a-college-math-class. … As opposed to pre-COVID: My dorm is a working space or, and so is the cafeteria. I've got spots and layouts that, like the only thing I ever knew in my dorm was working. So that was naturally a workspace. (Kaleb, Interview, April 17, 2020)

The ideas Kaleb, a mathematics undergraduate student, shared with us in an interview after instruction had shifted online in response to the coronavirus pandemic brought to our attention how space affects students' learning experiences. Upon directing our attention to matters of space, we noticed that other undergraduates majoring and minoring in mathematics whom we also interviewed after the shift to online instruction—as part of a longitudinal study of students' development of agency and autonomy—were telling similar stories.

To make greater sense of the stories that were shared with us, we draw on human geography to distinguish between *space* and *place*, that is, to distinguish between the purely physical and the physical married with human experience (Entrikin, 1991; Tuan, 1977). Although geographers are not in agreement as to the exact relation between space and place, we draw on humanistic geographers to define place as a space–time configuration with associated human activities and experiences (Agnew, 2011; Sack, 1997). This distinction allows us to situate the pandemic as: (a) a collapse of *spaces*, that is, describe how an abundance of class spaces (e.g., classrooms, help rooms, the instructor's office) became a single space (i.e., home), and (b) a collision of home and class *places* to capture how two worlds collided—as exemplified by Kaleb's words.

Using the terminology of space and place, we will share spatial and "platial" aspects of three undergraduate students' experience taking mathematics classes during the pandemic. In particular, we will illustrate how these students (a) found on-campus spaces important, (b) suffered from problems of motivation and concentration by the distractions offered by being home, and (c) engineered their homes to make them more conducive to their learning.

On top of adumbrating the utility of a spatial/"platial" lens for understanding important aspects of students' experiences during the pandemic, we will also discuss implications for instruction. Specifically, by posing three example questions, we intend to show how attending to space and place can offer valuable insights for equitable instruction during the pandemic: (a) Who has what access to synchronous online class places? (b) What other places are students in while in online class during the pandemic? and (c) What places are students in while taking monitored exams?

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Chapter 12:

Teaching & Classroom Practice

USING TEACHER AND STUDENT NOTICING TO UNDERSTAND ENGAGEMENT DURING SECONDARY MATHEMATICS LESSONS

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The purpose of this paper is to present a framework that illustrates teachers' and students' ways of noticing mathematical engagement. This framework offers clarity about the complexity of engagement, and it includes three elements: evaluations of the presence of engagement, descriptions of the nature of engagement, and features of the classroom that support or constrain engagement. We interviewed 30 sets of high school math teachers and focus groups of their students and asked them to reflect on students' engagement during a videotaped lesson from their classrooms. Results illustrate cases of how noticing of engagement between teachers and students can be shared. Cases of partially and minimally shared noticing of engagement suggest opportunities for teachers to learn about students' perspectives or how to communicate with students about their intentions to engage them.

Keywords: mathematics engagement, teacher noticing, student noticing, motivation

Engaging secondary students in mathematics lessons is an enduring challenge, as students' mathematics engagement has been found to decrease as they move through years of schooling (e.g., Collie et al., 2019). Engagement is a complex construct, involving affective, behavioral, cognitive, and social factors (Fredricks et al., 2004; Middleton et al., 2017), each impacting student learning and performance differently. In a nationally representative sample of high school students in the United States, behavioral and cognitive engagement explained more of the variance in students' mathematics achievement scores than affective or emotional engagement (Sciarra & Seirup, 2008). This study highlights that the nature of engagement matters for students' learning and performance, even though it was not specific to mathematics learning.

Teaching practice shapes the learning environment that students experience (Anderson et al., 2004), and the learning environment impacts students' engagement (Shernoff et al., 2017). However, according to Pedler et al., (2020), teachers face challenges understanding how to engage students because engagement is such a complex phenomenon. According to Erickson, to teach effectively, "one needed to 'learn' the children one was trying to teach" (Erickson, 2011, p.18). So, perhaps one approach for teachers to learn more about students' engagement is for teachers to become "students of our students" (Ritchart & Church, 2020, p. 11).

In this study, we investigated what mathematics teachers noticed about their students' engagement and how their students exhibited similar or different noticings about their engagement. Building upon research on specialized noticing practices of mathematics teachers (Jacobs et al., 2010; van Es et al., 2017) and research on students' noticing of mathematics (Hohensee, 2016; Lobato et al., 2013), the purpose of this study is to investigate what teachers and their students noticed about mathematics engagement while viewing video recorded events from their classrooms. We offer a framework to demonstrate that noticing of mathematics engagement involves evaluating whether or not students were engaged (presence of engagement), describing the nature of students' engagement (dimensions of engagement), and interpreting what happened in the classroom to elicit students' engagement (features of engagement).

Teachers' and Students' Noticing

Noticing is a process of identifying events as noteworthy, using evidence to discuss these events, and providing interpretations of these events (van Es & Sherin, 2002). What a person notices influences their reasoning about the event (van Es & Sherin, 2002; Lobato et al., 2013). Across teacher noticing literature (e.g., van Es & Sherin, 2002; Jacob et al., 2010; van Es et al., 2017), researchers consider *interpreting* to be higher quality noticing compared to *evaluating* or merely *describing*.

Different targets for noticing suggest a need for alternative frameworks for illustrating ways of noticing. Researchers have studied what teachers notice about their students' mathematical thinking (Jacobs et al., 2010) or equity in the mathematics classroom (van Es et al., 2017), as well students' different centers of focus of noticing during mathematics lessons (Hohensee, 2016; Lobato et al., 2013). For this study, we investigate a different target of teachers' and students' noticing: mathematics engagement. Additionally, previous research studies examined teacher and student noticing separately, while we investigate them in relation to each other.

Mathematics Engagement

Academic engagement is a psychological investment in and effort directed towards learning from academic tasks (Jansen, 2020; Newmann et al., 1992). Engagement is a complex meta-construct (Fredricks et al., 2004) that includes affective, behavioral, and cognitive dimensions (Appleton & Lawrenz, 2011; Bobis et al. 2016; Fredricks et al., 2004; Helme & Clarke, 2001; Middleton et al., 2017), and a social dimension (Middleton et al., 2017; Jansen & Bartell, 2013; van Uden et al., 2013; Wang, et al., 2016). Behavioral engagement includes effort or time on task. Cognitive engagement is concentration or connections made while learning. Affective engagement is an emotional state of investment, such as interest. Social engagement is participation in the learning process. When describing student engagement in the classrooms, teachers and students could focus on different *dimensions of engagement* (affective, behavioral, cognitive, or social).

Recent prior research reveals teachers' thinking about what students will find engaging in mathematics classrooms and how these interpretations may guide their instruction (Bobis et al., 2016; van Uden, et al., 2013). We named interpretations of what will engage students as interpretations of *features* that engage students. Researchers have explored whether teachers determine engagement to be present or not (Skilling et al., 2016), or teachers' *evaluations* of the presence of engagement (or disengagement). In terms of describing engagement, when reflecting on engagement generally, teachers tended to focus on relatedness or sense of belonging in the classroom (Herman, et al., 2000; van Uden et al., 2013). However, when teachers were asked to focus on particular students or to consider a situated case of engagement, they tended to focus primarily on behavioral or overt emotional engagement (Bobis et al., 2016; Skilling et al., 2016).

Previous research on students' perspectives have uncovered the motivators that drive their engagement (Daniels & Arapostathis, 2005; Jansen & Bartell, 2013; Middleton, 1995). These motivators range from interest and reward (Daniels & Arapostasis, 2005; Middleton 1995) to interpersonal relationships in the classroom (Daniels & Arapostasis, 2005; Jansen & Bartell, 2013). Cognitive and social engagement appear to be prominent in students' perceptions of their own engagement.

Research Questions

To understand teachers' and students' noticing of mathematics engagement, we designed this study to answer the following research questions: *What do teachers and their students notice about elements of engagement (presence, dimensions, features) when asked to reflect on a shared mathematical experience? In what ways do they agree or disagree on these elements?*

Methods

Context and Participants

To address our research questions, we analyzed student and teacher interview data collected for the Secondary Mathematics in-the-Moment Longitudinal Engagement Study (SMiLES). SMiLES is a three-year mixed-methods study funded by the National Science Foundation that explored engagement in high school classrooms across two states (one state in the Southwestern region and one in the Mid-Atlantic region of the United States). Data analyzed for this study comes from 6 high schools (3 in each state) collected during the Fall 2018 to Spring 2019 academic year. Data for SMiLES were collected during Algebra 1 or Integrated Math 1 lessons.

Regarding our participants, we recruited teachers for this study by soliciting nominations of teachers from district curriculum supervisors and mathematics coaches. We invited nominated teachers to participate in the study, and 16 teachers participated (11 female, 5 male) in the SMiLES project during the Fall 2018 to Spring 2019 academic year. Teachers self-reported their races: 14 identified as white, one identified as Asian-American, and one identified as Hispanic/Latinx. They averaged 10.8 years of teaching experience with a range of 1 to 27 years. Student demographics for the schools in the Southwest were: 85-94% low income, 2-5% white, 1-15% Black, 74-96% Latinx, and 0-5% Asian, Native American, or Multi-Racial, and student demographics for the schools in the Mid-Atlantic were: 9-30% low income, 24-57% white, 27-46% Black, 7-24% Latinx, and 0-5% Asian, Native American, or Multi-Racial.

Students were selected to participate in focus group interviews from each class period that we observed. The criteria we used to select students for the focus groups was based on an analysis of students' responses to a mathematics engagement survey administered by the research team at the beginning of the semester. A cluster analysis of this data identified motivational profiles of students (Tarr et al., 2019). Three clusters were identified across the sample and the students invited to participate in the interviews: (1) strongly aligned with one of the profiles (2) had parent consent, and (3) had given assent. The average number of students who participated in a focus group was 2.45 with a range from 1 to 3 students.

Data Collection and Analysis

The dataset for this analysis consists of 30 sets of interviews with teachers and their students, with multiple class periods studied for most teachers. Interviews were conducted one-on one with teachers and in a focus group for the students. Prior to the interview, the research team identified a video clip that showed a representative example of student engagement from an observed lesson in the SMiLES dataset for that class. Each video clip was between 90 seconds and three minutes in length and had been experienced by the teacher and students in the focus group. Interviews took place two to three weeks after each observation. We conducted these interviews as video viewing sessions (c.f., Erickson, 2007), during which participants commented upon what they noticed in a video regarding the nature of engagement during that activity. These interviews were not treated as stimulated recall (e.g., Lyle, 2003), as we did not expect participants to be able to recall their experience after multiple weeks and we did not intend to capture participants' decision making in those moments.

We employed a case study approach (Yin, 2017) when analyzing these sets of interviews. We defined a case as a teacher's noticing and their students' noticing of a recorded classroom activity. Data for the case consisted of a set of interviews: the teacher interview and corresponding student focus group interview about the classroom activity. We analyzed these interview episodes to identify three elements of engagement reported by teachers and their students: the *presence of engagement, dimensions of engagement* and *features which elicit engagement or disengagement*. We defined *presence of engagement* as the indication by the teacher and students of whether students were engaged or disengaged. We defined *dimensions of engagement* as the type of engagement reflected in how a teacher or student described the nature of engagement, according to six categories of dimensions: affective, behavioral, cognitive, instrumental, social, and relatedness. We defined *features that elicit engagement or disengagement* to be teachers' and students' self-reports of what appeared to support (or constrain) students' engagement (or disengagement). Features are interpretations that the teachers and students provided about what happened in the lesson that engaged or disengaged students. These features were coded using an emergent process (Saldaña, 2013) from listening to voices of both the students and the teachers. We also analyzed for the *presence of engagement*; whether teachers and students determined that students were or were not engaged during the event in the video clip. We then identified ways in which sets of teachers and their students agreed or disagreed about what they noticed with respect to presence of engagement, dimensions of engagement, and features which elicit engagement or disengagement.

The elements of engagement in this study (presence, dimensions, and features) parallel the noticing stances described by van Es and Sherin (2002): *describe, evaluate, and interpret*. The participants' characterization of presence of engagement is a form of *evaluation* of whether or not students were engaged. When teachers or students talked about the ways in which students were or were not engaged, this aligns with *describing* engagement. (We coded participants' descriptions of engagement according to dimensions.) We considered *interpretations* of these video clips to be when teachers or students reasoned about the features that brought about students' engagement or disengagement.

During analysis, it became clear that there were cases in which teachers and their students noticed and agreed about engagement in various intersections of these elements. Our stance was that high quality noticing of engagement between teachers and their students occurred when they shared perspectives on engagement. *Strongly shared noticing* occurred when a teacher and their students agreed on all three elements (presence, dimensions, and features). *Partially shared noticing* occurred when a teacher and their students agreed on any two of those three elements. *Minimally shared noticing* was agreement between a teacher and their students on any one of those three elements. A *disagreement* on noticing any of the elements of engagement could provide an opportunity for teachers' learning about how to engage their students. Either a teacher could learn more about their students in order to engage them or the teacher could communicate rationales more explicitly so that students could learn more about their teachers' intentions for engaging them in particular activities.

Results

Through the process analyzing interview data, we examined ways that elements of engagement (presence, dimension, and feature) intersected and what these intersections revealed about how teachers and their students thought about engagement in secondary mathematics classrooms. To this end, we built a framework (see Fig. 1) that organizes our results and helps to describe the ways our participants noticed engagement in their math class.

This framework, organized as a Venn diagram, recognizes each element of engagement -- presence, dimension, and feature -- as a set which can intersect and interact with the other elements. What teachers and their students notice can then be described through these elements and whether and how they intersect. Each of the outer circles (no intersections) represents when a teacher and their students notice and agree only on one element. We describe this as "Minimally Shared Noticing of Engagement." If a teacher and their students similarly noticed two out of the three elements of engagement, then they would fall into one of the intersections of two elements: "Partially Shared Noticing of Engagement." Finally, the innermost intersection ("Strongly Shared Noticing of Engagement") indicates that a teacher and their students exhibited shared noticing on all three elements of engagement. The complement of this Venn diagram also exists and would include cases where a teacher and their students did not notice similarly or agree on any of the three elements of engagement. We present two cases here: "Strongly Shared Noticing of Engagement" and "Minimally Shared Noticing of Engagement – Presence."

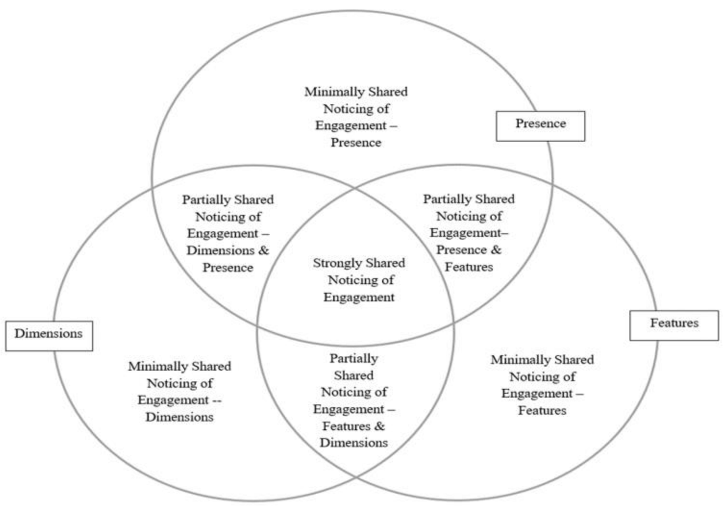

Figure 1: Framework for the Elements of Engagement

Case 1: Case of Strongly Shared Noticing of Engagement

Julie and her students represent a case of a teacher and her students expressing shared noticing of mathematics engagement according to all three elements in our framework: presence, dimensions, and features (see Table 1). In the activity captured on our video recording, we observed that Julie shifted out of a whole-group discussion and had students move into working in smaller groups during the mathematics lesson. When reflecting on the video, Julie and her students evaluated students' engagement similarly; they agreed that the students were engaged. They interpreted the opportunity to work in small groups as the feature which elicited this engagement and in describing this feature, they described engagement in terms of social engagement. Thus, we interpret this case as one of strongly shared noticing of engagement in a secondary mathematics classroom.

Julie attended to the social dimension of engagement (engagement through student interactions and discourse with and around mathematics) when she explained that she knew students were engaged because they were interacting with each other to make sense of the mathematics. She explained that by providing the opportunity for students to work in groups, she avoided a potential pitfall of whole-class discussion: the same students answering all the questions. She said, “That's why I try to throw it out to them, because they're not communicating with me. I don't want to know just what three people know -- I want to know what everybody knows.” She went on to explain that small group work allows her to engage in formative assessment by listening to conversations and to assess each student’s knowledge. She said, “That's why I kind of, like I said, threw it back to them, circulated so that I could hear and talk to each group.” In her reflection on the classroom video, Julie described students interacting with each other around the mathematics, and she interpreted the students’ interactions to mean they were engaged.

Table 1: Julie and her students, Case of Strongly Shared Noticing of Engagement

Elements of Engagement		Summary
Presence	Agreed: Engagement	This teacher and her students expressed shared noticings for all elements [presence, dimensions, and features]. They agreed students were engaged and have some agreement about what engages students and why.
Dimensions	Agreed: Social	
Features	Agreed: Students worked and talked together in groups about mathematics	

Julie’s students interpreted engagement in the video similarly to Julie. When reflecting on the video, the students said that the class was engaged, and explained that this was the case because of the interactions they were observing – indicating that they also connected social engagement with the presence of students being engaged. They also noticed instances of engagement similar to Julie’s. Katie noted, “I think that the whole class was into the activity just because of all the talking that was going on. We were all discussing what was going on the board and arguing over the correct answer, which is definitely our class.” Luna agreed with Katie, “[Student 1] was talking, but then [Student 2] started, and then people were there and over here started talking about what answer was right … yeah … that's when everybody was engaged.” Both Luna and Katie interpreted that engagement was evident through discussing and arguing over the answer indicating the social dimension of engagement.

This case is an example of a strongly shared noticing of engagement between an instructor and their students. Both Julie and her students interpreted the video as indicative of social engagement by focusing on the interactions that students had around the mathematics. This indicates that Julie and her students noticed the same elements of engagement: when students have the opportunity to work together on mathematics, the activity can be engaging. Julie’s decision to put students in groups suggested that she considered what her students needed to engage. It might be the case that when a teacher and her students have a shared perspective on what engages students (features) and why and how students engage (dimensions), students are more likely to be engaged.

Case 2: Case of Minimally Shared Noticing of Engagement – Presence

Jacob and his students represent a case of agreement on the presence of engagement with disagreement on the dimensions and features of engagement (See Table 2). When reflecting on the video clip, both Jacob and his students evaluated that students were not engaged; however, they associated this lack of engagement with different dimensions of engagement and features of the classroom. This video viewing session illustrates an opportunity for Jacob to improve his teaching practice, as both the teacher and the students agreed that students were disengaged.

Table 2: Jacob and his students, Case of Minimally Shared Noticing of Engagement – Presence

<table>
<tr><th colspan="2">Elements of Engagement</th><th>Summary</th></tr>
<tr><td>Presence</td><td>Agreed:
Disengagement</td><td rowspan="3">Although the teacher and students agreed that students were not engaged [presence], the teacher did not notice similarly to his students in terms of why and how students engaged [dimensions] or what engaged or disengaged students [features]</td></tr>
<tr><td>Dimensions</td><td>Disagreed:
Teacher – Cognitive
Students – Behavioral, Social</td></tr>
<tr><td>Features</td><td>Disagreed:
Teacher – Pressing for explanations
Students – Whiteboards</td></tr>
</table>

During the video recording, we observed that students first solved problems by writing on their desks with dry erase markers while Jacob and the classroom aide walked around to answer questions. Then Jacob brought the class together to discuss the answer to the problem they were working on: *Solve the system of equations: y = -4x -14 and y = 8x+2.* Students found an answer of (1.33, 8.67), but they observed that this ordered pair did not exactly satisfy the equations. In response to this, Jacob spent time explaining to the class that, when plugging an ordered pair in to check a solution, students should use a fraction representation rather than rounded decimals because the fractions are exact. During this whole class discussion, Jacob provided a few opportunities for students to call out answers, but primarily explained through direct instruction.

When Jacob described students' disengagement, his focus was on the challenges the students faced and how he handled them,

> ...they're willing to work until they're done with the problem and then they go away from the engagement. But I think in terms of just trying to explain with the fraction and things like that it was really … I was just challenging them to think on their own. … I was trying to challenge them to think about it and doing some prodding and things like that to steer them in the right direction.

Jacob explained that he noticed that student engagement was low, especially after they finished the problem they were working on. He described how he tried to engage students cognitively by pressing them to think about why the approximation did not yield the same answer as the exact fraction. When these justifications were not correct, he tried to steer them in the correct direction. Jacob's description of engagement focuses on the cognitive dimension; students could be engaged when they are asked to actively think about their own work.

When Jacob's students were interviewed about if they thought their classmates were engaged, they expressed that they did not think so, and both described evidence of this disengagement in terms of behavioral and social dimensions. Nikia said, "I feel like most of them wasn't [engaged]…you know how he said, 'So what'd everybody get?' Only one person answered. Then everybody was just looking down like this, playing with their markers." She decided they were not engaged because they were not paying attention. Ashiya agreed with Nikia on the presence, dimension, and feature of engagement, but provided an example from before the class discussion, "Whiteboards, yeah, it's a lot of talking. But you'll do the work. While you're doing the work, you'll talk to somebody. Then when you're done, you'll sit there and wait for [the teacher or the aide] to come around and say that's correct or they'll help you if it's not correct… but what Nikia was saying, it is a lot of talking." Although both students recognized their classmates' behavioral disengagement, Ashiya mentioned that sometimes when Jacob and the class aide were busy answering questions, students sometimes helped each other, recognizing the potential for social engagement, but she did not comment on whether or not helping each other was engaging.

Although Jacob and his students did not agree on the features or dimensions of engagement seen in the video clip, they did agree on the absence of student engagement. This indicates that Jacob and his students had a shared understanding of when students are not engaged, even if the dimensions and features were different. This is a case of minimal shared understanding, but we recognize the potential for teacher learning if there is a shared noticing about presence of engagement. If Jacob used this opportunity to learn about what students thought about how and why they were disengaged, he potentially could find ways to increase engagement in the classroom.

Discussion

We offer a framework for teacher and student noticing of mathematics engagement aligned with noticing stances of evaluating, describing, and interpreting (van Es & Sherin, 2002), as illustrated by these cases. Both teachers and students were capable of going beyond evaluating whether or not students were engaged (presence) to describe the nature of students' engagement. Both teachers and students also articulated features of engagement to interpret what may have elicited engagement in the classroom.

This study extends previous work on noticing by investigating noticing of engagement in contrast to noticing mathematical thinking (Jacobs et al., 2010; Hohensee, 2016; Labato et al., 2013) or noticing related to equity in mathematics teaching and learning (van Es et al., 2017). Additionally, previous research on teachers' noticing (e.g., Jacobs et al., 2010; van Es et al., 2017; van Es & Sherin, 2002, 2008) and students' noticing (Hohensee, 2016; Labato et al., 2013) did not compare what teachers noticed with what their students noticed. This study also demonstrates that teachers are capable of noticing a range of dimensions of engagement beyond behavioral and affective engagement, as seen in previous research studies (Bobis et al., 2016; Skilling et al., 2016), as these teachers also noticed cognitive and social engagement.

We conjecture that when a teacher and their students have a more strongly shared understanding of engagement, students' engagement is likely to be stronger, but this could be explored in future research. To establish a shared understanding of engagement, a teacher could (a) strive to understand their students' perspectives and adjust their teaching to align better with students' views or (b) more explicitly provide meaningful, explanatory rationales to students. When perspectives on engagement are not shared, this is an opportunity for teachers to learn about their students. When a teacher provides an explanatory rationale for their instructional choices, this can support students' autonomy and motivation (Reeve, 2009).

This study offers a framework for characterizing teacher and student noticing of mathematics engagement, and it investigates the potential for examining whether and how students and their teachers share noticing practices. The evidence provided in this study shows that teachers and students *can* share common descriptions, interpretations, and evaluations across this framework, and that differences in shared noticing can align with different elements of engagement. Our framework illustrates how elements of mathematics engagement can provide insight on the complex construct of engagement and how it may reveal opportunities for teachers to learn how to further engage their students in the future.

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CONVENTIONS AND CONTEXT: GRAPHING RELATED OBJECTS ONTO THE SAME SET OF AXES

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Several researchers have promoted reimagining functions and graphs more quantitatively. One part of this research has examined graphing "conventions" that can at times conflict with quantitative reasoning about graphs. In this theoretical paper, we build on this work by considering a widespread convention in mathematics teaching: putting related, derived graphical objects (e.g., the graphs of a function and its inverse or the graphs of a function and its derivative) on the same set of axes. We show problems that arise from this convention in different mathematical content areas when considering contextualized functions and graphs. We discuss teaching implications about introducing such related graphical objects through context on separate axes, and eventually building the convention of placing them on the same axis in a way that this convention and its purposes become more transparent to students.

Keywords: Mathematical Representations, High School Education, Undergraduate Education

Graphs are a foundational mathematical construct used ubiquitously across science, technology, engineering, and mathematics (STEM). Modeling quantitative relationships through graphs has been promoted as essential in STEM education (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010; National Research Council, 2012). Yet, there are two important issues that lie at the intersection of graphs and modeling contexts. First, there is a tension between (a) the power of *contextualizing* mathematics for conceptual understanding and quantitative reasoning and (b) the power of *abstraction* in mathematics to see general structures and underlying ideas (Freudenthal, 1968; Mitchelmore & White, 2007; van den Heuvel-Panhuizen, 2003). Second, there is a tension between (a) using conventions in displaying graphs for communicative or illustrative purposes, and (b) conflicts those conventions may have with perceiving and understanding deeper quantitative relationships (Moore & Silverman, 2015; Moore et al., 2014; Moore et al., 2019).

Previous research work, discussed in the next section, has made strides in bringing quantitative and covariational reasoning to bear on students' graphical thinking. In this theoretical paper, we contribute to this area by examining the convention of graphing related, derived objects (e.g., the graphs of a function and its inverse, or the graphs of a function and its derivative, or input and output vectors) onto the same set of axes in connection with contextualization versus abstraction. To do so, we use example cases from across different mathematical areas to describe how this convention, which is appropriate for abstract situations, conflicts with contextualization and quantitative reasoning. We then offer implications that our theoretical exploration has for teaching mathematical topics involving this graphing convention.

Literature Review

Graphs play a central role in representing quantities and quantitative situations mathematically, and they are used extensively across STEM fields to model a wide variety of phenomena (e.g., Angra & Gardner, 2017; Beichner, 1996; Planinic et al., 2012; Rodriguez, Bain, Towns, et al., 2019; Rodriguez, Bain, & Towns, 2019). Unfortunately, research from both mathematics and science education shows clearly that students have difficulties using graphs with quantitative situations (e.g., Bajracharya & Thompson, 2014; Beichner, 1994; McDermott et al., 1987; Testa et al., 2002; Woolnough, 2000). One prominent tendency is for students to see graphs as "pictures" or "shapes" rather than as a depiction of a relationship between quantities (Beichner, 1994; Leinhardt et al., 1990; McDermott et al., 1987; Moore & Thompson, 2015). Other challenges include confusing "slope" with "height" (Hale, 2000; McDermott et al., 1987; Planinic et al., 2012), making incorrect assertions based on the visual look of a graph (Aspinwall et al., 1997), understanding changing rates (Carlson et al., 2002), and making local versus global interpretations (Leinhardt et al., 1990; Monk, 1994). One key issue is that many conventions, such as x residing on the horizontal axis or a *vertical* line test, end up being considered as essential requirements for graphs by students (Moore & Silverman, 2015; Moore et al., 2019).

In order to promote better understanding, several researchers have focused on conceptually-rich and quantitatively-founded ways of approaching graphs in mathematics education. Some early work in this area focused on using videos and computers to track information such as distance and velocity to produce graphs (Beichner, 1996; Zollman & Fuller, 1994), or using motion detectors to embody graphical activity (Beckmann & Rozanski, 1999; Berry & Nyman, 2003). Since then, a significant portion of research has focused on re-imagining functions and graphs at the fundamental level through covariation (e.g., Carlson et al., 2002; Castillo-Garsow et al., 2013; Ellis et al., 2016; Moore & Thompson, 2016; Paoletti & Moore, 2018; Thompson & Carlson, 2017). Several studies have reported on improved student understanding and reasoning about functions, graphs, and coordinate systems when students develop these covariation-based ways of thinking (Ellis, 2011; Ellis et al., 2016; Moore, 2014; Moore et al., 2013).

In particular, Moore and Thompson (2015) contrast *static thinking*, in which students think of the graph as a static object (like a "wire"), with *emergent thinking*, in which students imagine the covariational relationship between x and y as tracing out the graph. In emergent thinking, a graph involving quantities communicates an evolving "story" between the quantities (Rodriguez, Bain, Towns, et al., 2019). However, a barrier to thinking this way is students' adoption of certain conventions as being necessary for graphs to be mathematically correct. For example, Moore et al. (2019) explain that students state that a sine curve snaking up the y-axis does not suggest a function, because of the vertical line test, despite the fact that the graph perfectly well represents the functional relationship $x = \sin(y)$. Moore and colleague's work suggests that confronting these conventions directly can help students develop a stronger sense of how graphs can portray quantitative relationships (Moore et al., 2014; Moore et al., 2019). Our paper builds on the existing literature by considering an important convention in mathematical graphing activity, described in the next section, and its conflict with representing quantitative relationships.

Theoretical Perspective: Conventions in Graphing

This study uses the lens of "conventions" in terms of graphical activity. Hewitt (1999, 2001a, 2001b) described what he called "arbitrary" aspects of mathematics, consisting of social conventions that do not necessarily have to be done that way, but on which some consensus has been reached. Some examples Hewitt provided were the names of shapes, the usage of the symbols x and y as coordinates, or terminology for operations (1999, p. 4). Other examples in the context of graphing could include using perpendicular axes (unlike Descartes' early conventions, Katz, 2009), having "up" be the positive direction, or using uniform scaling. Thompson (1992) proposed the importance of students becoming aware of these conventions they were using. He used the phrase "convention qua convention" to mean when one understands "that approaches other than the one adopted could be taken with equal validity" (p. 125). He explained that problems arise when conventions are not properly understood as conventions (see also Thompson, 1995). Moore et al. (2019) then built on these ideas by providing a definition of "convention," which we use in this paper: a convention is a combination of a concept, a community, and a representational practice.

The precise convention we highlight in this paper is the common practice of placing a graphical object and a related, derived graphical object on the same set of axes. Here, by "object" we mean the literal "thing" that is placed on the axes, such as a function's graph, a curve, or a vector arrow. Applying Moore et al.'s (2019) definition, we see the "community" as the mathematical community, the "concept" as graphical objects related in some key way, and the "practice" as placing these related objects on the same set of axes. As examples, it is common to graph a function and its inverse function on the same set of axes (e.g., Blitzer, 2018; Sullivan & Sullivan, 2020), or to place the graphs of a function and its derivative on the same set of axes (e.g., Hass et al., 2020; Rogawski et al., 2019; Stewart et al., 2021).

Conventions in mathematics also have connection to abstraction in mathematics (Dreyfus, 2020; Ferrari, 2003). A core practice of mathematics is abstracting similar structures from different contexts (Brousseau, 1997, 2002). This practice results in decontextualized objects, for which certain conventions might be adopted to track them in the absence of concrete quantities, such as using x primarily as the "input/independent" variable, or placing the output primarily on the vertical axis. Such conventions in communicating about abstract structures may be perfectly valid in the abstract space, but may conflict with reasoning within a quantitative context.

The goal of this theoretical paper is to examine cases of this convention across mathematics content areas to show how it may problematically intersect with contextual graphs that represent quantities. We explain how not understanding this convention *as* a convention can lead to issues in making sense of contextualized or quantitative interpretations of a graphical system. We then describe teaching implications based on our theoretical investigation and attempt to situate this convention more appropriately within learning about representing related graphical objects.

Conflicts between the Convention and Contextualization in Case Content Areas

In this section, we unpack the convention described in the previous section in three distinct mathematical content area cases. The primary purpose of this section is to show, theoretically, the conflicts that can arise between this particular convention and contextualization. The next section then discusses the pedagogical implications that may help address this conflict.

Case 1: Graphing Inverse Functions

One key concept in mathematics is inverse functions. Students are exposed to inverse functions each time they learn about a new function (e.g., exponential, trigonometric). A common convention of inverse functions is to graph the function and its inverse on the same set of axes, where the inverse is a reflection of the original graph over the $y = x$ line. Figure 1 shows common types of images students see in textbooks and in classrooms. In abstraction, this convention can make sense to examine how the features of the two graphs relate to each other.

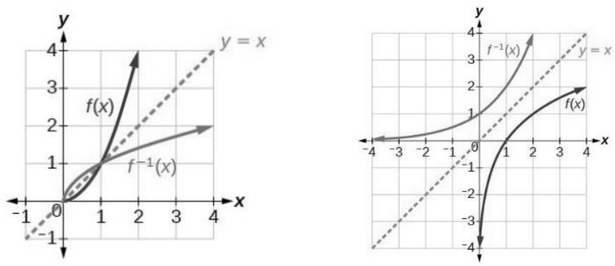

Figure 1: Common types of images of functions and inverses on the same set of axes

The problem with this convention is in moving to a contextualized situation. For example, if one examines a function converting Euros to U.S. Dollars, D(E) = 1.3E (Teuscher et al., 2018), one might create a graph with Euros on the horizontal axis and U.S. Dollars on the vertical axis. The point (50,65) on the graph suggests that an input of 50€ gives an output of \$65. However, if the inverse function is graphed as a reflection across the D = E line on the same axes, then the new graph conflicts with the quantitative meanings of the two axes. The point (50,65) reflects to the point (65,50), now suggesting, according to the axes, that 65€ corresponds to \$50, which is incorrect (Figure 2). To accurately interpret the placement of the two graphs on the same axes, a student must make *both* axes simultaneously represent *both* the D and E quantities, tracking which version of the axis label matches which graph is being examined. For novice learners of inverse functions, this may be quite sophisticated reasoning, obscuring the actual quantitative relationship that exists for a function and its inverse (see also Duval, 2016). Thus, the convention of placing two related graphical objects (graphs of a function and its inverse) on the same axes may conflict with contextualizing the functions and graphs through quantities.

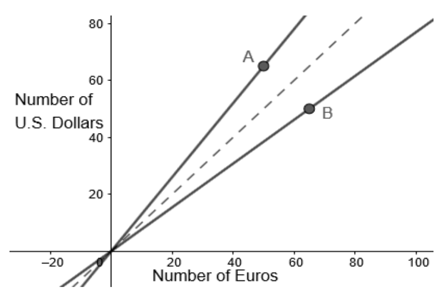

Figure 2: Graphs of function and inverse in the context of money

Case 2: Graphing Derivatives of Functions

This convention also appears in the practice of placing the graph of a function and the graph of its derivative on the same axes. Calculus textbooks and instructors often do this, as exemplified in Figure 3 taken from Stewart et al. (2021, p. 164) and Hughes-Hallett et al. (2012, p. 106). If the variables x and y represent nothing more than pure numbers, this practice is appropriate, since one can put the numeric outputs of f and f' on the same *unit-less* axes.

Olanoff, D., Johnson, K., & Spitzer, S. (2021). *Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Philadelphia, PA.

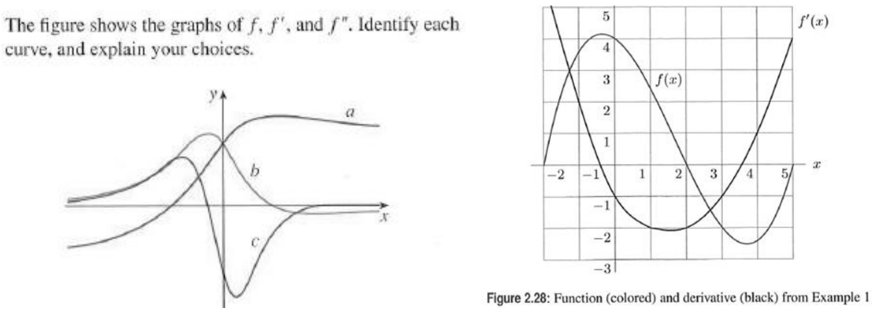

Figure 3: Function and derivative graphs on the same axes

As with inverse functions, the problem arises when trying to contextualize these functions. For example, suppose $f(x)$ represents the temperature in degrees Celsius as a function of time, with x in units of minutes. Then the output of the derivative, f', is the rate of change of temperature in degrees per minute, which is a fundamentally different quantity than the output of f as temperature. When placing f and f' on the same axes, some confusion arises: What quantity is represented by the vertical axis? Is it the temperature? The rate of change? Both? In fact, it would require *both* quantities being on the vertical axis simultaneously. For example, Figure 4, taken from Hass et al. (2020, p. 150), shows a position function (s) and its derivative function (velocity, v). Note the label "s, v" on the vertical axis, meaning that single axis is simultaneously representing two quantities. When looking at $s = 5\cos(t)$ or $v = -5\sin(t)$, one must constantly shift the vertical axis between position and velocity quantities and their associated units. While experts can likely make such subtle shifts without much problem, and it may even be a goal for students to eventually do such thinking, this convention again conflicts with contextualizing functions and their derivatives and examining them quantitatively, especially for novices.

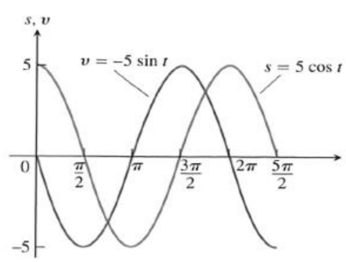

Figure 4: The vertical axis representing two quantities simultaneously

Case 3: Graphing Matrix Operations

One common way to think about matrix-vector multiplication in linear algebra is as a matrix performing a geometric transformation on a vector. In this perspective it is common to show the original vector and the transformed vector on the same set of axes (e.g., Poole, 2015), as in Figure 5. When working in the abstract world of vectors as geometric arrows or lists of unit-less numbers, this convention can help to illustrate the transformation that is represented by a particular matrix.

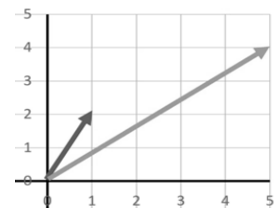
Figure 5: A vector and a transformed vector on the same set of axes

However, yet again, contextualizing vectors and matrices can possibly come into conflict with this convention. As an example, the second author collaborated with two secondary teachers to develop a set of tasks to help students better understand the structure of matrix multiplication and how matrix multiplication can be used to model real-world phenomena. The tasks used the context of two basketball players, Joaquin and Raul, who played the same position but averaged different numbers of points and rebounds each quarter, with Joaquin averaging 6 points and 3 rebounds per quarter and Raul averaging 2 points and 5 rebounds per quarter. The students were asked: The basketball coach wants his centers to combine for 20 points and 14 rebounds, because that might help them win the game. What number of quarters could the coach play Joaquin and Raul to average 20 points and 14 rebounds? This problem gives rise to the matrix equation: $\begin{bmatrix} 2 & 6 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} J \\ R \end{bmatrix} = \begin{bmatrix} 20 \\ 14 \end{bmatrix}$, with solution $\begin{bmatrix} J \\ R \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

From a mathematical standpoint, we can think about this as a 2-by-2 matrix that transforms the vector $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ to the vector $\begin{bmatrix} 20 \\ 14 \end{bmatrix}$. However, the vector $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is in units of *quarters played* for each player, $\begin{bmatrix} \text{1 J quarter} \\ \text{3 R quarters} \end{bmatrix}$, while the output vector has units of *points* and *rebounds*: $\begin{bmatrix} \text{20 points} \\ \text{14 rebounds} \end{bmatrix}$. When plotting these two vectors on the same axes, we again see a complication with what quantities the axes represent. The horizontal axis must simultaneously represent "Joaquin quarters" *and* "points" while the vertical axis must simultaneously represent "Raul quarters" *and* "rebounds" (Figure 6). Like with function inverses and derivatives, this is likely difficult for novices learning about matrix-vector multiplication, and we see that the convention of placing related graphical objects on the same axes once again can be at odds with contextualization.

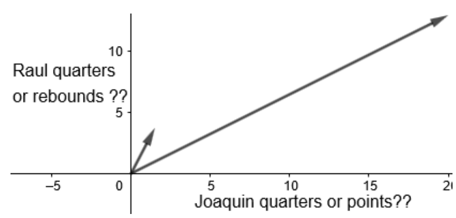

Figure 6: Conflict in the Quantities on Each Axis

When the Convention Does *Not* Conflict with Contextualization

To be clear, the convention of placing related, derived graphical objects on the same axes does not *always* conflict with contextualization. For example, consider certain graphical transformations, such as a vertical shift given by $f(x) + 4$ or a vertical stretch given by $2f(x)$ (Figure 7). If the function and graph represent a quantitative context, such as x representing the price of a good and $f(x)$ representing the amount sold of that good, then the transformations typically retain those same quantitative meanings. The function $f(x) + 4$ can represent that an additional fixed 4 units are sold at all price levels, and 2*f*(*x*) can represent that twice as many units are sold at each price level. In both of these cases, the vertical axis retains the same quantity "amount sold" and there is no conflict with the convention and the context.

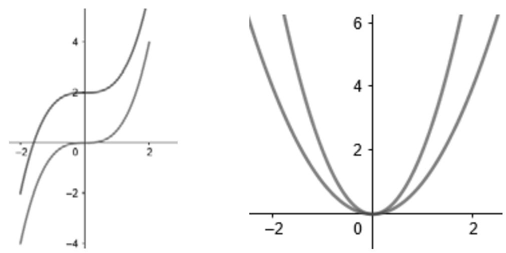

Figure 7: Transformations graphed on the same axes

Teaching Implications

We have explained how the practice of placing related, derived graphical objects on the same axes can often conflict with using quantitative contexts for functions and graphs. The answer should not be to avoid contextualization to sidestep the conflict and teach solely with abstract pure-numeric functions (Ferrari, 2003). But, it also cannot be to avoid the abstract convention, because it is a common practice. To address the dilemma, instruction should use context, but then help students see the convention *as* a convention (Moore et al., 2019; Thompson, 1992). To this end, we offer the following suggestions, based on Brousseau's (1997; 2002) idea of contextualizing mathematics in order to then re-decontextualize it. Thus, we take the stance that teaching using contextualized quantitative situations is crucial, but that students must be guided toward a comprehension and usage of the abstract convention we have described.

The first step must involve the teacher identifying whether a mathematical topic wherein two related graphical objects are commonly placed on the same axes actually conflicts with contextualization, or not, as we did in the previous section. Thus, the previous section contributes as a model for that type of conceptual analysis. If a potential conflict is identified, we recommend that the graphical representations of the two related objects initially be placed on *separate* axes. To use inverse functions as an example, if the class is learning that the function D(E) = 1.3E has an inverse function E(D) = D/1.3, then the two graphs should be placed initially on two separate axes (Figure 8). Doing so initially avoids the conflict between quantitative reasoning and using a single set of axes and permits the students to track how each graphical object (function or inverse, in this case) relates to the context's quantities. With each graph separate, students can develop quantitative reasoning about the overall context, without the difficulties described in the previous section, such as identifying the rate that one currency accumulates as the other currency accumulates (see also Moore, 2016; Moore & Thompson, 2015; Thompson & Carlson, 2017). Thus, in the first step, the focus is on quantitatively understanding each mathematical object *in its own right*, at the global level.

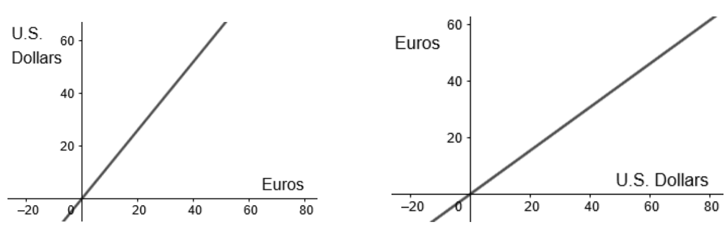

Figure 8: First, Begin with Related Objects on Separate Axes

The second step acts as an intermediate between the initial contextualization and the abstracted, decontextualized convention. In this step, students are asked to compare the two graphical objects to identify specific connections, differences, or relationships. A key part of this step is to explicitly ask the students to place the two sets of axes near each other in a way that facilitates comparisons between the two graphical objects. For example, suppose students are investigating the graphs of a function and its derivative, where $f(x)$ represents the temperature as a function of time. As per step one, the students have represented the graphs of f and f' on separate axes and have examined the behaviors of the temperature and the rate of change separately. The teacher could ask, "Compare the graph of f with the graph of f'. What connections do you see between them? I encourage you to put the derivative graph directly underneath the function graph to help compare them." Such placement allows some connections to become clearer, such as what behavior for f is associated with f' being positive, negative or zero (Figure 9). The key feature of this step is that, while still on separate axes, the students are now thinking about the benefits of placing the graphs more closely to each other. The physical placement gets the students one step closer to the convention, while still maintaining a focus on the quantitative relationships.

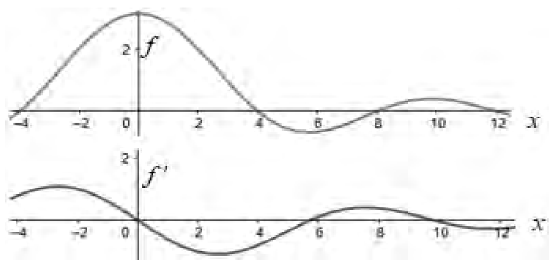

Figure 9: Second, Place Graphs in a Way to Examine Specific Connections

The third step then moves to the abstract convention and its purposes, while explicitly attending to quantitative difficulties associated with the convention. In this step, the teacher can place the two graphical objects on the same axes and ask the students what the advantages and disadvantages of doing so might be. To use matrix-vector multiplication as an example, suppose a teacher has used the quarters-played / points-rebounds context, has started with the vectors on two different axes (step one), and has had the students compare the two vectors by placing the two axes close to each other (step two). The teacher then places the two vectors on the same axes, and asks, "What might be the benefit of putting the two vectors on the same axes? But what might be confusing about doing that?" This step helps students identify the benefits of seeing directly how one vector can be thought of as a transformed version of the other. Yet, the discussion also helps students see the difficulties in thinking about one vector as representing one pair of quantities (quarters played) while the other vector represents a different pair of quantities (points and rebounds). By now seeing this convention *as* a convention (Thompson, 1992), the students can see why it is used in the first place, when they might choose to use it, and what cognitive work they need to do to make sure they can quantitatively interpret graphs superimposed on top of each other.

In conclusion, we believe these three steps address the problems described in the previous section by allowing the concepts to be taught with contextualization and quantitative reasoning, but then gradually transitioning to the abstract convention (Brousseau, 1997; 2002). Such a sequence of steps makes the convention of placing the graphical objects on the same axes more reasonable, but also helps students be aware of the challenges associated with it. We believe these steps put students in a better position to develop the sophisticated abstract thinking that goes along with assigning multiple (conflicting) quantities on the same axis. Students can use this convention appropriately both within and outside mathematics. That is, not only can they better appreciate the convention for abstract graphical objects, but they can also better reason about contexts that *are* represented with multiple quantities on the same axis, such as the position and velocity graphs in Figure 4, or the “double *y*-axis” graphs shown here in Figure 10.

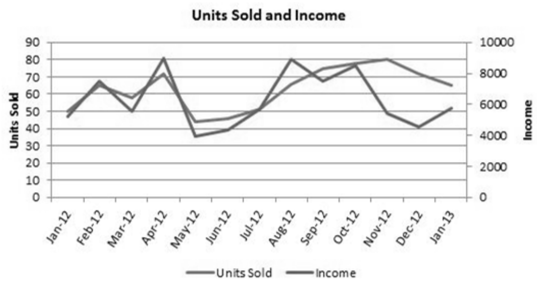

Figure 10: Double *y*-axis Graphs with Two “Vertical Axis” Quantities

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ESTABLISHING STUDENT MATHEMATICAL THINKING AS AN OBJECT OF CLASS DISCUSSION

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Productive use of student mathematical thinking is a critical yet incompletely understood dimension of effective teaching practice. We have previously conceptualized the teaching practice of building on student mathematical thinking and the four elements that comprise it. In this paper we begin to unpack this complex practice by looking closely at its first element, establish. Based on an analysis of secondary mathematics teachers' enactments of building, we describe two critical aspects of establish—establish precision and establish an object—and the actions teachers take in association with these aspects.

Keywords: Classroom Discourse, Communication, Instructional Activities and Practices

The Association of Mathematics Teacher Educators [AMTE] Standards (2017) argued that an important component of whole-class instruction is the "intentional discussion of selected and sequenced student approaches... to move students through a trajectory of sophistication toward the intended mathematics learning goal of the lesson" (p. 16). This argument is supported by other related recommendations (National Council of Teachers of Mathematics [NCTM], 2014) that have highlighted the importance of productively using students' mathematical thinking as part of whole-class instruction. There are many different ways that teachers can productively use students' mathematical thinking, however, and these ways are determined, at least in part, by the nature of the thinking itself (Stockero et al., 2020). It has been posited that some instances of student thinking are of particular importance and that using them productively can be especially advantageous (Leatham et al., 2015).

That said, taking advantage of such instances requires coordinating a complex collection of teaching practices, and there is evidence that certain aspects of these practices do not occur naturally in whole-class instruction (Stockero et al., 2020). To better understand and improve teachers' ability to engage in complex practices, Grossman and her colleagues (2009) suggested that practices be decomposed into their "constituent parts" (p. 2069) for the purpose of helping teachers develop these practices. We have previously conceptualized the teaching practice of *building on student mathematical thinking* and the four elements that comprise it (see Van Zoest et al., 2016). In this paper we begin to further decompose this complex practice by looking closely at its first element, *establish*.

Theoretical Framework

Before describing the teaching practice of building, we first introduce the type of instances of student thinking that this practice is intended to take advantage of. As we have described elsewhere in greater detail (Leatham et al. 2015), MOSTs (**M**athematically Significant Pedagogical **O**pportunities to Build on **S**tudent **T**hinking) occur at the intersection of three critical characteristics of classroom instances: student mathematical thinking, significant mathematics, and pedagogical opportunity. Particularly relevant to this paper, MOSTs are observable instances of student mathematical thinking that provide sufficient evidence to "make reasonable inferences about student mathematics" (Leatham et al., 2015, p. 92).

When we say *building on student mathematical thinking* we mean the teaching practice that takes advantage of the opportunity that a MOST provides (Van Zoest et al., 2016). More specifically, we define *building on a MOST* (hereafter referred to as *building*) as making a MOST "the object of consideration by the class in order to engage the class in making sense of that thinking to better understand an important mathematical idea" (Van Zoest et al., 2017, p. 36). As we unpacked that definition in the context of our collective experience with analyzing teaching (our own and that of others), we theorized that building is comprised of four elements: (1) *establish* the student mathematics of the MOST as the object to be discussed; (2) *grapple toss* that object in a way that positions the class to make sense of it; (3) *conduct* a whole-class discussion that supports the students in making sense of the student mathematics of the MOST; and (4) *make explicit* the important mathematical idea from the discussion.

As part of our ongoing research, we have been investigating these elements. The current paper focuses on the first element and addresses this research question: What are necessary components of the *establish* element of building as revealed through teachers' attempts to enact the practice?

Literature Review

Productively using student thinking during whole-class discussion involves teachers capturing the essence and relevant details of student contributions (a central purpose of the *establish* element of building). Thus, research on productive whole-class discussions sheds some light on this important facet of teaching, although it has seldom been the direct focus of studies. For example, van Zee and Minstrell (1997) described a *reflective toss*, which is a teacher response that "elicit[s] further thinking by *catching the meaning of the student's prior utterance* and throwing responsibility for thinking back to the students" (p. 241, italics added). Another example comes from the work of Webb et al. (2014), who identified teacher moves that facilitate students "referencing *the details of another student's idea*" (p. 88, italics added) as an important aspect of promoting students' productive engagement with their peers' mathematical thinking. Implicit in these findings is the need for the meaning and details of student contributions to be available for reference. Knowing more about capturing the essence and relevant details of student contributions (and thus about aspects of *establish*) is critical to understanding productive use of student mathematical thinking during whole-class discussion.

One significant contribution to understanding this preliminary facet of productively using student thinking is Staples' (2007) model of a teacher's role in supporting collaborative inquiry. A key component of this model, which was conceptualized through her longitudinal study of one high school teacher, is the work a teacher needs to do to *establish and monitor a common ground*. Staples identified a variety of instructional strategies that a teacher may use to establish student ideas as the common ground. One strategy was repeating student contributions and using multiple modes of communication (e.g., verbal, written) to provide students with a variety of opportunities to access one another's ideas. Another strategy involved publicly recording ideas in a structured way on the board to provide some permanence of student contributions and to facilitate students' development of an idea throughout an inquiry. Later she further elaborated by indicating that the goal of such practice is "not perfect use of vocabulary or formal sentences, but rather clear enough expression of ideas so that both the teacher and other students can consider the contribution" (Staples & King, 2017, p. 40). Our study builds on Staples' work in this area by investigating the *establish* element of building with multiple teachers who were conducting whole-class discussions around the same tasks and often the same student thinking. Broadening the pool of teachers and simultaneously focusing on comparable situations across them provided a rich data set that allowed us to more fully identify necessary aspects of *establish* and the subtleties that are involved in teachers accomplishing it.

Methods

In order to study our theorized practice of building we enlisted 12 teacher researchers—practicing secondary mathematics teachers who desired to more productively use their students' mathematical thinking. These teachers enacted the building practice in their classrooms using four mini tasks (see Figure 1) that were designed to elicit particular MOSTs, resulting in 27 building enactments. We compared these enactments to our initial conceptualization of building by coding transcripts of the enactments for actions that seemed to either facilitate or hinder the overall practice of building. Analysis of these coded data led to refinement of the four elements of building, including identifying necessary aspects of each and a variety of associated subtleties. With respect to the focus of this paper—*establish*—our analysis revealed both aspects of this practice and actions teachers might take to effectively position student contributions to become the object of discussion.

(a) Percent Discount The price of a necklace was first increased 50% and later decreased 50%. Is the final price the same as the original price? Why or why not?	**(c) Points on a Line** Is it possible to select a point B on the y-axis so that the line $x + y = 6$ goes through both points A and B? Explain why or why not. 
(b) Variables Which is larger, x or $x + x$? Explain your reasoning.	**(d) Bike Ride** On Blake's morning bike ride, he averaged 3 miles per hour (mph) riding a trail up a hill and 15 mph returning back down that same trail. What was his average speed for his whole ride?

Figure 1: The Four Mini Tasks Used in Creating Instantiations of Building

Results

We describe here two aspects of *establish*: *establish precision* and *establish an object*. (A third identified aspect, *establish intellectual need*, is beyond the scope of this paper.) In order to effectively position a student contribution (a MOST) to become the object of discussion teachers must establish (a) precision—the student contribution must be clear, complete, and concise so that the class can focus on making sense of that contribution, and (b) an object—the contribution must take on a measure of permanence and identity so that it can clearly be referred to during the remainder of building. In the following sections we elaborate on these two aspects, describing actions teachers take in association with each aspect. Note that although students might spontaneously take actions that contribute to making the contribution precise or an object, we focus here on the actions teachers take to ensure that these aspects are satisfied. Those teacher actions initiate the work, even though the actor could be students or the teacher.

Establish Precision

The first aspect of *establish* requires that the teacher ensure that the student contribution is clear, complete, and concise. Precision is important because making imprecise thinking the object of consideration is likely to hamper building (Peterson et al., 2020). By carrying out this aspect, the teacher establishes what it is the class is going to make sense of during the *conduct* element of building.

Of course, not all student contributions are imprecise; some are stated precisely to begin with. Precise contributions, however, were more the exception than the rule in our data. Analysis of the building enactments revealed a number of ways that student contributions were not precise enough for students to engage in making sense of them. In the following sections, we consider three actions that might be needed to establish the precision of a student contribution: clarifying, expanding, and honing.

Clarifying. Clarifying is about making clear WHAT the student has said. We discuss here two types of situations where clarifying actions may be needed. First, student contributions often need to be clarified because students use informal language or pronouns with vague referents (Peterson et al., 2020). For example, during a Percent Discount (Figure 1a) enactment, as a student was sharing their solution, they said, “Like you’re subtracting it.” The teacher followed up with, “Okay, subtracting what?” and the student replied, “50 percent.” We see here that the teacher’s question helped to make clearer this part of the student contribution by making the referent explicit. This type of clarifying also occurs when details that are naturally left out of a contribution due to conversational conventions, such as the prompt the student was responding to, are added in.

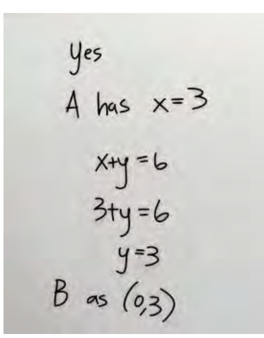
Second, student contributions need to be clarified when students share the substance of their reasoning, but the logical structure of those ideas are not clearly articulated. For example, consider this student contribution during a Variables (Figure 1b) enactment: “I believe that x plus x is larger because if x is just one value, x plus x would be double the value, which in this case makes it larger,” and the teacher’s response, “You were thinking x plus x is larger than x, because when you add the values, it makes it double, so it’s larger?” Without changing the logic of the student’s contribution, the teacher clarified the logical structure. By confirming with the student that the clarification was accurate, the teacher has clarified the contribution for other students in the class and kept the focus on the student contribution.

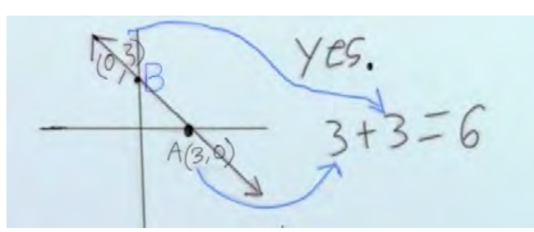
Expanding. Expanding is about making the contribution complete and involves adding something to the contribution that is needed to position the class to engage in making sense of it. The most common expansion situation that we saw in the enactments we analyzed was when a student provided an answer without reasoning. Student contributions that are merely an answer need to be expanded because the class will not be able to fully make sense of the contribution without the underlying reasoning behind that answer. For example, in response to Variables (Figure 1b), a student initially simply stated, “x + x is greater than x.” Although the teacher knew the student had reasoning for their answer from monitoring students’ work, without expanding the student’s contribution to include the reasoning, the class would be left guessing about what exactly they were to make sense of. When asked to share their explanation for their answer, the student elaborated, “So x plus x will be 2x, and x will be just 1x.” This expansion provided the necessary fodder for a sense-making discussion. This teacher expanded further when they responded, “So you’re saying 2x is bigger than 1x, is that what you’re saying?” This response seems to be important as it makes explicit the critical reasoning that was missing. As mentioned above, confirming that an inference is accurate keeps the focus on the student's contribution.

Student contributions also need to be expanded when reasoning is present, but a piece of information needed to make the contribution complete is missing. For example, during a discussion about Bike Ride (Figure 1d), a student wrote (a+b)/c on the board and explained that c represented the number of speeds. Although it is possible to infer what a and b represented, asking the student to explicitly define these variables made the contribution more complete. There was nothing unclear about what the student said, but their explanation did not provide all the information the class would need to make sense of it. In both these latter examples, rather than counting on students to guess the missing information or to infer the implicit information, the expansion made that information explicitly available, and resulted in a more complete contribution—one the class was better positioned to collaboratively make sense of.

Honing. Honing is about making the student contribution concise and involves reducing it to its essence. Sometimes a student contribution contains extra verbiage or extraneous information that is unnecessary for, and may even interfere with, making sense of the student contribution. Making these contributions concise requires removing unnecessary information that might distract students from the main sense-making opportunity. For example, during a Variables (Figure 1b) enactment, a student explained part of their reasoning as, "Because they have the same shirt, so they can be added together, so x plus x will be 2x." The teacher response honed the student contribution by taking up the "so x plus x will be 2x" piece of the student contribution and omitting the part about "the same shirt."

More often, honing is a matter of capturing ideas within a student contribution more succinctly by using symbols or other shorthand. For example, in a Points on a Line (Figure 1c) enactment, a student explained, "I put 'yes' because A has the point, like, its x equals 3," to which the teacher commented, "All right, so, 'Yes, A has x equals 3,'" as they wrote that same information on the board (Figure 2a, lines 1 and 2). The student continued their explanation, "And then the equation is x plus y equals 6, so then I just plugged in the x, which is 3, plus y, equals 6 and figured out y, it needs to be 3, and then just put point B as (0,3)." The teacher listened and carefully captured what the student was saying on the board (Figure 2a). In this case, the teacher made the student contribution more concise by the use of mathematical symbols. In a different Points on a Line enactment, part of a student explanation included, "3 plus 3 equals 6 where the first 3… come[s] from A and the second 3 comes from B." In Figure 2b, we see how, rather than writing down what the student said in words, the teacher concisely captured their contribution by drawing lines from the 3s in each of the two points on the line to the corresponding 3 in the equation. All of these honing actions contribute to making the student contribution more precise by making it appropriately concise.





(a) (b)

Figure 2: Two Illustrations of Teachers Establishing Precision through Honing

Establish an Object

Beyond establishing what the student contribution is (just described in Establish Precision), *establish* also entails the work of ensuring that the student contribution is established as an object, as a "thing" that can be considered. The initial goal of establishing a contribution as an object is to support making the *grapple toss* as efficient and effective as possible; it is much easier to toss an object—and for students to then engage with it during the *conduct* element—when the object is well-defined. We have come to see objects as well-defined when they have a high degree of both permanence and identity.

Our analysis of building enactments revealed a number of teacher actions that have the potential to contribute to establishing the student contribution as an object. In the following sections we describe two main sets of teacher actions that seem to contribute to this aspect of *establish*: re-presenting the object, which makes it more permanent, and referring to the object, which contributes to the identity of the object.

Re-presenting the Object. Re-presenting happened most frequently in the enactments we analyzed when student contributions were first made public orally (as opposed to students initially sharing their work at the board or on a document camera). In order to set these contributions apart from the ongoing conversation, teachers can re-present them. Re-presentation acts serve to demarcate a student contribution from the ongoing discussion and thus give it a degree of permanence, a staying power that often does not exist with the numerous passing comments of classroom discourse. These re-presentation acts signal a pause in the ongoing dialogue and begin to create space for a new kind of activity—one that will make the student contribution the object of consideration.

One option is for the teacher to re-present the oral student contribution orally. Two common forms of re-presentation occurred in our data: repeating and revoicing. Consistent with the definitions of others (e.g., Chapin et al., 2009; Forman et al., 1998), repeating is when the entire object is restated with no replacement in language and revoicing is when the student contribution is paraphrased without changing its meaning. One benefit of re-presenting through revoicing is that the re-presentation may be a more precise object than the original. One risk of re-presenting through revoicing is that a poorly executed revoicing may result in an object that is less precise.

Another way to re-present an oral student contribution is to switch to a written presentation (as the teachers did in Figure 2). Creating a public record of an oral student contribution by, for example, inviting the student to write what they said on the board or by acting as scribe themselves, is a way for the teacher to take the somewhat ephemeral spoken word and make it more tangible. That is, the student contribution becomes something the teacher and students can hold on to, can refer to, can operate on. It creates, in essence, a physical object that can be referred to in the *grapple toss* and pursuant discussion. Creating a public record sets the MOST apart from other verbal contributions during a whole-class discussion, giving it a permanence that is difficult to achieve otherwise.

Referring to the Object. Another way that teachers establish the student contribution as an object involves referring to the thinking AS an object. In other words, treating the student contribution as an object makes it more of an object. Such referring creates a sense of identity for the object. But student contributions are complex entities (often several sentences in length). We have found a number of ways that teachers refer to student contributions, some of which have more potential than others to contribute to making the student contribution an object.

One way a teacher may refer to a student contribution as an object is to use a pronoun (i.e., that, this, it) for which everyone would likely know that the referent is the student contribution. For example, during a Percent Discount (Figure 1a) enactment, a student contributed, “Because you’re adding fifty and then you’re taking away fifty percent,” and the teacher responded, “Say it again, what you just said.” Members of the classroom would likely recognize that the “it” in “say it again” was referring to the entirety of the students’ contribution, which helps to make the contribution an object. Furthermore, students would likely recognize that the phrase “what you just said” was also a somewhat generic way of referring to the student contribution.

A second way of referring to the object is to name it. We have seen teachers name student contributions by characterizing the nature of the thinking (e.g., this claim, this reasoning), attributing that thinking to the student by name (e.g., Tray’s thinking)—and sometimes by doing both (e.g., Jaden’s claim). Naming, a form of *metatalk* (Leinhardt and Steele, 2005), is a way of marking the student contribution so the class can access it again when the name is used.

A third way of referring to the object is for a teacher to point to or make a gesture toward a public record of the student contribution. The action of pointing at the board contributes to the student contribution being the object that the class is to focus on as the discussion continues.

The aforementioned ways of referring to a student contribution (pronouns, generic terms, naming, and pointing) vary in their potential to contribute to making a student contribution an object. Referring to the object by name seems to have the most potential for making the student contribution an object because a) the name reduces the potential for ambiguity in the referring, and b) because the name makes the student contribution easily identifiable for future reference.

Although we are not claiming that any particular subset of objectifying actions is necessary for “sufficiently” establishing the student contribution as an object, our analysis of teaching enactments suggests that re-presenting the contribution by creating a precise public record of it and referring to it by name (based on the nature and/or the contributor) provide a strong foundation for the *grapple toss*. Given the difficulties students have in focusing on making sense of a specific contribution (Franke et al., 2015; Webb et al., 2014), the more scaffolding we can provide in that regard, the more likely they are to maintain this focus.

Discussion & Conclusion

Establish is comprised of three aspects (see Figure 3), two of which were discussed in this paper: establish precision and establish an object. The goal of establish precision is to ensure that the student contribution is clear, complete, and concise, accomplished respectively by clarifying, expanding, and honing actions. The goal of establish an object is to ensure that the student contribution achieves a measure of permanence and identity, accomplished respectively by re-presenting and referring actions. In other words, we want the class to know exactly what the student contribution is and also position that contribution as an object that can easily be referred to and acted upon throughout the remaining elements of building.

Element	Establish						
Aspect	Precision			An Object		Intellectual Need	
Action	Clarifying	Expanding	Honing	Re-presenting	Referring		

Figure 3: Establish Broken Down by Aspects and Associated Actions

We conclude with several observations about these aspects of *establish*. First, whereas the work of establishing precision operates on the pieces of the student contribution in order to create a clear, complete, and concise object, the work of establishing an object operates on the object as an entity, re-presenting and referring to the entire object in order to make it more of an object. For instance, when expanding, one adds a piece to the object, and when honing, one removes a piece from or replaces a piece of the object. In contrast, when re-presenting through revoicing, one paraphrases the entire object, and when referring through naming, one names the entire object.

Second, the *establish* element of building is a teacher practice—it is always the teacher's responsibility to *ensure* that the student contribution is a precise object. The teacher does not always need to be the one who makes a student contribution a precise object, but they DO always need to consider WHETHER the contribution is precise and a well-defined object and take action if it is not. That said, there are many different ways that both the teacher and the contributing student carry out *establish* actions. For example, although the desired action might be "clarify," the teacher might invite the contributing student to clarify or they themselves might provide the clarification with a confirmation from the contributing student. As we have discussed elsewhere (Van Zoest et al., 2021), it is helpful to disentangle the actor from the action in order to unpack critical nuances of teacher responses to student mathematical thinking.

Third, although we have discussed these aspects and associated actions of *establish* discretely, generally they do not occur as such in practice. That is, teachers often accomplish multiple aspects of *establish* simultaneously. We see this in Figure 2a, where the teacher is engaged in honing (as discussed), as well as re-presenting by creating a public record of the student's oral contribution and clarifying the reasoning of the contribution by placing each piece of the logic on a separate line.

Finally, the individual actions we have identified are not new—they have been discussed to some degree in the literature. Furthermore, others have observed relationships between these actions and broader teacher practices, noting that it is valuable to consider actions (e.g., clarifying) with respect to "the purpose that those techniques are serving" (Boerst et al., 2011, p. 2854). Our work here illustrates the importance of coordinating a collection of actions in order to accomplish a particular purpose, in this case to *establish* a student contribution as part of the broader teaching practice of building on that contribution.

Unpacking *establish* has allowed us to better understand the complexity and craft of this critical element of building, better positioning us to work with teachers to develop their abilities to productively use student mathematical thinking.

Acknowledgments

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NOTICING FOR EQUITY IN WRITTEN WORK: EXPLORING ONE TEACHER'S STUDENT WORK ANALYSIS PRACTICES

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Attending to students' thinking and using it to inform instruction has been shown to be an effective teaching practice. Although research on teacher noticing has explored how teachers attend to and interpret thinking in the moment and through video, less is known about the ways in which teachers notice students' thinking in written work, as well as the cultural dimensions that shape noticing. While work on "noticing for equity" has begun to explore the latter, it focuses on noticing of participation. This qualitative case study asks if equitable noticing extends to students' work. Analysis of one equity-oriented math teacher's student work analysis practices revealed that she a.) attended to the details of students' strategies with a learner stance, b.) contextualized their understandings, c.) interpreted their understandings through a strengths-based lens, and d.) planned to respond by identifying aspects of work to share with the class.

Keywords: Teacher Noticing; Equity, Inclusion, and Diversity

Introduction

Research has shown that when teachers attend closely to their students' mathematical thinking, they can use it to inform instruction, leading to gains in achievement (Carpenter et al., 1989). When teachers deeply understand students' thought processes, they can build on their prior knowledge and leverage student-generated strategies for class learning. Studies on "teacher noticing" of thinking and of classroom activity have explored what teachers attend to and how they interpret it (Jacobs et al., 2010; Sherin & van Es, 2005). Although teacher noticing has been studied primarily in the midst of teaching or in video, teachers can also attend to students' thinking by examining their written work (Kazemi & Franke, 2004). Less is known about what and how teachers notice thinking in work, as well as the ways in which that noticing is shaped by teachers' pedagogical commitments (Erickson, 2011), dispositions (Hand, 2012), and immersion in dominant discourses about mathematics (Louie, 2018). While work on "noticing for equity" has begun to explore these cultural dimensions of noticing, it focuses on noticing of participation (van Es et al., 2017; Wager, 2014). This study asks if equitable noticing might extend beyond participation—as noticed in-the-moment or on video—to students' written work. Given that student work is accessible and can be looked at outside of instruction, it is an untapped resource for making sense of thinking. Understanding equitable ways of noticing thinking in written work may support teachers in their practice, teacher educators training novice teachers, and researchers documenting equitable math pedagogies. While noticing of thinking and noticing for equity have been theorized separately, illuminating connections between the two may support future study of their intersections. This qualitative case study uses think-aloud protocols to explore one equity-oriented math teacher's student work analysis process, investigating the following questions:

1. What and how does an equity-oriented teacher notice when examining their students' written work?
2. In what ways do a teacher's pedagogical commitments and dispositions inform their noticing when examining their students' written work?

Conceptual Framework

This study draws on three bodies of literature to examine the ways in which an equity-oriented teacher's pedagogical commitments and dispositions shape what and how they notice when looking at work (figure 1). The teacher noticing literature has included several studies involving student work analysis related to the construct of noticing children's mathematical thinking. Scholars have recently extended the noticing literature towards theories of equitable mathematics pedagogy, utilizing "noticing for equity" to document equity-oriented teachers' noticing of participation. In these studies, scholars argue that teachers' commitments to equitable pedagogy shape their equitable noticing of participation. In exploring noticing for equity in written work, this study suggests pedagogical commitments also shape noticing of thinking.

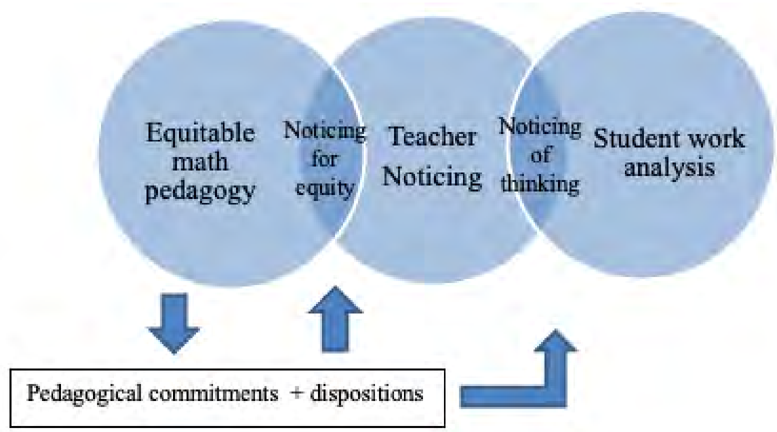

Figure 1: Conceptual Framework

Teacher Noticing

The teacher noticing literature builds on Goodwin's (1994) concept of "professional vision," defined as "ways of seeing and understanding events" that are distinct to a social or professional group (p. 606). In their study of noticing mathematical thinking, Jacobs and colleagues (2010) conceptualized noticing as three interrelated cognitive processes: attending to details of students' strategies, interpreting students' understandings, and planning to respond based on those understandings. The authors found that teachers improved their noticing through professional development in which they analyzed video and written work from their own students and anonymous students. The teachers attended to more details of a student's thinking and used robust evidence to interpret their understanding and to plan to respond (Jacobs et al., 2010). Similarly, Sherin and van Es (2005) found that teacher participants in video clubs shifted from an evaluative to an interpretive stance. Goldsmith and Seago (2011) found that teachers engaging in professional learning with video or work attended more deeply to mathematical details, used evidence from artifacts to support claims, and noticed more potential in students' thinking.

Student Work Analysis & Noticing of Children's Thinking

Other researchers have examined the affordances of student work analysis without using the noticing construct. In their study of a workgroup in which teachers collectively analyzed their own students' work, Kazemi and Franke (2004) found that teachers learned to attend more closely to their students' thinking, becoming more detailed in their descriptions of students' strategies, developing an appreciation for students' unique mathematical ideas, and finding ways to elicit and build on students' thinking. Additionally, researchers have found that student work can serve as a resource to deepen teachers' knowledge around student thinking and to strengthen instruction (Ball & Cohen, 1999). Despite these affordances, researchers have identified constraints in looking at student work. Goldsmith and Seago (2011) found that when looking at anonymous work, teachers attended closely to students' reasoning and remained open to multiple possible interpretations, but they focused more on lesson details rather than the specifics of thinking when looking at their own students' work. Accordingly, researchers advocate for an inquiry-based approach to looking at work to support teachers to focus on the details of thinking and to carefully draw on their knowledge of students and context (Little et al, 2003).

Equitable Mathematics Pedagogy & Noticing for Equity

The teacher noticing literature has focused on noticing as a cognitive process; however, Louie (2018) argues that scholars ignored two aspects of Goodwin's professional vision: noticing is culturally situated and is not politically neutral. Erickson (2011) identifies a teacher's "pedagogical commitments" as the tacit and explicit ontological assumptions about teaching and learning that shape noticing. Hand (2012) posits that what teachers notice is informed by their "dispositions," which are the perspectives they have developed through both their teaching experiences and life experiences. Building on Hand, Louie (2018) argues that teachers' immersion in dominant ideologies shapes what and how they notice. The dominant mode of instruction in math classrooms is knowledge transmission from teacher to student (Boaler, 2016), which corresponds to similarly narrow definitions of ability (Louie, 2017). Narrow notions of ability are not applied equally, as racialized discourses that position white and Asian students above Black and Latinx students persist in and beyond classrooms (Shah, 2017). Teachers, particularly white teachers, are immersed in these ideologies, which may shape their noticing.

Emerging work on "noticing for equity" considers these cultural dimensions of noticing. R. Gutiérrez (2007) argues that achieving equity means no longer being able to predict, based on group membership, students' achievement, participation, and ability to mathematically critique the world. One line of inquiry on equitable pedagogy works to expand conceptions of mathematical activity and ability (Louie, 2017) through practices like Complex Instruction—a form of groupwork that combines multi-dimensional content with attention to status (Cohen & Lotan, 2014), including teachers noticing students' strengths (Jilk, 2016). Building on this work, van Es and colleagues (2017) define "noticing for equity": "How mathematics teachers notice aspects of classroom activity that have consequences for whether or not particular groups of students feel more or less empowered to take up these practices [i.e. engagement in mathematical reasoning]" (p. 252). In their study, the authors found that teachers' equitable instructional practices were connected to particular forms of "noticing for equity" around participation, such as attending to issues of status and positioning, attending to individual students' histories to inform interactions, and attending to the energy and flow of the class (van Es et al., 2017). Similarly, Wager (2014) found that teachers' positionality toward equitable pedagogy was connected to noticing of participation. These studies examined noticing of participation in-the-moment or in video; less is known about equitable noticing of thinking in students' work.

Methods

Study Context & Participant

I identified Ms. D[1] as "equity-oriented" due to her pedagogical commitments (Erickson, 2011), her implementation of equitable math pedagogies, and her dispositions (Hand, 2012) toward pushing back on dominant discourses about mathematics for Black and Latinx students. Ms. D, who identities as a white woman, is National Board Certified and has taught at public and charter schools in California for the past 23 years. Originally trained as a science educator, she became fascinated with her students' thinking when she began teaching middle school mathematics 20 years ago and has since engaged in significant professional learning around listening to and learning from students' thinking. While Ms. D is not representative of the broader teaching force, she does represent, as Shulman (1983) writes, "images of the possible."

Ms. D's fascination with students' thinking and commitment to equity are integrated into her teaching philosophy. She believes it is her responsibility to "create the conditions that promote equity," defining equity as all students having access to multiple approaches to mathematical content, different ways of participating, a supportive relationship with their teacher, and collaborative relationships with classmates (Interview 1). This resonates with Ms. D's philosophies: 1.) mathematics consists of different ways of thinking; 2.) people learn through participation and interaction, and 3.) building relationships with and among students helps establish a learning community in which students' identities within and beyond the classroom are acknowledged. Relatedly, Ms. D aims to address power in her classroom, implementing Complex Instruction (CI) and working to center her Black and Latinx students' voices.

Data Collection

I used a case study design (Yin, 2009) and grounded theory analysis (Strauss & Corbin, 1994) to illustrate one teacher's practice of looking at student work. Data was collected during the 2019-2020 school year, which was Ms. D's first year at a public middle school, where she teaches sixth grade mathematics. The district in which she teaches is 8% Asian, 24% African American, 32% White, 21% Latinx, and 15% multi-ethnic/other. As a result of COVID-19 and the shift to remote learning, the majority of data collection took place over Zoom. Because this study centers around student work analysis, the primary data source consisted of three think-alouds, in which Ms. D made sense of her students' work, each followed by a short interview protocol. Students' work samples consisted of individually completed "Cool Downs" (i.e. exit tickets), which prompted students to represent their thinking in multiple ways. For the first think-aloud, which was video recorded in person, work samples came from the 20 students who were present and whose families had consented. For the second and third think-alouds, which were recorded via Zoom, work samples came from 19 (think-aloud 2) and 16 (think-aloud 3) of the 20.

Secondary data sources were used to contextualize Ms. D's noticing, given that teachers' philosophies and dispositions may shape their noticing (Erickson, 2011; Hand, 2012). Prior to the think-alouds, a semi-structured interview (Glesne, 2005) was conducted to gather information about Ms. D's context, philosophies, conceptions of equity, and experience with equitable pedagogies. The original design involved observing Ms. D's class the day of each think-aloud. One observation was conducted on the day of think aloud 1, during which fieldnotes were generated (Emerson et al., 2011). Due to the shift to online learning, no additional observations were feasible. Instead, Ms. D's weekly digital materials were consulted as artifacts.

Data Analysis

In the first phase of analysis, I identified Ms. D's pedagogical commitments. I engaged in line-by-line open coding of the interview transcript, from which bottom-up codes of Ms. D's pedagogical commitments and conceptions of equity emerged (Emerson et al., 2011). I refined these codes through visual diagraming and coding of observation fieldnotes, constructing a pedagogical commitments codebook which I then used to focus code the transcript.

In the second phase, I analyzed how Ms. D made sense of work in the think-alouds. I constructed time-indexed content logs (Derry et al., 2007) of the recordings and transcribed dialogue and movement of work. I broke the transcript into idea segments—separated by a change in an idea or turn—which served as the unit of analysis for three rounds of coding: open coding; a priori coding using Jacobs and colleagues' (2010) noticing framework; and focused coding for connections to equitable math pedagogy literature. I constructed a codebook based on commonalities across the rounds and used it for a fourth round of coding. To generate themes, I wrote analytic memos and constructed diagrams of Ms. D's think-aloud process and relationships among codes. I then mapped each theme back to its related codes and excerpts, confirming that each theme was supported by evidence from at least two think-alouds. Finally, I looked for counterexamples of themes, expanding one theme to account for its complexity.

Findings

My analysis showed that Ms. D noticed students' mathematical thinking in ways that potentially promote equity. Although she engaged in the three cognitive processes of the Jacobs et al. (2010) framework, she did so through the lens of her pedagogical commitments. As she attended to the details of students' strategies, she maintained a learner stance, acknowledging her uncertainty with their thinking. As she interpreted understanding, she contextualized it within the learning environment, drawing on her knowledge of students and critically reflecting on the opportunities to learn that she had provided. These noticings supported Ms. D to engage in a strengths-based interpretation of students' understandings, recognizing strengths and partial understandings. Finally, her expansive definition of mathematical understanding supported Ms. D to notice aspects of students' work to share with the class as part of her plans to respond.

As has been found in studies of noticing thinking, Ms. D engaged in three intertwined cognitive processes as she looked at students' work: she attended to the details of students' strategies, interpreted their understanding, and made plans to respond (Jacobs et al., 2010). She engaged in these three processes in all think-alouds and in at least two of the three for every piece of work. For example, when looking at student G's work in think-aloud 1, Ms. D described G's thinking in detail, noting how she broke the 12 apart, recognizing her expression as equivalent, and wondering if she meant 16 instead of 12 (figure 2). As she attended to these details, Ms. D interspersed interpretations of G's understanding, determining that she understands how to write an expression with parenthesis and how an expression connects to a rectangle's area. Finally, Ms. D identified areas of growth (e.g. understanding partial shading of a rectangle). This example reflects a pattern across think-alouds: Ms. D attended to the details of a student's work, moved between interpreting and attending, and then made a plan to respond.

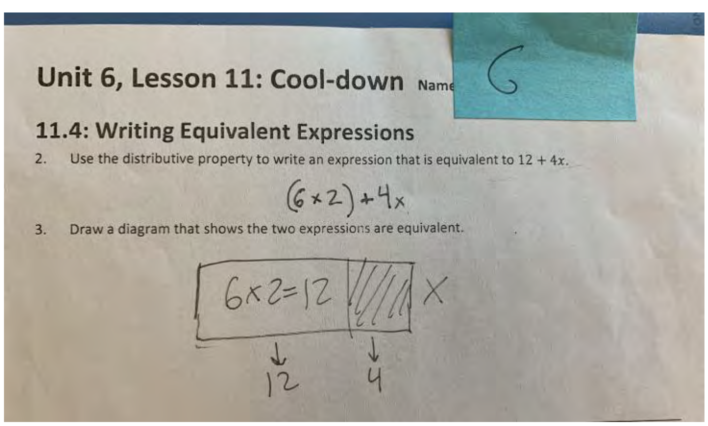

Figure 2: Photo of G's work

Attending to students' strategies: Maintaining a learner stance

As Ms. D attended to the details of students' strategies, she did so with a learner stance, in which she attempted to deeply understand students' thinking, expressed fascination with it, and acknowledged her uncertainty around the particulars of students' strategies. In the third think-aloud, Ms. D spent four minutes attempting to decipher how one student may have found the area of a parallelogram, testing out multiple possible strategies. She was fascinated by his work regardless of its accuracy, which was a trend throughout the think-alouds and which aligned with her assumption that all work showcased deep thinking and was worth paying attention to. It was common for Ms. D to comment that a student was "thinking about something," even if she wasn't clear on what that something was. This comfort with uncertainty, rather than seeking resolution, shaped Ms. D's process of attending to thinking. She consistently acknowledged her own uncertainty around a student's strategy, sometimes phrasing it as a question she planned to ask the student, such as, "but what does she mean by height and base?" (Think-aloud 3). Ms. D interacted with the work as a learner, naming uncertainties and framing them as wonderings.

Attending & interpreting: Contextualizing students' thinking

As Ms. D moved from (and between) attending to the details of students' strategies and interpreting which aspects of the concept they understood, she contextualized students' thinking in two ways. To make sense of their thinking and interpret their progress, she drew on her knowledge of her students as people and as mathematical thinkers. At times, she referenced a student's prior mathematical thinking, such as their facility with mathematical vocabulary or their mastery of particular strategies. For example, in think-aloud 1, Ms. D drew on her knowledge of student J's strengths (i.e. mastery of using tape diagrams to represent equations) and her areas of growth (i.e. area) to interpret her understanding. Additionally, Ms. D drew on her knowledge of students as people to understand their progress. In think-alouds 2 and 3, Ms. D spoke about J's challenges with distance learning—feeling overwhelmed by technology and missing interaction—and celebrated her completion and understanding amidst these struggles.

Importantly, Ms. D drew not only on her knowledge of students to interpret their work but also on her role in shaping their opportunities to learn. When noticing a student's partial understanding, Ms. D critically reflected on the extent to which she had provided that student access to the learning opportunities necessary to develop that understanding. In think-aloud 2, for example, Ms. D noted that students' struggles with language precision were likely related to lack of discussion during distance learning. Rather than attributing these struggles to individual students, Ms. D situated them within the learning environment and her role as an educator. Additionally, Ms. D referred back to the directions she had written for each problem as she processed students' work. For example, she acknowledged that the term "diagram" is vague, that there doesn't have to be a question in a student's word problem, and that describing a strategy doesn't require a numerical answer. In all three cases, Ms. D's critique of her directions widened the space of understanding, allowing for different kinds of representations and strategies.

Interpreting understanding: Applying a strengths-based lens

Ms. D's learner stance on and contextualization of thinking comprised an expansive notion of mathematical understanding, supporting her to interpret students' understanding through a strengths-based lens. For each problem on the cool down, Ms. D attended to each student's work and sorted them into two piles: understanding and partial understanding. Although she sorted along this binary, the piles were fluid and did not correlate with categories of "right" or "wrong." Instead, Ms. D sought out evidence that students understood the concepts—even if they had a computational error—and sometimes moved students across piles based on evidence from a later problem. This fluid and conceptual sorting enabled Ms. D to recognize partial understandings of the learning objective in each piece of work, articulating what that student understood and what they did not understand yet. Additionally, Ms. D recognized strengths outside of the objective, such as writing an equivalent expression (even if doesn't use the distributive property), writing a numerical expression (even if it doesn't have a variable), and drawing a tape diagram (rather than a rectangular diagram). For this last strength, Ms. D's critical reflection on the directions supported her to recognize this student's tape diagram as a strength. Ms. D's recognition of partial understandings and strengths thus enabled her to notice a range of aspects of work.

Plan to respond: Identifying aspects of work to share with the class

Although Ms. D's plans to respond included many typical of formative assessment (e.g. feedback and small-group instruction), a portion of her plans involved using students' work as a tool for learning. As Ms. D recognized different ways of thinking and partial understandings, she identified aspects of students' work to share with the class. Her practice of noticing work to share took on two forms: highlighting exemplars and leveraging mistakes for class learning.

In think-aloud 3, Ms. D commented on many aspects of student thinking that she planned to "highlight" in class the following week. For example, Ms. D noticed and planned to share K's use of units and A's use of mathematical vocabulary. Both noticings were supported by Ms. D's contextualization of students' thinking. For K, Ms. D recognized his precision with units, even though that wasn't part of the objective. For A, Ms. D critically reflected on her directions, noting that A did not need an exact answer. Additionally, Ms. D noticed and planned to highlight different ways of thinking. For example, Ms. D recognized two students' creative ways of finding a parallelogram's area: T cut a parallelogram in half and Si chose a base from which to draw a height. In planning to highlight T's and Si's work, Ms. D affirmed the use of approaches that differed from most of the class. Ms. D's practice of highlighting student work was observed prior to school closure. During the observation, eight students shared aspects of their cool down, ranging from using arrows to communicate thinking, substituting to test out possibilities, and incorporating vocabulary. When selecting work to highlight, Ms. D not only considered students' approaches, but also their identities within and beyond the classroom. In interviews, Ms. D expressed a desire to "elevate students' status" in the classroom, particularly her female students of color and her quieter students, as she was aware that students' identities may relate to their perceived status. The eight students who shared their work, for example, came from a range of identities. In recognizing her students as multi-layered people with different personalities and backgrounds, Ms. D worked to elevate those who may have less power in class or in the world.

In addition to noticing aspects of work to highlight, Ms. D also planned to leverage student work as a learning opportunity for the class. While highlighting involved students sharing exemplar aspects of their work, leveraging involved Ms. D organizing an instructional activity around student work that showed partial understanding. In think-aloud 2, for example, Ms. D planned to respond to students' word problems by repurposing them for the class's learning, saying: "Already as I'm reading these, I'm excited to use these as an assignment for next week, which ones make sense, which ones don't, and have the kids try to see if they can come up with an expression [for each]" (Think-aloud 2). Ms. D aimed to leverage students' work as a tool for learning, positioning them as mathematical thinkers whose ideas are worthy of discussion.

Taken together, Ms. D's approach to looking at work with a learner stance, a critical lens on context, and a strengths-based interpretation supported her to identify aspects of work to plan to share with the class (figure 3). This plan to respond by leveraging and highlighting students' work has the potential to expand students' ideas about mathematics and about each other.

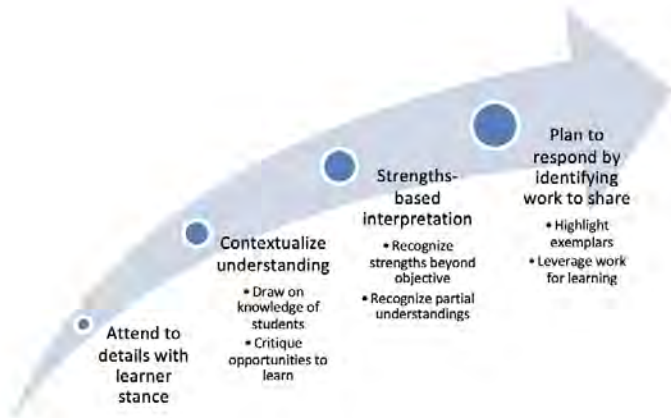

Figure 3: Visual representation of the major features of Ms. D's noticing process

Discussion and Implications

Ms. D's case illustrates that there are ways of attending, interpreting, and planning to respond to work that potentially promote equity. This case also shows that pedagogical commitments shape noticing, as Ms. D simultaneously utilized her expansive notions of mathematical understanding, as well as her ideas about status and positioning. This intertwining of multiple commitments supported her to a.) notice a range of thinking and b.) plan to highlight the work of students who may have been perceived as low-status in the classroom, the world, or both.

Ms. D's learner stance, her critical lens on context, and her commitments to expansive notions of mathematics supported her to notice a range of ways of thinking. Ms. D's inquiry lens resonates with studies showing this approach yields deep understanding of thinking, which can inform instruction and contribute to teacher learning (Kazemi & Franke, 2004). Despite being immersed in deficit discourses about students (Louie, 2018), Ms. D's commitments and her professional learning may have supported her to notice diverse strengths. The widening of understanding is important to equitable pedagogy, as math is accessible to more students when it is represented in multiple forms and different ways of thinking are valued (Boaler, 2016).

Noticing a range of thinking enabled Ms. D to plan to highlight multiple work samples, potentially expanding who is seen as competent. Ms. D's highlighting practice resonates with the CI routine of "assigning competence" (Cohen & Lotan, 2014), in which teachers position low-status students as competent. Importantly, Ms. D also attended to students' race and gender identities, which she believed may intersect with their perceived status. Put another way, Ms. D saw students' "social identities" in the world as potentially connected to the "practice-linked identities" they developed in the classroom (Esmonde & Langer-Osuna, 2013, p. 1). This practice of positioning students with attention to status and identity suggests teachers can "notice for equity" when looking at work, which resonates with van Es and colleagues' (2017) findings that attending to status and positioning was embedded in equitable noticing of participation.

This study has implications for practice and research. Teachers may take up these ways of attending and interpreting when looking at work to expand students' ideas about math and each other. Teacher educators may support novice teachers with attending to and selecting work outside of class as an entry point to implementing Smith and Stein's (2011) five practices. Future professional learning may support teachers to consider students' mathematical ideas and their statuses/identities when selecting their students' work. Finally, future research on student thinking might theorize the ways in which equity-oriented teachers simultaneously take into account the mathematical significance of students' ideas and students' statuses and identities.

Note

[1] All teacher and student names are pseudonyms.

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MOVES TEACHERS USE TO RESPOND TO STUDENTS' NON-CANONICAL APPROACHES FOR SOLVING EQUATIONS

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A historical review of mathematics textbooks suggests a canonical method to solving equations that teachers often see as "the" way to solve equations. In this paper, we examine data from a nationally-distributed sample of 524 secondary mathematics teachers who responded to scenario-based survey items that represent the instructional situation of solving equations. The items featured scenarios in which students presented non-canonical solution methods and asked participants to share how they would respond. Using a framework that draws on systemic functional linguistics, we describe the linguistic resources teachers used. While closed moves are frequently used to avoid discussion of non-canonical solutions, our results suggest that teachers find ways to make regular use of: (1) closed moves for accommodating non-canonical solutions and (2) open moves when steering the conversation back to the canonical method.

Keywords: Algebra and Algebraic Thinking, Classroom Discourse, Research Methods

Background and Framework

While policy documents have been crafted to provide numerous visions for mathematics instruction in the U.S. (NCTM, 1991, 2014)—such visions have yet to become a regular state of affairs in actual classrooms. This is nowhere less true than teachers' instructional practices of responding to students' mathematical contributions (Milewski & Strickland, 2016) where teachers tend to be overly evaluative and propagate standard teaching routines—praising only those contributions that correctly carry out previously-demonstrated procedures while dismissing contributions that do not use expected methods even if they present correct solutions (Ball, 1997; Crespo, 2002). Furthermore, when teachers demonstrate a stalwart commitment to a single procedure, they cue students to learn rotely—undermining the development of conceptual understanding and flexible thinking (Hiebert & Carpenter, 1992).

In the case of solving equations in Algebra 1, a historical review of the mathematics textbooks suggests a long-standing canonical method (Buchbinder et al., 2015) that teachers expect students to use to solve equations (Buchbinder et al., 2019a). This method has been described by scholars as containing the following steps: (1) use the distributive property to clear out grouping symbols (when applicable), (2) simplify expressions on each side of the equation, (3) use the addition and subtraction properties of equality to isolate the variable from the constants, and (4) use the multiplication and division properties of equality to solve for the unknown variable (Buchbinder et al., 2015; Star & Seifert, 2006).

While many teachers prefer to spend class time on the canonical method (Buchbinder et al., 2019a), they sometimes have to make on-the-spot decisions about how to handle non-canonical solutions offered by students (Mason, 2015; Schoenfeld, 2008). This study investigates the linguistic resources teachers use when responding to non-canonical solutions in the instructional situation of solving equations: including those responses that manage to make use of students' alternative contributions as well as those that do not. In this paper, we examine data collected from a nationally-distributed sample of 524 secondary mathematics teachers who responded to a set of scenario-based survey items that each featured an embedded, rich-media representation of the instructional situation of solving equations (Chazan & Lueke, 2009). Within these items, teachers were asked to share how they would respond to scenarios in which a student presents a non-canonical solution for an equation on the board if such a situation would occur in their class.

Theoretical Framework

While teachers' instructional decisions are commonly modeled as expressions of individual characteristics, such as a teacher's resources, orientations, and goals (Schoenfeld, 2010), other factors need to be taken into consideration. Phenomena such as cultural scripts (Hiebert & Stigler, 2000) and lesson signatures (Givvin et al., 2005) provide evidence that the norms of teaching can be distinguished across cultural lines, which suggests that teaching is as much a socially-shaped activity as it is individual.

The theory of practical rationality (Herbst & Chazan, 2012) accounts for teachers' decision making using both individual and social resources. It does this using the two primary building blocks of (1) Brousseau's (1997) notion of didactical contract, and (2) Herbst's (2006) notion of instructional situation. Brousseau's concept of didactical contract identifies relationships between the teacher, their students, and the content in ways that tacitly regulate the ways that the teacher and students are expected to act within instructional exchanges (Herbst, 2003). Author's notion of instructional situation takes note of the way the didactical contract is shaped within the set of recurring situations within a course of study. For example, the theory posits the set of norms for solving equations in algebra differs from the set of norms for doing proofs in geometry and these differences impact both the teachers' and students' understanding of what kind of work is necessary for the teacher to claim the student has learnt what is expected of them (Herbst, 2006; Herbst & Chazan, 2012). In this way, the normative and routine nature of these instructional situations create a stable social resource that can be used by teachers and students to know how to act within a given situation.

In the case of the instructional situation of solving equations, the canonical method represents or activates the norms of the situation (Buchbinder et al., 2019a; Chazan & Lueke, 2009). To be clear, the norms of the situation are not deterministic, even for teachers with strong preferences for the canonical method. For example, when faced with the circumstance of having a shy student at the board presenting a non-canonical solution, a teacher who might normally feel quite strongly about adhering to the situational norms may respond in ways that accommodate the student's work to avoid embarrassing the student. Teachers have resources they can use to navigate such circumstances. For example, at least some portion of the reform literature has aimed to delineate specific linguistic resources teachers can use to shift their practices of responding to supporting students' mathematical contributions (e.g., O'Connor & Michaels, 2019).

Research Questions

In our prior work (Buchbinder et al., 2019b), we have shown that when confronted with non-canonical student solutions in the instructional situation of solving equations, teachers' responses can be parsed into one of three broad categories—those responses where the teacher: a) ***complies with the norm*** by finding a way to move quickly back to the canonical method , b) ***repairs the task*** by finding a way to make slight accommodations for a non-canonical solution, for example, by ensuring each step of the students' solution was justified before moving on, and, c) ***repairs the situation*** by making large accommodations for a non-canonical solution such as switching the focus of the lesson towards that solution. In this paper, we ask: what are the various linguistic resources teachers use to: (a) ***comply*** with the norms of that situation?, (b) ***repair the task***?, and (c) ***repair the situation***?

Methodology and Data Sources

Participants

Data used in this paper come from a nationally-distributed sample of 524 secondary mathematics teachers from 47 states who were invited by email and received an honorarium for participation. The sample of teachers included 59.6% female, 40.1% male, and 0.36% other or no answer; 83.58% White, 7.3% Black, 2% Hispanic, 2.8% Asian, 0.89% Other. The teachers had an average of 14.32 years of experience (*SD*=8.68) ranging from 1 to 40 years. Participants were invited to partake in a total of 27 open-ended scenario-based instruments—one of which, the Algebra-Equations Decision Instrument, we focus on here.

Instrument

As part of their participation in the Algebra-Equations Decision Instrument, each participant was provided with four rich-media, scenario-based items; each containing a classroom scenario that played out across several storyboard frames. Such multimedia representations have been found effective at gauging participant teachers' decision-making (Herbst & Chazan, 2015). Each scenario begins with a teacher posing a solving equations task and includes a moment in which a student is called to the board to share their work and the student subsequently describes a solution. In all cases, the students' solution was both mathematically correct and non-canonical.

For example, in one item, the teacher poses the problem 4x + 2 = 5x - 3, and a student volunteer approaches the board to share their solution where they *solve by graphing* (see Figure 1a). In another item, the teacher poses the problem 5(x + 2) = 56 - 2 (x +2), and a student shares a solution in which they *attended to structure* of the equation, meaning that the student solves by treating the term (x+2) as a quantity, instead of distributing first (see Figure 1b).

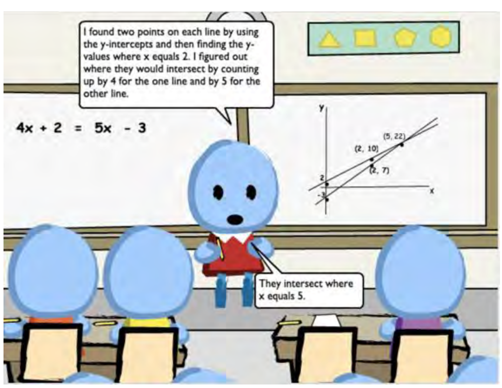

Figure 1a. **A frame from one of the items where a student elects to *solve by graphing***

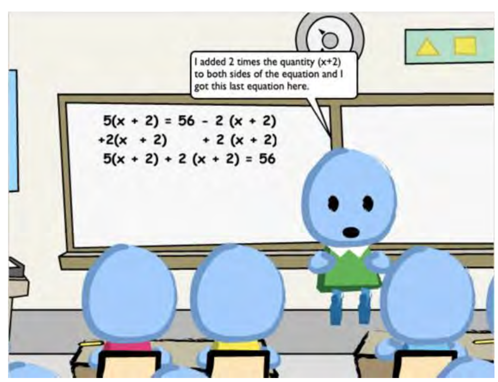

Figure 1b. **A frame from one of the items where a student elects to attend to structure.**

After viewing each scenario, study participants were asked to respond to the following open-ended prompt: "Please describe the action you would do next and your reasons for doing this action". Participants' open-ended responses are the focus of our analysis for this paper.

Data Corpus and Analytical Method

In total, the corpus contains 2,087 participant responses: some included a single "next action" (n=1,530), while others included a more detailed sequence of moves (n=463), or no action (n=94), i.e. restating of the scenario but not addressing the prompt. Among the single actions responses, some avoided addressing the students' solution (n=251) by naming some other action such as, "I would apologize to the class for my poor time management.". The present analysis focused on those responses that managed to address the students' solution with a single "next action" (n=1,279) and proceeded in two parallel phases. In phase one, we coded responses according to the degree the participant indicated they would direct the class towards the canonical method or towards the offered non-canonical solution provided by the student in each scenario: (a) ***comply*** with the norms of that situation?, (b) ***repair the task***?, and (c) ***repair the situation***?.

In phase two, we used a previously-established coding scheme that augments a framework developed by teachers, who were conducting action research, (Authors, 2020) with functional classifications drawn from the linguistic framework by Eggins and Slade (2005). The Eggins and Slade framework comprises two functional systems of choice which organize responding moves according to how they shape the discourse. The first functional system of choice (***open/close)*** distinguishes between moves that prolong or curtail the discussion of the prior contribution; the second distinguishes between moves that demonstrate a willingness to accept the contribution (***support***, ***confront***) or defers responsibility for responding by asking other students to react to the contribution (***invite***).

Altogether, the combination of these systems of choice produce the following six codes for actual utterances: curtail the interaction by supporting the student contribution (***close-support***), curtail the interaction by confronting the contribution (***close-confront***), defer responsibility for responding by suggesting other students curtail the interaction (***close-invite***), extend the interaction by supporting the contribution (***open-support***) extend the interaction by confronting the contribution (***open-confront***) and defer responsibility for responding by suggesting other students prolong the interaction (***open-invite***). Details about the first and second phases of the coding can be found in our earlier work (Buchbinder et al., 2019b & Milewski & Strickland, 2020), but will also be illustrated with examples in the results section. After both phases of coding were complete, we examined patterns in the frequency of overlap of codes to help answer the research questions.

Analysis and Results

From the 1,279 responses we coded, 599 (47%) contained descriptions of actions that ***comply with the norms*** of the situation—finding ways to move quickly back to the canonical method. Of these 599 responses, the majority (n=404, 67%) represent actions that could be coded as ***close-confront***. Some of these *close-confront* responses took on the form of telling (e.g., *I would work through it using another method that is more routine*) while others took the form of a negative evaluation (e.g., *Since the bell rang I would make a note to bring up the same problem next class period and start off by solving it the right way -- meaning the way the students were used to*). Still others took a softer form, soliciting the class for a different solution (e.g., *I would ask if anyone in the class solved the problem a different way so that we could discuss the more* *traditional method).* Of course, ***close-confront*** moves are not the only way that teachers can manage to comply with the norm (see Table 1).

Table 1. **Examples of responses distinct from *closed-confront* that teachers used to *comply with the situation***

Linguistic Code	Participant Response Example	% of responses
Close-Support	*I would explain while that works there's a much simpler way to solve the equation.*	*17%*
Close-Invite	*Have someone else share their method and show how it shows the same thing as what orange just did*	*5%*
Open-Support	*I would ask [the student]: 'why did you not divide (x+5) by 9 also on the right side?' ((common mistake))…*	*5%*

From the 1,279 total responses, 430 (34%) contained descriptions of actions that represent mild breaches of the norms of the instructional situation (repair the task)—providing some slight accommodations for the student's non-canonical solution. Nearly a third of those 430 responses (n=156, 36%) fit into the linguistic category of open-support. Some represented the teacher asking the student to clarify or justify aspects of the student's non-canonical solution (e.g., *Have the student explaining reiterate the step and make sure the class understands*) while others represented the teacher resolving the uncertainty in the room by some reassurance about the mathematical appropriateness of the method (e.g., *I would explain that as long as the same action is performed to each side of the equation that method is valid*). That said, teachers sometimes found other ways, beyond open-support moves, to repair the task (see Table 2).

Table 2. **Examples of responses distinct from *open-support* that teachers used to *repair the task***

Linguistic Function	Participant Response Example	% of responses
Open-Invite	*It's not clear what 'dividing everything by 9' means so prompt students to ask questions of the student.*	*27%*
Close-Support	*Go over how each term changes when you divide it by 9.*	*22%*
Close-Invite	*I would ask the students for homework to write down whether or not they thought the solution on the board was correct and if they could get the same solution algebraically.*	*7%*

From the 1,279 responses, the remaining 250 (20%) responses contained descriptions of actions that implied breaches of the instructional situation (repair the situation)—making large accommodations for the student's non-canonical solution. Nearly half (n=135, 54%) of those 250 responses were coded as close-invite. Some of these responses represented the teacher asking other students to evaluate the contribution (e.g., *I would ask the students to discuss at their tables what was on the board and see if they agree or disagree with what is on the board*) while others represented the teacher requesting other students or the class take up the strategy on another problem (e.g., *I would give them another problem similar to the one [that student] did and see if they can duplicate the process*). Again, not all of the responses describing actions that breach the situation were categorized as close-invite (see Table 3).

Table 3.* Examples of responses distinct from *close-invite* that teachers used to *repair the situation

Linguistic Function	Participant Response Example	% of responses
Open-Invite	*I would have students discuss in pairs what they think Blue did.*	*20%*
Close-Support	*I would answer the students questions about why certain procedures were done in the problem.*	*14%*
Open-Support	*I would ask the student (with help from the class) to justify using mathematical properties or concepts each step.*	*7%*

In this section, we have shown that the modal teacher response to students' non-canonical solutions ***comply with norms*** of the situation (47%) and the preponderance of those responses take up the form of moves that could be coded as ***close-confront*** (67%). We have also shown that teachers sometimes elect to make small accommodations for students' non-canonical solutions (***repair the task***, 34%), and when they manage to do so they tend to use moves that were coded as ***open-support*** (36%). That said, nearly half of the responses that ***repaired the task*** were accomplished with moves that were coded as ***open-invite*** (27%) or ***close-support*** (22%). Finally, in 20% of the responses, we see teachers make sweeping accommodations for students' non-canonical solutions by ***repairing the situation***; and in the majority of those responses, teachers elected to use moves that could be coded as ***close-invite*** (54%).

Discussion, Conclusion and Significance

Despite reformers' calls for teachers to embrace the open discussion of multiple students' solutions, our research has reported that teachers favor canonical solution methods over non-canonical one. The theory of instructional situations and practical rationality has suggested teachers are often operating in contexts in which they feel responsible for maintaining the norms of the situation, which favors the canonical method. That said, we see in this data some promise in that a small majority of teachers' responses (54%) deviate from the norms of the situation by making some kind of accommodations for students' non-canonical methods. Yet, teachers' willingness to use ***open*** and/or ***supportive*** moves is mostly restricted to those instances when they are making only slight accommodations of students' non-canonical solutions (***repair the task***). In contrast, when a teacher takes the risk of making a significant accommodation for a students' non-canonical solution (***repair the situation***), they tend to use ***closed*** moves—albeit they often elect to use ***closed invitations.*** Yet even the invitational nature of these more-accommodating moves allow the teacher to maintain some semblance of control of the situation by sanctioning a narrow platform from which students can react to the non-canonical solution presented (e.g., requesting students evaluate, add on to, or replicate the method). These results support our prior hypotheses (Chazan & Lueke, 2009) that even when teachers are willing to engage with students' non-canonical solutions, there are important tensions in doing that.

While the analysis we have reported herein focuses on the response set as a whole, we have reason to believe that the breaches to the instructional situation represented across these items, i.e., the types of non-canonical student solutions, are not equivalent in terms of their likeliness to be perceived by teachers as reasonable approaches to take up in whole class discussion (Buchbinder et al., 2019a). Drawing from a recent use of the instrument administered to a set of secondary teachers prior to their involvement in professional development focused on facilitating whole class discussion, we have noticed that when aggregating teachers' responses according to item, some items (such as the type of solution featured in Figure 1b) seemed to also have greater numbers of closed responding moves than others (such as the type of solution featured in Figure 1a). Further, such items also contained more comments like the following, in which teachers remark on the represented method in ways that suggest they have concerns about it.

> The approach [the student represented in Figure 1b] took may be a bit confusing for students (such as [those who used] order of operations) and may lead to more anxiety and apprehension ... I think [the teacher] did a nice job hearing [the student] out, but should also show the [order of operation] approach ...and see if that helps to clarify some confusion.

To further explore teachers' rationality about particular kinds of non-canonical work, future work could interrogate patterns that exist when looking across teachers' responses to different items.

In closing, one of the primary ways that reformers have sought to further teachers' openness towards student-generated solutions is by suggesting alternative discursive moves that encourage teachers to use more open or invitational responding moves. The results from the analysis of the second and third parts of the research question cast some suspicion on the efficacy of such prescriptions. These results suggest that teachers can and do find ways to make regular use of ***closed*** moves to make accommodations for the students' non-canonical solutions (***repair the situation***)—in which they, in some serious way, take the risk of abandoning the canonical solution method. These results also suggest that teachers make regular use of ***open*** moves to ***repair the task***—steering the conversation back to the canonical solution method. These findings are reminiscent of earlier work in the field that looked critically at reform recommendations (Chazan & Ball, 1999; Cohen, 1990). In closing, we suggest that more work is needed to understand teachers' practical rationality in order to better understand which suggestions teachers may be more inclined to take up.

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CLASSROOM SUPPORTS FOR GENERALIZING

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Generalization is a critical component of mathematics learning, but it can be challenging to foster generalization in classroom settings. Teachers need access to better tools and resources to teach for generalization, including an understanding of what tasks and pedagogical moves are most effective. This study identifies the types of instruction, student engagement, and enacted tasks that support generalizing in the classroom. We identified three categories of Classroom Supports for Generalizing (CSGs): Interactional Moves, Structuring Actions, and Instructional Routines. The three categories operate at different levels to show how teachers, students, tasks, and artifacts work in interaction to mutually support classroom generalizing.

Keywords: algebra and Algebraic Thinking, Classroom Discourse, Instructional Activities and Practices

Understanding Classroom Generalization

Generalization is a central component of mathematical learning, with researchers arguing that it serves as the origin of mathematical ideas (Vygotsky, 1986; Peirce, 1902). The importance of generalization is reflected in national standards documents across North America (Council of Chief State School Officers, 2010; Ontario, 2005; Secretaría de Educación Pública, 2017), as well as in curricular materials (e.g., Hirsch et al., 2007). However, research shows pervasive student difficulties in creating and understanding correct general statements (e.g., adez & Kolar, 2015; English & Warren, 1995), creating further challenges in fostering success in many domains, including function, geometry, and combinatorics (e.g., Ellis & Grinstead, 2008; Pytlak, 2015; Lockwood & Reed, 2016).

Although students' challenges with generalizing is well documented, less is known about how to better support generalization, particularly in classroom settings. The majority of research on generalizing has occurred in laboratory settings, such as clinical interviews and small-scale, researcher-led teaching experiments. The field knows less about how productive generalization occurs in school settings with practicing teachers teaching everyday topics. Furthermore, the limited research on teachers' abilities to foster generalization shows that effectively supporting generalization is challenging for teachers (e.g., Callejo & Zapatera, 2017; Mouhayar & Jurdack, 2012). Teachers need support in learning how to help students generalize, including increased access to research-based tools and resources that build on the field's knowledge of students' productive generalizing. In response to these needs, this paper investigates the state of student generalizing in middle-school and secondary classrooms. In particular, we addressed the following questions: What are the opportunities for generalizing in classroom settings? Specifically, what types of instructional moves, student engagement, and enacted tasks support classroom generalizing?

Literature Review and Theoretical Framework

Researchers have identified both cognitive activities and pedagogical strategies that can foster generalization. The cognitive activities include visualizing properties beyond what is perceptually available (Becker & Rivera, 2007; Yeap & Kaur, 2008), attending to particular characteristics or relationships above others (Rivera & Becker, 2007), identifying relationships between tasks, representations, or properties (Cooper & Warren, 2008; Johanning, 2004), and describing general relationships or processes verbally or in written form (Ellis, 2007; Rivera & Becker, 2008). Research on pedagogical strategies has identified potentially productive moves to foster generalization, which includes having students consider big numbers (Zazkis et. al, 2008), showing variation across tasks (Mason, 1996), guiding students to reflect on their mathematical operations (Doerfler, 2008; Ellis, 2007), providing access to physical or visual representations (Amit & Neria, 2008), emphasizing similarity across contexts (Radford, 2008), and ordering tasks in a progressive sequence (Ellis, 2011; Steele & Johanning, 2004).

There are two caveats to consider in relation to the above findings concerning teachers supporting generalizing. The first is that the bulk of these studies were conducted in small-scale laboratory teaching settings, and the degree to which their findings might translate to whole-classroom activity is not well understood. A couple of studies, however, did detail the classroom factors influencing how middle-school students engaged with a generalization problem (Jurow, 2004; Koellner et al., 2008). For instance, Koellner and colleagues found that working with an open-ended problem with multiple entry points, having opportunities to visualize a concrete representation, and being able to work collaboratively fostered students' generalizing, along with the teacher's discursive moves of pushing for algebraic generalizations without supplying answers. The second caveat is that although many of the above studies have addressed specific instructional moves, fewer have explicitly addressed to the role that interaction can play in fostering generalizing. There are two notable exceptions. Ellis (2011) identified a number of generalizing-promoting actions representing how teachers and students can interact to foster generalizing, including publicly generalizing, encouraging justification, building on ideas, and focusing attention on mathematical relationships. This study, however, was situated in a teaching experiment setting rather than a classroom setting. In a classroom-based study, Jurow (2004) introduced the notion of participation frameworks to account for how students generalized in small groups. Both studies suggest that generalizing can occur as a consequence of processes distributed across tasks, students, and tools.

Defining and Situating Generalizing

While definitions of generalization vary, most characterize it as a claim that some property holds for a set of mathematical objects or conditions larger than the set of original cases (Carraher et al., 2008). For instance, Radford (2006) described generalizing as identifying a commonality based on particulars and then extending it to all terms, and Harel and Tall (1991) characterized generalization as the process of applying a given argument to a broader context. These definitions situate generalization as an individual, cognitive construct, but as seen with Jurow's (2004) work, one can also consider generalizing as a collective act distributed across multiple agents (Ellis, 2011; Tuomi-Gröhn & Engeström, 2003). This perspective attends to how social interactions, tools, and classroom environments can shape students' generalizing actions, positioning generalization as a fundamentally social practice. We follow this perspective to define generalizing as an activity in which learners in specific sociomathematical contexts engage in at least one of the following actions: (a) identifying commonality across cases, or (b) extending reasoning beyond the range in which it originated (Ellis, 2011).

We use the symbolic interactionist perspective, considering classroom learning to be a social process that occurs in interaction (Bauersfeld, 1995). From this perspective, learning is examined through the lens of multiple processes of interactions, in which students' interactions with tasks, artifacts, one another, and with their teacher all co-contribute to the activity of generalizing. This can occur through conversation, shared problem-solving activity, and negotiated meaning of problems and solutions. We view the learning environment as a system, made up of mutually interacting agents, and then consider how that system supports students' shared construction of meaning as they generalize. Reflecting the foci of our research questions, the symbolic interactionist perspective enables us to privilege both individual students' reasoning and the processes of interaction that supported that reasoning (Blumer, 1969; Voigt, 1995).

Methods

We conducted a series of classroom observations in one middle-school and two high-school classrooms. Prior to scheduling the observations, we asked each teacher to choose a unit that they thought would offer opportunities to observe generalizing. Mr. J was a third-year teacher who taught advanced algebra and precalculus, Ms. R was a sixth-year teacher who taught high-school algebra, and Ms. N was a third-year teacher who taught sixth-grade mathematics. In each classroom we conducted videoed observations with two cameras. One camera focused on the teacher and whole-class setting, and the other recorded a focus group of three to four students, capturing the entirety of their engagement including conversations, gestures, and written work.

In Mr. J's tenth-grade advanced algebra class we observed a three-day unit on exponents and roots, culminating in the development of the rule $\sqrt[b]{x^a} = \left(\sqrt[b]{x}\right)^a$. In Ms. R's ninth-grade algebra class we observed a four-day unit on using algebraic symbols and equation solving techniques to represent word problems. In Ms. N's sixth-grade class we observed a four-day unit on the coordinate plane, basic properties of quadrants, determining horizontal and vertical distances between points, and determining reflections over the *x*- and *y*-axes. We also interviewed each teacher twice after the observed units in order to explore their definitions of generalization, their beliefs about generalization, and their beliefs about how to foster generalization in the classroom. For the purposes of this paper, we draw specifically on the classroom observation data in order to determine student opportunities to generalize in classroom settings.

To analyze the data, we relied on both transcripts and video recordings, considering the participants' talk, gestures, intonations, and use of tools, drawings, and physical objects. We first coded all instances of generalization using Ellis et al.'s (2017) RFE Framework, and then turned to Ellis' (2011) categories of generalizing-promoting actions as an initial scheme to code instances of classroom interaction that supported the generalizations. In addition to using the generalizing-promoting actions categories, we revisited all classroom interactions to identify those that potentially contributed to the generalizations but were not captured by existing codes. We coded actions as fostering generalizing if generalizing occurred in direct response to an action, if a generalization mirrored or responded to a new idea introduced by an action, or if we could identify a conceptual chain linking the ideas or structure introduced by an action and a generalization that followed it. A number of interaction instances yielded novel codes, which contributed to the Classroom Supports for Generalizing (CSG) presented in this paper. Three members of the project team then independently re-coded every transcript, collaboratively resolving any discrepancies through consensus. Following the approaches others have used to investigate discourse (e.g., Pierson & Whitacre, 2010), the codes do not distinguish between teachers' and students' utterances. This is consistent with the interactionist framework, in which the students and teachers jointly contribute to a shared understanding (Cobb & Bauersfeld, 1995).

Results: Classroom Supports for Generalizing

We found three major CSG categories: (a) Interactional Moves, (b) Structuring Actions, and (c) Instructional Routines (Figure 1). Interactional Moves refer to the questions, initiations, responses, or ideas that people, task prompts, artifacts, or representations can introduce into the conversation. These moves are not limited to teacher moves; students can also initiate questions, share ideas or strategies, or encourage one another to generalize, justify, or share. In addition, specific task prompts or even one's use of a representation can constitute an Interactional Move, if they play an in-the-moment role of fostering generalizing during a classroom conversation.

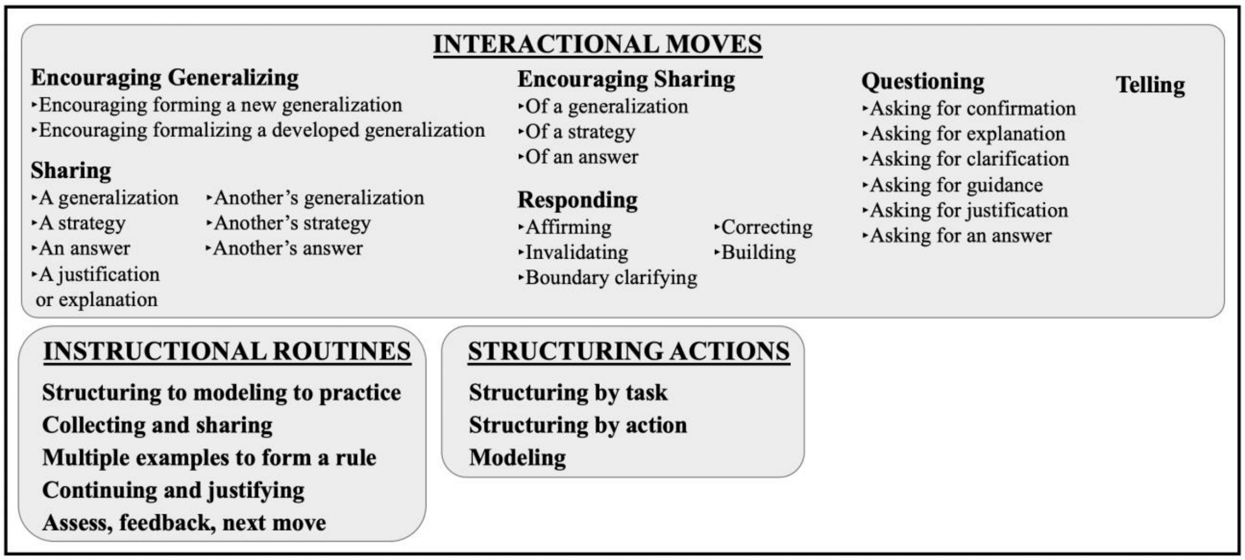

Figure 1: Interactional Moves, Instructional Routines, and Structuring Actions

In contrast to Interactional Moves, which are spontaneous and localized, Structuring Actions typically address the aspects of a teacher's instruction that are more systematic and intentional. They are the actions one employs to implicitly or explicitly structure students' activity in a manner designed to lead to a generalization. This can include developing and implementing task sequences with the aim of fostering a generalization, explicitly drawing students' attention to sameness across problem types or ideas, or choosing to organize a series of representations in a manner that highlights a generalizable feature. It can also include modeling the process of developing a generalization for other members of the community, an action that students may sometimes engage in as well as teachers.

The third category, Instructional Routines, depicts the patterned and recurrent ways that instruction unfolds in a classroom (Horn & Little, 2010). Following the work of those who have studied professional routines in teaching (e.g., Leinhardt et al., 1987; Rösken et al., 2008), we consider these routines to entail a stable schematic core with a more fluid shell, allowing for variable responses to demands of the moment. The Instructional Routines we identified were those stable, repeatable series of pedagogical moves that fostered student generalizing. These are processes such as collecting a range of student strategies to share for whole-class discussion and to serve as a source for forming a generalization (*collecting and sharing*), or visiting a small group, assessing their progress towards a generalization, providing feedback and guidance based on their progress, and then leaving them with a specific next step to achieve (*assess, feedback, next move*). Each of the routines we identified appeared repeatedly in one teacher's class but not in others', indicating that many routines may be somewhat teacher specific.

Developing a Generalization in Interaction: Horizontal Distance

Due to length constraints, rather than defining and discussing each CSG, we instead offer an extended data episode illustrating the manner in which multiple CSGs operate together in order to support the classroom development of a generalization. This episode draws from Ms. N's 6th-grade classroom and takes place during a lesson about the horizontal and vertical distance on a coordinate plane. The excerpt illustrates one of Ms. N's Instructional Routines, *multiple examples to form a rule*. In this routine, a teacher shares and discusses multiple examples of the same phenomenon, and then directs students to consider what remains invariant across the examples with the aim of developing a mathematical rule as an articulation of the invariance.

In launching the routine, Ms. N projected a coordinate plane on the board and placed a magnetic dart at the point (7, 5). She then asked a student to place a second dart a horizontal distance of 8 units from the first dart. The student placed the dart at the point (-1,5), and Ms. N encouraged the students to note the ordered pairs of the two points. She then repeated this process, placing a dart at (-1, 1) and asking a student to place a second dart at a horizontal distance of 3 units away. The student placed the dart at (-4, -1), and Ms. N again asked the students to attend to the ordered pairs of the two points. Ms. N then repeated this process a third time, placing the dart at (7, -4) and asking a student to place the second dart a horizontal distance of 10 units away. The student placed the dart at (-3, -4). At this point, Ms. N also engaged in the Structuring Action CSG of *structuring by action*: She wrote the three pairs of ordered pairs together on the board in a manner that made it visually salient that the *y*-values of each pair of ordered pairs was the same (Figure 2). The written representation itself played the role of *encouraging generalizing (forming)* by directing students' attention to the structure of each pair of points.

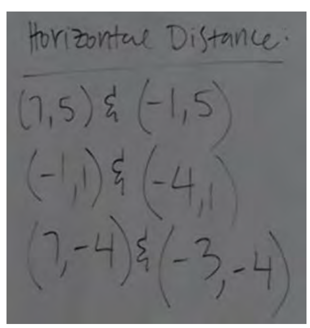

Figure 2: Ms. N's Representation of Three Pairs of Ordered Pairs

In the following table (Table 1), we provide each classroom member's utterance with the accompanying CSG it represents. The excerpt begins with Ms. N explicitly asking the students what the ordered pairs have in common:

Table 1: First Excerpt Utterances and CSGs

Utterance	CSG
Ms. N: Who can tell me what looked at these two ordered pairs to start [points to the first pair]. What do they have in common? What are these ordered pairs have in common?	Encouraging Generalizing (forming)
Ari: They both have the same *y*-axis coordinate?	Sharing (a generalization)
Ms. N: *y*-coordinate. Good. What is their *y*-coordinate?	Questioning (asking for answer)
Ari: Five.	Sharing (an answer)
Ms. N: Five. All right, what do these two points (points to the next pair) have in common? Rayna?	Encouraging Generalizing (forming)
Rayna: They have the same *y*-coordinate?	Sharing (a generalization)
Ms. N: What is the *y*-coordinate?	Questioning (asking for answer)
Rayna: One.	Sharing (an answer)
Ms. N: They both have a one in common in the *y*-coordinate place, and what do these two points have in common (points to the last pair)? Wesley.	Encouraging Generalizing (forming)
Wesley: They both have the same *y*-axis coordinate which is negative four.	Sharing (a generalization)
Ms. N: Perfect. So, what do they *not* have in common? What are they not sharing?	Encouraging Generalizing (forming)
Parker: *x*-coordinate.	Sharing (a generalization)
Ms. N: Their *x*-coordinates, right? So that is going to be a pattern that you will always notice whenever we are talking about horizontal distance between two points.	Sharing (a generalization)

Ms. N was *structuring by action* throughout the above exchange by explicitly drawing students' attention to sameness across the three pairs of ordered pairs. This occurred not only through the above exchange, by also by Ms. N's actions of finger pointing and underlining the *y*-coordinates of each ordered pair on the board. Those actions were to support the generalization that when determining a horizontal distance, each pair of points will have the same *y*-value. Ms. N then *encouraged generalizing* by asking the class, "Is it possible that I could look at these ordered pairs and without even plotting them, know the distance between them?" Jonah proposed the idea that you can simply take the sum of the absolute value of the *x*-values of each pair of points to find the difference:

Jonah: You just need to add them together. You can get how many things you go over. Because the top [pointing to (7, 5) and (-1, 5)] like if you, you add them together, but you get rid of the negative sign, it equals eight. Second [pointing to (-1, 1) and (-4, 1)] you move five.

Ms. N: Okay. So be careful with, with saying add them together. I think I know what you mean. But be careful with say add them.

With this proposal Jonah *shared* a generalization. He subsequently added that he meant the absolute value: "Absolute value. Just add them together." In response, Ms. N asked the students to consider the second case Jonah mentioned, with (-1, 1) and (-4, 1). In doing so, Ms. N engaged in a form of *responding* that was *boundary clarifying*: Her intent was to help the students determine when Jonah's generalization would work and when it would not. The students determined that it worked for the first and third pair, but not the middle pair of (-1, 1) and (-4, 1);

they concluded this by physically counting the number of units between the two points on the coordinate plane. In the next excerpt, the students and Ms. N together began with Jonah's incorrect generalization and transformed it into a correct one (Table 2):

Table 2: Second Excerpt Utterances and CSGs

Utterance	CSG
Ms. N: So what, what happened with your theory? I like the theory, it's almost there. But we need to tweak it a little bit going.	Encouraging Generalizing (forming)
Jonah: I think we are going negatives to positives. I think it only works with positive negative, positive positive.	Sharing (a generalization)
Ms. N: And try them if my two coordinates are not the same sign, you mean?	Questioning (asking for clarification)
Jonah: You change the negative, you just kind of do the opposite.	Sharing (a generalization)
Ms. N: Okay, cool, can be something to add to our rule.	Responding (affirming)
Riley: This one, like go, go ones that he's talking about adding. They start with the positive number. And when we, with this [(points to (-1, 1) and (-4, 1)], and it starts with negative number, you can subtract it from before, and equals three.	Responding (building)
Jonah: Yeah, that's what I mean by like negative, negative.	Responding (affirming)
Ms. N: Okay, so in general, what am I looking for? Absolute value is asking us for a, what do we say? What kind of measurement?	Questioning (asking for an answer)
Robin: Distance.	Sharing (an answer)
Ms. N: A distance. So in general, this is always going to be true. What am I looking for between the two points that aren't the same?	Questioning (asking for an answer)
Quinn: Positive number.	Sharing (an answer)
Ms. N: I'm looking for, the word you just said -	Questioning (asking for an answer)
Riley: (Interrupts) Distance.	Sharing (an answer)
Ms. N: I'm looking for the distance between them, right? So if I'm finding the distance, Jonah, between a positive number and a negative number, you're right, I am going to need to know their absolute value so that I can combine them. But if they're already on the same side of zero, I can literally just do what I can count one, two, I can just count the distance, right? Like I know from negative one to negative four. It's how far -	Telling
Jonah: (Interrupts) I think that only works when they are both at opposite sides.	Responding (building) Sharing (a generalization)
Ms. N: Yeah, I think that's true if they don't have the same sign, I like your strategy.	Responding (affirming)

The excerpt began with Jonah's initially incorrect generalization, that you add the absolute value of the x-coordinates for any two points. Through a series of transformations, Ms. N and the students built on one another's statements to develop a modified generalization, which was that if the two points are on the opposite side of the origin, the absolute values can be combined to determine the distance, but if they are on the same side of the origin, one can count the distance between them. Riley did propose a modification to Jonah's generalization, that one can subtract the absolute values for the pair of points that were both on the same side of the origin, the teacher did not take it up. In her interview, Ms. N shared that she did not do so because her students had not yet learned arithmetic with negative numbers. So, she instead highlighted that one could just count to determine the distance.

Note that all of the CSGs in each of the two excerpts were from the Interactional Moves category. The CSGs in this category are ones that lend themselves readily to teachers' and students' utterances in conversation, as well as particular task prompts or representation choices, such as Ms. N's organization of the three pairs of points in Figure 2. These Interactional Moves, however, occurred within the broader Instructional Routine of *multiple examples to form a rule*. Ms. N enacted a what was for her a common routine, that of sharing and discussing multiple cases of the same phenomenon, before then directing the students to consider what was the same across the examples in order to develop a general rule. Within this routine, she also engaged in a Structuring Action, structuring generalizing by drawing students' attention to sameness across the three ordered pairs. Within the Structuring Action and Instructional Routine, the Interactional Moves were the more immediate, localized moves made by both the teacher and the students that worked together to build up to the final generalization for determining the horizontal distance between two points.

Discussion

The three categories of CSGs enable attention to classroom interactions simultaneously at three different grain sizes. We found that the manner in which the Interactional Moves supported particular generalizations needed to be considered in light of the larger Structuring Actions and Instructional Routines in which they occurred. For instance, a specific move such as *sharing a generalization*, *boundary clarifying*, or *asking for an explanation* may or may not be effective in supporting generalizing depending on the immediate structure of interaction in which it takes place, as well as the larger structure of pedagogical actions and routines that form the sociomathematical milieu of the classroom. By considering the classroom environment to be a system of mutually interacting agents (Voigt, 1995), we have been able to identify simultaneous levels of support in order to better understand how generalization emerges in classroom contexts.

Similar to other studies attending to aspects of interaction in supporting generalizing (Ellis, 2011; Jurow, 2004), we found that the teacher, the students, the enacted tasks, the students' use of tools and artifacts, and the nature of representations worked in concert to support generalizing. Ms. N's representation of the pairs of points on the board worked together with her guiding remarks and the students' contributions to build up to the final generalization for determining horizontal distance. This illustrates the collective nature of generalizing, and the manner in which members of the classroom community can collaboratively build upon one another's ideas to introduce, reflect on, and refine generalizations.

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INVESTIGATING WHAT MAKES BEGINNING TEACHERS' ENACTMENT OF NUMBER TALKS MORE OR LESS AMBITIOUS

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Number talks, a popular mathematics teaching routine in the United States, may offer supports for beginning teachers (BTs) to engage in ambitious instruction. BTs' enactments of number talks, however, are varied, and there are few empirical studies that explore how BTs' enactment of number talks could be more (or less) ambitious. This paper draws on classroom observation data from a large investigation of BTs' enactment of ambitious instruction in elementary mathematics across five teacher preparation programs. We analyzed 19 transcripts of number talks enacted by seven BTs to investigate what makes number talks more or less ambitious. Findings illustrate three patterns among number talks that were categorized as approaching ambitious using the M-Scan (Berry et al., 2013) instrument. Discussions and implications are offered in relation to mathematics teacher education and research.

Keywords: Elementary School Education, Instructional Activities and Practices, Teacher Educators

Beginning teachers (BTs) are expected to engage in ambitious mathematics instruction (Kazemi, Franke, and Lampert, 2009; Lampert et al., 2013) that is also equitable for all students (e.g., Jackson & Cobb, 2010). This kind of teaching is inherently complex. For example, it requires teachers to be responsive to their students relative to both mathematics content and pedagogical methods, while also attending to unique social management demands required for productive mathematical classroom discourse (Lampert, Beasly, Ghuousseini, Kazemi, & Franke, 2010).

Number talks, instructional routines in which students use mental mathematics to solve computational problems, offer one way for BTs to engage in ambitious and equitable instruction. Number talks create a participation structure that may support "students to take back the authority of their own reasoning" (Humphreys & Parker, 2015, p. 1). The *routinized* nature of number talks can offer strong support for BTs to engage in aspects of ambitious instruction as novices. Despite the growing popularity of number talks in elementary classrooms there is a need for more empirical evidence about their enactment (Matney, Lustgarten, & Nicholson, 2020). In our recent work (Authors, 2021a, 2021b, and 2021c), we observed variation in BTs' enacted number talks in both structure and equitable opportunities offered to students. We continue this work to explore the features of BTs' enactment of number talks to answer the research question: What makes BTs' number talks more or less ambitious?

Theoretical Framework

To frame BTs' enactment of number talks, we build from the perspective that *ambitious instruction* in mathematics consists of teaching practices that foster students' deep, conceptual understanding of content (Lampert et al., 2013). Rigor, a hallmark of ambitious instruction, is evident in teachers' selection of tasks, supports for students, and how they respond to what students think and do (Kazemi et al., 2009). Further, when teaching ambitiously, teachers focus students on intellectual processes that support them to tackle demanding tasks, pushing students to justify their approaches, and pressing them to elaborate on their explanations and to clarify their thinking. Since the number talk routine provides space for students to engage in these kinds of mathematical practices, we see enacting number talks as a support for BTs to engage in ambitious teaching practices.

Additionally, we draw on Cazden's (2001) perspective on classroom interaction, which is shaped by two dimensions (*sequential* and *selectional*) of teaching. The *sequential dimension* describes relatively stable, cultural structure and routines. In contrast, the *selectional* dimension describes how individuals navigate such structures and routines. The *selectional* dimension highlights teachers' agency within a set of (taken for granted-or-not) constraints. Similar to Ehrenfeld and Horn's (2020) utilization of these two dimensions to describe teachers' group work monitoring routines, we view number talk routines as shaped by both *sequential* and *selectional* dimensions of teaching. For example, Parker and Humphrey's (2018) description of the "Revised Number Talk Routine" (p. 40) highlights the *sequential* dimension inherent in number talks. They found a strong correlation between teachers' decision-making within the number talk routine and the quality of teachers' number talks. Similarly, we found that BTs followed a predictable structure of enacting consistent phases in their number talks, *introducing*, *collecting*, *idea sharing*, and *closing* (See Figure 1; Cavanna et al., 2021; Pak et al., 2021a; and Pak et al., 2021b). Additionally, we identified important variations with the *idea sharing* phase on this structure (Pak et al 2021a), which highlights the *selectional* dimension of NT routines. Specifically, we found that number talks that included particular types of segments during the idea sharing phase created more opportunities for multiple students to engage with mathematical ideas than were available in number talks with only simple strategy segments shared by a single student. In this study we further explore the *selectional* dimension to understand the implications of BTs choices on the ambitiousness of their lessons.

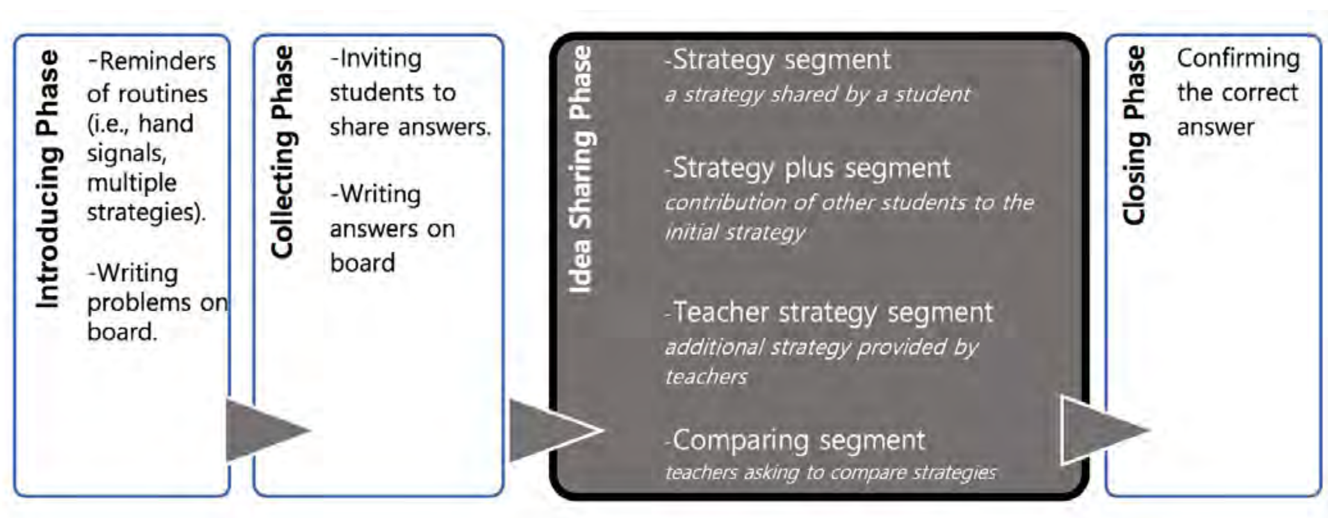

Figure 1: Number Talks Routine Phases

Methods

Context and Data Sources

Data for this study was gathered as part of a large research project that investigated BTs' enactment of instructional practices in elementary mathematics and English/language arts (ELA). Research project members video recorded BTs (n=69) three times per year as they taught mathematics and ELA. For this study, we draw on video data from a subset of 16 purposefully selected case-study participants. We reviewed 144 videos from these case-study participants (three mathematics lessons per year) over the course of three years. Of these, we identified 19 videos in which seven case-study teachers enacted Number Talks. The Number Talks occurred at the beginning of each lesson, with an average duration of ten minutes per number talk. These 19 number talk transcripts comprise the data for this investigation.

Data Analyses

As part of the larger project, all video recorded lessons were scored using the Mathematics Scan (M-Scan) instrument (Berry et al., 2013). This instrument offered an assessment of the ambitiousness of the mathematics instruction (Berry et al., 2010). Of the nine elements assessed by M-Scan, we focused on six categories of ambitious teaching: cognitive demand, problem solving, use of representations, mathematical discourse community, explanation and justification, and mathematical accuracy. These six dimensions capture key elements of ambitious mathematical instruction (Kazemi et al., 2009; Lampert et al., 2013). Scores on the M-Scan instrument range from 1 to 7, with scores of 6 or 7 considered to be demonstrating the most ambitious mathematical instruction. We considered mean scores across the six focal dimensions that were greater than 5 to be approaching ambitious instruction. Although the M-Scan scores were evaluated based on the full lesson and the number talks typically comprised less than that, we considered the instruction of the number talks portion as having a significant effect on the overall lesson score.

Our analyses of the number talk transcripts involved iterative stages of qualitative coding by the three authors of this study. First, to understand the *sequential* dimension of the number talks, we divided transcripts into phases (see Figure 1; Pak et al., 2021a). Next, we explored the *selectional* dimension of the number talks. We divided the *Idea Sharing* phase into segments, which were separated by the span in which teachers allow students to talk about a particular mathematical idea. We closely examined each segment in this phase because our prior works suggested that a systematic analysis of this *Idea Sharing* phase might show what made BTs' number talks more (or less) ambitious. The authors iteratively coded 19 transcripts, comparing individual codes until we reconciled all coding across the team. Table 1 outlines the segment types identified as a result of this process. We recognize the similarity of the *Strategy Plus* subcodes to well-known mathematical talk moves (e.g., Chapin, O'Connor, Anderson, 2009). In our presentation we expand on these points of convergence and divergence from existing literature.

Table 1. Codes to represent various teacher moves within Idea Sharing phase

Segment Types	Descriptions and subcodes	
Strategy	Asking students to volunteer talking about their strategies or calling on a specific student to explain a strategy to solve a problem.	
Strategy plus	Inviting, or calling on, students to share their ideas related to how they make sense of the initial strategy.	
	(1) Inviting	Directly inviting students to question the strategy sharer.

	(2) Interpreting	Asking another student to offer his/her own reasoning regarding how the initial strategy works.
	(3) (Dis)agreeing	(Dis)agreeing with follow-up (Not related to an error)
	(4) Detecting	Detecting error and may challenge the strategy sharer
	(5) Guiding	Generating the reasoning to draw attention to something specific in the strategy (e.g., funneling).
	(6) Repeating	Asking students to revoice what another student says
Teacher strategy	Initiating and providing additional strategies to solve the problem.	
	(1) Feeding a strategy	Making moves to push students to consider thinking in a certain way, but don't set up the problem in a way to use a certain strategy. (e.g., What if…Could I…?)
	(2) Do this strategy	Setting up the problem to be carried out a particular way.
	(3) Call and response	Walking a student (or the class in choral response) through a logical progression.
	(4) Guess my strategy	After asking students to guess, teachers explain what he/she did and ask students to unpack the reasoning.
Comparing Strategy	Asking students to compare similarities and differences between strategies.	

Lastly, in order to answer the question of what makes number talks more (or less) ambitious, we examined the segment types present across the set of 19 lessons and compared those to the M-Scan scores on those lessons. Specifically, we developed matrices that recorded the types and frequencies of segments within each number talk. We then looked for patterns across the lessons related to the M-Scan scores and patterns within those number talks.

Results

Our analyses of the number talk transcripts and M-Scan scores revealed six of 19 number talks as approaching ambitious mathematics instruction. Figure 2 lists these lessons and the mean scores of the M-Scan dimensions we utilized. Across this set of lessons, we identified three salient patterns that seem to be associated with approaching ambitious number talks. In the following sections we draw on excerpts from these six more ambitious lessons and compare to the other 16 less ambitious number talks to help us understand what makes some number talks more or less ambitious.

The first pattern we observed relates to the prevalence of *Strategy Plus* segment types across the more ambitious number talks. As shown in Figure 2, *Strategy Plus* segments appeared frequently across the *Idea Sharing* phase of the approaching ambitious number talks. Within the strategy plus segments, we also identified further variation across different teacher moves, including *Inviting, (Dis)agreeing*, *Interpreting*, *Guiding*, and *Detecting* (See Table 1). For example, Teacher B, in lesson Y1 M2, used both the *inviting* and *interpreting* moves in two adjacent *Strategy Plus* segments. Teacher B began the segment by inviting a student Jason to share an initial strategy. Jason responded to the invitation and explained his strategy for solving the addition problem 470 + 450. Next, the teacher *invited* the class to question Jason's work, saying "Does anyone have a question about that?". This question prompted two students to pose questions about the mathematics in Jason's strategy. Following this exchange, the teacher used an *interpreting* move to support one of the questioning students, Landon, to go further in unpacking Jason's initial strategy. The short excerpt below illustrates this *interpreting* move and the subsequent exchange.

> Landon: 450 plus 400 which is 850
> Teacher B: [writing on board] 850, okay.
> Landen: So, we're kind of back where we started up here a little bit? Right? Wait, why would we plus 30?
> Teacher B: We would plus 50, right? Why? Landen?
> Landen: Because you can benchmark.

This exchange from Teacher B is characteristic of the kinds of inviting and interpreting moves we observed in five of the six ambitious number talks. As illustrated in this exchange, BTs tended to use multiple *Strategy Plus* talk moves within these more ambitious lessons to move support multiple students to engage with the mathematical strategies being shared.

Lesson (Year, Obs)	M-Scan Means[2]	NT Idea Phase Segments
Teacher A (Y2 M1)	5.0	Strategy; Strategy Plus [Interpreting]; Strategy Plus [Detecting]; Strategy
Teacher B (Y1 M2)	5.3	Strategy; Strategy; Strategy; Strategy; Strategy; Strategy; Teacher Strategy [Do This Strategy]; Strategy Plus [Inviting]; Strategy Plus [Interpreting]; Teacher Strategy [Guess My Strategy]; Strategy; Strategy
Teacher B (Y1 M3)	5.2	Strategy; Strategy; Strategy Plus [Interpreting]; Strategy; Strategy; Teacher Strategy [Do This Strategy]; Strategy Plus [Inviting]; Strategy
Teacher C (Y2 M1)	5.2	Strategy Plus* [Guiding]; Strategy Plus* [(Dis)agreeing]; Strategy; Strategy Plus* [Interpreting]; Strategy Plus* [Detecting]
Teacher C (Y2 M2)	5.3	Strategy; Strategy Plus* [Detecting]
Teacher C (Y3 M2)	5.3	Teacher Strategy* [Do This Strategy]; Strategy Plus* [Inviting]; Strategy Plus* [Inviting]; Strategy Plus [Inviting]; Comparing

Note. * indicates the segment contained a mathematical error

Figure 2. Idea Sharing Phase Segments of Ambitious Number Talks

The second pattern we observed across this set of number talks was related to the ways BTs inserted their own strategies into number talk discussions. We call this kind of move *Teacher Strategy* segments. While not all examples of teacher strategy segments were associated with approaching ambitious instruction, we observed two subtypes of teacher strategy segments within the ambitious number talks: (1) *Do This Strategy* and (2) *Guess My Strategy.* The excerpt below illustrates how Teacher C set up a problem to be carried out in a specific way through an example of the *Do This Strategy* teacher move.

Teacher C: First of all, before we start sharing out strategies and agreeing and disagreeing with people. I'm wondering how we would put this into a bar diagram [pointing to diagram on board] because a bar diagram is a super important tool. Why is a bar diagram such an important tool? John, why is it an important tool?
John: Because you can use it to know parts and wholes.
Teacher C: Yeah, you can use it to understand your parts and your whole. And if you're going to add or you are going to —
Class/Teacher C: subtract.
Teacher C: And if this isn't a word problem, we know right away we are going to do what to solve this?
Class: subtract.

Following this excerpt, students worked the number talk routine, with many students utilizing bar models in their solutions. This excerpt shows how Teacher C intentionally encouraged students to use a bar model to solve a problem. Prior to solving the problem, however, Teacher C supported students to understand why a bar model might be helpful in solving this particular problem. We observed the *Do This Strategy* move in three number talks with approaching ambitious M-Scan scores (i.e., Teacher B (Y1 M2), Teacher B (Y1 M3), Teacher C (Y3 M2)).

In contrast, we observed a different teacher strategy, which we called, *Feeding Strategy*, only in number talks that were not ambitious according to their M-Scan scores. Due to space constraints, we did not present the less ambitious number talks from our dataset in Figure 2. We will expand upon the contrasting features of the more and less ambitious Number Talks in our presentation. To illustrate the differences between more and less ambitious teacher strategies, we offer an excerpt from Teacher D (Y2 M1), which includes a *Feeding Strategy*.

Teacher D: Did anybody use the distributive property, but you split it up in a different way? Or can you think of another way that would make sense to split it up? Ally?
Ally: I split up the 12 into 6 and 6. And did 6 times 3 and 6 times 3.
Teacher D: Yeah - So, if you split the 12 into 6 plus 6, then you have 6 times 3 plus 6 times 3. If you love your six facts. Maybe you don't know your 6 times 6. What's 6 times 3?
Class: 18.
Teacher D: 18 plus 18 is?

In this excerpt, Teacher D pressed students to think about how to solve the problem in a certain way (e.g., using the distributive property). This teacher, however, did not support students to reason about why this specific solution strategy might be helpful in this problem. We found it somewhat surprising that these *Feeding Strategy* appeared only in number talks that were not ambitious, while the *Do This Strategy*, another type of teacher strategy, appeared in ambitious number talks. We wonder if perhaps the nuances between the BTs setting students up with a possible solution strategy in advance, as was the case for the *Do This Strategy* move, offered more space for teachers to build from students' thinking than when BTs funneled students towards a particular strategy later in the number talk, as in the *Feeding Strategy* move. Further investigation is warranted.

The third and final pattern we observed relates to the role of mathematical errors in number talks. As noted in Figure 2, segments with an asterisk symbol (*) contain an error; there are three ambitious number talks in which students made mathematical errors. All three were facilitated by Teacher C. Across these number talks, we observed errors in both *Strategy Plus* segments and *Teacher Strategy* segments. Teacher C used a range of moves to respond to errors made by students, *Inviting, Guiding, (Dis)agreeing*, and *Interpreting.* Importantly, Teacher C demonstrated ambitious mathematics instruction in the ways she supported students to make sense of the mathematics related to the errors. For example, in Teacher C's (Y2 M1) lesson, many students shared incorrect answers to the subtraction problem 74–36. The excerpt below shows how Teacher C (Y2 M1) used a *Guiding* move to respond to these errors.

> Teacher C: So, it's already up here, alright, excellent, we have four different ideas, we have four different ways. What were we just doing when we were just thinking about that?
> James: Most of us were using that.
> Teacher C: Maybe we were using our algorithm. Who could walk me through what you did? [Many hands in the air] Who can walk me through it? Talk it up. Alex, what did you do?
> Alex: First, I did regrouping.
> Teacher C: Okay. How did you do that?

In this lesson, Teacher C received three different incorrect answers and one correct answer. The teacher guided the students to use "our algorithm" to solve the problem. The teacher asked Alex, one of the students who shared an incorrect answer, to walk the class through what he did to get the answer ("ideas"). This excerpt is characteristic of Teacher C's student-centered approach to handling students' incorrect answers.

Discussions and Conclusion

Our analyses revealed three patterns related to BTs' number talks being more ambitious, all of which related to teacher moves occurring within the *Idea Sharing* phase of the number talks routine. Specifically, we observed patterns related to the nature of *Strategy Plus* segments, *Teacher Strategy* segments*,* and the ways BTs responded to student errors. These findings offer potential insights for the field of mathematics teacher education as we seek to support BTs to engage in ambitious mathematics instruction.

First, these findings suggest that number talks offer a transportable container for BTs to engage in ambitious instructional practices early in their teaching career. At the same time, we found that only six of 19 number talk lessons could be considered examples of ambitious mathematics instruction. Therefore, we posit that the number talk routine itself– the *sequential* dimension– does not necessarily result in ambitious instruction. Instead, systemic analysis of BTs' number talks in terms of the *selectional* dimension (Cazden, 2001) offers insights into the role of teacher choice within the number talk container. It is within the *selectional* dimension that BTs exercised their instructional agency that created opportunities for the number talks to be characterized as more or less ambitious. This suggests that mathematics teacher educators must not only work with BTs to understand and use the overall number talk structure, or container, but also to consider the way certain segments are used within the *Idea Sharing* phase. This relates to our next point of discussion.

Second, these findings suggest that if we want BTs to engage in more ambitious number talks, then they need to move beyond strings of *strategy* segments, in which students simply share out one strategy after another. Instead, we offer that BTs need to integrate intentional talk moves (e.g., Chapin, O'Connor, Anderson, 2009; Kazemi & Hintz, 2014) to bring in multiple students into the conversation and shift towards *Strategy Plus* segments, as well as offer *Teacher*

Strategy segments. Such findings offer implications for mathematics teacher education. Specifically, mathematics teacher educators need to provide opportunities for BTs to learn to infuse *Strategy Plus* segments into their number talk lessons. Further research into the role that particular arrangements of talk moves play in the ambitiousness of number talks is warranted.

Third, our findings do not highlight *Comparing* segments as clearly tied to the ambitiousness of number talks. Building on prior literature, we anticipated that *Comparing* segments, which offer opportunities for students to build off of another's reasoning (e.g., Herbel-Eisenmann, Steele, Cirrillo, 2013; Wagganer, 2015) would be associated with ambitious number talks. Interestingly, we observed only one instance of a *comparing* segment, in Teacher C (Y3 M2). Since creating opportunities for students to build on one another's reasoning is a potentially difficult move, perhaps this is not so surprising given our dataset was lessons enacted by teachers within their first three years of teaching. At the same time, these findings may suggest that BTs could benefit from more opportunities to learn about moves to support students to compare mathematical reasoning. Such support could, in turn, contribute to more ambitious number talks.

Lastly, number talks are increasingly popular; and due to their portability and ease of use, they will likely continue to be a staple of mathematics classrooms. The elements highlighted in this study offer possibilities for enhancing the power of these flexible routines. Specifically, number talks that are more ambitious offer greater opportunities for engaging students in critical disciplinary practices of mathematics, such as the Standards for Mathematical Practice (NGA & CCSSO, 2010). At the same time, in number talks that are more ambitious, we see more space for equitable mathematical opportunities, as well. There is a need for further research that examines BTs enactment of number talks over time in order to understand how the ways students' mathematical identities develop and the ways number talks affect the nature of power structures within the mathematics classroom. Number talks that are both ambitious and equitable offer a route for teachers to provide meaningful opportunities for *all* students to engage in critical mathematical practices.

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INSTRUCTORS' FACILITATION OF STUDENT PARTICIPATION IN ADVANCED MATHEMATICS LECTURES: A CASE STUDY OF TWO INSTRUCTORS

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Literature portrays advanced mathematics lecturing as a uniform teaching style, often criticized for offering minimal opportunities for student participation. In this paper we present results from a comparative case study of two instructors' facilitation of student participation in Real Analysis lectures. Analyzing fieldnotes from several observed lectures of each instructor, we found that the two instructors' facilitation of student participation during lectures consistently differed in (1) the participation structures were used, (2) the types of questions asked, and (3) how instruction responded to students' contributions. Our findings show that lecturing in advanced mathematics is not a uniform style and that active student participation in lectures is possible. We interpret the potential impact of observed differences on students' learning and experiences in terms of the Teaching for Robust Understanding (TRU) framework (Schoenfeld, 2018).

Keywords: Instructional Activities & Practices, Undergraduate Education, Classroom Discourse

The undergraduate math education literature tends to portray lecturing in advanced mathematics as a single teaching style, in which instructors engage in "chalk talk" (writing formal mathematics on the blackboard while providing oral commentary) and student participation is minimal (Artemeva & Fox, 2011; Lew, Fukawa-Connelly, Mejía-Ramos, & Weber, 2016; Paoletti et al., 2018). Here, we join other recent scholarship (e.g. Pinto, 2019; Viirman, 2015) in problematizing this homogenous picture of advanced mathematics lectures. We report on variability in two instructors' practices of facilitating student participation during Real Analysis lectures, discussing potential implications of each approach for student experience in terms of the Teaching for Robust Understanding (TRU) framework, which delineates five dimensions of classroom practice important for learning (Schoenfeld, 2018).

Active student participation leads to more robust learning outcomes than passive observation (Chi & Wylie, 2014), specifically in the context of tertiary STEM education (Freeman et al., 2014). Furthermore, agentic participation in classroom discourse provides students with opportunities to develop productive disciplinary dispositions and identities (Gresalfi, Martin, Hand, & Greeno, 2009). Because of this and based on the shared assumption that lectures involve minimal student engagement, many mathematics education researchers argue against lecturing, and instead advocate for student centered teaching approaches such as Inquiry Based Learning (IBL: Laursen & Rasmussen, 2019). Yet others argue that it is neither realistic nor desirable to abandon lectures altogether (Sfard, 2014). Most advanced mathematics courses are still taught in a lecture format (Johnson, Keller, & Fukawa-Connelly, 2018), and mathematicians – many of whom claim that lectures are "the best way to teach" – are not likely to abandon lecturing in favor of radically different teaching approaches (Woods & Weber, 2020).

Instead of a complete overhaul of lecturing, several scholars suggest "tilting the classroom" (Alcock, 2018) – incorporating minimally invasive active-learning strategies into traditional lecturing (Woods & Weber, 2020). Suggestions include using participation routines such as Think-Pair-Share and classroom polls (Braun, Bremser, Duval, Lockwood, & White, 2019). However, there is little research on how implementation of such strategies actually looks like in the context of advanced mathematics lectures (Woods & Weber, 2020), and what such practices may be good *for*. Our paper begins to address this gap in that it provides an analysis of two distinct examples of how instructors facilitate active student participation during lectures and discusses the relative affordances of each for student experience and learning.

Participation in Classroom Practices and the TRU framework

We draw on sociocultural theories that conceptualize disciplinary knowledge as participation in disciplinary practices (Lave, 1996). Within this tradition, participation in classroom practices is seen as a central mechanism by which students both learn disciplinary content and develop disciplinary identities and dispositions (Cobb, Stephan, McClain, & Gravemeijer, 2001; Esmonde, 2009). Hence, an important aspect of teaching is how instructors organize classroom environments to support student participation, including the specific moves they make to invite and facilitate such participation. What matters is not just *that* students participate, but also *what* they get to participate in, and *how*; the nature of classroom practices shapes the kinds of content, skills, and dispositions students develop through their participation (Gresalfi et al., 2009).

The Teaching for Robust Understanding (TRU) framework is a synthesis of such research on aspects of classroom practice associated with robust learning outcomes (Schoenfeld, 2018). The framework delineates five dimensions – *(1) the content, (2) cognitive demand, (3) equitable access, (4) agency, ownership and identity, and (5) formative assessment* – meant to comprehensively characterize a classroom environment. *The content* captures the extent to which the enacted mathematics is coherent, connected and centered on important practices and ideas. *Cognitive demand* refers to the degree to which classroom activities provide opportunities for "productive struggle", hitting the sweet spot of accessibility and challenge. *Equitable access* refers to classroom practices that ensure that *all* students have opportunities for meaningful engagement with disciplinary activities. *Agency, Ownership, and Identity* refers to the extent to which students get to make significant choices, generate mathematical content themselves, and get positioned as mathematically competent by having their ideas built on in the classroom. *Formative assessment* refers to the extent to which student thinking – including productive beginnings and possible misunderstandings – is surfaced and responded to in instruction.

In this paper, we use TRU to organize our findings about instructors' practices of facilitating student participation in ways that directly connect them to central mechanisms of learning in classrooms. We focus on dimensions 2-5 of TRU, for which student participation is relevant. Our focal research question is: *how did the observed lecturers facilitate student participation, and what significance might these moves have for student learning and experience?*

Methods

Context & Data Collection

The data our project is grounded in are observational fieldnotes of Real Analysis lectures taught in a large research university in the western U.S. during Fall 2019. We observed three instructors teaching the same elective upper division undergraduate course in parallel during a period of 3 weeks in the first half of the semester. Due to space limitations, here we focus only on Dr. A and Dr. B. We observed eight 50-minute lectures taught by Dr. A and four 80-minute lectures taught by Dr. B (a total of 400 and 320 minutes, respectively). Each lecture-section was attended by approximately 30 students.

Each author took detailed fieldnotes using a note-taking instrument specifically designed for our analytic purposes. The top third of each page was used to document blackboard inscriptions verbatim. The bottom two-thirds were devoted to descriptions of participants' speech and actions. This section was further divided into two columns: the left column for general actions made by the instructor, the right column for events related to student participation. Board inscriptions and speech/action descriptions were linked to one another by number indexing, allowing us to reconstruct the sequential organization of each lecture, coordinate speech and inscriptions, and systematically document student participation – the analytic focus of this paper.

Data Analysis

Our analysis proceeded both inductively and deductively. Each author reviewed their own fieldnotes independently to identify participation facilitation routines. In subsequent analytic discussions, we consolidated our independent characterizations, and organized them into three aspects of facilitation practice: what participation-structures were used, what kinds of questions instructors asked, and how instructors responded to student contributions. Next, we took a deductive approach. We turned to the literature to learn about how these three aspects (participation structure, teacher questions, response to students) were defined in prior work. This process helped us refine and operationalize our initial analytic categories. With refined definitions, we returned to the data, and coded all instances of student participation in relation to these categories. What follows is an explanation of the analytic categories we used.

Participation structures are defined as "the organization of persons' reciprocal rights and obligations in social interaction" (Erickson & Mohatt, 1977, p. 139). Common classroom structures are: whole-class discussion, group work, and individual work. One can, however, further specify "persons' reciprocal rights and obligations" within each broad category. Thus, in addition to a general *participation structure* category, we looked at *initiation type*, where type refers to who has the right and obligation to respond to a particular initiation (some student, a specific student, all students), and *who participated* as a result (all or some students).

In the literature, teacher questions are typically coded for the kind of responses they require from students. While precise category-labels vary, most studies use a coding scheme that organizes question-types in an hierarchical progression, from low- to high-order questions (DeJarnette, Wilke, & Hord, 2020). We adopted a similar approach, and categorized instructors' mathematical questions broadly as either *low, medium or high level* in terms of the question's cognitive demand (e.g. definition recall vs. proof idea) and openness (a single correct answer vs. many possible answers). In addition to mathematical questions and tasks, we also coded instances of: non-mathematical questions, solicitations of questions from students (e.g. "do you have any questions?") and comprehension-monitoring questions (e.g. "does that make sense?").

Finally, to characterize responses to student contributions, we first noted the *extent* (in terms of length and complexity) of student contributions. We then noted whether instructors' initial *response to student contributions* was encouraging or evaluative, as well as the extent to which subsequent lecture-talk *builds on student ideas*. Finally, we noted whether response episodes involved a single exchange sequence (a student asks a question, the instructor answered) or several back-and-forth exchanges, as these are indicative of the extent to which classroom discourse is dialogic and ideas are co-constructed (Wells & Mejia Arauz, 2006).

The last step of our analysis involved connecting these observables to the dimensions of the TRU framework. The same behavioral indicator can be implicated in more than one TRU dimension. For example, a student's opportunity to explain their thinking in relation to a conceptually rich task involves *cognitive demand,* disciplinary *agency*, and constitutes a *formative assessment* opportunity for the teacher. Thus, for each dimension of TRU, we used the definition of the dimension to read across the coding described above and synthesized all aspects of instructors' facilitation of student participation that contribute to that dimension.

Findings & Discussion

The table below summarizes our findings. The two sections that follow provide detailed descriptions and examples of each instructor's facilitation practices, unpacking the codes and descriptions in the table. In the last findings section, we interpret the potential impact of identified practices in terms of the four selected TRU dimensions: *(2) cognitive demand, (3) equitable access, (4) agency, ownership and identity, and (5) formative assessment*.

Table 1: Facilitation of Student Participation - Dr. A and Dr. B

	Dr. A		Dr. B
Participation Structure	Whole-Class	Think-Pair-Share	Whole-Class
Initiation Type	Voluntary Q & A Poll	Pair: *"Discuss with your neighbor..."* Share: randon calling using name-cards	Voluntary Q & A Poll
Who participated?	A few students	All students	A few students
Types of Tasks	Math: low-level	Math: med-level	Math: low, med & high-level
Types of Questions	Open solicitation: *"What are your questions so far?"*	Open solicitation: *"X, what are your thoughts, ideas, and questions here?"*	Non-mathematical, Comprehension monitoring: *"Does that make sense?"*, Open solicitation: *"Any questions here?"*
Length of student contributions	Short and quick take-over		Some short & some extended
Instructor's response to student contributions	Encouraging (all contribution types): *"That's a great question!"*		Endorsing (correct answers): *"You got it!"* ; *"Perfect!"*
Lecture building on student ideas	Short		Some short & some extended
Extent of dialogue	Single exchange		Some single exchange some back-and-forth

Facilitation of Student Participation:

Dr. A. Dr. A utilized two distinct participation structures in his lectures: a traditional whole-class format and a participation structure known as "Think-Pair-Share" (Braun et al., 2019).

During whole-class (i.e., outside of Think-Pair-Share), Dr. A invited students to participate by asking questions addressed to all students, to which individual students could respond verbally on voluntary basis. Dr. A rarely asked "known-answer-questions" in the whole-class format. The vast majority of questions Dr. A posed were open solicitation questions such as "what do you notice?" or "what are your comments so far?". *Any* thematically relevant thoughts, comments or questions could count as a legitimate student response, and indeed, students utilized these open-solicitation prompts as invitations to ask questions and clarify confusions. Whenever Dr. A asked specifically mathematical question during whole-class, the question required only short responses from students. Several times during our observation period, Dr. A initiated polls; prompts involving several specified answer-options (e.g., "should this be $\leq$, $\geq$, or $=$?") on which *all* students were asked to vote, though often only a subset of them did. Occasionally, Dr. A initiated short IRE exchanges in the middle of an explanation (e.g. "what is $cos\ cos\ \pi/3?$"). The purpose seemed to be ensuring that all students are "on the same page".

Think-Pair-Share was a salient participation structure in Dr. A's class. Several times each lecture (at least 2-3 times), Dr. A introduced a short mathematical task and asked students to discuss it with their "neighbor." Students were seemingly used to this routine by the time of our observations; all seemed to work on the task and lively talk could be heard. Meanwhile, Dr. A left his front-of-the-room position, circulated among the desks, and stopped to talk with students (typically called-on by the students themselves). After a few minutes, Dr. A called the class back to a whole-class format and nominated a specific student to “share.” The choice of student to call on was done through a randomized selection mechanism: a stack of cards with students’ names. Students cooperated with the “cold-calling” approach, and to the extent evident from our observations, seemed to be comfortable with it. In the public “share” phase, Dr. A initiated student contributions with open-solicitation prompts such as “what are your thoughts, ideas, questions here?” The format implied that it was acceptable to “share” by stating a confusion, a hesitation or asking a question rather than providing a direct answer to the original task. Indeed, students often responded by asking a question and sometimes by saying “I don’t know”.

The tasks Dr. A used for pair activities incorporated an assortment of mathematical practices. To name a few examples, students were asked to: determine if an example satisfies a definition, complete a few steps in a proof, interpret a theorem-statement in terms of a diagram, and complete a proposition claim. One overarching characteristic of such tasks is that they are all at a medium-level in terms of task-openness and cognitive demand; they involve non-trivial mathematical engagement from the students, yet were structured and accessible. A similar approach to selecting classroom activities for Real Analysis lectures was articulated by Alcock (2018), where she described such activities as “short-and-snappy” and geared toward “conceptual understanding” rather than calculations (p. 24).

Whether in the whole-class format or the “share” phase of Think-Pair-Share, students’ verbal contributions in the public sphere of the classroom were short. In whole class, Dr. A’s initiations typically called-for short responses only. In the “share” part of pair-share, students sometimes began articulating what could potentially be a longer contribution, but in all of our observations, Dr. A quickly took over. The impression we had was that Dr. A took over to make the explanation clearer to the whole class and cut students’ own explanations short to “save time”. In general, there was a sense of fast pace in Dr. A’s class, both in terms of the speed of Dr. A’s “chalk-talk” and because of swift transitions between many planned activities.

Whenever a student asked a question or offered an idea, Dr. A responded encouragingly (e.g. “great question!”), with level of enthusiasm independent of the contribution’s sophistication or correctness. Whenever a student’s question generated new mathematical content, Dr. A took it up and responded to it (e.g., by answering the question). However, in all of our observations, such contributions never led to an extended whole-class discussion or significantly altered the lecture. Furthermore, they rarely involved a back-and-forth of ideas between Dr. A and the student, or other students. For example, once a student asked why two quantities are equal. Answering that question prompted Dr. A to write a sketch of a proof for “why it should be equal” on the board. Thus, the student’s question altered the course of the lecture slightly, and prompted Dr. A to generate new content, both verbally and in writing. However, the episode was short and neither the student who initiated the question nor other students made any follow-up contributions.

Dr B. Dr. B’s lecturing was “traditional” in that it was conducted through a whole-class participation structure, “chalk-talk” was a pervasive discourse genre, and student participation was organized primarily through voluntary question-answer exchanges with the instructor.

A salient feature of Dr. B’s lecturing was frequent use of short IRE sequences. This involved both low-level math tasks, such as simple recall, and tasks we considered to be medium-level, such as discerning proof-structure. A common routine in Dr. B’s class featuring low-level IRE sequences can be broadly described as scaffolded "chalk-talk". Dr. B often prompted students to verbalize the precise formulation of a definition, and then used students' responses to "dictate" her board writing. As a mathematical task, this routine is closed and involves low demand. Yet, given that verbalizing formal mathematical texts can be difficult for newcomers (Shepherd & van de Sande, 2014), such a scaffolded version of "chalk-talk" (in which the instructor does the "chalk", and a student does the "talk") could have important benefits. A medium-level IRE routine Dr. B often used was to engage students in discerning logical-structure. She prompted students to recognize assumptions, or givens, by asking questions such as "What else do I know?" or "What is my claim?". While such questions follow a relatively narrow path (a student responded, Dr. B endorsed it), discerning proof-structure is a non-trivial skill for students to learn in proof-based courses (Selden & Selden, 1995). Thus, we considered such tasks to be at a medium-level, as they required actions that went beyond recall and recitation.

Though less frequently, Dr. B also posed questions that can be considered high-level in terms of openness and cognitive demand. On several occasions, Dr. B invited students to suggest mathematical examples, contribute central steps in an argument, or generate proof ideas. Occasionally, Dr. B made use of classroom voting, or polls. This seemed to primarily function as a formative assessment mechanism, that is, as means for Dr. B to gauge whether all students are following and could discern the correct option. Dr. B also routinely asked short comprehension monitoring (e.g. "does that make sense?") and open-solicitation (e.g. "any questions?) questions. Students rarely volunteered verbal responses to such questions, though we presume that Dr. B read some non-verbal feedback from students. Notably, Dr. B also regularly asked questions engaging students in decisions pertaining to mathematical conventions such as notation (e.g. "what letter should I used?"), as well as non-mathematical questions (e.g. "what's that French word for combining two words together?"). Such questions invited broad student participation.

Students' mathematical contributions, whether as responses to Dr. B's initiations or initiated by the students themselves, varied in extent and complexity, ranging from single phrase responses to lengthy articulations of mathematical scenarios and ideas. Furthermore, extended student contributions were often situated within longer classroom episodes that involved substantial building on student ideas and several back-and-forth exchanges between students and Dr. B. The following vignette serves as an example of one such case:

> Dr. B introduced a definition for *limit points* of a set in a metric space and offered an illustrative context: R (the set of real numbers) as the metric space, the interval $S = [0,1]$ as a set under consideration, and the 1 as a possible *limit point* of $S = [0,1]$. Later, she wrote the following statement on the blackboard: $1 \in lim\ (S) \Leftrightarrow \exists (x_n)$ in S s.t. $x_n \to 1$ as $n \to \infty$, which can be read as: the number 1 is a limit of point S if and only if there exists a sequence of numbers all of which are within the interval $[0,1]$ such that this sequence approaches 1 at infinity. Referring to the sequence x_n, Dr. B asked students "What's an example?", and a student suggested the sequence $x_n = 1 - \frac{1}{n}$. Dr. B picked up and elaborated on this suggestion by writing several elements in this sequence, and indicating that each belongs to the interval S: $x_1 = 0 \in S$, $x_2 = \frac{1}{2} \in S$, $x_3 = \frac{2}{3} \in S$ $\ x_4 = \frac{3}{4} \in S \ldots x_n = 1 - \frac{1}{n} \in S$. The last equality refers to the general pattern: *any* element in the sequence x_n belongs to the interval $S = [0,1]$. She then turned to the class and asked, "how do we prove it?". Another student (not the same one that originally suggested the sequence), provided an elaborate answer

constituting steps of a proof. Dr. B picked-up and "re-voiced" the student's idea by producing a short proof-text mirroring the argument the student described verbally.

This episode illustrates several of the general trends mentioned above. High-level mathematical questions from Dr. B, while not the most frequent form of questions used in her class, prompted students to contribute ideas, both short (e.g. suggesting an example sequence "$x_n = 1 - \frac{1}{n}$") and extended (e.g. verbally describing a proof-argument). When ideas were suggested, Dr. B took them up by writing them on the board, and further elaborated and explained them. Dr. B often posed further questions in the context of an initial student-suggestion. In this case, the question "how do we prove it?" prompted another student's response. Such moves initiated extended episodes that featured back-and-forth exchanges between instructor and students, and at times, allowed students to build on one another's ideas.

Interpretation of instructors' facilitation of student participation using TRU

Cognitive Demand. In Dr. A's class, Pair-Share activities provided students with routine opportunities to engage in mathematical tasks. The tasks Dr. A used were at a consistent medium-level of demand; they involved non-trivial mathematical engagement such as interpreting a proposition statement, or completing steps in a proof, yet were concrete and accessible. In Dr. B's class, there was greater variability in terms of tasks' cognitive demand. Dr. B's questions ranged from basic recall and verbalization questions (e.g., dictate a definition), to medium-level proposition-structure tasks (e.g., "what is my claim?"), and up to open and cognitively demanding questions (e.g., "how do we prove this?"). Both approaches – Dr. A's keeping cognitive demand at a consistently moderate level, and Dr. B's varying demand level from low to high within a single lecture – provided "productive struggle" opportunities.

Equitable Access. Several of Dr. A's teaching routine supported *equitable access* to mathematical content and practices. His frequent use of the Pair-Share participation structure provided opportunities for *all* students to actively engage with non-trivial mathematical tasks. In addition, the open-format of the questions Dr. A posed ("what are your thoughts?"), and his explicitly affirmative responses ("that's a great question!"), reduced access barriers for students' verbal participation during whole class, since all types of contributions (whether questions, suggestions or articulated confusion) were framed as legitimate and valuable. In Dr. B's class, we identified few explicit mechanisms that supported equitable access to mathematical content. Dr. B asked questions frequently, and several students participated verbally, but this participation was not distributed equally and the experiences of students who did not participate verbally is difficult to gauge (though we do not assume silent students did not participate, see e.g. (O'Connor, Michaels, Chapin, & Harbaugh, 2017)). Dr. B's non-mathematical questions seemed to encourage broader participation. However, her closed-form mathematical questions were typically responded to by a single student (once a correct response was given, no further were needed). Most notably, extended discussions (as describe above) seemed to engage a select few.

Agency, Ownership, Identity (AOI). In Dr. A's class, it was not clear to what extent students had opportunities to see themselves and peers as creators of mathematics. The selected tasks supported students' engagement with important mathematical practices. However, given that they were relatively narrowly circumscribed, we might ask to what extent engagement with the tasks allowed students to exercise mathematical *agency* and take *ownership* of the content. For example, a fill-in the blank task to complete a proposition text engages students in important mathematics. However, does it make students feel *ownership* over the resulting mathematical text and ideas? Similar questions can be raised about students' participation in whole class too.

Given that students' verbal contributions in whole class were short and not significantly built upon, it is not clear to what extent participation in Dr. A's whole class discussions provided students with opportunities to feel *ownership* of the content, exercise mathematical *agency*, and be positioned as competent doers and creators of mathematics. In Dr. B's class, extended episodes provided participating students with ample opportunities to exercise *agency*, feel *ownership* of ideas and be recognized as mathematically competent. Yet, as described above, only a few students in Dr. B's class had this experience. Silent observers of extended dialogic interactions are afforded valuable learning opportunities (O'Connor et al., 2017). However, the impact on students' negotiated *identities* may be more problematic. As observers in Dr. B's class, we could "read" a mathematical hierarchy among students. In contrast, in Dr. A's class, we could not discern a similar pattern: it was not easy to tell who "the smart students" were.

Formative Assessment. Dr. A's practices afforded ample opportunities to surface and notice student thinking, at all levels of correctness and completion. When walking around during pair work, Dr. A could observe and respond to students' ideas. Also, Dr. A's explicit framing of all contribution types as legitimate through open solicitations and encouragement, ensured that students voiced confusions and partial understandings, not just confident correct answers. Dr. B's facilitation afforded less systematic surfacing of student ideas. Dr. B asked many questions, yet most were responded to by one student at a time, so a range and variety of student ideas were not easy to pick up on. Importantly, students rarely voiced incorrect ideas, and mistakes and confusions were not part of the whole-class discussion in Dr. B's class.

Conclusions

Our results indicate that the observed instructors used distinct approaches to facilitate student participation. Dr. A and Dr. B utilized different participation structures, posed different kinds of questions, and used contrasting approaches in responding to students' verbal contributions. Similarly to Pinto (2019) and Viirman (2015) we found that lecturing is not a uniform teaching style. This paper contribute to the field's understanding of the nuances of mathematics teaching practices at the university level (Speer, Smith, & Horvath, 2010). However, how widespread the observed facilitation moves are among mathematics instructors remains an open question.

By interpreting observed variations using the TRU framework (Schoenfeld, 2018), we further suggested that Dr. A's and Dr. B's approaches to facilitating student participation have different consequences for learning. Dr. A's frequent use of Pair-Share activities and non-evaluative questioning routines, afforded *all* students with consistently moderate level of cognitive demand and provided ample opportunities to surface and respond to partial understandings. Students could not hide in Dr. A's class; everyone's name was called-on at some point. Yet, no student "out-shined" others; identities of competence (Gresalfi et al., 2009) were not a salient aspect of the classroom stage. Content learning opportunities were distributed more equally, but we suspect classroom interactions may have not supported students in developing strong disciplinary identities. Dr. B's diverse questions provided students with a range of "productive struggle" opportunities and contributed to the enactment of multifaceted mathematical practice. Extended, student-centered episodes provided a few students with opportunities to generate and refine mathematical ideas on a public stage in ways that positioned them as mathematically competent. For observers, this was an "existence proof" that mathematics *can* be generated by a student, and thus constituted an important socio-mathematical norm (Yackel & Cobb, 1996). But, did all students emerge to see themselves as mathematically competent? We suspect not. The extent to which each approach to facilitating student participation (Dr. A's or Dr. B's) actually contributed to longer term outcomes for learning and identity remains an open question for future research. Coordinating such analyses with student assessment and interview data could be a productive direction for future research.

When debating the effectiveness of different teaching approaches, it is important to keep in mind there might not be a single "best" way. Here, we showed that distinct approaches to incorporate "minimally invasive" active learning in lectures are possible and may have different affordances in terms of learning and identity. Thus, negotiating our values and ultimate goals for advanced mathematics education is an important part of the conversation about what is "best".

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EXPLORING TEACHERS' RESPONSIVENESS TO CHILDREN'S FRACTION THINKING AND RELATIONSHIPS TO FRACTION ACHIEVEMENT

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Identifying components of teaching that make a difference in children's learning is an ongoing challenge in our field. Focusing on teaching that is responsive to children's fraction thinking, we decomposed responsiveness into the instructional practices of questioning to support and extend children's thinking, noticing children's thinking, and anticipating children's thinking. We worked with 49 teachers in grades 3–5 in multiyear professional development and assessed their expertise in each of the practices. We also assessed their students' fraction achievement at the beginning and end of the school year. Correlational analyses revealed significant moderate relationships among teachers' expertise in the three practices, and a multilevel regression analysis revealed significant positive relationships for both expertise in teacher questioning and years of professional development with children's fraction achievement.

Keywords: Instructional Activities and Practices, Professional Development

An ongoing challenge in mathematics education is identifying components of teaching that make a difference in children's learning (Hiebert & Grouws, 2007). We take up this challenge guided by a vision of teaching in which children's mathematical thinking is centered and teaching is responsive to that thinking. By *responsive to children's mathematical thinking,* we mean teaching that elicits children's ideas and takes up and builds on those ideas as an integral feature of instruction (Richards & Robertson, 2016). This kind of responsiveness involves the continuous adjustment of decisions during instruction about what to pursue and how to pursue it in response to children's ideas.

A small number of studies have documented that teachers' instructional practices related to responsiveness to children's mathematical thinking are linked to children's achievement. For example, Webb and colleagues (2014) found that the more teachers engaged children in each other's thinking during instruction, the higher children's achievement was on a story-problem assessment. Bishop (in press) found that the more teacher discourse reflected uptake of children's ideas, the higher the gains were in children's proportional reasoning. (See also Fennema et al., 1996; Howe et al., 2019; Ing et al., 2015; Saxe et al. 1999).

We contribute to this body of work by presenting findings from our analyses of relationships between teachers' responsiveness to children's fraction thinking, which we decomposed into three instructional practices, and children's fraction achievement. The three practices were selected as a focus of our study because we considered them foundational to teachers' expertise in responding in the moment to children's fraction thinking. They include teacher questioning to support and extend children's mathematical thinking (Jacobs & Ambrose, 2008; Jacobs & Empson, 2016), teacher noticing of children's mathematical thinking (Jacobs et al., 2010), and teacher anticipating of children's mathematical thinking (Smith & Stein, 2018). Throughout the paper for the sake of brevity, we refer to these practices in shorthand as questioning, noticing, and anticipating, with the qualification that each focuses on children's mathematical thinking. We recognize other important aspects of teachers' responsiveness, such as to children's cultural, social, and linguistic identities (Parsons et al., 2018) but do not address them here.

Data for our study were drawn from a professional development (PD) design project in which we enrolled three successive cohorts of teachers to participate in up to three years of professional development (Jacobs, Empson, Pynes, et al., 2019). The PD was designed to support the development of teachers' expertise in responsiveness to children's fraction thinking, informed by research on the instructional practices and on children's fraction thinking (Empson & Levi, 2011). Our goals in this study were to explore relationships among the three practices comprising this expertise and between this expertise and children's fraction achievement. We collected data at the end of the third year of the project, in which teachers were enrolled in either their first, second, or third year of PD, to capture variation in expertise in questioning, noticing, and anticipating. Assessments given near the beginning and end of the school year documented children's fraction achievement at each point. These data allowed us to answer two questions: *(1) Do the instructional practices of questioning, noticing, and anticipating children's fraction thinking relate to each other? (2) Does teachers' expertise in questioning, noticing, and anticipating children's fraction thinking relate to children's fraction achievement?*

Conceptual Framework

Our conceptualization of responsiveness to children's mathematical thinking is informed by a theoretical view of teaching that foregrounds the work of teaching and its decomposition into instructional practices that are complex enough to authentically represent teaching but simplified enough to be accessible to teachers who are developing expertise (Grossman et al., 2009; Hiebert & Morris, 2012). The work required to enact a vision of teaching as responsive to children's mathematical thinking has been increasingly parsed by researchers (Boerst et al., 2011; Jacobs & Spangler, 2017; Franke et al. 2009; Munson, 2019). Although researchers have identified a number of potential practices, we selected three based on their connections to teachers' capacity to be responsive to children's mathematical thinking in the moment.

The first practice, *questioning to support and extend children's mathematical thinking,* involves making children's thinking visible during instruction and responding to that thinking in ways that consider children's existing understandings (Fraivillig et al., 1999; Jacobs & Ambrose, 2008; Jacobs & Empson, 2016). We have conceptualized the essence of this questioning as embodied by a teacher in conversation with children to explore their thinking about a mathematics problem—often a story problem—by posing questions to elicit children's thinking and pressing children for explanations of specific parts of their problem-solving processes (Jacobs, Empson, Jessup, & Baker, 2019). In these conversations, a teacher may also question to ensure children are making sense of a problem, link children's representations to the story context (if one exists), encourage children to consider other strategies, connect children's thinking to mathematical notation, or pose a related problem linked to children's understandings. Questioning is customized with respect to children's thinking and can be enacted during instruction in both one-on-one conversations with children as well as in conversations with groups of children, such as during whole-class discussions of children's strategies.

The second practice, *noticing children's mathematical thinking,* involves attending to and making sense of children's thinking in the moment. We have previously conceptualized noticing as a set of nested skills that are temporally and conceptually linked, which include attending to the details of children's strategies, interpreting children's understandings reflected in those details, and deciding how to respond on the basis of those understandings (Jacobs et al., 2010). Noticing is an invisible practice, in that it occurs prior to a teacher's observable response. Thus, teacher noticing is foundational for teacher questioning because, without noticing, teachers would not be able to question in ways that were customized with respect to children's thinking.

The third practice, *anticipating children's mathematical thinking*, involves envisioning how children might engage in solving a problem. We draw on a conceptualization of anticipating as teachers' consideration of the array of strategies that children would be likely to use for a problem prior to posing that problem. Anticipating orients teachers to possible conversations with children during instruction and can inform selecting and adapting problems, interacting during circulating, and planning for and facilitating discussions (Simon, 1995; Stein et al., 2008). Thus, teacher anticipating prepares teachers to notice and question.

This set of practices has three qualities which we argue are useful for enhancing teacher learning in PD and beyond. First, the practices are organized around a specific focus to lend coherence to the set. In our study, this focus was responsiveness to children's fraction thinking. Second, the practices are accessible to teachers as they are beginning to learn but also offer room for growth. They were therefore usable in all three years of our PD. Third, the practices are generative with respect to teachers' continued learning in that as teachers enact the practices in their classrooms, they have opportunities to not only support children's thinking but also improve their understanding of children's thinking and use this understanding to further develop expertise in the practices. Based on earlier findings about the generative nature of practices used by teachers to engage with young children's mathematical thinking (Franke et al., 2001), we conjectured that our practices would create similar opportunities for the teachers who completed our PD and continued to use the practices. In selecting a set of practices, we drew inspiration from a well-known precedent in mathematics education: the "5 practices," which are focused on the expertise needed to facilitate whole-class discussions of children's solutions to cognitively demanding tasks, and were also designed to support teacher learning (Stein et al., 2008).

Methods

Participants

The study was situated in three demographically diverse neighboring districts in a state in the southern region of the United States. A total of 49 teachers and 876 children were included in the analysis. The teachers represented a subset of the 92 teachers who were participating in the larger PD design project and were selected because they were working as classroom teachers in an upper elementary grade (3–5), were available to have one of their mathematics lessons observed, and had at least a third of the children in their classes who completed both the fall and spring administrations of the fraction assessment. Data were collected during one school year, when teachers were at the end of their first ($N = 15$), second ($N = 20$), or third ($N = 14$) year of PD.

The 49 teachers (42 females, 7 males) ranged in teaching experience from 2–36 years, with a mean of 12.4 years. The mean number of children per class who completed the fraction assessment was 18 (72% of the class) and ranged from 8–27 (33%–96% of the class).

Professional Development

The PD in which teachers were participating at the time of data collection was focused on teachers' responsiveness to children's fraction thinking, conceptualized in terms of the three instructional practices described above combined with research-based frameworks of children's fraction thinking (Empson & Levi, 2011). It included over 150 hours of face-to-face workshops over three years. Workshop activities involved working with children, analyzing children's written work, and discussing videos of math instruction focused on classroom instruction, small group instruction, and one-on-one conversations with children. These experiences provided teachers with opportunities to reflect on their teaching, explore new practices, and collaborate with colleagues (Jacobs, Empson, Pynes, et al., 2019).

Teacher Assessments and Scoring

We assessed teachers' expertise in each of the practices separately. The questioning assessment was based on a lesson observation and the noticing and anticipating assessments were written assessments. The questioning data were independently scored by at least two researchers and all disagreements were resolved through discussion, a process described as a consensus method for reliability (Goldsmith et al., 2014). Noticing and anticipating data were blinded so that teacher identities were hidden, and all data were at least double-scored. Interrater reliability was 80% or higher and discrepancies were resolved through discussion.

Questioning assessment. We asked teachers to plan a lesson that included at least one Equal Sharing story problem with a fractional answer (e.g., 6 children sharing 10 pancakes equally), using whatever lesson format they would normally use for story problems. All lessons were videorecorded by a member of the research team using a camera that followed the teacher, to capture all mathematical conversations between the teacher and children and as many details as possible of children's mathematical thinking. After the lesson, teachers were interviewed about the lesson. We used the videorecorded observations, supplemented by the interviews, to determine the level of responsiveness in teacher questioning. All parts of the lesson that focused on fraction story problems were considered, including launch, circulating, and discussion phases.

Rather than consider the lesson representative of teachers' typical instruction, we considered it evidence of teachers' capacity for questioning to support and extend children's fraction thinking. To indicate the extent of evidence of expertise in questioning in the midst of instruction, we assigned a holistic score of 1 ($N = 8$), 2 ($N = 18$), 3 ($N = 15$), or 4 ($N = 8$) to each teacher, with 4 representing the most evidence of expertise. We developed our scoring using an iterative process, which started by adapting prior research on teachers' engagement with children's thinking (Franke et al., 2001) and incorporating findings from earlier research on questioning (e.g., Franke et al., 2015; Jacobs & Ambrose, 2008; Jacobs & Empson, 2016).

Broadly, our scores reflected a continuum. We were not looking for "perfect" questioning but rather evidence that questioning made room for children's existing understandings and building on those understandings. At the high end, teachers actively explored children's thinking. Their questioning was customized with respect to the details of children's thinking and persistent in eliciting the details of that thinking. If there was a group discussion, children's thinking and talking predominated. When children shared their thinking, teachers followed up to support and extend that thinking, and children were regularly given opportunities to describe their thinking and engage with the thinking of others. In short, there was room for children to work from their existing understandings, and teachers positioned children as having authority for sensemaking. At the low end, teachers tended to question to evaluate the correctness of children's thinking and often took over children's thinking, especially when children had incorrect responses. If there was a group discussion, the teacher's thinking and talking predominated and children were provided few opportunities, if any, to describe their thinking or engage with the thinking of other children. In short, there was little room in these lessons for children to develop their existing understandings, and teachers positioned themselves as the authority for sensemaking.

Noticing assessment. Building on our earlier work on professional noticing of children's mathematical thinking (Jacobs et al., 2010), we assessed teachers' expertise in noticing with a written assessment structured around three instructional scenarios linked to solving fraction story problems. The scenarios were conveyed by strategically selected artifacts (video or children's written work), and for each scenario, teachers were asked to notice children's thinking and respond, in writing, to four categories of prompts. These prompts were related to the component skills of noticing: (a) attending to children's strategy details (*Describe in detail what you think each child did in response to this problem.*), (b) interpreting children's understandings (*Explain what you learned about these children's understandings.*), (c) deciding how to respond via follow-up questions (*Describe some ways you might respond to this child's work on the problem and explain why.*), and deciding how to respond via next problems (*What problem or problems might you pose next? What is your rationale?*).

We scored teachers' responses for the extent to which we had evidence for their engagement with children's fraction thinking. We then conducted a latent class analysis on the scores to empirically identify groups of teachers who displayed similar patterns of responses across the noticing assessment. This analysis yielded a 3-profile solution that was ordered in terms of overall noticing expertise. We assigned the profiles a score of 1 ($N = 25$), 2 ($N = 33$), or 3 ($N = 14$) for use in our multilevel model, with 3 representing the highest level of expertise. At the high end, teachers showed consistently strong expertise across the noticing component skills. They centered children's thinking in all their responses and the details of children's strategies were consistently visible. At the low end, teachers showed consistently weak expertise across the noticing component skills. They provided fewer details in their strategy descriptions and those details played a smaller role in other responses. Further, teachers sometimes privileged their own strategies over children's strategies by describing how they would funnel the children's thinking toward the teachers' preferred strategies (Wood, 1998). (See Jacobs & Empson, 2021, this volume, for more information on the noticing assessment and analysis.)

Anticipating assessment. In contrast to questioning and noticing, the construct of teacher anticipating was exploratory in that we had little prior empirical research on which to base our assessment. We created a written assessment with two open-ended items that asked teachers to anticipate a range of valid strategies that elementary-grades children might use to solve two fraction story problems. One was a Partitive Division (Equal Sharing) problem and the other was a Measurement Division problem, although neither was labeled as such for the teachers.

We scored each item on a 0–4 scale, for a total maximum score of 8. Teachers' scores on the assessment ranged from 2–8, with a mean of 5.8 ($SD = 1.7$). At the high end of the scale, teachers anticipated a variety of distinct strategies that were consistent with typical strategies children have been documented to use, spanning multiple levels of understanding and showcasing variety within those levels. At the low end, teachers tended to anticipate a smaller number of strategies that showed less variety and were sometimes accompanied by strategies that were inconsistent with research findings about children's fraction thinking and its development.

Fraction Assessment for Children

We assessed children's fraction achievement with a written assessment teachers administered to their students in the early fall and late spring. Teachers were told to allot 45 minutes but encouraged to allow extra time for children who wanted it. The assessment consisted of 7 items—5 fraction story problems and 2 fraction comparisons (see Table 1 for sample story problems). All items were open response and children were simply instructed to solve each problem. The story problems were designed to assess children's understanding of fraction quantities and relationships in story situations, whereas the comparison problems were designed to assess children's understanding of ordering relationships without the support of a story situation. The assessment was developed using an evidence-centered design approach and included protocols to ensure content-related validity (Mislevy & Haertel, 2006).

There were two versions of the assessment—one for grade 3 and one for grades 4 and 5. The versions were parallel, with simpler numbers in the fraction story problems for grade 3. Due to the parallel nature of the assessments, they were treated as equivalent in the analyses. For all grades, the fall and spring forms of the assessment were identical.

Assessments were blinded for scoring. Scoring took place in teams of 3–5 researchers, who were trained on using a code book developed for the assessment. Total scores for the assessment ranged from 0–12, with the 5 fraction story problems each scored 0–2 and the two fraction comparisons each scored 0–1. When scoring the 5 story problems, we considered both the correctness of the children's answers and the validity of their strategies. When scoring the 2 fraction comparisons, we considered both the selection of the greatest number and children's rationales. Interrater reliability was at or above 80% for all items and the internal consistency of the assessment was adequate, as indicated by a Cronbach's alpha of .80. The grand mean of the 49 class means improved from fall ($M = 2.2$, $SD = 1.9$) to spring ($M = 5.8$, $SD = 1.9$), showing that learning did occur. The grand mean of 5.84 (out of 12) in the spring suggests that the assessment was challenging.

Table 1: Sample Story Problems from the Fraction Assessment for Children

Mathematical Focus	Grades 4/5 items (*Grade 3* number adjustments)
Equal Sharing	Mr. Lara gave 3 children 5 oranges to share so that each child got the same amount. If the children shared all of the oranges, how much orange did each child get? (*Grade 3:* 4 children, 9 oranges)
Multiplication	It takes 1/5 of a block of cheese to make a pizza. How much cheese do you need to make 17 pizzas? (*Grade 3:* 1/4 of a block of cheese, 6 pizzas)
Missing Addend	Allie has 1 6/8 sticks of butter. She needs a total of 5 1/8 sticks of butter to make cookies. How much more butter does Allie need so that she can make cookies? (*Grade 3:* 1 2/3 sticks of butter, 4 sticks of butter)

Findings

Research Question 1: *Do the instructional practices of questioning, noticing, and anticipating children's fraction thinking relate to each other?*

To explore the relationship among the instructional practices, we began by examining the three pairwise correlations, including the correlation between questioning and noticing (r (47) = .56), between questioning and anticipating (r (47) = .50), and between noticing and anticipating (r (47) = .54). All pairwise correlations were significant ($p < .05$) and of similar, moderate strength, suggesting that the practices are related, but distinct.

We were also interested in the relationships among the three practices when they were considered as a set. We therefore conducted a partial correlation analysis to identify the strength of the relationship between any two practices when all three were included but the effect of the third practice was removed. Again, all three partial correlations were significant ($p < .05$) and of moderate strength: .39 for questioning and noticing, controlling for anticipating; .29 for questioning and anticipating, controlling for noticing; and .36 for noticing and anticipating controlling for questioning. These findings suggest that the three practices together have interrelated features reflective of a coherent set of practices.

Research Question 2: ***Does teachers' expertise in questioning, noticing, and anticipating children's fraction thinking relate to children's fraction achievement?***

We conceptualized teachers' responsiveness to children's fraction thinking as the collection of our three practices—questioning, noticing, and anticipating children's fraction thinking. We were interested in the relationship between teachers' expertise in these practices, as captured by our three teacher assessments, and children's achievement. We used the children's spring scores on the fraction assessment as our measure of children's achievement and included children's fall scores on the identical assessment as a covariate. One-tailed tests were conducted because we hypothesized based on prior research that more questioning, noticing, and anticipating expertise should increase children's achievement (Carpenter et al., 1989; Jacobs et al., 2007).

We began by examining the three practices independently in unadjusted bivariate models. All three practices were significantly related to children's spring achievement ($p < .05$) and so were included in the multilevel model. We next constructed our multilevel model with the practices as independent variables, the children's spring achievement as the dependent variable, and the children's fall achievement as a covariate. The overall multilevel model (see Model 1 in Table 2) was significant (Wald χ^2 (4, $N = 876$) = 425.10, $p < .05$). Children's fall achievement was significantly related to their spring achievement, as expected. At the teacher level, questioning was the only practice that remained significantly related to children's spring achievement. Given that our PD was designed to help teachers develop expertise in the three practices, we were also interested in the relationship between the number of years of PD teachers had completed and children's fraction achievement. Further, we were interested in whether the relationship between teacher questioning and children's achievement would remain significant even when years of PD was included as a teacher-level variable. We therefore extended our multilevel model to include teachers' years of PD (see Model 2 in Table 2). As before, the overall model was significant (Wald χ^2 (5, $N = 876$) = 435.40, $p < .05$). In addition, the number of years of PD was significantly related to children's spring achievement, and again, teacher questioning remained significantly related to children's spring achievement.

Table 2: Models Relating Instructional Practices and Children's Spring Achievement

	Model 1		Model 2	
	Coefficient (*SE*)	*z*	Coefficient (*SE*)	*z*
Children's Fall Achievement	0.66 (0.03)	20.18*	0.66 (0.03)	20.26*
Teacher Questioning	2.59 (1.04)	2.50*	1.99 (1.02)	1.95*
Teacher Noticing	0.04 (0.37)	0.11	– 0.08 (0.35)	– 0.23
Teacher Anticipating	– 0.48 (1.15)	– 0.41	– 0.91 (1.11)	– 0.82
Years of PD			0.66 (0.29)	2.31*

*$p < .05$, one-tailed

In summary, increased expertise in teachers' questioning and increased years of PD were directly linked to children's higher achievement on the spring assessment, after adjusting for fall achievement. Expertise in teacher noticing and teacher anticipating were not significantly related to children's spring achievement after adjusting for fall achievement. However, as seen in the analyses for Research Question 1, expertise in these practices was significantly related to expertise in questioning. In other words, they appear to be necessary but not sufficient for teachers' questioning expertise and its direct link to children's achievement.

Discussion

Our first research question focused on investigating the relationships among teachers' expertise in three instructional practices representing responsiveness to children's fraction thinking. Our findings provide evidence that teachers' expertise in each practice is positively related to expertise in the others. We conjecture that these positive relationships may be due to the joint knowledge base on which the practices draw—namely, knowledge of children's thinking, which includes general knowledge of children's thinking and specific knowledge of individual children's thinking. However, each practice also captures a distinct piece of responsiveness, connected with how this knowledge is used in the work of teaching represented by each practice. For example, when teachers notice, they start with a specific instance of a child's thinking and connect it with what they know, but when teachers anticipate, they start with what they know and use it to generate possible instances of a child's thinking.

Our second research question focused on investigating relationships between teachers' expertise in instructional practices and children's fraction achievement. Our findings provide empirical support for the direct link between the two, and we add to a small but growing body of evidence of positive relationships between practices that are responsive to children's mathematical thinking and children's mathematics achievement (see, e.g., Bishop, in press; Webb et al., 2014). We highlight in particular the significant positive relationship in our final multilevel model between teacher questioning and children's achievement. Children in the classrooms of teachers with higher questioning scores tended to have higher fraction achievement, indicating their greater capacity to apply their understandings of fractional quantities and relationships to solve story problems and compare fractions. This finding provides evidence of the power of questioning to support and extend children's fraction thinking, which we conjecture resides in creating ongoing opportunities for children to articulate, consider, coordinate, and refine their fraction understandings in conversations with the teacher during instruction. We assessed teachers' capacity to create such opportunities and future research should directly assess the opportunities created over time in teachers' classrooms.

With this study, we identified three practices—questioning, noticing, and anticipating children's mathematical thinking—that comprise a set of related instructional practices for teaching that is responsive to children's thinking. We intentionally focused on a small number of practices because it offered a manageable way for teachers to engage with the complexity of responsiveness. Our findings suggest that teachers were indeed able to engage with this complexity. A focus on a manageable, but coherent, set of instructional practices defined by teachers' work with children's thinking thus offers a way to decompose teaching expertise so that it is accessible for teachers and can make a difference in children's learning.

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AN ANALYSIS OF PATTERNS OF PRODUCTIVE AND POWERFUL DISCOURSE IN MULTILINGUAL SECONDARY MATHEMATICS CLASSROOMS

UN ANÁLISIS DE DISCURSO PRODUCTIVO Y PODEROSO EN AULAS MULTILINGÜES DE MATEMÁTICAS DE SECUNDARIA

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We present an analysis of three 9th grade integrated mathematics lessons in which a group of teachers and researchers redesigned a sequence of lessons with the goal of engaging a linguistically diverse group of students in productive and powerful discussions (Herbel-Eisenmann et al. 2013). The three lessons were part of a design experiment. Two lessons were observed during regular school day instruction, and the other lesson was part of an after-school teaching experiment. Drawing on a sociocultural framework and methods of classroom discourse analysis (Cazden, 2001; Pierson, 2008), we analyze how the teachers in the three settings structured whole-class discourse to create opportunities for a multilingual group of students to participate in the discussion and to appropriate mathematical tools for thinking.

Keywords: Equity, Inclusion, and Diversity; Design Experiments; Algebra and Algebraic Thinking; Classroom Discourse

In this paper, we describe results from a design research effort in which a group of teachers and researchers redesigned a sequence of lessons with the goal of engaging ninth graders in academically productive whole-class discussions (Herbel-Eisenmann et al., 2013). The research was situated in a US school where nearly all students were from minoritized communities, most students were multilingual, and about 30% of students were designated as English Learners (ELs). Drawing on the Academic Literacy in Mathematics framework (Moschkovich, 2015), project-specific design principles (Zahner et al., 2021a, 2021b), and research on student learning of linear rates of change (Lobato & Ellis, 2010; Thompson, 1994), we created a sequence of lessons specifically designed to promote student participation in classroom discussions about linear rates of change. In the empirical results below, we show how, in comparison to student engagement before the design intervention, the redesigned lessons led to increased student participation and higher levels of cognitive work in whole class mathematical discussions.

Theoretical Framework & Prior Research

This study is rooted in a sociocultural perspective on learning, where mathematics learning is conceptualized as appropriating problem-solving tools and developing participation in valued mathematical practices, including mathematical discourse practices (Forman, 1996; Moschkovich, 2002). In alignment with this theoretical focus, our analyses focus on classroom discussions and patterns in teacher and student discourse. We also drew upon the Academic

Literacy in Mathematics (ALM) framework (Moschkovich, 2015) to create study-specific design principles (Figure 1). In the ALM framework, developing academic literacy in mathematics includes developing forms of mathematical *proficiency*, engaging in mathematical *practices*, and participating in mathematical *discourse*.

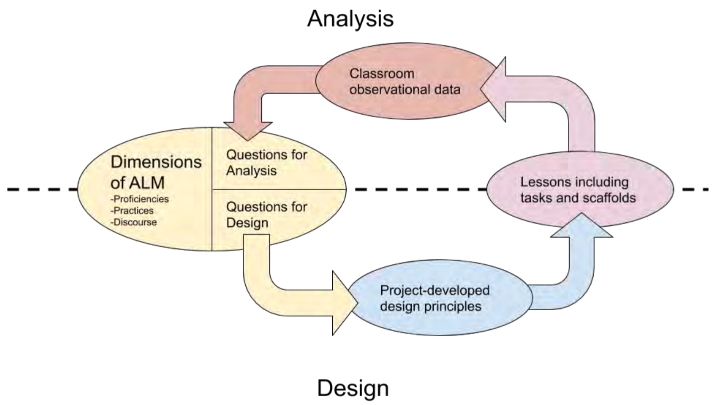

Figure 1: Overview of the Design Cycle. (Image credit Authors year, used with permission)

Multilingual students, including those learning the language of instruction, can learn critical mathematical concepts and reasoning practices through engaging in productive and powerful discussions (Chapin & O'Connor, 2012; Erath et al., 2018; Erath et al., 2021; Gutiérrez, 2002; Khisty & Chval, 2002; Moschkovich, 1999). Following Herbel-Eisenmann et al.'s (2013) definitions, *productive* discussions are those that lead students to appropriate mathematical tools for thinking (i.e., develop mathematical proficiencies and practices in the ALM framework). *Powerful* discussions are those that promote students' participation in and identification with mathematics (i.e., engage in the disciplinary discourse). One way to foster productive and powerful discussions is for students and teachers to engage in authentic dialogue focused on important mathematical concepts (e.g., O'Connor, 2001). Yet, most multilingual students who are classified as English Learners have very limited opportunities to engage in productive and powerful classroom discussions. Instead, these students are often tracked into low-level classes where they have little access to either rich content learning opportunities or the discourse of the discipline (Callahan, 2005; Kanno & Kangas, 2014). Thus, one critical question facing the field is how to transform patterns of classroom discourse in multilingual settings.

The default template for most classroom discussions is the triadic Initiation-Response-Evaluation (IRE) sequence (Cazden, 2001). Initiations are questions or statements. Responses occur after, and in response to, an initiation. Evaluations are moves that offer judgement–either explicit or implicit–on the response and end the IRE sequence (Cazden, 2001; Mehan, 1979). IRE-dominated instruction typically positions the teacher as the authority, thereby limiting students' opportunities to engage in productive and powerful discussions. In this project our goal was to transform patterns of discourse. Yet, we found that, while the ideal transformation of classroom talk may be to create dialogic discussions like the one in O'Connor (2001), achieving such transformation is challenging, possibly due to institutional constraints and the deeply ingrained patterns of discourse in school mathematics (Herbel-Eisenmann et al., 2010).

Short of a wholesale transformation, there are subtle ways to document transformations in classroom talk while still within the IRE framework. For example, it is possible to distinguish between evaluation and follow-up moves in the "third slot" of the IRE sequence (Wells, 1993).

Follow-up moves include asking students to expand on their reasoning, presenting new examples to build on their contributions, and asking for clarification (Pierson, 2008; Wells, 1993). While evaluation moves typically close a sequence, follow-up moves can extend the discussion. When students respond to a teacher's follow-up move, they often provide a justification or explain the reasoning behind their answer – two examples of disciplinary discourse practices (Moschkovich & Zahner, 2018).

Pierson's (2008) analysis of teacher and student talk in 13 seventh-grade mathematics classrooms offers an avenue for unpacking more and less productive uses of talk within the triadic IRE framework. Pierson (2008) developed two coding schemes: (a) one to capture the level of a teacher's *responsiveness*, the extent to which the teacher focused on student thinking, and (b) a second focused on the level of *intellectual work*, the kind of cognitive effort imposed on or requested from students within a teacher's move. Pierson found a positive relationship between more responsive teacher moves and higher levels of intellectual work in teacher talk with growth in students' mathematics achievement as measured by a curriculum-aligned assessment.

The constructs of intellectual work and responsiveness connect to this project's goal of promoting productive and powerful discussions. In a productive discussion, we would expect to see higher levels of intellectual work. In a powerful discussion, we would expect to see higher levels of responsiveness as teachers take up and build upon students' ideas. Thus, we adopted Pierson's (2008) coding schemes to explore whether our design efforts were effective in promoting powerful and productive talk in linguistically diverse classrooms.

In the analysis that follows, we address the following research question: To what extent did each lesson engage multilingual students in productive and powerful discussions? Specifically,

1. What were the levels of intellectual work and responsiveness in teacher moves?
2. What was the distribution and frequency of student participation in whole-class discussion?

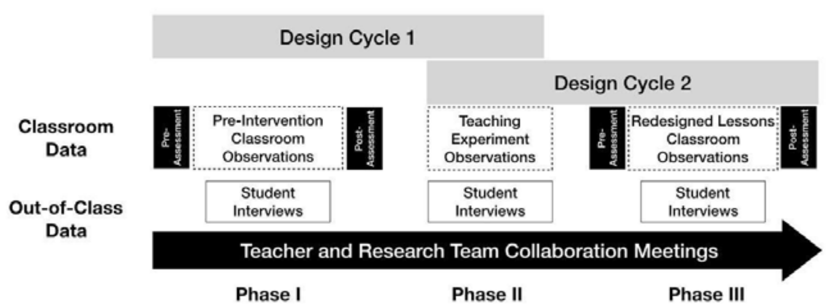

Figure 2. Design Cycles (Image credit Authors year, used with permission)

Methods

The overall framework for this research arose from design research (Cobb et al. 2003). While researchers have identified productive practices in multilingual classrooms (e.g., Chval & Chávez, 2012; Chval et al., 2021), these productive practices appear to be relatively rare in linguistically diverse mathematics classrooms (Callahan, 2005). Therefore, design research was

Olanoff, D., Johnson, K., & Spitzer, S. (2021). *Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Philadelphia, PA.

chosen as a method for studying phenomena unlikely to arise without intervention (Cobb et al., 2003). The project included two design cycles spread across three academic years (Figure 2).

Setting

We present data from three lessons recorded in ninth grade Integrated Mathematics 1 (IM1) classes at City High, an urban high school located near the US-Mexico border serving linguistically diverse students. The school was chosen as a research site in order to situate this research in a setting that parallels the inequitable educational experiences of minoritized students in US schools, particularly students who are classified as English Learners (Gándara & Contreras, 2009). At City High, 77% of students were identified as Latinx, 12% Asian, 7% African American, and 4% other. About 89% of students were from low-income families. Thirty percent of all students at City High were classified as ELs. Three City High IM1 teachers joined our research group in redesigning a unit on linear rates of change for Design Cycles 1 and 2.

The Phase I data includes eight class meetings recorded during regular school hours. These lessons were taught by the students' regular teacher, Mr. S, who was certified to teach mathematics and who had taken teacher education coursework related to teaching ELs. Mr. S was bilingual in Spanish and English. He primarily spoke in English during class, and he talked to some students in Spanish during small group work. The Phase II data includes ten lessons taught after school in a Teaching Experiment (TE) setting. The TE lessons were designed by the teachers and researchers. A bilingual researcher with experience facilitating classroom discussions with linguistically diverse students taught the Phase II TE lessons while Mr. S and the other teachers and researchers served as observers. The Phase III data includes ten redesigned lessons taught by Mr. S during regular school hours. In the analysis presented below, we focus on one lesson from each phase, each chosen for analysis because they feature a pivotal concept in the design experiment unit—introducing average rate of change. All whole-class discussions and talk among one small group of students were transcribed. Further, students were invited to participate in the language of their choice across all three phases. This invitation was made explicit in Phase II. In Phases I and III, students could use the language of their choice, but this option was not emphasized. In this analysis, we narrow our focus to whole-class discussions.

Redesign

The design principles and illustrations of the redesigned lessons are presented in Authors (Zahner et al. 2021a, 2021b). In brief, the main foci of the redesign effort were developing student participation in productive and powerful discussions through (a) adopting a coherent mathematical focus across the unit and strategically using problem contexts, (b) designing a unit of lessons with intentionally integrated mathematical and language development goals, and (c) integrating language and discourse supports including technology and mathematical language routines (Zwiers, 2017) throughout the unit.

Analysis

To start our analysis, we coded the transcripts as whole-class and small group interactions, noting the time spent in each participation structure. We noted that in the Phase I lesson (Pre-intervention), 41 of 48 minutes (~85%) were whole-class interactions, 36 of 63 minutes (~57%) in Phase II (TE), and 46 of 77 minutes (~60%) in Phase III (Redesigned lesson). We then examined discourse patterns during these whole-class interactions, coding each teacher- and student-turn of talk as I, R, E, or F, allowing up to two codes per turn of talk since teachers often offer an evaluation and then initiate a new question in one turn.

Next, we used Pierson's (2008) responsiveness and intellectual work coding schemes for analyzing each teacher's talk during whole-class discussions. Pierson's (2008) responsiveness coding scheme categorized each teacher follow-up move (F) into one of four levels of responsiveness: Low, Medium, High I, or High II. The level of each follow-up was determined based on whether the move addressed a student's comment, whose idea (teacher's versus student's) was the focus, and whose reasoning (teacher's versus student's) was displayed. Pierson's (2008) coding scheme for intellectual work had four categories: Low Give, High Give, Low Demand, and High Demand. The two Give codes were for teacher moves in the third slot (E/F) of an IRE/F sequence of talk that provided information, whereas the two Demand codes were for teacher moves that requested information from the students. Teacher moves that both supplied and requested information were double-coded with a Give and a Demand category. The designations Low and High for Give and Demand codes depended on the type of information being supplied or requested in a teacher's move. Low was for basic information, whereas High was for more elaborate information intended to extend mathematical reasoning.

Consistent with Pierson's coding mechanism, we only coded talk with a math focus (e.g., we did not code segments of classroom management). We coded every instance of talk in the third slots (E/F) of the IRE/F sequences as either Low, Medium, High I or High II responsiveness. We expanded Pierson's intellectual work coding scheme to include both the first or third slots (I or E/F) of the IRE/F sequences as either Low Give, High Give, Low Demand, or High Demand, allowing for double-coding of single turns with both Demand and Give codes when applicable. We chose to include the first slot (I) of the sequence because we were not working with predetermined questions as the teachers were in Pierson's study. At each stage, coding was done by one researcher and then the research team met and reviewed the coding to discuss each code, consider questions, and reach consensus.

Results

Evaluation and Follow up Moves

During the Phase I lesson, Mr. S's most common move during the third slot of the IRE sequence was evaluation, occurring in 94% of coded moves. After noticing this trend, we made transforming the pattern of IRE discourse a target of our design efforts. Our aim was to encourage teachers of the Phase III lessons to use more follow-up moves, such as pressing for reasoning or asking students to elaborate on an idea (Chapin et al., 2009). This pattern of discourse was modeled during the Phase II TE lessons, during which 53% of the researcher's E/F moves were follow-ups. As indicated in Figure 3, this form of discourse appeared to be taken up by Mr. S in the Phase III lesson, where 36% of Mr. S's turns in the third slot of triadic IRE discourse were follow-up moves rather than evaluations.

The level of intellectual work coded in the teachers' talk was relatively consistent across Phase I, Phase II, and Phase III. In general, the majority of teacher moves (both Gives and Demands) were coded as Low. Many of the Low Demand turns were questions that had a known answer and that could be answered without offering an explanation or justification. The Low Give moves included providing information without explanation or justification. Figure 4 shows a summary of Intellectual Work across the three classrooms.

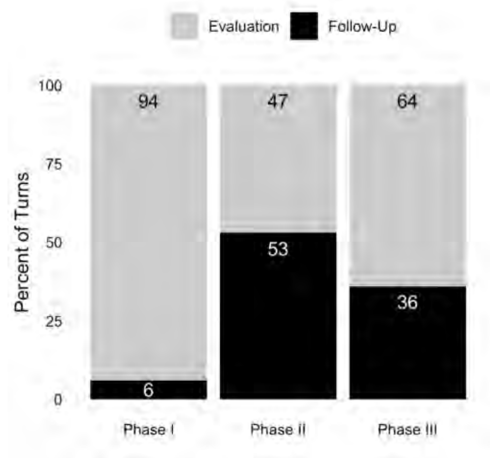

Figure 3. Percent of "third slot" turns that were evaluation or follow-up

Intellectual Work

Despite this consistency, there was one notable shift in the level of intellectual work in the Demand category. During the Phase I lesson, 14% of the teacher's questions were coded as High Demand. This increased to 29% in the Phase III lesson. Thus, while the overall proportion of High Demand turns in each lesson remained under 30%, the proportion of High Demand moves doubled from the Phase I to the Phase III lesson.

There was a complementary shift in the Give category. The proportion of High Give moves decreased from 13% in the Phase I lesson to only 4% in the Phase III lesson. One possible interpretation of this unexpected decrease can be attributed to the teacher using talk moves (Chapin et al., 2009) such as rebroadcasting student input rather than providing high-level explanations. Following the coding scheme, rebroadcasting moves were coded as Low Give. In a sense, Mr. S may have been trying to shift the authority to the students by not engaging in lecture, which decreased the level of Give moves.

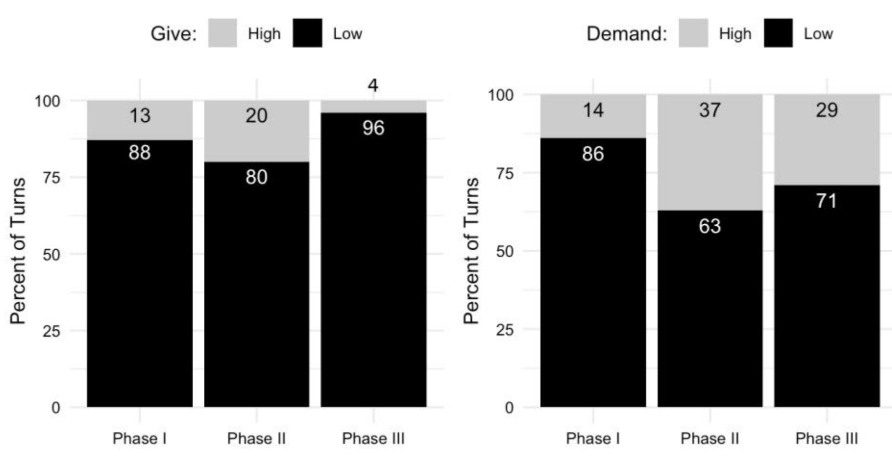

Figure 4. Intellectual Work (Give and Demand) across the three learning environments

Responsiveness

In parallel with the results for intellectual work, the coding for the level of responsiveness of the teachers' discourse showed two trends. First, across all three lessons, the majority of the teachers' talk was coded at a low level of responsiveness. Second, despite this trend, there was also a notable increase in the proportion of turns coded at the high level of responsiveness. Over half of Mr. S's talk in the Phase I and Phase III lessons were coded Low in responsiveness (see Figure 5). However, the proportion of teacher talk that was coded as high in responsiveness (combining the categories High I and High II) increased from 14% to 25%. With this increase, the pattern of responsiveness in the teacher's talk in the Phase III lesson was relatively similar to the pattern of discourse in the TE lesson.

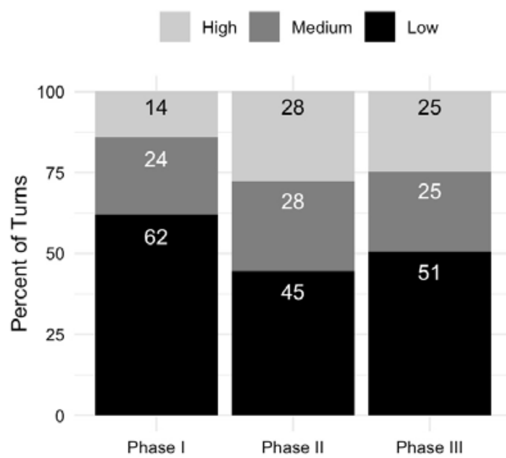

Figure 5. Coding Results for Responsiveness

Distribution and Frequency of Student Participation

Recall that powerful discussions are those which build students' identification with mathematics and broaden student participation (Herbel-Eisenmann et al., 2013). Thus, we were curious about who was contributing and how often in the whole-class discussions. Looking at the patterns of who talked, we were able to characterize the proportion of the talk by the teacher versus students in the whole-class setting. We were also able to identify how many unique students made a contribution to the whole-class discussion. Table 1 shows a total count of the number of coded teacher and student turns, the unique number of students who were called upon by name to contribute to the whole-class discourse, and the number of times each of the called upon students contributed to the whole-class talk.

Table 1. Patterns in Classroom Discussion Participation

	Phase I	Phase II	Phase III
Coded Turns	345	497	447
Students present	28	12	33
Number of unique student speakers called by name	7	12	10
Average turns per identified student	2.5	24	7.3
Number of choral responses	55	7	40
Unidentified student turns	109	0	89
Teacher turns	163	202	230

Looking across the two lessons recorded during regular school hours (Phase I and Phase III), we saw approximately the same proportion of students who were called upon to participate: seven out of 28 students (25%) in the Phase I lesson and 10 out of 33 (~30%) in Phase III. Yet, comparing these lessons, one striking trend was that the students who contributed spoke an average of 7.3 times in the Phase III lesson and only an average of 2.5 times in the Phase I lesson. Thus, for the students who were called on to participate, the average number of turns per student was higher in the Phase III lesson than in the Phase I lesson. As might be expected for the much smaller class in the Phase II TE, all 12 of the students who were present were called upon to contribute to the whole-class discourse, and students tended to have more frequent contributions.

In addition to the identified student turns in whole-class talk during the Phase III lesson, students also had multiple opportunities to talk during small group discussions: a substantially higher proportion (38%) of class time was devoted to group work (compared to only 15% in the Phase I lesson). Therefore, the counts of student turns presented in Table 1 underreport the amount of student talk in the Phase III lesson. Our analysis also revealed that during the designated small group times in the Phase I lesson, very little time was dedicated to group discussion and was actually used for individual seat work. Therefore the counts in Table 1 are more likely reflective of the total number of student turns in the Phase I lesson.

Discussion

The trends we identified in the levels of intellectual work in these three lessons lead to questions about what may explain the relatively frequent incidence of low give and low demand during Phases II and III. One possible explanation is that repeating, one of the talk moves from Chapin et al., (2009), was coded as a low level of intellectual work. Yet, we wondered if this coding accurately captured the effects of these moves. Alternatively, the teachers of these lessons used fewer High Give responses to provide space for the students to supply explanations and needed information to each other. In this respect, we consider this trend from decreasing High Give evaluations or follow-ups to increasing the number of High Demand evaluations or follow-ups as a signal of providing students with more mathematical authority. Finally, an additional alternative to consider is that, in our redesigned lessons, some evaluation and feedback was built into the Desmos activities we created for these lessons, which may have reduced the need for the teacher to voice these moves and be picked up in our analysis.

Considering teacher responsiveness to student thinking, we found that our Phase II and Phase III lesson designs helped the teachers increase the proportion of medium and high levels of responsiveness to student thinking. While we would like to have seen this improve even more, we found that change to classroom discourse in this setting has been gradual. Recall that each phase occurred in a different school year. The ability to document a change in teaching practice even after several months had passed since the TE intervention is noteworthy, and speaks to the potential of our redesigned lessons to support teachers in engaging multilingual students in mathematical discourse.

In our next analyses, we plan to look more closely at the small group interactions that take place both during designated group work segments and those side conversations that take place in small groups during the whole-class discussion. Our preliminary analyses indicate that much more mathematical discussion is happening student-to-student than one finds when focused on the whole-class discussions.

Acknowledgments

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SEMANTICALLY LINKED SYNTACTIC LITERACY AFFORDANCES IN SECONDARY MATHEMATICS

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This report details a literacy affordance framework for describing and connecting the ways in which teachers focus their students on the syntactic structures of reading, writing, speaking, and listening in mathematics. This framework is intended to serve as a critical access point for connecting and moving broader research in secondary mathematics teaching towards a sociolinguistic perspective. The framework is applied to a sample of teachers from two U.S. states to indicate ways in which these secondary mathematics teachers currently attend to such literacies in otherwise dialogically orientated lessons. Findings indicate the applicability of the framework as well as the opportunities and shortfalls in how such teachers currently attend to language in secondary mathematics.

Keywords: Classroom Discourse, Communication, Mathematical Representations, Instructional Activities and Practices

It is impossible to disentangle the use of language from the learning of mathematics. Reeves (1990) states, "language is the essential vehicle for transmitting and understanding mathematics in school, for turning experience into thinking and learning" (p. 213). Pimm (1987) goes further, declaring that mathematics *is* a language, and if mathematics is a language then the teaching of mathematics *is* the teaching of language. Research on how reading, writing, speaking, and listening relate to mathematics teaching in sequester is abundant (e.g., Österholm [2006] for reading; Resnick [1982] and Shield and Galbraith [1998] for writing; Chapin and O'Connor [2007] for speaking; Hintz and Tyson [2015] for listening) but, as Gutiérrez and colleagues (2010) explain, studies on how all four such modalities of language *intertwine* to mediate the teaching of mathematics are lacking. Research on multilingual or English learners in mathematics education has striven to promote a multimodal and resource-oriented perspective to the topic of teaching such students (see de Araujo et al., 2018), but broader research in mathematics education is fraught with culturally neutral (at best) or deficit-oriented (at worst) perspectives towards language (Moschkovich, 2010).

My aim with this study is to describe the opportunities which secondary teachers do (or do not) afford students to grapple with the multimodal, multisemiotic language of mathematics. The present study thus describes a literacy affordance framework which recognizes and connects the multimodal dimensions of language in mathematics teaching. Further, the study demonstrates the utility of this framework in the context of twelve secondary mathematics lessons. Specifically, this study seeks to answer the following questions:

1. In what ways do secondary mathematics teachers enact affordances addressing their students' use of syntactic structures to read, write, speak, and listen mathematically?"
2. In what ways do such teachers' instructional affordances semantically link the syntax of these different modes of language?

Texts, Literacy, and Literacies in Mathematics

One of the challenges in addressing the role of language in mathematics education is the limited definition of literacy within the field. For instance, Draper and Siebert (2010) describe how teachers may not recognize *reading a graph* or *writing an equation* as literacy practices if their conception of literacy is confined to only "fluency in reading and writing [with] print texts" (p. 23). Such restrained conceptualizations of literacy mask opportunities to recognize and study the use of language in mathematics education. This study addresses this concern by integrating consistent and inclusively defined definitions which are meant to better connect ideas of language, literacy, and mathematics.

At the core of this study are the ideas of *texts*, *literacy*, and *literacies*. Although traditional definitions of texts and literacy are limited to a focus on reading and writing printed text (Draper & Siebert, 2010), this work recognizes a more inclusive understanding of such terms. In the present study a *text* is considered any representational object which is intended by its creator to communicate a meaning (Draper & Siebert, 2010; Wells, 1990). Literacy, in turn, can be considered "the ability to negotiate (e.g., read, view, listen, taste, smell, critique) and create (e.g., write, produce, sing, act, speak) texts in discipline-appropriate ways" (Draper & Siebert, 2010, p. 30).

Literacies then are the multiple modes (or "meaning-making systems"; Kress, 2001, p. 11) of texts through which students must navigate during the learning process. These include both primarily receptive (reading and listening) and primarily expressive (speaking and writing) literacies (Aguirre & Bunch, 2012; Bloom, 1974; Draper & Siebert, 2004). Meaning can also be communicated in other ways such as gesture (Arzarello et al., 2009). However, because this study adopts the four primary language demands of reading, writing, speaking, and listening which students face in school mathematics (Aguirre & Bunch, 2012), such modalities fall beyond the focus of this study.

Syntactic Literacy Affordances

Given the current study's definition of texts as representational objects, and literacies as different modes of texts, literacies themselves can be considered representational systems. Goldin (2002) came to a similar conclusion in recognizing the development of representational systems in mathematics as akin to language learning. The current study adopts Goldin's conception of representational systems and flips the focus back to the realm of literacy in mathematics teaching. Of particular relevance is Goldin's (1998, 2002) recognition that representational systems have internal *syntactic* configurations as well as *semantic* relations with other representational systems.

Regarding the syntactic nature of representational systems, Goldin (1998) explains that "To know and be able to construct the configurations formed from characters, and to use the relationships among configurations established by higher-level structures, is one way of giving *meaning* to the characters and configurations in a representational system" (p. 144). Representational systems are not immaculately bestowed with meaning. Rather, understanding and using the syntax of the system fosters that meaning. In the present study this indicates the importance of affording students' opportunities to grasp the syntax of reading, writing, speaking, and listening.

Syntax is traditionally defined as "grammatical relationships among words in a sentence or the structural arrangement among sentences in a passage" (Vacca & Vacca, 2002, p. 381). Given this study's broader definition of text, syntax also refers to valid ways in which symbols or objects that hold mathematical meaning can be procedurally manipulated or configured (Bayaga

& Bossé, 2018; Goldin & Kaput, 1996; Kaput, 1987). A *syntactic literacy affordance* thus occurs when an instructional activity supports students with developing their understanding or use of syntactic structures within the representational systems of reading, writing, speaking, or listening. Specifically, corresponding definitions of such affordances within each relevant literacy are drawn from this overarching definition to form the crux of the literacy affordances framework:

- A **syntactic reading affordance** is when a teacher focuses students on interpreting the syntactic structures of *already-constructed written texts* (representational objects such as written language, graphs, tables, equations, charts, etc.). This instructional move emphasizes how attending to such structures helps to uncover mathematical meaning. Ambiguities of a construct*ed* written text are addressed.
- A **syntactic writing affordance** is when a teacher focuses students on the syntactic structures of *their own written texts*. This instructional move emphasizes how attending to such structures helps to communicate mathematical meaning. Ambiguities while construct*ing* written text are addressed.
- A **syntactic speaking affordance** is when a teacher focuses students on the syntactic structures of *their own spoken texts* (representational language such as explanations, justifications, clarifications, etc.). This instructional move emphasizes how attending to such structures helps to communicate mathematical meaning. Ambiguities while construct*ing* spoken texts are addressed.
- A **syntactic listening affordance** is when a teacher focuses students on interpreting the syntactic structures of *others' spoken texts* (representational language such as explanations, justifications, clarifications, etc.). This instructional move emphasizes how attending to such structures helps to uncover mathematical meaning. Ambiguities of a construct*ed* spoken text are addressed.

Semantically Linked Syntactic Literacy Affordances

Lingering beyond this focus on syntactic literacy affordances are the (previously noted) semantic aspects of representational systems. Indeed, the above definitions of syntactic literacy affordances, with their emphasis on addressing ambiguities within each literacy, are semantic in nature. It would make sense, for instance, for a mathematics teacher to help students *read* the syntax in a graph of a linear function by *speaking* with students about the representation.

This is intentional, as the literacy affordance framework does not describe the modalities used to enact all instruction (which certainly extends beyond the scope of reading, writing, speaking, and listening) but rather which syntax the teacher is *focusing student attention towards* in its relation to mathematical meaning. I fully recognize that teachers and students may be *using* semantic cues as well as syntactic structures within and across literacies throughout their instruction. This emphasis on literacy affordances shifts the conversation away from literacies as they arise with or without the teacher's intention to instead emphasize instructional aspects of literacy that are more directly within the teacher's control.

However, semantic aspects can still be considered through the literacy affordances framework. Kaput (1987) astutely describes the mathematical power in "applying the syntactical properties of a given symbol system's symbol scheme to a new field of reference" (p. 181). In other words, corresponding the syntax of one representational system onto that of a new representational system is "among the key ways that mathematics evolves, both historically and within individuals" (Kaput, 1987, pp. 180-181). This study's framework provides a window into how teachers might promote such correspondences from a literacy standpoint by identifying when teachers address the syntax of *multiple* literacies in relation to a single mathematical text. Such groupings of syntactic literacy affordances are thus considered *semantically linked*.

The Literacy Affordances Framework

When combined, the literacy affordances framework situates syntactic reading, writing, speaking, and listening affordances along similar dimensions as Aguirre and Bunch's (2012) visualization of language demands of reading, writing, speaking, and listening. However, this model stands apart in focusing on the teacher's role in attending to language rather than on the demands of the language itself. Given the importance of semantically linking such affordances, this aspect is centered on the representation of the framework, shown in Figure 1.

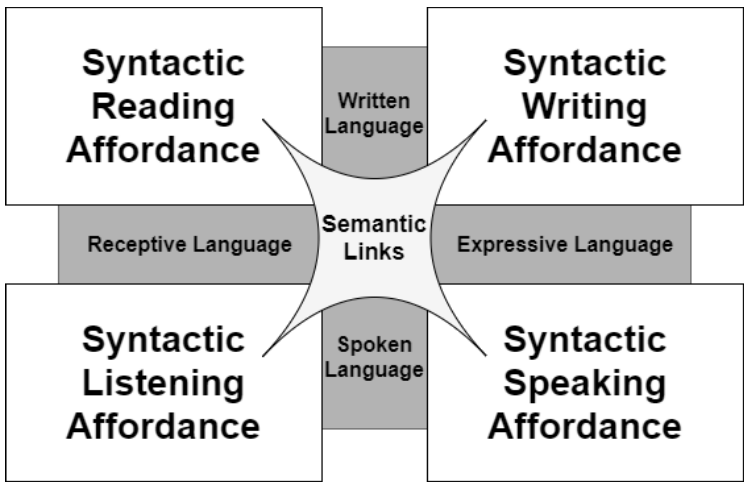


Figure 1: The literacy affordances framework adapted from Aguirre and Bunch (2012)

Methods

Research Setting and Participants

This study is based on recordings collected from 9 secondary mathematics teachers' lessons in two U.S. states. 6 of these teachers taught in a mid-Atlantic state and 3 in a southwestern state. 6 of these teachers identified as white while the remaining 3 teachers identified as Black, Hispanic, and white/Asian, respectively. These recordings were captured as part of the SMiLES project (Secondary Mathematics in-the-moment Longitudinal Engagement Study), which collected student survey data, classroom observations, and teacher and student interviews to understand the role of engagement in secondary mathematics classrooms.

These nine teachers were chosen for the present study because their instruction appeared highly dialogic. Three such qualities of dialogic instruction are the use of high-level tasks (Henningsen & Stein, 1997), opportunities for sharing multiple representations or strategies (e.g., graphs, tables, etc.), and student discourse (Munter et al., 2015). Each of these dialogic qualities had previously been captured in qualitative coding of the classroom videos as part of the SMiLES project (Jansen et al., 2021). High demand tasks were hypothesized to present more opportunities for reading comprehension, while sharing multiple representations and mathematical talk were conjectured to afford more opportunities for writing, speaking and listening. Thus, teachers who enacted dialogically focused instruction were hypothesized to be ideal candidates for this investigation.

The mean of these three qualities was taken for all 156 lessons analyzed for SMiLES and lessons with a mean greater than 2 (out of 3) and with no individual dialogic quality rating lower than 2 were selected for inclusion in this study. Additionally, for teachers who had more than two lessons that met these criteria, only the two lessons with the highest mean were selected so that any one teacher's instruction would not dominate the focus of the results. This left 12 lessons to analyze for the present study.

Classroom Observations

Observations were conducted during the 2018-2019 and 2019-2020 school year. The observed lesson episode for SMiLES was an activity which the teachers believed would be most engaging for students. When teachers would attempt to engage students in learning mathematics, it was hypothesized that they would be likely to also provide a greater number of opportunities for students to engage with or across different literacies. However, the absence of such literacy affordances in these activities would also be valuable, as such results would indicate that these teachers do not necessarily include literacy as part of their conceptions of engaging mathematics instruction. As such, the activities captured in these classroom recordings were well suited for the present study.

Unit of Analysis

Given this study's focus on syntactic literacy affordances – including those which are semantically linked – it was critical to define a unit of analysis that would capture affordances which genuinely correspond with one another and to not confuse these with semantically isolated affordances. For instance, enacting a syntactic reading affordance to support students in interpreting the features of a linear function represented in a graph and then later providing a syntactic writing affordance to support a student in revising a written function equation would not inherently link the two affordances. However, if both the reading and writing affordance attended to how the *slope* of the same function manifests (in the graph and in the equation), then these two affordances would be (from an instructional standpoint) semantically linked.

As such, this study delineated its unit of analysis not only by the overarching mathematical ideas that constitute an instructional task (i.e., Stein & Lane, 1996), but also what Gresalfi et al. (2009) refer to as the task *affordances*, which includes "the ways that mathematical knowledge got constructed – individually, in pairs, with the entire class, and with the teacher" as well as "the ways that the teacher engaged with students around the task as they completed it" (p. 56). The unit of analysis for this study can thus be considered a *textual affordance*, or an instructional moment where one or more texts are being used to communicate meaning about a particular mathematical idea in a particular social context.

Reliability of the Framework

To establish reliability for this study, the author met with two colleagues to test the unit of analysis and the types of syntactic literacy affordances. In the first round of checks, the author identified distinct textual affordances within a sample activity from SMiLES and asked the colleagues to replicate the procedure. Together, these two colleagues correctly identified 100% of the textual affordances that the author had previously identified.

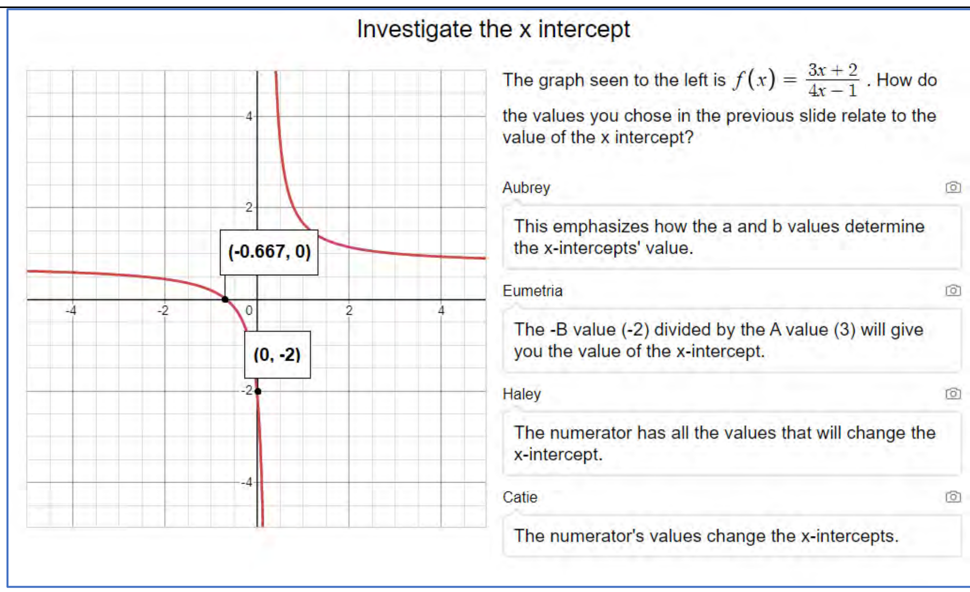
Examples of different syntactic literacy affordances (reading, writing, speaking, and listening) were then identified by the author from a sample observation recording and transcript, including examples where multiple, semantically linked syntactic literacy affordances were present. These same examples were sent to the two colleagues for them to replicate the process of identifying the type of syntactic literacy affordance. Interrater agreement from this process was approximately 85%, indicating a sufficiently reliable framework.

Results

The 12 analyzed observation recordings ranged in length from approximately 15 to 28 minutes. Every activity investigated included examples of syntactic literacy affordances. Fifty-two enactments of syntactic reading affordances and 44 enactments of syntactic writing affordances were found. Eleven enactments of syntactic speaking affordances were present in the analyzed activities as well as one syntactic listening affordance. In addition, 16 examples of semantically linked groups of affordances were also found, although 73 of the 108 syntactic literacy affordances were not linked. Although different amounts of literacy affordances would be expected given that the length of teacher-selected activities varied, these data show the prevalence of such affordances throughout such activities regardless of the length of the activity.

Use of Technology for Whole Group Writing Affordances

Syntactic writing affordances were intentionally defined as focusing students on their own constructed texts. As such, a teacher merely asking a student to describe a mathematical text whole group could not be considered a syntactic writing affordance since the syntax of the written text is not at play but rather the syntax of the student's spoken interpretation of that text. This could have potentially limited whole group syntactic writing affordances to instances where students construct (or reconstruct) their texts in a public space (e.g., a white board) or the teacher has the means to share individual work publicly (e.g., a document camera). For the latter option, the results showed that several teachers used virtual learning platforms (e.g., GeoGebra and Desmos) to enact whole group syntactic writing affordances.



Mrs. Hudson: So a and b were the coefficient of x and the constant term in the numerator. All right. And when you all started to analyze that, there were a couple of you are like -- I'm going to point out -- this one here, where it said, "the -B value, which is (-2), divided by the A value (3) will give you the value of that x-intercept." So, there were a couple of us -- we explained that -- or some of you all even said "if you set the [denominator] equal to 0, and solve for x." I saw some of that. So I'm glad that you all saw the patterns there, but at least you all saw that it is in the numerator and it is the values of the coefficient and the constant.

Figure 2: A whole group syntactic writing affordance

For example, Figure 2 shows how Mrs. Hudson (all names are pseudonyms) provided feedback on students' written explanations of how features of a rational function relate to the x-intercept. She contrasted some students who simply referenced "the numerator" against those who described the actual a and b values and the procedure of solving for x. She noted how these students "saw the pattern there, but at least you *all* saw that it [the values used to find the x-intercept] is in the numerator and it is the values of the coefficient and the constant." The ambiguities in some students' displayed writing, such as a response which stated "The numerator's values change the x-intercepts" were drawn out through this syntactic writing affordance.

Semantically Linked Syntactic Literacy Affordances

There were 16 examples of semantically linked literacy affordances in the data. Figure 3 shows an example from Mrs. Barnett's classroom, where she enacted a semantically linked syntactic speaking and listening affordance during a lesson about maximization with systems of inequalities. She posed a question to Jimmy about how it can be known that one combination (or solution) is "the best." She then engages in a revoicing talk move (Chapin & O'Connor, 2007), having other students respond to Jimmy's answer. She focuses students on *how* to effectively listen during this talk move ("the exercise here is, can we restate or rephrase what Jimmy just said? We need to listen carefully.") allowing both Jimmy and his peers to attend to the clarity of his spoken response. Her insistence that Jimmy must "tell me the math that supports that" with his justification further cements the syntactic speaking affordance within this exchange.

Mrs. Barnett:	I've written my question quite large here. You need to convince me that this is the very best combination. We've talked about the fact that, okay, everywhere within this region…is possible, but Jimmy told us that there are better combinations than others. Of course we want to carry the most passengers. But then how do I know?
Jimmy:	If you were to do all the other combinations, you would still find that with that combination of vans, you would not be able to carry as many senior citizens as the (0,5) combination, which would carry 75.
Mrs. Barnett:	Do you guys hear what Jimmy had to say? Can someone restate or rephrase what Jimmy just said? Say it one more time, Jimmy, because now the exercise here is, can we restate or rephrase what Jimmy just said? We need to listen carefully.
Jimmy:	There's no other way to get more than 75 people due to the budget. There are good combinations. (0,5) is the best combination because it gives you the most people that you can carry within the vans.
Mrs. Barnett:	Okay. Can someone restate or rephrase what Jimmy just said?
Michael:	Based on the money you have, you can't get any more 75 senior citizens to be taken in the vans.
Mrs. Barnett:	Jimmy, did Michael adequately rephrase what you said?
Jimmy:	I mean, yeah. It's kind of the main idea of that.
Mrs. Barnett:	There's no larger amount of passengers that can be carried. You're telling me that it's what, 75 people?
Jimmy:	Yes.
Mrs. Barnett:	Can you tell me the math that supports that?
Jimmy:	Five times 15, because each large van carries 15 passengers. If we purchased five large vans, five times 15 is 75.

Figure 3: Semantically Linked Syntactic Literacy Affordances

Nonexamples Syntactic Literacy Affordances

Many instructional moves fell short of focusing students on the syntactic structures of reading, writing, speaking, and listening mathematically. The key definitional piece of the literacy affordances framework that held back additional moves from being classified within it was the need for the move to *emphasize **how** attending to such structures helps to communicate mathematical meaning*. While the teachers in the study often alluded to the fact that different texts do communicate mathematical meaning, these conversations did not always describe how. Indeed, many such interactions between teachers and students could be described as funneling rather than focusing patterns (Wood, 1998). For example, one teacher (Ms. Ellis) asked a student "what direction would this graph open" for a quadratic function written in expanded form ($f[x] = ax^2+bx+c$). The student believed it would open down and when asked "How do you know?" they ambiguously said because it was negative. Ms. Ellis then responded with "What is a negative?" and realized that the student was referring to the c value. At this point Ms. Ellis told him that "c tells you the y-intercept letters. That is correct. What tells you the direction of the graph?" and proceeded to funnel the student towards recognizing that his a value, with the value of 1, would mean that the graph of the function would open upwards.

Although this paper does not stake any claims on the efficacy of this instructional move for supporting the student in matching equations of functions with their graphical representations, the lack of emphasis on the student's mathematical speaking syntax is apparent. Instead of focusing the student on the ambiguity of their one-word responses (e.g., "what do you mean by 'it's negative?' I'm not sure what you are referring to in the equation when you just say 'it.'") the teacher ignored the ambiguity of the spoken syntax (or at most implied its ambiguity by asking "What is a negative") and moved on without addressing how she interpreted it to be so.

Discussion

This study investigated how secondary mathematics teachers support student meaning-making by attending to and linking the syntactic structures involved in reading, writing, speaking, and listening to mathematical ideas. This first required the construction a literacy affordance framework which described – in corresponding terms – how teachers can attend to these four literacies. Such a framework on its own represents a critical step in drawing research on mathematics teaching towards a sociolinguistic perspective (Moschkovich, 2010) by providing a basis upon which to describe teachers' attention to the syntax of language.

This framework was validated and applied across a diverse set of secondary mathematics classrooms, providing an exploratory glimpse into the ways that teachers do (and do not) attend to and connect these literacies. Some findings indicate concerning trends. For instance, the limited findings of spoken language affordances could indicate that, despite the critical role of mathematical discourse and argumentation in mathematics reform movements (see CCSSI, 2010; NCTM, 2014), dialogic instruction (Munter, Stein, & Smith, 2015) is still limited in these classrooms.

However, the results also indicate that these teachers *are* attending to literacy, and that over 30% of the time these affordances semantically link multiple literacies in relation to a particular mathematical idea. The use of technology to attend to student writing in a whole-group setting is also notable. Such whole group opportunities could expose more students' writing to feedback and validation from the teacher and position such students as competent participants in mathematical discourse (Gresalfi et al., 2009).

If, as Pimm (1987) says, the teaching of mathematics is the teaching of language, then the opportunities which mathematics teachers afford for students to engage with literacy warrant the upmost attention. Language, as the core means of our ability to communicate mathematical meaning, dictates not only *what* mathematical meaning is elevated in the classroom but also *who* plays a part in constructing that meaning. Affording students opportunities to develop their language of mathematics is thus a critical piece of affording them the means to mathematical power.

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DEVELOPMENT OF TEACHERS' PRESS FOR CONTEXTUALIZATION TO GROUND STUDENTS' UNDERSTANDING

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We investigate the development of a teaching practice called pressing for contextualized operating, which is teacher questioning aimed at grounding students' mathematical activity to their understanding of the problem context. We found significant differences in two teachers' facilitation of class discussions after making the teachers aware of the learning opportunities that pressing for contextualized operating provides. The teachers demonstrated similar patterns in their development of their practice, both exhibiting a phase where contextual backing for mathematical activities were elicited through leading questions. We also provide descriptions of more student-centered approaches to contextual press that the teachers developed, as they experimented with the practice.

Keywords: classroom discourse, professional development, problem-based learning, instructional activities and practices

In response to reforms in math education, teachers and curriculum designers frequently aim to develop students' mathematical understanding by having them solve contextual problems (CPs). To realize the potential of teaching through problem solving, mathematics teachers press students to explore the conceptual meaning underlying their mathematical work (Kazemi & Stipek, 2001; Thompson et al. 1994). To elaborate the construct of conceptually-oriented teaching, Reinke et al. (under review) described *pressing for contextualized operating*, or teacher questioning aimed specifically at grounding students' use of representations to their understanding of the problem context. We extend this research by asking, how does teachers' press for contextualization develop after co-analyzing video of a teacher pressing for contextualized operating and transcripts of their own classroom discussions?

Theoretical framework

In contrast to traditional instruction, which positions CPs as opportunities to apply mathematics at the end of instructional sequences, Freudenthal (1973) advised that problem contexts should anchor students' mathematical development from the start of instruction when possible. Through progressive formalization (Freudenthal 1973, 1991), students' concrete, context-bound activities can slowly build toward formal reasoning with conventional representations. The goal is for students' formal reasoning to be grounded in their common sense understanding of real-world phenomena (Gravemeijer and Doorman, 1999).

Researchers demonstrate, though, that without proper support, students tend to solve problems in procedural ways that suspend, rather than rely on sense making (Verschaffel et al., 2000). Because of this, teachers must intentionally support students in using their everyday experience to make sense of problems and the mathematical activity used to solve these problems. Thompson et al. (1994) describe a conceptual orientation toward teaching, that pushes students to understand and work deeply within problem situations. Other researchers describe teachers' press for conceptual thinking (Doerr, 2006; Henningsen and Stein, 1997; Kazemi and Stipek, 2001). Elsewhere, we elaborated on these ideas to identify a particular type of press for conceptual thinking called *pressing for contextualized operating*, aimed at grounding students' calculations in their common sense understanding of the problem context (Reinke et al., under review). Such press is particularly important during discussions in which the reason for a calculation is difficult to explain in the abstract. When a meaningful context is used, it is often easier to understand why an operation is used by appealing to actions on realistic objects. As an example, we describe a teacher-facilitated discussion of solutions to a problem related to the teachers' fictional dream, wherein she discovered that (1) she could pacify invading aliens by feeding them candy bars and (2) one bar was sufficient for pacifying three aliens. Figure 1 shows a students' strategy for solving the problem "If one food bar feeds three aliens, how many food bars are needed to feed 39 aliens".

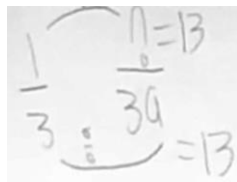

Figure 1: A students' strategy using formal proportional notation

We documented how the teacher, Ms. Kent, engaged in contextual press by asking students to explain the contextual meaning of dividing 39 by 3. She did so through a series of questions to focus students on the meaning of division in the aliens/candy bar context, including "why would he divide thirty-nine by three? and "why would we use division?" More importantly, the teacher was able to surface the idea that it makes sense to divide 39 aliens by 3 because they are grouping the 39 aliens into groups of 3.

The development of productive norms of discourse takes time, as teachers try new practices and respond to feedback from students and students adapt to expectations that may be different than they have experienced in the past (Hufferd-Ackles et al., 2004). As Staples (2008) points out, researchers and practitioners "need detailed accounts of the development of these communities for teachers who face the challenge of transforming their students' participation, and potentially their own as well" (p. 164-165). In this study, we seek to understand how two teachers' practice of press for contextual operating developed with support from university-based coaches.

Methods

To determine how teachers' press for contextualization developed over time, we observed two teachers teach the same unit over two consecutive years: a 7th grade ratio unit that uses contexts in its design. During the first year, we joined planning meetings to help the teachers anticipate student responses and plan for productive mathematical discussions (Smith and Stein, 2011; Stephan et al., 2016). Before observing during the second year, we met with the teachers specifically to share our findings related to press for contextualized operating. First, we described our finding that their beliefs about the benefits of teaching through contextual problem solving had evolved from a focus on engagement to a focus on how the contexts helped students better understand mathematical concepts (Reinke and Casto, 2020). Then, we contrasted the typical language around using contexts as "hooks" with an alternate analogy: contexts as anchors for understanding (Stephan et al., 2020). We then presented transcript excerpts from their classroom the previous year, including a case where students described their work with a ratio table but provided a justification that was not contextualized. In this case, students were shown a table relating the number of candy (Snickers) bars to the number of aliens:

Table 1: Ratio table relating the number of candy bars to number of aliens.

Snickers	1	2	3	4	5	6
Aliens	3	6	9	12	15	?

Teacher: Yes? So, what did you do to get 18? Green group? What did you do to get 18?
Student(s): Six times three.
Teacher: Six times three. That gave us 18? Why did we multiply by three?
Student(s): Because one times three equals three, two times three equals six, three times three equals nine, four times three equals 12, and five times three equals 15.

We collectively brainstormed what an appropriate contextualized backing would be. We also presented a video case where a teacher pressed for contextualized operating and students had difficulty providing that contextual backing. We discussed ways she supported students in making connections between their work with informal, pictorial solution strategies, long tables (as shown in Table 1) and formal proportion representations.

Following these coaching sessions, we then observed a number of lessons from both teachers as they taught the unit again. We were able to record four of the same lessons we had recorded in Ms. Kent's classroom the previous year and three of the same lessons that we had recorded in Mr. Jones' classroom. We transcribed the portions of the recordings that contained whole class discussions of students' strategies. We then coded the transcripts with an a priori coding scheme that had been developed during a previous study. Specifically, we coded each teacher turn for the presence of press for contextualized operating, press for a conceptual explanation (but not specifically contextualized operating), press for contextualized operating through leading questions, and instances where the teacher provided the contextualized backing for the mathematical activity. We coded the student turns for the presence of unprompted contextualized operating and contextualized operating prompted by the teacher.

Results

Across the two teachers, we found a similar pattern in the evolution of the manner in which they pressed for contextualized operating in the second year compared to the first year. We focus on one teacher here, due to limited space.

Year One

During the first year, Ms. Kent frequently engaged in conceptual press that had the potential to surface contextualized operating, but she regularly accepted explanations for operations with ratio tables and formal proportions that did not include contextual justification, (as in the excerpt above related to Table 1). The press through funneling described above (in figure 1) was one of two instances of press for contextual operating with preformal (ratio tables) or formal (proportions) representations in the dataset from 2018.

Year Two

After we made Ms. Kent aware of the construct of contextualized operating and brainstormed different ways to support students in providing contextualized operating, we observed Ms. Kent engaging in significant press for contextualized operating in a variety of ways: by referring to their initial pictorial strategies, by asking students to identify the contextual backing for mathematical activity in their small groups, and by consistently checking for understanding of the contextual meaning of their activity with ratio tables and formal proportional notation.

For example, in the second investigation, a student introduced a formal proportion as a way of describing whether 12 bars will be enough to feed 36 aliens and multiplied both the numerator and denominator of the original 1 bar: 3 alien ratio by 12. Ms. Kent asked "What is that 12?" referring to the multiplier. When students struggled to explain what multiplying by 12 meant in terms of the food bars and aliens, she highlighted a students' pictorial strategy and, after identifying where the various quantities showed up in the picture, she asked:

Kent: Why did we multiply by 12? What was that 12?
Student 3: scale factor
Kent: You're saying scale factor but what does that mean? Think about it. Talk it out with your group. Talk it out. What does that 12 mean? I saw it also on the top and bottom? Why? Where do we see that in the picture?

Throughout the initial lessons in the unit, Kent consistently pressed for the contextual meaning of additive build-up strategies using ratio tables and multiplying by a scale factor and the coefficient of proportionality. While engaging in this press, she frequently checked for understanding from multiple students.

Implications and Limitations

Comparing across the two cases, we found that, after making the two teachers aware of the construct of contextualized operating and brainstorming ways of supporting students to make connections between formal representations and informal, situation-specific strategies, both teachers exhibited a tendency to press for contextualized operating. We also note that both teachers progressed through similar stages as they developed the practice. As Ms. Kent and Mr. Jones first demonstrated an inclination to press for contextualized operating, they elicited this backing through a funneling pattern (Bauersfeld, 1988; Wood, 1998), or a series of leading questions, when students struggled to produce the intended backing. At the beginning of year 2, Ms. Kent demonstrated other forms of press, including connecting back to pictures and providing small group time to encourage students to identify contextual meanings for operating with ratio tables and proportions.

We hypothesize that the changes in practice were significant because the intervention was very specific, focused on a particular practice and a particular mathematical topic, using data from their own classroom. This likely helped support teachers in this unit, but also it remains to be seen how these practices transferred to other units. Furthermore, the teachers in the study knew that we were observing for the presence of press for contextualized operating during year two, so it is unclear the extent to which the observations we recorded were due to this observer effect. However, it is clear that making teachers aware of the possibility of press for contextualized operating supported them in developing the capacity for this practice, which is important for supporting students' meaning making through problem-solving, particularly early in instructional sequences when the contextual grounding of their operating is not yet established.

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CHARACTERISTICS OF TWO OUTSTANDING ELEMENTARY TEACHERS OF MATHEMATICS: IMPLICATIONS FOR TEACHER EDUCATION

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This study identifies characteristics of two "outstanding" elementary teachers of mathematics who were different in style and who taught in different settings. The intent is to determine what characteristics make these different teachers outstanding in hopes of helping preservice teachers improve their teaching of mathematics. Preliminary findings indicate that both teachers were (1) focused on children's learning of mathematics; (2) focused on the mathematical solution methods used by students; (3) believed that all of their students could learn mathematics; (4) were enthusiastic and dedicated to the profession of teaching; and (5) cared deeply about their students and emphasized the necessity of building relationships with them.

Keywords: Instructional Activities and Practices, Algebra and Algebraic Thinking, Elementary School Education, Professional Development, Preservice Teacher Education

Two teachers in the Conceptual Algebra Readiness for Everyone (CARE) Project for students in grades 3 to 8 distinguished themselves as "outstanding" elementary teachers of mathematics. Each teacher had different styles of teaching and they taught in different classroom settings. Both of the teachers in this study had CARE training, became CARE workshop trainers, and presented project work at national conferences. CARE is a curriculum development project for grades 3 to 8 in partnership with a high-needs school corporation (Feikes, Pratt, & Griffiths, 2012). CARE includes professional development for teachers around helping students develop conceptual algebra readiness and curriculum use. In interviews, both teachers described how CARE shaped their views of teaching mathematics. The goal of this study is to explore commonalities and differences in characteristics of these two teachers and discuss implications for teacher education.

Methodology

This study employs a multiple-case study design (Baxter & Jack, 2008; Merriam, 2009) with a thematic analysis approach. In our analysis, we identified emerging themes which we understood to be "an extended phrase or sentence that identifies what a unit of data is about and/or what it means" (Salda a, 2013, p. 175). Correspondingly, we grouped quotes from the interview transcripts to identify similarities, differences, and themes. The voices of the teachers are used to add understanding to outstanding teaching of mathematics. The two teachers were recruited to participate based on their work with CARE. They participated in structured interviews about their teaching practice and the recordings were transcribed. Additionally, the teachers were observed multiple times and one or two observed lessons were video-taped. This paper reports the analysis of the interviews, using the observations to support the analysis.

Theoretical Framework

A review of studies that deal with excellence in teaching mathematics helped identify some characteristics of outstanding teachers (Hinz, Walker, & Witter, 2019; Lim, Tang, & Tan, 2013). One of these characteristics is building rapport with students and making strong connections.

Building rapport and making connections includes caring about students and the students recognizing that the teacher cares. A second characteristic is focusing on conceptual understanding while recognizing the importance of procedural knowledge. Additional characteristics noted in the research included demonstrating enthusiasm, showing respect for students, being prepared for instruction, and striving to be a better teacher. The literature review produced few current studies that focus on outstanding teaching of mathematics in the elementary school. A key premise of this paper is interpreting what has been learned by these two elementary teachers to help preservice teacher education.

Commonalities and Differences in Characteristics

The first teacher, Mr. Francis (pseudonym), taught for forty years in a Midwestern city with a population of 20,000. A majority of his teaching was in fourth grade. Mr. Francis taught in the school with the highest state standardized test scores in the district. The students in the school were predominantly middle class and White. The school had a 12% minority population and a free and reduced lunch rate of 24%. In recognition of his excellent teaching, Mr. Francis was the winner of the Presidential Awards for Excellence in Mathematics and Science Teaching.

The second teacher, Mr. Marker (pseudonym), taught in a Midwestern city with a population of 40,000. Mr. Marker taught several different primary grade-levels over fifteen years, most recently sixth grade. The school where he taught had a 55% minority population and a free and reduced lunch rate of 76%. Mr. Marker was also recognized as an excellent teacher as a finalist for the Presidential Awards for Excellence in Mathematics and Science Teaching.

Observations and interviews found that the teaching styles and personalities differed for each teacher. Mr. Francis was a composed and caring teacher, describing one of his classroom interactions as "… without being angry or loud." His students never misbehaved. Mr. Francis was very organized, as demonstrated by his calendar of academic standards to be addressed for the entire year. Mr. Marker was higher-energy and sarcastic. The students enjoyed their relationships with Mr. Marker, as evidenced by the number of students who came up to him before school to joke around and share stories. Observed math lessons were energetic and nonstop. Despite the high number of students that lived in poverty, Mr. Marker's sixth-grade students demonstrated success on the state standardized mathematics test, with all but one student earning a passing score.

Differences between the teachers help put their commonalities in starker contrast. Mr. Francis was a planner. When comparing himself to other teachers he noted, "I spend more time planning and creating my own curriculum, making sure I have all the standards outlined." In contrast, when Mr. Marker was asked to compare his planning to Mr. Francis he said, "I go more spontaneously in the classroom." Another difference between the two teachers was their level of confidence about their own mathematical ability. Mr. Francis described a lack of confidence in his mathematical ability. "By the time I got to high school I was not confident. … I did not feel like I was a great math student." His lack of confidence motivated him to make mathematics more meaningful for his students. In contrast, Mr. Marker was very confident in his mathematical ability. Referring to his high school math classes, he said, "At the time it was being taught, I could go the process of it and do the work fine."

Despite these differences, analysis of the interviews found characteristics that the teachers had in common. The first characteristic described by the teachers during the interviews was focusing on children's learning of mathematics. When asked what aspects of mathematics were personally interesting, Mr. Francis talked about children's learning. "It is interesting to see how children think and to see how they solve certain problems and to see how they develop from one stage to another in certain areas." When asked the same question, Mr. Marker also talked about children's learning of mathematics.

> It is trying to get kids to be problem solvers. It's the most challenging because it is so broad. You can find 5 or 6 approaches to solve any problem using computational strategies. … They do it a different way because they are smart enough, because they have built the skills.

In the mathematics education community, we categorize all the different types of mathematical and pedagogical knowledge, but these teachers did not make that distinction in the interviews. For them the interesting aspect of "mathematics" was how children learn mathematics.

The second common characteristic was that each teacher focused on the solution methods used by students when working on mathematics problems. During the interviews, both teachers mentioned that they encouraged the students to share a variety of solution methods. Mr. Francis said, "You wouldn't think it would be so diverse, to see the different types of thinking going on." He relished the enthusiasm students had when they explained their ways of thinking and noted that they especially enjoyed describing different ways to think about a problem. Mr. Marker said, "Some of the coolest moments in the classroom are when kids show different ways that they figured it out and they can't wait to express it. They do not want to be like anyone else."

The third common characteristic was the belief that all students could learn mathematics at a meaningful level. Mr. Marker regularly commented that all his high-poverty students could be successful in mathematics. Mr. Francis expressed similar sentiments about his middle-class students. Both teachers commented on the importance of the solution process and on the value of enabling students to develop their own mathematical thinking.

The fourth common characteristic from the interviews was that each teacher was enthusiastic and dedicated to the profession of teaching. Mr. Francis said "I love it when you see students getting something and you see them excited." Mr. Francis built his enthusiasm off of children's excitement of learning. Mr. Marker was also enthusiastic about teaching, he said, "I really care about [teaching]. I care about the kids being successful, I love to teach. I get excited about it every day, like every day is a fun day for me." These comments demonstrate their enthusiasm for and dedication to teaching.

The interviews and observations provided evidence of a fifth characteristic shared by the teachers; both teachers cared deeply about their students and emphasized the necessity of building relationships with them. This finding is consistent with prior research (Lim, Tang, & Tan, 2013). For example, Mr. Marker stated:

> Once you are able to build that relationship with the kids, once they know that you care about them and they can trust you, they will do anything for you in the classroom. … I think the kids have a good respect for us and we do for them. And that goes a long way in how you manage your classroom and build relationships with students.

When asked what made him an outstanding teacher Mr. Francis referred to his organizational ability and his relationship with children. "I think with the planning and wanting to know the whole child and caring about them and wanting to know their whole picture."

Discussion

The analysis of the data demonstrated that these outstanding teachers of mathematics focused on children's mathematical learning, focused on students' solution methods, believed that all of their students could learn mathematics, were enthusiastic and dedicated to the profession of teaching, and cared deeply about their students and emphasized the necessity of building relationships with them. These findings have implications for mathematics teacher education.

Research has shown that preservice teachers benefit from learning how students learn about mathematics (Feikes, Pratt, & Hough, 2006; Philipp, Thanheiser, & Clement, 2002). In preservice teacher education, we can emphasize the value of knowing children’s thinking and focusing on children’s solution methods to develop conceptual understanding and assessment. Both mathematical content and methods of teaching in the education of preservice teachers should foster a focus on students’ learning of mathematics.

In order to help preservice teachers focus on solution processes or take an inquiry approach in their teaching (Richardson & Liang, 2008), similar approaches should be modeled in college courses. Field experiences that focus on the process and not the product should be available for prospective teachers. Experiencing a focus on process in their college courses and in the field will help preservice teachers adopt this approach to teaching mathematics.

Mr. Francis supported students who were two or three grade levels behind. Mr. Marker taught in a high poverty school and in one year, 55 out of 56 students passed the state accountability test in mathematics. A point to emphasize with preservice teachers is that teachers can make a difference when their teaching embodies the idea that all students can learn mathematics.

As professionals participating in ongoing teacher education, these two teachers regularly participate in professional development. They were interested in what other teachers do and often tried to adapt their teaching based on conversations with other teachers. Preservice teacher education should encourage this type of professional collegiality. Similarly, instructors of preservice teachers need to model enthusiasm for the teaching profession and provide field experiences where prospective teachers see this in action.

Both teachers in this study described developing positive teacher-student relationships with their students. These relationships added to the learning of mathematics and promoted a safe learning environment. Preservice teachers need to learn about developing teacher-student relationships with students based on respect so that the students know they care about them.

Conclusion

This paper describes commonalities and differences in characteristics of two outstanding teachers of elementary mathematics. The teachers taught in different instructional settings, one was a planner and one was not, and one was very confident in his math abilities and the other had significant reservations. However, both teachers focused on children’s learning, the solution methods used by students, and all students being able to learn mathematics. Both teachers were enthusiastic about being teachers and dedicated to professional growth. They cared deeply about their students and building relationships with them. These characteristics have implications for higher education teacher education programs. Preservice teachers need experiences that help them consider student solution methods and student learning about mathematics. Teacher education programs should provide opportunities for collegial interactions around mathematics education and building enthusiasm for mathematics teaching.

Because the characteristics identified in this research are limited to the case study of two teachers, there are limitations to these findings. Interviewing additional outstanding teachers of mathematics and comparing findings to existing research (e.g., Hinz, Walker, & Witter, 2019) could provide additional insights, especially for the teaching of mathematics in the elementary school. Identifying key characteristics and incorporating them into teacher education programs can help advance the effectiveness of future teachers of mathematics. Additional research on how these characteristics can positively impact mathematics teacher education is also needed.

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ELEMENTARY MATHEMATICS TEACHERS' FEEDBACK PRACTICES: A MULTIPLE CASE STUDY

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Feedback is essential for helping students move forward in their learning, and the beliefs teachers hold could potentially affect the way they provide information to their students. However, the ways in which teachers provide feedback during mathematics instruction and their own implicit beliefs are often overlooked as contributors to the various types of feedback they provide. The purpose of this study was to examine the ways in which elementary teachers provide feedback during mathematics instruction. The results of this study revealed that (1) although both participants ascribed to an incremental theory, they demonstrated varying commitments to providing self-level feedback, (2) one participant provided all three types of feedback within one classroom observation on multiple days, and (3) both participants provided little to no feedback directed at the process and self-regulation levels overall.

Keywords: Teacher Beliefs; Teacher Noticing; Instructional Activities and Practices; Communication

Introduction

Students develop mathematical problem-solving skills through interactions with teachers as well as other students (Jung, Diefes-Dux, Horvath, Rodgers, & Cardella, 2015; Slavin, 1996). Through these interactions students gradually increase their understanding of the content as well as their ability to solve problems (Jung et al., 2015). This information transferred between teachers and students, particularly through questions asked and informational responses, is known as feedback and can help advance students' mathematical knowledge towards a conceptual understanding of the mathematics (Brookhart, 2008; Hattie & Timperley, 2007; Slavin, 1996). Feedback is essential for helping students move forward in their learning (Brooks, Carroll, Gillies, & Hattie, 2019), and the beliefs teachers hold could potentially affect the way they provide information to their students (Rattan, Good, & Dweck, 2012). Thus, the purpose of this study was to examine how elementary mathematics teachers provide feedback. To this end, the research question was: In what ways do elementary teachers provide feedback during mathematics instruction?

Feedback

Feedback, defined as the information conveyed to learners about their actions (Hattie & Timperley, 2007; Shute, 2008), is intended to make a connection between what students understand and what is meant for them to understand (Sadler, 1989). This information educates students as to how they are doing relative to the learning goals of the lesson (Hattie & Timperley, 2007) and how they might modify their work to reach these goals (Jung et al., 2015). Based on numerous observations and analyses of hundreds of studies on feedback, Hattie and Timperley (2007) found that in order for feedback to be effective, it must answer three questions: "Where am I going?, How am I going?, and Where to next?" (Hattie & Timperley, 2007, p. 86). These questions correspond to the three types of feedback: feed up, feed back, and feed forward, respectively (see Figure 1). Additionally, Hattie and Timperley (2007) identified four feedback levels at which each type of feedback can be given. The levels are intended to help facilitators provide “specific feedback to individual learners dependent upon their learning needs” (Brooks et al., 2019, p. 19). The feedback model shown in **Error! Reference source not found.** displays t he four levels at which the feedback can operate, including feedback at the task, process, self-regulation, and self levels.

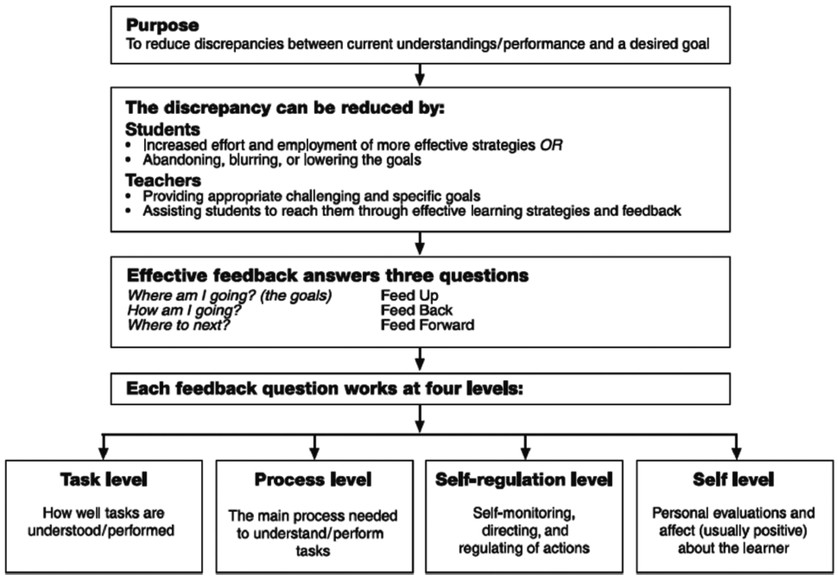

Figure 1: Model of Feedback to Enhance Learning. From Hattie, J., & Timperley, H.

Theoretical Framework

There were two theoretical frameworks that combined to form the conceptual framework for the study. Models of feedback and implicit theories were used as theoretical frameworks to understand the distinct teacher feedback practices in two elementary mathematics classrooms. Observed feedback was categorized according to Hattie and Timperley’s (2007) Model of Feedback to Enhance Learning including the types (i.e., feed up, feed back, and feed forward) and levels (i.e., task, process, self-regulation, and self) in which the feedback was directed. The second model, implicit theories (Dweck, Chiu, & Hong, 1995), was used to select the participants and served as a potential basis for understanding the individual differences that may contribute to the ways elementary mathematics teachers provide feedback.

Methodology

My initial intent was for the two participating teachers to offer contrasting conditions (i.e., one who held an incremental theory and one who held an entity theory); however, due to the unavoidable issues during the participant selection process, both of my participants ascribed to the same subgroup (i.e., an incremental theory). Continuing my study with two participants who ascribed to the same implicit theory was beneficial to see whether there were similarities in the feedback practices within the subgroup of elementary teachers who ascribe to an incremental theory. To this end, I used an exploratory multiple-case study with two elementary mathematics teachers within the same subgroup: one who ascribed to a strong incremental theory and one who ascribed to a weak incremental theory (Dweck et al., 1995).

Participants, Instruments, and Data Sources

The first participant was Ian Smith, an African American male in his late twenties, who ascribed to a strong incremental theory. The second participant was Ellie Jones, an African American female in her mid-forties, who ascribed to a weak incremental theory. I gathered data from multiple sources to fully describe each case: an implicit theories survey, observational protocol, audio recordings, video recordings, semi-structured interviews, and participant reflective journals. To select my participants, I administered all teachers at an urban elementary school (Byron Elementary, a pseudonym) a modified version of Dweck et al.'s (1995) Implicit Theories Measures. The modified Implicit Theories Survey contained an additional domain of mathematical ability (Willingham et al., in press). Once the two participants were selected based on their implicit theory, I conducted daily observations for approximately 11 days during their mathematics instruction. Following each daily observation, I interviewed the participants individually and all data were transcribed and coded for analysis based on the analytical framework of Hattie and Timperley's (2007) characterizations of feedback types and levels. I conducted a cross-case analysis of the two participants to determine any patterns, similarities, or differences across the cases (Yin, 2014).

Results and Discussion

I will organize the results and discussion of the data according to my three key findings. First, although both Ian and Ellie were identified as holding an incremental theory, there were varying commitments to providing self-level feedback during the observed mathematics instruction. Second, while both participants provided all three types of feedback (i.e., feed up, feed back, and feed forward) throughout the 11 days of classroom observations, Ian provided all three types of feedback within one classroom observation on multiple days. Last, feedback at the process and self-regulation levels are necessary for helping students to move towards the Goal of Mathematical Proficiency (NRC, 2001); however, the data showed that Ian and Ellie provided little to no feedback in these ways.

Self-Level Feedback

Although Ian and Ellie were classified as incremental theorists based on their overall average choices on the Implicit Theories Survey (Willingham et al., in press), they demonstrated varying commitments to providing self-level feedback (i.e., praise) during mathematics instruction. Ian, who ascribed to a strong incremental theory, provided self-level feedback directed at the person by praising students for their intelligence, bravery, confidence, and actions, and at the process by praising students for their thinking. He explained that by providing feedback in these ways, he wanted to instill self-confidence and a growth mindset in his students to build a safe and comfortable classroom community of learners. Ian's actions aligned with Hattie and Yates's (2014) idea that some praise could be valuable for establishing relationships within the classroom. These relationships are particularly essential for teachers who ascribe to an incremental theory for building a supportive learning community (Willingham, 2016).

Feedback Type

My second key finding was that although both participants provided all three types of feedback (i.e., feed up, feed back, and feed forward) throughout the 11 days of classroom observations, Ian provided all three types of feedback within one classroom observation on multiple days. This aligns with Hattie and Timperley's (2007) model of feedback to enhance learning where the authors described effective feedback as feedback which answers all three questions (i.e., Where am I going? How am I going? and Where to next?). Effective feedback provided in this way should help students move forward from their current understanding to their desired goal and increase student achievement overall (Hattie & Timperley, 2007). However, Hattie and Timperley (2007) also noted that the degree of effectiveness depended on the level at which effective feedback should be directed and described the specific details of effective feedback between the types and levels as "fuzzy" (p. 103).

It is important to note that although Ian answered all three questions on multiple days, Ellie had no instances where she answered all three questions (i.e., provided feed up, feed back, and feed forward) within one classroom observation. This result, however, only shows that Ellie did not provide effective feedback during the 11 classroom observations according to Hattie and Timperley's (2007) definition of effective feedback. An area of future research would be to closely examine the feedback practices, particularly the types and levels of feedback, of more mathematics teachers in their own classrooms and the impact on student learning as a result of the observed feedback to determine the effectiveness of providing feedback of various types and levels in a natural setting and the direct impact on student learning.

Feedback Level

The last key finding from my data showed that Ian and Ellie provided little to no feedback directed at the process and self-regulation levels overall. When breaking down how each participant provided feedback by level, the results showed that Ellie provided task-level feedback (63 instances) approximately twice as many times as she provided process (25 instances) and self-regulation (two instances) levels combined. Ian provided the most feedback at the task level (56 instances); however, he provided more feedback at the process level (36 instances) and self-regulation level feedback (33 instances) combined. The result that Ellie provided little feedback at the process and self-regulation levels is important given that students use feedback at these levels to build a deeper understanding and take more control of their learning (Hattie & Clarke, 2019; NRC, 2001). Although task-level feedback is the most common form of feedback and necessary in the mathematics classroom to build a strong surface-level understanding (Brooks et al., 2019; Hattie & Clarke, 2019), it is not sufficient for helping students move towards the Goal of Mathematical Proficiency (NRC, 2001).

The result that Ian provided more process-level and self-regulation level feedback is important as feedback directed at the process and self-regulation levels are essential for fostering independent students and helping students in moving forward with their thinking (Hattie & Clarke, 2019; Hattie & Timperley, 2007). In addition, self-regulation level feedback supports students in monitoring and assessing their own learning, and often leads to the detection of their own errors (Hattie & Timperley, 2007). Although feedback at the process and self-regulation levels are essential in the mathematics classroom for supporting students, teachers do not often recognize what "higher levels of feedback look like, and are, thus, unable to use them to enhance learning" (See, Gorard, & Siddiqui, 2016, p. 69). Thus, the results of my study suggest the need for preservice/in-service training to address the different types and levels of feedback in the mathematics classroom.

Conclusion

This study examined how two elementary mathematics teachers, both who ascribed to an incremental theory, provided feedback during mathematics instruction. Teacher feedback has been established as one of the most important influences on learning and student achievement (Hattie & Timperley, 2007; Hattie & Yates, 2014; Wisniewski, Zierer, & Hattie, 2020). However, the ways in which teachers provide feedback during mathematics instruction (Jung et al., 2015; Li et al., 2016) and their own implicit theories are often overlooked as contributors to the various types of feedback they provide (Rattan et al., 2012). Students must be given opportunities to engage with meaningful mathematics through effective feedback that supports their efforts in moving forward with their learning and understanding (Boaler, 2015).

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REVEALING THE PEDAGOGY OF DISCUSSION IN MATHEMATICS METHODS

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Although characteristics of discussions have been identified in the mathematics education research literature, pedagogy of MTEs' discussion practice is underexplored. Using a self-study methodology, we characterize three MTEs' pedagogy of discussion practice for teaching about teaching mathematics in methods courses. Data sources include recordings and transcripts of weekly critical friends' meetings, MTE facilitated discussions, and instructional artifacts. We identify commonalities in our pedagogy of discussion practice: anticipating PTs' thinking and layering instructional activities. We argue that scaffolding informs MTEs' discussion practice.

Keywords: Classroom Discourse, Preservice Teacher Education, Teacher Educators

Discussion is a core practice in mathematics teaching (Grossman et al., 2019). Although teaching about mathematics teaching involves engaging teachers in discussions (e.g., Steele, 2005), mathematics teacher educators' (MTE's) pedagogy that informs such discussions has received less research attention. We address this gap by describing pedagogy of practice and identifying scaffolding as a concept with potential to inform MTEs' pedagogy of discussion practice. Findings from qualitative self-study of MTEs' discussion practice are used to describe two processes: anticipating prospective teachers' (PTs') responses to discussion prompts and layering instructional activities. We argue that scaffolding (Bakker et al., 2015) is a pedagogy that informs MTEs' discussion practice in teaching about teaching.

Background and Literature

Pedagogy is a theory of teaching and learning such that together "they lead to growth in knowledge and understanding through meaningful practice" (Loughran, 2006, p. 2). Grossman et al. (2009) described three concepts teacher educators use to teach about practice: representations, approximations, and decomposition of teaching practices. These concepts are useful in the design and analysis of pedagogy in mathematics teacher education (e.g., Ghousseini & Herbst, 2016). Yet Grossman et al. (2009) caution that instructors mediate the "learning potential" (p. 2089) of representation or approximation of practice used in professional education through "planning, modeling, and feedback" (p. 2090). This caution raises questions about pedagogies that inform MTEs' practices. Thus, we explored MTEs' pedagogy of discussion practice guided by the research question: What is our pedagogy of discussion practice in mathematics methods?

Discussion is defined as an interaction in which people address "a question of common concern" (Dillon, 1994, p. 8) through an exchange of ideas (Alexander, 2019) and examination of differing views to share ideas or solve a problem (Kim & Wilkinson, 2019). Drawing from this definition we describe discussion as a talk strategy MTEs use to support PTs' development of concepts and perspectives on concepts relevant to mathematics teaching practice. Discussion is informed by teacher educators' use of "knowledge, theories, and understandings" (Pinnegar & Hamilton, 2009, p. 16) in practice to develop "knowledge from practice" (Pinnegar & Hamilton, 2009, p. 17) as a way of "knowing to" (p. 18) engage PTs in a given practice.

Scaffolding is a process teachers use to support students' concept development. Bakker et al. (2015) assert that scaffolding can inform instructor pedagogy, pedagogies which Grossman et al.

(2009) claim vary and may produce different outcomes. Smit et al. (2013) described whole class scaffolding as involving the characteristics of diagnosis of understanding, responsiveness to current understandings and needs, and fostering independence (Visnovska & Cobb, 2015). Whole class scaffolding is "layered, distributed and cumulative" (Smit et al., 2013, p. 829) suggesting that these characteristics may take place during and outside whole-class interactions.

MTEs draw from research on the role of PTs' mathematics experiences on their practices. PTs' experiences learning mathematics have been shown to inform their practice (Drake, 2006; Mewborn, 1999; Mewborn &Tyminski, 2006, Towers et al., 2017). As such, MTEs assume such experiences inform PTs' responses to discussion questions. MTEs' experiences facilitating mathematics discussions inform tacit knowledge of productive discussions (Munby & Russell, 1994) in mathematics methods. MTEs' facilitation of discussions of mathematics teaching and learning requires understanding that teachers' pedagogical claims draw from values and experience (Steele, 2005) unlike their mathematics claims.

MTE's discussion practice in mathematics methods is impacted by phenomenological factors (Dillon, 1994). Such factors are necessary but not sufficient conditions for discussions (Kastberg et al., 2020) that support concept development and perspective taking (Lischka et al., 2021). PTs need a sense of community and relevant common experiences as a foundation for discussion. MTEs' facilitation of discussions, including structuring discussion questions, facilitating the interpretation of the question, and constraining evidence PTs use to address the question, impacts the form (i.e., IRE, recitation, discussion) and content of the PTs' talk (Lischka et al., 2021).

Institutional Contexts and View of Learning that Informs Pedagogy

We teach in different institutional contexts including elementary and secondary mathematics methods, and a range of institutions from teaching-focused to research-intensive. Signe's study of discussion practice focused on opportunities for PTs to address the question: "How do children learn mathematics?" Alyson's discussion study was guided by the question: "How is effective feedback practice enacted?" Susan's study of discussion practice was guided by the question: "How does cognitive demand of tasks and knowledge of children's mathematical thinking inform planning instruction?" We draw from a constructivist epistemology in our pedagogy (Steffe & D'Ambrosio, 1995) of teaching about teaching to create and use models of PTs' knowledge of mathematics teaching that inform instructional decisions such as the development of discussion questions. We create models of our practice and identify "living contradictions" (Whitehead, 1989, p. 41) between the models and our practice.

Methodology and Methods

Self-study methodology as characterized by LaBoskey (2004), is a form of practitioner research (Borko et al., 2007) uniquely positioned to support inquiry into pedagogy of practice. Evidence of such pedagogy spans the dynamic experience of MTEs and artifacts from such experience. During Fall 2020, we engaged in weekly critical friend conversations in the form of analytical dialogues (Guilfoyle et al., 2004) focused on facilitating discussions in mathematics methods. Support for trustworthiness was attended to by including course artifacts from teaching. Within the methodology of self-study, we used three qualitative analytic methods: dialogic analysis (Guilfoyle et al., 2004), evidentiary maps, and descriptive coding (Saldana, 2016). Ongoing dialogic analysis of our discussion practice in Fall 2020 semester supported our process of "coming to know" that served as the basis for "action (in knowing, understanding, and doing)" (Guilfoyle et al., 2004, p. 1111). Ideas contributed in our weekly discussions were unpacked, analyzed and critiqued. Analysis of transcripts from seven meetings during Fall 2020, unearthed the development of "coming to know" that served as the basis for our action. We created evidentiary maps of the "structure of events" (Jordan & Henderson, 1995, p. 57) from courses relevant to our discussion practice using teaching artifacts. Evidence from artifacts were analyzed and critiqued regarding the extent to which they supported or contradicted initial findings from the dialogic analysis. Descriptive coding (Saldana, 2016) of transcripts from our dialogic conversations used the ways of knowing for discussion practice identified during the dialogic analysis. We looked for evidence of ways of knowing relevant to describing our pedagogy of discussion practice. Important in these methods is the movement from dialogue to course artifacts and back to dialogue. These three analytic methods created an evidentiary basis for findings common across three MTEs' discussion practice and related contexts.

Findings

We used two processes during Fall 2020 to inform our discussion practice: anticipating PTs' thinking and layering instructional activities. Shifts in one process resulted in shifts in the other. Examples from Signe's pedagogy of discussion practice are used to illustrate our findings.

Anticipating PTs' Thinking

Signe originally planned to have a whole-class discussion of the question "How do students learn mathematics?" around the middle of the term. We agreed that PTs' thinking about students' learning of mathematics influenced curricular decisions, yet we struggled during early conversations to anticipate how PTs might address this question:

> Signe: . . . we're trying to build this idea of knowing each other as teachers and what kinds of things we would do as teachers and where that comes from. . . . That's going to be a big question that I'm leading up to, . . . but I'm really nervous about sustaining it and facilitating the discussion without meddling in it. . . . so how do I make sense of my development of knowledge for teaching? Not me, but them. How are they aware of it?
>
> Alyson: So you are seeing these different perspectives that might come out from the [PTs]. . . Would it help you to sustain the discussion, if you thought through those different perspectives, and … what are the positive things of those different perspectives that you might draw out or probe more deeply? … it's like you're anticipating different directions that the discussion might go and being ok with different outcomes based on what they're bringing to the table. (Conversation 08-31-2020)

Our dialogue of anticipating PTs' responses to planned discussion questions illustrates this difficulty. Across the semester Signe created opportunities to support PTs' understandings of the word "learning." Such opportunities led to diagnosing PTs' responses to determine next steps in a cycle of anticipating and creating learning opportunities. Signe's evidentiary map illustrated a shift from the language of "conceptual knowledge" used in classes during the first few weeks of the course to "learning" used in a course assessment prompt at the end of the course. We continued to have conversations about how we were getting to know our students, anticipate their ideas about teaching mathematics, and how to provide learning opportunities that would support discussion. As a result of these conversations about anticipating PTs' ideas, our conversations revealed how initial target dates for discussions were shifted back again and again. The discussion Signe aimed for was moved from the middle of the term to the end of the semester and evolved from multiple layers of evidence. In her effort to support PTs' understanding of the terms "conceptual knowledge" and "procedural knowledge," Signe's intended discussions became dialogues and recitations. To have a discussion regarding conceptual knowledge Signe needed PTs to gain facility with the terms in the context of their student work. As our conversations continued, we realized that layering experience and learning over time helped us reconsider and expand the time frame of discussions we had planned.

Layering Instructional Activities and Situations

Instructional activities designed to gather evidence of PTs' ideas and experiences resulted in diagnosis of PTs' responses that informed anticipating PTs' understanding of key ideas relevant to our planned discussion questions. Focus was maintained or we designed and implemented new activities for PTs to build understanding. This cycle created layers of instructional activities. As we reflected on events toward the end of the semester, we gained clarity on the importance of the layers for supporting discussions, which served to reframe the timeframe of our discussions:

> Signe: So, going iteratively, over the course of the semester, shows me that those seemingly divergent points of view, actually are convergent in the way that many of them are talking about it. . . . hearing this discussion go on, incrementally over the course of the semester, has given me pause to say maybe they understand more about the nuance than I thought . . .
>
> Alyson: I think that is the thing I keep coming back to, is it's the idea of the experience. . . . So, if we're trying to force a discussion into one class period, are we really allowing them the opportunity to draw on their experience, particularly if we don't prime it ahead of time in some way? . . .
>
> Signe: . . . I think we understood intuitively not explicitly that it's going to take space and time, for the students to be ready to have a discussion.
>
> Alyson: We decided early on, there was work to be done before you could have a discussion. . . . But we still came into this semester saying, if I've done that prep work, then I can have this discussion. And it's this enclosed thing that happens within one class period. I feel like right now, we're not even saying that. (Conversation 11-10-2020)

A few weeks later, the conversation continued but with a focus on what the layers of opportunity were fostering in relation to the discussion question. Although our conversations over time shifted from talking about anticipating to talking about layering of instructional activities, the interactions between anticipating PTs' understanding and layering of activities was a cyclical process in which we engaged while pursuing our study of discussion practice.

Discussion

Study of our discussion practice unearthed two processes: anticipating PTs' thinking and layering instructional activities. These processes, identified as contributing to the integrative concept of scaffolding (Bakker et al., 2015), maintain the relationship between teaching and learning (Loughran, 2006) while informing our discussion practice. Thus, MTEs' pedagogy (Loughran, 2006) of discussion practice is informed by the concept of scaffolding and as such constitutes a key concept "for understanding pedagogies of practice in professional education" (Grossman et al., 2009, p. 2058). Each instructional activity resulted in diagnosis to anticipate PTs' sense-making of ideas foundational to the discussion. Layers of opportunity to learn were distributed over time and accumulated to support understanding of the discussion question and significance of that question. The layers of instructional activity are created by a cycle of instructional design initiated by the MTEs' needs to anticipate PTs' thinking, be responsive to PTs' thinking, and support PTs toward independence (e.g., Smit et al., 2013). This paper addresses in part how MTEs' pedagogy plays a key role in the "learning potential" (Grossman et al., 2009, p. 2089) of representations or approximations of practice used in professional practice by outlining how scaffolding informs MTEs' discussion practice.

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FACTORS INFLUENCING THE INSTRUCTIONAL PLANNING OF SECONDARY STATISTICS TEACHERS

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An increasing number of secondary teachers are being asked to teach standalone statistics courses. In planning for these courses, teachers draw upon their beliefs and are influenced by the context in which they teach. In this collective case study, seven high school statistics teachers participated in a series of interviews and observations designed to assess their decision-making processes, the beliefs they draw upon to make these decisions, and influential contextual factors.

Keywords: Instructional Activities and Practices, Teacher Beliefs, Data Analysis and Statistics.

Recent technological advances and world events make it imperative that every student acquires a basic understanding of statistics (Wild et al., 2018). Often, statistics is integrated throughout other mathematical courses, but increasingly at the secondary level, it is taught as a separate course. With this rapid rise in the number of statistics courses being offered comes an increased need for teachers to teach these courses. However, inservice and preservice secondary teachers often feel unprepared and lack confidence to teach statistics (Banilower et al., 2018; Lovett & Lee, 2018). This paper examines the process by which secondary statistics teachers draw on their beliefs to make instructional decisions by addressing the following two research questions: *What beliefs do secondary teachers draw on when planning statistics instruction? What contextual factors impact whether secondary teachers' beliefs are put into practice?*

Theoretical Framework

Teachers' beliefs have a significant impact on the decisions they make in the classroom, but this relationship is often mediated by the context in which they teach. Teachers' beliefs regarding statistics and teaching statistics have been shown to be quite varied (Chick & Pierce, 2008; Lee et al., 2017; Umugiraneza et al., 2016), and many of these beliefs are at odds with recent recommendations and guidelines for teaching statistics (e.g., Bargagliotti et al., 2020). Schoenfeld (1998) proposes a model of *teaching-in-context*, in which teachers' instructional decision-making depends, in part, on a certain subset of their beliefs that are activated by the context in which they are teaching.

Previous studies (e.g., Zieffler et al, 2018) have shown a large variation in the content and setting of statistics courses being offered to secondary students in the United States. This study examines how teachers' beliefs may impact their teaching of statistics and which contextual factors may influence how and whether these beliefs are put into practice.

Methods

Data and Participants

Participants in this study consisted of a volunteer and convenience sample of seven secondary mathematics teachers who were currently teaching at least one section of a statistics course. The teachers taught in a variety of contexts, including a traditional public high school, selective private and public schools, a private religious school, and a charter school across five counties in a state in the southeastern United States. Participants' years of teaching experience ranged from 4 years to 16 years, with the participants having taught between 1 and 32 prior sections of statistics. Five of the teachers were teaching Advanced Placement (AP) Statistics.

As part of a larger study, each participant was observed for three consecutive lessons (one teacher was observed for two lessons). Prior to the first observation, each teacher participated in an interview designed to identify factors that influenced the teacher's statistics instruction and to identify their learning goals and planned activities for the upcoming lessons. After each observed lesson, an additional interview was conducted with the participant, which included questions asking the participant to discuss reasons behind observed practices that were not discussed in the prior interview and to discuss the planning of the next observed lesson, if any. Part of the post-observation interviews consisted of a stimulated recall (Calderhead, 1981), in which teachers were shown video recordings or were read a description of an instructional practice, and were asked to recall reasons behind decisions that were made. The seven pre-observation interviews and 20 post-observation interviews serve as the data corpus for this study.

Analysis of Data

Passages from the 27 interviews were assigned codes using the constant comparative method (Glaser, 1965; Kolb, 2012) indicating specific types of beliefs that teachers discussed. Passages were also coded to indicate when a teacher discussed their instructional plan for the lesson, the context in which they taught, or previous experiences.

To answer the research questions, passages were identified that contained one of the following combinations of codes: *instructional plans/beliefs*, *instructional plans/contexts*. For each such passage, a short summary was written that captured the essence of the relationship between the two constructs. Each of these summaries were then analyzed along with summaries from other passages containing the same pair of codes to identify common themes, first for each individual teacher and then across all teachers.

Findings

Five different areas of beliefs were identified that impacted teachers' statistics instruction. The three areas most pertinent to statistics will be discussed here: beliefs about learning, about statistics, and about technology. Influential contextual factors that were discovered include limited planning time, limited instructional time, the presence of external assessments, large class sizes, and others. What follows elaborates on these findings.

Influential Beliefs about Learning

For all participants, beliefs about how students learn impacted how participants structured their classroom, norms they established for interactions, as well as larger curricular structures and goals of a lesson. Beliefs about learning that impacted instruction included beliefs about the role of discourse in learning, about needed supports for learning, and about the types of activities that best supported students' learning.

All seven teachers valued student-student discourse, but teachers had varying beliefs about the value of teacher-student discourse that impacted their instruction. Four teachers consciously tried to limit the amount of teacher-student discourse in favor of student-student discourse. These teachers held beliefs similar to those of Mr. Dennis (all names are pseudonyms):

> It's through that conflict that they have--of reading the question, not understanding it, and asking their neighbor--that they start to really solidify their understanding, much more so than me just talking and doing it. And so I found that they can internalize the concepts at a

> much deeper rate, at a much faster rate, if they go through that productive struggle in their groups without me interjecting the answer.

Other teachers felt that teacher-student discourse was invaluable to student learning. These teachers typically had discussions with as many students or groups as possible, in order to assess their understanding and to help them progress in their thinking. Large class sizes and limited class time often made this difficult for these teachers.

All of the teachers in the study valued "hands-on" experiences for their students. They believed that when learning was situated in active experiences, it was more likely to lead to students retaining information and making connections in later units between old material and new. However, limited instructional time often impacted how often these activities were used and resulted in teachers modifying activities to take less time. For example, several teachers believed that having students collect their own data would help them build a deeper understanding and establish a personal connection with the data. However, a lack of instructional time, either for a single class period or over an entire course, often caused these teachers to omit or shorten student data collection. A lack of access to effective hands-on activities and a lack of planning time to search for such activities also caused some teachers to use less hands-on activities than they believed was ideal.

Influential Beliefs about Statistics

Only three of seven teachers regularly expressed that their beliefs about statistics impacted how they taught statistics. These teachers believed that statistics was different than mathematics and thus effective statistics instruction looked different than it did for mathematics. Teachers' belief about the importance of communication in statistics, rather than the following of procedures, impacted these three teachers' instruction, as expressed by Ms. Greene:

> I think just, like, algebraically working through problems is very different than manipulating data….There's not but so much stats you can do, there's a lot of stats that you have to talk about….I have stressed to my students that stats is not about following procedures. Procedures will only get you so far. Whereas a lot of, I think, algebra and calculus is, like, building procedures that you can then use to solve different problems. But statistics is a little bit different from that.

These beliefs resulted in Ms. Greene using less direct instruction than in her other mathematics courses. However, like other teachers, she also felt pressures to cover all the material that was on the AP exam, and felt direct instruction was often a quicker way to cover this material. Because of this as well as the fact that she teaches four different courses and has limited planning time, she sometimes falls back on direct instruction despite her beliefs.

Influential Beliefs about Technology

Four participants expressed concerns that certain types of technology, if not used carefully, could hinder students' learning of statistics. These teachers would sometimes include technology-free portions of a lesson to ensure students would have a better understanding of what the technology was doing once it was introduced. Some of these teachers regularly expressed that technology such as applets and simulations could better support students' learning than graphing calculators did. However, contextual factors often made teachers utilize graphing calculators more often than they may have otherwise. A lack of sufficient instructional time resulted in some teachers using graphing calculators for the sake of efficiency, despite believing that the use of calculators was not the most effective way for students to learn. For those teachers that taught AP statistics, the fact that graphing calculators were the only technology allowed on the AP exam caused some teachers to avoid or reduce the use of other technologies.

The other three teachers believed that technology allows students to focus less on calculations, and more on interpreting results and understanding underlying statistical concepts. For some of these teachers as well, limited available class time impacted how frequently students would use technology other than calculators. Some would make compromises in how technology was used for the sake of time. For example, Mr. Fahey and Ms. Greene would often manipulate data or perform simulations on their own computers and project it to the students, despite beliefs that having students work with the data themselves would likely serve their learning better.

Discussion

It is noteworthy that only three of seven teachers readily discussed how they approached the teaching of statistics different from mathematics. Teachers were not directly asked if they approached these subjects differently. However, for these three teachers, these differences emerged in their interviews rather frequently, while for the other four teachers, these differences were almost never mentioned. There are some key differences between statistics and other mathematical topics (Rossman et al., 2006) that are reflected in recommended guidelines for teaching statistics (e.g., Bargagliotti et al., 2020). Considering these differences is an important part of planning effective statistics instruction. Most participants reported receiving little to no instruction on teaching statistics during their preservice education. Introducing this instruction into preservice teacher education could help teachers recognize some of these differences and positively impact their teaching of statistics.

Regardless of whether beliefs about statistics were influencing teachers' instruction, many of the planning decisions teachers made are particularly meaningful in statistics classrooms. Soliciting two different solutions to a problem, for example, can play a very different role in a statistics course than it can in many other mathematics courses—the non-deterministic nature of statistics allows for two different solutions to both have merit, whereas in mathematics, one of two competing solutions is likely to be incorrect. Choosing contexts for problems that are relevant to students can have a larger impact in statistics, where context plays a more vital role in the problem-solving process (Cobb & Moore, 1997; delMas, 2004). Giving students the opportunity to engage in non-procedural tasks, though important in other mathematics courses, is crucial in statistics courses to allow students' statistical thinking to develop (Bargagliotti et al., 2020). As technology has a significant impact on statistical analyses that can be performed and on the learning that can result, the choice of which technology to include in the classroom is also an important one (Bargagliotti et al., 2020). These decisions are among the many that teachers of statistics must consider if they are to meet the learning needs of their students.

All seven participants had to continually grapple with contextual factors, ranging from limited planning time, large class sizes, short class periods, external assessments, and limited access to technology. Even though these teachers may have had the knowledge needed to effectively teach statistics under ideal circumstances, these factors often resulted in the teachers having to either compromise their beliefs or be inventive in the ways in which they dealt with these factors. However, preservice teachers are often tasked with lesson planning without regard to these contextual factors. They are often given weeks to plan a lesson, given access to technology that may not be accessible in many schools, and are given the freedom to choose their class length and characteristics of students. These utopian conditions unfortunately do not match the reality that these teachers will likely face in real classrooms. Making preservice teachers aware of contextual factors that they will have to contend with and giving preservice teachers practice planning in more realistic situations (e.g., an hour to plan a 50-minute lesson for a class of 25 students) can better equip these teachers with the skills they need to succeed in the classroom.

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UNDERSTANDING SPECIFIC STRATEGY DETAILS TO SELECT STRATEGIES FOR WHOLE-CLASS DISCUSSIONS

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Whole-class discussions have become an integral part of mathematics lessons that focus on teaching that is responsive to students' thinking. Central to these discussions are the student strategies, thus it is beneficial to understand the criteria teachers consider when engaging in the practice of selecting strategies for discussions. In particular, understanding the specific strategy details for fraction story problems can make selecting strategies for whole-class discussions more purposeful. This study explored 3 teacher cases and 41 teachers participating in professional development activities who were engaged in the practice of selecting. Examination of the data revealed specific strategy detail categories which help characterize the practice of selecting strategies. This article illuminates the specific strategy details teachers can consider to make selecting strategies for whole-class discussions more purposeful.

This study is grounded in a type of high-quality instruction called *responsive teaching*, which encompasses three characterizing features identified by Richards and Robertson (2016): (a) attending to the substance of students' ideas, (b) recognizing the important connections within those ideas, and (c) taking up and pursuing those ideas. One way that teachers are able to learn about their students' reasoning in mathematics classrooms is by providing opportunities for students to articulate their mathematical ideas during whole-class discussions. In these discussions, teachers typically showcase some strategies and then facilitate a conversation by posing questions based on the mathematical ideas within and across strategies. Smith and Stein (2018) highlighted the complexity of orchestrating productive discussions in their seminal book that identifies five practices in which teachers must engage: (a) anticipating strategies, (b) monitoring strategies, (c) selecting strategies, (d) sequencing strategies, and (e) connecting strategies in whole-class discussions.

Despite the widespread appreciation of the five practices, I argue that there is limited specific guidance on purposefully selecting strategies based on the details for whole-class discussions. *Selecting* strategies refers to the practice in which teachers determine the pieces of student work to be shared in whole-class discussions. Given the importance of selecting strategies to orchestrate productive discussions and the limited information about how to specifically engage in this practice, this study was designed to illuminate teachers' criteria for purposefully selecting strategies based on specific strategy details in the context of fraction instruction in the upper elementary school grades.

Literature

The practice of selecting strategies for whole-class discussions is one of the five practices identified by Smith and Stein (2018) to orchestrate productive discussions. There have been some criteria identified by research to purposefully select strategies for discussions (Cirillo, 2013; Kazemi & Hintz, 2014; Kersaint, 2017; Meikle, 2016; Smith & Stein, 2018). These criteria have been general in nature and provided a starting point for teachers working to improve their selecting expertise. Hewitt (2020) built upon the information to create a framework for selecting strategies. The framework for selecting strategies for whole-class discussions identifies three main criteria (Hewitt, 2020) applicable in any content area. The three main criteria for selecting strategies include: (a) the mathematics of the strategy, (b) the author of the strategy, and (c) the class engagement with the strategy. Teachers can base their strategy selections for whole-class discussions on one or more of three main criteria. These criteria do not serve as a checklist, but rather provide different considerations for teachers when selecting strategies. This article builds on the specific strategy details, which are a sub-criteria of the mathematics of the strategy criteria, by exploring the following research question:

- *How do elementary school teachers, who are working toward being responsive to students' thinking, select strategies based on specific strategy details for fraction story problems for whole-class discussions?*

Methods

In this study, I collected and analyzed data from two data sets that come from a larger set of data, *Responsive Teaching in Elementary Mathematics*, in which teachers participated in three years of professional development (PD) focused on developing teachers' responsiveness to students' thinking. The first data set included six observations and 12 interviews about the practice of selecting strategies with three teachers (grades 3-5) who had demonstrated expertise in teaching that is responsive to students' thinking. The second data set included PD conversations of 41 grades 3-5 teachers as they collaboratively engaged in the practice of selecting strategies during workshop activities in year two or three of a PD. Data included four PD activities and included a total of 17 video-recorded conversations among small groups of teachers. My analyses involved a constant-comparative method (Glaser, 1965) to make sense of the data and identify the range of criteria teachers considered. I used descriptive coding (Miles et al., 2014) to capture the specific strategy detail teachers were discussing when selecting strategies. To be considered a code, teachers had to mention it more than once if it was an individual teacher or by more than one teacher. I then synthesized these criteria to create the categories based on specific strategy details for selecting strategies for whole-class discussions.

Findings

In selecting fraction strategies to share, teachers considered one or more of three main categories of specific strategy details: (a) final answer, (b) representation, and (c) use of quantities (Hewitt, 2020). Each of these detail categories includes multiple sub-categories, which were determined as a result of this study and will be described in the remainder of this article.

Teachers' Decision Making Linked to the Final Answer

Teachers considered the specific strategy detail of the final answer when selecting strategies to share during whole-class discussions. *Final answer* refers to the teachers' consideration of how the final answer was included in the strategy, including the correctness, the form, or the visibility in the strategy.

Teachers considered the *correctness of the final answer* when selecting strategies to share with the class to broaden assessment of problem solving. Specifically, the final answer was considered in conjunction with the validity of the strategy yielding multiple possibilities: correct answer with a valid strategy, correct answer with an invalid strategy, incorrect answer with a valid strategy, or incorrect answer with an invalid strategy. Because many teachers emphasize helping students reach correct answers, one might assume teachers would typically select strategies that have correct answers with valid strategies or incorrect answers with invalid strategies. The sub-category of correctness of the final answer underscores that there is more complexity to this decision making.

Teachers selected strategies based on the *form of the final answer* to showcase the different ways the answer can be written and still refer the same quantity. Answers could be written in word form, using fractions that are not combined, or using proper or improper fractions.

Teachers selected strategies based on the *visibility of the final answer in the strategy* to promote understanding of how an answer can be reached using a drawing-based strategy. When the symbolic notation used to identify the final answer is consistent with the drawing, students are supported in seeing how the final answer was reached. In contrast, at the end of a strategy, students sometimes transform the final answer into an equivalent fraction (e.g., a fraction in lowest terms) and this transformed answer may make it harder to see the connection between the drawing and the final answer.

Teachers' Decision Making Linked to the Representation

Another specific strategy detail category teachers considered when selecting strategies for whole-class discussions was the representation. *Representation* refers to how the student used drawing (i.e., pictures), symbolic notation, or words in solving the story problem. In particular, teachers considered the overall form of the representation, the use of word labels in the strategy, the extent of the representation, and the specific shapes used in the strategy.

Teachers selected strategies based on the *overall form of the representation* to highlight the variety of ways story situations could be represented. In particular, teachers distinguished three types of strategies: strategies based solely on drawing, strategies that relied exclusively on symbolic notation, and strategies that involved a mixture of drawing and symbolic notation. Drawing strategies refer to all of the fractional quantities being represented with a picture and no use of symbolic notation except for the answer. Symbolic-notation strategies refer to strategies in which the amounts are only represented numerically, and no drawings are used. Strategies that involved a mixture of drawing and symbolic notation refer to strategies that have a drawing representing part of the strategy, but other parts of the strategy or the fractional amounts are symbolically notated.

Teachers selected strategies based on the *use of word labels in the representation* to explicitly connect the story problem to the strategy. Teachers considered whether or not any labels existed as well as what types of labels. Labels typically identified problem quantities (e.g., children, pizza, etc.) or pieces of the story situation (e.g., amount needed, amount already have, etc.).

Teachers considered the *extent of the representation* to refer to the way students represented various problem quantities when they were selecting strategies for whole-class discussions. Each story problem involved multiple pieces (e.g., items to share, items to group, sharers, etc.), and teachers considered how these pieces were or were not represented in strategies. This criterion was important because when students initially start solving problems, they generally represent all of the problem quantities or at least all of the quantities that need to be manipulated (Empson & Levi, 2011).

Teachers selected strategies based on the *specific shapes used in the representation* to highlight the importance, or lack of importance, of the shape being used. Specifically, teachers considered whether rectangles, circles, or other shapes were used to represent the quantities in the story situation. Students typically start by matching shapes to the shapes of the objects described in the story problem (e.g. circles for pizza and pancakes, or rectangles for sticks of wood) and eventually move to using shapes that are easiest for them to draw and partition regardless of what those shapes are representing. Research has shown that the specific shape used does not reflect different levels of fraction understandings (Empson & Levi, 2011) but students may initially hesitate to move away from the real-world representations.

Teachers' Decision Making Linked to the Use of Quantities

The third specific strategy detail category teachers considered when selecting strategies for whole-class discussions was the use of the quantities to highlight the ways quantities were manipulated in the strategy. *Use of quantities*, refers to how the quantities were manipulated in the strategy. In particular, teachers considered equivalent quantities, benchmark quantities, and operations used on quantities. Additionally, teachers considered the fractional quantities students created, most often for equal sharing story problems.

Teachers selected strategies based on the inclusion of *equivalent quantities* to highlight important comparisons of fractions of different sizes or ways to combine fractional quantities of different sizes.

Teachers selected strategies based on the inclusion of *benchmarks quantities* (also called landmark numbers) to emphasize the power of using familiar amounts (e.g., one whole) when solving problems. Benchmarks play a role in strategies using a drawing, as well as those that relied exclusively on symbolic notation.

Teachers selected strategies based on the *operations used on quantities* so that they had the opportunity to connect students' reasoning to formal operations. Teachers also connected operations across strategies to help students make sense of the relationship between operations or to link use of formal operations to drawings.

Teachers selected strategies based on the *fractional quantities created* in equal sharing problems to give students familiarity with a variety of fractions (e.g., thirds, fourths, eighths, etc.). Teachers often posed story problems that would likely result in the use of different fractional quantities, and then, depending on the goal for the lesson, teachers would select strategies because certain fractional quantities were created. This consideration was most prevalent in equal sharing problems.

Teachers also selected strategies based on the inclusion of an explicit link between the number of sharers and number of partitions to highlight the power in considering this relationship in equal sharing problems. Sometimes the number of partitions corresponded to the number of sharers (e.g., using sixths with 6 sharers) and other times factors of the number of sharers were involved (e.g., using thirds with 6 sharers).

Conclusion

The specific strategy details described in this article illuminate what teachers might consider when purposefully selecting fraction strategies for whole-class discussions. These detail categories are not meant to be a checklist to be executed, but instead they are meant to give a sense of the range of what specific strategy details can inform the selection of strategies for whole-class discussions. The specific strategy detail categories provide benefits to both researchers and practitioners. For researchers, the specific strategy details elaborate on criteria already identified in the literature, while also incorporating new criteria. In particular, the specific strategy details elaborate on the selecting strategies framework (Hewitt, 2020) by providing more nuanced ways to think about selecting fraction strategies. For practitioners, the specific strategy details can inform teachers' decision making and help them become more purposeful when selecting strategies, thereby providing students with more opportunities to learn in whole-class discussions.

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ILLUMINATING PURPOSES OF GROUP WORK THROUGH TEACHERS' LANGUAGE IN EVERYDAY MATHEMATICS LESSONS

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Teachers have responded to NCTM's (2014) charge to enact student-centered pedagogy by having their students work together on mathematical problems in small groups. To investigate how teachers enact group work in everyday mathematics lessons, we analyzed 33 video recorded 4th-5th grade mathematics lessons using an inductive qualitative approach. Our preliminary findings report on 115 instances of group work across the lessons, revealing multiple simultaneous cognitive and social purposes of group work evident in teachers' language. We share a variety of examples from our data to illustrate the interaction between these purposes.

Keywords: Classroom Discourse, Elementary School Education, Instructional Activities and Practices

Many teachers have responded to the National Council of Teachers of Mathematics' (NCTM; 2014) charge to shift to more student-centered pedagogy by having their students work together on mathematical problems in groups (Featherstone et al., 2011). While researchers have documented productive conditions for cooperative small-group learning (Cohen, 1994), less is known about how teachers enact small group work in everyday mathematics lessons (Webb et al., 2019). Moreover, teachers' facilitation of small group work influences how students engage with each other's mathematical ideas (Yackel et al., 1991; Webb et al., 2006; Franke et al., 2015), students' dispositions toward mathematics (Jansen, 2012), as well as equitable participation (Cohen & Lotan, 2014; Featherstone et al., 2011). The goal of this study is to examine teachers' language as they initiated small group work to illustrate multiple simultaneous purposes of group work. Our research question is: What purposes of group work are conveyed by teachers' language?

Theoretical Perspectives

The perspective that learning and talking are inextricably linked underlies our work. Drawing from Vygotskian-inspired theories of learning (Sfard, 2015; Cazden, 2011), learners construct their own knowledge via language (both verbal and non-verbal) either internally or with others. Cognition (or thinking) cannot exist without communication, implying that communication is a necessary condition for thinking, and, in turn, learning. From this perspective, teaching and learning transpire through talk in the classroom (Mercer, 1995; Resnick et al., 2010; Michaels et al., 2008; Hufferd-Ackles et al., 2004). For our study, we are concerned with talk as it pertains to the immediate learning environment. Since we are interested in how talk influences classroom interactions, we operationalize teachers' language as *tools* (Michaels & O'Connor, 2015) for structuring classroom discourse. That is, teachers' utterances surrounding group work become tools that communicate explicit and implicit purposes for group work.

Methods

Data were collected from 2014 to 2016 as part of a larger project. Participants in the study were 33 4th and 5th grade teachers in one mid-sized urban school district in the Pacific Northwest with teaching experience ranging from 1 to 30+ years (averaging 15.3 years). One full mathematics lesson from each of the 33 teachers was analyzed. The 33 lessons were sampled based on variation of Mathematical Quality of Instruction (MQI; Hill, 2014) into three categories: high, medium, and low (4,5=high, 3=mid, 1,2=low). This data was selected because it represents variation in mathematical quality of instruction across one school district. Therefore, we anticipated that teachers' language surrounding group work would vary across these lessons.

Our first analytic phase consisted of creating transcripts from the group work portions of the video recorded lessons. The criteria identifying instances of group work were: 1) students were prompted by the teacher to work in groups or with partners and/or 2) there was evidence that students talked to each other in pairs or small groups. Two coders independently viewed each lesson and identified *group work segments* (unit of analysis); 115 such group work segments were identified across the lessons. Next, we created segment memos (Creswell & Poth, 2016), then iteratively read through the data and memos to develop initial codes related to cognitive purposes (drawing on cognitive demand; Stein & Smith, 1998) and social purposes of group work that began to emerge (see Tables 1 and 2). One researcher coded all 115 segments, and a second researcher served to challenge interpretations. Any disagreements were resolved through discussion.

Table 1: Cognitive Purposes for Group Work

Cognitive Purposes	Description	Percentage
Facts/Answers	Recall facts, or share/tell, compare, check answers.	16%
Procedures	Complete procedural problems or talk about procedures without connections (Stein & Smith, 1998).	17.9%
Sense-making	Make sense of mathematical words/language, symbols, procedures with connections (Stein & Smith, 1998), contexts, representations, and relationships.	38.4%
Problem-solving	Analyze strategies, solve mathematical problems, pose problems, compare solutions or strategies.	21.4%
Justify/Generalize Math Claims	Construct mathematical arguments to justify a claim or statement; generalize a pattern or use a counterexample to disprove a claim.	6.3%

Table 2: Social Purposes for Group Work

Social Purposes	Description	Percentage
Optional	No clear purpose or prompt to talk, students can choose to work individually or with others.	11.6%
Sharing	Talk (e.g., share, tell, explain) with no additional social purpose.	43.8%
Sharing and Listening	Get help from each other, share ideas while others listen without defined roles, or read/interpret ideas without further engagement.	15.2%
Sharing and Listening +	Share and listen with defined roles or read/interpret ideas with further engagement (e.g., agree or disagree, why, think of a question).	16.1%
Comparing/Connecting	Compare or make connections between each other's ideas or other students' ideas.	4.5%
Interdependent	Co-construct ideas (e.g., come to an agreement, persuade each other).	8.9%

Preliminary Results

Our research question aimed to identify purposes for group work evident in teachers' language. The data displayed in Figure 1 illustrates the interconnection between cognitive and social purposes across the 115 coded group work instances (data points are scattered for clarity). We share examples from the data that highlight different interactions between cognitive and social purposes.

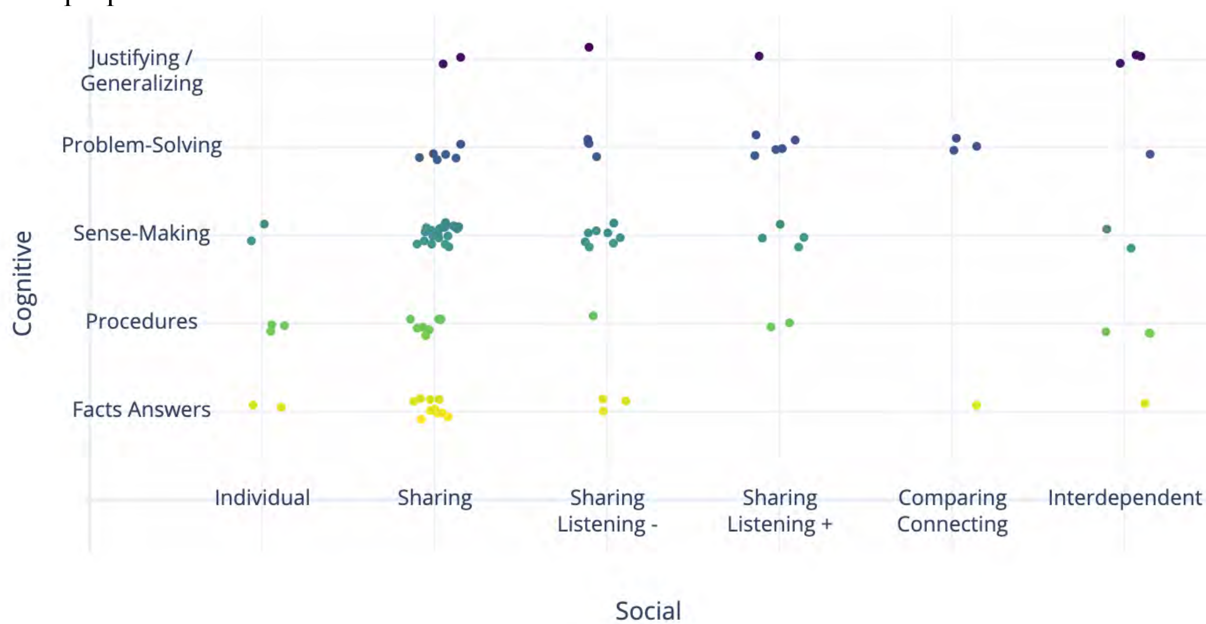

Figure 1. Interaction between Cognitive and Social Purposes of Group Work

Example 1: Sense-making and Optional Purposes

In a 4th grade lesson about modeling fraction multiplication problems with manipulatives, students sat together at table groups talking and exploring the manipulatives. The teacher addressed the whole class, “a nice thing to do might be to see if you can put them together from the largest fractions to the smallest … and then do you know what each fraction is worth.” Here the teacher’s language indicated that the cognitive purpose was to order the fraction pieces while exploring the size or value of the pieces (Sense-making). However, students could have done this individually or together since there was no explicit prompt to talk to each other (Optional).

Example 2: Sense-making and Sharing Purposes

At the beginning of a lesson, a teacher asked students to first solve the problem 4 divided by 1/4, then prompted, “Why would I be dividing and then all of a sudden, it's multiplying […] why would I do that? Does that even make sense? What do you think? […] turn to your partner and talk about it.” The teacher’s language suggested that the purpose of group work was to “turn” and “talk about” “why” a procedure makes sense moving beyond just sharing procedures to making sense of why a procedure works (Sense-making), but without additional social structure for how to talk about it with their partners (Sharing).

Example 3: Comparing/Connecting and Problem-solving Purposes

A 4^{th} grade teacher made two students’ work publicly available for the whole class. After providing individual time to study and think about the two students’ strategies, the teacher then initiated small group work: “Since some of you are on the floor and stuff with your elbow partner or in a small group of 3… talk about what you notice about the two works, about how they're similar and how they're different, go.” The language here is reminiscent of the Sharing social purpose (“talk about what you notice” with a partner), however the social purpose went beyond sharing *their own* strategies (Problem-solving) to comparing *two different* student strategies with someone else (Comparing/Connecting).

Example 4: Interdependent and Sense-making Purposes

During a lesson about solving equations, a teacher wrote 5 = 3 on the board, and asked students a warm-up question to learn about what they think an equal sign means: “So what does that equal sign *mean* when you see that? I would like you to turn to your color partner and see if you can decide– agree on a different definition for an equal sign.” The social purpose here was not just to share ideas, but to *agree on* a definition with their assigned partner (Interdependent), while the cognitive purpose was to come up with a definition for an equal sign – making sense of the meaning behind a common mathematical symbol (Sense-making).

Conclusion

Teachers’ enactment of small group work in everyday mathematics lesson has received little attention in the literature. The contribution of our preliminary analysis revealed multiple simultaneous cognitive and social purposes of group work evident in teachers’ language. We shared a variety of examples from our data to illustrate the interaction between these purposes. We argue that by making subtle purposes of small group work visible, the research community can better understand how teachers’ language shapes student-student interactions and impacts students’ opportunities to access and engage in mathematical discourse with their peers. This work provides further insight into the complexities of enacting group work to achieve simultaneous goals during mathematics lessons, and has the potential to inform existing teacher professional development programs focused on talk in the classroom (Michaels & O’Connor, 2015).

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AN EXPLORATION OF TEACHERS' WHY-QUESTIONS IN THE MATHEMATICS CLASSROOM

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Teacher's why-question can press students toward justification, make sense of mathematical structure, and make students' thinking visible to others. However, the productivity of why-questions hinges on their underlying purpose. In this brief report, we illustrate our framework of underlying purpose of why-question by examining 152 why-questions from 49 classroom videos (grade 4th-8th). While a particular question can appear similar in content, the expected student responses ranged significantly and thus we argue for two implications. First, as researchers, coarsely defining question types by categories such as "why" may be insufficient to tie a teacher move to a particular functional purpose. Second, if we want why-questions to cue students to provide mathematical domain explanations (justifications), there is a need to better understand what classroom/discourse factors lead to productive why-question use.

Keywords: Classroom Discourse, Instructional Activities and Practices

The questions that a teacher poses to students can shape mathematics classrooms in terms of opportunities for students to reason mathematically (e.g., Boaler & Brodie, 2004; Franke, et al. 2009; Sahin & Kulm, 2008). One particularly powerful type of question is the why-question which is often associated with higher order questioning (Kawanaka, & Stigler, 1999) which can serve to press students towards justification (e.g., Conner, et al., 2014), make sense of important mathematical structures (e.g., Jones & Bush, 1996), and make students' thinking available for the teacher and other students (e.g., Sahin & Kulm, 2008). However, the productivity of such questions hinges on their underlying purpose. The same question, "Why?," can lead to a substantially different level of student engagement in varying classrooms.

In a larger project classifying teacher prompts (Melhuish, et al., 2020), we discovered that why-questions were particularly anomalous when compared to other moves that can be productive in engaging students in rich mathematical discourse and reasoning. When we created profiles of various types of classes via a cluster analysis of teaching moves, we found that "why" questions did not serve as a marker differentiating classes that were more focused on mathematical reasoning from classrooms where the teacher did the majority of the mathematical work (Author, year). In fact, the "generic why" prompt was the most prevalent of any of our codes and spanned the majority of the lessons in the project. As a result, we conjectured that why-questions were likely serving substantially different roles for different teachers.

In this paper, we share an analysis of the why-questions found in a corpus of 64 video-taped mathematics lessons spanning grades 4-8. For each instance, we considered the nature of the why prompt, conjectured an instructional purpose, and considered how the students responded to the request. As a result of this analysis, we developed a framework to classify the mathematical why's of instruction. We share this framework and discuss the implications for instruction.

Problematizing Why-Questions

The mathematics classroom reflects a community of a teacher and students where norms shape the overall activity (e.g., Cobb, 2002). And so, we argue there is a need to go beyond categorizing teacher questions based on linguistic form, but to situate their meaning in the larger classroom domain and likely functional outcome. This is especially true for "why-questions" which have often been differentiated from other types of requests because of their context-sensitivity (Cox, 2019). We define a why-question as "some proposition *P* along with the request that *P* be explained" (Temple, 1988, p. 141). Often these questions will be of the form "Why *P*?" such as "Why did you add five to the nine?"; however, such a request for explanation could be asked with an implicit why such as, "How come you added five to the nine?" Thus, why-questions can be operationalized as any prompt for an explanation that could be formulated into a "Why *P*?" question without changing the intended meaning.

Why Questions are Context-Dependent

Temple (1998) elaborated that the "assumption that lies behind the [why-question] seems to combine a *motive* for asking the question in this way with an *expectation* about the sort of answer that is likely to be given" (p 150). The motive may not be immediately apparent as why-questions are particularly context-sensitive (Cox, 2019; van Fraassen; 1980) relating to both contrast (why this and not that) and domain sensitivity (what is an acceptable explanation in the relevant domain). Take the example from above. Depending on contrast, this question could be implicitly asking why "five" was added (rather than another number) or why the numbers were "added" (rather than another operation). Further, why-questions are domain dependent where adequate explanation depends on the relevant domain. While all of the why-questions in this project are in the context of a mathematics classroom, it is quite likely that the domain of explanation could vary based on norms (such as a procedure or conceptual focus).

Why-Questions as Implicit Requests for a Mathematical Justification

In mathematics education literature, why-questions are often treated as serving a particular motive: requesting a justification or proof which we call a *domain explanation.* A proof or justification can be thought of as a mathematical argument for why a particular mathematical claim is true using accepted premises, structures, and modes of argument (Stylianides, 2007). If we consider tools focused on teaching, we find "why" often plays this role explicitly or implicitly such as in EQUIP where teacher questions are categorized as *why*, *how*, *what*, or *other* (Reinholz & Shah, 2019) or Conner et al.'s (2014) collective argumentation framework where the questions: "Why?" or "Why doesn't that work?" are used to exemplify a request for a justification. Similarly, educators like Jones and Bush (1996) illustrate that why-questions are fundamentally linked to exploring mathematical structure. However, we conjecture that why-questions may not always be linked to the expectation of a mathematical domain explanation.

Why Questions are Not Always Requests for a Domain Explanation

While some philosophical (Sandborg, 1998) and empirical attempts (Stacey & Vincent, 2008) have been made to operationalize explanation in the domain of mathematics, they tend to stem from mathematician communities or mathematical text. Such explorations are likely to idealize mathematical explanation in ways that do not fully account for the types of explanations requested during conversation in a K-12 classroom. The literature about why-questions outside of the classroom point to a number of ways they are used in conversation including: serving the role of critiquing (Bolden & Robinson, 2011), rhetorical (Larrivée & Levillain, 2019), requesting a fact or process (Faye, 1999), or requesting an opinion (Mishra, & Jain, 2014). Further, the work on mathematical domain explanations (Sandborg, 1998; Stacey & Vincent, 2008) focus on mathematical claims rather than why a student did a particular thing, a second-person perspective (Roessler, 2014) likely to exist in a classroom. Such why-questions in mathematics classrooms could reflect legal (why are we allowed to do that?) and strategic (why did you make that choice?) decisions (Chazan & Sandow, 2010).

The Project and Analyzing Why-Questions

In this report, we share an analysis of the why-questions teachers ask in mathematics lessons. Our data corpus includes 64 lessons (49 which had why-questions) from distinct teachers spanning two school districts in the United States. In district 1, a midsized urban district in the Pacific Northwest, we selected a stratified random sample of 33 4th and 5th grade lessons based on their Mathematical Quality of Instruction (Hill, 2014) score. In the second district in the Southwest, a large urban district, we included 31 middle school (5th-8th grade) mathematics. For each video, two coders identified any instance of a "generic why" – that is a why question coming from the teacher-to-student(s). The coders met and reconciled any differences. From this process, we identified a set of 152 instances of why-questions. For each why-question, a member of the research team wrote a memo containing context leading up to the why, a transcript of the why-question, and the student response. From the first twenty videos, two researchers took notes describing the evidenced purpose of the why-questions eventually leading to a framework including a number of dimensions: type of why (why, why not, why or why not; strategic, legal, peer-evaluation, or claim), conjectured expected student response (elaborated below), focal mathematical object, and who introduced the mathematical object (teacher, student, peer, class). The initial framework and categories were tested and refined based on the remaining data.

The Why of the Whys in the Mathematics Classroom

An overview of expected student responses can be found in Table 1. Notice that of the why-question, 55% aligned with an expected response in the mathematical domain—evidencing that why-questions do often serve the motive implicitly assigned to them of seeking a mathematical justification. However, 45% of the why-questions did not appear to seek mathematical-explanations reflecting substantial variation.

Table 1: Expected Student Responses

Non-Explanation (32%)		Non-Domain Explanation (13%)		Domain Explanation (Justification) (55%)			
No response (Rhetorical)	3%	Explain the process of arriving at an answer or step	6%	Argue for representational or numerical equivalence	9%	Argue that a mathematical claim (propositional) is true or false	8%
Correct an error or mistake	13 %	Explain a strategic choice or efficiency of approach	3%	Argue that an instantiation meets a definition	14 %	Argue for the Reasonableness of an Answer	3%
Refer to a rule/fact	16 %	Explain a linguistic or task context choice/feature	4%	Argue that a strategy is appropriate conceptually	15 %	Critique or debate a peer's contribution	7%

Table 2 illustrates several examples to contextualize some of these variations. The first example illustrates a why-question whose purpose aligns with students generating a mathematical explanation (justification) -- which is consistent with the implicit treatment of why questions in the literature. In the second example, we see a teacher asking students to provide a fact that makes a procedural option invalid (a legal request). In the third example, the why-question does not seem to be requesting an explanation but rather is serving the purpose to notify the student of a mistake.

Table 2: Illustrative Examples

Description and context	Interpretation
The teacher asked whether each piece of a given shape is equal to 1/4 and students said, "No." So, the teacher said, "why not?" It was followed by the student's explanation, "because they are not even, these don't make a square."	**Type:** why not & claim **Expected response:** argue an instantiation meets a definition **Focal object:** concept
During whole class discussion, the teacher asked whether they can add 5/7 and 2/3 and students said "No." So, the teacher asked, "why not?" A student answered, "because they don't have the same denominator."	**Type:** why not & legal **Expected response:** refer to a rule/fact **Focal object:** procedure
Students are working in small groups on a problem involving elephants eating 150lbs per day, determining how much they eat in April. The teacher asks one of the groups, "So why did you divide these two numbers?" The student then attempts to explain their process, to which the teacher responds, "Are you sure you want to divide, though?"	**Type:** why & strategic **Expected Response:** correct a mistake **Focal object:** strategy to solve a problem

Discussion

In this brief report, we share our analysis of why-questions spanning a diverse set of mathematics classrooms. We found that the context-sensitivity of why-questions is apparent in the mathematics classroom discourse. While a particular question can appear similar in form, the expected student responses ranged significantly. With these results in mind, we argue for a few implications. First, as researchers, coarsely defining question types by categories such as "why" may be insufficient to tie a teacher-move to a particular purpose. Second, if we want why-questions to lead to students providing mathematical domain explanations (justifications), there is a need to better understand the necessary components for a why-question to be productive. Finally, the ambiguity of why-question can also lead to situations where students interpret why-questions as different from the teacher's intent. Such mismatches could lead to student responses being assessed as incorrect or incomplete (on exams or in conversation). As educators, we should be attentive to the very valid alternative way students can understand these types of questions, which this data suggests has its roots in how teachers likely vary in their intentions while using the same linguistic form.

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MATHEMATICAL AUTHORITY IN AUTHORING, ANIMATING, & ASSESSING MATHEMATICAL IDEAS

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In this paper, we describe how mathematical authority is enacted in middle-grades mathematics classrooms where authority, for us, is dynamic, negotiated, and discursively enacted. To operationalize authority, we considered (a) the mathematical activity for which one was claiming authority, and (b) the holder of authority (who deferred to whom for a given activity). We accounted for three broad categories of mathematical activity: Authorship, Animation (oral and written communication), and Assessment of mathematical ideas. The results shared here provide an overall characterization of mathematical authority for the classrooms in our study.

Keywords: Classroom Discourse, Communication

How often do students see themselves as a mathematical authority in their classroom? Are students positioned as important contributors to the mathematics under investigation? How regularly do students assess the correctness or validity of mathematical ideas? We view these kinds of activities as important indicators of mathematical authority in a classroom. Moreover, we contend that it is necessary and productive for students to have mathematical authority.

Classroom environments in which students have opportunities for increased mathematical authority are not only difficult to create and maintain, but they also produce new challenges. For example, Stein, Engle, Smith, & Hughes (2008) identified the importance of balancing student authority and disciplinary accountability when facilitating student-driven discussions. Students should have authority to solve problems in ways that make sense to them and be positioned as mathematical authorities within their classroom community. But students' work also needs to be accountable to the discipline and consistent with accepted knowledge and practices of the larger field. This tension looms large as teachers and students navigate new responsibilities in contexts with shared authority. And, yet, too often students are not given these opportunities. Much of the way we organize and structure classrooms (Pimm, 1987) as well as the words we use positions students with little authority (Drew & Heritage, 1992; Herbel-Eisenmann & Wagner, 2010; Wagner & Herbel-Eisenmann, 2009). But it is possible to create classrooms in which students have mathematical authority (Cobb et al., 2009; Gerson & Bateman, 2010; Langer-Osuna et al., 2020; Wilson & Lloyd, 2000). And a growing body of literature suggests that when students have authority to engage in problem-solving and sense making, they develop deeper understanding of concepts (Boaler, 2002; Wilson & Lloyd, 2000), more productive identities and dispositions toward mathematics (Bishop, 2012; Boaler & Greeno, 2000; Cobb et al., 2009; Gresalfi et al, 2009;), and may become more responsible, empowered, independent learners (Wilson & Lloyd, 2000). Thus, in this paper, we consider how mathematical authority is enacted in middle-grades mathematics classrooms, with an eye toward students' mathematical authority.

Review of the Literature on Authority in Mathematics Education

Scholars use various conceptualizations to study authority in mathematics classrooms. The primary conceptions we draw from are: (1) authority as a relation between groups or individuals, (2) authority as an opportunity to make a choice, and (3) authority as discursive, each of which we discuss below. Benne (1970) defined authority as a triadic relationship between a bearer of authority, their subject (i.e., someone in need of guidance or direction willing to defer to the bearer), and the field (context in which the subjects operate and in which determinations of competence are made). Authority, unlike power, involves a decision for one to acquiesce to another in service of a common purpose. Pace and Hemmings (2007) frame authority as relational, socially constructed, mutable, and dependent on contextual factors. These relational conceptions of authority highlight its dynamic, negotiated, malleable, and social nature. Some scholars do not focus on relationships per se, but instead foreground the idea of opportunities for choice as a core component of authority. For example, Cobb, Gresalfi and Hodge (2009) defined authority as "the degree to which students are given opportunities to be involved in decision making about the interpretation of tasks, the reasonableness of solution methods, and the legitimacy of solutions." (p. 44) While a relationship between individuals may be inferred in this definition, the focus of Cobb et al.'s definition is on characterizing authority by looking at what students are allowed to do with respect to the mathematics in the classroom (see also Gresalfi et al., 2009). And still other research demonstrates how authority is discursively constituted. For example, there is a growing body of work that uses positioning theory to highlight certain types of discursive moves (such as invitations for students to consider and respond to a peer's idea and directing a peer's problem solving) that position students as a mathematical authority or competent (Bishop, 2012; Engle, et al., 2014; Langer-Osuna, 2016; Turner et al., 2013). Others focus on the structural organization of classroom discourse and relate observed discursive patterns to authority. For example, Herbel-Eisenmann, Wagner, and Cortes (2010) used the construct of a lexical bundle to identify repeated phrases (e.g., "I want you to") that structured authority relations in classroom settings by, for instance, obligating students to engage in particular mathematical activities (see also Herbel-Eisenmann & Wagner, 2010).

Our operationalization of authority draws from all three of these conceptualizations. We define *authority* as a dynamic and negotiated relationship between people (or groups or organizations) where one party defers to another within a mathematical situation. We account for the source (or bearer) of authority by focusing on who makes decisions about what is permissible or allowable during various mathematical activities and how those activities are discursively constituted moment to moment. The research questions guiding our work were: (1) In what ways and by whom is mathematical authority enacted during whole-class interactions within and across multiple middle grades classrooms? (2) What patterns and variations in authority relations exists across classrooms in our study?

Methods

Participants, Data, & Analytic Framework

This study is part of a larger research program investigating mathematics discourse in middle-grades classrooms. The participants are eleven grades 5-7 classrooms across four US states. Data was comprised of video recordings and transcripts of at least four lessons on algebraic reasoning in each classroom for a total of 57 algebra lessons. The unit of analysis for our coding was a segment. We define a segment as a series of turns of talk with a common focus (e.g., activity or strategy) and a consistent form of participation (whole-class, independent work, etc.). Boundary markers for segments were indicated by changes in a problem, task, or topic often indicated by changes in intonation, resources, physical orientation, or linguistic markers. To analyze authority relations, we considered the activities in which students *collectively*, as a group, had the authority to engage, during a given segment. Thus, our analysis focused only on whole-class interactions around a bounded activity. To operationalize authority, we considered (a) the mathematical activity for which one was claiming authority, and (b) the holder of authority (who deferred to whom for a given activity). We accounted for three broad categories of mathematical activity: Authorship, Animation, and Assessment of mathematical ideas which are defined in Table 1. In our analytic framework, the categories of Author and Animator are consistent with Goffman's (1981) forms of participation, though we created separate subcategories for oral (speaking) and written (scribing) animation.

Table 1: Authority Relations AAA Coding Categories

Category	Description
Author	**Who generated the main mathematical idea that was the focus of the segment.** Identifies the source of the mathematics that was taken up by the class.
Animator[1]	*Speak*: **Who orally communicated the mathematics.** Identifies who verbally (and publicly) uttered the mathematical ideas in the segment. Speaking includes asking probing questions, clarifying, or adding on to a previously-stated idea. These contributions, on their own, are not sufficient for authorship.
	Scribe: **Who publicly contributed to the mathematics through written or gestural communication.** Identifies who was responsible for public inscriptions or gestures such as inscribing math content using a document camera, or pointing to written work while standing at the board.
Assessor	**Who explicitly judged or validated the mathematics under consideration.** Assessments could take multiple forms: expressing an opinion or conclusion about information; determining the value or quality of a mathematical object or process; and expressing disagreement or agreement. We did not include implied evaluations (i.e., echoing a correct answer, repeating a question) as assessments.

For each segment we assigned a code for *who* enacted authority for each of the four mathematical activities described in Table 1. Mathematical authority for Authorship, Animate Speaking, Animate Scribing, and Assessment might lie with the Teacher, Student(s), Both, or None (i.e., did not occur). Codes for Authorship, Animate Speaking, and Animate Scribing were assigned *holistically* by looking across the entire segment and assigning a single code that best characterized the overall discourse in terms of who authored and animated the mathematical ideas. We did not code Assessment holistically; instead, we aggregated individual assessments within a segment by identifying every assessment in a segment and who made it. We assigned codes of None, Teacher, Student, and Both when, respectively, no assessments were present, only teachers assessed, only students assessed, or both teachers and students assessed. Examples of the Authority Relations AAA (read 'triple A') Framework are presented in the Results.

Results

The main result of this study is the development of the Authority Relations AAA Framework. The example and coding explanation in Figure 1 below illustrate how we used the AAA Framework to account for mathematical authority across the activities of authorship, animation, and assessment. Further, this example illustrates how students and teachers can work together to productively share mathematical authority within a bounded mathematical activity.

Figure 1: Illustrating Mathematical Authority with the AAA Framework

Illustrative Segment: Which is larger, *x* or -7?	**Coding: Both-Both-Teacher-Student**
Students in Teacher L's classroom were working on a series of comparison tasks to determine which algebraic expression was larger or if they had enough information to determine. Here, the students were comparing *x* and -7. *Anuj:* So I think it has to be *x*. [For problem 9, *x* and -7] *Student 1:* No. *T:* Why? *Anuj:* … Because um, because um, um *x* is a positive number. *Student 2:* How do you know? *Anuj:* Wait, that's where I'm kind of getting confused. … *T:* So it didn't, what does *x* mean? What's a variable mean Anuj? We could replace the *x* with? *Anuj:* Any number but I didn't [trails off]. *T:* So you guys were asking him why do you know? Why'd you ask him that? *Hays:* Because he doesn't know what number *x* is. *T:* Okay so you're saying *x* could be what? *Multiple Students:* Be a negative or a positive. Or a fraction. Anything. … *Anuj:* But wait. I thought if they replace a negative number they have to put something like a negative *x*. Right? *T:* Well if we go back to looking at this [Points to expression -*x* on board from previous problem]. This is the part that we're just replacing [circles *x* in expression -*x*] is this. This [points to *x*] is where the number is. *Student 3:* Then it could be a negative. *Anuj:* Yeah but if it was a negative number, then uh, I'm just kind of confused. Because then if it was a negative number I thought they have to put a negative *x*. *T:* So look right here. So here's your *x* [points to *x* in the expression -*x*]. And you're saying if it's a negative number, if it's a negative five. Do you see how these we just took this *x* and we said okay we're just going to make it a negative five [gestures to – (-5) already written on board]. But we still have to keep this negative in front [points to negative sign on -*x* and negative sign to left of (-5) in expression – (-5)] . And that's how we end up with that negative, negative. … *Anuj:* Well in that case it's a mystery. … Because uh, there's a negative seven, but then you don't know what *x* is. *T:* So if *x* were a negative number what would happen? If *x* were a negative three, which one would be larger? *Anuj:* It would be negative three. *T:* If *x* were a negative seven which one would be larger? *Goldie:* They're equal. *T:* What if *x* were a negative ten? *Anuj:* Then it would be negative seven.	**Author:** Anuj and other students share their understanding of the problem as they consider whether the sign of the variable determines its domain (i.e., is it valid to consider $x < 0$ for the expression *x*) and reason about the conditions under which *x* is greater, equal, and less than -7. The teacher demonstrates how to substitute values for *x* and selects three cases of $x = -3$, $x = -7$, and $x = -10$ for Anuj to compare to 7. Thus, in this segment **Both** the teacher and students had authority for authorship. **Animator (Speaking): Both** the teacher and students had authority to verbally present mathematical ideas. **Animator (Scribing):** The teacher is directing the students' attention by gesturing to written board work. Thus, the **teacher** had authority for scribing. **Assessor:** Student 1's disagreement ("No") and Anuj's statement of confusion ("I'm just kind of confused") are explicit **student** assessments. There are no explicit teacher assessments in this segment.

After developing the AAA framework, we also used this analytic tool to explore trends across the classrooms in our study and generated authority profiles to describe important differences in those classrooms. Though we do not have space to discuss these findings here, we found, in general, that teachers were more likely to share authority for authorship of ideas, but maintain authority for scribing and assessing. In closing, we hope this framework can provide a lens for teachers, teacher educators, and researchers to reflect on the ways teachers and students enact mathematical authority.

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Note

[1] The codes of Author and Animator differentiate the roles of generating the mathematics under consideration and communicating the mathematics. In many cases the Author and Animator coincided, but not always. For example, if a teacher revoiced a student strategy, the student was the Author and the teacher the Animator.

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EXAMINING THE RELATIONSHIP BETWEEN AMBITIOUS INSTRUCTION AND CULTURALLY RESPONSIVE TEACHING IN ELEMENTARY MATHEMATICS

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Mathematics education research has yet to reach a consensus on what ambitious instruction looks like in practice with historically marginalized learners. This mixed methods study examines the relationship between ambitious instruction and culturally responsive teaching (CRT) in elementary mathematics classrooms. The first phase of this study incorporated a multi-case study to understand how three teachers who have been certified in CRT by professional development opportunities within their district are implementing CRT. In the second phase of the study, a quantitative observation measure was used to examine the standards-based mathematics teaching practices of the three teachers. The findings suggest that effective mathematics teaching practices are foundational to implementation of CRT and examine dimensions of ambitious instruction that support CRT in practice.

Keywords: Culturally Relevant Pedagogy, Elementary School Education, Equity, Inclusion, and Diversity, Instructional Activities and Practices

Purpose & Frameworks

The construct of ambitious instruction is well supported in the literature on teaching (Franke, Kazemi, & Battey, 2007; Grossman, Cohen, Ronfeldt, & Brown, 2014; Lampert, Boerst, & Graziani, 2011; Thompson, Windschitl, & Braaten, 2013). Ambitious instruction is broadly defined as a set of teaching practices that foster students' deep conceptual understanding of standards-based mathematics concepts (Newman & Associates, 1996). However, Smith et al. (2017) claim that "ambitious mathematics teaching must [also] be equitable" and responsive (p. 5). Although there are numerous practitioner resources on equitable teaching practices for all learners, there continues to be a lack of research on the prevalence and practice of ambitious instruction with historically marginalized learners (Gutierrez, 2013). Therefore, mathematics education research has yet to reach a consensus regarding what ambitious mathematics instruction looks like in practice with historically marginalized learners. Combined with heightened awareness of social injustices, this lack of a strong research base has revived researchers' commitment to critical pedagogies.

While the theoretical framework for CRT has informed the educational community for some time, scholars (e.g., Hammond, 2015) continue to discuss the challenges of how to operationalize CRT. Mathematics education has produced limited research examining the teaching practices of culturally responsive teachers in preK-12 (Thomas & Berry, 2019). Bonner (2014) offers three reasons for why this might be the case, including: the majority of the works are specific to one population such as African American learners (e.g., Ladson-Billings, 1994); there is a broad focus on content and practice, making it non-mathematics-specific (e.g., Gay, 2010); and, the works remain largely theoretical (e.g., Greer et al., 2009).

The purpose of this study is to examine the relationship between CRT and ambitious instruction through the lens of standards-based instruction in elementary mathematics. A mixed methods comparative case study design (Creswell & Clark, 2018) has been used in which qualitative and quantitative data were collected concurrently and analyzed in phases. Qualitative data were collected to analyze the CRT of the three participating teachers and quantitative data were collected to examine dimensions of ambitious mathematics instruction.

Theoretical Framework

Gay (2010) introduced a CRT framework for "using the cultural knowledge, prior experiences, frames of reference, and performance styles of ethnically diverse students to make learning encounters more relevant to and effective for them." (p. 31). Gay (2010) defines CRT as being: validating and affirming, comprehensive, multidimensional, empowering, transformative, and emancipatory. While Ladson-Billings' (1994) framework of culturally relevant pedagogy (CRP) differs in some ways from the CRT, Gay (2010) stated, "Although called by many different names, including *culturally relevant*, … and *responsive*, the ideas about why it is important to make classroom instruction more consistent with cultural orientations of ethnically diverse students, and how it can be done, are virtually identical" (p. 31). Thus, both CRT and CRP influence the work being done in the focal district in this study. Furthermore, the district draws upon the work of Hammond's (2015) *Ready for Rigor Framework* in which she claims to make explicit, "the natural intersection between so called brain-based-learning and CRT" (p. 4).

Conceptual Framework

Ambitious pedagogies are often used interchangeably with *standards-based teaching practices* (e.g., Lampert et al., 2010). *Standards-based* refers to teaching practices that provide learners with opportunities to engage in mathematical practices or behaviors as outlined in the National Council of Teachers of Mathematics (NCTM, 2000) process standards that focus on problem solving, reasoning and proof, communication, connections, and representation (Walkowiak et al., 2018). Such teaching practices also capture more recent standards (e.g., mathematical modeling and argumentation) in the U.S. released by the National Governors Association Center for Best Practices and Council of Chief State School Officers (2010; Walkowiak et al., 2018). These standards focus on teaching practices that support conceptual understanding. Numerous observation measures have been developed to measure for standards-based mathematics teaching practices including Mathematics-Scan (M-Scan) (Berry et al., 2017).

Research Question

1. How do three elementary teachers, who have been certified in CRT, implement mathematics teaching practices? How does the mathematics instruction support CRT?
2. What are the Mathematics-Scan scores of the three teachers' elementary mathematics lessons? How is the instruction similar or different across the cases?
3. What is the relationship between ambitious instruction and CRT in these elementary mathematics classrooms?

Methods

This mixed-methods study draws on observations and interviews from three teachers in Wayne (all names and locations are pseudonyms) school district in the South East. The school district is large, serving 14,000 elementary students. The district is known for its diverse student population, and for the past four years has been offering teachers CRT certification. All three teachers were part of the 2019 cohort to receive CRT certification.

Participants

The three teachers in this multi-case study are Sophia, Ava, and Chris. Sophia is a pre-kindergarten teacher at River Elementary. She is a Black woman in her mid-30s, and she has been teaching for six years. There were 18 students in her classroom including: 11 Latinx, 5

African American, and two students of Middle Eastern descent. Ava is a third-grade mathematics teacher in a second and third-grade multiage classroom at River Elementary. She is a Black woman in her late-20s who has been teaching for five years. Ava described the students in her class by saying that nine students are Black, one is White, two are multi-racial, one is Asian, and seven are Latinx. Chris is a fourth-grade teacher at Willow Elementary in a Spanish immersion program. He is a White, Jewish man in his late-40s who has been teaching for 11 years. Chris had 20 students, 13 of whom are Latinx and seven are White.

Data Collection

In the first phase of the study, I conducted a semi-structured interview with each of the participants to understand how they perceived their CRT in mathematics. The interview was followed by a questionnaire about each of the participant's cultural reference points (Hammond, 2018). Next, I spent approximately 20 hours in each of the classrooms conducting observations. Data were collected using double-column field notes and all of the lessons were video-recorded. Additionally, the teachers completed journal prompts for each of the observations. Following observations, I conducted a second-round of semi-structured interviews. The second phase of the study involved going back through the recorded video footage to score 32 mathematics lessons using the observation measure M-Scan.

Analytical Strategies

In Phase 1, the information gathered from the mapping of cultural reference points questionnaire (Hammond, 2015) served as preliminary data and led toward the development of other methods. Teacher journal reflections were re-read and compared to data from the corresponding classroom observations to examine each teacher's awareness. Both rounds of semi-structured interviews were recorded and transcribed to allow for member checking. I transferred all fieldnotes into write-ups and wrote analytic memos intermittently to document emerging themes and inferences from transcripts, write-ups, journal reflections. Dedoose was used to support the coding process, using a combination of inductive and deductive coding (Corbin & Strauss, 2015). I compared confirming and disconfirming evidence and continued to adjust the findings until all of the evidence was accounted for.

In Phase 2, I used M-Scan to analyze observation data. M-Scan represents a schema of instruction used to observe teacher' implementation of standards-based teaching practices. The instrument is used in research that focuses on ambitious mathematics instruction because it captures differences between teaching for conceptual understanding and teaching for acquisition of procedural knowledge (Berry et al., 2017). M-Scan has nine dimensions that measure teaching practices in four domains: task selection and enactment, use of representations, the use of mathematical discourse, and lesson coherence. See Berry et al. (2017) for a conceptual model and definitions of each domain and dimension, and Walkowiak et al. (2014) for the measure's validity and score reliability. When using the scoring rubric, the dimensions are each coded on a scale of 1 to 7 with descriptors of low (1-2), medium (3-5), and high (6-7).

In Phase 3 of data analysis, strategies (Creswell & Clark, 2018) were used to merge the two sets of results, such that cases were compared based upon the criteria outlined and results were integrated. In Phase 4, interpretations were made based upon the merged results.

Results

In addressing the first research question, it became evident that the teachers' conceptions of CRT were highly influenced by the work of Hammond's (2015) *Ready for Rigor* framework. However, some practices of CRT exemplified across the three cases are more thoroughly captured in other literature (e.g., Gay, 2010; Ladson-Billings, 1994). The findings for CRT focus on *Quadrant 1: Awareness* (recognizing cultural reference points and reflecting upon teaching practices), *Quadrant 2: Learning Partnerships* (partnerships with families, partnerships with learners, care, and high expectations), *Quadrant 3: Community of Learners* (classroom environment, cultural competency, and power), and *Quadrant 4: Information Processing* (growth mindset, relevance, mathematical representations, and mathematical discourse). The quadrants are oriented to mathematically model the coordinate plane and the ways in which the teachers went about building CRT at the beginning of the school year. However, after the initial phase (of consecutive order), this model is viewed as a continuous cycle without particular attention to order and the quadrants are not mutually exclusive. *Gaining knowledge* has been placed at the center of this model or at the origin of the quadrants. Lastly, within each quadrant, there are nested circles such that the first indicator (in parentheses above) represents the outer circle and each succeeding indicator is nested within the previous.

Initial data from M-Scan scoring revealed similarities and differences across the cases. For example, all three teachers have consistently scored in the medium-high (5) range on use of mathematical representations and mathematical discourse and for encouraging multiple strategies for problem solving. Furthermore, while the cognitive demand of the tasks were often low-medium (3), the teachers' enactment of such tasks were most often medium-high (5). An exemplar of differences identified across cases relates to the teachers' scores for connections and applications of mathematical tasks. In general, these scores were fairly low which certainly speaks to opportunities for improvement in making the mathematics more relevant, but Chris' scores were higher in this dimension across the lessons than those of Sophia and Ava.

When making comparisons across the two data sets, conclusions can be drawn about the relationship between CRT and ambitious instruction with these three teachers. The teachers were all engaging in elements of both ambitious instruction and of CRT, but at times in their lessons, vital elements were missing from each. For instance, in ambitious instruction, the teachers were engaging in strong use of mathematical representation and of mathematical discourse but all struggled with connecting and applying the mathematical tasks to their students' lived experiences. This parallels the teachers' use of CRT in which they were implementing strategies that drew upon students' funds of knowledge to help students process information while neglecting to realize that the mathematical tasks were rarely culturally relevant. On the other hand, the presence of dimensions of ambitious instruction seem to relate to tenets of CRT. For example, the teachers' scores in mathematical accuracy speak to their mathematics content knowledge and their comfort with giving students choice and power in their learning.

Discussion & Significance

In mathematics education, there is still uncertainty regarding how to operationalize CRT (e.g., Bonner, 2014) and other pedagogies focusing upon equity and social justice, indicating that more work is needed in this space (Thomas & Berry, 2019). Researchers in mathematics education continue to wrestle with the relationship between ambitious instruction and CRT. When mathematics education researchers focus upon issues of equity, they are often met with criticism surrounding the lack of emphasis on the mathematics content (Foote & Bartell, 2011). Yet work that focuses on content often neglects to address learners' funds of knowledge and cultural backgrounds. This work looks at both CRT and ambitious instruction, bridging the divide. The finding that teachers' strengths and weaknesses across CRT and ambitious instruction are parallel has profound implications for how we think of these two constructs.

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TEACHERS' INSTRUCTIONAL RESPONSES TO THE COVID-19 PANDEMIC

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In March 2020, the COVID-19 pandemic drastically impacted schooling across the United States. Many schools closed rapidly giving teachers little time to prepare. In May 2020, as part of an ongoing study on Algebra instruction, we interviewed teachers to understand how their instruction changed as a result of the switch to emergency remote teaching. Using a framework of practical rationality, we consider the breach of norms due to the COVID-19 pandemic, a natural breaching experiment, in examining three teachers' instructional responses during that time. Our findings suggest that while teachers often cited similar norms being breached, their responses to those breaches and the reasons behind those responses varied.

Keywords: Instructional Activities and Practices; Online and Distance Education; Middle School Education; High School Education

Introduction and Background

When the COVID-19 pandemic began to impact the United States in March 2020, schools rapidly transitioned to remote instruction. For many schools, this enormous shift to schooling occurred with little time to prepare. We describe instruction during this time as *emergency remote teaching* (ERT) (Hodges et al., 2020). ERT differs from typical online instruction in that it is temporary, rapidly deployed, and, in contrast to instruction designed to be online from the start, shifts in-person instruction to remote delivery (Hodges et al., 2020). Though there have been other recent crises that have impacted schooling in the United States (e.g., wildfires, hurricanes), the scope of COVID-19 has been unprecedented in terms of reach and duration.

Despite its recency, there have been studies on teachers' actions during the pandemic. Horn and McGugan (2020) studied secondary mathematics teachers' transition to ERT. They found that the teachers encountered obstacles surrounding time management, centering student thinking, and maintaining student relationships. They also suggested strategies for supporting teachers in these areas. Whittle et al. (2020) also examined the shift to ERT. Working with teachers and instructional designers, researchers documented challenges in moving to ERT. In doing so, they developed a framework that included eight dimensions related to the design of ERT environments: (1) critical learning goals, (2) pedagogy and the student social role, (3) ratio of teacher to students, (4) social role of the instructor, (5) building agency, (6) communication methods, (7) assessments, and (8) feedback. In the present study, we too examine teachers' move to ERT and focus specifically on the rationales for their instructional decisions.

The notion of practical rationality (Herbst 2010; Herbst & Chazan, 2011) has been used to describe the way that teachers justify instructional actions. Herbst and Chazan (2011) explained that teachers' actions, "are not mere expressions of their free will and personal resources; rather their actions also attest to adaptations to conditions and constraints in which they work" (p. 407). Webel and Platt (2015) used the construct of practical rationality to understand conflicts between teachers' goals and professed obligations and discussed the ways that disciplinary obligations impacted two teachers' decision-making. In these ways, practical rationality can be used as a means of studying changes in teacher practice.

The COVID-19 pandemic acted as a natural breaching experiment (Herbst, 2010), whereby classroom norms were suddenly violated, and teachers were required to adjust their instruction. Under practical rationality, when a norm is breached, teachers can use one of three *repair strategies* that "not only confirm the existence of that norm, but also elaborate on the role that the norm plays in the situation" (Herbst, 2010, p. 52). In response to this breach of norms, teachers can engage in actions that allow them to reject, repair, or accept the situation (Herbst & Chazan, 2011). Rejecting norms would imply a teacher chooses to continue classroom instruction as though the norms had not been breached. In contrast, teachers who accept the breach in norms will align their instruction with the new condition. Finally, teachers might opt to repair the situation by offering a "softer version of rejection" (Herbst & Chazan, 2011, p. 438) whereby teachers approach the breach with a repair strategy that indicates acknowledgment of the breach but adhere closely to teaching practices prior to the breach. In deciding upon a response to the breach of norms, teachers might draw from four categories of obligations: *individual* (individual learners' needs)*, institutional* (policies and practices related to the profession of teaching)*, interpersonal* (social interactions of the classroom)*, and disciplinary* (mathematical principles and practices) (Herbst & Chazan, 2011; Shultz et al., 2019).

Past studies that drew on practical rationality have mainly focused on how teachers would respond to predetermined scenarios. Little is known about how teachers respond to rapid, radical environmental changes in their classrooms. In this study, we seek to understand such an instance in answering the following research question: In the abrupt move to ERT due to the COVID-19 pandemic, what norms were breached, and how did the teachers respond?

Method

This study is part of a larger study examining teachers' instruction in algebra. During the final year of the larger study, the teachers were forced to move to ERT. Prior to that shift, we had observed the teachers' instruction 3 times throughout the 2019-2020 school year. We interviewed 11 teachers in Spring 2020 to understand how their instruction changed. For the present study, we focus on 3 of those participants: Ms. B, Ms. N, and Mr. J. We purposefully selected these participants because they varied in their instructional approaches (Table 1).

Table 1: Participant Information

Teacher	Grade	Location	Original Teaching Mode
Ms. B	8th	Suburb	Flipped
Ms. N	9th	Suburb	Non-flipped
Mr. J	9th	Rural Fringe	Individually Paced

The main data source for the present study is a semi-structured, video-recorded interview. Each interview lasted between 30-60 minutes. The interview aimed to understand the teachers' transition to ERT, and the questions centered on how the transition occurred and what their current instructional practices entailed. We also included the teachers' pre-ERT classroom observation and survey data as secondary data sources.

We first transcribed the video data verbatim. The authors then independently read through Ms. B's transcript to summarize the norms that were breached, her response to the breach, and the obligations on which she drew in determining that response. We met as a team to discuss our initial summaries. This meeting resulted in our development of a coding scheme that we used to consistently identify the breaches, responses to the breaches, and changes to instruction. Our resultant scheme required us to first identify instances in which the teacher explicitly described a breach of norm, then we identified the teacher's response to that breach, and we then classified the obligation the teacher drew on in deciding their response (Table 2). Lastly, we independently re-coded Ms. B's data to establish the reliability of the scheme, and then two authors coded the remaining two teachers' data independently and met to confirm the results.

Table 2: Coding Excerpt from Ms. B

Norm	Breach	Response to Breach	Obligation
Teachers deliver new content to students regularly.	Principal told teachers to only teach one topic per week.	*Reject:* Ms. B continued to teach new content.	*Individual:* "I just didn't think it was fair to teach them one topic per week when I knew that they were the accelerated or advanced students."

Findings

We found four norms that were breached across all three teachers in their move to ERT: 1) students complete assigned work, 2) teachers deliver new content to students regularly, 3) teachers consistently meet with students, and 4) teachers assign grades. Due to the brief nature of this report, we present our findings related to the first two norms and the teachers' responses to their breaches in the following sections.

Students Complete Assigned Work

"But just a handful [of students], for whatever reason, like haven't done anything." (Mr. J)

In discussing their move to ERT, the teachers consistently mentioned a significant decline in students' completion of assignments, suggesting a breach of a previous classroom norm. In discussing the reason for this breach, Mr. J and Ms. N cited their district's "hold harmless" policy that prevented teachers from assigning new grades that would lower students' overall course grades. Given that the teachers described grades as motivation for students to complete their work, the teachers perceived the inability to assign grades as a loss of a key incentive for students. In addition, because there were no set, compulsory class meeting times, the teachers had limited connections or space to motivate students to work on the assignments.

To encourage students to complete classwork, the teachers tried different strategies in seeking to repair the breach. Ms. N and Mr. J offered one-on-one virtual meetings with individual students either by appointment or dedicated virtual office hours. Ms. B and Ms. N set more flexible assignment deadlines as they understood students may have had difficulty completing the work on time due to the situations at home. In addition, all three teachers sought to consistently communicate students' progress with them and their families. In enacting these repair strategies, the teachers described individual students' needs in learning mathematics during ERT, thus evidencing their adherence to individual obligations.

Teachers Deliver New Content to Students Regularly

"I tried to reassure them that this was not something that I was going to throw brand new units out to them and expect them to be able to gain the knowledge and be assessed on that content on their own." (Ms. N)

In contrast to their instruction pre-ERT, the teachers discussed being constrained in their ability to move forward in the curriculum during ERT. Ms. N and Mr. J discussed a school policy that explicitly limited the amount of new content they were permitted to deliver to students during ERT. Ms. N's administration encouraged teachers to review prior content rather than move forward in the curriculum. Thus, rather than introducing new content, she provided her students with review materials to complete practice on topics they had already learned.

The three teachers responded to their new content policy to varying extents and drew on different obligations to justify their instructional decisions. Ms. N accepted the school's policy of not moving forward in the curriculum, suggesting adherence to institutional obligations.

> We basically took the sections in a [prior] unit and took the big ideas and put them together in basically very similar to what we would do for a "note section." Practice problems, but a lot more visuals for them. (Ms. N)

As she followed her school's new policy, she also discussed her obligation to individual students' needs by creating the review materials similar to what they previously did before the pandemic so the students would feel less anxious after they moved to the online setting.

Mr. J taught in an individually paced program in which students could move ahead through algebra and into the next course at their own pace. However, after the move to ERT, he was told only to allow students to finish their current coursework in algebra, but not permit them to move to the next course. As he already took out some sections (e.g., a project on sequences and series) due to either its difficulty or hard to manage in a virtual setting, he accepted this school's policy. He said, "I feel like you can't guarantee that they're learning the proper way" In adhering to the administration's policy, these teachers evidenced adherence to their institutional obligations.

Unlike the other two teachers, Ms. B rejected her administration's edict that she introduce no more than one new topic per week. She said that she did so because she did not feel it was fair to teach her 8th graders one topic per week since they are "advanced students." In making this decision she drew on her obligation to individual students' needs.

Conclusion and Discussion

We found that although teachers perceived many of the same norms being breached, they often responded differently to the breach and/or used different justifications for their actions. Most often, teachers cited individual and institutional obligations. We rarely saw teachers cite disciplinary obligations as a factor in their decision-making. This finding is in keeping with the obstacles that remained most salient to teachers in Horn and McGugan (2020), but different from the professional obligations that the teachers used differ from the obligations that teachers used for justifying their actions in changing teachers' teaching practices (Webel & Platt, 2015) or in dealing with students' mathematical contributions in whole class discourses (Herbst, 2010). This could be because, in the context of an ERT environment, the teachers not only respected their school's regulative decisions but also attended most to student needs when they addressed the breached norms. Additionally, "during COVID-19, participants experienced a focus on the method of delivering instruction rather than the learning goals" (Whittle et al., 2020, p. 315), which could have impacted the obligations teachers drew on when making decisions. In future research, it will be important to build from what we know about ERTs and how we can improve and synchronize directives from administrators regarding these changes to instruction.

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SUPPORTING GENERALIZING IN THE CLASSROOM: ONE TEACHER'S BELIEFS AND INSTRUCTIONAL PRACTICE

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The purpose of this case study is to understand how one teacher supports generalizing in her classroom by examining her beliefs about generalization and how to support generalizing in classrooms. We found that the teacher held numerous beliefs about generalization and these beliefs influenced her beliefs about how to support generalizing in the classroom. Moreover, we found that her beliefs about generalization and how to support generalizing formed a system of beliefs that were consistently evidenced in her classroom instruction. Practical implications of the findings, particularly for mathematics teacher educators, are discussed.

Keywords: Teachers' Beliefs, Instructional Practice, Generalizations

Generalization is an important aspect of learning mathematics; researchers have argued that the development of generalizations is essential to all mathematical activity (Becker & Rivera, 2006; Pierce, 1902). Researchers have investigated different types of generalizations students make (e.g., Ellis, 2007; Radford, 2006; 2008), the mental activities required to generalize (e.g., Amit & Neria, 2008; Becker & Rivera, 2006), and the types of instructional activities that support generalizing (e.g., Doerfler, 2008; Steele & Johanning, 2004). Still, studies investigating what teachers do to foster generalizations in the classroom are scant (Mata-Pereira & da Ponte, 2017) and the field lacks research that considers the *teacher's* perception of generalizations. Because understanding teachers' beliefs is an integral part of fostering substantive, lasting change to their practice (Pajares, 1992), the purpose of this case study is to examine one teacher's beliefs about generalization and how to support generalizing in classrooms as well as how these beliefs relate to her instructional practice.

Theoretical Framework and Literature Review

Generalization and How They Are Developed

Researchers investigating how instruction can foster generalization have identified a number of specific recommendations. These include techniques such as showing variation across tasks (Mason, 1996), emphasizing similarity across tasks (Radford, 2008), and ordering the structure of tasks in a progressive sequence (Ellis, 2011). Other recommendations address pedagogical moves (e.g. Amit & Neria, 2008, Koellner et al., 2008), yet research investigating teachers' efforts to foster generalizing at the classroom level is limited. Overall, however, recognizing, elucidating, and encouraging appropriate generalizations remains challenging for teachers (Callejo & Zapatera, 2017). Given these challenges, it is critical to identify teachers' perceptions of generalization and their beliefs about how to teach for generalization, in order to better support their ability to foster generalizing in the classroom.

Defining A Belief

For the purposes of this study, we use the definition provided by Rokeach (1968), who defined a belief as “any simple proposition, conscious or unconscious, inferred from what a person says or does, capable of being preceded by the phrase, ‘I believe that…’” (p. 113) and added that “all beliefs are predispositions to action” (p. 113). Another construct that was influential to this study is that of a belief system. Green (1971) proposed a theory of how beliefs are held in a system, and described three characteristics of a belief system. First, individuals hold beliefs in a manner that is logical and consistent to them. The second characteristic is that beliefs can be held with varying psychological strength. Thirdly, beliefs can be held in clusters and these clusters may be isolated from other clusters.

Mathematics Teachers’ Beliefs Research

The types of beliefs mathematics education researchers have most commonly investigated can be grouped into three categories: beliefs about *mathematics*, *teaching mathematics*, and *learning mathematics* (e.g. Conner et al., 2011; Thompson, 1984). Of the three categories, beliefs about mathematics may be most influential, as some researchers (e.g. Cross, 2009; Thompson, 1984) have argued that these beliefs influence a teacher’s beliefs about both learning and teaching mathematics. Regarding the consistency of teachers’ beliefs, a number of researchers have found teachers’ beliefs to be consistent with their practice (e.g. Conner & Singletary, 2021; Cross Francis, 2015). Yet, others have claimed inconsistency between the beliefs teachers hold and their practice (e.g. Raymond, 1997). Consistent with our definition of beliefs, we agree with Leatham (2006) and Philipp (2007) who claimed that researchers should not assume that a teacher holds beliefs that are inconsistent.

Although there is extensive research on teachers’ beliefs, investigations into what teachers believe about generalization and how to support generalizing in the classroom are scarce. Due to our view that beliefs have profound influence on one’s actions, we believe investigations into teachers’ beliefs about generalizations can yield novel insights into their instructional practices and can aid educators as they help teachers support generalizing in classrooms.

Methods

The present study was a case study (Merriam, 1998) of one teacher’s beliefs about generalizations and how to support generalizing in the classroom. Ms. N, the participating teacher, was a third-year teacher who taught sixth-grade mathematics. We conducted four classroom observations during one week of instruction and recorded each observation with two cameras. For the observations, Ms. N chose lessons in which students explored various properties of ordered pairs in the coordinate plane and how to choose appropriate axes scales.

Because we believe a teacher’s beliefs must be inferred from both their words and actions, we also conducted two interviews with Ms. N after the observations. The first interview was semi-structured and provided opportunities for Ms. N to discuss her beliefs about mathematics, generalizations, and how she supported generalizing in her classroom. The second interview was a videoclip interview (Speer, 2005). Two clips which Ms. N chose from one of the observations formed the basis of the interview. Both interviews were video recorded and transcribed.

The data from each observation was transcribed. We began data analysis by coding the two interviews with the broad codes of *generalization* and *classroom supports for generalizing* (CSG). We then determined emergent themes within each code and created subcodes according to these themes (Strauss & Corbin, 1998). With the new subcodes, we then analyzed Ms. N's classroom data and determined if there were any themes not captured with the current codes. After analyzing the observation data and revising our codes, we re-coded the interview data, looking for evidence for the existing codes and asking if new codes needed to be included. Once no new codes emerged from this iterative process, we began to write narratives for what Ms. N believed about generalizations and how to support generalizations in the classroom.

Results

In the following sections we discuss Ms. N's beliefs about generalization and her beliefs about supporting generalizing with examples of how she enacted those beliefs in her instruction.

Beliefs About Generalizations and How They Are Developed

The beliefs we inferred Ms. N held about generalization form a subset of her beliefs about mathematics. We identified two primary beliefs and three derivative beliefs (Green, 1971) Ms. N held about generalization. One of her primary beliefs about generalization was that a *generalization is an always true statement*. On multiple occasions she described a generalization as something that will always be true. The transcript below captures Ms. N's response when asked to contrast a generalization with a strategy. We underscore that Ms. N described generalization as the "one true thing" that she wanted students to walk away having developed.

Interviewer:	Why was this a strategy and not necessarily a generalization?
Ms. N:	Because…when I think of a generalization, I'm thinking of…one truth.
Interviewer:	Say that again, you cut up a little bit.
Ms. N:	Sorry, I'm thinking of, like, there's one truth. Like, there is one true thing that I want them to get. That's the generalization in my mind.

One of Ms. N's derivative beliefs was that *a theory is an idea that is not yet a generalization*. She described a theory as an idea that is either untested or true in some cases, but not in others. In one instance, Ms. N described a student's initial idea as a theory because it was based off one example and "it was true for that example". Implied in the way Ms. N described the student's theory was that it worked in some cases but not all cases. Ms. N also described theories as being untested. After being asked when a theory becomes a generalization, Ms. N responded by saying "Yeah. When…have you seen enough different types that you always trust it?"

Another belief Ms. N held that appeared to be derivative to her belief about generalizations was that *generalizations are tested theories*. In one instance from her classroom Ms. N claimed that she knew one student's initial theory "wasn't always true". However, after testing his theory and engaging in a discussion about the theory, the student modified his statement so that it was true for all cases. Ms. N claimed it was important for the student to refine his theory into "a more accurate statement that he could cling to." In this instance the student was able to develop a generalization as his theory went through a process of refinement.

Another notable and primary belief Ms. N held about the development of generalizations was that generalizations are actively developed rather than passively received. Throughout both interviews, Ms. N continually described generalizations as something her students would "discover", "develop", and "notice".

Beliefs About How To Support Generalizations In The Classroom

Ms. N believed it was important to engage students in examples that were "easy" and accessible for all students. The purpose of the easy examples was for students to "clearly *see* that it (the pattern she wanted students to notice) works". For instance, in a lesson in which students were to determine when two points were reflected across the x- or y-axis, Ms. N began the lesson by placing a red dart on the front board, then had a student place a yellow dart where the point would be if reflected across the y-axis. After discussing the coordinates of these two points and writing them on the board, Ms. N repeated this sequence for a second pair of reflected points. Rather than writing the two ordered pairs on the board and telling students that one point is the reflected image of the other, students were able to see this and presumably trust it to be true.

As alluded to in the previous paragraph, Ms. N believed it was important to leverage these easy examples to lead students to notice and discuss relationships that emerged from the initial examples. She did this by asking students a focusing question highlighting certain relationships. Ms. N also seemed to indicate that the relationships students noticed and discussed from the initial examples were not yet generalizations, but theories. Hence, Ms. N believed that accessible examples could be leveraged in a way that students could notice patterns and develop theories which then could be cultivated into generalizations.

After noticing and discussing the patterns, or theories, that emerged from the initial examples, Ms. N believed that engaging in additional examples helped students refine or reinforce their theories. For instance, after discussing the relationships between the coordinates of a point with the coordinates of its reflection, Ms. N gave additional examples for students to work in small groups. Ms. N claimed such examples were important because if "(you) do practice with them, and then they can do it independently on their own, they've like, got it (the intended generalization)".

Ms. N's System of Beliefs

Together, Ms. N's beliefs about generalization and how to support generalizing in the classroom form a system of beliefs (Green, 1971) that appear to be internally consistent and consistent with her practice. Specifically, we infer her beliefs that generalizations are developed through refining or reinforcing a theory to be consistent with and related to her beliefs about what constitutes a generalization and a theory.

Ms. N's beliefs about generalizations and how they are developed also appear to influence her beliefs regarding how to support generalizing in the classroom. Her beliefs about refining or revising theories to develop generalizations appear to influence her beliefs that students need to engage in easy, accessible examples first, then discuss patterns salient across those examples, then do additional examples so that students develop the intended generalization. As described in the previous section, Ms. N's beliefs about how to support generalizing in the classroom seem to be consistent with her instructional practice, and these beliefs also appear to be consistent with her beliefs about generalizations and their development.

Discussion

This study reveals one teacher's beliefs about generalization and how to support generalizing in the classroom and how those beliefs relate to her instructional practice. Most notable is that her primary belief that a generalization is "an always true statement" appears to heavily influence her beliefs concerning how generalizations develop and, as an extension, how to support students' in generalizing in the classroom. As we continue to work with teachers, in our own on-going project, their beliefs about generalization and how they are developed have been a paramount consideration in our interactions with each teacher. Moreover, the teachers' beliefs about generalization have been critical as we plan professional development aimed to co-construct a vision of productive generalizing in classrooms and generate instructional strategies that foster this type generalizing.

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SUPPORTING IMPROVEMENT EFFORTS IN TERTIARY INSTRUCTION: THE CASE OF CHARLES AS A BRICOLEUR

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The French anthropological notion of bricolage has been used as a research framework to describe various phenomena, but rarely to describe teaching mathematics. In this study, one college mathematics instructor, Charles, positions himself as an expert when lecturing, but acts as a bricoleur, or do-it-yourself craftsperson, when teaching via active learning. The case of Charles illustrates the obstacles that many experienced lecturers encounter as they attempt to transition to evidence-based instructional practices and reframes the efforts of teachers as bricoleurs with an anti-deficit perspective. This paper provides implications for tertiary institutions wanting to support instructors who wish to improve their pedagogy.

Keywords: Instructional Activities and Practices, Undergraduate Education, Precalculus

Levi-Strauss' (1974) notion of *bricolage* concerns the manipulations and resources used to engage in scientific sense-making. Bricolage could be defined colloquially as "do-it-yourself crafting" and refers to the intellectual and hands-on work of "crafts-people who creatively use materials left over from other projects to construct new artifacts" (Rogers, 2012, p. 1); in other words, these are creative, versatile workers who often use indirect means to accomplish a task.

Hatton's (1989) positioning of teachers as *bricoleurs*, or colloquially, "professional do-it-yourself" people (p. 75), provides a lens to view the efforts of teachers as they work toward multiple objectives with a fixed pool of resources. Toward humanizing the work of college mathematics instructors and supporting their pedagogical efforts, I present an interpretation of Hatton's (1988) established framework that aligns with anti-deficit discourses (Adiredja, 2019) and celebrates their productive struggles toward improving tertiary instruction.

Conceptual Framework

Internationally, institutions and instructors are moving toward active learning approaches to teach mathematics (AAU, 2017; CBMS, 2016) due to evidence that it is more effective than traditional lecture approaches (e.g., Freeman et al., 2014); however, lecture remains the norm (Stains et al., 2018), and the feasibility of implementing interactive strategies is often challenged by faculty and instructors (Le et al., 2018; Michael, 2007).

The lack of research about the *practice* of teaching (Rasmussen & Wawro, 2017; Speer et al., 2010), specifically those who teach via active learning (Bennett, 2020), leads to limited research-based support for instructors who are beginning to shift toward evidence-based teaching practices. Furthermore, institutional factors such as the physical learning space can influence pedagogy (e.g., Haines & Maurice-Takerei, 2019), and instructor self-efficacy to lecture in traditional spaces does not always translate to implementing innovative norms, such as groupwork, in collaborative spaces (McDavid et al., 2018).

Instructors make pedagogical decisions based on demands imposed on them by external stakeholders and structures. The actions of teachers can be understood via the practical rationality of mathematics teaching framework (Herbst & Chazan, 2003, 2012), which describes the *practice* of teaching, and thus shifts the unit of analysis to the external factors and justification sources that influence instruction, rather than internal factors of the instructor.

In this paper, the data were analyzed through the lens of practical rationality (Herbst & Chazan, 2012); however, the case of Charles as an instructor journeying toward an active learning approach is structured and presented via the tenets of teachers' work as bricolage (Hatton, 1988). Similar to other frameworks that describe teachers' pedagogical decisions and justifications, the positioning of teachers as *bricoleurs* is a way to examine the factors that influence, support, and constrain their work. Hatton (1989) restructured the tenets of bricolage to draw parallels between the work of teachers and that of bricoleurs. The six main characteristics are: conservatism, limited creativity, *ad hoc*-ism, teacher's use of theory, indirect means, and repertoire enlargement. Here, I briefly explain three of these characteristics.

The notion of *limited creativity* acknowledges that teachers' responses to many (often contradictory) obligations and goals can be restricted by external factors, such as content coverage in a curriculum or a lack of time and resources to demonstrate complete, unhindered creativity. A key parallel between teachers and bricoleurs is the *ad hoc* nature of their work (Hatton, 1989). The necessary improvisation of teaching highlights the versatility and competence of teachers to tackle situational challenges; namely, they use ad hoc-ism to make in-the-moment decisions and solve problems. Teachers' *use of theory* plays a key role in their pedagogy and "intellectual bricolage" (Hatton, 1989, p. 80), referring to how they order, justify, and explain their decisions. Similar to the bricoleur, teachers are expected to borrow knowledge from other disciplines and interpret it to fit their situation.

Here, I reiterate that the purpose of framing an instructor (Charles) as a bricoleur is to humanize his efforts and experiences as an instructional faculty member shifting toward a completely new teaching approach with the fixed pool of resources available to him.

Methods

This study took place at a large, public, research-oriented university in the U.S. that had recently created collaborative learning spaces as part of an initiative to improve undergraduate STEM teaching (AAU, 2017). This paper focuses on one non-tenure-track, instructional faculty member, Charles, in the Mathematics Department. Charles had over 40 years of experience teaching mathematics at the tertiary level, mostly at community colleges, and he had recently started teaching at this university. During the semester of data collection, Charles taught one section of College Algebra in a collaborative learning space with adaptive furniture and one section of Pre-Calculus in a traditional classroom with individual desks in rows (see Table 1).

Table 1. Information about Charles and His Courses

Course	Classroom	Class Size	Experience
College Algebra	Collaborative Space	72	Over 40 years of experience teaching college-level math
Pre-Calculus	Traditional Classroom	35	

The data presented here are part of a larger qualitative study. In this study, I observed Charles' teaching six times, three times for each class, using the Observation Protocol for Active Learning tool (Frey et al., 2016), which was validated for recording teacher and student actions in undergraduate STEM courses that utilize an active learning approach. I also took qualitative field notes during observations and immediately followed up some observed lessons with a debriefing conversation, which resembled an informal interview (Hatch, 2002) and helped to clarify my interpretations of Charles' actions during class. Toward the middle and the end of the semester, I conducted two semi-structured, in-depth interviews with Charles, which were guided by instructional situations from the observed lessons.

For data analysis, I coded qualitative interviews, lesson debriefs, and field notes using a combination of data-driven and theory-driven coding methods (Gibbs, 2007; Saldaña, 2016). I organized these data into teaching-related themes and goals as a single case study to describe Charles' pedagogy and experiences with active learning during this semester (Maxwell, 2012; Stake, 2006). Later, I realized that the central themes in Charles's case aligned with Hatton's (1988) theorizing of teachers' work as *bricolage* and restructured the case to present Charles' initial experiences of and experiments with active learning through the lens of a *bricoleur*.

Findings

Here, I describe Charles' experience (the successes and struggles) with active learning via the tenets of teachers as bricoleurs. Due to space constraints, I present only two of the six tenets.

Limited Creativity

Soon after arriving to the university mathematics department, after years of lecturing in community college settings, Charles was encouraged and inspired by colleagues to teach via an active learning approach. Charles taught College Algebra in a flexible learning space, with desks on wheels intended for both individual and group work. When this space was filled with the 72 students enrolled in Charles's section, it became a "swirling mass of chairs" and was challenging for Charles to maneuver between groups. Even though the "chaotic" arrangement of furniture in this space caused participation issues, Charles still preferred a collaborative-type room to promote active learning. He thought his Pre-Calculus students, currently in a traditional classroom, would be "more comfortable" in a collaborative space, admitting, "I like it when they have tables. If we had tables everywhere, that would be helpful, but I really don't mind."

Charles acknowledged that he could do more groupwork and even adapt how he structured collaborative activities if he only had different resources, such as a collaborative classroom with tables. His ideas for shifting instruction strategies in his Pre-Calculus class was limited by the lack of practical furniture. However, he was willing to work with what he was given.

During an interview, I asked Charles if he reflected on his own instructors for inspiration or guidance for teaching via active learning. He said that he had had "none" and that "working together [during class] was not an option in any environment" when he was a mathematics student. He agreed that he was trying to do something that he had never seen before – to create an image of a teacher that he had never experienced. He seemed to have a realization during our conversation: "Yeah, how do I do [collaborative learning]? I've never seen it before!" His lack of a role model or consistent mentor for shifting his teaching approach led to limited creativity in visualizing a collaborative learning environment for his classes.

Teacher's Use of Theory

Even after 40 years of teaching mathematics at the college level, Charles called himself a "novice" with respect to active learning. He acknowledged that many of the collaborative activities he tried in the classroom were "experiments," and later realized that having students work in groups sporadically is different from taking an active learning approach to teaching:

> I've done some group stuff in my class, but this is a different thing I'm finding out. …It's not just, 'I'm going to do a group activity every now and then.' This is a whole model of a

> pedagogical technique that we can use in the classroom that might be more effective. So I'm in the learning stage. …I feel like I don't really know what I'm doing.

This uncertainty led to a persistent theme in our conversations: his desire to understand the learning theories in education and psychology research. He admitted, "A big part of my anxiety about [teaching via active learning] is I'm not a psychologist." He felt that facilitating groups and letting students work on problems on their own required a deep understanding of how students learn. Lacking this knowledge, he experienced "anxiety" about teaching via active learning.

To better understand the mathematics education research, Charles initially sought out mentors at his university. He spoke about reading research articles written by his mentors to learn about collaborative learning techniques. Charles also read the Instructional Practices guide published by the Mathematical Association of America (Abell, 2018), hoping it would help him understand how students best learn mathematics and thus, how to teach effectively.

Charles was concerned about making groups work effectively, not simply ensuring that groups stay on task but also that they cooperate, participate equitably, and learn the mathematical concepts. At the end of the semester, Charles reflected on his educational background:

> There's so much psychology involved, and college teachers aren't necessarily trained in psychological aspects of teaching. …I've just pretty much modified whatever I started with over the years. I didn't really have any formal teacher training. And I guess now I'm kind of realizing, I might want to try to look into some formal aspects of how [teaching] works.

After a semester of modifying active learning strategies to align with a teaching approach he used for decades, he realized that a "formal" teacher education would be the next step in understanding how to implement the best teaching practices. Charles consistently demonstrated his thoughtfulness and dedication to improving his pedagogy by trying to understand the theory behind new teaching practices before implementing them.

Discussion and Conclusion

The case of Charles describes the ongoing journey of an experienced lecturer attempting to shift to an active learning approach to teaching introductory-level college mathematics. Positioning Charles as a bricoleur highlights the resourcefulness of his practices (*ad hoc-ism* and *indirect means*) and his dedication to improving his teaching regardless of the obstacles he encountered (*use of theory* and *repertoire enlargement*). Although Charles initially reached out to mentors and thoughtfully reflected on improving his pedagogy, he viewed the wide gap between understanding theory and using it in practice as very daunting without "formal" support structures. In other words, a key contributor to his *conservatism* and *limited creativity* could be the lack of formal support structures available to him. Charles' accomplishments during just a single semester can be celebrated, but it is also important to acknowledge his struggles in order to understand how they can be mitigated for other tertiary-level instructors.

Finally, I emphasize that the case of Charles illustrates an anti-deficit interpretation and re-presentation of Hatton's (1988) framing of teachers as bricoleurs. Recognizing the skill that bricoleurs bring to their craft, I argue that for an instructor who is completely changing their teaching approach, exhibiting the adaptability and resourcefulness of the bricoleur is commendable, given the many obligations and institutional constraints they must navigate (Bennett, 2020; Mesa et al., 2019). Charles made substantial progress toward an active learning approach within one academic year and planned for future improvements. He was working toward becoming an expert craftsman in active learning, rather than a bricoleur, striving to learn the theory and demonstrate unlimited creativity. But acting as a bricoleur was a critical part of his pedagogical journey, as it likely is for many tertiary-level instructors, who typically do not have formal teaching preparation (Laursen, 2019). In research, anti-deficit framings of teachers should be explored, elaborated, and utilized across all levels. As a field, we can study and promote institutional policies and sustainable systems that minimize the tensions in teaching obligations and support improvement efforts in tertiary instruction.

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SIMULATING MORE EQUITABLE DISCUSSIONS: USING TEACHER MOMENTS AND PRACTICE BASED TEACHER EDUCATION IN MATHEMATICAL PROFESSIONAL LEARNING

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Digital simulations have become an increasingly popular approach to practice-based teacher education. In this paper we report on a professional learning intervention where we used digital clinical simulations to help mathematics teachers' fluency in facilitating both small group and whole group discussions. Further we discuss implications of digital clinical simulations as a tool that can help mathematics educators develop, practice, and further support their teaching.

Keywords: Equity, Inclusion, and Diversity, Professional Development

Purpose of Study

Imagine a classroom is split into small groups and one of the students raises their hand to have the teacher address a mathematical dispute. The students share their answers and rather than elicit more information about their different understandings, methods, and perspectives, the teacher smiles and only invites the student with the correct answer to elaborate on their response. Just as other researchers, we view classroom interactions as powerful sites of learning for educators (Hiebert et al. 1997; Mellone, Jakobsen & Ribeiro, 2015). These interactions illustrate the nuanced relationship between power, identity, and participation in mathematical discussions. These interactions send powerful messages about what and who is valued in mathematics, and how students orient themselves to these messages is consequential for how students see themselves as mathematics doers. Teachers need to be mindful not to position students or their ideas as outsiders to the mathematical conversation (Amit & Fried, 2005).

In light of the complexities of orchestrating inclusive math discussions, practice-based teacher education (Grossman et al., 2009) presents an opportunity for mathematics teachers to develop sensibilities toward facilitating inclusive mathematical discussion. Through iterative cycles of practice, whether it be modeling (e.g., McDonald et al., 2013), rehearsals of "approximations" of practice (Grossman et al., 2009; Lampert et al., 2013), or video recording (e.g., Ball, 2013; Schoenfield, 2017), educators have the opportunity to deconstruct their actions in teaching and try new skills and routines in low stakes settings, while receiving feedback and support (Grossman et al., 2009). Increasingly, digital simulations have become a popular approach to practice-based teacher education (Driver, Zimmer & Murphy, 2018; Cohen, Wong, Krishnamachari, & Berlin 2020) as they provide the opportunity to distill the complex task of ambitious teaching into smaller, distinct, manageable approximations (Grossman et. al, 2009). As a community, educators and researchers collectively reflect on teachers' moves in a low-stakes setting i.e. before stepping foot in front of a classroom full of students (Dieker Hynes, Hughes, & Straub, 2017; Thompson et al., 2019).

In this study, we investigated the effectiveness of our efforts to help teachers 1) choose to include a range of student perspectives in their discussions and 2) to see these contributions not solely as errors to be rectified, but as sources of mathematical thinking to be valued and explored. We used a digital clinical simulation called *Teacher Moments*, in our 2019-2020 fellowship program supporting 19 in-service math teachers in grades 3 through 9. The platform engaged participants in vignettes representative of a teachers' classroom experience and then called on participants to respond, "in the moment" (Thompson et al., 2019). The simulations required participants to check-in with three simulated small groups and then discuss how they would lead a whole group discussion. In the small group interactions, participants were presented with student work and scripted student responses. The three simulated groups displayed four distinct approaches or attempts to solve the task. One group presented an *incomplete student response* (e.g., students who had difficulty solving the problem and stated they needed help); another group presented an *unconventional student response* (e.g., students used guess and check to correctly solve the problem); and the last group presented both, a *sophisticated student response* (e.g., correct algebraic equation) and an *incorrect student response with a misconception,* a designed mathematical dispute for the teacher to settle. At the close of each of these interactions, participants were purposefully asked the open question "how would you respond to this group" to capture a wide range of participants' responses (see Figure 1 for selected interactions in the scenario. The full scenario can be played https://teacher-moments-production.herokuapp.com/run/fb24f9ea87/slide/0) After checking in with all three groups participants had to decide which group they would call on to start the ensuing class discussion and why.

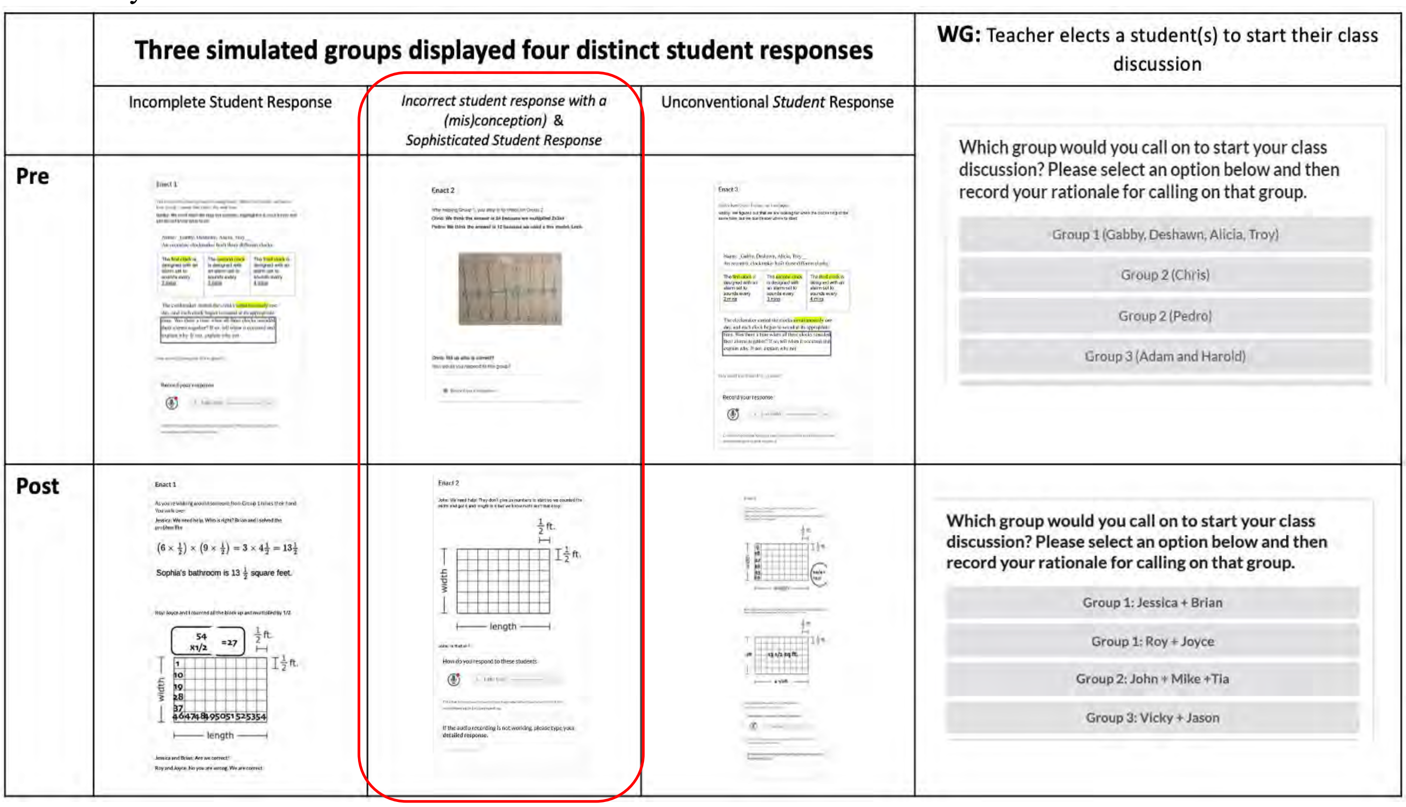

Figure 1: Teacher Moments Simulation Design

We address two research questions to understand what teachers learned and how their simulated behavior changed during this process: 1) In the post-test simulation, did teachers include diverse perspectives while initiating small group discussions, and did they recognize the potential value of contributors without correct, efficient answers? 2) In the post-test simulation, did teachers similarly include diverse perspectives and recognize their potential value while initiating whole group discussions?

Methods

As part of the research design teachers engaged in a pre-test simulation administered in February 2020, in which participants' responses and selection choices were recorded. Participants then engaged in monthly professional development workshops (two four-hour in person sessions, four 90-minute virtual sessions), where participants explored excerpts and readings geared towards the affirmation, validation and incorporation of multiple students in math discussion. Teachers engaged in simulation play that required them to rehearse and reflect on their instructional moves as well as debrief various sample responses in the Teacher Moments platform. In June 2020 participants then completed a post-test simulation, which had a parallel structure to the pre-test simulation.

Data analysis

Data analysis began with a systematic review of all participants' responses from the pre-test simulation's small and whole group discussion (Miles & Huberman, 1994). Researchers individually listened to participants' responses paying close attention to whether the teacher provides space for students with misconceptions to share their approach and/or explain their thinking. Researchers also listened to how students with misconceptions were treated and positioned; were they viewed from a deficit lens? Researchers then used the constant comparison method (Strauss & Corbin, 1998) to systematically examine all participants' responses in small and whole group discussion respectively. Discrepancies in coding were settled by another trained member of the research team. Data from the post-test simulation was handled in a similar manner. Further data from participants' pre-test simulation and post-test simulation were analyzed separately at first and then together to examine any changes in teacher responses.

Results

Small Group Discussion

Due to interruptions from COVID and data collection issues with the technology, fourteen teachers completed the pre-test simulation, and nine completed the post-test simulation. Overall, in the small group task of the simulation, participants chose more inclusive starting points for conversation following the intervention. Participants were more attuned to including all students. In Table 1, we show Teacher 14's small group response in the pre-test and post-test simulations as representative of a teacher whose views on including a range of student voices in discussion changed following the intervention.

Table 1: Teacher 14 SMG Simulation Response

Participant	Pre-Test Simulation	Post-Test Pre-Test Simulation
Teacher 14	"Okay, Chris, so you multiplied 2x3x4, and Pedro, to summarize, you made this drawing and found that it was 12.	Okay, so I see Jessica and Brian have one answer, Roy and Joyce have another, different answer. First, I'd like to hear

	Pedro, do you want to explain to Chris how you thought to do this, why you decided to do it this way?”	Jessica and Brian walk through their process so I could understand. I think I see what you did, but I'd like to hear from you, how you did it, and Roy and Joyce, I'd like you to then walk me through your process to see what you did, and then we'll go from there.

During the pre-test simulation, Teacher 14 elects to only invite the correct student (Pedro) into the conversation to voice their opinion, ignoring other ideas and possible contributions. Additionally, Teacher 14’s response illustrates how positions are verbally constructed in classroom interaction. Her response suggests one student (Pedro) can provide a meaningful mathematics experience for the other (Chris) but disregards what Chris’s conception adds to the conversation. The roles that were assigned within this interaction left little to no room for Chris to be a contributor in this interaction. While probably not intentional, this interaction could leave others not involved to feel excluded and discouraged from the learning process. Though this interaction may seem subtle and insignificant to teachers, students may have a different perspective.

During the parallel interaction in the post-test simulation, Teacher 14 now addresses the small group interaction by providing the opportunity for both students to further explain their thinking. In this remark, we first notice that Teacher 14 is oriented to understanding students’ processes and creates space for both students to share their ideas. Secondly, she does not give directive authority to any student. She uses questions to continue to engage all students as sense-makers and provide space for both students to continue exploring the problem rather than evaluating and designating one student(s) to lead the mathematical experience.

Whole Group Discussion

In contrast to the small group discussion simulation task where we saw participants adopt new strategies to facilitate more inclusive discussions, we saw very little change from pre- to -post in the whole group discussion task. Even when participants did invite non-correct perspectives into the discussion, simulated students were not positioned as sense makers or contributors. For instance, Teacher 11 elected to start the WG discussion with the *incorrect student response with a misconception*. In describing her approach to introducing this incorrect response, she states,

> I will show a drawing on the board of the length of 1/2 foot and 1 foot) Are these two measures the same? Is 1/2 foot the same as 1 foot? So, the length of 9 of 1 foot each, the same as 9 of 1/2 foot each? And the width of 6 of 1 foot each, the same as 6 of 1/2 foot each? So then, do you think this group's answer makes sense?

This response is focused on remediating their initial conception rather than encouraging students to think more deeply or identify strains of thinking that might be useful in other contexts. She publicly positioned the student as not knowing and took the mathematical authority way, funneling the student to a particular understanding. Students are guided on a directive path of information gathering to the answer, steering them further away from their own idea. This brief interaction is focused on accuracy of procedures and answers and is not situated to be a discussion but rather a list of commands ending with what could be considered a rhetorical question.

Discussion

Digital clinical simulations hold great promise for helping teacher educators both support and measure changes in teacher's practice. We found evidence that after our professional development workshops, participants espoused more inclusive discussion practices. Qualitative evidence showed that participants changed their simulated behavior when facilitating small group discussions: more teachers invited students to share their ideas regardless of correctness and/or efficiency and teachers included these perspectives for evaluating and consideration, not just for remediation. We did not, however, find any evidence that teachers substantially changed their practices in facilitating whole group discussion. In the final task of the simulation, our participants continued to choose to primarily feature students with correct, efficient answers. Why might that be? One possibility is that teachers agreed with our instruction that emphasized inclusiveness in small group discussions but did not believe that these principles should be applied to whole group discussion. Another possibility is that teachers continue to feel the pressure of the larger system that evaluates their effectiveness by student test scores, supporting a desire to simply tell students how to solve problems and have discussions about answers instead of ideas. A third possibility is that teachers did not know how to facilitate whole group discussion that start with diverse students' conceptions, though we saw evidence that participants knew how to do so with small groups. Though teachers may have agreed with our inclusive principles and knew how to employ strategies to initiate a discussion with diverse perspectives, they may have had concerns about sustaining such a discussion. While research recommends teachers frame their discussions around student-created strategies and suggest engaging in open discussions as a way for deeper mathematical understandings (Hiebert et al. 1997), this is no easy task as it requires teachers to have content knowledge and/or pedagogical content knowledge expertise. In a real classroom setting, this might be expected—teachers might legitimately fear that they might not be able to navigate a whole group discussion from an unfamiliar starting point to a collective class understanding that advances mathematical goals. In a simulated setting, however, we anticipated that the very low-stakes of the situation would allow participants to take pedagogical risks, and try new strategies with the confidence that no real student learning will be harmed.

More research is needed to understand whether any of these possibilities are manifested in our particular participants or in future participants, or if other alternative explanations better explain our mixed results.

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COMPARING MOTIVATIONS FOR FLIPPED INSTRUCTION TO DATA ON FLIPPED IMPLEMENTATIONS IN ALGEBRA

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Flipped instruction, where videos are assigned for students to watch as homework in lieu of problem sets, has been implemented by an increasing number of mathematics teachers. Their reasons for flipping range from a desire to have less time spent in class on lectures and more time on collaboration (and greater engagement) to a hope that videos increase completion rates for homework compared to problem sets. This study examines the extent to which those reasons for flipping instruction are reflected in observation data from 22 flipped and 25 non-flipped Algebra 1 classes. Flipped classes were found to have less lecture time in class but the increased student work time was not necessarily collaborative. During student work time engagement levels were higher in flipped classes than non-flipped classes but the reverse was true during whole-class discourse. Homework completion rates were not significantly different.

Keywords: Flipped instruction, Instructional activities and practices, Algebra, Technology

Motivations for Flipped Instruction

The reliance on video lectures during the COVID-19 pandemic has highlighted a pre-existing trend which is the use of videos as homework in flipped mathematics lessons (Zainuddin et al., 2019). Although flipped instruction is defined by the presence of video or multimedia homework, reducing the need for content delivery during in-class time, it is not a unified instructional model. Many varieties of flipped implementation have been documented (e.g., de Araujo et al., 2017a; Muir, 2019). There are also a variety of reasons that teachers or school leaders have for choosing to flip instruction. One primary reason may be that they have seen reports of flipped instruction having positive outcomes with regard to student achievement (e.g., Amstelveen, 2019; Bhagat, Chang, & Chang, 2016; Overmyer, 2015). But other studies have found mixed results (e.g., DeSantis et al., 2015; Lo & Hew, 2017), which perhaps is to be expected given the variety of implementations just mentioned.

Beyond achievement, there are other reasons educators have for flipping instruction that potentially address phenomena that undergird not only achievement but also other important outcomes like identity, agency, and attitude. These reasons below, documented by de Araujo and colleagues (2017b) and also our own surveys and interviews with teachers implementing flipped instruction, are not exhaustive but do constitute the focus of the present brief report:

- Flipping will **increase the amount of in-class time available for students to work on problems or exercises**, which can lead to more support from the teacher, more collaboration with peers, or possibly more opportunities to engage in cognitively-demanding tasks;

- Flipping will **increase student engagement** throughout the in-class portion of the lesson by reducing the time the teacher spends lecturing in class;
- Flipping will **increase homework completion** because it is easier to watch a finite-length video than it is to solve a set of problems (where there is a risk of getting stuck or struggling for an inordinate amount of time); and
- Flipping will **provide an accessible version of the lesson material** via the video recording rather than a live lecture and students can refer to the video as needed.

Our guiding question was: To what extent do these reasons for flipping instruction materialize in implementations of flipped instruction? We explored this question using observation data from a large-scale study of 47 Algebra 1 classes from a diverse set of school districts.

Conceptual Framework

The framework (Figure 1) of this study draws upon curriculum enactment theories (e.g., Remillard & Heck, 2014; Stein, Grover, & Henningsen, 1996), observation instruments (e.g., Walkowiak et al., 2014), and advice from experts in different fields (e.g., educational technology, mathematics education). The framework's scope is a lesson and it distinguishes between structural features of the at-home and in-class portions while also attending broadly to the activity formats (Otten et al., 2018). For the purposes of this brief report, we focus on the time spent in various in-class formats (whole-class, group work, independent work), behavioral engagement within those formats, homework types and homework completion rates, and, for flipped classes specifically, the involvement of videos during the whole-class discourse and the rate of video use during group or independent work time in class.

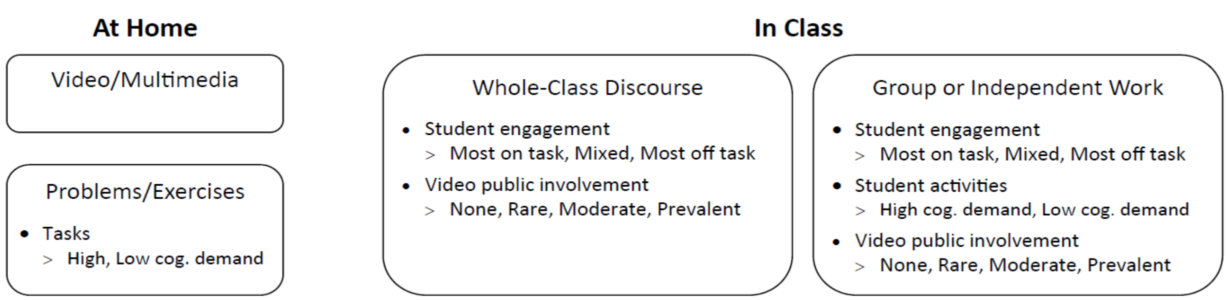

Figure 1: Simplified Flipped Mathematics Lesson Framework (Otten et al., 2018)

Method

Setting and Participants

Data were collected from 541 Algebra 1 students and their teachers. There were 22 flipped (338 students) and 25 non-flipped classes (203 students). The teachers who flipped chose to do so on their own and the project was observational in nature (no intervention). Most students were in the 9th grade but many were in 8th grade and a few were in 7th or 10th grade. The classes were located in 20 different districts ranging from rural to urban. The 34 participating teachers (some taught multiple classes in the data set) were varied in their background and experience.

Data Collection and Analysis

The primary data source for this study was field notes from lesson observations. Each class was observed three times throughout the school year with two observers for at least the first two observations. If reliability was high and a clear lesson pattern had been established, then in some cases a single coder conducted the third lesson observation. Field notes were coded using a protocol based on the Flipped Mathematics Lesson Framework (Otten et al., 2018). In particular, discourse formats were recorded to the nearest minute, behavioral engagement was coded on a 3-point scale from "most students off task most of the time" to "most students on task most of the time", and video involvement (in the flipped classes) was coded on a 3-point scale from rare to prevalent use of homework video in class and also a 4th code option of no video use. The three lesson observations for each class were discussed and condensed into a lesson profile.

A teacher survey and a student survey were also administered to gather information about their perspectives on the algebra instruction, attitudes toward the class, and, importantly for the present study, students' self report of homework completion and their perceptions of peers' homework completion (to attempt to account for desirability bias in the self-report data). A pre- and post-test containing measures of procedural understanding and conceptual understanding was also administered but not reported in this article. To answer our research questions, we conducted a series of Analysis of Variance (ANOVAs) to examine whether there were statistically-significant differences between flipped and non-flipped classes.

Findings and Brief Discussion

More Class Time was Used for Student Work in Flipped Lessons than Non-Flipped Lessons

Aligning with some teachers' motivation for flipping, there were statistically-significant differences in the use of class time. Flipped lessons had significantly more time allotted to student work compared to non-flipped lessons, but this difference came in the form of more independent work time, not more group work time. Non-flipped classes spent 3.4% of class time in group work (SD=12%) and flipped classes spent 4.6% of class time in group work (SD=12%); this difference was not statistically significant ($p=0.731$). With regard to independent work time, non-flipped classes spent 31% of time in that format (SD=17%) and flipped classes spent 64% of time (SD=31%) in independent work, which is statistically-significantly greater ($p<0.001$).

A purported advantage of flipped instruction is that the extra work time can be used to enact cognitively-demanding tasks. Not enough lessons involved group work for us to compute meaning statistics on that activity format, but for the independent work time, non-flipped classes had 5.0% of tasks that were cognitively demanding tasks (SD=6.9%) and this was not significantly different ($p=0.661$) from the 4.2% of cognitively-demanding tasks in flipped classes during independent work time (SD=5.5%). So although flipped lessons incorporated more independent work, the cognitive demand of the tasks in our data set was not different.

Was the independent work time nevertheless worthwhile? Our pre/post-test analysis does suggest a positive correlation between independent work time and the learning measures. But for the present report, we instead look to the engagement levels, which was another motivation for flipped instruction and which relates to independent work time because teachers have expressed that having students solve problems is more active than listening to a lecture during class.

Student Engagement was Not Different Between Flipped and Non-Flipped Lessons

Looking first at engagement levels during whole-class discourse, it averaged 3.0/3.0 in the flipped lessons (SD=0) and 2.84/3.0 in the non-flipped lessons (SD=0.37), which was not a statistically-significant difference ($p=0.068$). Recall that whole-class discourse was briefer in flipped lessons which may explain why it maintained high levels of on-task behavior. In the non-whole-class discourse (group work and independent work), engagement in flipped lessons averaged 2.55/3.0 (SD=0.50) and in non-flipped lessons averaged 2.64/3.0 (SD=0.49), which was not a statistically-significant difference (p=0.291). Conversely to the whole-class discourse, the non-whole-class discourse was briefer in non-flipped classes than in flipped classes. Overall, although lessons had slightly higher engagement levels in the activity formats that were relatively brief (whole-class discourse in flipped lessons and work time in non-flipped lessons), these differences were neither meaningful nor significant. What may be more meaningful is not the behavioral engagement as coded here but rather the nature of the activity format itself, that is, students engaged in solving problems or completing exercises versus listening or discussing.

Homework Completion was Reported Similarly for Flipped and Non-Flipped Lessons

Homework completion rates were calculated for non-flipped classes as the percentage of students who reported usually or always finishing the homework. The mean homework completion rate was 0.90 (SD=0.14). Video watching completion rates were calculated for flipped classes as the percentage of students who reported usually or always watching the assigned videos. The mean video watching rate was 0.83 (SD=0.22). This difference in completion rates was not statistically significant (p=0.182). We also asked students whether their peers completed the homework. The mean peer homework completion rate was 0.44 (SD=0.28) in non-flipped classes. The mean peer video watching rate was 0.35 (SD=0.25) in flipped classes. Again, this difference was not statistically significant (p=0.252). As with engagement, the benefit (if any) of flipped instruction may not be the rate of homework completion but rather the type of activity done outside of class: completing exercises or receiving content delivery.

Videos were Accessible to Students but Not Often Accessed During the Flipped Lessons

A motivation for flipping was the accessibility of videos and for this feature we looked only at the flipped classes. Of the 22 flipped classes, 12 consistently had both a video associated with the lesson and a whole-class discourse segment within the lesson. For these 12 classes, the video involvement level on the 0–3 scale was 0.92 (SD=0.95) which means that, on average, the homework video was briefly referenced in class but not shown or discussed substantially. With regard to non-whole-class discourse, during which students might pull up the video on a computer or phone while they are working, 19 flipped classes had videos and non-whole-class discourse and the video involvement level was 1.63 (SD=1.26), which means that, on average, a few students might access the video but not usually a majority of students. The standard deviation indicates that in several of the flipped classes, no students accessed the videos in class.

In survey responses, teachers reported appreciation for videos being accessible to absent students and for content review, but we do not have data of students' delayed video access behaviors. Future research may investigate how students rewatch lecture videos compared to students in non-flipped classes who review lecture notes or ask the teacher to reteach material.

Conclusion

Many existing studies of flipped instruction use student achievement as the measuring stick (e.g., Ichinose & Clinkenbeard, 2017), but in this study we compared implementations of flipped instruction to some of the primary motivations that teachers have for flipping instruction in the first place. Is this instructional model providing the learning opportunities and taking advantage of the affordances as intended? Note that we were not comparing a teacher's flipped instruction to their own instruction prior to flipping, but rather to other non-flipped teachers in the same or similar school contexts. The broad reach and the variation in our data set, however, still make the comparison meaningful.

We found that flipping did provide substantially more time in class for student work but that this time was not necessarily used for more student collaboration or more cognitively-demanding tasks. We found that student reports of homework completion were not higher than those in non-flipped classes but the videos do provide options with regard to accessibility and rewatching prior lectures. These results imply that additional support (new curriculum materials, strategies for collaborative practices, guidance for video creation and access) may be required for teachers to take full advantage of the affordances of flipped instruction to meet their intended goals.

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ANALYSIS OF TEACHER ACTIONS TO PROMOTE GENERALIZING

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This study analyzes the instruction of one teacher in two classroom episodes. We analyzed the teacher's instruction using a framework for whole-class discourse and a framework for identifying activity that supports generalizing. Across both episodes when priming and particularly generalizing-promoting activity increased, students' generalizing activity also increased. An increase in the number and quality of questions and student contributions occurred with more student generalizations. Similarly, the responsibility for questioning and thinking shifted from teacher to student as more students responded to requests for justification. Future research should identify productive small group practices to support generalization.

Keywords: Algebra and Algebraic Thinking, Preservice Teacher Education, Discourse

Algebraic thinking appears as early as Kindergarten in the Common Core State Standards (National Governors Association Center for Best Practices and Council of Chief State School Officers, 2010) and is often linked to the development of critical thinking and problem-solving skills (Schoenfeld, 1995). We focus on the algebraic practice of generalizing, the situated activity of "lifting" and communicating reasoning to a level where the focus is no longer on a particular instance, but rather on patterns and relationships of those particular instances (Kaput, 1999, p. 137). Algebraic thinking involves reasoning with generalizations--constructing generalizations and justifying and applying them (Blanton et al., 2011; Cooper & Warren, 2011; Kieran, 2007; Kieran et al., 2016; Mason, 1996). We consider how instruction can support students in engaging in the mental activity that leads to generalizing about functional relationships represented in a pattern task. Prior research has mapped out how students develop functional thinking (e.g., Stephens et al., 2017), but we do not have a clear understanding of general ways to support students in engaging in generalizing about functions. By examining the actions of one preservice teacher as she worked to support students' generalizations of a visual pattern, we have identified features of teacher actions and classroom activity that support students in generalizing.

Conceptual Framework

Central to fostering algebra learning is understanding how to support students in generalizing, thus we conducted our analysis using a framework (viz., Strachota, 2020) built on a teaching experiment (Ellis, 2011) that identified seven types of generalizing-promoting actions. The framework (Strachota, 2020) used here was designed to identify students' generalizing and the activity that supported them in developing those generalizations, which is described as *generalizing-promoting activity*, the moves and interactions that promote generalizing, and *priming activity*, the moves and interactions that typically prepare students to engage in a later generalizing-promoting activity (see Table 1 for a description of priming activities and Strachota (2020) for descriptions of generalizing promoting). Understanding the context of generalizing is needed to support students in developing and refining generalizations. Ellis (2011) argues that generalizing is tied to a specific socio-mathematical context through which people construct generalizations. Generalizing is demonstrated through an individual's activity and discourse (Ellis, 2011; Kaput, 1999), so we use the Math-Talk framework (Hufferd-

Ackles et al., 2004) to capture how students and teachers use discourse to support mathematics learning. We focused on questioning and mathematical explanations to capture the back and forth of teacher-student interactions to see how teachers' questions drew students' attention and explanations towards aspects of the mathematics that may have contributed to their generalizing.

Table 1: Priming Activities

Priming Activities	
Naming a phenomenon, clarifying critical terms, reviewing critical tools	Offering a common way to reference a phenomenon or emphasizing the meaning of a critical term or tool.
Constructing or encouraging constructing searchable and relatable situations	Creating or identifying situations or objects that can be used for searching or relating. Situations that can be used for searching or relating involve particular instances or objects that students can identify as similar in some way.
Constructing extendable situations	Identifying situations or objects that can be used for extending. Extending involves applying a phenomenon to a larger range of cases than that from which it originated.

Methods

Ms. Patton was enrolled in a one-year master's program for individuals with STEM degrees and was in the first semester of a year-long placement in an Algebra II classroom. She planned, taught, and reflected on her video data from two episodes of teaching a pattern task with grades 9-10 students as an assignment for her mathematics teaching methods course. Data included planning documents, video recordings, and reflections of Ms. Patton's teaching episodes of the pattern task. With her mentor and supervisor, Ms. Patton co-reflected on her teaching, and then retaught the same task. She then analyzed the video of her teaching using several frameworks, described in the data analysis section, to determine the effectiveness her promoting students' generalizations. In our analysis, we coded the video data in 15-second segments by noting: the types of interaction; who was directing the activity; questioning and mathematical explaining (Hallman-Thrasher, 2017; Huffered-Ackles et al., 2004); and instances of priming activity (PA), generalizing-promoting activity (GPA), or generalizing activity (GA). Each instance of generalizing was coded when it initially occurred only even though a single instance may have spanned multiple segments. Interrater reliability was established by all members of the research team reviewing and coding all video data and transcripts. Disagreements were discussed until consensus was reached (Syed & Nelson, 2015). The quantity, density, and distribution of codes across a lesson helped us determine the relationship between PA and GPA to GA, and connections among generalizing and math-talk.

Results

We compare two episodes of Ms. Patton teaching the same lesson. In the first generalizations were more sparse than in the second. We focus on what different activities occurred during these episodes and how those activities may have been linked to or contributed to students' generalizations. Hereafter, we refer to the generalization sparse class as Episode 1 and the generalization dense class as Episode 2. We highlight Episode 2 because Ms. Patton presents a positive example, especially for a novice, that is worthy of investigation. Overall, Ms. Patton used more instances of priming activity (PA) and generalizing-promoting activity (GPA) and, as noted, had more generalizing activity in Episode 2 (Table 2). While this finding validates prior work (e.g., Strachota, 2020), we also identified patterns in the ways the practices were used. Moreover, while the difference between six and ten generalizations may seem insignificant, we highlight that these kinds of sophisticated statements are fairly rare and result from intentional instruction and students engaging in highly complex cognitive mental activity. Across both episodes, we observed that when priming and particularly generalizing-promoting activity increased students' generalizing activity also increased. In Episode 2, there were repeated instances of GPA-PA, PA-GPA, or PA-GPA-GPA, before a generalization occurred. Further once student generalizations were made, they were often closely followed by additional generalizations. Repeated instances of GPA and PA were nearly always necessary to produce GA. For example, Ms. Patton asked students to share expressions to represent the fourth image in Figure 1. Ms. Patton elicited several examples of expressions and represented them on the board, which was one way she constructed relatable situations (PA). Once students provided an example she followed up with a GPA, such as encouraging justification or encouraging relating. In one instance when she encouraged relating and in turn justification, she asked the student to explain how their expression was represented in the fourth image. In response, one student explained "(you) have two 4 by 4 perfect squares." When Ms. Patton helped the student clarify the student concluded, "(they) have two 4 by 4 squares, plus 2." In response Ms. Patton encouraged reflection (GPA) by asking, "How would you represent that algebraically?" The student who described seeing two 4 by 4 squares said, "It would be $2 \times 4 \times 6 + 2$."

Table 2: Summary of Priming, Generalizing-Promoting, and Generalizing Activities

	Teaching Episode 1*			**Teaching Episode 2****		
	Instances	% of Episode	% of Discussion	Instances	% of Episode	% of Discussion
PA	9	5%	9%	11	7%	9%
GPA	17	10%	17%	27	18%	21%
GA	6 St, 2 Tr	5%	8%	10 St, 2 Tr	8%	9%

*Note episode 1 was 43 minutes, 26 of which were discussion. **Episode 2 was 38 minutes, 32 of which were discussion.

We share this excerpt to show how Ms. Patton set students up for situations that could be built on in a way that supported generalizing. It succinctly illustrates the generative nature of priming, generalizing-promoting, and generalizing activity.

Ms. P: What about another one? **(PA)**
Jen: 2s and 4s.
Ms. P: Where's that at? **(GPA)**
Jen: Huh? I mean split in to 32 with 2s and 4s.
Ms. P: How would you do that? So you're talking about like this 32 [points to 8 by 4 rectangle]? **(GPA)**
Jetta: Yes.
Ms. P: What do you mean by 2s and 4s? Like how would you split the picture? **(GPA)**
Jen: Columns of twos.
Addie: $2 \times 4^2 + 2$

Ms. P: How would you show that in the picture? **(GPA)**
Addie: Just have two 4 by 4 perfect squares.
Ms. P: So you have two 4 by 4s, plus 2. Lets go back to what you were just saying Jen.
Jen: Yeah. Would it be rows of two and columns of four?
Ms. P: So how would you represent that algebraically? **(GPA)**
Addie: It would be $2 \times 4 \times 6 + 2$.

The quality of students' contributions, including when students explain their ideas to each other, and the quality of teachers' questions seemed to play a critical role. We noted an increase in instances and quality of teacher questioning and eliciting students' ideas (Table 3) that we identified using the modified Math Talk rubric (Author, 2017). Within the rubric, lower levels are associated with teacher-generated and answered questions (levels 0 and 0.5), whereas the higher levels are associated with richer justifications that are prompted by the teacher (levels 1 and 1.5). As the levels increase, the responsibility of generating questions shifts towards the students, in turn student-to-student dialogue increases (level 2 and above).

In Episode 2, when students' generalizing increased, there was a shift of responsibility for questioning and thinking from teacher to student. The number of questions and student contributions of thinking increased as did the quality of those interactions (Table 3). The number of questions Ms. Patton posed doubled in the second episode and Ms. Patton used more follow-up questions that pressed students to justify their ideas (level 1.5). Similarly, there were more than double the instances of eliciting student contributions in the second episode and those instances involved a greater number of students sharing their thinking (level 1) and responding to follow-up requests for justification (level 1.5 and 2). The increase in frequency and quality of questions and explanations provided opportunities for generalizing in the second episode. The increase in the quality of student contributions and teacher moves to facilitate those contributions made student thinking and ideas more visible to the class so that ideas with the potential to be generalized were public to the work of the class.

Table 3: Instances of Questioning and Explaining for each Level of the Math Talk Rubric

	Teaching Episode 1		**Teaching Episode 2**	
	Questioning	Explaining	Questioning	Explaining
Level 0	6	8	9	11
Level 0.5	6	0	9	6
Level 1	4	1	7	30
Level 1.5	18	6	43	15
Level 2	0	4	4	13
Level 2.5	1	0	0	0

Conclusion

Within the complex task of teaching, this study provides more evidence that intentionality leads to better outcomes for students, specifically in supporting students in developing mathematical generalizations. We argue that teachers should be purposeful in the questions they ask and the structure of interactions. Our study highlights that small pedagogical moves can have a big impact, and we illustrate some of those moves in practice. Future research should aim to better understand supporting teachers in implementing those practices.

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DIMENSIONS OF SUCCESSFUL ELEMENTARY MATHEMATICS TEACHERS' EFFECTIVENESS DURING PROFESSIONAL DEVELOPMENT

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This 5-year mathematics professional development project involves 27 elementary teachers being prepared and supported as Elementary Mathematics Specialists (EMSs) through completion of a university's K-5 Mathematics and Teacher Supporting & Coaching Endorsement programs, as well as participation in Professional Learning Communities and individual mentoring. Across the project, data are gathered to examine changes in mathematical content knowledge, instructional and coaching practices, beliefs, and teacher leader skills of the EMSs. Described here are Year 1 data from the participants, who have been identified as successful, experienced teachers, focusing on specific aspects of teacher effectiveness. The findings illuminate their classroom instructional practices, including those that are learner-centered and equitable, along with their early histories as learners of mathematics.

Keywords: Instructional Activities and Practices; Teacher Knowledge; Teacher Beliefs; Equity, Inclusion, and Diversity

Purpose of the Study

This study's context is a 5-year mathematics professional development project involving 27 elementary teachers who are prepared and supported as Elementary Mathematics Specialists (EMSs). They complete a university's K-5 Mathematics and Teacher Supporting & Coaching Endorsement programs and participate in Professional Learning Communities and individual mentoring. Across the 5 years, data are collected to examine how the intentional and continuous project components affect the mathematical content knowledge, instructional and coaching practices, beliefs, and teacher leader skills of the EMSs. Described here are specific data collected in Year 1, with these questions guiding the inquiry:

- To what extent do experienced, successful elementary teachers implement instructional practices that foster standards-based learning environments in mathematics?
- What are the relationships between these instructional practices and their mathematics content knowledge and beliefs?
- What are their early histories with mathematics as learners?
- What are their views on equitable mathematics instruction and their own enactment of equitable teaching practices in mathematics?

Perspectives

Teachers should implement effective and equitable instructional practices in mathematics (NCTM, 2014) that support *standards-based learning environments* (SBLEs). They should use instructional tasks with high levels of cognitive demand that support students' reasoning and problem solving, and facilitate productive discussions that elicit student ideas, attend and respond to student thinking as it unfolds during a lesson, and use that thinking to guide instructional decisions. These practices place children's thinking and learning at the center of classroom activity and instructional decision-making, leaning heavily on developed teacher identity and agency, and provide fruitful opportunities for students to develop positive identities as mathematics doers and learners (Aguirre, Mayfield-Ingram, & Martin, 2013). Teachers must navigate many constraints when it comes to implementation of these learner-centered practices, including those that are contextual and their own divergent past experiences as a learner of mathematics (Bartell, Cho, Drake, Petchauer, & Richmond, 2019).

Teachers' content knowledge and beliefs are also related to their support for children-centered learning environments. Teachers require deep and broad knowledge of mathematics to be effective in their teaching (Hill, 2010), including *specialized content knowledge* characterized as "mathematical knowledge needed to perform the recurrent tasks of teaching mathematics to students" (Ball, Hoover Thames, & Phelps, 2008, p. 399). This depth of understanding equips teachers to navigate children's mathematical thinking during instruction, including misconceptions, and the continuous decision-making processes required for responsiveness to this thinking. Also, teacher beliefs shape classroom instruction. Two important teacher beliefs constructs include pedagogical beliefs (i.e., beliefs about teaching and learning) and teaching efficacy beliefs (i.e., beliefs about capabilities to teach effectively and influence student learning).

Methodology

The design of this study includes a descriptive, holistic singular-case approach (Yin, 2014). The case is experienced elementary teachers who have been identified as effective teachers of mathematics and teacher leaders; all were employed in one urban-situated school district and teaching in high-need schools with diverse student populations. Multiple sources of data, both quantitative and qualitative in nature, were collected to form the descriptive findings. Participants were 27 elementary teachers in a large, urban school district in the southeastern USA. Their schools (n=22) served 91% students of color and 69% students eligible for the free/reduced lunch program. The participants identified as 24 females and 3 males and 70% teachers of color. They are a highly educated group, with 100% having a master's degree and 33% holding an educational specialist degree; further, they are experienced teachers, on average having 10.5 years of teaching experience. Teaching positions vary widely and include: three kindergarten, one first grade, two second grade, five third grade, one fourth grade, seven fifth grade, four STEM/Math Specials, one English to Speakers of Other Languages, one Special Education, one Early Intervention Program, one Accelerated Content, and two Dual Language Immersion.

The teachers had recently been selected to participate in a federally-funded, 5-year professional development project focused on developing EMSs. EMSs are generally considered to be teachers, teacher leaders, or coaches with the expertise to support effective elementary mathematics instruction and student learning (Association of Mathematics Teacher Educators, 2013). The project's recruitment efforts had concentrated on the highest need elementary schools in the district, as determined by free/reduced lunch program rates. The teachers were chosen based on criteria that identified them as successful, experienced teachers of mathematics and teacher leaders. Their application materials and interviews were reviewed for meritorious professional achievement, academic accomplishment, knowledge of mathematics, commitment to teaching mathematics, and evidence of/desire for teacher leadership. These criteria, plus consideration of race/ethnicity, gender, grade level, and school site with the aim of assuring participation of underrepresented groups and diverse school sites and grade levels, informed the selection of the 27 teachers in the project.

Quantitative data were collected from all participants via a classroom teaching practices observation protocol (i.e., Standards-Based Learning Environment Observation Protocol [SBLEOP], Tarr et al., 2008), *specialized content knowledge* assessment (i.e., Learning Mathematics for Teaching [LMT], Hill, Schilling, & Ball, 2004), background and practices survey, and two belief surveys (i.e., Mathematics Beliefs Instrument [MBI], Peterson, Fennema, Carpenter, & Loef, 1989, as modified by the CGI Project; Mathematics Teaching Efficacy Beliefs Instrument [MTEBI], Enochs, Smith, & Huinker, 2000). Qualitative data were gathered through individual interviews of all 27 participants, as well as three focus group interviews with nine participants in each group. The interview protocol includes questions related to their histories with mathematics and their mathematics instructional practices, particularly equitable mathematics instruction. Data were collected using virtual means at the start of the professional development project. This collection occurred during the COVID-19 health pandemic, and all teachers were providing instruction via different hybrid models with a mix of face-to-face and virtual delivery. Both descriptive and inferential statistics were used for analysis of the quantitative data. Relationships between scores from the different instruments were analyzed using Pearson Correlation. Analysis of the qualitative data involved constant comparative methods (Lincoln & Guba, 1985).

Results

Table 1 displays data from the SBLEOP used to assess the extent to which participants enacted learner-centered SBLEs during their classroom observations. The SBLEOP evaluates specific classroom events on a scale of 1-3, with higher scores indicating more alignment with a SBLE. For example, across the SBLEOP rubric criteria a score of 2 indicates partial evidence of a classroom event (e.g., “students had *some* opportunity”, “the teacher *sometimes* encouraged students to orally explain how they arrived at an answer”, and “different perspectives or strategies were *occasionally* elicited from students”). Shown are the mean scores on eight classroom events, or dimensions of facilitating a SBLE, and the overall mean score across classroom events. Lesson structures were somewhat consistent across all observations per school district guidelines, with teachers beginning with an activation activity, followed by a whole group mini-lesson and gradual release model, and ending with small group instruction based on ability grouping.

With an overall mean score of 1.5, the participants’ implementation of SBLEs was less than partially evident. Teachers were rated the highest on the *Mathematical Connections* indicator, suggesting that they were observed making some connections among mathematical topics during the lesson, though those connections were not typically discussed in detail. *Conceptual Understanding* (i.e., how the lesson fostered the development of conceptual understanding) was the next highest rated indicator. Both of these mean scores, though comparatively higher than other events, still fall below a 2. Teachers scored the lowest on the indicators *Making Conjectures* (i.e., observed opportunities for students to make conjectures about mathematical ideas) and *Reflecting on Reasonableness*, suggesting that teachers were rarely asking students whether their answers were reasonable and when students gave incorrect responses, another student was asked to provide a correct answer.

Table 1. Means on the SBLEOP

Classroom Event	**Mean Score (1-3)**
Making Conjectures	1.3
Fostering Conceptual Understanding	1.7
Making Mathematical Connections	1.9
Connections with Daily Lives	1.6
Students Explaining Strategies	1.6
Valuing Multiple Perspectives	1.4
Using Student Statements	1.4
Students Reflecting on Reasonableness	1.2
All Classroom Events	1.5

The analysis of the quantitative data reveals several other key findings. Notably, the extent of teachers' implementation of SBLEs was related to the depth of their content knowledge and strength of their pedagogical beliefs, as the correlational analysis shows a significant positive relationship between scores on the SBLEOP and both the LMT and MBI. When it comes to content knowledge (LMT), the participants' understandings of number and operations were the strongest compared to the two other content areas measured (i.e., algebra and geometry). All three of the subscales evidence considerable variability in scores. Further, when considering beliefs about the teaching and learning of mathematics (MBI), they were largely uncertain about cognitively-oriented pedagogy. And, while they were confident in their capabilities to teach mathematics effectively (MTEBI), they were less confident that this effective teaching would influence student learning in positive ways.

The analysis of the individual and focus group interview data provides insights into the participants' histories with mathematics and how their early experiences as a learner shape their instructional practices. They also described views on equitable mathematics instruction and specific practices they use with their students to support access and equity. Participants expressed a variety of firsthand experiences involving marginalization as mathematics learners and doers, sometimes as early as kindergarten, and how those early occurrences shaped their mathematical teacher identity and trajectory. Participants recalled experiencing inequities, though they recognized not having that language or awareness at the time, and how finding that language and awareness in adulthood as teachers has impacted their practices and relationships with their students. Participants are committed to providing equitable instruction, and the interview data show a range of enactment of those equitable practices with a consistent focus on learning new and better ways to teach mathematics equitably.

Discussion

The quantitative findings give us a distinct picture of these participants, who have been identified as successful, experienced teachers, at the very beginning of a lengthy, rigorous professional development project. The details and nuances of this picture are provided by the interview data, telling a story of themselves historically as mathematics doers and learners, and how those impact their practices, especially in addressing issues of equity and agency. The project's continual data collection and analyses across 5 years provide a unique and exciting opportunity to follow the trajectory of the participants as teacher leaders in high-need schools serving student populations rich in diversity.

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PATTERNS OF REASONING: WARRANTS IN ELEMENTARY MATHEMATICS AND CODING ARGUMENTS

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Argumentation is widely used in teaching mathematics, but little research has been done on argumentation in teaching integrated mathematics and coding. As part of a larger study investigating collective argumentation in teaching mathematics, science, and coding, we classified the warrants given by elementary age students who were engaged in argumentation in mathematics and coding. Three major categories – calculation, visual, and unformalized knowledge – accounted for the majority of warrants given. Further analysis revealed differences in types of warrants when the primary focus of the argument was coding versus when the primary focus of the argument was mathematics. Our results suggest that expecting students to provide reasons for modifying their code, similar to what is expected in mathematics arguments, helps move them away from a trial-and-error to a more structured approach to coding.

Keywords: Classroom Discourse, Elementary School Education, Integrated STEM/STEAM

Background

Reasoning in Mathematics and Coding

Reasoning is important in the teaching and learning of mathematics. Research suggests students should develop an understanding of mathematics beyond a collection of facts and procedures (Cuoco et al., 1996; Goldenberg, 1996; Kilpatrick et al., 2001). Building on this research, national policy documents in the United States have highlighted the importance of reasoning in K-12 mathematics (National Council of Teachers of Mathematics [NCTM], 2000; National Governors Association Center for Best Practices and Council of Chief State School Officers, 2010). Together, these policy documents suggest K-12 mathematics instruction should enable students to recognize the importance of reasoning in mathematics, make and explore mathematical conjectures, construct and critique mathematical arguments, and use various types of reasoning and proof.

Although coding/computer science/programming is a relatively new area of instruction, the K-12 Computer Science Framework ("K-12 Computer Science," 2016) recognizes communication as one of the seven core practices. This practice requires students to describe and justify their processes and solutions, promoting a more structured approach to coding rather than the trial-and-error approach commonly used by novice programmers (Lye & Koh, 2014; see recommendation by Fessakis et al., 2013). Thus, reasoning and the ability to communicate rationales are valued in both coding and mathematics.

Collective Argumentation in Mathematics and Coding

One lens that provides insight into the reasoning practices of students and teachers is that of collective argumentation. We define collective argumentation as teachers and students working together to establish or reject claims. There are multiple examples in the mathematics education literature of teachers facilitating collective argumentation to support student learning, reasoning, and sense making (e.g., Forman et al., 1998; Krummheuer, 1995, 2007; Yackel, 2002).

Although argumentation has not been widely used in teaching and learning coding, the larger project from which this study originates proposed that teaching coding through argumentation has several benefits, including (1) constructing and critiquing arguments provides a more structured approach to coding than trial-and-error and (2) teaching coding through argumentation allows teachers to use methods they already use in teaching mathematics, thus making it more likely for them to teach coding in conjunction with mathematics. One of the goals of the larger project was to provide teachers with strategies to help improve students' abilities to construct and critique arguments. The present study builds on the collective argumentation literature by focusing on one component, the warrant, of Toulmin's (1958/2003) model for argumentation.

Conceptual Framework

In mathematics education research, argumentation is often studied using Toulmin's (1958/2003) structure of an argument which includes data, claims, warrants, qualifiers, rebuttals, and backings. Toulmin argued that although what is accepted as valid for each component is discipline dependent, the structure of arguments is the same across disciplines. This structure can be expanded to include sub-arguments and the contributor(s) of each component (see Conner, 2008). Although entire arguments can provide insights into the reasoning that occurs in classrooms, warrants can provide a clearer understanding of what students and teachers use and accept as rationales. According to Toulmin (1958/2003), a warrant in argumentation serves as a bridge that explains how a person got from the data to a claim. The types of warrants provided during collective argumentation can illuminate the ideas on which teachers and students base their reasoning. Existing research classifies warrants in multiple ways (Conner, 2012; Inglis et al., 2007; Nardi et al., 2011). In this study, we adapted and expanded Conner's (2012) framework for analyzing warrants. The initial framework identified 29 types of warrants that were collapsed into ten major categories.

Methods

The larger study from which these data were analyzed included two phases of data collection with 32 elementary school teachers. During the first phase, teachers participated in a semester-long professional development course that included block-based coding content across multiple platforms and discussions about using collective argumentation across multiple disciplines. In the second phase, ten teachers were selected for classroom observations and coaching sessions. This paper is focused on analysis of video recordings of classroom observations. Participating teachers selected the topics for the observed lessons, focusing on integrating multiple disciplines and using argumentation during their teaching. Videos of classroom observations were reviewed and episodes of argumentation were identified via identifying main claims and associated argument components. Episodes of argumentation from each teacher's class were selected for analysis through a random sampling process, diagrammed by pairs of researchers using Conner's (2008) modified diagram structure, and then compared until consensus was reached. A total of 222 arguments were diagrammed across ten participants. We labeled the primary and secondary focus of each argument as mathematics, coding, science, literature, or social studies. For this study, we analyzed 108 warrants from 35 arguments with a primary focus of mathematics (secondary focus coding) and a primary focus of coding (secondary focus mathematics) from four teachers' classrooms. We inserted all of the information from each argument into a spreadsheet, noting whether the warrant was implicit or explicit. Implicit warrants were identified when a warrant was not explicitly stated or written but seemed to be understood by at least part of the group. We categorized the types of warrants provided by students and teachers according to the framework developed by Conner (2012). However, due to the context of our data – mathematics and coding arguments in elementary classrooms – we made adaptations to this framework, which was originally developed from high school algebra and geometry arguments. Using this adapted framework, we examined the types of explicit and implicit warrants contributed by students and teachers to make sense of the kinds of reasoning that were evident in mathematics and coding contexts. Analysis of these data is ongoing.

Results

In our initial analysis, we identified 21 different types of warrants with 15 of these having been identified in Conner's (2012) initial framework and six being newly identified from our data. By examining our data using Conner's framework as a starting point, we collapsed the 21 types of warrants into 11 major categories (Table 1).

Table 1: Relative Frequencies and Types of Warrants

Categories	Types of Warrants	Coding Focus	Math Focus	Total
Authority	External Authority, Given, Mathematical Convention	2 (7.4%)	1 (1.2%)	3 (2.8%)
Given	Given	0 (0.0%)	1 (1.2%)	1 (0.9%)
Interpretation	Interpretation of Problem*, Interpretation of Written Code*	4 (14.8%)	0 (0.0%)	4 (3.7%)
Method	Procedure-General	2 (7.4%)	4 (4.9%)	6 (5.6%)
Patterns	Patterning, Pattern Noticing	1 (3.7%)	4 (4.9%)	5 (4.6%)
Preference	Personal Preference*	0 (0.0%)	3 (3.7%)	3 (2.8%)
Visual	Appearance, Observation*, Observation with Quantification*, Visualization	8 (29.6%)	14 (17.3%)	22 (20.4%)
Calculation	Procedure-Calculation	0 (0.0%)	29 (35.8%)	29 (26.9%)
Unformalized knowledge	Informal Understanding, Number Sense, Previous Experience	5 (18.5%)	19 (23.5%)	24 (22.2%)
Knowledge	Definition, Prior Knowledge	3 (11.1%)	0 (0.0%)	3 (2.8%)
Reasoning	Interpretation of Definition, Calculation-Why*	2 (7.4%)	6 (7.4%)	8 (7.4%)
Total		27	81	108

An asterisk (*) indicates a newly identified type of warrant.

We first examined all of the warrants aggregated across both foci: mathematics and coding. Of the 108 warrants, approximately 70% of the warrants were classified into one of three categories: *calculation*, *unformalized knowledge*, or *reasoning*. More than a quarter (26.9%) of all warrants analyzed were classified as *calculation*; these were warrants in which a student or teacher provided a mathematical process or set of steps that produced a solution to a specific problem. For example, a student offered the warrant "Because 6 x 4 = 24" to justify the claim that a square with side length 6 would have a perimeter of 24. It is unsurprising that the largest category of warrants was *calculation* because each of these arguments included mathematical operations familiar to elementary students. Warrants classified as *unformalized knowledge* made up the second largest category (22.2%). For example, when investigating the relationship between time and distance, one student explained, "When I go to the gas station, it's really close to my house so [I] have a shorter time to go. But when I go to [the grocery store], it's like in the city, it takes a way longer time because it's more farther." The large proportion of these warrants that were *unformalized knowledge* suggests that students reasoned intuitively or based on ideas that had not yet been formalized in class. The third largest group of warrants (20.4%) were classified as *visual*, as students based their reasoning on physical representations they could see. For instance, when justifying why the robot needed to travel a longer distance along one side of a rectangle that was taped on the tile floor, a student offered, "This side looks longer. This [side] is 4 squares width and then this [side] has 3 squares width."

Although the remaining categories made up only 30% of the warrants, there are interesting things to note in this smaller group. Warrants were classified as *reasoning* when a student or teacher provided evidence for a claim based on the *interpretation of a definition* or when they provided a rationale for performing a calculation (*calculation-why*). Even though this category makes up a small percentage of all warrants analyzed (7.4%), the idea that one should give a reason for a calculation was evident in these elementary classrooms. In addition, the category of warrants that were based on some *external authority* made up one of the smallest percentages overall (2.8%), indicating that students were not relying heavily on what the teacher said when providing evidence for claims.

When we examined the warrants according to their primary focus, we found that almost 30% of the warrants in coding-focused arguments were classified as *visual*, compared to only 17.6% of warrants in mathematics-focused arguments. The higher proportion of *visual* warrants in coding-focused arguments could be due to students working with robots that students could observe carrying out their written code. For example, the reasoning students provided for their claim "we halved one second" to adjust how far the robot should travel was "one second got us two times too far." This warrant was classified as *observation with quantification* because students noticed the robot traveled too far and they used mathematical ideas ("two times") to describe what they observed.

Although more than a third (35.8%) of warrants provided in mathematics-focused arguments were *calculation,* none of the warrants provided in coding-focused arguments involved only calculation. Warrants involving a calculation in coding-focused arguments included a reason for doing the calculation (*calculation-why*). When students were attempting to code a robot to go a certain distance, they often related it to a previously established distance and time: "Because the length is doubled and 12 inches is doubled so I should double the delay." When focused on coding, it is reasonable that students include justifications related to the task in their warrants.

Discussion

Understanding the patterns of reasoning from these elementary mathematics and coding arguments provides insight into what teachers and students accept as appropriate justifications. Although some research exists on what types of warrants are acceptable in mathematics classrooms, little is known about what is considered valid reasoning in coding-focused arguments. Understanding reasoning patterns in coding contexts can help us support teachers in engaging students in argumentation in learning coding. Additionally, none of arguments in this analysis showed students used a trial-and-error approach to coding, which is commonly used by novice coders (see Lye & Koh, 2014). This is likely because the teachers insisted that students provide reasons for modifying their code, promoting a more structured approach to coding. This gives us reason to believe that argumentation is a promising approach for teaching students to code. And, the coding context, with expectations of argumentation, provided a way to access students' reasons for their calculations.

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A SERENDIPITOUS MISTAKE: HOW ONE TEACHER'S BELIEFS AND KNOWLEDGE MEDIATED HER IN-THE-MOMENT INSTRUCTION

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In this case study, we report how a ninth-grade mathematics teacher's beliefs about her students and knowledge influenced her decisions and actions. We first illustrate how the teacher retrospectively interpreted her response to an unexpected incident in the classroom as a mistake mediated by her beliefs about the students' abilities (i.e., "let's just focus on the basics"). Then, we illustrate how the teacher's response to that moment played a role in enabling her to leverage meaningful mathematical discussion. Results showed that, although the teacher was aware of different conceptualizations of slope, she wanted to promote a particular meaning of slope (i.e., slope as a formula) in the moment guided by her mathematical knowledge for teaching slope and beliefs about her students' mathematical knowledge.

Keywords: Algebra and Algebraic Thinking, Mathematical Knowledge for Teaching, Teacher Beliefs

Researchers have long argued that a teacher's belief system and mathematical knowledge for teaching have significant influence on instructional practice and student learning (e.g., Ball, Thames, & Phelps, 2008; Campbell et al., 2014; Wilkins, 2008). Moreover, teachers' beliefs about their students play a key role in designing and implementing instruction (Skott, 2001; Sztjan, 2003). In this paper, we illustrate the role that one teacher's beliefs about her students' ability and mathematical knowledge played in influencing her teaching decisions and actions.

Literature Review and Theoretical Framework

Teachers' Beliefs about Students

The types of beliefs typically investigated in the field fall into three categories: beliefs about mathematics, about teaching mathematics, or about learning mathematics (e.g., Conner et al., 2011; Liljedahl, 2009; Thompson, 1984). Of these three types, beliefs about mathematics are particularly important, given their potential to profoundly influence teachers' beliefs about teaching and learning mathematics (Cross, 2009). A number of researchers have also studied more specific or nuanced teacher beliefs. For instance, Skott (2001) investigated how one teacher's beliefs influenced his interactions with student groups, finding that the teacher's beliefs about raising students' self-confidence influenced how he interacted with different groups in different ways. Similarly, Sztjan (2003) studied two teachers' enactment of recommendations for change in mathematics teaching and found that they implemented these recommendations differently as a consequence of how each teacher perceived students who came from different socioeconomic backgrounds. The teachers' practices were ultimately mediated by their "value-laden visions of students, of parents, and of society" (Sztjan, 2003, p. 70). Together, these studies suggest that mathematics teachers may hold beliefs about their students that may not fit neatly into the categories of mathematics, teaching mathematics, and learning mathematics, and that these beliefs can have great influence on their practice.

Teachers' Meanings for Slope

Researchers (e.g., Byerley &Thompson, 2017; Stump, 2001; Nagle & Moore-Russo, 2013) have characterized a variety of teachers' meanings for slope including, but not limited to, (i) slope as an index of the steepness of a line, (ii) slope as a procedure, and (iii) slope as a ratio. *Slope-as-steepness* involves conceiving a line as a physical object and making perceptual associations between its steepness and a particular numerical value. For example, when considering the graphs of two lines, that person might conclude the slopes are necessarily the same if they have the same steepness visually despite being graphed in coordinate systems with different scales. *Slope-as-procedure* involves either the idea of "rise over run" (i.e., slope as a direction on how to move up and over on a Cartesian coordinate plane) or the slope formula (i.e., vertical change/horizontal change). Stump (2001) and Byerley and Thompson (2017) showed that most teachers' meanings of slope include the slope formula. In our study, we focus on a teacher who intended to promote an understanding of slope in terms of formula and prevent the slope-as-steepness meaning.

Methods

The present study is an investigative and descriptive case study (Merriam, 1998; Yin, 2003;) of Ms. R's beliefs about her students and the knowledge influencing her instruction. Ms. R was a sixth-year high-school algebra teacher. We conducted classroom observations in her ninth-grade algebra class, where we observed a four-day unit on equations and inequalities. We recorded all classroom sessions using two video cameras, one focused on the teacher and one focused on a focus group of three to four students, capturing the entirety of their engagement including conversations, gestures, and written work. We also conducted two interviews with Ms. R after the observations. The first interview was semi-structured (Roulston, 2010), and its goal was to discuss Ms. R's beliefs about mathematics and how she supported student learning. The second interview was a videoclip interview (Speer, 2005), and its goal was to further explore Ms. R's beliefs. Prior to the videoclip interview, we asked Ms. R to watch her lesson on Systems of Equations and Inequalities, choose clips where she thought there was evidence that her students had developed a generalization, and describe the generalizations in each clip. Ms. R chose clips depicting students' engagement in the pet sitter task, where they were asked to represent a situation both algebraically and graphically (Figure 1). In this task, Ms. R also provided students with two pieces of graph paper to represent the solution set for each constraint.

Carlos and Clarita have been worried about space and start-up costs for their pet sitters business, but they realize they also have a limit on the amount of time they have for taking care of animals they board. To keep things fair, they have agreed on the following time constraints.

<u>Feeding Time:</u> Cats will require 12 minutes to eat per day. Dogs will require 20 minutes to eat per day. Carlos can spend up to 8 hours each day to feed the animals.

<u>Playing Time:</u> Cats need 16 minutes each day to be brushed. Dogs will need 20 minutes each day playing with the ball. Clarita can spend up to 8 hours to play with the animals.

Write inequalities for each of these additional time constraints. Shade the solution set for each constraint on **separate** coordinate grids.

Figure 1: The pet sitter task.

Including a multi-phased qualitative process, we analyzed both transcripts and video recordings from the interview. In the first pass, we coded each interview using an open and axial coding approach (Strauss & Corbin, 1998) to identify and characterize the teacher's beliefs about students as she reflected on the video clips that she chose. In this paper, we report a portion of the interview data in which Ms. R reflected on her actions facilitating the discussions of the graphical representation of the solution set for the feeding time constraint.

Results

In this section, we report on how Ms. R's beliefs and knowledge influenced her actions. We begin by illustrating how an unexpected incident happened and how Ms. R responded to it.

Unexpected moment. Recall that there were no explicit directions in the task regarding the axis orientation. While the students worked in their small groups, Ethan asked Ms. R if they should assume that the number of cats should be represented on the vertical axis of the coordinate plane. Ms. R responded, "Yeah, I guess we should make that a general thing with everybody," and she asked the whole class "Alright, just so we're all staying consistent. What do you guys want to be your *y*-axis? Let's just all keep it consistent." After Ms. R noticed that the students had different opinions regarding axis orientation, she said to the whole class, "Actually, let's just see who comes up with what. I think that'll be better. …That'll be cool to see." In the second interview, we played the short video clip from the classroom data in order to get Ms. R's insights into the decision that she made. She began stating, "Oh Lord! [sighs]. That is an exact example of a moment like I don't know what I want to do, so let's just bring both ways [laughs]." She further explained that her student teachers had taught a similar version of the task the day before, and she could not remember what they had done in terms of the axis orientation. Therefore, she did not want to ask the students to graph in a certain way that might be different from what they had done the day before.

As the lesson progressed, Ms. R viewed this decision to let her students choose the axes orientation as a mistake. As she rotated from one group to another in the class, she said "Man, I wish I had never said anything about y'all's axes." In the interview, she added that, except for a couple of students, most of her students would find the lack of direction difficult. Although Ms. R considered the idea of graphing the same relationship in two differently oriented axes to be an important discussion, she believed that this was too advanced or difficult for many of the students in an on-level class to understand. We infer that Ms. R's regret about letting the students choose the axes' orientation was a consequence of her beliefs about her students' abilities.

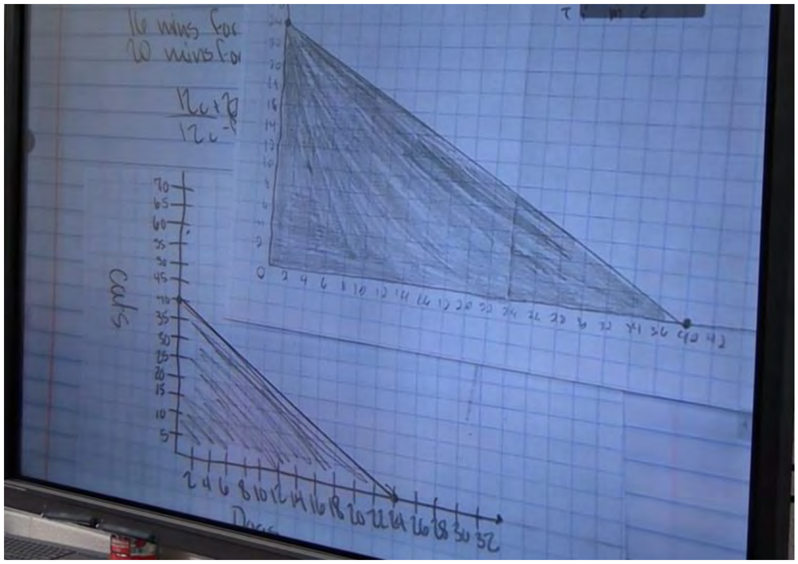

Figure 2: Two differently oriented graphs for the feeding time.

Pivoting at this unexpected moment. Although Ms. R regretted letting the students do what they wanted, she still capitalized on this moment by bringing two students' graphs to the whole class, each graph with a different orientation (see Figure 2). She put the two graphs under the document camera, saying, "I thought it was interesting, um, how you guys graphed. So, the cats and the dogs, the axes were different and similar."

Ms. R asked the whole class, "Alright, so do you guys see the difference between these two graphs?" Several students responded that the dogs and cats were "flip-flopped" on the axes of the coordinate plane. Then, Ms. R asked, "What about key features wise? You guys notice anything?" There were different opinions regarding the slopes as the students began answering "same slope," and "different slopes." The teacher then drew the students' attention to the fact that "they [both graphs] represent the feeding time … the one has cats as a y-axis, one has dogs as a y-axis," and asked "So, would the slopes be the same or different?" After students discussed "the rise and run would be like switched," most of the students concluded that the slopes were different, in fact, they were "flip-flopped" meaning that two slopes were reciprocal.

In the second interview, we played a short video clip depicting Ms. R's move to bring the two graphs to the whole class. We asked Ms. R what would be "cool to see" about the two different orientations, and why she chose to show these two graphs to the whole class. Ms. R stated that she wanted to get the students to discuss about the slopes being the same or not, and to realize that the slopes were reciprocal. She believed that this can be done if the students use a procedure (i.e., apply the slope formula) that they learned to a new situation. For her, if the students are not able to extend the use of this procedure to a new problem (i.e., the other graph), they might think the slopes are the same. Relatedly, Ms. R spoke about "a common misconception" regarding the shape of the graphs, which is that if "both of them are decreasing, and they look about the same amount of spread," then the slopes are the same. Ms. R thought that showing different orientations could help "to show that oh, we are just looking at how this vertical distances is changing over this horizontal distances is changing." We infer that Ms. R wanted to create an opportunity for her students to calculate the slope for the two different axes orientations, compare the slopes, and in turn, make sense why the slopes were reciprocal.

Discussion

In this paper, we illustrated that a teacher's beliefs about students and mathematical knowledge for teaching can operate together to guide her actions and decisions in the classroom. We showed that the teacher's beliefs about students made her regret her move in the moment. That is, she perceived herself making a mistake by letting her students choose the axis orientation in their graphing activity. However, her mathematical knowledge for teaching came into play and supported her decision to bring the two different oriented graphs to the board. Ms. R then was able to turn her mistake into an opportunity for her students to develop a particular meaning of slope. We interpret that Ms. R's mathematical knowledge for teaching slope productively informed her ways of designing her instruction in the moment and responding to the students in her classroom. It was this knowledge that allowed her to become *pedagogically powerful* (Diamond, 2020).

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EXAMINING PRODUCTIVE STRUGGLE IN MATH 1 CLASSROOMS USING MATHEMATICAL LANGUAGE ROUTINES

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This study examined how implementing mathematical language routines affected multilingual learners in Math 1 classrooms engaged in productive struggle. We used the theoretical framework of productive struggle to investigate two co-taught special education Math 1 classes, qualitatively analyzing video data from one student in each class. We noted a difference in students' productive struggle between lessons taught prior to the introduction of the mathematical language routines and lessons in which teachers implemented the mathematical language routines. This study illustrates how teachers participating in professional development promoted more frequent and deeper productive struggle in lessons utilizing mathematical language routines.

Keywords: Problem Solving, High School Education, Instructional Activities and Practice

The population of multilingual learners in U.S. schools is increasing (NCES, 2020). Across the United States, it is expected that multilingual learners will make up one-quarter of the total number of students in K-12 settings by 2025 (National Education Association, 2005). In addressing this trend in relation to mathematics classrooms, this study used a "studio days" model of professional learning (Von Esch & Kavanagh, 2018) to introduce teachers to mathematical language routines that could engage multilingual learners in rich mathematics content (Zwiers et al., 2017). We examined the differences in teachers' instructional practices, and more critically, the difference in multilingual learners' mathematical processes during observed lessons from these professional learning cycles. This study sought to answer the following research question: How did implementing mathematical language routines affect how multilingual learners in Math 1 classrooms engaged in productive struggle?

Theoretical Framework

This study is framed by the construct of productive struggle**.** We draw our definition of *productive struggle* from Hiebert and Grouws (2007) and Grandberg's (2016) work around this concept. Hiebert and Grouws described productive struggle as the intellectual efforts that students expend to make sense of mathematical concepts. This sense-making includes making connections between facts, between concepts and procedures, and most importantly, assimilating new information with prior knowledge (Granberg, 2016; Hiebert & Grouws, 2007). Specifically, productive struggle occurs when present comprehension is insufficient, and students engage in cognitive processes that restructure prior knowledge and construct new knowledge in relation to what is already known. Granberg (2016) further situated productive struggle in the learning of mathematics with understanding. In addition to student struggle, student interactions with teacher responses can alter the possible outcomes of the student struggle (Warshauer, 2015). Teachers play an important part in supporting the students' productive struggle (Smith et al., 2017). This study utilizes Granberg's (2016) definition for *productive struggle*, considering viewing activities that afforded students' deeper insight or that were needed to solve (all or parts of) a given problem as *productive*.

Methods

Our study was situated in one school district in Central California that included a substantial number of multilingual learners. Teachers engaged in a two-year professional development program organized around mathematics "studio days" (Von Esch & Kavanagh, 2018) for multilingual learners, in which teachers developed and studied a single lesson focused on one mathematical language routine and one mathematics language principle during each cycle. The findings of this paper come from the first studio day cycle of the larger study.

Participants

Nine mathematics teachers from three high schools participated in this study. Of the nine teacher participants, five were female and four were male. Seven were White/Caucasian, one was Latinx, and one was Asian American. One teacher was bilingual (Polish), and the rest were monolingual English-speakers. Four of these nine served as focal teachers.

For this study, the research team then chose to focus on two of the classrooms featuring focal teachers – Ms. Frasca's and Ms. Parker's– due to similarities in the classroom environments. Both classes utilized an inclusive co-teacher model with a special education teacher (Rimpola, 2014). We used data from two students' cameras, Chris, from Ms. Frasca's class, and Jorge, from Ms. Parker's class.

Chris and Jorge were each chosen due to the similarities of their demographics and availability of data. Each student was designated as an "English learner" (we use the state designation here rather than multilingual learner). Additionally, each student was designated as a special education student. Both Chris and Jorge also volunteered to wear the "student cameras" during the lessons, allowing for an intimate look at their processes during the observed lessons.

Professional Development Cycle

The data in this study were situated in a professional development cycle. Using Von Esch and Kavanagh's (2017) professional development model of studio days, we created a cycle of three professional development meetings for our participants. The studio day cycle of interest paired the mathematical language routine "Three Reads" (Kelemanik et al., 2016), which provides students access to rich text, with the principle *disciplinary language demands and supports* (Aguirre & Bunch, 2012).

Mathematical language routines support students to engage productively with content, providing them with tools that they can grow familiar with and return to regularly when engaging in cognitively demanding mathematics work (Kelemanik et al., 2016). Mathematical language routines are effective instructional practices when working with multilingual learners, due to the simultaneous foci of developing mathematics skills and language use in mathematics classroom (Zwiers et al., 2017). Utilizing the mathematical language routine, Three Reads, students become familiar with reading mathematical text for three different purposes – to understand the context, interpret the task, and identify information requisite for completing the task (Kelemanik et al., 2016).

The professional development cycles were guided by five key principles of reform-based instructional practices for multilingual learners in mathematics classrooms (Roberts & Bianchini, 2019). These principles grounded our work with teachers and framed the conversations with teacher participants about the teaching and learning of mathematics for multilingual learners. The fourth principle, *identifying disciplinary language demands and supports for multilingual learners* (Aguirre & Bunch, 2012; Lyon et al., 2016), focuses on language demands typical for critical thinking in mathematics and the challenges and supports teachers utilize in facilitating mathematics language use.

Data Collection

The research team observed the teachers four times across the study – once before beginning the professional development, twice on studio days, and once more after the first studio day cycle. We use student data for this data: video cameras in the form of eyeglass frames (Estapa et al., 2016), referred to as "student cameras" in this study, to capture audio and video data of the teacher's instructional practices. This study utilizes data from the first studio day cycle – the initial observation prior to professional development, in which no mathematical language routines were utilized, and the first studio day, which incorporated the Three Reads routine.

Data Analysis

The research team coded instances in which students engaged in struggles and activities that afforded them deeper insight needed to solve (all or parts of) a given problem (Granberg, 2016). The research team identified all such instances; for each, we reconciled whether or not it was an example of *productive struggle*; and we discussed whether the instance showed *productive*, *low-level productive*, or *unproductive struggle* (Warshauer, 2015).

Findings

Our findings noted a difference in productive struggle between lessons taught prior to the introduction of the mathematical language routines and lessons in which teachers implemented mathematical language routines.

Initial Observation

In Ms. Frasca and Ms. Lacrosse's Math I classroom, they taught transformations primarily utilizing direct instruction during our initial observation. One teacher lectured in the front of the room, while the other teacher circulated through the class and modeled participation by asking clarifying questions and encouraging students to participate (they swapped these roles interchangeably). Throughout the lesson, Chris engaged with the mathematics using a task-oriented focus, meaning that he recorded responses verbatim to his notes, but there was no evidence that he worked toward a conceptual understanding. Chris' interactions with his peers and teachers did not prompt additional insights for Chris, but rather, led to Chris' completion of the class tasks, meaning the data collection lacked evidence of any productive struggle

Ms. Parker and Mr. Neuman's initial lesson focused on linear relationships. Their instructional sequence and task was also more task-oriented than conceptual. Jorge worked throughout the lesson, fixing responses once he identified them as incorrect, such as when he restarted a task during class after graphing a line incorrectly. However, Jorge appeared to be more focused on completing the task than on increasing his conceptual understanding of linear relationships, as he showed frustration by crumpling his paper and restarting his work after making a graphing mistake—indicating unproductive struggle, because while he eventually corrected his mistake and completed the assigned task, Jorge failed to communicate his understanding verbally or in writing.

Studio Day Observation

Ms. Frasca and Ms. Lacrosse designed a lesson utilizing the Three Reads mathematics language routine to introduce a lesson on polynomials involving an algebra tiles task for their studio day lesson. Students engaged in the Three Reads routine prior to collaborating with peers and teachers in the classroom to represent the area of a specified rectangle. Similar to the initial observation, direct instruction proceeded the routine and collaborative opportunity; however, in this studio day lesson, a different peer interaction occurred. Chris and his group engaged with the task and discussed the various configurations available with the algebra tiles. The group struggled with the task, but Chris engaged authentically with the manipulatives. A teacher present for the observation prompted the group to follow Chris in his exploration. Chris and his peers showed evidence of *productive struggle*, as their conceptual understanding of area was recorded through the representation of the algebra tiles and the resulting equation for the rectangle's area was recorded on the worksheet.

Ms. Parker and Mr. Neuman's studio day lesson attended to parallel and perpendicular lines, while utilizing the Three Reads routine. Unlike the lesson observed in Ms. Frasca and Ms. Lacrosse's class, evidence of productive struggle was seen during the implementation of the mathematical language routine, as opposed to after the fact. Jorge began to show evidence of conceptual understanding of linear relationships during the second read of the routine, in which he interpreted the purpose or task in the problem as he worked at his own pace through the routine. He stopped the task after graphing one line, appearing to struggle with graphing a second line that was supposed to be parallel and transformed five units higher. After hearing the discussion of a teacher with another student, clearly audible in the student camera, Jorge then completed the task. When prompted on his worksheet, "What is the relationship between $y = \frac{3}{5}x - 4$ and its image?", Jorge wrote, "They have the same slope they look parallel." This showed evidence of *low-level productive struggle*, as he advanced his understanding of linear relationships, but did not clearly grasp an understanding of parallel lines.

Discussion and Conclusions

We found lessons involving the mathematical language routines engaged the two multilingual learners followed in this study in more opportunities for productive struggle. In both initial classroom observations, students did not appear to engage in advancing their conceptual understanding of the mathematics content. However, lessons utilizing mathematical language routines appeared to engage students in productive struggle in two ways. First, teachers provided more opportunities for students to engage in productive struggle. Second, through the use of mathematical language routines, teachers facilitated spaces in their Math I classrooms to allow for deeper exploration and understanding of the mathematics content, and ultimately, to lead to productive struggle among their students. This study provides evidence of another avenue by which researchers and practitioners may utilize professional development to improve instructional practice and student learning in mathematics classrooms for multilingual learners.

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IMPACT OF LESSON DESIGN ON MATHEMATICAL QUESTIONS

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How does the design of lessons impact the types of questions teachers and students ask during enacted high school mathematics lessons? In this study, we present data suggesting that lessons designed with the mathematical story framework in order to elicit a specific aesthetic response ("MCLEs") have a positive influence on the types of teacher and student questions asked during the lesson. Our findings suggest that when teachers plan and enact lessons with the mathematical story framework, teachers and students are more likely to ask questions that explore mathematical relationships and focus on meaning-making. In addition, teachers are less likely to ask short recall or procedural questions in MCLEs. These findings point to the role of lesson design in the quality of questions asked by teachers and students.

Keywords: classroom discourse, mathematics curriculum, aesthetic, mathematical story.

There is a broad consensus that the quality of the questions asked in mathematics classrooms influence student learning as it can constrain or broaden learning opportunities (Doyle, 1983; Chapin, O'Connor, & Anderson, 2009; Smith & Stein, 2011; Sullivan & Clarke, 1991). Despite this importance, there is evidence that mathematically rich questioning in high school mathematics lessons is rare (e.g., Boaler & Brodie, 2004; Hiebert & Wearne, 1993).

Responses to this challenge have been two-fold. First, one approach is to provide professional learning opportunities for teachers to learn more about productive mathematical discourse (e.g., Breyfogle & Herbel-Eisenmann, 2004; Chapin et al., 2009). A second approach has focused attention on incorporating teacher questioning as a part of designing lessons with high cognitive demand tasks (e.g., Smith & Stein, 2011) since such tasks provide potential opportunities for questions that involve meaningful mathematical connections and reasoning.

In this study, we describe how the mathematical questions posed by teachers and students in enacted lessons shifted through a third approach, namely, by having teachers design mathematically captivating learning experiences ("MCLEs") using the mathematical story framework (Dietiker, 2013, 2015). What is novel about this approach is that the professional learning and lesson design, which focused on how the ideas unfolded across a lesson as a form of narrative, included no attention toward the types of questions that would be asked during the lesson. This study is a part of a larger research project where lessons that students identified as highly interesting on a post-lesson survey were compared with those described by the same students as low interest in order to identify the characteristics of lessons that students find captivating. Since captivating mathematical stories should, in theory, provide opportunities for curiosity and questioning, the current study compares the types of questions asked during MCLEs and non-MCLEs. We report the ways in which this approach shifted the mathematical questions of both teachers and students, asking *How are the questions asked during MCLE* *enactments the same or different than those posed during non-MCLE enactments?* We end with a discussion on how this lesson design approach likely led to these improvements.

Theoretical Framework

This study interprets a sequence of mathematical events (e.g., tasks, discussions) that occur across a lesson as a *mathematical story* (Dietiker, 2015). Similar to literary stories, mathematical stories can differ in ways the content unfolds throughout the story and impacts the experiences of the audience (i.e., a student experiencing the lesson). Through the release and withholding of information, a mathematical story dynamically shifts a reader's recognition of what they know and enables them to recognize what they do not yet know. This shift in tension can compel a reader to wonder how the story will progress and end. For example, a mathematical story can provide a hint of a future revelation, thereby supporting the formulation and pursuit of questions (e.g., "Can leading coefficients of a polynomial help to identify its roots? How?"), similar to how a reader of a literary story might wonder how the story will advance. Although the MCLEs in this study were designed using the mathematical story framework, the non-MCLEs can also be interpreted for how the content unfolds in ways to inspire inquiry (or not).

Methods

The data for this study includes transcripts of lessons that were designed and taught during the 2018-2019 school year. Immediately after each lesson, students individually rated their interest in the lesson. The analysis of this survey data enabled the identification of lessons for each teacher that students described as most (and least) interesting. The six lessons with the highest interest measures were MCLEs, and the six lessons with the lowest interest measures were all non-MCLEs. These 12 lessons were taught by six experienced teachers with a minimum of four years of experience, and who taught in three high schools in New England with different curricula and demographic settings. The lessons were designed by teachers, with researchers, and represent a range of mathematical topics for both honors and non-honors courses spanning from Integrated Math 1 to calculus. To support the teachers' designing process of MCLE lessons, the teachers attended a two-week professional development during the summer of 2018 to learn about the mathematical story framework and begin the design process.

Data Analysis

Each of these lessons was transcribed and later coded for the mathematical plot by a team of researchers. The team identified the mathematical questions that were raised, explicitly or implicitly, by teachers and students throughout the lesson. Any questions that were non-mathematical, such as "Can you show it under the document camera?", were not included. We also did not formulate questions that clearly involved content from earlier grades, as we interpreted these as "checking answers" (e.g., "Is 18 times 3 is 54?"). Also, repeated questions were not counted as additional questions in this study. After identifying all the questions, we identified the acts during which each question was open and unanswered.

Then, each of the three researchers independently coded each question raised in the mathematical plots for its mathematical qualities. To distinguish the types of questions, we adapted Boaler and Brodie's (2004) categories of questions. We began with including six categories, namely *gathering information: procedural and factual* (GIPF), *inserting terminology* (IT), *exploring mathematical meanings and/or relationships* (EMMR), *probing for an explanation of thinking* (PET), *linking and applying* (LA), *extending thinking* (ET). We excluded three categories because either these categories are non-mathematical and thus not part of the mathematical story (in the case of *establishing context)* or they could be merged with other categories (in the case of *generating discussion* and *orienting and focusing*).

This coding framework was used to distinguish the qualities of the questions raised explicitly or implicitly by teachers and students. An important distinction of our coding scheme is that we coded questions based on how they were taken up and addressed within the story arcs as opposed to deciding the intent of the question independent of how the question was answered. Based on our initial analysis of questions in non-MCLEs and MCLEs lessons, we added two new categories: struggling with recently learned procedure and facts (SIPF) and problem-solving without known procedures (PSWP). SIPFs also are recall type questions; however, unlike GIPFs, students do not provide quick responses but instead struggle to recall the recently learned facts and perform the procedures. PSWPs a range of problems; both novel (unfamiliar to students) and challenging (perhaps familiar, but students choose to reason their way through the problem instead of applying a familiar procedure).

After questions were coded, the researchers met to resolve differences and to find consensus. To learn whether MCLEs have different proportions of each type of question when compared to non-MCLEs, we conducted a paired samples t-test for each teacher (pairing their MCLE and non-MCLE). Significance was determined when $p < 0.05$.

Findings

Overall, there were 417 teacher questions and 176 student questions in the 12 lessons, out of which 182 teacher and 99 student questions were from six non-MCLEs and 235 teacher and 77 student questions were from six MCLEs. On average, MCLEs have 30% more teacher questions in comparison to the non-MCLEs. In contrast, students asked approximately 27% more questions in non-MCLEs in comparison to MCLEs. However, the numbers of teacher and student questions were not significantly different between these two types of lessons. Note that we found only one *extended thinking* question across all the lessons. It was asked by a student in an MCLE lesson. Because of the lack of this type of question this category was excluded from further analysis. Following are our findings on the shifts of teacher and student questioning when comparing MCLEs to non-MCLEs.

Shifts in Types of Teacher Questions

The data (Table 3) show a stark difference for the types of questions that emerged in lessons that were MCLEs, as compared to those that were non-MCLEs. Overall, in MCLEs lessons, approximately one-fourth of the teachers' questions were for encouraging students to explore the mathematical meaning and reasoning (EMMR) in comparison to their non-MCLEs with only 1.6% of such questions. In fact, only two teachers asked either one or two EMMR questions in their non-MCLEs. These preliminary findings suggest that teachers tend to ask significantly more EMMR questions in their MCLEs (M=23.8, SD=10.4), which encourage students to explore underlying mathematical meaning and relationships, as compared to their non-MCLEs (M=1.3, SD=2.1) lessons (t (5) =5.6, p=0.003). On the other hand, in non-MCLEs, teachers tend to ask twice as many recall questions (i.e., GIPF and SIPF types) (M= 27.6, SD=12.5), which is significantly different as compared to their MCLEs (M= 56.6, SD=18.9) lessons (t (5) = -3.5, p=0.018).

Our data also suggests that teachers probe students to explain their thinking more often in their MCLEs (M= 18.6, SD=16.2) as compared to their non-MCLEs (M=6.4, SD=5.2), though this difference is not statistically significant (t (5) =1.97, p=0.11). The "problem solving with logic and unspecified procedures" (PSWP) type of questions were slightly higher in non-MCLEs

(M=35.6, SD=19.2) in comparison to MCLEs (M=29.6, SD=15.7). However, there was no significant difference between MCLEs and non-MCLEs for this question type, (*t* (5) = - 6.8, *p*=0.53), as both types of lessons have questions that require problem-solving where students felt challenged and used multiple strategies (e.g., logic, guess and check).

Table 3. Proportions of Student and Teacher Questions in MCLEs and non-MCLEs

	Teacher Questions		Student Question	
	MCLE mean (SD)	non-MCLE mean (SD)	MCLE mean (SD)	non-MCLE mean (SD)
No. of questions/ lesson	39.2 (5)	30.3(8.5)	12.8 (7.3)	16.5 (6.8))
EMMR	23.8 (10.4) *	1.3 (2.1)	25.8 (24.1) *	0.06 (1.6)
GIPF	27.2 (12.7) *	50.7 (16)	19.4 (13.5)	38.9 (17.8)
SIPF	0.4 (0.97) *	5.9 (5.8)	6.2 (13.4)	11.6 (18)
PET	18.7 (16.2)	6.4 (5.2)	18.5 (14.1)	23.5 (15.8)
PSWP	29.6 (15.7)	35.6 (19.2)	23.4 (14.4)	22.4 (14.8)
IT	0.38 (0.9)	0.0 (0.0)	6.7 (10.3)	3.02 (5.7)

Note: *Reflects a statistically significant difference (alpha < .05)

Shifts in Types of Student Questions

Similar to teachers, students also asked a higher proportion of exploring the mathematical meaning and reasoning (EMMR) questions in MCLEs (M=25.8, SD =24.1) in comparison to non-MCLE lessons (M=0.06, SD=1.6), and this difference was significant (*t* (5) =2.6, *p*=0.048). In contrast, the proportion of student recall questions (GIPF and SIPF, combined) in non-MCLEs (M=50.6, SD=31) was nearly twice that of MCLEs (M=25.6, SD=19.5). However, this difference was not significant (*t* (5) = - 1.8, *p*=0.14).

Discussion

We are encouraged to find that when teachers design lessons as MCLEs, it also results in a richer and wider variety of teacher and student questions during enacted lessons. This unexpected benefit of designing lessons with the mathematical story framework raises new questions for mathematics teacher education; namely, rather than training teachers what types of questions to ask during instruction, might it be better to prepare teachers to design lessons that encourage student curiosity and inquiry? We suspect that the teachers' intentional focus on how and when to enable certain mathematical ideas to emerge throughout a lesson in order to spur student curiosity and inquiry in MCLEs likely supported both teachers and students asking a rich and wide variety of questions.

Across all our lessons, we note the lack of *linking and applying* and *extended thinking type* questions in both types of lessons. This finding is supported by other studies (e.g., Kosko, Rougee & Herbst, 2014) where secondary school teachers did not include these types of questions in their planned lessons. One possibility is that the data of this study did not include any consecutive lessons, so therefore these question types did not appear in these lessons. The reason for non-significant differences in question types, such as teachers' *probing* questions, was likely due to the limited number of lessons in our study. Future research is needed to better understand the impact of designing lessons using a mathematical story framework on teacher and student questioning during enacted lessons.

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SPIN-UPS: HOW TEACHERS SCAFFOLD GROUPWORK WITH WHOLE CLASS PROMPTS AND THE MESSAGES THEY CONTAIN

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Teachers often employ groupwork to actively engage students in mathematical activity. While groups work, teachers may support groups in a number of ways. We extend the metaphor of a "launch" to define a "spin-up" as an instance in which teachers scaffold groupwork with whole class prompting. We examined an AP Calculus AB classroom in which the teacher often used spin-ups for a variety of purposes to support groupwork. We describe our findings from analyzing the occurrence of each spin-up during the lesson, the content of each spin-up instance, and the messaging around each spin-up. These findings help highlight the complex decision-making involved in supporting students' more autonomous work in the classroom.

Keywords: Classroom Discourse, High School Education, Instructional Activities and Practices

As educators seek to involve students in mathematical activity, groupwork has become increasingly important for both researchers and educators (Cohen & Lotan, 2014; Dunleavy, 2015; NCTM, 2018). The groupwork format shifts the responsibility of problem-solving and confirming solutions from the teacher to students. Using groupwork in the classroom creates opportunities for the unpredictable, often requiring teachers to improvise how they build on students' thinking or give feedback to groups. To begin to account for the pedagogical judgments that teachers use and their impacts, researchers have documented the ways in which teachers monitor and decide to intervene in individual groups (Ehrenfeld & Horn, 2020) as well as students' uptake of such scaffolds (van de Pol et al., 2019). Yet, teachers may also choose to provide scaffolds to the entire class, rather than an individual group, during groupwork time.

We refer to such scaffolding as a *spin-up.* We build on the metaphor of a rocket *launch* (Jackson et al., 2012) to describe teachers' work to initiate an activity, and extend this notion to a *spin-up* which stabilizes rockets while in flight to describe teachers' support to stabilize groupwork. This notion of a *spin-up* builds on the practice of monitoring (Stein et al., 2008). Rather than monitor for students' ideas to build on in a discussion to follow groupwork, *spin-ups* are used to support and sustain the continued activity of groups. A spin-up may not only consist of additional content information, it may also convey messages to students about how knowledge is built and their role in that process.

In this brief research report, we describe the preliminary findings from our investigation of one teacher's use of spin-ups and its impact on her class, by answering the following questions: (1) When do spin-ups occur in class during groupwork? (2) What is the content and purpose of each spin-up offered? (3) What messages are conveyed via the teacher's framing of the spin-up?

Theoretical Framing

We consider classroom learning to be a *situated interactional activity* among students and teachers and the content. Effective teaching that centers students is highly improvisational, context-dependent, and responsive to activities that happen within the classroom (Robertson et al., 2017). Especially in the enactment of groupwork, in which students take on a central role, outcomes may be unpredictable. Yet, patterns still exist in these interactions that emerge as routines, which we can characterize and investigate (Horn & Little, 2010).

Within this framing of classroom interaction, we adopt a lens of framing and messaging (Scherr & Hammer, 2009; Russ, 2018) to characterize teachers' use of spin-ups as whole-class scaffolds to support groupwork. Framing and messaging are one dimension of teachers' behavior that impacts student-teacher interaction by communicating to students their roles in the learning process and notions about the content itself. This communication is not direct, but embedded in the routine moves of the teacher and interactional routines the students and teacher negotiate together (Kelly, 2020). Literature on framing and messaging does not attempt to make inferences about teachers' judgements or purposes for enacting instructional moves. Instead, the theoretical lens of framing and messaging recognizes the ways that meaning about "what is happening here?" are socially constructed within classroom interactions among the teacher and students.

Methods

Using a grounded approach (Corbin & Strauss, 2014), we analyzed video and audio recordings of one lesson from an AP Calculus AB teacher, Barbara, from the 2014-2015 academic year. Barbara taught in a racially diverse, high-performing, suburban school. In this school, teachers were beginning to consider access and representation of students who had historically not taken advanced mathematics classes, which Barbara discussed in reference to her AB Calculus students. These data come from a larger study (Dyer, 2016), which investigated 10 high school mathematics teachers aiming to become more attentive and responsive to student thinking in their teaching. The lesson video was captured with several cameras placed around the classroom, and audio recorders placed at the center of desks that were grouped together in the room, and included all five groups of students in the classroom. The content of the lesson was the use of integration to calculate volumes of solids of revolution. Specifically, the students were given a warm-up task, which they worked for the first 15 minutes of the lesson. Then, the students worked on a related task, which was given as "team practice," described in Table 1.

Table 1: Groupwork Tasks Used in Lesson

	Class Time	Task
Warm-up Task	~00:30-15:00	1. Make a sketch of the region bounded by y=2x+4, y=4, and x=5. 2. Rotate this region around the line, y=4. What shape have you formed? What is the formula you learned in Geometry to find its volume? 3. Use the formula from #2 to calculate the volume of this solid. 4. Use calculus to find this same volume.
Team Practice	~16:00-42:00	Find the volume for each solid of revolution described below: 1. Region created by y-axis, x-axis, y=sin(x)+1 and x=3pi/2, revolved around x-axis (three more solids of revolution are given)

We first used *Datavyu* software to code the activity formats used in the lesson according to the codes: launch, content discussion, groupwork, and other for unstructured time. Portions of the lesson were coded as content discussion when the teacher addressed the class as a whole regarding content, for a duration longer than twenty seconds. From our analysis, a notion of spin-up emerged as we attempted to classify the teacher's interaction with the class as students worked in groups. We classified an instance of teacher intervention as a spin-up when she offered a supporting statement, question, or direction to the class as a whole, meant to sustain the activity of students in groups. We then coded instances of spin-ups by marking the beginning of the spin-up. Through a process of open and axial coding (Corbin & Strauss, 2014) to analyze the content of the spin-ups, we developed definitions of six types of spin-ups. Finally, we began a preliminary analysis of messages embedded in the framing of each spin-up according to three dimensions: epistemic, social, and disciplinary framing. Throughout this process, we listened to the audio recordings of each group the teacher interacted with before and after each spin-up to better contextualize the spin-up, alongside the videos.

Results

We identified 12 instances of spin-ups in the calculus lesson that we analyzed. The time series in Figure 1 shows the occurrence of each spin-up as a blue asterisk overlayed on the varied types of activity formats used in the lesson. The official lesson ran for approximately 42 minutes of the video recording, and three different activity formats were used in the lesson: launches (l), content discussions (cd), and groupwork (g). Unstructured time at the end of the lesson was coded as other (o). The density and spacing of the spin-ups in the lesson provide more insight into how the teacher made use of them. Nine of the twelve spin-ups occurred in the first fifteen minutes of the lesson, when students worked on the warm-up task and were given in closer proximity the earlier in the lesson they occurred. Furthermore, the first six spin-ups occurred in very close proximity to each other, within a four-minute window, shown in portion of the time series magnified in Figure 1. Only three of the spin-ups were given in the portion of the lesson when students worked the team practice tasks. Additionally, four of the spin-ups led to content discussions, shown in green on the time series, which also became increasingly spread apart as the lesson progressed.

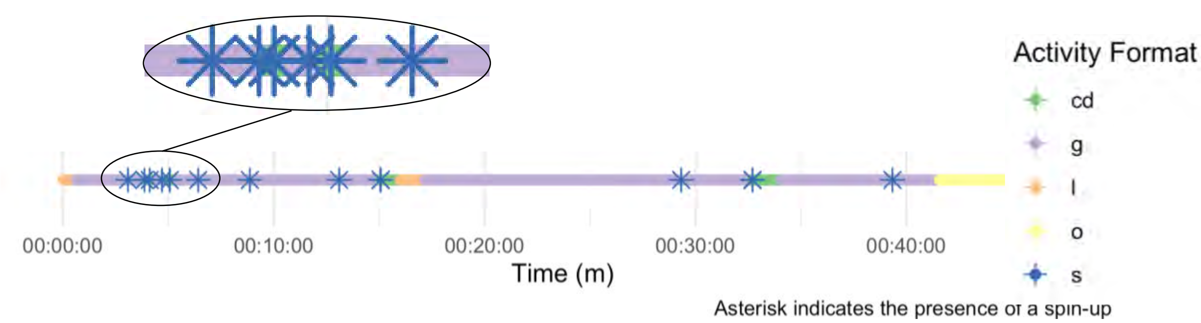

Figure 1: Time Series of Lesson's Activity Formats and Instances of Spin-ups

To better understand the messaging contained in the spin-ups, we first analyzed the content of each spin-up Barbara offered. We developed codes to classify the spin-ups according to their content and purpose and found six distinct purposes to the spin-ups, shown in Table 2. These purposes are: restating a question, referencing previous work, offering a method to check a solution, giving a directive, asking for a status check, and offering content information. We note that a single spin-up may be classified as multiple types, as it may contain multiple statements with distinct purposes given in the same talk turn of the teacher. The six distinct types of spin-ups center students, support students' group activity, and frame mathematics in strategic ways through the messages they contain. Analyzing the content and type of spin-up through a framing lens revealed three categories of messages: (1) framing students as epistemic authorities, (2) framing learning as social, and (3) framing the discipline as coherent and cohesive.

Table 2: Six Types of Spin-Ups Identified in this Lesson

	Spin-Up Type	Definition	Example (Instance #)	Instances
1	Restate a Question	The teacher restates a question asked by a student	Restating a question that a student asked the teacher in his group: “I don’t know, *is* the radius $2x+4$?” (#1)	1, 2, 5
2	Reference Previous Work	The teacher references previous work as guidance	“Isn’t that the one Megan asked us yesterday?” (#3)	3, 6
3	Offer Method to Check Solution	The teacher suggests a method for students to check their solution	“So [number] 4 is you get the integral…you should get an answer that matches the one you know from Geometry” (#7)	7, 8
4	Give Directive	The teacher directs students to do a specific step of the task	“Do me a favor, make a little 10-second sketch of the revolution…” (#4)	4, 7, 8, 12
5	Status Check	The teacher asks about or provides expectation of student progress through tasks	“What’s the consensus on the radius?... So, make sure you have someone in your team can convince why the radius is $2x$” (#6)	6, 9, 10
6	Offer Content Information	The teacher provides mathematical information	“What’s 2x+4 measuring? The height to the axis, right? …The 2x+4 is this height” (#5)	5, 11

The messaging in spin-up types 1-4 frame students as owners of their mathematical activity, communicating to students that they are central to the knowledge-building process. Restating a question (type 1) to the whole class amplifies an individual student’s thinking, and gives groups the chance to hear others’ thoughts beyond those of their own group. Referencing previous work (type 2), offering a method to check a solution (type 3), and giving a directive (type 4) all serve to give students a path forward without directly giving content knowledge. Some spin-ups contained messages that framed learning as social. For instance, in Spin-up 6, Barbara asked students to make sure someone in their team could convince them why the radius was $2x$. The message embedded in this spin-up is that students can and ought to rely on each other for knowledge building. Finally, some spin-ups framed the discipline of mathematics in a particular way. For instance, by referencing previous work, Barbara sent the message that the current content (volumes of solids) was directly related to other topics and tasks (area between curves), framing mathematics as cohesive and coherent.

Discussion

The findings of this study further characterize how teachers orchestrate group work and whole class activity. With the use of spin-ups, Barbara supported her class with various scaffolds to advance groups’ activity while also signaling about students’ roles in learning, supporting each other, and mathematics as a discipline to the class as a whole. Future work in this study will follow each groups’ activity throughout the lesson, note instances in which a teacher scaffolds a group without using a spin-up (not speaking to whole class), the content and messaging of these scaffolds, and what precedes each intervention type for any discernable patterns. More broadly, future research into the messaging of spin-ups in mathematics classrooms may analyze connections between such messaging and larger discourses and ideologies, such as those researchers have identified in other settings (Louie, 2018; Louie et al., 2021; Philip et al., 2018).

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A FRAMEWORK FOR ANALYZING THE USE OF LANGUAGE IN UNDERGRADUATE MATHEMATICS

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Keywords: Classroom Discourse; Undergraduate Education; Equity, Inclusion, and Diversity

While there is a common notion that mathematics is a universal language, on closer examination we find that doing mathematics also involves engaging in mathematical practices that require sophisticated uses of language (Moschkovich, 2002). For instance, the authors of the Common Core State Standards for Mathematics state that students should "communicate precisely with others… using clear definitions in discussion" (NGACBP & CCSSO, 2010, p. 7). The issues related to language and mathematics have received less attention at the undergraduate level, though some research has considered the linguistic demand of undergraduate mathematics (Barton et al., 2005; Cornu, 1981; Kaplan, Fisher, & Rogness, 2009; Lavy & Mashiach-Eizenberg, 2009; Parker, 2011; Tall, 1993). Yet, much of this work focuses on "language as a problem" rather than as a resource (Planas, 2018; Ruiz, 1984). These observations warrant a careful analysis of the ways teachers and students use language in undergraduate mathematics classes. To pave the way for this analysis, this poster develops a conceptual framework of language use in undergraduate mathematics.

Drawing primarily on a situated sociocultural theory of learning (Moschkovich, 2015), the proposed framework is organized based on the use of several language components (and their relationships). I briefly elaborate on four such components: lexical ambiguity, students' linguistic resources, multiple-semiotic systems, and mathematical practices. *Lexical ambiguity* highlights that certain words may share the same form but have different (and sometimes even conflicting) meanings within and outside mathematics, whereas other words may have two or more different but related meanings (e.g., Kaplan, Fisher, & Rogness, 2009). For example, I have seen linear algebra students use everyday meanings of dependence (e.g., reliance on something or someone) to make sense of linear dependence as a relationship. Related to this component is the use of *students' linguistic resources*: Not only is it important to build on the students' everyday understandings in the language of instruction (e.g., English in most US undergraduate mathematics classrooms), but also on their understanding in their additional language(s) (Erath et al., 2021). *Multiple semiotic systems* is the use of three interrelated mathematical language systems (natural language, visual displays, and symbol system), each with different affordances and limitations (O'Halloran, 2000). For example, when teaching linear independence in linear algebra, a teacher might describe a linearly independent set of vectors in terms of a world problem context as well as using a graph or a vector equation. By doing this, the teacher could engage students in connecting multiple representations -- one important *mathematical practice.*

For researchers, this framework could function as an organizing tool for analyzing uses of language and refining a theory on language use. It could also be used to design professional development or student support workshops that promote intentional and strategic language use. This could increase multilingual students' access to mathematics and ultimately promote equity.

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HOW LANGUAGE SWITCHING FACILITATES FOLDING BACK TO COLLECT

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My ongoing interest in researching the role of 'language switching' – described by Baker (1993) as the way bilingual individuals alternate between two languages, whether in words, phrases or sentences[1] – dates back to my mathematics and bilingual upbringing in Tonga, a small island country in the South Pacific. This study focuses on Tongan bilingual teachers in a mathematics classroom setting and explores the connection between their use of *language switching* and *folding back to collect* – a form of 'thickening' students' mathematical understanding by recalling, accessing and collecting their prior or existing knowledge. The contention here is that if interventions for the purpose of folding back ought to be intentional, explicit or stimulating, then language switching could be a powerful way to invoke or facilitate folding back to collect in bilingual situations. This study further articulates an adaptation into the bilingual domain of the theoretical foundation, the Pirie-Kieren or P-K Theory (1994), which has been presented and discussed previously at a number of PME meetings (Martin, 2008).

Folding back, a central key construct of the P-K Theory, occurs when a learner is faced with a challenge, one that is not immediately solvable, and he or she is prompted to return to an inner mode of understanding in order to reconstruct, and to extend his or her currently inadequate inner-layer understanding (Pirie & Martin, 2000). In a classroom situation, teacher interventions are sometimes required to stimulate the process of re-collection and especially important in its ability to *invoke* folding back to collect by *facilitating* a learner's awareness of what he or she already knows and the recognition of the need to fold back and collect a relevant piece of information whenever he or she is confronted with a mathematical obstacle (Martin, 2008). If and when this process is initiated or engaged through language switching within a bilingual setting, the choice of which language is more accessible or appropriate becomes quite potentially significant to the 'thickening' effect of one's mathematical understanding (Manu, 2005).

This study is based on video recordings, followed up with interviews, of two experienced bilingual mathematics teachers within one high school in Tonga during the second half of the 2019. Two excerpts were highlighted to explicate some of the ways in which Tongan-type bilingual teachers use language switching to facilitate students in *folding back to collect* from prior knowledge or from existing understanding. The first shows the role of a teacher in choosing her first or native language and also the 'thickening' effect of folding back that engaged the students to return to a more specific and local understanding at an inner layer in order to support and extend their understanding at the outer layer. The second illustrates how a teacher and her students both used *substitution* and *borrowing* of equivalent and non-equivalent mathematical words as forms of language switching to facilitate their existing or prior knowledge.

In short, this study finds that when the instructional transition moves from one language to another, the continued learning relies upon *what the students already know*, including their knowledge of, and proficiency in, both languages. This inner layer knowing is significant and easily receptive whenever 'collecting' is initiated or called upon. It supports both the use of the students' first language and the view that effective instruction involves identifying clues that can help students draw upon both their entire linguistic repertoire and their *primitive knowing*.

Note

[1] Manu (2020) discusses further the flip-flop nature of 'language switching' and its relation to *translanguaging*.

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THE EVOLUTION OF ROUTINE TASKS IN THE COLLECTIVE

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Keywords: Problem solving, problem-based learning, instructional activities and practices

Which fraction is represented by this drawing? Can you believe that this task led to a deep mathematical investigation of about fifty minutes in a Grade 6 classroom (Proulx & al., 2019)? While it is recognized that teachers play an important role regarding the students' mathematical activity, numerous research on problem-solving has focussed on what could be "good problems" for the mathematics classroom (Hoshino & al., 2016; English & Gainsburg, 2015). However, some experiences that I had shown myself that routine tasks have the potential to become good problems through the mathematical interactions in the classroom. Could the notion of "good problems" be more complex than what it appears at a first glance? Could it be broadened? Does the problem in itself play such an important role? These concerns can find echo in Beghetto's work (2017) on lesson unplanning, and in Mason's (2019) work where he states that "it is not the task that is rich, but whether the task is used richly" (p. 146).

In this regard, in this poster presentation, I propose a theoretical framework to study the evolution of routine tasks through the mathematical activity of the classroom. This framework is built from enaction theory (e.g., Maturana & Varela, 1992), a biological cognition theory, and from works being conducted in the "teaching of mathematics *via* problem-solving" area of research (e.g., Borasi, 1996; Lampert, 2001). A particularity of the proposed framework is to consider the classroom as a collective entity, a collectivity, who bring forth a mathematical activity together; meaning that the classroom is the unit of study. The framework allows to study the nature of the evolution of the task through the mathematical practices that are put forth to solve it. The nature of the evolution of the task is also characterized as implicit or explicit: the implicit evolution referring to the evolution of the task through the steps being posed to solve it, and the explicit evolution to new mathematical (sub)tasks that emerge from the collective mathematical activity and on which the collectivity tries to solve. In this poster presentation, examples are given to illustrate the proposed framework that studies the evolution of routine tasks in the collective mathematical activity that takes place.

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WHAT IS AMBITIOUS MATHEMATICS TEACHING? A LITERATURE SYNTHESIS

¿QUÉ ES AMBITIOUS MATHEMATICS TEACHING? UN SÍNTESIS DE LA LITERATURA

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Keywords: Systemic Change; Equity, Inclusion, and Diversity; Instructional Activities and Practices

In a systematic review of the literature, we find that the concept of ambitious mathematics teaching (AMT) has evolved over the past two decades. This poster summarizes this evolution and considers implications for teacher educators and researchers. Early descriptions of AMT (which did not use the term ambitious as it is used currently) focused on aligning teaching and learning in school mathematics with disciplinary practices (Lampert, 1990).Teachers facilitate student engagement in problem solving activity and discussion. This "ambitious" vision was contrasted with "traditional" math classes. The disciplinary practice vision of AMT permeates standards documents and associated guidance (NCTM, 2014), and some recent research equates standards-aligned teaching and AMT (e.g., Coburn et al., 2012; Litke, 2020).

Since the introduction of AMT into the research lexicon, mathematics educators have investigated how to make this vision a reality in multiple settings with a diverse array of teachers. Currently, several related constructs appear in the literature alongside AMT, including classroom discourse, tasks, teacher noticing, and equity. Classroom discourse is connected to AMT because student engagement in disciplinary practices often requires a teacher to skillfully solicit verbal and/or written contributions by students (Chapin et al., 2009; Hufferd-Ackles et al., 2004). Using instructional tasks with higher levels of cognitive demand can support student engagement in classes characterized by AMT (Boston & Candela, 2018; Tekkumru-Kisa et al., 2020). But, in order to facilitate discussions of such high cognitive demand tasks, teachers must notice (Jacobs et al., 2010) and build on student ideas (e.g., van Es et al., 2017). Finally, equity has grown as a central area of focus in the body of research concerned with AMT. Early conceptualizations of equity in research on AMT focused on the extent to which discipline aligned learning was available to "all students" (Lampert & Graziani, 2009) or in schools serving students from minoritized communities (Boaler & Staples, 2008; Jackson & Cobb, 2010). More recently, mathematics educators have focused on articulating how a more specific equity stance toward minoritized students (e.g. building on students' funds of knowledge) is integral to learning to engage in AMT (Kinser-Traut & Turner, 2020).

For teacher educators, one implication of this shift is that while the original vision for and definition of AMT remains viable, the field is expanding this concept to address multiple overlapping concerns that arise in teacher education including (equity, teacher development, and supporting a shared vision across contexts). For mathematics education researchers, the evolving definition of AMT implies we should be careful to specify what aspects of AMT are the focus of each investigation and how the definition matches prior research and each study's conceptual framework.

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STUDENT-TEACHER INTERACTIONS DURING STUDENT PRESENTATIONS THAT CONTAIN MATHEMATICAL ERRORS IN SECONDARY CLASSROOMS

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There has been a push towards increasing student-centered learning within mathematics classrooms (NCTM, 2000; 2014; NGA & CCSSO, 2010), which has been shown to improve both mathematics achievement and overall attitudes (Zarkaria et al., 2010). Student presentations of their mathematical work allow teachers to build on student thinking to optimize learning for the class as a whole. The presentation of student work can sometimes give rise to errors. Rather than these instances being avoided or dismissed, presentations of incomplete or incorrect student work can be opportunities for pivotal teaching moments (Stockero & van Zoest, 2013; van Zoest et al., 2017). When teachers discuss and build upon incomplete or incorrect student work, they may do so in a variety of ways. Previous studies have documented how teachers may build on incorrect student thinking (van Zoest et al., 2017) and authority relations among teachers and students during class presentations (Byun et al., 2020). In this study, we detail how teachers and students interact when incorrect mathematics is presented to better understand this particular aspect of teaching practice. Our study investigates the question: How do students and teachers interact when students present incorrect or incomplete mathematics in secondary classrooms?

To answer this question, we analyzed video of nine 100-minute lessons that contained student presentations, three each from three secondary mathematics classrooms. With the use of *Datavyu* software, we first coded the various activity formats used within these lessons to identify instances of student presentations. We closely analyzed these specific episodes to capture the variety in student-teacher interactions when incorrect mathematics was presented. Through a process of open and axial coding (Corbin & Strauss, 2014), we analyzed relevant aspects of student-teacher interactions within these episodes.

We found three distinct ways in which the teachers brought other students into the conversation. These other students advanced the presentation in different ways. One way involved the teacher restating other students' comments or questions to include them in the discussion. A second way that teachers involved other students was to ask the class follow-up questions in order to fill in the gaps throughout the presentations. In the third way, teachers invited other students to help the presenter, even allowing the presenter to choose which student supported them.

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A REVIEW OF A FUNCTION-BASED APPROACH TO ALGEBRA AT THE SECONDARY SCHOOL LEVEL

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Keywords: Algebra, Secondary School Education, Systematic Review

The purpose of this study is to conduct a systematic review of the empirical studies that explore a function-based approach to algebra at the secondary school level. The author reports on the various ways this approach has been implemented: the types of teaching experiments used, the types of technology used, and the study's outcome. As reported in the *Common Core State Standards for Mathematics* and by the U.S. Department of Education, school algebra is foundational to mathematics in grades K to 12 (NGA & CCSSO, 2010; U.S. Department of Education, 2008). All students should have the opportunity to be successful in school mathematics (NCTM, 2000), and consequentially be successful at school algebra. Thorpe (1990) argues that "functions should form the backbone of a first course in algebra" (p. 17). A function-based approach to algebra is defined as "an approach that assumes the function to be a central concept around which school algebra curriculum can be meaningfully organized" (Yerushalmy, 2000, p. 125).

Methods

A systematic review provides a means to aggregate, interpret, explain, or integrate related existing research on a topic over a period to describe the trends in that topic (Xiao & Watson, 2017). In this study, a systematic review was conducted to examine each included study's teaching experiment, use of technology, and outcome to investigate for secondary students under what conditions a function-based approach to algebra was implemented. The data collection occurred in two steps: screening and inclusion (which includes searching the literature) and coding of studies. The inclusion criteria for this systematic review were as follows: only empirical studies: (i) with primary student data; (ii) explore the use of functions to teach algebra; (iii) the unit of analysis is students in grades 6 to 12; and (iv) reported in English.

Discussion

The studies in this analysis varied across the type of publication, the decade of publication, the vicinity where the study was implemented, the research approach, the methods for data collection, the grade level, the sample size, and the algebra level. This systematic review showed that technology, that is, some type of graphing tool, seems to play an important role for successfully introducing a function-based approach to algebra with secondary school students. Also, the outcomes seem to indicate that a function-based approach to algebra has been effective in developing secondary students' understanding of variables, functions, problem solving strategies, use of multiple representation, and algebra, across the years. Therefore, given the difficulties students experience when transitioning from learning arithmetic to learning algebra (Sharpe, 2019), a function-based approach to algebra instruction may improve the educational foundation students need for the transition to learning algebra.

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TEACHERS' REFERENCING OF PUBLIC RECORDS OF STUDENT MATHEMATICAL THINKING

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Public records of student contributions are a common resource mathematics teachers use to make student thinking accessible in whole class discussions. Studies have implicitly addressed the use of public records in whole class discussions, such as to scaffold students' engagement with each other's ideas (e.g., Webb et al., 2014), but the detailed nature of how teachers use public records is largely unknown. Our research on whole class discussions that leverage student thinking (Van Zoest et al., 2016) surfaced the ways teachers regularly used public records throughout such discussions. In our work, we consider public records of student mathematical thinking to be physical and visual representations of student mathematics that are accessible to all participants within a classroom. This poster will present findings from our analysis of 12 secondary teachers' *explicit referencing* of public records. Our analysis focused on how referencing helped teachers to build on student thinking throughout whole class discussions.

Explicit referencing is a teacher's physical or verbal actions or combination of actions that draws attention to the public record or a portion of the public record. Physical actions include gestures towards specific or general parts of the public record; verbal actions include words that draw attention to the public record in some way (e.g., naming student thinking, using locator words). Early in the discussions, explicit referencing is needed to make a student contribution an object so that there is no ambiguity regarding the contribution as a discussion develops around it. Additionally, explicit referencing can focus and engage the class with specific mathematical ideas of the student's contribution. As discussions continue, explicit references to additions or edits of the public record help the class to identify and coordinate the mathematical ideas emerging in the conversation.

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ELEMENTARY PRESERVICE TEACHERS INTEGRATING MULTICULTURAL LITERATURE TO DESIGN CULTURALLY RELEVANT MATHEMATICAL TASKS

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Teaching with a commitment to culturally relevant pedagogy recognizes the need for preservice teachers (PTs) to have structured learning opportunities to explore culture in the context of mathematics pedagogy and content. For PTs to engage in such practice, PTs must have mathematics methods courses that enable them to learn about and build on the cultural assets and identities that students bring to the classroom (Association of Mathematics Teacher Educators, 2017; Ladson-Billings, 1995). One way to recognize culture in mathematical studies is by integrating multicultural children's literature to make connections between mathematics and students' cultural experiences (Mendoza & Reese, 2001). Research suggests that such texts can be essential resources to situate story problems and encourage reflection in terms of mathematical identity, cultural competence, and critical consciousness (Chappell & Thompson, 2000; Harding et al., 2017; Iwai, 2013; Leonard et al., 2014; Moldavan, 2020). Motivated by the need to further explore how PTs might work with multicultural literature in a digital context, the researchers designed an instructional activity during a mathematics methods course to get PTs working with such texts in conjunction with mathematical tasks. In this study, we examine the following research question: What are elementary PTs' experiences integrating multicultural literature with mathematical concepts to design culturally relevant mathematical tasks?

Research Design

We used a qualitative case study design conducted in an online elementary mathematics methods course. The PTs engaged in an instructional activity facilitated by the researchers that modeled culturally relevant mathematical tasks using multicultural literature to teach mathematical concepts. Following the activity, the PTs worked in groups to design similar tasks using multicultural texts of their choosing. Accompanying the tasks, the PTs submitted reflections on how the tasks used the texts to portray cultural awareness and elicit culturally relevant mathematical thinking. Data also included survey responses comparing the PTs' experiences integrating such texts into their tasks before and after their participation in the instructional activity. The collected tasks, anecdotal notes, reflections, and survey responses were analyzed using open coding to discern emerging themes (Grbich, 2013; Saldaña, 2016).

Summary of Findings

We report on the PTs' experiences using multicultural literature to make connections to mathematical concepts for purposes of designing culturally relevant mathematical tasks. We note two themes that emerged from the data. The first theme addresses the PTs' growth in awareness of and confidence in using multicultural texts to explore mathematical concepts. The PTs described how various texts were used to create a cultural context that prompted reflection and appreciation for cultural diversity. The second theme looks at how the tasks elicited culturally relevant mathematical thinking with targeted concepts (e.g., multiplication, geometry, patterns). Recommendations are made for how others can build on this work in mathematics education.

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INSTRUCTORS' MODIFICATIONS OF ANALYSIS PROOFS INTO PEDAGOGICAL PROOFS

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Despite the fact that mathematical proofs are central to the field of mathematics, proofs are difficult for many undergraduate students to understand (Almeida, 2000; CadwalladerOlsker, 2011; Hanna, 1990; Hersh, 1993; Mills, 2011; Weber & Mejia-Ramos, 2014). Given the significant role that proofs play in the field of mathematics, we believe that it is critical to investigate ways to improve the teaching and learning of mathematical proof at the undergraduate level. The work described in this paper is part of a larger study and focuses on how mathematics professors modify proofs in order to make them more understandable to undergraduate students.

Using the framework of communities of practice (Lave & Wenger, 1991; Wenger, 1998), we view mathematicians as forming their own community of practice, and through participation in community activities, mathematicians reify ideas and experiences into physical artifacts such as textbooks, journal articles, lecture notes, and solutions to problems. Proofs are the most prolific of these objects, being used in nearly every mathematical task that a mathematician performs. Mathematics students are peripheral members of the community, similar to apprentices, who are given small tasks that simulate parts of the practice within the community. When reading a proof in a textbook or written by a professor, students can see the way a mathematician thinks about the task of proving a theorem, and thus learn more about theorems and proofs, in general.

Since students spend a significant amount of time reading proofs, it is important to study how those proofs are modified for presentation to students. A prior study (Lai et al., 2012) asked several instructors to modify two proofs to improve student understanding and then invited 110 mathematicians to provide feedback on the original and modified proofs. Lai and colleagues found that the mathematics community agreed that the changes the professors made to the proofs should improve student understanding of those proofs. One goal of our study is to better understand how instructors modify textbook proofs in real analysis to make them clearer to undergraduate students. While similar to the work of Lai et al. (2012), our study has significant differences in environment and context.

For this portion of the study, four proofs were selected from undergraduate analysis textbooks. These proofs were then given to three mathematics faculty members at a given university in the Midwest, who were asked to modify the proofs over the course of a week to make them more understandable to students. These modified proofs were coded based on the codes developed in Lai et al. (2012). Then, each faculty participant engaged in a 90-minute interview. These interviews were recorded, transcribed, and coded inductively using an open coding method (Corbin & Strauss, 2014).

Thorough results will be discussed during the presentation. However, initial findings suggest that faculty participants chose to add significant portions to the proofs, often turning proofs that were originally 5 lines into a page-long proof. The most common addition to the proofs was to follow symbol-heavy sentences with a colloquial English explanation. It was also common to add sentences that draw the readers' attention to the goals of small parts of the proofs, such as explicitly stating that they were proving the injective or surjective part of a bijective theorem.

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IMPLICATIONS OF THE UNSTANDARDIZED NATURE OF STANDARDS-BASED GRADING IN MIDDLE SCHOOL MATHEMATICS CLASSROOMS

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Keywords: Instructional Activities and Practices, Standards

In recent years, there has been movement towards implementing an alternative to traditional grading practices called standards-based grading (e.g., Vatterott, 2015). Critics of traditional grading practices claimed that such practices focus too much attention on rote memorization and procedural skills (Gentile & Lalley, 2003). In contrast, advocates of standards-based grading argued argue that grades are determined "by the complexity of tasks and the level of mastery of higher-level thinking skills that students can attain" (Vatterott, 2015, p. 28).

Research Design

Currently, there is a lack of qualitative research focused on implementation of standards-based grading (Brodersen & Randel, 2017). Using a multicase study design (Stake, 2006), this research study sought to fill this gap by focusing on implementation of standards-based grading practices within middle school mathematics classrooms. Data collection consisted of a combination of interviews and classroom observations. The data were analyzed at the both the case and cross-case levels for themes related to the purpose of this study. Peer examination and member checking where among the strategies used to ensure increased trustworthiness and rigor.

Results and Discussion

Mx. Brown claimed they did not receive "much formal training" and, instead, graded by using their "mental calibration for assigning the scores.". Mx. Johnson "love[d]" standards-based grading. They believed that the practice helped them better target student strengths and weaknesses with respect to students' understanding. They had a general perception of what each score, on a four-point scale, meant; however, the requirements for a score depended on the specific competency they were measuring. Mx. Williams perceived the district rubric as a "loose" description of each level on a four-point scale and there was "no expectation" that teachers use the rubric when determining students' grades. According to Mx. Miller, the overarching purpose of standards-based grading was "to give more specific feedback to students and parents about specific skills on what, where students are proficient or not." When assigning standards-based grades, they considered the whole body of evidence related to a student's work focused on a single learning target based on a one-time-only summative assessment.

While all four teachers used a four-point grading scale to evaluate and grade students, the teachers differed on how they (a) defined and interpreted that grading scale, (b) supported and allowed students to improve their grade, and (c) communicated and calculated the students' grades. These differences greatly influenced to type of mathematical learning opportunities they provided to their students. Furthermore, the discrepancy in grading reduces the possibility of determining the mathematical understanding the students' hold. The evidence suggested the need for improved standards documentation, resource development, and professional development both at the preservice and inservice levels to better achieve the recommendations of the standards-based grading literature.

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MAKING 'TALK' A (NON)MATHEMATICAL ACTIVITY

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Utilizing classroom talk and discussion-intensive pedagogies are seen as essential to 'reform-oriented' mathematics instruction (Staples, 2007; Walshaw & Anthony, 2008). However, teachers often struggle to engage and support student participation while facilitating mathematical discussions (O'Connor et al., 2017), despite spontaneous inclinations of many students to casually converse with peers about non-mathematical topics. In this research project, we explore the complex intersection between mathematical talk and classroom norms. Specifically, we seek to understand the following research question – how do teacher messages about norms and behavior complicate student perceptions of talk as a mathematical activity?

We use the distinction between social and sociomathematical norms to animate our argument. While the former involve generally acceptable forms of activity in the classroom, the latter are more specific to mathematical aspects of student activity (Yackel and Cobb, 1996; Yackel et al., 1991). Yackel and Cobb note that students develop their personal perspectives based on the continuous negotiation of these norms with the teacher. Put differently, the kinds of actions permitted in the mathematics classroom influence what students consider as acceptable mathematical activity (Wagner & Herbel-Eisenmann, 2008).

Methodological Notes

We observed video recordings of over 20 mathematics lessons in Algebra I courses from a large high school in Northeastern United States as part of a larger project that investigated teachers' use of discussion in their mathematics teaching (Durkin et al., in press). For this study, we selected three classroom scenes to be presented as vignettes, having analyzed them using a line-by-line coding of their transcripts.

Findings and Implications

In our vignettes, teachers convey discipline-related social norms that prohibit talking in the classroom (Vignette 1), permit talking only if it is relevant to mathematical problem solving (Vignette 2), or permit talking only if supervised by the teacher (Vignette 3). At the same time, the students show a spontaneous tendency to socially interact with each. Though the students are encouraged to talk to each other due to the discussion-oriented nature of these classrooms, the 'talk' is permitted to be only about mathematical topics, and additionally, only under teacher supervision. These scenarios show how teachers create boundaries between mathematical and 'non-mathematical' talk - rendering the former as a special *kind* of talk that is unrelated to the latter. However, this distinction is complicated for students, and we speculate whether the fuzziness of these boundaries makes it difficult for students to switch between these two *kinds* of talk in various situations. As a result, students might possibly begin seeing their spontaneous inclination to converse as a non-mathematical activity – which might subsequently influence their motivations to instinctively participate in mathematical discussions. We call for further explorations of how teachers distinguish mathematical and non-mathematical talk, how students perceive social norms in talk-centered mathematics classrooms, and how the facilitation of mathematical discussions conflict with teacher beliefs about discipline and order.

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USING PROCESS MINING TO ANALYZE TEACHER-STUDENT INTERACTION

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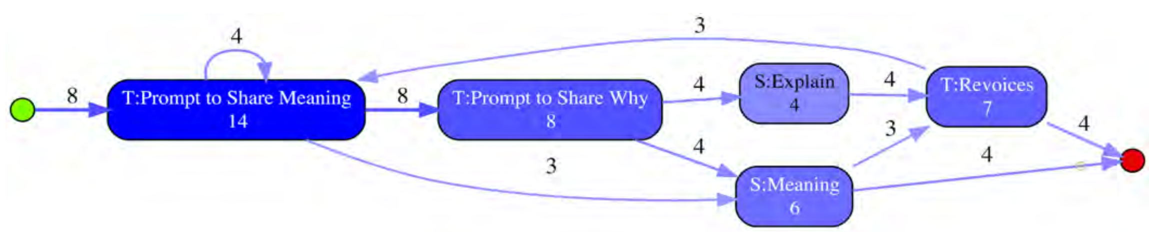
Prevalent mathematics classroom observation tools do not currently capture "interactive nature of leading discussions—the timing of teaching moves is not always considered, and teacher actions and student actions are often coded separately so that teaching moves are not always linked with specific student responses" (Jacobs & Spangler, 2017, p. 784). In this poster, we share some of our exploratory work adapting process mining (Van der Aalst, 2012) into the educational context to address this need.

Underlying our project is the assumption that the classroom is a social system where the learning of mathematics depends on interactions between teacher, students, and content. The data analyzed in this report stems from a larger project analyzing student-teacher activity in the mathematics classroom (Melhuish, et al., 2020). We analyzed lessons from 31 middle school teachers from a large, urban school district in the Southwest United States. Each lesson was coded for: *teacher moves* that can serve to engender students in rich mathematics and *student contributions* reflecting standards-based mathematical practices (National Governors Association Center for Best Practices) along with corresponding time stamps to create an *event log*.

Educational process mining (EPM) "uses log data gathered specifically from educational environments in order to discover, analyze, and provide a visual representation of the complete educational process" (Bogarín, et al., 2018, p.1). The techniques involve analyzing logs of events (*activities*, *timestamp,* and other information such as actor or resource) to capture the most frequent events and paths. See Figure 1 for an example of a process (along with frequencies) mined from our data.



In this poster, we will share several common processes and illustrate how this research methodology has the potential to provide unique insights into classroom discourse analysis by unearthing processes beyond the traditional Initiate-Respond-Evaluate/Feedback patterns (Cazden, 2001).

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IN-SERVICE TEACHERS' PERCEPTIONS ABOUT VIRTUAL COLLABORATION IN MATHEMATICS CLASSROOMS: CHALLENGES AND POSSIBILITIES

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K-12 instruction took a drastic turn during the global pandemic in early 2020. As the pandemic continued, questions arose about the affordances and constraints of virtual mathematics instruction (Carius, 2020; Iuvinale, 2020; Khirwadkar et al., 2020; Mulenga & Marbán, 2020; Reimers et al., 2020) on how to provide continued instruction and maintain best teaching practices in a virtual environment. We aimed to learn about mathematics teachers' experiences planning for, designing and implementing lesson plans to allow student collaboration. Our research question is: *How can in-service mathematics teachers be supported to foster virtual collaboration in their mathematics classrooms?*

To answer the research question, we used a teacher engagement, challenge, & opportunities for learning framework (Rahman, 2018). The framework allows for analysis of teachers' learning opportunities emerging from challenges they face while planning, implementing, collaborating and reflecting about their teaching practice. In this study, two middle school and one high school teacher from a professional learning community worked together on a Desmos activity focusing on transformations and then prepared a shared document that explained their thinking. Data consisted of (a) video recording(s) of the teachers' engagement with online mathematics collaboration and their reflections on their experience especially describing the challenges they would face, (b) their lesson plans for implementing this activity, and (c) one teacher's reflection on actual implementation of their lesson. We used open coding (Strauss & Corbin, 1998) to analyze the data to learn about the teachers' experience.

The teachers first developed group norms and communicated that these norms were different for them (as compared to students), because they were adults; they had their cameras on and so used hand gestures to communicate. During the task itself, the high school teacher took the lead in explaining their understanding of the task and all three teachers worked together in developing the final document. In terms of connecting their collaborative experience to their own classrooms and their students, the teachers expressed several possibilities as well as perceived challenges: the benefits of a collaborative space to encourage student communication, the importance of helping students develop group norms or assigning roles to help them collaborate, the ability to monitor student work simultaneously, the challenge of not being able to send students to breakout rooms due to district policies, providing explicit directions to the students, and assisting students with technology tasks like taking a screenshot.

A real challenge emerged when one teacher designed and implemented a lesson in their classroom. The students did not have their cameras on (as per district policies teachers are not allowed to ask the students to turn their cameras on) and the teacher could not see student reactions and get a sense of any challenges they were experiencing. Findings show there are possible benefits for virtual collaboration as implemented in K-12 classrooms. More research is needed to learn about perceived and experienced teacher challenges to design effective tasks.

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INTERACTIONS IN BLENDED MATHEMATICAL LEARNING ENVIRONMENTS

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In examining interactions in an online environment, we utilize Moore's (1989) framework in which he outlined three types of interactions. In his work, he purports that to determine appropriate levels of interactions for effective learning, it is first important to be able to classify the types of interactions taking place in distance learning environments. He identified three interaction types: learner-content, learner-instructor, and learner-learner. We add to that, the work of Hillman et al. (1994), who added a fourth interaction, learner-interface.

We received a single video from each of 19 high school mathematics teachers with between six and 34 years of experience, representing seven different school districts. Three levels of coding were applied to the analysis of each video. First, the type of interaction was identified (learner-content, learner-instructor, learner-learner, learner-interface). Then the activity (level 2) within that interaction was coded. Finally, it was noted if technology was used during the activity. The researchers, who are the authors of this paper, met to view a video together, clarify the unit of analysis, and define each of the codes. A unit of analysis was considered a level 2 interaction type. When there was a change in the activity type, that was treated as a new unit to code. A video was coded together and then six videos were assigned to pairs of researchers to determine inter-rater reliability. Agreement on units of analysis was determined (87.1%) and agreement on codes on the units that were in common was calculated (86.7%). Once inter-rater reliability was established, the remainder of the videos were coded by one researcher.

In their online teaching, mathematics teachers are overwhelmingly utilizing learner-instructor interactions (75.97%), which are predominately expository. Of the learning-instructor interactions, 52.14% are expository and 29.91% are explanatory. Overall, 23.38% of the interactions were learners engaging with mathematics content. The majority of the learner-content interactions were students solving a problem (58.33% of learner-content interactions). Instances of problem-solving were observed across all learning types: hybrid classes (n=5), pre-recorded videos (n=10), and synchronous classes (n=6). In addition, we found evidence that teachers provided a few opportunities for their students to conjecture (n=8), confirm their answers (n=4), predict (n=2), and interpret their solutions (n=1). We found no learner-learner interactions of any type across the videos of our 19 participants.

The teachers were facilitating high school mathematics classes across different modalities of online teaching. Seven of the 19 videos were prerecorded (36%) included students in a hybrid setting or synchronous classroom. While there was a high percentage of learner-instructor interactions there were no learner-to-learner interactions observed across all videos. In hybrid and synchronous settings where learners were present with the instructor, typical interaction occurred between the teacher and the students. Could this be because of Covid-19 which requires social distancing among in-person students? Although, we noticed it also when students were remote. We may want to provide additional professional development to teachers, particularly those teaching online, about ways they can engage students in conversations with each other to support mathematics learning.

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SUPPORTING THE CONSTRUCTION OF VARIABLES IN AN INVERSE FUNCTION CONTEXT THROUGH TARGETED QUESTIONS

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While international and U.S. mathematics standards expect secondary students to develop connected meanings of inverse functions (Bergeron & Alcantara, 2015), previous research has found post-secondary students, pre-service teachers, and practicing teachers often struggle to construct these productive meanings (Brown & Reynolds, 2007; Engelke, Oehrtman, & Carlson, 2005; Even, 1992; Lucus, 2005; Paoletti, et. al, 2017; Vidakovic, 1996). In three implementations of a problem-based lesson on inverse functions, we observed the teacher place value on asking students to consider "any value" when learning about inverse functions. After previously solving for a team's fee based on its number of players, students are asked to find the number of players on a team based on its fee. While the students are given a table of values to work with, the teacher asks them to consider "any receipt," "because there might be more receipts than [they] … have" in the table. The teacher continued to ask for a method that works for "any value" throughout the lesson. We connected this idea to the notion of conceptualizing quantities as variables (e.g., Thompson & Carlson, 2017). We present what we have learned about how teachers can support the construction of variables in an inverse function context.

We designed the problem-based lesson in collaboration with a teacher from a large midwestern public high school (Stevens et al., 2020). We then recorded and qualitatively coded three lesson implementations (Corbin & Strauss, 2008). We coded for inverse function conceptions based on the findings of Stevens et al. (2020), and problem statements and questions based on the teacher moves identified by Milewski and Strickland (2020) and our own observations. Next, we created a table that matches the questions and restatements of the problem launch used to get students to consider "any value" with the inverse conceptions expressed by the student the teacher is speaking with. From the results of this table, we believe the teacher selectively asked these questions to students he viewed as holding inadequately generalizable inverse conceptions. We see this as a potential strategy for supporting students to construct the conception of variables in an inverse function context. To learn how teachers view this way of launching an exploration of inverse function, we designed three ways of launching the problem (solving for an unknown value, generalized number, variable). We then asked eight teachers to discuss the alternative launches and to depict the remainder of classroom discussions started with each alternative. Teachers preferred the "any value" generalizable number framing employed by the lesson we studied, citing, for example, the benefit of allowing students "to experiment with concrete scenarios before … discussion of the abstract" concept of inverse.

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MATHEMATICS VOCABULARY PRACTICES IN EARLY CHILDHOOD CLASSROOMS

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Mathematics vocabulary plays a key role in the learning and understanding of mathematics concepts and is a critical component of communicating mathematical thinking (Monroe & Panchyshyn, 1995; National Council of Teachers of Mathematics [NCTM], 2000; Olander & Ehmer, 1971). Learning mathematics vocabulary, however, can be challenging. In addition to technical vocabulary with precise meanings (e.g., polynomial), mathematics consists of words from everyday language that have different meanings in mathematical contexts (e.g., plane), as well as symbols (e.g., +) (Capps & Pickreign, 1993; Raiker, 2002; Schell 1982). The majority of research studies on mathematics vocabulary instruction focused on the teaching of general academic vocabulary in the upper grades (Fisher, Blachowicz, & Watts-Taffe, 2011; Scott, Jamieson-Noel, & Asselin, 2003). This study investigated the extent to which mathematics vocabulary instruction occurred in early childhood (Grades K-2) classrooms and the instructional practices teachers utilized. The following research questions were examined: 1) *How is mathematics vocabulary taught to young children during mathematics instruction?* 2) *How often does mathematics vocabulary instruction occur and how many words are taught?*

An observational research design was used to investigate how three teachers from an urban independent school in the northeastern United States taught mathematics vocabulary to their students during mathematics lessons in a summer school program. Each teacher was observed during seven mathematics lessons for the entirety of each lesson, for a total of 21 observations. The lessons were audio recorded, and for each, I also completed an observation protocol, which was utilized to take field notes and dictation of the teachers' vocabulary instruction. Transcripts of the audio recordings as well as the observation protocols were analyzed using a general inductive approach (Cresswell, 2012; Thomas, 2003).

The findings of this study revealed that the three teachers taught 486 mathematics vocabulary words during their mathematics instruction with students during 1,254 episodes of vocabulary instruction. This indicates that some words were taught repeatedly, providing students with multiple exposures to a number of the words. Across the episodes of instruction, the teachers employed three categories of instructional practices, which were explanations, visuals, and demonstrations, which represent forms of direct instruction methods advocated for the teaching of academic vocabulary, including mathematics vocabulary (Flevares & Perry, 2001; Good & Grouws, 1979; Stahl & Fairbanks, 1986; van Oers, 2013). All three teachers employed multiple instructional practices during some episodes of vocabulary instruction.

Absent from the teachers' repertoire of instructional practices were strategies particularly supported for young children's conceptual understanding and building mathematics language, such as utilizing rich discussions and children's literature (Hong, 1996; Schiro, 1996). Teachers can purposefully plan discussions for particular mathematics concepts and the related vocabulary, providing students with explicit vocabulary instruction as they explain the meanings of targeted words, and encourage the use of precise terminology in interactive and meaningful contexts (Carrison & Muir, 2013; Hassinger-Das, Jordan, & Dyson, 2015).

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TEACHER CANDIDATES' TEACHING FOR CONCEPTUAL UNDERSTANDING THROUGH THE LENS OF REPRESENTATIONS

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Keywords: Mathematical Representations; Preservice Teacher Education; Assessment

It has been agreed that conceptual understanding is a key tenet of current mathematics education reform efforts for both prospective and practicing teachers (AMTE, 2017; National Council of Teachers of Mathematics [NCTM], 2014). Utilizing multiple representations within instruction can help develop rich mathematical understandings and flexibility with mathematical structures (Dreher et al., 2016; Stylianou & Silver, 2004). While these practices are certainly supported by recent reform efforts (AMTE, 2017; NCTM, 2000, 2014), the enactment of teaching for conceptual understanding and for flexibility when dealing with representations is not an easy feat, especially for novice teachers (Eisenhart et al., 1993; Lampert et al., 2013).

The purpose of this embedded multiple case study was to investigate the following research question: During mathematics instruction, how do teacher candidates (TCs) implement and reflect on the inclusion of multiple representations in ways that support conceptual understanding? This study's conceptual framework was situated around qualities of teaching mathematics for conceptual understanding (Hiebert & Grouws, 2007; Jansen et al., 2017). The TCs in this study were selected from a stratified random sample of completed edTPA portfolios of recent graduates from elementary and secondary mathematics teacher preparation programs (TPPs) at a large university in the southeastern United States. These TPPs maintain preparing teachers to teach mathematics for conceptual understanding as a key tenet of their program design and delivery.

Document analysis was completed on edTPA artifacts, including written commentary responses, lesson plans, instructional materials, student work samples, and transcripts of video recorded teaching segments. The commentary responses provide evidence regarding TCs' reflection of representation inclusion and conceptual understanding. Through the lens of the study's conceptual framework, data were analyzed utilizing *a priori* coding, along with open coding for recurrent themes. Codes such as explicit connections between symbolic and visual representations, student to student talk, and probing questions were included within the *a priori* codes.

Data analysis suggest that TCs are planning for the inclusion of multiple representations in ways that support conceptual understanding, such as questioning and explicit connections; however, these are not always realized during lesson implementation. Additionally, assessments captured in the edTPA artifacts include opportunities for students to engage with multiple representations and explanations of solution strategies. Cross-case analysis provides evidence to understand how the elementary and secondary cases incorporate representations in ways that support conceptual development. Data support that TCs were able to harness qualities of teaching mathematics for conceptual understanding during a snapshot of their student teaching placement. This study adds to the scholarship around how practices learned within TPPs are realized within TCs' classrooms during student teaching. Much of the current literature is situated around a single grade band (e.g., Jansen et al., 2017; Yang, 2012), thus examining any relationships between TCs from various grade level programs has the potential to inform TPPs' practices.

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A TEACHER'S STRUGGLES TO TEACH CONDITIONAL PROBABILITY FOR UNDERSTANDING: THE CASE OF MR. KANTOR

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Teachers' knowledge is considered as one of the most important factors that influences teachers' instructional decisions and their long-life journey of learning to teach (Da Ponte & Chapman, 2006; Franke, Kazemi & Battey, 2007).

In a previous research study, Borko et al. (1992) examined from several perspectives a classroom lesson in which a student teacher failed to provide a conceptual explanation for the standard division-of-fractions algorithm. While the student teacher believed that good teaching involves making mathematical meaningful to students by explaining the reasons behind the procedures, the researchers concluded that those beliefs were difficult to achieve in practice because the student teacher lacked strong content and pedagogical content knowledge about division of fractions.

In the present study, I examined the pedagogical events (explanations, representations, and questions) that a middle school mathematics teacher, Mr. Kantor, provided when teaching the concept of conditional probability to eight graders. The main sources of data were videotaped lessons supplemented with questionnaires, open and semistructured interviews, and written documents.

The analysis of the teaching episode revealed that Mr. Kantor struggled to help his students understand why $P(A \cap B) = P(A) \times P(B \text{ given } A)$. However, unlike the student teacher in Borko et al.'s study, Mr. Kantor's difficulties were unrelated to his mathematical and pedagogical content knowledge. His explanation for the topic of conditional probability was conceptually based, indicating that his content knowledge was strong. This finding suggests that other factors, such as the intrinsic difficulty of a topic and students' cognitions about the topic, play an important role in the success of teachers' explanations.

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¿CÓMO JUSTIFICAN ESTUDIANTES DE BACHILLERATO LA PROBABILIDAD DE UN EVENTO EN UNA ACTIVIDAD DE MODELIZACIÓN?

HOW DO HIGH SCHOOL STUDENTS JUSTIFY PROBABILITY OF AN EVENT IN A MODELING ACTIVITY?

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Este trabajo es un primer acercamiento para responder la pregunta: ¿Cómo justifican estudiantes de bachillerato la probabilidad de un evento en una actividad de modelización? Observamos que la modelización matemática en educación ha ganado relevancia a nivel internacional y que se ha enfocado en áreas como cálculo, geometría, estadística y álgebra, con poca atención a otras áreas tales como probabilidad (Preciado Babb et al., en prensa).

Aquí presentamos algunos resultados preliminares del análisis de las justificaciones que dieron estudiantes de bachillerato sobre el resultado más probable de la suma de los puntos que se obtendría al lanzar dos dados en el contexto de modelización matemática.

Este trabajo se enmarca en la perspectiva *educativa* de la modelización matemática (Kaiser & Sriraman, 2006), en la cual el objetivo es enseñar un contenido específico, en este caso una introducción a la probabilidad con el uso de un simulador (Batanero, 2003, Koparan, 2021). Blum y Leiß (2006) describieron la modelización mediante un ciclo en el que se parte de un fenómeno para construir un modelo matemático que permita su estudio. Un paso importante en el ciclo es la validación del modelo, incluyendo modificaciones si fueran necesarias.

En este estudio participaron cuatro grupos de 50 estudiantes cada uno. Los alumnos usaron un simulador para obtener resultados del lanzamiento de dados en muestras de al menos 30 lanzamientos. Después de representar sus resultados con gráficas de frecuencia, se les pidió a los estudiantes que indicaran cuál sería el resultado más probable y que justificaran su respuesta. Un diagnóstico previo a la actividad mostró que solo un 8% de los estudiantes tenía conocimiento de la actividad y sus posibles resultados. Los datos se recabaron de forma individual por medio de un formulario de Google y a través de un archivo que enviaron de su trabajo por equipo.

Al revisar las justificaciones de los estudiantes notamos que, si bien usaron distintas representaciones, estas se podían clasificar en argumentos teóricos o experimentales (Gómez et al., 2013). Es importante mencionar que, hasta el momento de esta actividad, no se había abordado en la clase la probabilidad teórica.

Los resultados indicaron que la mayoría de los estudiantes (más del 60%) respondió utilizando cálculos de probabilidad teórica, sin considerar los resultados del simulador, aproximadamente el 11% tiró los dados varias veces o revisó los resultados del equipo, casi 20% contestó conforme al número de tiradas (25, 100 y 1500) en el juego y un 3% decidió continuar usando el simulador hasta obtener una muestra más grande.

El hecho de que la mayoría de los estudiantes justificaron sus respuestas con argumentos teóricos sugiere que le asignan mayor importancia al modelo matemático sin considerar la validación del este, como sugiere el ciclo de modelización propuesto por Blum y Leiß (2006).

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HOW DO HIGH SCHOOL STUDENTS JUSTIFY PROBABILITY OF AN EVENT IN A MODELING ACTIVITY?

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This study is an initial approach to answer the question: How do high school students justify the probability of an event in a modelling activity? We have noticed that mathematical modelling in education has increased relevance internationally and has focused subjects such as calculus, geometry, statistics and algebra, with less attention to other subjects such as probability (Preciado Babb et al., in press).

Here, we present preliminary results from the analysis of high school students' justifications on the most probable outcome for the sum of the numbers obtained by throwing two dice in the context of mathematical modelling.

This work is framed within the *educational* perspective on mathematical modelling (Kaiser & Sriraman, 2006), in which the goal is to teach a specific content, in this case an introduction to probability with the use of a simulator (Batanero, 2003, Koparan, 2021). Blum y Leiß (2006) described modelling through a cycle that begins with a phenomenon to build a mathematical model for its study. An important step in the cycle is the validation of the model, including modifications, if they were required.

Four groups of 50 students, each, participated in this study. Students used a simulator to obtain the results of throwing the dice with a sample of 30 throws. After representing the results in a frequency graph, students were asked to indicate which would be most probable outcome, justifying their answers. A prior diagnostic to this task showed that only 8% of the students had prior experience with this activity and its possible outcomes. Data were collected individually using a Google form and through the files with the work they did as a team.

After reviewing students' justifications, we noticed that while they used different representations, their justifications could be classified in theoretical and empirical arguments (Gómez et al., 2013). It is worth mentioning that by the time students engaged in the task, theoretical probability was not introduced yet to the course.

Results showed that most of the students (more tan 60%) answered using calculations for the theoretical probability, without considering the results from the simulator, 11% of the students, approximately, throwed the dice several times or reviewed the results of the team, almost 20% answered based on the number of throws (25, 100 and 1500) in the game and 3% decided to continue using the simulator until they obtained a larger sample.

The fact that most of the students justified their answers with theoretical arguments suggests that they gave more relevance to the mathematical model, without considering its validation, as suggested by the modelling cycle proposed by Blum y Leiß (2006).

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"WHAT GOT FLIPPED?": A TEACHER'S USE OF CONTRASTING CONCEPTIONS TO SUPPORT STUDENTS' DEVELOPMENT OF INVERSE FUNCTIONS

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This study builds on literature on teacher noticing (e.g., Sherin et al., 2011) and management of the complexity in teaching (e.g., Lampert, 2001). In attending to the complexity of a lesson, teachers may elect not to attend to different conceptions among students. Yet, doing so can be instrumental to achieving instructional goals that might be more difficult to reach by following on single approaches. In this study, we focus on how a teacher might support students with differing conceptions of inverse function in discussing ideas central to an instructional goal.

We started by identifying different student conceptions These conceptions were created by considering prior research on students' conceptions of inverse (e.g., Paoletti, 2020; Paoletti et al., 2017; Stevens et al., 2020; Teuscher et al., 2018; Vidakovic, 1996) and then refining descriptions by representation by analyzing student work (Strauss & Corbin, 1998). In this report, we focus on two meanings students could have when representing an inverse function on a graph: switch-graph and undo-graph. In switch-graph, students switch the independent and dependent variables while keeping the same variable labels; they anticipate that one graph can represent both directions of a relationship. In undo-graph, students think about an undoing process; after representing one direction of the relationship with a graph, they create a new graph to represent the inverse relationship by switching the coordinate values and their corresponding labels.

We used these conceptions to code video of a teacher using a contextualized problem-based lesson in a high school algebra class in the midwestern U.S. In this poster, we focus on one instance in which the teacher was talking with two students who were working on the graph of the inverse relationship during small group time. We identified these students as having two different conceptions for inverse- an undo-graph and a switch-graph. Rather than telling the students how to reconcile their differences in order to produce a graph with conventional inverse notation, the teacher asked the following questions: "Why are they different?", "Which one do we want or does it not matter?", and, "So what got flipped?". In asking these questions and listening to the students' responses, the students had the opportunity to exchange around the different ways they were representing their inverse relationship, explicitly discussing matters of the labeling, independent/dependent quantities, and axes orientation in their representations.

We conclude by noting that the features that the teacher decided to have the two students attend to were directly related to topics that moved the students closer to the instructional goal of the lesson. For example, the focus was on variable roles rather than discrepancies in scale on axes. We argue that this illustration of a teachers' interaction with students with different conceptions opens up a way for us as researchers to attend more to the conception-specific ways in which teachers can promote productive ways for students with differing conceptions to learn and build their own mathematical understandings.

Acknowledgements

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REFLEXIONES ACERCA DE LAS IDEAS MATEMÁTICAS EN EL PROCESO DE DISEÑO

REFLECTIONS ON THE MATHEMATICAL IDEAS IN THE DESIGN PROCESS

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Palabras clave: Educación primaria, Experimentos de diseño, STEM/STEAM integrado, Conocimiento del profesorado

Perspectiva teórica

La educación integrada STEM postula que al abordar las disciplinas STEM de una manera conectada pueden hacer que las materias sean más relevantes para los estudiantes y profesores, y se pueda aumentar el interés y los logros en el aprendizaje (National Rearch Council, 2014). Por otro lado se sustenta que el proceso de diseño puede ser una actividad central al implementar actividades STEM debido a que "el proceso de dise mplia y refuerza la forma de pensar en los problemas y ofrece herramientas que pueden ayudar a los estudiantes a ampliar de forma creativa su pensamiento" (NCTM y NCTM, s/f, p. 3).

Para esta investigación se llevó a cabo un taller de diseño con estudiantes para profesor de nivel primaria. Se adoptaron algunas actividades que sugieren Benenson y Neujahr (2002), otras se diseñaron en función del interés de la investigación. Para organizar y analizar los resultados de las actividades de diseño y responder a las preguntas de investigación, se retoman las categorías que integran la estructura de la reflexión en acción (Schön, 1983; 1987).

El diseño y las preguntas de investigación

El diseño de la investigación fue de corte cualitativo, con enfoque fenomenológico. El propósito de la investigación fue distinguir la forma en la que los profesores resolvían los problemas que les planteaba el diseño y las ideas matemáticas que ponían en obra. Las preguntas fueron ¿Cómo resuelven los profesores el problema de diseño de objetos del entorno en un escenario STEM? ¿Qué tipo de reflexiones tienen lugar en los profesores acerca de los conceptos matemáticos que utilizan para resolver el problema de diseño de objetos de su entorno?

Técnicas de recolección y análisis de datos. En la fase de diseño de las actividades del taller se crearon formatos para recuperar información, los formatos incluían cuadros de doble entrada, preguntas, espacio para hacer operaciones, etc. También se usaron técnicas cualitativas: diario de campo, observación participante, charlas informales. Las herramientas con las que se hizo el análisis de la información fueron el registro ampliado (Bertely, 2000), la construcción de categorías y la triangulación (Woods, 1978), la escritura de memos (Corbin y Strauss, 2015).

Resumen de los hallazgos. Se encontró que al enfrentar el diseño de un objeto, los profesores recurrieron al repertorio de conocimientos que poseen, no sólo ideas matemáticas, lo hicieron en un ambiente de colaboración, de discusión de sus ideas y de negociación para seleccionar la propuesta de diseño a cumplir en el prototipo. Las ideas matemáticas que utilizaron inicialmente para plantear un cambio en el diseño estaban asociadas a las características físicas, se referían a un cambio en el ta "al doble" "aumentar los espacios" " ande", pero en las siguientes fases del diseño se involucraban en reflexiones sobre conceptos y operaciones abstractas: proporcionalidad, regla de tres, escala 1:2, si $\frac{36.5}{x} = \frac{1}{2}$.

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REFLECTIONS ON THE MATHEMATICAL IDEAS IN THE DESIGN PROCESS

REFLEXIONES ACERCA DE LAS IDEAS MATEMÁTICAS EN EL PROCESO DE DISEÑO

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Theoretical perspective

Integrated STEM education postulates that approaching STEM disciplines in a connected way can make the subjects more relevant to students and teachers, and can increase interest and learning achievement (National Research Council, 2014). On the other hand, it is argued that the design process can be a central activity when implementing STEM activities because "the design process broadens and strengthens problem thinking and provides tools that can help students creatively extend their thinking" (NCTM and NCTM, n/d, p. 3).

For this research, a design workshop was conducted with students for elementary level teacher. Some activities suggested by Benenson and Neujahr (2002) were adopted, others were designed according to the research interest. To organize and analyze the results of the design activities and answer the research questions, the categories that integrate the structure of reflection in action (Schön, 1983; 1987) are taken up.

The research questions and design

The research design was qualitative, with a phenomenological approach. The purpose of the research was to distinguish the way in which teachers solved the problems posed by the design and the mathematical ideas they put into action. The questions were How do teachers solve the problem of designing objects of the environment in a STEM setting? What kind of reflections take place in teachers about the mathematical concepts they use to solve the problem of designing objects of their environment?

Data collections techniques and analysis. In the design phase of the workshop activities, formats were created to retrieve information; the formats included double-entry boxes, questions, space to do operations, etc. Qualitative techniques were also used: field diary, participant observation, informal talks. The tools with which the analysis of the information was done were the extended register (Bertely, 2000), the construction of categories and triangulation (Woods, 1978), the writing of memos (Corbin and Strauss, 2015).

Summary of findings. It was found that when facing the design of an object, teachers resorted to the repertoire of knowledge they possess, not only mathematical ideas, they did so in an environment of collaboration, discussion of their ideas and negotiation to select the design proposal to fulfill in the prototype. The mathematical ideas they initially used to propose a change in the design were associated with physical characteristics, they referred to a change in size: "double" "increase the spaces" "bigger", but in the following phases of the design they were involved in reflections on abstract concepts and operations: proportionality, rule of three, scale 1:2, if 36.5/x=1/2 .

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Chapter 13:

Technology

PRESERVICE SECONDARY TEACHERS' REASONING ABOUT STATIC AND DYNAMIC REPRESENTATIONS OF FUNCTION

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This study aims to describe how preservice secondary mathematics teachers (PSMTs) reason about different function representations. The study focuses on two PSMTs' reasonings across static and dynamic representations of functions. Sfard's (2008) Theory of Commognition guided our analysis. Findings indicate that while static representations restrict attention given to covariation, dynamic representations support PSMTs' reasoning about covariation including making connections to how covariation is represented in static graphs.

Keywords: Mathematical representations, algebra and algebraic thinking, technology

The concept of function permeates all levels of mathematics and is a large focus of the high school curriculum. Central to the treatment of functions in high school is attention to characteristics of families of functions given their usefulness for mathematical modeling. This attention means that significant emphasis is placed on graphical representations of functions (i.e., static graphs on a Cartesian plane). Research has shown that when analyzing graphical representations of functions, students and teachers alike often attend to perceptual cues rather than the relationships between the variables the perceptual cues are representing (e.g., Moore & Thompson, 2015; Sinclair et al., 2009). The coordination of two quantities represented in a graph of a function and the ways they change in relation to each other is called *covariational reasoning* and has been identified as foundational for mathematical modeling as well as many calculus concepts (e.g., Carlson et al., 2002). Given the role that functions play in the high school curriculum, it is essential that preservice secondary mathematics teachers (PSMTs) develop covariational reasoning skills. Carlson et al. pointed to the potential of dynamic technologies to support those learning to apply covariational reasoning. Recent work with a particular dynamic representation of functions in one dimension, the dynagraph (Goldenberg et al., 1992), has pointed to its potential to elicit student reasoning about the ways in which independent and dependent variables vary and covary (e.g., Antonini et al., 2020; Sinclair et al., 2009). To that end, the purpose of this study was to examine the similarities and differences in the ways PSMTs reasoned about different representations of functions–static and dynamic.

Background Literature

Reasoning about Static Graphs

Static representations of functions include tables, lists of ordered pairs, equations, and graphs. There is evidence that students' limited experience with graphical representations constrains them from making meaningful connections among algebraic and graphical representations (Knuth, 2000). Static graphical representations also conceal the dynamic aspects of function, such as the rate of change and relative position, that are essential for forming a robust understanding of function (e.g., Antonini et al., 2020; Carlson, 1998; Confrey & Smith, 1995; Ng, 2016). Research has shown that when reasoning about static graphs, it is not unusual to pay attention to shape and perceptual cues rather than the ways the graph represents how the variables change together (e.g., Moore & Thompson, 2015; Oehrtman et al., 2008; Weber, 2012). Moore and Thompson (2015) have also shown that it is not uncommon for a graph to be interpreted as the function itself, rather than a representation of the function. They distinguished *static shape thinking* from *emergent shape thinking* and described static shape thinking as considering a graph as an object, reacting to perceptual cues and the perceived shape of the graph rather than perceiving a graph as a trace and a representation of covarying quantities. Students with emergent shape thinking interpret features of the graph as properties of covariation. Thompson and Carlson (2017) noted that covariational reasoning happens "most strongly when a person is strategizing how to keep track of quantities' values simultaneously" (p. 438). Thus, dynamic representations where one can act on one quantity and track the concurrent relationship with another quantity might support the development of covariational reasoning skills.

Reasoning about Dynamic Representations of Functions

Digital technology provides many affordances related to the ways in which one can interact with functions and their graphs (Drijvers, 2015). Such technologies allow for the use of tables, expressions, and graphs to be dynamically linked to animated motion (e.g., Johnson et al., 2020; Kaput Center, 2016). There is evidence that engaging in activities of these types can support the development of reasoning about varying quantities and the ways in which they are represented in graphs (e.g., Johnson et al., 2020). Another way of leveraging the dynamic affordances of digital technologies to represent function is to use parallel axes rather than perpendicular axes, typically referred to as a dynagraph. Goldenberg et al. (1992) introduced dynagraphs to draw attention to dynamic function behavior and to help students focus on the function by eliminating complex information shown in a Cartesian graph. Students can test their conjectures of the relationship between input and output by dragging the input and observing the resulting change in the output. With the movement of this interactive representation, identification of invariants and covariation become central to one's exploration. Antonini et al. (2020) described dynagraphs as dynamic interactive mediators because students can engage in discourse with a dynagraph, "asking" questions of the tool and engaging to receive an "answer". Research has shown that use of dynagraphs can support the teaching of functions (e.g., Antonini et al., 2020) and can foster covariational reasoning by eliciting attention to movement, time, and space (Lisarelli, 2017).

There is little research comparing student reasoning when engaging with static and dynamic representations of function (exceptions include Antonini et al., 2020; Ng, 2016, and Sinclair et al., 2009). Given the potential of dynagraphs to support the development of reasoning about variables both separately and together, it is of interest to compare the ways in which one would reason about static and dynamic representations of the same functions. Our aim is to describe how PSMTs reason about functions with different visual mediators (i.e., static and dynamic).

Theoretical Framework

Sfard's (2008) Theory of Commognition unites cognitive and communicational processes to explain student thinking as "an individualized version of interpersonal communication" (p. 81). For Sfard, communication is a back and forth (action/reaction) that includes all communication, including with oneself. From this perspective, Antonini et al. (2020) explained "doing mathematics means engaging in the type of communication defined as mathematics and learning means becoming able to access and express this discourse" (p. 4). So, to study students' learning, we must attend to the words, visual mediators, narratives, and routines that form their discourse.

Mathematical discourse is characterized by specific words that are used by experts in specific ways. In the context of this study, that might include words like function, quadratic, increasing, rate of change, domain, or it might include informal language that is clear enough for an expert to understand the mathematics one is referring to. Visual mediators are objects that can be seen and operated on in the communication process. In mathematics, visual mediators can include, but are not limited to, symbols and graphs. Visual mediators of this type are static in that they can be seen and operated on, but not interacted with. In contrast, Antonini et al. (2020) introduced a dynamic interactive visual mediator (referred to as a DIM) described as a mediator that is "both *dynamic* - they change over time - and *interactive* - they respond to a person's manipulations" (p. 5). A dynagraph is an example of a DIM. In this study, we attended to the ways students communicate with and about different types of visual mediators (i.e., static vs. DIM).

Context of the Study

Since our goal was to compare and contrast the ways in which PSMTs reasoned about static graphical representations of functions and dynamic interactive representations of functions, asking students to compare and contrast within each representation type met our needs. To this end, we decided to use the instructional routine of Which One Doesn't Belong (WODB) (Danielson, 2016). A typical WODB task includes four objects and students are simply asked, which one doesn't belong? The main characteristic of a WODB task is that all of the options within the task can be considered correct which shifts students' focus away from trying to obtain the "correct" answer to distinguishing attributes of the objects presented in each option. This study used two WODB tasks, one with four static graphs of functions – each from a different function family (referred to going forward as the static task) and one with four dynagraphs of the same four functions (going forward referred to as the dynamic task).

In the static task, PSMTs considered four graphs (Figure 1) of functions and were asked to decide WODB and why. Once the PSMTs explained their choice, they were then asked to explain why someone else might argue that each of the other remaining graphs does not belong.

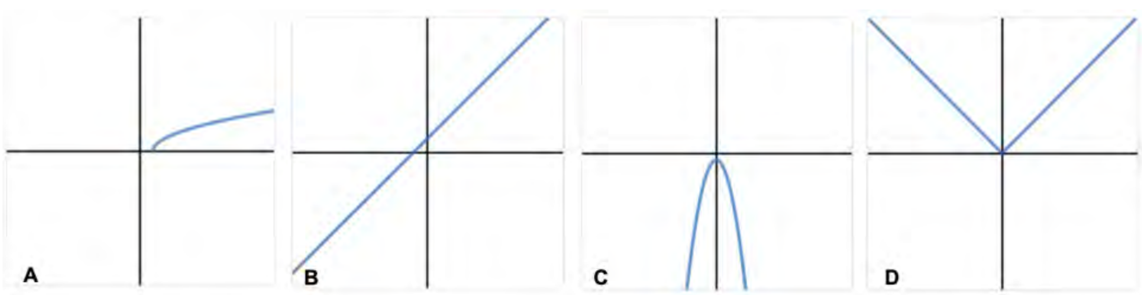

Figure 1: The static WODB task

In the dynamic task, the same four functions were presented (in a different order) but this time represented using dynagraphs (Note: PSMTs were not told they were the same). As a reminder, a dynagraph consists of two parallel number lines (function input on one, output on the other), and as the input is dragged, corresponding changes to the output will result (Figure 2). Just like the static task, PSMTs were asked to decide and explain WODB. Then they were asked why someone else might argue that each of the other remaining dynagraphs does not belong.

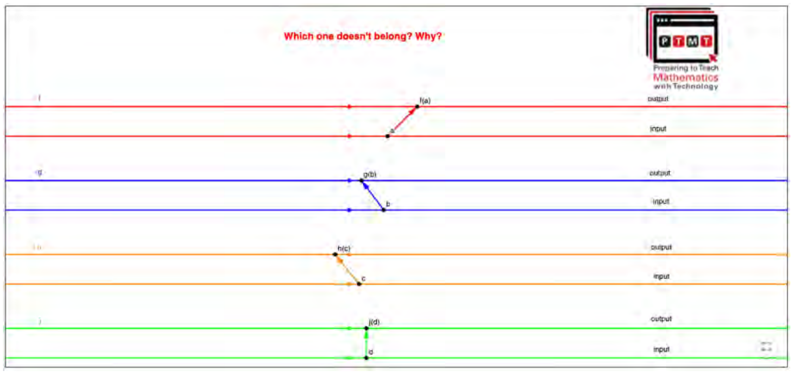

Figure 2: The dynamic WODB task–linear, square root, quadratic, and absolute value functions from top to bottom (https://www.geogebra.org/m/wjecnfev)

Methodology

This study was situated within the context of a larger study investigating how PSMTs reasoned across static and dynamic representations of function. Here we used a multiple case study design (Yin, 2017) to explore two cases, where each case was defined by the type of visual mediator (i.e., static and DIM) with which the students interacted. Our overarching research question was: What is the nature of students' discourse about function as they interact with different visual mediators (i.e., static and dynamic interactive mediators)?

The full study included 25 PSMTs attending six universities. Here we focus on two female participants; neither had experience with dynagraphs before. They were secondary mathematics education majors attending different universities. Both participants were enrolled in a math methods course, prior to student teaching, at the time of the study.

Video screen capture recordings of semi-structured interviews (Goldin, 2000) served as the main data source. One interview posed the static visual mediator first, and the second began with the DIM. Interviews were transcribed verbatim and uploaded in Atlas.ti to assist with coding.

Similar to Antonini et al. (2020), we used Sfard's (2008) Theory of Commognition to guide our analysis. We attended to words, discourses, and narratives to code the PSMTs' comparing and contrasting of the mediators presented in the tasks. In our consideration of the scholarly mathematical discourse (words), we were specifically interested in the characteristics of function that were elicited. We read the transcripts for the specific characteristics of function being described (e.g., domain, range, increasing, decreasing, maximum), created quotations for each chunk of transcript referring to a specific characteristic, and applied labels. A full list of these characteristics is included in the findings.

Next, we coded for the discourse about and with each mediator. To do so, all team members watched each video and reviewed the transcript to become familiar with each participant's discourse. This was followed by full team meetings to develop short codes describing the discourse using the protocol set forth by DeCuir-Gunby et al. (2011). We compared the data and emerging codes using a constant comparative method to create our final version of the codebook. From there two researchers coded each interview, and any discrepancies were discussed amongst them to come to a consensus. Finally, the researchers read the coded data again, condensing codes into categories and then identifying themes in the participants' narratives about the different types of mediators. These themes are presented in the findings section.

Findings

The PSMTs engaged with two different visual mediators, a set of four static Cartesian graphs and a set of four dynagraphs. Regardless of the order of engagement (i.e., static or DIM first), the characteristics of function they attended to when comparing and contrasting the functions in each activity were similar. The characteristics elicited included: differentiability, domain and range, function families, function vs. non-function, increasing and decreasing, independent and dependent variables, local and global extrema, rate of change, and symmetry. Representative examples from our data for the most commonly noted characteristics are shown in Table 1. The only characteristics not included in the communication with both mediators were function/non-function and symmetry. In both cases, these characteristics were mentioned by only one of the two PSMTs and on only one of the four functions being compared. The other seven characteristics were routinely included by both PSMTs in their discourse about both visual mediators. Given the similarities in the mathematical focus of their discourse related to both visual mediators, we next present findings related to the nature of the discourse related to the mathematical focus for each of the two visual mediators.

Table 1: Representative examples of discourse related to characteristics of functions

	Representative Examples	
	Static	DIM
Domain/Range	PSMT 1: "The domain [of A] is only from one to infinity."	PSMT 1: "The domain [of h(c)] is going to go on forever here."
Increasing/ Decreasing	PSMT 1: "[A's] just going to keep going up."	PSMT 2: "And *b*, I'm noticing that as you increase the input of *b*, more and more like the like *g(b)* increases less."
Rate of Change	PSMT 1: "The slope of the lines [of absolute value function] was one or negative one."	PSMT 2: "And it looks like so as *d* is approaching negative infinity *j(d)* increases at the same rate that *d* is decreasing."

Nature of Discourse with and about Static Visual Mediators

When engaging with the static version of the WODB task, the nature of the PSMTs' discourse was routinely focused on using formal mathematical language (though not always precisely) and what we referred to as "describing the image". Representative examples for each of these types of narratives are provided in Table 2.

As they compared and contrasted the four static graphs, the PSMTs consistently used the formal language to typically describe characteristics of functions – e.g., domain, range, increasing, decreasing, slope, concave up. For example, PSMT 2 described graph A using the term "restricted domain" (see row 1 of Table 2). This is to be expected given the years of experience they have of being asked to identify characteristics of functions based on their graphs. Of course, their discourse would include the words used by the mathematicians they have learned from over the years. However, the PSMTs did not always use these words in precise ways. For example, when trying to describe the non-constant rate of change of the function in graph C, PSMT 1 refers to it as "average rate of change" with uncertainty, as she knows it is not constant, but is not sure what to call it.

Table 2: Representative examples of discourse about the static visual mediators

Features of PSMTs' narratives	Representative Examples
Use of formal and precise mathematical language	PSMT 2: "[A's] the only one that has like a restricted domain. Because it like, it doesn't have any inputs, that work for the, they give, like a real value lower than one."
Use of formal and imprecise mathematical language	PSMT 1: "The slope [of D] from zero to infinity and the range is going to be positive one. But I wouldn't say it's that's not really an av- (pause) that's not really an average rate of change."
Describing the image	PSMT 1: "Well, [C's] concave down, it's opening down. ... [D's] the only one that goes through the origin. ... B's the only one that doesn't belong because it's the only one that touches all or touches three quadrants while the rest touch one or two." PSMT 2: "[C's] only one where it doesn't go above zero. It doesn't have an output above zero, right."

Whether their formal mathematical language was precise or not, the PSMTs consistently compared and contrasted the static graphs by describing the presented images as if they were pictures. For example, while PSMT 1 (see row 3 of Table 2) correctly described graph C as "concave down", the follow up phrase "it's opening down" reveals she is describing the image rather than the increasing and then decreasing rates of change that the term concavity is intended to describe. The routine of describing the image in the static graphs is consistent with what Moore and Thompson (2015) refer to as "static shape thinking" or "treating a graph as a piece of wire ... attending to perceptual cues and the perceptual shape of a graph" (p. 784). The attention to perceptual cues (e.g., "goes through the origin", "touches three quadrants") and shape (e.g., "doesn't go above zero") is evident in both examples presented in row 3 of Table 2.

Nature of Discourse with and about Dynamic Interactive Mediators

When engaging with the dynamic version of the WODB task, the nature of the PSMTs' discourse was routinely focused on describing relative direction, distance, and/or speed and connecting their noticing to imagined graphs of known functions. Representative examples for each of these routines are provided in Table 3.

The PSMTs interacted with each dynagraph by dragging the independent variable and examining the resulting reaction of the dependent variable. As they explored dynamically, they described the dynamic characteristics they noticed. Both PSMTs consistently noted the variables' relative direction and speed. For example, as PSMT 2 dragged *d* to the far left, she saw *j(d)* move to the right and stated, "as *d* is approaching negative infinity, *j(d)* increases". In addition, she noted that "*j(d)* is increasing constantly or very close to the same amount that *d* is". Relative distance and speed were also often discussed. For example, when describing her exploration of *g* alongside of *h*, PSMT 1 noted their relative speed but used distance to make sense of it, referring to the length of the connector between *c* and *h(c)* to describe that distance, she explains "so, the outputs, the *g(b)* aren't moving as fast as this one" and then describing *h* she said, "it's like the arrow from the inputs, the outputs, gets larger and larger and larger." The arrow getting "larger" corresponded to the increasing rate of change. In the PSMTs' discourse about the relative speed, direction, and distance, there is evidence of attending to not only how each of the input and output are varying, but also the ways in which they are covarying.

Table 3: Representative examples of discourse about the dynamic interactive mediators

Features of PSMTs' narratives	Representative Examples
Describing relative speed, direction, and/or distance	PSMT 1: "So the outputs, the *g(b)* aren't moving as fast as if like if I move this one [talking about *h*]. And I move it, it's like the arrow from the inputs, the outputs, gets larger and larger and larger. But here [talking about *g*] it does get larger. But it's like it takes a longer time for it to get as long, like cause see, like here I have to move *b* all the way to the right for it to get really long."
	PSMT 2: "It looks like *j(d)* is increasing constantly or very close to the same amount that *d* is and then when *d* is less than zero *j(d)* is increasing, oooh. And it looks like so as *d* is approaching negative infinity, *j(d)* increases at the same rate that *d* is decreasing. And so that reminds me of the absolute value function."
Imagining a Cartesian graphical representation of a known function	PSMT 2: "Because, so like, if I'm picturing, like, the function, like absolute value of *x* and to the right of zero, *d* would be, or to the right of zero, *x* and *f(x)* would be the same or *d* and *j(d)* would be the same. But then to the left of zero, as *d* approaches negative infinity *x* would approach or I feel like a mixing of other notation like as *x* approaches negative infinity, *f(x)* would approach infinity at the same rate. If that makes sense, like it's just the *x* function on both sides. But. Yeah."
	PSMT 1: "[h(c)'s] probably like (pause) a quadratic that is facing down or maybe even an absolute value, because if I move. The *x* values from negative to positive, the *y* values look like they're staying. (pause) They're kind of repeating themselves, the outputs are similar."

Another routine that emerged in the PSMTs' discourse was connecting the relationships they were observing among the variables with Cartesian graphs of function families with which they were familiar. They tended to describe images of static graphs of functions on Cartesian planes they envisioned as sharing characteristics with the dynamic representations they were investigating. For example, as PSMT 2 dragged *d* from the far right to the far left and noticed *j(d)* moving at a constant speed and the same direction until a certain point and continuing at a constant speed but moving the opposite direction she said, "like if I'm picturing, like, the function, like absolute value of *x* and to the right of zero, *d* would be, or to the right of zero, *x* and *f(x)* would be the same ... as *x* approaches negative infinity, *f(x)* would approach infinity at the same rate. If that makes sense, like it's just the *x* function on both sides." She is connecting what she is seeing in the dynagraph with a Cartesian graph that she is familiar with. PSMT 1's description is similar; she starts by connecting the relative direction of *d* and *j(d)* to "probably like (pause) a quadratic that is facing down", but then considers the constant relative distance and says "maybe even an absolute value ... *y* values look like they're staying. They're kind of repeating themselves". This routine of connecting the dynamic movement of the dynagraph representation to an imagined Cartesian graph of a known function is consistent with what Moore and Thompson (2015) referred to as emergent shape thinking, "understanding a graph *simultaneously* as what is made (a trace) and how it is made (covariation)" (p. 785). Here the PSMTs were imagining the one-dimensional trace as a two-dimensional trace and in doing so demonstrated their understanding of the ways in which the quantities were covarying.

Discussion and Conclusion

In this study, we investigated the similarities and differences in the ways PSMTs reasoned about static and dynamic representations of functions. Findings from this study suggest that static representations of function limit students' attention to covariation; this is consistent with prior research that showed students pay attention to shape and perceptual cues rather than the ways the graph represents how the variables change together (Moore & Thompson, 2015; Oehrtman et al., 2008; Weber, 2012). On the other hand, we found evidence of emergent shape thinking when students engaged with the DIM. They were imagining how their action would be represented in a static graph (a trace) which led them to reason covariationally. This attempt to make a connection raises a question: Do PSMTs not naturally tend to reason covariationally when presented with static representations in the first place or did the DIM support this connection making? Further research is needed to gain more insight to the reason behind this connection.

Results also suggest that different representations of function influenced PSMTs' use of mathematical language. The DIM seemed to disrupt PSMTs' reliance on formal mathematical language since they did not have a formal language to attach to the non-traditional function representations (i.e., dynagraphs). This is consistent with Ng's (2016) findings that students demonstrated increased reliance on verbs of motion and less reliance on formal mathematical language in the dynamic environment. In both representations, PSMTs described dynamic situations; however, they used dynamic language only with the DIM. It is possible that PSMTs had to think differently than they are used to, and thus did not have an expert's discourse at hand to describe what they observed. Whereas the familiarity of the static representation caused them to draw upon formal language (sometimes imprecisely), either because they are accustomed to doing so or because they felt it was expected. The fact that the DIM elicited more informal language may benefit the development of precise use of formal language. PSMTs tend to attend to dynamic features of the function as they are describing what they see. This will eventually evolve into formal mathematical language with the support of mathematics teacher educators.

Given the promise of the use of dynagraphs in supporting PSMTs' use of dynamic language and expressing emerging shape thinking, we plan to scale up the study with more PSMTs to determine if the patterns we saw here are consistent. In addition, we plan to consider how the use of DIMs might support the development of emergent shape thinking about static representations.

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EXAMINING PRESERVICE TEACHERS' PROFESSIONAL NOTICING OF STUDENTS' MATHEMATICS THROUGH 360 VIDEO AND MACHINE LEARNING

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Preservice teachers (PSTs) often demonstrate difficulty learning to attend to content-specific student actions in-the-moment. However, machine learning algorithms applied to PSTs' viewing of 360 videos provides a potentially useful tool for teacher educators. In this paper, we describe the initial development of such a tool and the implications for its use.

Keywords: Teacher Noticing; Technology; Preservice Teacher Education.

"Effective teaching requires attending to students' mathematical thinking and reasoning during instruction" (AMTE, 2017, p. 16). These skillsets of attending, interpreting, and responding to students' mathematical reasoning encapsulate *professional teacher noticing* (Jacobs et al., 2010; van Es & Sherin, 2002). When observing mathematics classrooms or viewing video of such contexts, novice teachers often attend to the teacher's actions or describe students' non-mathematical activities, whereas more experienced teachers focus on a specific set of students and describe their mathematics in detail (Huang & Li, 2012; Jacobs et al., 2010). There is clear evidence that, with appropriate scaffolds, preservice teachers (PSTs) can progress to more specific, focused professional noticing (Schack et al., 2013; Teuscher et al., 2017), with mathematics teacher educators continuing to pursue improved techniques and technologies to facilitate such pedagogy. Typically, standard video has been used as the technological medium for facilitating professional noticing in mathematics methods courses (van Es et al., 2017). However, recent technological advances have made certain tools and mediums more commonly available. One such medium is 360 video, which is a version of virtual reality that records video omnidirectionally (see Figure 1). Specifically, PSTs viewing a 360 video can choose which direction to look in the recording, whereas standard videos (e.g., camcorders, Swivl cameras) select what is viewable a priori (Balzaretti et al., 2019; Kosko et al., 2021).

Evidence suggests that 360 video may facilitate PSTs' professional noticing, by creating a viewing context more representative of being in the classroom (Kosko et al., 2021; Roche & Rolland, 2020). Beyond this, 360 video allows for PSTs' choices of *where* they look in a classroom to be measured by recording their selected field of view (FOV). Extending the potential of this technological affordance, recent advances in machine learning, or artificial intelligence (A.I.), allow for examination of patterns in what PSTs focus in their recorded FOV. The purpose of this paper is to examine the efficacy of a machine learning algorithm in identifying the kind of mathematical actions that students engage within a video, and use of such a tool to examine PSTs' professional noticing. This paper reports on our initial efforts to align attending behaviors of PSTs (observed by teacher educators) with patterns recognizable from a machine learning algorithm.

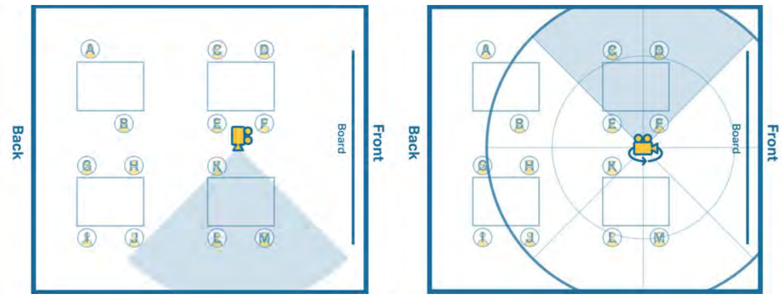

Figure 1: What is viewable from a standard video (left) vs. a 360 video (right).

Background Literature & Theoretical Perspectives

Attending as part of Professional Noticing

Professional noticing involves identifying key aspects in a pedagogical context, interpreting those aspects to one's professional knowledge and norms, and then applying this reasoning to decide how to engage next (van Es & Sherin, 2002). As noted by Scheiner (2016), scholars examining PSTs' act of identifying key aspects have often focused on PSTs' perceptions rather than examining the constructs of attention or awareness. However, "attention selects certain stimuli of a perceived scene for detailed analysis, while perception goes to build up a certain visual experience" (Scheiner, 2016, p. 231). Furthermore, attention involves coordination between various elements of one's professional knowledge and contextual resources. Studies including eye-tracking to examine professional noticing provide insight in how attending is actualized by teachers. Comparing 40 inservice and preservice teachers, van den Bogert (2014) found that more experienced teachers focused their gaze on more students but spent a majority of time attending to a smaller, select set of students in the video examined. By contrast, PSTs scanned the room for larger swaths of time and focused their gaze on relatively few students for any meaningful duration of time. Expanding upon such findings, Dessus et al. (2016) observed that more experienced teachers tend to identify a focal sub-group of students that allows them to attend to more specific, fine-grained events. By contrast, more novice teachers were observed to scan a wider range of events and students, thus limiting their ability to focus on more specific events. Studying this phenomenon using 360 video, Kosko et al. (2021) found that PSTs with less variance in where they attended also had more specific descriptions of children's mathematics. By focusing more attention on two front tables in the classroom, certain PSTs were able to describe more specific aspects of the lesson that occurred. Such findings resemble those of Dessus et al. (2016) and suggest there may be several ways to examine teachers' attending behavior.

As noted by various scholars, professional noticing in general, and attending in particular, are complex skills, but can be taught and learned (Jacobs et al., 2010; Schack et al., 2013). Key commonalities in many of these successful approaches include numerous interactions with videos of students engaging in mathematics, and a focus on giving "opportunities to recognize the power of attending to the subtle details in individual children's strategies" (Jacobs et al., 2010, p. 176). Analysis of such teacher education initiatives have yielded frameworks for specificity of teachers' articulated noticing. As noted by Barnhart and van Es (2015), teachers may initially describe classroom management events and/or focus on the teacher with little focus on students' content-specific actions. As teachers begin to attend to and interpret students' actions, they may describe them from a procedural perspective before eventually learning to describe them from a more conceptual view. Jacobs et al. (2010) provide one example of such progression noting that more conceptual-based attending involved specific descriptions of a student's decomposition of numbers by place value and use of benchmark numbers. By contrast, a more procedural attending included descriptions of the numbers the child wrote down and that they added them, but concepts of place-value were absent.

The preceding paragraphs describe how teachers' attending has been examined from data of where and how they look in a recorded scenario and the specificity of how they describe such events in written or spoken noticings. Different scholars have examined the overlap in these sources of data using 360 video (Ferdig & Kosko, 2020; Kosko et al., 2021), wearable cameras (Sherin et al., 2008), and through analysis of teacher discussions while viewing videos (Jacobs et al., 2010; Schack et al., 2013). Each approach has demonstrated capacity for facilitating teachers' professional noticing, but they are often time-intensive and become less practical when considering large cohorts of PSTs in a teacher education program. This limitation motivated the need for applying machine learning to the study of PSTs' attending, with a long-term hope of applying this technology in pragmatic contexts (i.e., mathematics methods courses). Before describing our use of machine learning, however, we provide a brief overview of this technology and our vision for applying it to study professional noticing.

Applying Machine Learning to Study Professional Noticing

Machine learning is an artificial intelligence (A.I.) subdomain that relies on the ability of a machine to learn from an external source and develop and refine its own algorithms and routines toward a given goal. This goal may descriptive (describing a phenomenon), predictive (predicting a phenomenon), or prescriptive (suggesting how a phenomenon should occur). This technology has been used in education in two different ways. First, it has become a STEM field itself to explore and investigate in primary education – particularly high school (e.g., Korkmaz & Correia, 2019; Mariescu-Istodor & Jormanainen, 2019). Second, it has been deployed for staging the so-called "precision education" to develop personalized instruction from individual academic performances (Luan & Tsai, 2021). Recently, there has been increasing attention on machine learning for teacher education, targeting instructional videos. For instance, Goldberg et al. (in press) validated a manual approach for guiding machine learning algorithms in evaluating videos of three university lessons. Specifically, videos of instruction were examined to code for recorded students' visual engagement in class. Nückles (2020) explored eye tracking and related machine learning processes in video professional development for teachers, claiming that more efforts based on computation are needed for understanding how educators deal with lesson recordings and their elements. In particular, Nückles (2020) questioned the relevance of some eye tracking and machine learning approaches to video analysis in teacher education as focusing too much on teachers' perceptions of on/off task student behavior. Instead, there is a need to address how such technologies may be used to facilitate PST education. In line with Nückles (2020) view, we argue that such tools can be used to provide PSTs feedback in how they attend within a classroom, and this feedback can be used to improve their practice. Yet, to reach this eventual application to teacher education, examination and piloting of machine learning must take place. Thus, the purpose of this paper is to examine the efficacy of a machine learning algorithm to examine PSTs' professional noticing.

Method

Participants & Data

Analysis in this paper focused on six PSTs' viewings of a 360 video focusing on 4th graders' solving fraction equivalence tasks (2 minutes, 49 seconds). Data represents a subsample of 70

PSTs who participated in a larger study. Specifically, the analysis presented here reports on the training process for the machine learning algorithm used to analyze PSTs' 360 video viewing experiences. All six participants were preparing to become elementary teachers in a Midwestern U.S. teacher education program. The program included two focused mathematics methods courses. A focus of the second methods course is fraction pedagogy, including several video-based assignments focused on improving PSTs' professional noticing of students' mathematics (two such videos focus on fractions). Participants included a junior enrolled in their first mathematics methods courses (Nate), three seniors enrolled in their second mathematics methods course (Lynn, Aubrey, & Brie), and two seniors completing student teaching (Anna & Nash).

PSTs participated in the study near the end of their Fall 2020 semester (after Lynn, Aubrey, & Brie had viewed prior videos on students' fractions). As part of a larger study, participants were asked to watch the 360 video focusing on 4th grade students solving a task to determine equivalent fractions. Within the video, students were initially asked to use pattern blocks to find how many red trapezoids covered the shaded region of given shape ($\frac{3}{4}$). Next, they were asked to use green triangles to find the equivalent fraction ($\frac{9}{12}$). Towards the end of the video, student I suggested the answer was $\frac{8}{12}$, to which student G disagreed. Following the teacher's press for student G to "prove it," G demonstrated that there were three triangles for every trapezoid.

Prior to watching the scenario, PSTs were prompted to take notes on any 'pivotal moments' regarding students' mathematics they noticed in the video. Following viewing the scenario, participants were asked to transcribe (type) their notes and then to select one moment to describe as the most important and explain why it was significant. Participants were also prompted to describe what should happen next in the lesson, but the preliminary nature of this analysis, we currently focus on PSTs' attending and interpretations in this paper. In addition to written noticings, participants' viewing sessions of the 360 videos were recorded to allow for analysis of where PSTs' turned their perspective in the 360 video, and what in the scenario they focused on at specific timepoints in the video.

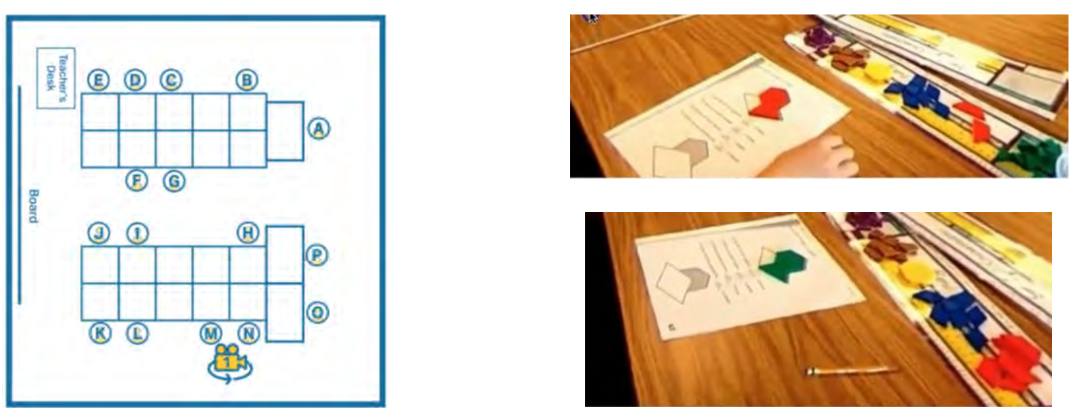

Figure 2: Classroom map with camera positioned between students M & N (left) and screenshots of student M's equivalent fractions (right).

Analysis & Findings

Machine learning in the context of video-based data involves identifying specified visual patterns and teaching the computer-based algorithm how to find the same visual patterns and provide feedback for when the A.I. provides false-positives or false-negatives. To facilitate this process in our current work, we incorporated an iterative analytic process. First, participants' screen recordings of their 360 video viewing experiences were collected from the Praxi platform. Next, body tracking was used to overlay digital "skeletons" of recorded students and teacher to identify their torso, orientation of their arms, and direction of where recorded individuals' heads were turned (see Figure 5). The first, third, and fifth authors then analyzed PSTs' screen recordings second-by-second to identify observable student actions present or absent from participants' viewings. Given prior evidence that suggests PSTs' written noticings are related to what and where they attend when watching 360 video (Ferdig & Kosko, 2020; Kosko et al., 2021), PSTs' written noticings were also examined as a way of ensuring that analyzed videos were more likely to include relevant patterns to train the A.I. The analysis and findings of each stage is provided in the sections that follow.

PSTs' written noticings. Analysis of PSTs' written noticing was conducted using Systemic Functional Linguistics (SFL) (Halliday & Matthiessen, 2014, Eggins, 2004). SFL is a methodology for examining how participants' use of grammar conveys meaning. In the present study, we examined PSTs' conveyed meaning through use of reference (i.e., PSTs' use of grammar to refer to grammatical objects). In particular, *reference chains* are formed by the repeated incorporation of references throughout a written text. As a referent continues to be used, the writer may provide additional information, thereby transforming, expanding, or clarifying the meaning of this referent. To analyze for reference chains, and how a referent's meaning was conveyed, the first and third author analyzed written text for PSTs' use of nominal groups and transitive processes. Figure 3 illustrates a snapshot of this process for two students, Nash and Anna. Nominal groups referring to pivotal moments are underlined, where a nominal group "is the part of the clause [that] contains nouns and the words that accompany nouns" (Eggins, 2004, p. 96). Each clause established by the user is separated by "//". Transitive processes, bolded in Figure 3, represents how the participants conveyed meaning for referent nominal groups.

Nash

Some pivotal moments I noticed were from
student M and N. //
When the teacher **asked** the questions about
what shapes to **fill** in their diagram and //
find the answer.
Some pivotal moments **were** when the
students **were** able to immediately **find** the
answer and // were eager to raise their hands.
Also, they quickly noticed the
teacher's "error" and //
wanted to address //
and // **fix it** because they **knew** the
answer.

Anna

There **were** a few pivotal moments in the video. //
One of the first ones **was** when the teacher has the
children define how many green triangles were needed to
fill the WHOLE shape. //
The student, I, was **given** the opportunity
to **think** and **respond** to both the questions //
how many to fill the whole and //
how many to fill the shaded. //
It is interesting that this student **responds** with a fraction,
8/12's. //
A key pivotal moment in the middle **is** also when the
other students announce they disagree with that answer. //
The teacher **says** there is a debate //
and challenges the student, G, to **prove** it.
G then goes to **show** //
how she **counted** the six in the top part and
then the three in the bottom part
to get 9/12's not 8/12's.

Figure 3: Example of written noticings of Nash and Anna.

Notable in Figure 3, Nash identifies the pivotal moments in the lesson as focusing on students finding an answer. The referents “what shapes to fill in their diagram” and “find the answer” both point toward this. Further, finding the answer is continuously referenced throughout the text. Towards the end, this manifests in a judgment of “the teacher’s error” and students fixing “it” (the answer) because they knew the correct “answer.” Similar to Nash, analysis of Brie and Aubrey’s reference chains also indicate a focus on students finding the answer. By contrast, Anna’s written noticing (see Figure 3) focuses more on fraction-based references. Initially, Anna references the pivotal moment as children “define how many green triangles…to fill the WHOLE shape.” This referent is clarified by the referents “fill the whole” and “fill the shaded” and then later with students’ responses of “8/12” and “9/12.” More than providing a math-specific referent, Anna’s response differs from Nash (and Brie & Aubrey) by the transitive processes used to convey the referents’ meaning. Nash continuously uses processes like **find**, **fix**, and **knew** “the answer” whereas Anna uses **think**, **respond**, **prove**, **show**, and **counted** to refer to the parts in relation to the whole. So, while Nash, Brie, and Aubrey’s reference chains focused on “the answer,” Anna, Nate and Lynn’s reference chains focused on children’s actions on and with fractions. Referencing Figure 3, We used these findings to help triangulate results of our preliminary video analysis and that of the A.I..

PSTs’ 360 video viewings. Following an analytic approach we have previously used (Kosko et al., 2021), the first, third, and fifth authors examined each participants’ screen recorded viewing second-by-second to identify which recorded students were in their field of view (see Figure 4). We then used findings from analysis of their written noticings to look for differences between PSTs’ viewing patterns. Figure 4 provides one example comparison between Anna, who attended to the mathematics, and Brie, who attended to students’ finding the answers (but no explicit reference to mathematics). Notably, between 44-88 seconds in the video, Anna tends to switch her field of view focusing on student M and students I and J, and these two sets of students are along the same line of sight from the camera perspective (see Figure 4). By contrast, Anna includes students M, J and I, H, and P in her field of view during the same timeframe. Notably, these students are not in the same line of sight but require the viewer to turn the camera perspective as much as 110 degrees from one moment to the next. At around 90 seconds in the video, Anna and Brie’s viewing patterns appear similar. This is when a class discussion begins regarding the task involving a fraction of $\frac{8}{12}$ or $\frac{9}{12}$. Thus, it appeared that any significant differences in viewing patterns were within the first 1.5 minutes of the video.

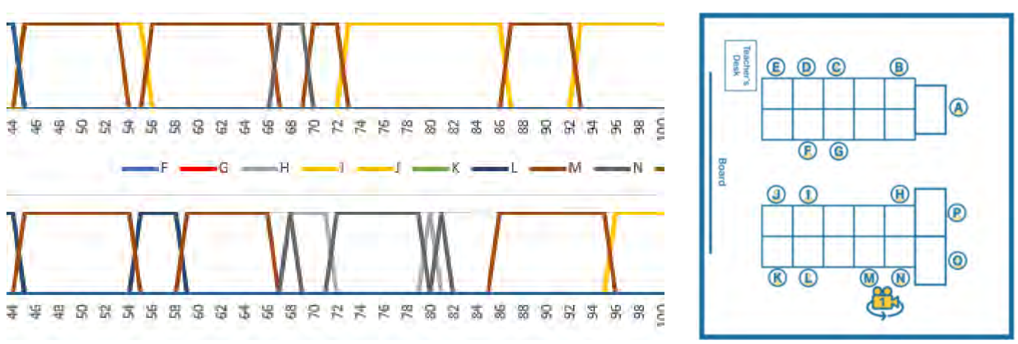

Figure 4: Anna (top) and Brie’s (bottom) student focus.

Next, videos were reexamined using the skeleton wireframes as a guide while focusing on specific intervals identified in the initial video analysis. Specifically, we used AlphaPose (Fang et al., 2017; Li et al., 2019; Xiu et al., 2018) to estimate skeleton points of students in the videos, and attached these wireframes to the video, with the goal of using a machine learning approach to analyze them. This allowed for several particular patterns to emerge from the data, but we discuss one such pattern for sake of space and focus. Common in screen recordings of PSTs like Anna was a focus on attending to students' working with the pattern block manipulatives. This was characterized by the skeleton wireframes when students' arms were both pointed inward and their head-gaze was directed downward (where their arms meet). Such instances were present across all screen recordings, but at varying frequencies. Figure 5 provides a comparative example of Nash and Anna's viewing patterns at two instances in the recorded scenario. At 38 seconds, Anna is focusing on student N's manipulation of green triangles onto the figure while Nash is adjusting his field of view from one end of the table to another (back-and-forth). At 50s, both PSTs are attending to student M, but Anna is also attending to students I and J (within the same line of sight). The first and fifth author coded for presence of these skeleton wireframes (K=0.87) and found that Anna attended to students' use of manipulatives for 51 seconds in the first 90 seconds of the video, while Nash did so for 28 seconds in the same interval (no such moments occurred after 90s in the video).

Figure 5: PSTs' attending captured, with wireframes overlain, at 38s (left) and 50s (right).

Initial machine learning results. Based on the initial analysis of participants' screen recordings and written noticing, a machine learning algorithm was developed to identify whether participants attended to students' manipulating fractions. We proposed three layers of a neural network model, which has an input layer, a hidden layer, and an output layer. We used categorical cross-entropy loss to update the parameters in the model, and trained the model for 20 epochs to achieve better performance. Developing and teaching a machine learning algorithm takes multiple iterations, and we report only on the initial run of the model. In training the algorithm, 42 skeleton wireframes were extracted from the sample videos and assessed in comparison to examples provided through the human-coded video analysis. To help train the algorithm further, an additional action was included (students raising their hand) to help the A.I. discern one action from another. The algorithm reached an accuracy of 75.86% in an initial training run (n=29) and then 69.23% on a test run (n=13) of the A.I. Results of the initial training and test run are positive and encouraging, but do call for the need for additional attending elements be included, and additional data be collected from participants' videos. Fortunately, our current dataset includes additional 360 video screen recordings of 70 PSTs, and there are several other attending elements (e.g., teacher within FOV, student(s) counting blocks on paper) that will be used to train the machine learning algorithm further. As additional attending elements are included, and more examples are extracted from participants' videos, the A.I. will improve in accuracy and provide a report comparable to human coders (but in a fraction of the time).

Discussion

Similar to prior findings examining more and less sophisticated noticing (Barnhart & van Es, 2015; Jacobs et al., 2010), we found that certain PSTs referenced children's mathematics-specific actions (Anna, Lynn, & Nate). By contrast, others focused on more general (not mathematics-specific) events. Nash, Brie, and Aubrey each attended to how and whether people in the recording found the correct answer. Interestingly, this focus on "the answer" infrequently referred to a numeric fraction. Corresponding to research on eye-tracking (Cortina et al., 2015; Dessus et al., 2016), analysis of PSTs' 360 viewing indicated participants with more sophisticated noticing (via writing) had more focused attention than their counterparts. For example, Anna's focus on three students within the same line of sight contrasted Brie's shifting from one length of the table to the other, and back (see Figure 4). This corresponds to Dessus et al.'s (2016) observation that more experienced teachers focused on subsets of students and examined more specifics, but more novice teachers scanned the room more frequently. However, findings here do not compare expert and novice teachers, but PSTs at similar levels of experience. Thus, findings presented here suggest that PSTs' embodied attending behavior may be due less to level of experience and more to some underlying professionalized knowledge.

This paper includes a diverse set of authors spanning mathematics education, computer science, and educational technology, with each area of expertise represented in the development and application of this new tool for teacher education. Thus, beyond the implications for our specific machine learning A.I., an additional implication is the benefit and need for cross-disciplinary collaboration. As mathematics educators seek to incorporate more 21st century technologies into teaching and teacher education, there is a critical need for such collaboration. This paper serves as an example of what such collaborative efforts can yield, as well as providing a description of how one such technology (machine learning) is developed in such contexts. Specifically, applied machine learning to PSTs' attending in 360 video. Findings are preliminary, but suggest that nuanced student actions relevant to pedagogical content-specific noticings can be detected by A.I. This is highly significant, since prior applications of machine learning have focused on more generic student behaviors (Luan & Tsi, 2021; Nückles, 2020). As the accuracy and breadth of our machine learning A.I. improves, it has potential not only for improving capacity for research of PSTs' professional noticing, but in providing timely feedback for PSTs in mathematics methods courses. Some of the attending element patterns detected with the A.I. described here can be applied to other videos (360 or standard), but such application is likely context specific and require additional training of the A.I. However, such additional validation of these machine learning algorithms will likely yield more robust tools for mathematics teacher educators and teacher education researchers.

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expressed in this paper are those of the authors and do not necessarily reflect the views of NSF.

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THE INFORMAL COVARIATIONAL STATISTICAL REASONING: FOCUS ON THE NOTION OF AGGREGATE USING DIGITAL TECHNOLOGY

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We report the results of a study on informal covariate statistical reasoning conducted with 22 students (aged 16 and 18 years). We designed and implemented a task in a digital technology environment to introduce the line of best fit. The task design having elements that foresee misconceptions reported in the literature, and by focusing on four statistical ideas that we consider being central to the development of informal reasoning about the line of best fit. After having used the digital technology environment, students transitioned from viewing points of a scatterplot as individual points or fragmented into subsets to viewing the scatterplot as an aggregate from a mathematical mechanism that links them through the notion of distance from a point set to a right line.

Keywords: informal covariational reasoning, line of best fit, aggregate, digital technology

Introduction

In statistics, covariation is the variation of two statistical variables that take numerical values (Moritz, 2004).The values for each variable are obtained from the same observation unit and expressed as an ordered pair; observations compose a set of pairs called bivariate data. The graphical tool used to represent a bivariate data set in a plane is a scatterplot. The most used techniques to investigate statistical covariation are correlation and regression. Correlation quantifies the strength of the linear relationship between a pair of variables, while regression expresses the relationship as a mathematical model (linear equation).

The correlation and regression are statistical objects that express global properties of a data set, properties that do not belong to isolated points of the data set but all of them. An isolated individual data does not contain the properties that will emerge when a data is associated with other data, namely the data viewing as an *aggregate*. Stigler (2016) used the term aggregation to designate the first pillar of statistical wisdom. For him, aggregation is the mechanism by which can provide more information of a data set, there is a loss of information of the individual data for retaining global properties of the data set.

Statistical educators have mentioned the aggregate to highlight a recurrent phenomenon in the learning of statistical concepts. Hancock et al. (1992) were the first to raise the problem that in data analysis, students are prone to focus on the characteristics of individual data without making sense of the aggregate properties of a data set, such as the mean. Other authors have mentioned the same problem concerning the notion of distribution (Bakker & Gravemeijer, 2004), in group comparison (Ben-Zvi & Arcavi, 2001), and the concept of the sample (Saldanha & Thompson, 2002). The correlation and regression result from an aggregation process and require that a point could be conceived of as an aggregate.

With the availability of specialized statistical applications and software, opportunities open up in teaching for students to look at data sets as an aggregate; the possibility of multiple representations, dynamic trawling, real-time data updating, and performing tedious calculations are ideal to show how statistical summaries are related to the data set from which they originate (Biehler et al., 2013). This paper informs about a design research that use the online data analysis platform CODAP, and an applet designed in GeoGebra with the aim to promote the development of covariational reasoning in high school students; In particular, we are interested in observing if the design of the task and the use of the software allow students to begin to see the points of scatterplot as an aggregate or, more precisely, the process of finding the line of best fit as an aggregation process.

Background

Many researchers have been interested in the problems of the correlation and regression teaching and learning. The first studies were about the conceptions of university students, such as of Truran (1995), who was interested in the detection and characterization of the interpretations of university students about the correlation coefficient and the determination coefficient. Sánchez-Cobo et al. (2000) studied verbal, graphical, and numerical representations of correlation, and Sorto et al. (2011) studied students' conceptions of the line of best fit. At the pre-university school levels, there is the study by Watson and Moritz (2007) analyzed students' reasoning when making graphical representations about covariation, and Casey (2015, 2014) studies of students' conceptions of what the line of best fit is. Regarding high school students' conceptions, two studies related to the graphical representation of covariation stand out: the study by Watson and Moritz (1997) analyzed the graphical representation established by students about the covariation present in non-symbolic contexts, and the study of Estepa and Batanero (1996) established some conceptions of covariation in students when they judge the relationship of two variables based on scatterplots.

A current trend in statistical education is to conduct research that investigates the relationships between teaching design and progress in student learning. In this way, studies have begun in which the design of the intervention in the classroom is an important component highlighting technology as an element that can help students make more accurate covariation judgments (Batanero et al., 1998; Cobb et al., 2003; Inzunza, 2016). At the high school level, research that includes teaching intervention is still scarce and scattered; we found only three studies at this level, each paper covering one topic: covariation in big data contexts (Gil & Gibbs, 2016), scatterplots, and the line of best fit (Medina et al., 2019), visualization and trend in the data (Dierdorp et al., 2011).

Framework Conceptual

The following four subsections present the concepts we consider central for understanding the research from which this report was done. We defined the conceptual framework used here as a set of concepts that clarify the key aspects to be studied. We aligned this notion of the conceptual framework with the one presented by Miles and Huberman (1994, p.18).

Definition of Aggregate and Aggregation in Statistics

From examining how researchers use the term aggregate, we define it as a set of data belonging to a larger whole with global properties. An aggregation process comprises an object produced from an aggregate that highlights properties common to all data, properties that individual data does not have. Thus, the mean and the line of best fit result from an aggregation process. A condition for the mean of a data set or the line of best fit of a "cloud"[1] of points to be seen as representatives of their respective data sets is that we must conceive them as aggregates. This characterization tries to synthesize the comments of Hancock et al. (1992), Konold and

Higgins (2002), Bakker and Gravemeijer (2004), Casey (2015), and Stigler (2016). From an educational perspective, the question arises: How does a student come to conceive that a data set is an aggregate? We hypothesize that there are two sources: context and mathematics. We can perceive that all the data belong to a phenomenon or associate them with a causal relationship. Here, we see the data within a context that unifies them. Also, we can perceive the data as part of an aggregate if it belongs to a set linked by a mathematical property, for example, even numbers in a set of numbers, cloud points aligned in a bivariate data set, etc. In both cases, the totality transcends the given set such that we can imagine the existence of other data that could be added to the given set.

Beliefs, Conceptions and Difficulties about Covariation

The literature reports that students do not separate their previous beliefs for observing and evaluating the behavior of two quantitative variables, and they do not include the word variation in their vocabulary (Moritz, 2004). Also, the previous concepts of a linear function in mathematics can interfere in their ability to make sense of determining the line of best fit (Casey & Nagle, 2016). Estepa and Batanero (1996) established some conceptions in the students when they evaluate the covariation and the line of best fit. Deterministic conception when students considered the relationship between the variables from a functional point of view (a line that passes through all the points), they expect a correspondence where each value of the dependent variable correspond to another value of the independent variable when this is not the case, consider that there is no dependence between the variables, local conception when they use only a part of the data and they generalize conclusions to the entire data set. Casey (2015, 2014) establishes the following strategies for students to draw the line of best fit: draw a line that divides the data points so that half of the points are at the top of the line and half are below the line, draw the line through the midpoints of different cloud groups. In addition, in the students may emerge the concept of "closeness" between the line and the point cloud but there is a lack of understanding of other elements such as the error that corresponds to the sum of the squares of the vertical distances between the observed and predicted values.

Informal Covariational Statistical Reasoning

Reasoning refers to the processes of obtaining and verifying propositions (conclusions) based on evidence or established knowledge or assumptions. Reasoning can take many forms, ranging from informal argumentation to deductive demonstration (National Council of Teachers of Mathematics, 2009, p.5). Informal statistical reasoning is related to data, samples, chance, inference, and relationships between statistical variables. Informal statistical reasoning about covariation is related to bivariate data sets and relationships between statistical variables. In the present research, the purpose is to develop students' informal statistical reasoning about the line of best fit. For this purpose, we define informal notions of linear (instead of quadratic) distance and line of best fit, which does not coincide with the formal concepts but is not inconsistent with them and has the advantage of being closer to students' intuitions.

The Influence of Digital Technology on Reasoning about Statistical Covariation

Research suggests that technology can help students make more accurate covariation judgments (Batanero et al., 1998; Cobb et al., 2003; Inzunza, 2016). Technology plays a very important role in statistics since it makes them visual, interactive and dynamic, allowing a focus more on concepts rather than algorithms and calculations, where interactivity and the quality of use of graphs allow conducting experimentations with data; this allows engaging students in productive activities (Biehler et al., 2013). Specifically, the topics of regression and correlation with the technology possess the following relevant features: 1) The possibility to form scatterplots and fit a regression line by visually showing the changing quadratic deviations of the line as it fits the cloud of points. 2) Obtain the numerical value of the correlation coefficient and determine the algebraic expression of the regression line. 3) Dragging points from the scatterplot and observing in real-time the effect of their location within the cloud on the strength (correlation coefficient) and direction (regression line) of the relationship, allowing one to see the interactions between the elements dragged and the statistical measures. 4) Linking multiple representations to discover and observe patterns and trends in data simultaneously from different perspectives (the graph, the summary measures, the regression line). In the present study, we show how with the help of technology, students can conceive of the line of best fit as the result of an aggregation process.

Methodology

The study participants were 22 high school students around the age of 17 who had not studied the topics related to correlation and linear regression. The application of the task took place in a computer classroom during a two-hour class session. The author of the present work carried out the implementation of the task. The data obtained were the worksheets developed by pairs of students.

We follow the principles of the design experiment of Cobb & McClain (2004) for the design and implementation of the task: the *use of technology*, we used *CODAP* and *GeoGebra* software because their features allow designing elements that we consider relevant for students to observe and interact with. For the *classroom discourse*, the teacher oversaw, monitoring, coordinating, and making sense the interaction between the students and between students and the technological tool. The *structure of the task* in the classroom is collaborative work students had the opportunity to explore solutions, compare them with those of their peers, and clarify them in a group meeting. To establish the *central statistical ideas*, we analyzed the difficulties and conceptions reported in the literature to promote elements of the task that anticipate the difficulties and students' conceptions. Also, we reviewed the content stipulated by the NCTM (2000, p.327-328) for the study of bivariate data (correlation and regression) in the last school level of high school and the bivariate data unit of the program of the College of Sciences and Humanities of the UNAM, Mexico.

The task starts from the graphical view in GeoGebra, where the data in Table 1 are displayed. The data correspond to the measure in fat gain (in kilograms) and change in energy use (in calories) from other "non-exercise activity" (NEA) (restlessness, daily life, and the like) of 12 young adults who overfed for eight weeks. We suggest that the teacher conduct a discussion asking whether changes in restlessness and other non-exercise activities explain weight gain in overeaters, guiding students to study the relationship between the variables on pencil and paper, followed by making the scatter plot in *CODAP* and leading them to describe the behavior of the point cloud (intensity and direction).

Table 1: Measures of Change in NEA and Fat Gain in the 12 Young Adults

NEA change (cal)	-94	-57	-29	135	143	245	355	486	535	571	620	690
Fat Gain (kg)	4.2	3	3.7	2.7	3.2	2.4	1.3	1.6	2.2	1	2.3	1.1

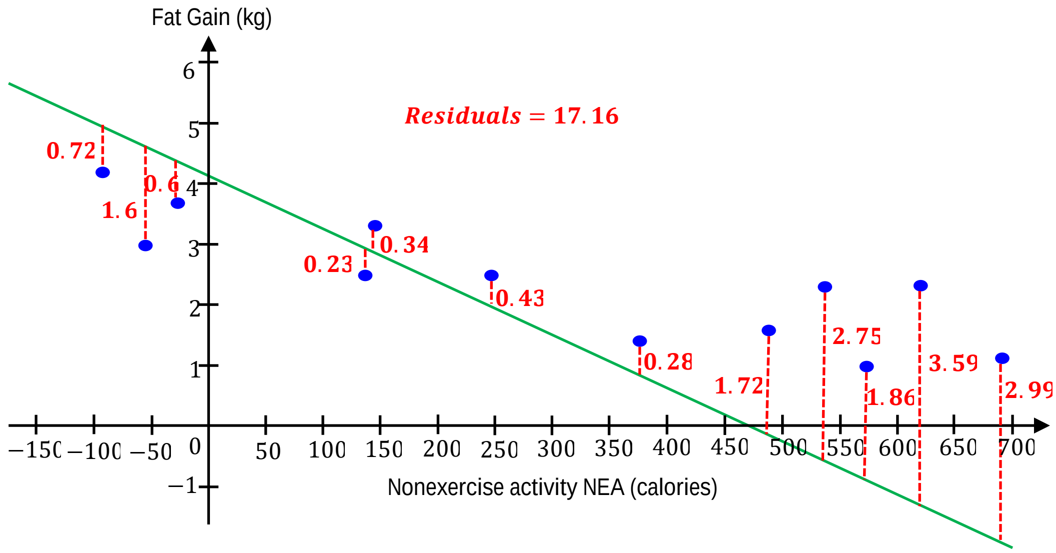

Figure 1: Screenshot of the Graphical View of the Task in GeoGebra

The *central statistical ideas* related to the line of best fit are present in the elements that make up the task and are:

1. The possible line always depends on all points in the scatterplot, i.e., all points influence the determination of how close or far away a line is.

The graphical view of GeoGebra shows a line that is movable by holding the click anywhere on it this line moves in different ways as desired, varying its slope or varying its point of intersection with the axis. The intrinsic characteristic of the movable line of being connected to the points (data in Table 1) focuses on the idea of conceiving a point cloud as an *aggregate*, i.e., all points in the cloud influence the determination of the best fit line.

2. Given a cloud of points and a line, we define the error of a pair of points (called residual) as the absolute value of the difference between the ordinate of the point and the ordinate of the projection of the point on the line of fit.

For each dashed segment in the graphical view, we show the numerical value called residual that corresponds to the difference between the ordinate of each data and the ordinate of the point that belongs to the moving line. What happens to the value of each residual if you move the line near or far from the point cloud? We intend to focus on the statistical idea of error, seen as a distance between a moving line point and a cloud point (datum).

3. Adding all the residuals, we obtain the measure (residuals) of closeness or remoteness between the cloud of points and the moving line.

In the file, there is a value called residuals. The teacher should mention that the residuals correspond to the sum of all residuals, and we calculate them by adding the absolute values of the difference between the points of the cloud and the possible points of the movable line. The objective is to provide a notion of distance from a line to a point cloud, and with this, to define a measure of the closeness of the line to the point cloud; this distance is the sum of all residuals. The student will explore how the distance from the line to a cloud changes by freely moving the line and observing the corresponding distance value. We suggest asking the students: Where do you think you should place the line so that the value of the residuals is the minimum? Can you be sure that the location you consider is unique, yes, or no? why?

4. The line that best fits the data is the one where the sum of residuals is minimal.

With the idea of defining the line that best fits the data as the one where the errors (residuals) or vertical distances are the smallest possible in some average sense in general, we suggest that the teacher indicate how to determine the line of best fit that GeoGebra yields and ask Is the way you placed the movable line the same as the line yielded by GeoGebra, yes, or no? How do they differ? What do you think is the criterion that GeoGebra uses to determine the line of best fit? We use a notion of linear distance and not quadratic distance because the former is more intuitive for the students., however, it seems clear to us that, understanding the idea with linear distance, it can be easily generalized considering the quadratic distance, arguing the reason for the advantages of this one.

Findings & Discussion

The analysis of the students' responses in judging the relationship between the variables shown in the problem situation together with the table of values without the use of technology provides the following reasoning as a result:

The *functional covariation* strategy comprises searching for and isolating bivariate data that adhere to a mathematical model, i.e., they focus their attention on the points they locate on a line. Students whose solutions fall into this category divide the data set into two parts, those that correspond to a linear model and those that do not. So their description of the data refers only to the subset of data that corresponds to the model and ignores those that are left out, and based on the data they selected, they describe the general trend, but their statements, often coinciding with the trend of the entire cloud, state, for example: "the more fat you consume, the more calories you increase", "the less you change in calories through movement, the less fat you will decrease". They also involve in their description's characteristics of the context of the problem*:* "fat increases because it remains encapsulated", "the less activity you do, the more fat", "the fats remain in the body". Their beliefs may influence their choice of the point cloud data set other than following the criterion that they lie on a straight line.

When students focus their attention on what happens in the passage from one point to the next within the values table, they present the strategy of *randomness.* They tracked the differences between successive points as they review the points from left to right, and they do not notice any predictable patterns. Sometimes, the difference is positive or negative, and the size varies. Thus, they conclude: "the fat increases or decreases depending on how many calories are burned in ANE", "if the calories decrease or increase depending on the ANE, the fat will also increase or decrease", "the calories that are burned will increase the kg or calories that are burned will decrease the kg", the students who follow this strategy, it is evident that there is no correlation since it is not possible to know if from one value to another value fat levels will go up or down. In both strategies, they visualize the data by paying attention to their individual or partial characteristics, and not as a whole. Thus, the analysis according to the first strategy consists of separating the data into two, in the second, of going through them one by one and seeing if and by how much they increase or decrease. When they use the CODAP platform to enter the table of values and make the scatter diagram and again to judge the relationship between the variables, the *inverse covariation* arises, which globally describes the behavior of the cloud, for example: “when increasing calories there is more fat-burning”, “the more change in ANE, the less fat increase there is”, “when the change in ANE is greater, the increase in fat is less”, “the more increase in ANE, the more fat loss there is. The less increase in ANE, the less fat loss”, the *context of the problem* also influences, for example: “the fat does not increase because, although you do not exercise if you keep moving”, “the increase in fat depends on the amount of ANE that is carried out daily”. Without using technology, students focus their attention on the variation from point to point (Randomness: fat sometimes increases and sometimes decreases), while that using the technology influences to make a description of the general trend the cloud, ignoring particular fluctuations (*inverse covariation:* fat values decrease as ANE values increase).

We see that treating the situation with the help of technology and the use of some of its possibilities influences students to abandon their tendency to highlight in their description of the cloud to a set of individual points that they compare with each other, to highlight a global property which becomes evident when you have an aggregate view of the set of data. In addition, with and without technology, the description that they did of the relationship between variables is influenced by the contextual content of the problem. Students do not see the data as simply numbers, but numbers with a context, which for Moore (1990) is what it implies to establish the statistical association.

Regarding the analysis of the students' responses when asked to establish the line of best fit to the data using the GeoGebra applet, the following reasoning emerged:

In many student responses when using technology, the intuition of the *closeness* of the set of points to a line is revealed; for example: "that the blue points were closer to each other to the line", "I think it is the line that passes near all the points and does not join them, so each one has a certain distance". This intuitive idea becomes operative with a notion of *distance* from a line to a set of points. The software allows to calculate the residuals, that is, the differences of the ordinates of each pair of points with the same abscissa, one belonging to the cloud and the other on the line. Each residual can be viewed as a distance from the point of the cloud to the line (for the present purpose it does not affect the fact that strictly the distance from a point to a line is defined as the distance of the segment that passes through the point perpendicular to the line) and the sum of residuals as the total distance of all residuals from the cloud to the line. As this sum can be seen in the software and is updated in real-time as the line moves, the students manipulate and see a real function determined by the point cloud and whose independent variable is the lines in the plane; then for them, the line that minimizes the function is the line closest to the points, that is, the one with the best fit. The following expressions of the students indicate some of their ideas in this regard: “the less distance there is between the points and the line, the remainder will change”, “the further the line of adjustment of the points in the table is in the graph is greater the value of residuals since there is a greater distance", " when the line is better centered, the level of residuals is lower, that is, it passes centering in the middle of the points.

One condition for viewing a set of data as an aggregate is to imagine a rational mechanism that unites them all into a totality that represents them, even if individual information is lost. When students choose a subset on a line from the point cloud, they consider the subset as an aggregate but ignore and discard some data; they do not see the cloud as an aggregate. The process of looking for the line closest to the point cloud with a notion of distance shows that each one of the points contributes to determining said line. This fact is significant because the literature has highlighted that one problem for students to understand the statistical notions of centers, variation, data comparison, and data distribution, is that they see the budding data set as an added (Bakker & Gravemeijer, 2004; Konold & Higgins, 2002; Roseth et al., 2008). Also, the context is an important factor to see a data set as an aggregate since in this case, it is the existence of a *causal mechanism* (the term is from Zimmerman, 2007) that suggests that all the data are related, in our case, that ANE and fat accumulation are part of a causal process.

Conclusions

A necessary condition for understanding the topics of regression and correlation is conceiving a set of bivariate data as an aggregate. However, students do not reach this conception spontaneously and probably not with traditional methods as they have been taught the topics at the high school level. There are two levels in which it is convenient to analyze the appearance and development of a set of points as an aggregate: the mathematical level and the contextual level. At the mathematical level, technological resources allow students to move from seeing a cloud of points as individual points or fragmented into subsets to seeing it as an aggregate based on a mathematical mechanism that links them with the notion of distance from a cloud to a straight line. At the context level, it is the existence of a causal mechanism that helps to imagine a unity in the data set. It would remain to work on the sources of variability for students to explain why the data differ from the probable causal model.

Note

[1] We use the term "cloud" of points or simply "cloud" to refer to bivariate data represented in the scatterplot.

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DYNAMIC SPATIAL DIAGRAMS AND SOLID GEOMETRIC FIGURES

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This paper reports on a study of learners' use of immersive spatial diagrams to make arguments about three-dimensional geometric figures. Immersive spatial diagrams allow learners to use the movement of their bodies to control their point of view, while immersed in three-dimensional digital renderings. We present analysis of two pairs of pre-service elementary teachers' argumentation about the shearing of pyramids, using the ck¢-enriched Toulmin Model of Argumentation (Pedemonte & Balacheff, 2016) to link the affordances of immersive spatial diagrams to the learners' mathematical reasoning. We share how one pair of learners took points of view bending beside and standing within the pyramid to describe how the space inside is transformed without reference to one- or two-dimensional components of the representation.

Keywords: Measurement, Technology, Geometry and Spatial Reasoning

Diagrammatic Representations of Three-Dimensional Figures

Three-dimensional geometric figures are often represented with diagrams on two-dimensional canvases in school geometry (Clements et al., 2017; Dimmel & Herbst, 2015; Dorko & Speer, 2015, 2013; Duval, 2006; Pittalis & Christou, 2010; Stevens et al., 2015), mediated through projection or cross-section. These frozen perspectives split learners' attention between spatially navigating the diagram and attending to the features of the mathematical figure. While 3D dynamic geometry software (*e.g.*, Cabri3D) allows learners to make visual observations from many points of view (Mithalal & Balacheff, 2019), these points of view are often controlled by two-dimensional (*e.g.,* touch, mouse) or keystroke-based input systems. Learners have difficulty working with two-dimensional representations of three-dimensional figures without spatial observation of the 3D shape (Pittalis & Christou, 2010).

Further, it is often impracticable to change the perspective *while* continuously manipulating the figure. Continuous manipulations of a diagram are important because they support learners' reasoning in school geometry. The continuous manipulation of dragging can support learners noticing the spatial properties of the diagram that are mathematically necessary (Clements, 2003; Laborde, 2005) and dragging can also allow for geometric transformations to be represented as "continuous and temporal" processes (Ng & Sinclair, 2015, p. 85). Observations of two-dimensional representations of three-dimensional figures may focus learners' struggles on navigation and manipulation of the diagram rather than on discerning which spatial properties of the diagram are incidental or mathematically necessary.

Physical spatial inscriptions (*e.g.*, 3D pens) are one alternative to two-dimensional renderings of three-dimensional figures. Using physical materials (*e.g.,* extruded plastics), diagrams can take up space and be manipulated by the learners' grasp (Ng & Sinclair, 2018). Further, learners can vary their point of view as they might with any other physical object – by walking, turning and bending their body and turning their head. However, physical spatial inscriptions have material constraints and are not generally able to be manipulated continuously with nonrigid transformations.

Immersive spatial diagrams are digitally rendered diagrams that share the learners' spatial environment, like physical spatial inscriptions, but also offer the digitally rendered flexibility of

3D dynamic geometry software (Bock & Dimmel, 2020; Dimmel et al., 2020). Immersive spatial diagrams can be rendered using various consumer-ready devices: virtual-reality head-mounted-displays (*e.g.,* HTC Vive, Oculus Rift*),* augmented-reality head-mounted-displays (*e.g.,* Microsoft Hololens*),* and mixed-reality head-mounted-displays (*e.g.,* Varjo XR-3). These diagrams bring learners into a world where space-occupying objects can have the dynamic properties of digital renderings or bring those dynamic spatial objects onto the learners' physical world. Immersive spatial diagrams offer learners an opportunity to explore the properties of spatial representations governed by mathematical laws rather than the laws of physics.

By combining embodied control over point of view, continuous transformations, and a three-dimensional visual experience of the diagram, immersive spatial diagrams offer learners new modes of interactions with representations of three-dimensional figures. In this study, we explored how learners interacted with an immersive spatial diagram — a dynamic, digital, three-dimensional representation of a pyramid bound between parallel planes, focused on the affordances that allow learners to use their body to access multiple points of view. We asked: How do the points of view that learners take while immersed in a spatial diagram shape their argumentation about geometric transformations?

Theoretical Framework: ck¢-enriched Toulmin Model of Argumentation

We used the conception-knowing-concept (ck¢) enriched Toulmin model of argumentation (Pedemonte & Balacheff, 2016) to analyze the arguments that learners constructed while using an immersive spatial diagram. The ck¢-enriched Toulmin model of argumentation situates Balacheff & Gaudin's (2010) conception-knowing-concept models' rich description of learners' reasoning about a mathematical context within the Toulmin (1958) model's transformation of observed data into a claim through inference. The ck¢ model describes mathematical conceptions in terms of observable components of the interactions between learners and their environment (Balacheff & Gaudin, 2002, 2010; DeJarnette, 2018; Herbst, 2005; Mithalal & Balacheff, 2019). We chose this model to highlight how the points of view available to learners, as a constraint on their observations used for data in their argument, shaped their claims and inferences with a rich mathematical characterization. Components of the (ck¢) enriched Toulmin model are explained in greater detail with application to spatial diagrams in Bock and Dimmel (2020).

Mathematical Context: Shearing of Pyramids

The mathematical context for the study was the *shearing* of a pyramid between parallel planes. In plane figures, a transformation is a shearing transformation if a figure can be bound between parallel lines such that the lengths of the parallel cross-sections of the figure are preserved by the transformation (Ng & Sinclair, 2015). Shearing can be extended as a volume-preserving transformation of 3-dimensional figures bound between parallel planes. Consider a pyramid whose apex is bound to a plane parallel to its base (Figure 1: *ABCDE*).

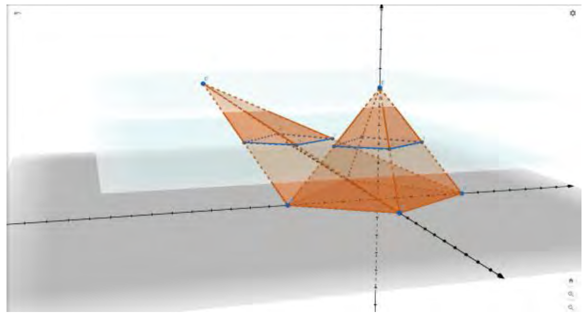

Figure 1: Pyramid *ABCDE* is sheared to form Pyramid *ABCDE'*

Any non-empty intersection of a pyramid and a plane parallel to its base is either the apex of the pyramid or a dilated copy of the base of the pyramid. In the case of a dilated copy of the base (Figure 1: *FGHI*), this cross-section is translated as the apex of the pyramid moves along a plane parallel to its base (Figure 1: *F'G'H'I'*). Then the area of the cross-section and the volume of the pyramid are preserved under the shearing transformation.

Figure 1 illustrates some difficulties with visual observations of three-dimensional geometric transformations mediated through a two-dimensional canvas. It is not immediately obvious if the cross-sections of the pyramid are (at least approximately) congruent. While a diagram, on its own, is not sufficient to prove geometric relationships, accurate diagrams are powerful heuristics that can suggest what relationships one ought to try to prove (Larkin & Simon, 1987). However, the two-dimensional diagram of a pyramid is caught in a conflict between seeing and knowing (Parzysz, 1988): perspectives that allow the pyramid to be *seen* as a figure that occupies space distort the polygonal cross-sections of the pyramid. As a result, it is difficult to show how the cross-section is transformed while showing how the cross-section relates to the volume of the pyramid.

Design of the Virtual Environment

We designed a virtual environment (Bock et al., 2020) where learners could explore the shearing of a pyramid with an immersive spatial diagram (Figure 2A). Learners could use pinch, drag and throw gestures to manipulate the apex of the pyramid, which was bound within a plane parallel to its base (Figure 2B). An open-palm gesture, parallel to the pyramid's base controlled the position of a cross-secting plane (Figure 2C). Instead of offering numeric representations of measure, a cube with volume equal to the volume of the pyramid and a square with area equal to the surface area of the pyramid could be loaded into the environment (Figure 2D). Finally, the participants had previously explored an analogous case of a triangle bound to parallel lines (described in Bock & Dimmel, 2020), which was also available for reference.

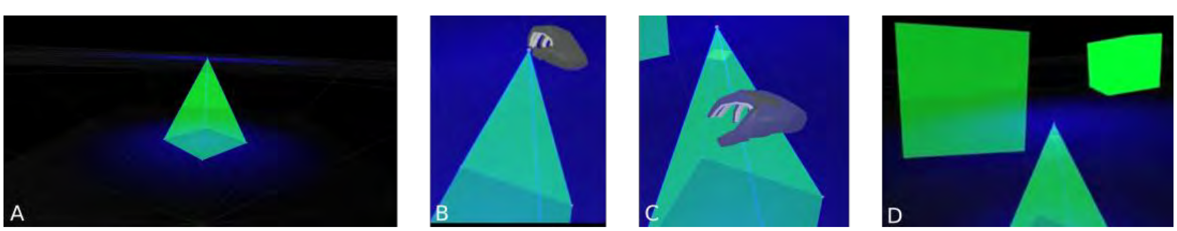

Figure 2: The Virtual Environment

Pedagogical Rationale

The environment was designed to provide immersed participants with direct embodied control over their point of view; it featured a gesture-based interface that allowed immersed participants to move the apex of the pyramid and investigate its cross-sections. By excluding numeric measures, we hoped to discourage learners from using routine empirical calculations (*e.g.,* calculating the area of the base as the square of the length of the sides, calculating the volume as the product of the area of the base and the height) to reason about the effects of the shearing transformation on the pyramid. We hypothesized that learners would instead make arguments using the congruence of stacks of cross-secting planes to make sense of the shearing of the pyramid – either as units of area, or as approximations to small units of volume.

Methods

This study used a case-study methodology (Yin, 2012), where a set of arguments made using a diagram was the unit for a case. We analyzed a set of arguments constructed by pre-service elementary teachers using an immersive spatial diagram at a public university in the United States. Participants worked collaboratively with asymmetric roles. One participant was immersed in the environment, via a head-mounted display, the other participant, who was not immersed in the virtual environment, viewed a real-time projected video of the immersed participants' interactions on a television screen. Though the learners' views of and roles in the environment were different, collaborative interactions have been analyzed in other settings with similar immersed and non-immersed roles where the non-immersed learner's view was mediated through a two-dimensional projection (Price et al., 2020, p. 216). We considered their co-constructed arguments as the unit of analysis.

Participants

Below, we analyze the argumentation of two pairs of participants. All four participants identified as female. The first pair of participants were a junior and senior pre-service elementary teacher with concentrations in mathematics and art, respectively. The second pair of participants were two first-year pre-service elementary teachers without selected concentrations. Each participant is referred to using a pseudonym.

We archived the participants' experiences using first person composite, mixed-reality composite (BluePrint Reality, 2017; Sheftel & Williams, 2019), and third-person physical views as well as a microphone for recording dialog between participants and interviewers (see Bock & Dimmel, 2020, p. 13). The mixed-reality view blends together the virtual with the actual, offering an observer's perspective on how the immersed participant navigated the immersive environment. We used these video records to identify episodes where participants made geometric arguments about the effects of shearing on the measures of the pyramid. We then used these episodes to construct ck¢-enriched Toulmin models of each argument.

Example of Analysis

We analyzed three arguments from two pairs of participants using the ck¢-enriched Toulmin model of argumentation. We report here on one excerpt of one of those analyses, as a means of illustrating how we applied the ck¢-enriched Toulmin Model. Each of the components of the ck¢-enriched Toulmin models are developed from the video records and transcriptions. Figure 3 shows an enriched Toulmin model for an argument made by Emily and Olivia.

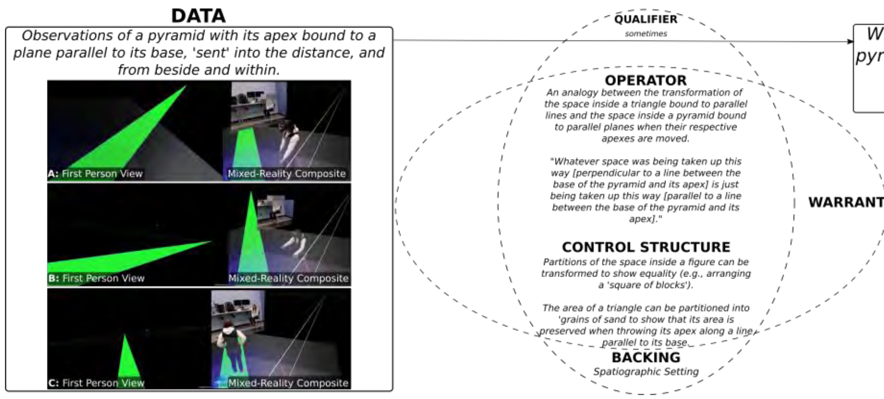

Figure 3. Model of Emily and Olivia's Second Argument

This excerpt explains the *data* element of the model in Figure 3, beginning after Emily and Olivia had experimented with manipulating the pyramid but had not begun developing an argument. Emily "sent" the apex of the pyramid into the distance by pinching, grasping and throwing the vertex. Emily waved goodbye as the apex was thrown along a line in a plane parallel to the pyramid's base.

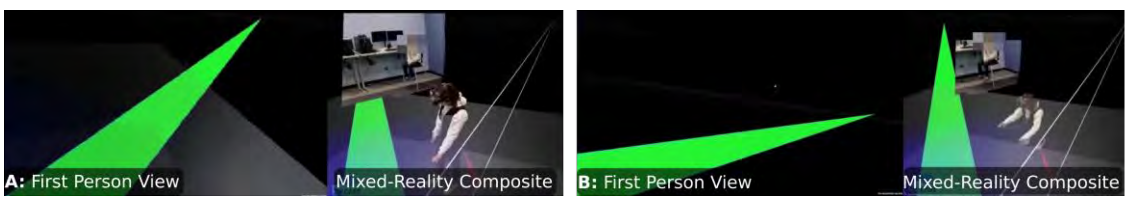

Figure 4: Emily's Points of View

As the apex continued to move away, Emily remarked that the pyramid "looks like its getting bigger" while her gaze looked along the length of the pyramid's nearest face (Figure 4A). Emily then walked beside the pyramid (Figure 4B), bent down, and remarked that "if you look at it...then it's getting so thin?". While in that position, Emily explained "whatever space was being taken up this way...it's just being taken up this way [gesturing along the length of the pyramid's face]." In this excerpt, Emily describes two visual observations: the pyramid "looks like it's getting bigger" and "if you look at it...then it's getting so thin." Emily's explanation of how the space inside the pyramid is being "taken up" showed that these observations serve as *data* (feedback from the virtual environment) to be transformed into their *claim* about the pyramid (see Figure 3).

Results

For each argument, a ck¢-enriched Toulmin model of argumentation was developed; these models are presented below and are accompanied by brief narratives. In our analysis, we were interested in how the feedback from the learner's environment – the *data* – shaped the mathematically rich descriptions of their conceptions in the *operator* and *control structure*. The data component of the model informs how the participants' interactions with the environment might have shaped their argumentation. The operator and control structure help to understand whether the learners used their interactions with the diagram to understand the shearing transformation differently then they might in other contexts.

A Point of View Within the Spatial Diagram: Emily and Olivia's Arguments

Emily and Olivia made two arguments about the shearing of a pyramid. In their first argument Emily and Olivia claimed that when you "send" or throw the apex of the pyramid into the distance within its plane, its volume will be conserved. Emily wore the head-mounted-display, while Olivia observed a mixed-reality third-person view on a large television screen and took three points of view within the virtual environment – standing beside the pyramid and dragging the apex of the pyramid locally (Figure 3A), bending down alongside the pyramid after the apex had been thrown (Figure 3B), and standing with her legs intersecting the pyramid (Figure 3C). While standing inside the pyramid, Emily remarked "this kind of looks like a road, I feel like Dorothy... I wish I could see my legs and see them being chopped off by the planes [faces of the pyramid]." Emily and Olivia used these observations from these three points of view to develop an argument that explained how it would be plausible that the space inside the pyramid is redistributed by the shearing transform such that the volume is conserved.

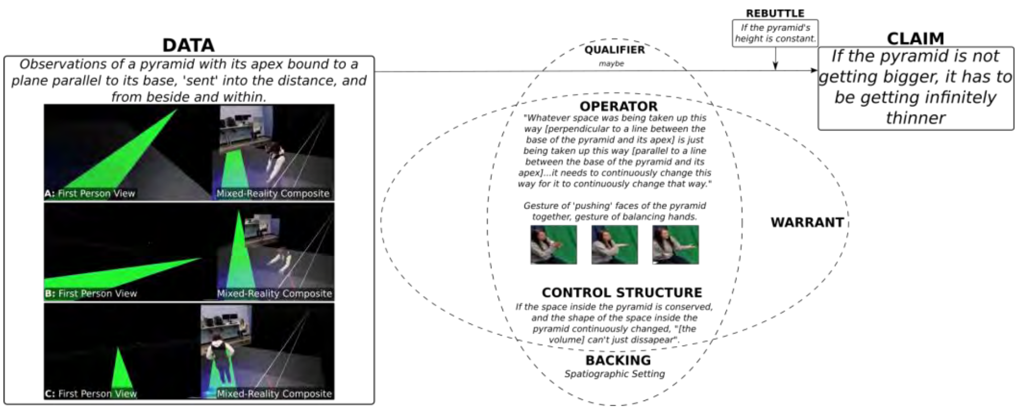

Figure 5. Model of Emily and Olivia's First Argument

After Emily and Olivia constructed this argument, the interviewers prompted: "is there anything else about the pyramid?... is there anything else about the pyramid changing?" Emily noted that the "length" [altitude] of the sides is becoming "super, super long... this looks infinitely long." While not infinite, the apex of the pyramid continued to move indefinitely into the distance. The interviewers prompted "so are you saying that a pyramid with infinitely long sides [faces] can have a finite volume?" Emily and Olivia then constructed another argument (Figure 5) to describe how the space inside the pyramid would need to change if the pyramid's volume is constant. Olivia described how "it would have to also get infinitely thin [as it is sheared], if it's not flattening out then I don't know where the space inside would like go," repeating gesture where she had her palms facing together and then pushed her palms together while tilting horizontally (Figure 5A). In this argument, Emily and Olivia reframed the warrant and control to be in terms of continuous and temporal transformations – describing how the pyramid is "flattening out", "getting infinitely thinner", and would need to "continuously change this way for it to continuously change that way". Emily and Olivia added a rebuttal that the height of the pyramid must be constant, however it was not clear why they attended to this measure.

Points of View from Above and Beside the Diagram: Abigail and Madison's Arguments

Abigail and Madison made an argument about the unbounded shearing of a pyramid where the only points they used were above and beside the pyramid. Abigail and Madison's argument included an argument about the area and perimeter preserving properties of shearing on a triangle to conclude that the volume and surface area of the pyramid would be analogously preserved under shearing (Figure 6). Abigail and Madison used a set of visual observations of the measures of the angles of the vertices at the base of the triangle and their line segments (Figure 6B) and visual observations of the behavior of the faces of the pyramid (Figure 6A) as they sheared each of the figures. While Abigail and Madison's warrant would not be understood to support their claim in a school mathematics context, the ck¢-enriched Toulmin model situates their conception within the feedback – or data – from their environment. During these observations, Abigail and Madison did not 'send' the triangle or the pyramid, so they did not have disconfirming feedback for their conclusions.

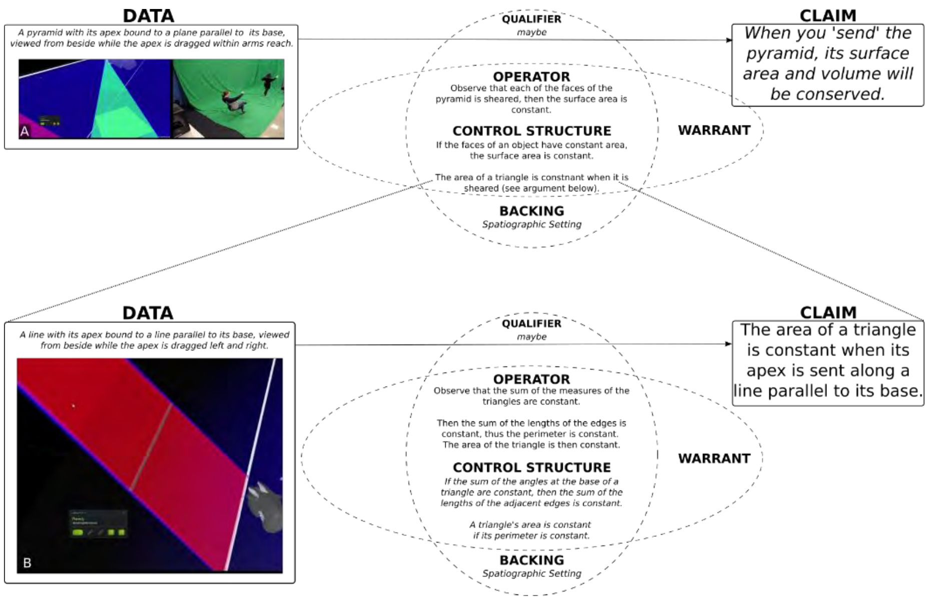

Figure 6. Model of Abigail and Madison's Argument

Discussion

In a school mathematics setting, the arguments developed by each pair of participants might feel incomplete – their arguments would need to be refined to be a rigorous explanation of the properties of the shearing transformation. With the lens of the ck¢-enriched Toulmin model we can look past a superficial evaluation of correctness to understand how the affordances of immersive spatial diagrams and the environment design supported their arguments, and the contexts where these diagrams might be useful in a less exploratory pedagogical setting.

Points of View

Both pairs of participants used their control over the point of view in the environment in ways that would be impracticable to replicate outside of immersive spatial diagrams: they used gestures to manipulate diagrammatic representations of pyramids and triangles while walking, bending, and turning their heads to make visual observations. Emily and Olivia took two points of view that would be difficult to replicate with two-dimensional diagrams: bending down beside and standing inside the pyramid. Emily took these points of view in order to share visual observations with Olivia and the interviewers as they constructed their argument. While we anticipated that participants might put their heads inside the pyramid, we did not anticipate the use of these points of view in the environment design. In contrast, Abigail and Madison engaged with the diagram using each of the novel affordances of immersive spatial diagrams but their visual observations of the triangle and pyramid are practicable to recreate with a two-dimensional dynamic representation of the figures.

Dimensionality

Emily and Olivia's argument had another unique feature – they described the continuous transformations of the space inside the pyramid and the triangle without describing lower-dimensional quantities. The points of view Emily shared beside and within the pyramid and Emily and Olivia's description of cross-sections of the pyramid through Emily's legs suggest that immersive spatial diagrams offer learners an opportunity to engage with three-dimensional mathematical figures without reconstruction from two-dimensional or one-dimensional elements. In contrast, Abigail and Madison's argument might have been better supported by the environment if measurement tools for angles, lengths and areas were available in the environment. Measurement of the triangle's angles or segments might have suggested that they do not correlate as Abigail and Madison suspected; measure of the faces of the pyramids might have suggested a changing surface area as the pyramid was sheared. This is a constraint of the environment design – not of immersive spatial diagrams – but also a feature that is easily accessible in many traditional dynamic geometric environments.

Existing research on immersive spatial diagrams has focused on representing mathematical figures with numeric representations of length and symbolic representations of area and volume (Lai et al., 2016), rigid transformations of static shapes (Gecu-Parmaksiz & Delialioglu, 2019), and gesture-based construction (Dimmel & Bock, 2017). This study was designed to explore how learners used points of view with immersive spatial diagrams to reason – and struggle with – the properties of continuous geometric transformations of three-dimensional figures. The results of this study explore two cases where learners investigated the shearing of three-dimensional figures, an extension of research on how learners reason about the shearing in plane geometry (Bock & Dimmel, 2020; Ng & Sinclair, 2015). In one case, the pair of participants identified mathematically relevant spatial invariants (volume of the pyramid, height of the pyramid) and described how these properties might relate to the shape of the pyramid for the spatial invariants to be plausible. However, the participants' argument did not relate properties of the figure to explain why the shearing transformation *necessarily* preserves volume. This is consistent with expectations from learners use of two-dimensional dynamic geometry environments, where dragging affordances have been linked to identification of spatial invariants (Clements, 2003). This process of "learning [to] identif[y] of visually relevant spatiographic invariants attached to geometrical invariants" is an important to the learning of geometry, alongside deductive reasoning from theoretical statements (Laborde, 2005, p. 177).

Conclusion

Emily and Olivia struggled productively to describe continuous transformations of volume without reducing to lower-dimensional elements, confidently reasoning from visual observations. This addresses a key constraint of two-dimensional representations of three-dimensional geometrical objects — that the figures must be analyzed through reconstruction from lower-dimensional components of the representation (Mithalal & Balacheff, 2019). Further research is needed to explore how spatial diagrams can be designed for learners to *see* or attend to one-, two-, or three-dimensional elements of figures, analogous to diagrammatic representations of two-dimensional figures (Duval, 2006, p. 116). Finally, there is an opportunity to explore how learners' analysis of three-dimensional figures without reconstruction from one- and two-dimensional elements might be designed into geometric construction environments and where this might best support learners' reasoning in school geometry.

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THE GEOMETRY OF SUNLIGHT: CONTINUOUS MULTIPLICATION WITH NATURALLY OCCURRING PARALLEL LINES

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We report the design of an analog technology, what we refer to as a SunRule, that uses sunlight to model multiplication. Physical models that explore multiplication are fixtures in elementary mathematics classrooms. Our interest in physical models of multiplication was driven by an overarching design problem: How could a physical tool realize a continuous model of multiplication? That is, how could we represent continuous, variable quantities with physical things? We identify specific challenges the SunRule was designed to solve. We explain the mathematical underpinnings of the device and report a teaching experiment during which pre-service teachers explored the device in small, socially-distanced groups. We consider how explorations with the SunRule create opportunities for mathematically rich instructional activities that are also essentially connected to being outside.

Keywords: Technology, Measurement, Geometry and Spatial Reasoning, Design Experiments.

Introduction

Sunlight provides an abundant, renewable, accessible source of naturally occurring parallel lines. It is the rare example of a mathematical contextualization with which nearly all children are familiar. Despite its familiarity and universality, sunlight plays almost no part in K-12 mathematics classrooms. Furthermore, while sunlight is among the closest physical realizations to the Euclidean ideal for parallel lines, there is scant research about how K-12 students might use sunlight and the real-world parallel rays it provides to engage in mathematical activities. But in the shadow of the global pandemic, when so much of schooling has moved to screens, there is an urgent need for outdoor, socially-distance-able activities that have robust mathematical designs – i.e., designs where mathematical concepts are intrinsic to the activity. To respond to this need, we report the design of an analog technology, what we refer to as a *SunRule*, that uses sunlight to model multiplication. We explain the mathematical underpinnings of the device and report an initial teaching experiment where pre-service teachers explored the device in groups. We consider how investigating multiplication with the *SunRule* can challenge familiar notions of contextualized mathematics.

Background & Design Problem

The sun shadows phenomenon

Sunlight and what has been described as the *sun shadows phenomenon* was used as a tool by Garuti and Boero (1992) to investigate geometric proportionality as a physical phenomenon with 11 and 12 year olds. This study offered promise that embedding problem situations in a context in which directly experiencing the geometrical-physical aspect is paramount may move students from an additive model to a multiplicative one. Building on this early success, the Genoa Group

(Boero, Garuti, & Mariotti, 1996a; Douek, 1999) used the overarching approach of examining heights of objects and the lengths of the shadows they cast to pursue a wide range of research questions within the sun shadows field-of-experience (Boero, 1989). Topics included argumentation, conjecture, proof construction, and angle concepts, among others. They all shared the structure of teaching experiments that capitalized on shadows cast by the sun (either imagined or observed and recorded experimentally) to explore problem situations in different dynamical ways (Boero, Garuti & Mariotti, 1996b). Students produced, through open problem solving situations, meaningful conjectures from a space geometry point of view. Douek (1999) also demonstrated the link between context-related arguments, mathematical modelling, and conceptualization of geometric ideas. The present study extends this work to use geometric proportionality of shadows cast by the parallel rays of the sun (Decamp & Hosson, 2012) to generate products of real numbers.

Models of multiplication

Various physical and visual aids are used to model multiplication in elementary classrooms (Kosko, 2019). There are discrete models that involve arranging things, such as playing chips, into equal-sized groups. For example, the problem (2)(3) could be represented as two groups of three things each or else three groups of two things each. Discrete models frame multiplication as a kind of repeated addition, and this is one of the most widely-used models to conceptually define multiplication (Hurst, 2015; Vest, 1985). But discrete models are harder to physically realize with fractions and decimals. Visual models that use area to represent multiplication are an alternative. For example, the numbers to be multiplied could be arranged as the length and width of a rectangle, and the area of the rectangle would be the product (Reys et al, 2014; National Governors Association 2010, 25). An advantage of this continuous model is that it applies to any of the kinds of numbers children encounter in school (Kosko, 2019). A drawback is that it models unidimensional numbers—that is, single points on a number line—as areas, thereby misrepresenting products as two-dimensional (McLoughlin & Droujkova, 2013).
Physical models are pedagogically compelling because they can create diverse avenues for exploration and learning (Clements, 2000; Domino, 2010). Our interest in physical models of multiplication has been driven by an overarching design problem: How could a physical, manipulable tool realize a continuous model of multiplication? That is, how could we represent continuous, variable quantities with physical things?

Design Framework

Diagrammatic multiplication

Our answer to this question was inspired by a geometric interpretation of multiplication that is predicated on the following observation:

> the hypotenuse of the right triangle determined by an object and its shadow must be parallel to the hypotenuse of any other object and its shadow. Hence, knowing the shadow of one object (we call this object the unit) gives us a way to deduce the shadow of any other object. (McLoughlin & Droujkova, 2013, p. 2)

From this observation, McLoughlin and Droujkova (2013) developed a diagrammatic definition that models multiplication as *continuous directed scaling*—*i.e.*, the length of one segment is a positive or negative multiplier that stretches the length of another segment in the positive or negative direction (Dimmel & Pandiscio, 2020). We initially realized this geometric definition of multiplication in a dynamic diagram that had draggable points (see Figure 1).

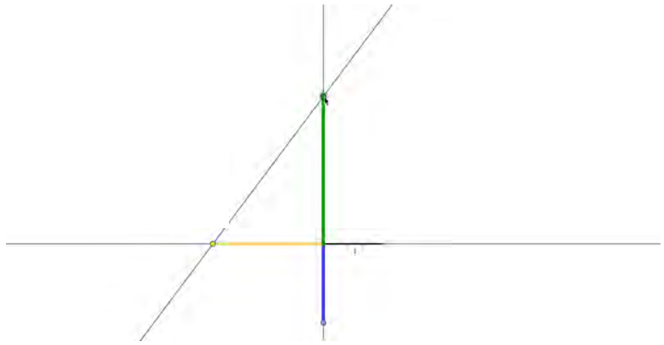

Figure 1. A dynamic diagram that realizes a geometric definition of multiplication. The yellow and blue points can be dragged along their respective axes to increase or decrease the lengths of the yellow and blue segments. The line that intersects the vertical axis determines the product of these lengths, which is represented by the green point.

In these diagrams, the "parallel to the hypotenuse" condition described above was satisfied by constructing a parallel line whose point of intersection with the *y*-axis would specify the product. The dynamic diagram we developed allowed students to use continuous transformations to explore ranges of products, such as products for pairs of numbers that are between 0 and 1 (Thompson & Saldanha, 2003). Because physical models for exploring arithmetic are common in elementary mathematics classrooms (e.g., Base Ten blocks, Cuisenaire rods, Pattern Blocks, Unifix cubes, chips and counters), we sought to design a physical embodiment of the multiplication diagram.

The variable altitude and variable length design problems

The keystone of the geometric definition of multiplication is parallel lines. Fortunately, sunlight offers a readily available, renewable, and abundant supply of naturally occurring parallel rays. The problem with using the sun as the source for parallel lines is that, at any time, the sun appears in one (and only one) position in the sky, and this position determines the proportion between an object's height and the length of its shadow (Douek, 1999). Thus, to multiply numbers in general requires control over the position of the sun. We refer to this as the *variable altitude* design problem.

Of course the sun cannot be moved, but there is nevertheless a solution to the variable altitude problem: We can change the *apparent* altitude of the sun by varying the angle of inclination of a surface onto which shadows are cast. By increasing the angle of inclination of a surface (i.e., the *shadow plane*), we decrease the lengths of any shadows falling upon it; by decreasing the angle of inclination, we increase the lengths of those shadows. Thus, by varying angles of inclination, it is possible to control the apparent altitude of the sun from 90 degrees (directly overhead, no shadow) to 0 degrees (sun on the horizon, undefined/infinite shadow).

The inclined plane provides control over the multiplier in a multiplication product – by varying the angle of inclination of a shadow plane, it is possible to stretch or shrink the length of the shadow of whatever object has been determined to be the multiplicative unit. What remains is a means to vary the multiplicand. This requires some method for increasing/decreasing length. We refer to this as the *variable length* design problem. The historical solution to this problem was the *slide rule,* an arithmetic aid that reigned from the 17^{th} century until it was abandoned for electronic calculators in the 1970s (Cajori, 1909; Tympas, 2017, 7-8). We adapted the sliding action of a slide rule, though not its logarithmic scales, to solve the variable length design problem.

Figure 2 shows a prototype of a device that embodies solutions to the *variable length* and *variable altitude* design problems. We refer to this device as a *SunRule*. It is not, strictly speaking, a combination of a sundial and slide rule; however the name is apt because it combines essential elements of each device (e.g., gnomons[1], adjustable scales) in novel ways.

Figure 2. A *SunRule* consists of a ruled board (shadow plane) and rods that are orthogonal to the ruled board. By changing the angle of the shadow plane, one changes the length of the shadow of the shorter rod, which serves as the multiplicative unit. The height of the longer gnomon represents the multiplicand. The device works because rays from the sun are parallel.

Method

Our initial plan was to analyze how pairs of elementary mathematics teacher candidates explored the *SunRule*. That plan is on hold until it becomes safe for pairs of students to interact in close proximity. In an effort to persevere through the challenge of data collection during the pandemic, we developed a handheld version of a *SunRule* that could be constructed from common household items (Figure 3). Thus, multiple devices could be built, which allowed students to interact at safe distances.

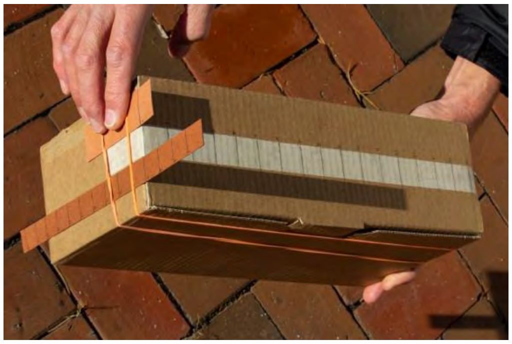

Figure 3. A handhels *SunRule*, constructed by elementary teacher candidates. The *SunRule* shows that (3)(4) = 12. Photo by Meg Pandiscio (2020).

In Figure 3, there is a longer gnomon (bottom) and a shorter gnomon (top). The shorter gnomon functions as a unit length. The unit length and the factor by which its shadow is stretched define a multiplier; in this case, that multiplier is (3), since the device has been inclined so that the length of the unit shadow extends (3) units. The height of the longer gnomon can be adjusted by sliding it up or down; the height of this gnomon specifies the multiplicand, which in this case is (4). The product is (12), shown here by the length of the shadow of the adjustable gnomon.

We report here an initial teaching activity where elementary teacher candidates built *SunRules* and used them to explore multiplication. We frame the initial teaching activity as a teaching experiment (Steffe & Thompson, 2000), where the second author was in the role of researcher-teacher. The purpose of the experiment was to generate hypotheses about how interacting with a *SunRule* creates opportunities for pre-service teachers to explore multiplication conceptually.

Context

During Fall, 2020, the second author taught an elementary mathematics methods course that convened in a hybrid in-person/online format. To comply with limits on indoor gatherings, the in-person students were split across two, five-student sections of the course that met on different days. The *SunRule* activity was planned as a two-class lesson that would allow elementary mathematics teacher candidates to explore a physical model of multiplication. For the first part of the activity, students worked with the second author to build *SunRules*. For the second part of the activity, students explored the *SunRules* outside, in small groups, while wearing masks and maintaining social distance. Both in-person sections of the course completed the first part of the activity. Students were told that the device had something to do with mathematics and that it needed to be used outside, on a sunny day. Figure 4 shows a selection of student-constructed *SunRules*.

Figure 4. *SunRules* constructed by elementary mathematics teacher candidates. Photo by Meg Pandiscio (2020).

Data collection

For the second part of the activity, five students from one section of the course[2] explored the *SunRules* in groups of two and three. The students within each group maintained social distancing throughout the activity, and the groups were separated by approximately twenty feet. Fixed video cameras recorded the activity of each group. The second author moved back and forth between the groups to facilitate their explorations of the device, following a semi-structured protocol. The protocol was designed to provide gradually more directed guidance to the groups of students. An example of a minimally directed question is, “What does the tool do?” An example of a more directed question is, “What are the ways that the lengths of the shadows of the gnomons could be varied?" The second author posed questions from the protocol to each group, as needed, to keep the students from getting stuck and spur them toward investigations of its mathematical opportunities. Below, we describe two episodes that capture how students explored and interacted with the *SunRule*. Episodes were identified by reviewing the video records of the teaching experiment and looking for instances where the tile of the shadow plane or the height of the gnomon was adjusted.

Episode 1: Sara's initial encounter with the *SunRule*

One group consisted of two students, Zak and Sara[3]. The second author launched the exploration activity for them by asking, "Any idea what this box does?" Sara replied, "Not yet", though as she said this, she had positioned the *SunRule* so that it was aligned with the azimuth of the sun, which caused the shadows of the gnomons to fall in parallel along its ruled surface (Figure 5).

Figure 5. While Sara declares that she does not know what the device does, she has oriented the device the way that it was designed to be oriented.

In this instance, Sara has guessed – in the technical sense of Wobbrock et al (2005) – how to interact with the device. She may not know what it does, but she already knows how it must be positioned in order to do it. Her next moves were to change the angle of inclination of the device. She tilted the device toward and then away from the sun, which caused the shadows of the gnomons to shorten and then lengthen (Figures 6, 7). As she varied the angle of inclination, she and Zak speculated that the device indicated a relationship between the sun and the shadows.

Figure 6. Sara inclines the device more toward the sun, which increases the sun's apparent altitude and causes the shadows of the gnomons to shorten.

Figure 7. Sara inclines the device less toward the sun, which decreases the sun's apparent altitude and causes the shadows of the gnomons to lengthen.

Sara noted the significance of the angle of inclination to the length of the shadows, "It really depends on how you hold it, like, if you tilt it towards (*sic*) the sun, then the shadows become very short, if you tilt it away from the sun the shadows get a lot longer." These initial interactions that varied the lengths of the shadows by changing the angle of inclination are the core of the mathematical design of the *SunRule*. This feature was salient for Sara almost immediately and suggests that the grounding predicate for the geometric definition of multiplication (block quote, above) is a natural and potentially powerful embodiment for a continuous scaling conception of multiplication.

Episode 2: Modeling division with the *SunRule*

After 10 minutes of open-ended exploration, both groups had zeroed in on the idea of the shadows varying in a constant ratio as the angle of inclination of the device was increased or decreased. As neither group had connected their observations about ratio to the operation of multiplication, the second author assembled the groups in a socially-distanced semicircle. He summarized the ratio ideas each group had discussed, and then stated that a mathematical operation the device could model is multiplication. Sara then demonstrated how the device could be used to show that (2)(3) = 6. The second author adjourned the groups to their respective places and asked them to continue exploring how the device could be used to model products.

In their discussion of multiplication, Zak and Sara realized that the device could also be used to represent division. Zak demonstrated this idea, which he attributed to Sara, by showing how the multiplication problem (5)(2) = 10 could be interpreted as the division problem (10)/(5) = 2 (Figure 8).

Figure 8. Zak positions the *SunRule* to show the quotient that (10)/(5) = 2.

To multiply with the *SunRule*, the angle of tilt varies the length of the shadow of the unit gnomon. This increase/decrease in the length of the unit shadow amounts to a scale factor that is applied to the length of the shadow of the other gnomon. To complete the product, one sets the height of the longer gnomon equal to the number that is being multiplied. The shadow of this gnomon is the answer. Zak and Sara realized that to represent division with the *SunRule* would require reversing this process; or, as Zak said: “there is an inverse relationship between multiplication and division.”

To use the *SunRule* to divide two numbers, set the height of the adjustable gnomon to be the divisor. Then, vary the angle of inclination of the *SunRule* so that the length of the adjustable gnomon’s shadow is the number that is being divided. The quotient will then be given by the length of the unit gnomon’s shadow. Sara made the connection to division within moments of demonstrating how the device represented multiplication. She and Zak conjectured that division should be possible to represent with the device and then worked out how that was possible. Zak demonstrated this, while narrating his *SunRule* manipulations. Zak pointed to the long shadow and said “10 divided by 5” and then pointed to the adjustable gnomon. He then said “equals 2” as he pointed to the shadow cast by the unit gnomon. He further noted that “2 is then the answer and it's the short shadow.” Sara and Zak's explorations of the connection between multiplication and division underscore the rich pedagogical opportunities of the SunRule.

Discussion and Reflections

The teaching experiment reported here documented pre-service teachers’ initial encounters with a physical device for modeling multiplication through continuous movements – e.g., tilting the device more or less, sliding the gnomon up or down. The movements made by Sara to vary the angle of inclination of the shadow plane and Sara and Zak’s linking of multiplication to division offer preliminary indications that the device worked as it was designed to work. Zak’s and Sara’s explorations of the *SunRule* suggested that it can be used to explore how multiplication and division are conceptually linked; we plan to develop and explore this hypothesis in follow up teaching experiments.

The *SunRule’s* connection to the real world is immediate, rather than applied or abstracted. The *SunRule* doesn’t apply mathematics to explain the world, rather, it uses an affordance of the world (sunlight) to model a mathematical operation (multiplication). Simultaneously, it shares a mathematically valid and robust representation of multiplication that is often missing in elementary school classrooms—that of multiplication as continuous scaling (Dimmel & Pandiscio, 2020; Kosko, 2019). By using a feature of the world to build a robust mathematical model, the *SunRule* represents an inversion of what is typically encountered in authentic/contextualized/real world mathematics.

The COVID-19 pandemic has triggered a reconsideration of how we gather. For schools, this has meant adapting instruction to remote, hybrid, or outdoor modalities, among other innovations, some of which will (hopefully) endure even when it is safe again to gather indoors. The *SunRule* provides a concrete material context for doing a mathematical activity outside—not simply for the sake of being outside, but because being outside is essential to use the device to do mathematical work. It provides a variable, tangible device for modeling families of multiplication problems and probing their mathematical structure. Beyond arithmetical utility, activities with the *SunRule* could pull students away from screens and create opportunities for students and teachers to reflect on how the geometry of sunlight is integrated with its design. These would be enviable outcomes at any time, and they are especially urgent in the face of the disruptions to teaching and learning brought on by the pandemic.

Notes

[1] This is the name for the part of a sundial that casts a shadow.
[2] The other section's opportunity was precluded by inclement weather.
[3] All names pseudonyms.

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RESOLUCIÓN DE PROBLEMAS VERBALES CON GEOGEBRA: UNA FUENTE DE POSIBILIDADES EN EL ESTUDIO DE RELACIONES

WORD PROBLEM SOLVING WITH GEOGEBRA: A SOURCE OF POSSIBILITIES IN THE STUDY OF RELATIONSHIPS

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El propósito de este estudio fue analizar cómo influye en las formas de razonamiento de estudiantes de bachillerato el uso sistemático de GeoGebra cuando lo incorporan en la resolución de problemas verbales. El estudio se realizó con un grupo de 20 estudiantes que cursaba la materia de Matemáticas I. Los resultados muestran que los estudiantes se apropiaron de recursos del sistema de geometría dinámica (GeoGebra) que, al utilizarlos de forma sistemática, les permitió implementar estrategias para representar geométricamente los conceptos involucrados en los problemas, explorar y analizar relaciones entre los elementos de los modelos dinámicos construidos y hallar las soluciones. Así, gracias a la exploración y análisis de relaciones, fue posible discutir con los estudiantes conceptos como: razón, dominio, lugar geométrico, variación.

Palabras clave: Resolución de Problemas, Problemas Verbales, Tecnología Digital, Álgebra

Introducción

El Covid-19 ha provocado un cambio sustantivo en el uso de tecnologías para el ámbito educativo. Se ha producido una rápida difusión de estas para mitigar las problematicas provocadas por el confinamiento. En particular, en lo que a Educación Matemática se refiere, el uso de tecnologías digitales está en un estatus marginal, pues su implementanción siempre va por detrás de la velocidad de la evolución digital, incluso antes de las modificaciones forzadas por la pandemia Covid-19, y a pesar de que es una necesidad en el siglo XXI para la enseñanza de las matemáticas (Cevikbas & Kaiser, 2020). Por ello, es importante que los profesores conozcan y utilicen estratégicamente diversas tecnologías, para que todos los estudiantes tengan las mismas oportunidades y posibilidades de acceder a las matemáticas (National Council of Teachers of Mathematics, 2011).

Implementar tecnología digital en las clases de matemáticas puede ayudar a aminorar los aspectos técnicos del álgebra que surgen cuando se resuelven problemas (Arcavi et al., 2017). Esto permite a los estudiantes enfocarse en desarrollar recursos y estrategias que sean útiles para describir relaciones y resolver problemas que involucren la construcción y uso de relaciones funcionales, que son objetivos principales en el estudio del álgebra porque favorecen la comprensión conceptual de los procesos algebraicos (Kieran, 2020). En este sentido, es válido reflexionar si debieran modificarse tanto el contendido como la forma en que se imparte álgebra; es decir, cuestionarse si hoy en día son adecuados los temas que se enseñan en álgebra y la manera en que se enseñan (Thomas, 2017).

Un hilo conductor en el estudio del álgebra es la resolución de problemas verbales, que van desde la educación básica hasta la superior (Amado et al., 2019). A través de estos, se espera que los estudiantes experimenten, articulen y debatan diferentes acercamientos a la solución que promuevan el análisis y comprensión de los conceptos e ideas principales del álgebra. De hecho, una propuesta bien recibida entre los investigadores y profesores para enfrentarse a estos problemas ha sido desarrollar los episodios y etapas del marco de resolución de problemas (Blum y Niss, 1991; Verschaffel et al., 2000), porque favorece el uso de estrategias y el trabajo colaborativo en los estudiantes.

No obstante, se han realizado investigaciones sobre propuestas de enseñanza que incluyen el uso de tecnología en la resolución de problemas verbales para conocer el impacto que tienen en los estudiantes. Por ejemplo, Amado et al. (2019) presentaron resultados de un estudio donde, además de trabajar bajo el marco de la resolución de problemas, utilizaron hojas de cálculo con el objetivo de observar cómo sus herramientas pueden ayudar a estudiantes de secundaria (entre 13 y 14 años) a representar y resolver problemas verbales y a introducirlos a métodos algebraicos formales. Ellos encontraron que, aunque los estudiantes todavía no aprendían a usar representaciones algebraicas (sistemas de ecuaciones de dos o más incógnitas) para resolver los problemas verbales, la hoja de cálculo no solo les ayudó a resolverlos, sino también a que interpretaran la solución como un valor que satisface un conjunto de condiciones que están asociadas a las ecuaciones.

En este contexto, el uso de un sistema de geometría dinámica (SGD) puede ser clave para el desarrollo del pensamiento algebraico y funcional en la resolución de problemas verbales, porque permite representarlos, explorarlos y resolverlos desde un enfoque geométrico. Bozkurt y Uygan (2020) explican que los SGD permiten a los estudiantes manipular objetos geométricos y explorar relaciones entre ellos. Además, identifican el arrastre como una estrategia eficaz de estos sistemas, que es el resultado de mover elementos de las configuraciones dinámicas sin cambiar sus relaciones geométricas subyacentes. Finalmente, reconocen su potencial didáctico, y la importancia de no usarlos de forma convencional, como sistemas de softwares estáticos.

Aunque se han hecho estudios que involucran el uso de la tecnología digital en la resolución de problemas verbales, todavía falta más evidencia que refleje una forma eficiente y adecuada de su implementación (Verschaffel et al., 2020). Entonces, con el objetivo de aportar más evidencias a esta discusión, se planteó la siguiente pregunta que guio la investigación: ¿Qué tipo de razonamientos construyen y exhiben estudiantes del nivel medio superior cuando resuelven problemas verbales con el uso de un sistema de geometría dinámica (GeoGebra) bajo el enfoque de resolución de problemas?

Marco Conceptual

Arcavi et al. (2017) identifican cinco puntos que son claves en la enseñanza del Álgebra. El primero está relacionado con enseñar álgebra a través de situaciones o problemas contextualizados, que tomen en cuenta las experiencias y conocimientos preliminares de los estudiantes. El segundo es fomentar prácticas que sean productivas: orientadas hacia actividades o tareas que requieran habilidades del pensamiento de mayor orden, como la búsqueda de diferentes formas de resolverlas, evaluación de la efectividad de los procedimientos, participación en las discusiones de clase, y reflexionar sobre los métodos o acercamientos mostrados. El tercero es reconciliar a los procedimientos de rutina con el entendimiento, porque, a pesar de que hay una amplia discusión de si se oponen o complementan, se necesitan ambos para potenciar el pensamiento algebraico. El cuarto es ver los errores de los estudiantes como una oportunidad para comprender de dónde o por qué surgen y, así, prevenir que sigan ocurriendo. Y el quinto punto es buscar formas de hacer accesible e involucrar a los estudiantes en pruebas o argumentos matemáticos, aun si estas se perciben abstractas y formales para ellos.

Por otra parte, Kieran (2020) identifica tres tipos de actividades en el estudio del álgebra que son esenciales para el desarrollo del pensamiento algebraico: (a) interpretación y representación algebraica de situaciones, propiedades, patrones y relaciones; (b) manipulación simbólica que permita el desarrollo de habilidades y aspectos conceptuales e; (c) implementación del álgebra como una herramienta para modelar situaciones, justificar y probar, hacer predicciones y conjeturas, buscar relaciones y resolver problemas. De cierta manera, estas acciones están comprendidas en los puntos que se mencionaron anteriormente.

Ahora, respecto al uso de tecnología digital, Santos-Trigo (2019) ha identificado que, cuando se involucra el uso de un SGD en la resolución de problemas, hay cuatro tipos de tareas que se pueden llevar a cabo, las cuales se caracterizan por las representaciones, estrategias y formas de razonamiento que surgen en los procesos de solución: (a) enfocarse en las figuras. Son tareas que utilizan un SGD para reconstruir las figuras que están descritas en los enunciados de los problemas o que aparecen como una imagen que acompaña al enunciado. Su valor está en la necesidad de identificar los elementos que componen a las figuras y explorar las formas en las que se relacionan; (b) tareas de investigación. Transforman problemas rutinarios, como los que se encuentran en los libros de texto, en una serie de actividades de investigación y reflexión matemática; (c) tareas de variación. En estas, interesa representar y analizar problemas que involucren fenómenos de variación mediante un modelo gráfico sin tener que recurrir a un modelo algebraico; (d) configuraciones dinámicas. El objetivo es formular problemas a partir de configuraciones dinámicas y buscar argumentos que validen las relaciones matemáticas encontradas.

Estas tareas no solo aportan información a la resolución de los problemas, sino también ayudan a comprender cómo se relacionan los datos y conceptos involucrados. Además, pueden realizarse de manera simultanea cuando se resuelve un problema. Por ejemplo, en los problemas verbales los estudiantes podrían concentrarse en *tareas de variación*, ya que la mayoría describen situaciones que involucran fenómenos de variación, sin embargo, también podrían dirigir su atención a tareas de *construcción de figuras* cuando el contexto de los problemas sea además geométrico.

Considerando este contexto, es importante un marco que permita planear, organizar y analizar las formas en que los estudiantes resuelven los problemas con el uso de tecnología digital. En este sentido, Santos-Trigo y Camacho-Machín (2013), basados en el marco de resolución de problemas propuesto por Schoenfeld (1985) y en el método de Polya (1945) para resolver problemas, articulan un marco que engloba la resolución de problemas y el uso de tecnología digital. Este consta de cuatro fases que se caracterizan por el tipo de preguntas que se plantean en cada una: (1) comprensión del problema. ¿Cómo representar la situación descrita en el problema en términos de las herramientas digitales disponibles?; (2) exploración. ¿Qué estrategias pueden implementarse con las herramientas digitales disponibles que permitan explorar las formas en que se relacionan los datos explícitos e implícitos del problema?; (3) búsqueda de distintos acercamientos a la solución. A partir de las exploraciones realizadas en la segunda fase, ¿cómo pueden aprovecharse para obtener la solución?; (4) integración y reflexiones. ¿Cuáles fueron las ideas principales durante el proceso? Y ¿qué aportó el uso de la tecnología digital a la resolución del problema?

Con base en estas ideas pueden evaluarse y caracterizarse las formas de razonamiento que exhiben los estudiantes durante la resolución de los problemas verbales con el uso de un sistema de geometría dinámica.

Metodología

Esta investigación es cualitativa, mediante la observación e interacción controlada se caracterizaron las formas de razonamiento que exhibieron los estudianes en el desarrollo de las tareas. Se seleccionó un grupo de Matemáticas I de nivel medio superior, el cual estaba conformado por 20 estudiantes (entre 15 y 17 años), y se trabajó con este por 24 sesiones, donde cada sesión tenía una duración de dos horas.

Para el desarrollo de las sesiones se dispuso de 11 iPads con la aplicación de GeoGebra instalada (10 para los estudiantes y una para el investigador) y un proyector. Debido al número limitado de iPads, se formaron equipos de dos estudiantes y se asignó un iPad a cada uno. La dinámica que se siguió durante la implementación de los problemas, después de haber utilizado las primeras cuatro sesiones para introducirlos al SGD, fue utilizar una o dos sesiones para que los equipos los representaran, exploraran, resolvieran y, después, discutieran sus resultados con todo el grupo mediante el proyector.

Para que los estudiantes pudieran resolver los problemas con el uso del SGD, se propuso el siguiente esquema (Figura 1):

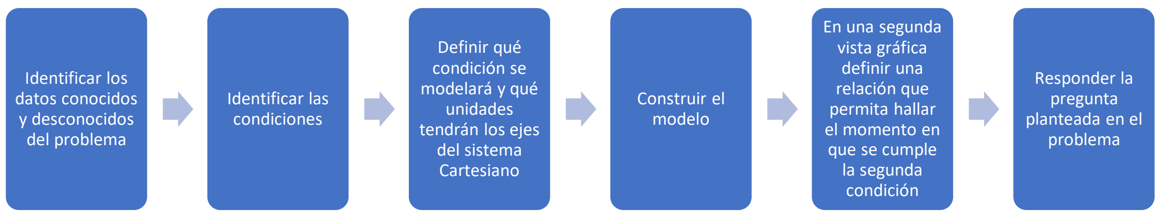

Figura 1: Esquema para resolver problemas verbales con el uso de GeoGebra.

Y para que pudieran representar algebraicamente la solución geométrica hallada con el SGD, se propuso el siguiente esquema (Figura 2):

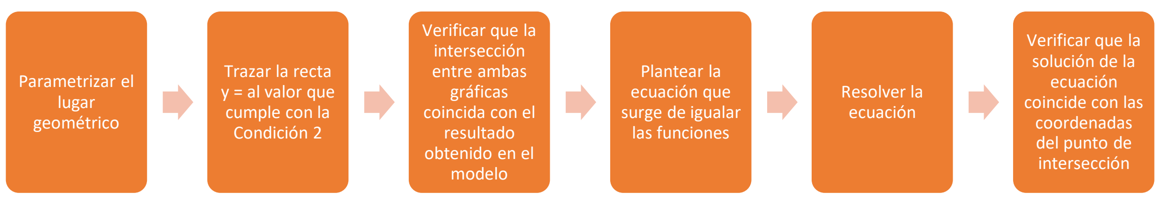

Figura 2: Esquema para algebrizar la representanción geométrica.

Los esquemas se constuyeron con base en los elementos del marco conceptual, y surgieron en respuesta a resultados obtenidos en investigaciones previas (Gómez-Arciga et al., 2018; Gómez-Arciga & Reyes-Martínez, 2019).

Para la selección de los problemas se hizo una lista donde se categoraizaron por su contexto, se trabajaron en las sesiones de un seminario de resolución de problemas, y se identificaron los más apropiados para alcanzar los objetivos del estudio. En los resultados se reportan los desarrollos que mostraron diferentes equipos en la resolución de dos problemas verbales.

Los datos se recolectaron a través de archivos de GeoGebra, videograbaciones y notas de campo. Estas últimas se utilizaron para destacar algunas ideas que, en un primer momento, se consideraron importantes en el desarrollo de las sesiones. Así, al momento de revisar las videograbaciones, se analizaron secciones específicas que ya se habían detectado en clase.

Resultados

En cada problema de esta sección se muestran dos acercamientos a la solución. Estos fueron desarrollados por diferentes equipos.

Problema 1: El perímetro de un triángulo isósceles es de 48 cm. Si el lado diferente equivale a 2/3 de la medida de los lados iguales, ¿cuál es la medida de los lados del triángulo?

La idea inicial del primer acercamiento fue construir una familia de triángulos isósceles con perímetro de 48 cm. Para lograrlo, trazaron lo siguiente (Figura 3): un segmento AB en el eje horizontal, con A en el origen y B un punto móvil; una circunferencia centrada en A y con radio AB; una circunferencia centrada en B con radio $r = 48 - 2f$, donde f era la longitud del segmento AB; y el triángulo ABD, donde D era una de las intersecciones entre las circunferencias. Así, el triángulo ABD cumplía la condición de ser isósceles con $AB = AD$ y tener perímetro de 48 cm.

Cuando el equipo utilizó la prueba del arrastre para mostrar que su modelo era robusto, observó que el triángulo solo existía si B se movía en el intervalo abierto (12, 24), lo cual derivó en una discusión con todo el grupo sobre el dominio del problema y la relación que guardaban los lados del triángulo para que pudiera construirse.

Luego, definieron el punto $\mathrm{E} = \left(a, \frac{2}{3}b - a\right)$ que relacionaba al lado desigual con la diferencia de las dos terceras partes de uno de los lados iguales y el desigual (a era la longitud del segmento BD, y b, la longitud del segmento AD), con el objetivo de encontrar las dimensiones del triángulo que cumpliera con la condición restante del problema. La solución la hallarían cuando al mover B la ordenada de E fuera cero, que gráficamente significó hallar la intersección del lugar geométrico descrito por E y el eje horizontal (Figura 4).

Finalmente, el equipo intentó, pero sin éxito, parametrizar el lugar geométrico para obtener la solución algebraica.

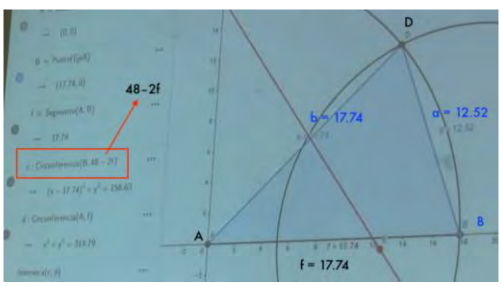

Figura 3: Construcción de la familia de triángulos isósceles con perímetro de 48 cm.

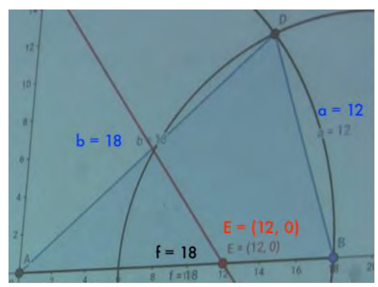

Figura 4: Dimensiones del triángulo que cumple con las condiciones del problema.

El segundo acercamiento, mostrado por otro equipo, consistió en modelar la relación entre uno de los lados iguales y el diferente (no construye el triángulo). Para ello, trazó un segmento AB, con A en el origen y B un punto móvil sobre el eje horizontal, y un segmento AC, donde C fue el punto de intersección entre el eje vertical y la circunferencia con centro en A y radio $r = \frac{2}{3}f$ (f era la longitud del segmento AB) (Figura 5). Con estos trazos, el equipo aseguró que la longitud del segmento AC fuera *2/3* de la longitud del segmento AB para cualquier posición del punto B. Entonces, la longitud del segmento AC se asoció a la medida del lado diferente del triángulo, y la longitud del segmento AB, a la medida de uno de los lados iguales.

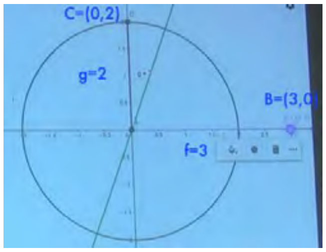

Figura 5: Modelo del segundo acercamiento.

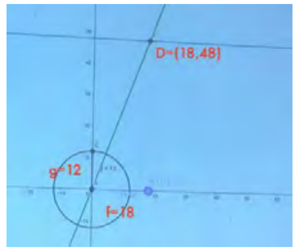

Figura 6: Solución gráfica.

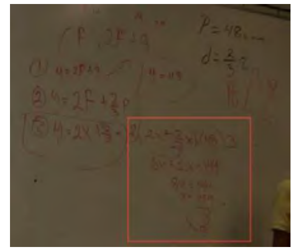

Figura 7: Solución algebraica.

A partir del modelo, definió el punto $D = (f, 2f + g)$ que relacionaba la medida de uno de los lados iguales del triángulo con su perímetro. La solución la obtuvo cuando la ordenada del punto *D* fue 48, porque cumplió con las condiciones descritas en el enunciado. En la Figura 6 se observa que cuando la medida de los lados iguales es 18 *cm* y la del diferente es 12 *cm*, el perímetro es de 48 *cm*, y que la solución gráfica se halla en la intersección del lugar geométrico descrito por *D* y la recta $y = 48$.

La parametrización y solución algebraica fue desarrollada adecuadamente por el equipo. En la Figura 7 puede observarse que planteó la ecuación $2x + \frac{2}{3}x = 48$, que fue resultado de igualar la función que surge de parametrizar el lugar geométrico y la función constante (*y = 48*), y la resolvió correctamente.

Problema 2: En cierta competencia de atletismo el corredor *A* se encuentra a 30 metros adelante del corredor *B*. El corredor *A* lleva una velocidad constante de 7 *km/h* y el corredor *B* lleva una velocidad constante de 8 *km/h*. Si los dos salen al mismo tiempo, ¿después de cuántos metros el corredor *B* alcanzará al corredor *A*?

El primer equipo que mostró su acercamiento seleccionó las unidades del eje horizontal como segundos (tiempo), y las del eje vertical, como *m/s* (velocidad). Eligió las unidades de los ejes de esta forma porque el problema pedía hallar una distancia en metros. En consecuencia, el equipo hizo las conversiones de las unidades de las velocidades de los corredores: la velocidad del corredor *A* fue de $1.9\overline{4}$ *m/s*, y la de *B*, de $2.\overline{2}$ *m/s*.

De este modo, como la distancia es el resultado del producto del tiempo y la velocidad, el equipo representó las distancias recorridas por los corredores mediante áreas de rectángulos: en la Figura 8 la longitud del segmento *AB* (*f = 2*) representó el tiempo, en segundos, transcurrido de cada corredor; las longitudes de los segmentos *AD* y *AC*, las velocidades de los corredores *A* y *B*, respectivamente; y las áreas de los rectángulos *ABED* (*c2 = 3.88*) y *ABFC* (*c1 = 4.4*), los metros recorridos por los corredores *A* y *B*, respectivamente. Así, la Figura 8 muestra el valor numérico de la distancia recorrida de cada corredor en dos segundos.

El equipo analizó, mediante el lugar geométrico que describía el punto $G = \left(c1, \frac{c2+30}{c1}\right)$, para qué distancia c1 se cumplía que $\frac{c2+30}{c1} = 1$ (Figura 9); dicho de otra forma, analizó, a través de una razón, qué distancia recorrió el corredor *B* para alcanzar al corredor *A* (quien, en el mismo tiempo que B, ha recorrido $c2 + 30$ metros).

En la Figura 9 se observa que, a los 253 metros, aproximadamente, el corredor *B* alcanzó al corredor A, ya que es la intersección entre el lugar geométrico y la recta $y = 1$. A partir de este resultado se preguntó al equipo sobre el significado de la razón cuando era menor o mayor a 1: el equipo identificó que cuando la razón era menor a 1, significaba que el corredor *B* había rebasado al corredor *A*; en caso contrario, *A* mantenía el primer lugar.

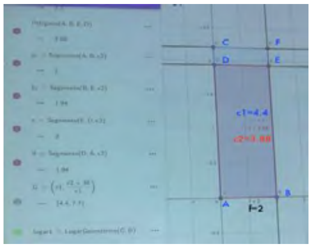

Figura 8: Modelo del primer acercamiento del problema 2.

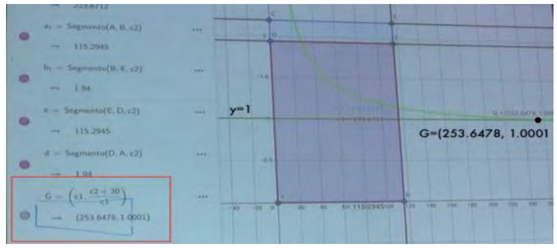

Figura 9: Exploración de la relación que hay entre las distancias recorridas por los corredores.

Figura 10: Solución gráfica robusta.

Al momento de parametrizar y resolver algebraicamente el problema, hallaron que la solución era 240 metros, lo que evidenció el margen de error provocado por no robustecer el lugar geométrico cuando lo intersectaron con la recta $y = 1$. Una vez corregida esa imprecisión en el modelo, observaron también que transcurrieron 108 segundos para que el corredor *B* alcanzara al corredor *A* (Figura 10).

Otro equipo, en su acercamiento, representó a las velocidades de los corredores como pendientes de rectas (Figura 11). Para ello, asignó al eje horizontal unidades de hora, y al eje vertical, unidades de kilómetro. Entonces, para construir las rectas con pendientes de 7 y 8 unidades (*km/h*), definió los puntos *B* y *C* sobre el eje vertical en 7 y 8, el punto *D* sobre el eje horizontal en 1, trazó perpendiculares a los ejes pasando por estos puntos, y con las intersecciones entre estas (puntos *E* y *F*), trazó la recta *AF* con pendiente 7 y la recta *AE* con pendiente 8.

Para representar el resto de los datos, el equipo ubicó un punto móvil sobre el eje horizontal (punto *G*), trazó una perpendicular al mismo eje pasando por este punto, marcó las intersecciones de esta con las rectas (puntos *H* e *I*) y, mediante perpendiculares al eje vertical que pasaban por los puntos *H* e *I*, marcaron las intersecciones de estas con el eje vertical (puntos *J* y *K*) (Figura 12). Así, el segmento *AG* representó el tiempo transcurrido (en horas) de la competencia, y el segmento *JK* o segmento a la distancia (en kilómetros) entre los corredores en el tiempo *AG* (pues *AK* era la distancia recorrida por el primer corredor, y *AJ*, la distancia recorrida por el segundo).

De esta forma, el equipo definió el punto $L = (AG,\ a)$ que relacionaba el tiempo transcurrido en la competencia con la distancia entre los corredores (Figura 12); y cuando su ordenada fue 0.03 (30 metros), obtuvo la distancia que recorrió el corredor *B* para alcanzar al corredor *A* en la longitud del segmento *AJ* (o en la ordenada del punto *J = (0, 0.24)*). Aunque el equipo identificó que la solución geométrica estaba en la intersección del lugar geométrico de *L* y la recta $y = 0.03$, no desarrolló el acercamiento algebraico.

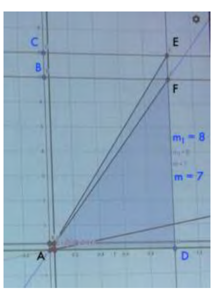

Figura 11: Velocidades representadas como pendientes.

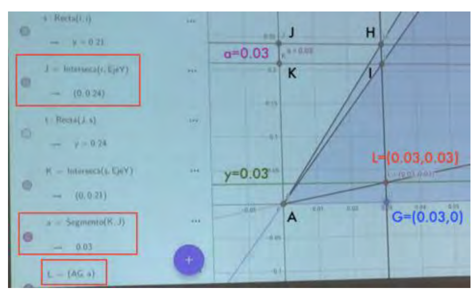

Figura 12: Solución geométrica basada en el modelo de pendientes.

Olanoff, D., Johnson, K., & Spitzer, S. (2021). *Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Philadelphia, PA.

Discusión de los resultados y conclusiones

En los procesos de solución de los problemas, se observó que los equipos realizaron distintas actividades: seleccionaron unidades para los ejes, identificaron el dominio, determinaron elementos fijos y móviles para los modelos, graficaron y analizaron relaciones, buscaron soluciones geométricas y algebraicas. Estas fueron resultado de implementar los esquemas que se les propusieron para trabajar con GeoGebra.

En el problema 1, el primer equipo mostró un acercamiento donde se enfocó en construir la familia de triángulos de 48 *cm.* de perímetro para después analizar la relación entre sus lados. Es decir, se centró en dos tareas: (1) construir la figura y (2) analizar la variación de sus elementos para llegar a la solución sin necesidad de utilizar ecuaciones algebraicas. El recurso clave de la primera tarea fue el uso de circunferencias para trasladar medidas, y la estrategia para hallarlas fue analizar su comportamiento o relación a través del lugar geométrico (que dependía del movimiento ordenado del punto B) que corresponde al desarrollo de la segunda tarea. De este proceso se dio la posibilidad de discutir sobre el dominio del problema y de determinarlo, ya que no es común prestarle atención a este concepto cuando se resuelven este tipo de problemas.

En el caso del segundo equipo, llevó a cabo una tarea de variación sin construir la figura. La relación que exploró y analizó fue la del comportamiento del perímetro del triángulo cuando variaban las longitudes de sus lados, las cuales conservaban una relación de proporcionalidad. De esta manera, pudo representar y resolver algebraicamente el problema.

Para el segundo problema, en los dos acercamientos mostrados, a diferencia de los anteriores, se fijaron los mismos datos en los modelos (las velocidades) y se analizaron las relaciones en función del tiempo. Sin embargo, las representaciones de los datos y las formas de explorar y resolver el problema fueron distintas.

El lugar geométrico que exhibió el primer equipo fue el resultado de analizar cómo cambia la razón de las distancias recorridas por los corredores respecto al tiempo. En cambio, el segundo equipo analizó cómo se relaciona la diferencia entre las distancias recorridas por los corredores con el tiempo. El acercamiento algebraico de este problema solo lo desarrolló el primer equipo.

En general, el uso sistemático de GeoGebra permitió resolver los problemas de distintas formas y visualizar el tipo de relaciones que hay entre los datos. Además, aunque no todos pudieron resolver los problemas algebraicamente, valoran este resultado como una prueba que sustenta su solución gráfica. De hecho, si se presta atención a los acercamientos de los equipos que no mostraron una solución algebraica, puede notarse que sus modelos son más elaborados que los otros. Por lo tanto, parametrizarlos es más difícil.

Así, las formas de razonamiento que exhibieron los estudiantes durante este proceso se caracterizaron por:

- Interpretar los conceptos de un problema en términos de sus propiedades y atributos geométricos (en términos de la herramienta).
- Cuantificar los elementos del modelo.
- Identificar el dominio del problema.
- Definir una relación que permitirá hallar la solución en una infinidad de casos.
- Determinar la solución en términos del modelo.

A pesar de que el uso sistemático de un sistema de geometría dinámica promueve el estudio de relaciones entre los conceptos involucrados en los problemas verbales, sigue habiendo dificultades para conectar estas representaciones geométricas con las representaciones

algebraicas. No obstante, trabajar bajo este enfoque parece ofrecer una serie de posibilidades para el desarrollo del pensamiento algebraico y funcional.

The purpose of this study was to analyze how the systematic use of GeoGebra influences the ways of reasoning of high school students when they incorporate it in the resolution of word problems. The study was carried out with a group of 20 students who were studying Mathematics I. The results show that the students appropriated resources from the dynamic geometry system (GeoGebra) which, when used systematically, allowed them to implement strategies to represent geometrically the concepts involved in the problems, explore and analyze relationships between the elements of the dynamic models built and find the solutions. Thus, thanks to the exploration and analysis of relationships, it was possible to discuss with the students' concepts such as: ratio, domain, locus, variation.

Keywords: Problem Solving, Word Problems, Digital Technology, Algebra

Introduction

Covid-19 has caused a substantive change in the use of technologies for education. There has been a rapid dissemination of these to mitigate the problems caused by confinement. As far as Mathematics Education is concerned, the use of digital technologies is in a marginal status, since their implementation always lags the speed of digital evolution, even before the modifications forced by the Covid-19 pandemic, and even though it is a necessity in the 21st century for the teaching of mathematics (Cevikbas & Kaiser, 2020). Therefore, it is important that teachers know and strategically use various technologies, so that all students have the same opportunities and possibilities to access mathematics (National Council of Teachers of Mathematics, 2011).

Implementing digital technology in math classes can help lessen the technical aspects of algebra that arise when solving problems (Arcavi et al., 2017). This allows students to focus on developing resources and strategies that are useful for describing relationships and solving problems involving the construction and use of functional relationships, which are main objectives in the study of algebra because they favor the conceptual understanding of algebraic processes (Kieran, 2020). In this sense, it is valid to reflect on whether both the content and the way in which algebra is taught should be modified; In other words, questioning whether the topics taught in algebra and the way they are taught are adequate today (Thomas, 2017).

A common thread in the study of algebra is the resolution of word problems, ranging from basic to higher education (Amado et al., 2019). Through these, students are expected to experiment, articulate, and debate different approaches to the solution that promote the analysis and understanding of the main concepts and ideas of algebra. In fact, a well-received proposal among researchers and teachers to deal with these problems has been to develop the episodes and stages of the problem-solving framework (Blum and Niss, 1991; Verschaffel et al., 2000), because it favors the use of strategies and collaborative work in students.

However, research has been carried out on teaching proposals that include the use of technology in solving word problems to find out the impact they have on students. For example, Amado et al. (2019) presented results of a study where, in addition to working under the framework of problem solving, they used spreadsheets with the aim of observing how their tools can help high school students (between 13 and 14 years old) to represent and solve word problems and introduce them to formal algebraic methods. They found that, although the students had not yet learned to use algebraic representations (systems of equations of two or more unknowns) to solve word problems, the spreadsheet not only helped them to solve them, but also to interpret the solution as a value that satisfies a set of conditions that are associated with the equations.

In this context, the use of a dynamic geometry system (DGS) can be key for the development of algebraic and functional thinking in solving word problems, because it allows representing, exploring, and solving them from a geometric approach. Bozkurt and Uygan (2020) explain that DGS allow students to manipulate geometric objects and explore relationships between them. In addition, they identify drag as an effective strategy of these systems, which is the result of moving elements of the dynamic configurations without changing their underlying geometric relationships. Finally, they recognize their didactic potential, and the importance of not using them in a conventional way, as static software systems.

Although there have been studies that involve the use of digital technology in the resolution of verbal problems, there is still more evidence that reflects an efficient and adequate way of implementation (Verschaffel et al., 2020). So, with the aim of providing more evidence to this discussion, the following question was posed that guided the investigation: What type of reasoning do students at high school construct and exhibit when they solve word problems with the use of a dynamic geometry system (GeoGebra) under the problem-solving approach?

Conceptual Framework

Arcavi et al. (2017) identify five points that are key in teaching Algebra. The first is related to teaching algebra through contextualized situations or problems that consider the preliminary experiences and knowledge of the students. The second is to encourage practices that are productive: oriented towards activities or tasks that require higher-order thinking skills, such as finding different ways to solve them, evaluating the effectiveness of procedures, participating in class discussions, and reflecting on the methods or approaches shown. The third is to reconcile routine procedures with understanding, because although there is extensive discussion of whether they oppose or complement each other, both are needed to enhance algebraic thinking. The fourth is to see student mistakes as an opportunity to understand where or why they arise and thus prevent further occurrence. And the fifth point is to look for ways to make accessible and involve students in mathematical proofs or arguments, even if they seem abstract and formal to them.

On the other hand, Kieran (2020) identifies three types of activities in the study of algebra that are essential for the development of algebraic thinking: (a) interpretation and algebraic representation of situations, properties, patterns and relationships; (b) symbolic manipulation that allows the development of skills and conceptual aspects and; (c) implementation of algebra as a tool to model situations, justify and prove, make predictions and conjectures, look for relationships, and solve problems.

Now, regarding the use of digital technology, Santos-Trigo (2019) has identified that, when the use of an SGD is involved in solving problems, there are four types of tasks that can be carried out, which are characterized by the representations, strategies and forms of reasoning that arise in the solution processes: (a) focus on the figures. They are tasks that use an DGS to reconstruct the figures that are described in the problem statements or that appear as an image that accompanies the statement. Its value is in the need to identify the elements that make up the figures and explore the ways in which they are related; (b) research tasks. They transform routine problems, such as those found in textbooks, into a series of mathematical research and reflection activities; (c) variation tasks. In these, it is interesting to represent and analyze problems that involve variation phenomena by means of a graphic model without having to resort to an algebraic model; (d) dynamic configurations. The objective is to formulate problems from dynamic configurations and look for arguments that validate the mathematical relationships found.

These tasks not only provide information to solve problems, but also help to understand how the data and concepts involved are related. In addition, they can be done simultaneously when a problem is solved. For example, in word problems, students could focus on variation tasks, since most describe situations that involve variation phenomena, however, they could also direct their attention to *figure construction* tasks when the context of the problems is also geometric.

Considering this context, a framework that allows planning, organizing, and analyzing the ways in which students solve problems with the use of digital technology is important. In this sense, Santos-Trigo and Camacho-Machín (2013), based on the problem-solving framework proposed by Schoenfeld (1985) and on Polya's (1945) method to solve problems, articulate a framework that encompasses problem solving and the use of digital technology. This consists of four phases that are characterized by the type of questions that are posed in each one: (1) understanding the problem. How to represent the situation described in the problem in terms of the digital tools available?; (2) exploration. What strategies can be implemented with the available digital tools that allow us to explore the ways in which the explicit and implicit data of the problem are related?; (3) search for different approaches to the solution. From the explorations carried out in the second phase, how can they be used to obtain the solution?; (4) integration and reflections. What were the main ideas during the process? And what did the use of digital technology contribute to solving the problem?

Based on these ideas, the forms of reasoning that students exhibit during the resolution of word problems can be evaluated and characterized with the use of a dynamic geometry system.

Methodology

This research is qualitative, through observation and controlled interaction, the forms of reasoning exhibited by the students in the development of the tasks were characterized. A group of Mathematics I of high school was selected, which was made up of 20 students (between 15 and 17 years old), and it was worked with this for 24 sessions, where each session lasted two hours.

For the development of the sessions, there were 11 iPads with the GeoGebra application installed (10 for the students and one for the researcher) and a projector. Due to the limited number of iPads, teams of two students were formed and an iPad was assigned to each. The dynamic that was followed during the implementation of the problems, after having used the first four sessions to introduce them to the DGS, was to use one or two sessions for the teams to represent them, explore, solve and, later, discuss their results with everything the group using the projector.

So that student could solve problems with the use of the DGS, the following scheme was proposed (Figure 1):

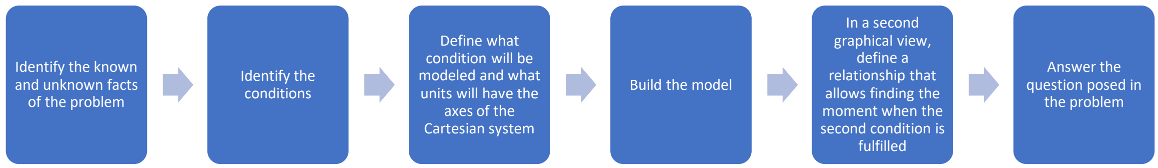

Figure 1: Scheme to solve word problems with the use of GeoGebra.

And so that they could represent algebraically the geometric solution found with the DGS, the following scheme was proposed (Figure 2):

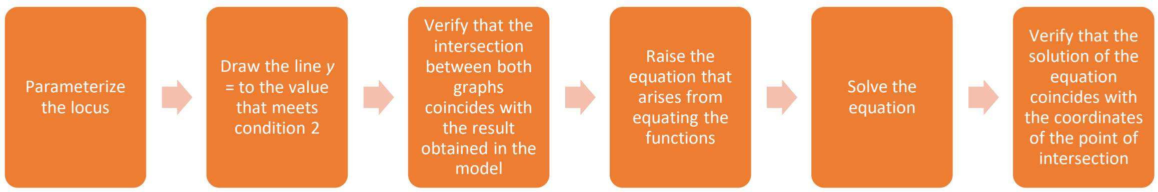

Figure 2: Scheme to algebrize the geometric representation.

The schemes were built based on the elements of the conceptual framework and emerged in response to results obtained in previous research (Gómez-Arciga et al., 2018; Gómez-Arciga & Reyes-Martínez, 2019).

For the selection of the problems, a list was made where they were categorized by their context, the sessions of a problem-solving seminar were worked on, and the most appropriate ones were identified to achieve the objectives of the study. In the results, the developments shown by different teams in solving two word problems are reported.

Data was collected through GeoGebra files, video recordings, and field notes. The latter were used to highlight some ideas that, at first, were considered important in the development of the sessions. Thus, when reviewing the video recordings, specific sections that had already been detected in class were analyzed.

Results

Two approaches to the solution are shown for each problem in this section. These were developed by different teams.

Problem 1: The perimeter of an isosceles triangle is 48 cm. If the different side is 2/3 the measure of the equal sides, what is the measure of the sides of the triangle?

The initial idea of the first approach was to construct a family of isosceles triangles with a perimeter of 48 cm. To achieve this, they drew the following (Figure 3): a segment AB on the horizontal axis, with A at the origin and B a moving point; a circle centered at A and with radius AB; a circle centered at B with radius $r = 48 - 2f$, where f was the length of segment AB; and the triangle ABD, where D was one of the intersections between the circles. Thus, the triangle ABD fulfilled the condition of being isosceles with $AB = AD$ and having a perimeter of 48 cm.

When the team used the drag test to show that their model was robust, they observed that the triangle only existed if B moved in the open interval (12, 24), which led to a discussion with the whole group about the domain of the problem and the relationship between the sides of the triangle so that it could be built.

Then, they defined the point E $= \left(a, \frac{2}{3}b - a\right)$ that related the unequal side to the difference of two thirds of one of the equal sides and the unequal one (a was the length of segment *BD*, and b, length of segment *AD*), to find the dimensions of the triangle that meets the remaining condition of the problem. The solution would be found when moving *B,* the ordinate of *E* was zero, which graphically meant finding the intersection of the locus described by *E* and the horizontal axis (Figure 4).

Finally, the team tried, but without success, to parameterize the locus to obtain the algebraic solution.

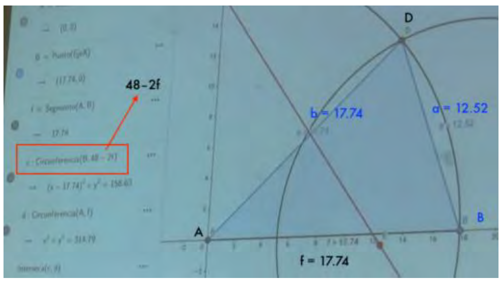

Figure 3: Construction of the family of isosceles triangles with perimeter of 48 *cm*.

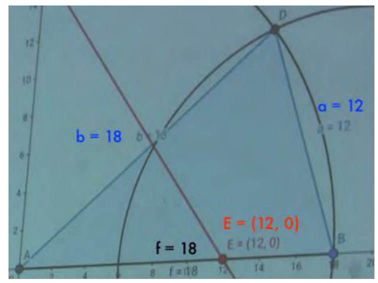

Figure 4: Dimensions of the triangle that meets the conditions of the problem.

The second approach, shown by another team, consisted of modeling the relationship between one of the equal and the different sides (it does not construct the triangle). To do this, the team drew a segment *AB*, with *A* at the origin and *B* a mobile point on the horizontal axis, and a segment *AC*, where *C* was the point of intersection between the vertical axis and the circumference with center at *A* and radius $r = \frac{2}{3}f$ (f was the length of segment *AB*) (Figure 5). With these traces, the team ensured that the length of segment *AC* was *2/3* of the length of segment *AB* for any position of point *B*. Then, the length of segment *AC* was associated with the measurement of the different side of the triangle, and the length of segment *AB*, measured by one of the equal sides.

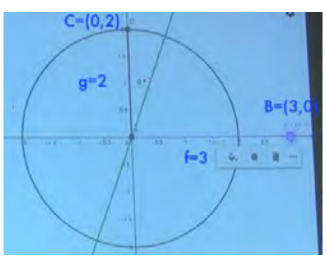

Figure 5: Model of the second approach.

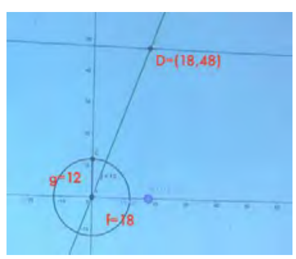

Figure 6: Graphical solution

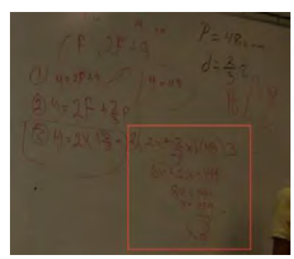

Figure 7: Algebraic solution.

Using the model, the team defined the point $D = (f, 2f + g)$ that related the measure of one of the equal sides of the triangle to its perimeter. The solution was obtained when the ordinate of point *D* was 48, because it met the conditions described in the statement. In Figure 6 it is observed that when the measure of the equal sides is 18 *cm* and that of the different one is 12 *cm*, the perimeter is 48 *cm*, and that the graphical solution is found at the intersection of the locus described by *D* and the line $y = 48$.

The parameterization and algebraic solution were adequately developed by the team. In Figure 7 the team raised the equation $2x + \frac{2}{3}x = 48$, which was the result of equating the function that arises from parameterizing the locus and the constant function (*y = 48*) and solved it correctly.

Problem 2: In a certain athletics competition, runner *A* is 30 meters ahead of runner *B*. Runner *A* has a constant speed of 7 *km/h* and runner *B* has a constant speed of 8 *km/h*. If they both start at the same time, after how many meters will runner *B* catch up with runner *A*?

The first team to show their approach selected the units on the horizontal axis as seconds (time), and those on the vertical axis as *m/s* (velocity). The team chose the units of the axes this way because the problem asked to find a distance in meters. Consequently, the team converted the units of the runners' speeds: runner *A's* speed was $1.9\overline{4}$ *m/s*, and *B's* was $2.\overline{2}$ *m/s*.

In this way, since distance is the result of the product of time and speed, the team represented the distances traveled by the runners using areas of rectangles: in Figure 8 the length of segment *AB* (*f = 2*) represented time, in seconds, elapsed from each runner; the lengths of segments *AD* and *AC*, the speeds of runners *A* and *B*, respectively; and the areas of the rectangles *ABED* (*c2 = 3.88*) and *ABFC* (*c1 = 4.4*), the meters covered by runners *A* and *B*, respectively. Thus, Figure 8 shows the numerical value of the distance covered by each runner in two seconds.

The team analyzed, using the locus described by the point $G = \left(c1, \frac{c2+30}{c1}\right)$, for what distance *c1* it was true that $\frac{c2+30}{c1} = 1$ (Figure 9); In other words, the team analyzed, through a ratio, how far runner *B* traveled to reach runner *A* (who, in the same time as B, has traveled $c2 + 30$ meters).

In Figure 9 it is observed that, at approximately 253 meters, runner *B* reached runner *A*, since it is the intersection between the locus and the line $y = 1$. Based on this result, the team was asked about the meaning of the ratio when it was less than or greater than 1: the team identified that when the ratio was less than 1, it meant that runner *B* had passed runner *A*; otherwise, *A* held first place.

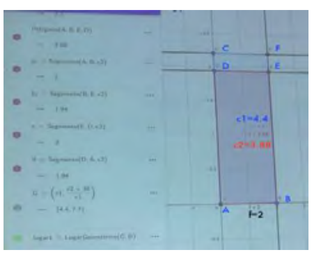

Figure 8: Model of the first approach of problem 2.

Figure 9: Exploration of the relationship between the distances covered by the runners.

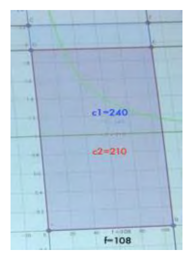

Figure 10: Robust graphical solution.

At the time of parametrizing and algebraically solving the problem, they found that the solution was 240 meters, which evidenced the margin of error caused by not strengthening the locus when they intersected it with the line $y = 1$. Once this imprecision in the model was corrected, they also observed that 108 seconds elapsed for runner *B* to reach runner *A* (Figure 10).

Another team, in its approach, represented the speeds of the runners as slopes of straights (Figure 11). To do this, he assigned units of hours to the horizontal axis and kilometer units to

Olanoff, D., Johnson, K., & Spitzer, S. (2021). *Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Philadelphia, PA.

the vertical axis. So, to construct the lines with slopes of 7 and 8 units (*km/h*), the team defined points *B* and *C* on the vertical axis at 7 and 8, point *D* on the horizontal axis at 1, plotted perpendicular to the axes passing through these points, and with the intersections between them (points E and F), the team drew the line *AF* with slope 7 and the line *AE* with slope 8.

To represent the rest of the data, the team located a mobile point on the horizontal axis (point *G*), drew a perpendicular to the same axis passing through this point, marked the intersections of this with the lines (points *H* and *I*) and, using perpendicular to the vertical axis that passed through points *H* and *I*, they marked their intersections with the vertical axis (points *J* and *K*) (Figure 12). Thus, the AG segment represented the elapsed time (in hours) of the competition, and the *JK* segment or a segment the distance (in kilometers) between the runners in AG time (AK was the distance traveled by the first runner, and AJ, the distance traveled by the second).

In this way, the team defined the point $L = (AG,\ a)$ that related the time spent in the competition with the distance between the runners (Figure 12); and when its ordinate was 0.03 (30 meters), it obtained the distance traveled by runner B to reach runner A in the length of segment *AJ* (or in the ordinate of point $J = (0,\ 0.24)$). Although the team identified that the geometric solution was at the intersection of the L locus and the line $y = 0.03$, they did not develop the algebraic approach.

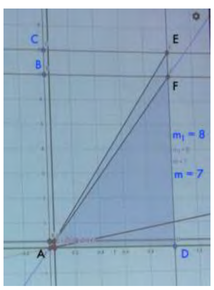

Figure 11: Speeds represented as slopes.

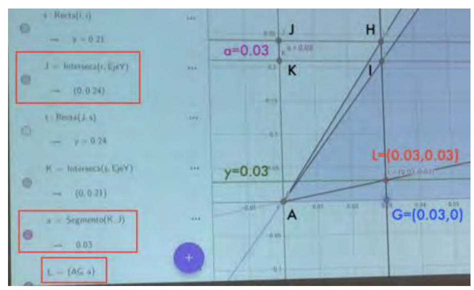

Figure 12: Geometric solution based on the slopes model.

Discussion of the results and conclusions

In the problem-solving processes, it was observed that the teams carried out different activities: they selected units for the axes, identified the domain, determined fixed and mobile elements for the models, plotted and analyzed relationships, and searched for geometric and algebraic solutions. These were the result of implementing the schemes that were proposed to them to work with GeoGebra.

In problem 1, the first team showed an approach where they focused on building the family of triangles with 48 *cm* perimeter and then analyzing the relationship between their sides. That is, he focused on two tasks: (1) constructing the figure and (2) analyzing the variation of its elements to arrive at the solution without using algebraic equations. The key resource of the first task was the use of circles to transfer measurements, and the strategy to find them was to analyze their behavior or relationship through the locus (which depended on the ordered movement of point *B*) that corresponds to the development of the second task. This process gave the possibility of discussing the domain of the problem and determining it, since it is not common to pay attention to this concept when solving this type of problem.

In the case of the second team, they carried out a variation task without building the figure. The relationship that the team explored and analyzed was that of the behavior of the perimeter of the triangle when the lengths of its sides varied, which conserved a proportional relationship. In this way, the team was able to represent and solve the problem algebraically.

For the second problem, in the two approaches shown, unlike the previous ones, the same data were set in the models (the speeds) and the relationships as a function of time were analyzed. However, the representations of the data and the ways to explore and solve the problem were different.

The locus exhibited by the first team was the result of analyzing how the ratio of the distances covered by the runners changes with respect to time. Instead, the second team looked at how the difference between the distances covered by the runners is related to time. The algebraic approach to this problem was only developed by the first team.

In general, the systematic use of GeoGebra allowed solving the problems in different ways and visualizing the type of relationships between the data. Furthermore, although not all of them were able to solve the problems algebraically, they value this result as a proof that supports their graphical solution. In fact, if you pay attention to the approaches of the teams that did not show an algebraic solution, their models are more elaborate than the others. Therefore, parameterizing them is more difficult.

Thus, the forms of reasoning that students exhibited during this process were characterized by:

- Interpret the concepts of a problem in terms of its geometric properties and attributes (in terms of the tool).
- Quantify the elements of the model.
- Identify the problem domain.
- Define a relationship that will allow finding the solution in an infinity of cases.
- Determine the solution in terms of the model.

Although the systematic use of a dynamic geometry system promotes the study of relationships between the concepts involved in word problems, there are still difficulties in connecting these geometric representations with algebraic representations. However, working under this approach seems to offer a series of possibilities for the development of algebraic and functional thinking.

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THE RELATIONSHIP BETWEEN CONFIDENCE, ACCURACY, AND DECISION MAKING IN A CALCULUS SKILLS REVIEW PROGRAM

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Just like physical skills, cognitive skills grow rusty over time unless they are regularly used and practiced so academic breaks can have negative consequences on student learning and success. The Keeping in School Shape (KiSS) program is an engaging, innovative, and cost-effective intervention that harnesses the benefits of retrieval practice by using technology to help students maintain proficiency over breaks from school by delivering a daily review problem via text message or email. A growth mindset is promoted through feedback messages encouraging students to try again if they get a problem wrong and to take on a challenge problem if they get a problem correct. This paper reports on the relationship between confidence, accuracy, and decision making during the implementation of the KiSS Program at a large university during winter break for students enrolled in an engineering introductory Calculus course sequence.

Keywords: Calculus, Metacognition, Technology, Undergraduate Education

Introduction and Theoretical Framing

Many Science, Technology, Engineering, and Mathematics (STEM) topics require proficiency in previously learned skills and concepts. Introductory STEM course sequences mimic this structure so that foundation courses feed into subsequent closely-related courses. Students finish the foundation course with skills and confidence that are critical to their success in the target course. However, this growth erodes in the time between the courses, especially if there is a prolonged gap in academic engagement such as a lengthy academic break (e.g., summer slide) (Cooper et al., 1996), or if the way students absorbed the information was not conducive to retention (e.g., stress-induced intentional forgetting) (Ramirez et al. 2017). In order to address this loss of proficiency, many faculty, departments, and institutions of higher education would like to see students engage with course content outside of class and during academic breaks. Regrettably, this is not likely to happen spontaneously, so the issue is how to reach students and prompt them, in a non-threatening way, to regularly review things that they have learned and need to maintain for future learning. One way of doing this is to deliver review opportunities, along with encouragement to confront deficiencies and meet potential, in a location that students are unlikely to miss, namely on their mobile phones or via email.

This paper discusses the implementation of an engaging, innovative and cost-effective program that uses technology to help students maintain proficiency over breaks from school, while also promoting a growth mindset (van de Sande, 2019a, 2019b). Theoretically, the Keeping in School Shape (KiSS) program draws on the well-documented benefits of regular retrieval practice, namely recalling previously material as an effective way of maintaining cognitive performance (Butler et al., 2014; Roediger & Butler, 2011; Rohrer & Pashler, 2007). The KiSS Program embodies retrieval practice by sending students a multiple-choice mathematics question daily via text messaging or email. The problems are chosen specifically to be skills that are requisite for success in the mathematics course following the break from school.

Retrieval practice delivered daily is also consistent with the growth mindset metaphor of the brain as a muscle that grows stronger with exercise (Yeager et al., 2019). Many students approach their studies convinced that intelligence is fixed, so those who have to put forth effort lack natural talent and the ability to succeed (Boaler 2010, 2013; Dweck, 2006). This failure to believe that, through effort, the brain can grow stronger negatively affects achievement (Boaler, 2013), especially for more underrepresented groups (Aronson et al., 2002; Blackwell et al, 2007). When students engage in retrieval practice by testing themselves regularly to see if they can perform a previously learned skill or concept, they are essentially flexing and toning their "brain muscle" and keeping it from atrophying with disuse. The design of the KiSS Program further promotes a growth mindset by including features such as hints, the option to attempt more challenging problems, and feedback messages that praise and encourage effort so that students have opportunities to see mistakes and difficult problems as profitable opportunities to engage in productive struggle.

Mindset not only plays a role in broad measures of math achievement and but can also influence how students behave in a specific problem-solving context (Shen, Miele, & Vasilyeva, 2016). Students with a fixed mindset are less likely to persevere in a challenging math task than students with a growth mindset, and also have lower confidence in their ability to do math after being confronted with challenging problems. Given that what counts as challenging is a subjective judgment, we use technology to explore the relationship between student confidence in the ability to perform a task and subsequent navigation through a review activity.

Methods

The KiSS program was designed to encourage students to regularly connect with their studies and to gauge their ability to perform previously learned material over break by providing easily accessible review opportunities for requisite skills. At the same time, the configuration of technology allowed us to unobtrusively collect information on confidence, accuracy, and decision making as students engaged with the review activities.

Context

Each problem in the KiSS program was designed as an independent Qualtrics (https://www.qualtrics.com/) survey since Qualtrics allows surveys to be distributed as sms messages or emails. Figure 1 shows a schematic of how the daily problem link was pushed to students' phones as a text message or sent to their email addresses. Clicking on the link took students directly to the daily question survey. Before attempting the daily review problem, students were first asked to use a 5-point scale (ranging from "not at all" to "Super Duper") to show how confident they were that they could answer it correctly. Informal rating labels and accompanying emojis were used in an effort to make this self-assessment less threatening.

After that, as shown in Figure 2 which depicts a flowchart of the regular daily agenda, students responded to the question by selecting one of the answer options, which opened up a sequence of possible paths and opportunities. On certain days of the week, the agenda included additional paths (not shown here): On Tuesdays (aka "2's-days"), students could choose to do an additional problem and on Sundays (aka "Trivia Days") students could choose to do the daily calculus problem followed by a calculus trivia question or just respond to the trivia question. Figure 3 depicts the opportunities for engagement that stem from getting the daily problem incorrect, namely getting a hint and trying again (encouraged), seeing the solution, or exiting. Students who got the daily problem correct had the option of trying a related more challenging problem (encouraged), seeing the solution, or exiting. Whether they got the daily review problem incorrect or correct, students were prompted with messaging to adopt a growth mindset by persisting ("Let's rethink this!") or pushing themselves ("Let's push ourselves!").

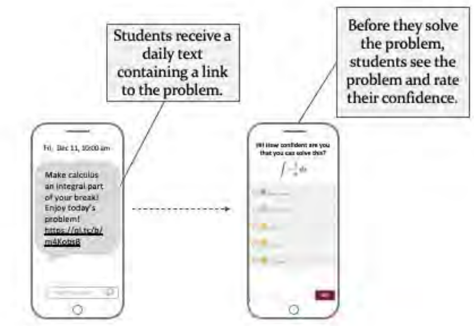

Figure 1: Schematic of review problem delivery and confidence rating

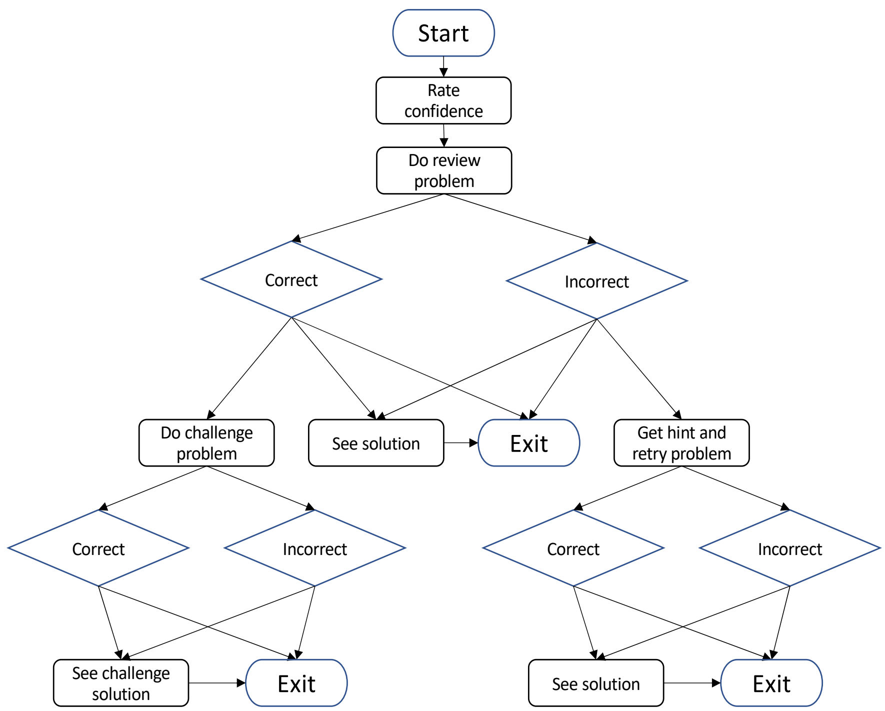

Figure 2: Flowchart showing various paths and opportunities within the daily review

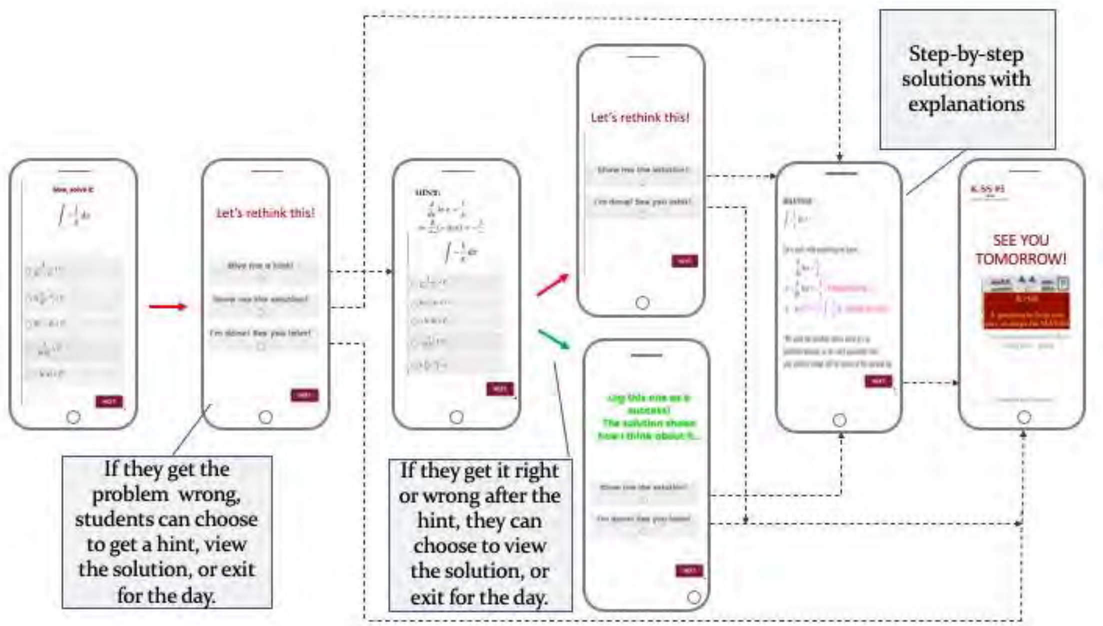

Figure 3: Schematic of possible paths following an incorrect response

Participants

This paper reports on select results from an implementation of the KiSS Program at a large university in the Southwest for students who were enrolled in an introductory Calculus course sequence for engineers. The program was designed for students who had successfully completed the first course of the sequence and were planning on taking the second course in the upcoming spring semester. At the end of the fall semester, students who were enrolled in the second course in the following spring semester were invited to participate in the KiSS Program over winter break via email notifications and posts by instructors on course websites. Students who responded by texting a self-selected code name were enrolled in the program and received a problem daily (with the exception of holidays) for each of the 33 days of break. 357 students signed up to participate in the KiSS Program, and 307 of these opened at least one of the 33 problems.

Data Collection

On any given day of the program, students could choose whether or not to respond to the daily problem and could also exit the daily problem at any stage (e.g., after rating their confidence but before answering the problem). The following data was logged for each participant who opened the daily problem: time and duration of participation, confidence rating (for the daily problem), answer choice, and problem path (e.g., whether or not they accessed the hint, viewed the solution, or opted for a more challenging problem). Answer choice and resource use was also logged for any second attempts at the daily problem following hint use, as well as for any challenge problem attempts. In an effort to engage students in the KiSS Program as a fun review tool (rather than as a research study), demographic data was not collected prior to participation. Data from entrance and exit surveys, along with more in-depth interviews on program experience are discussed elsewhere (author2 and author 1, 2021).

Results

In this paper, we report on how student confidence relates to performance and engagement in the program on a daily basis. In particular, we look at relationships between confidence and accuracy, and how confidence in the ability to solve the daily problem plays out in the decisions students make for navigating through the review activity.

Confidence Ratings

Figure 4 shows the percentage of confidence ratings of each level (n=5273). Students were generally very confident in their ability to be able to solve the daily problem, with 73% of the ratings being either "somewhat" or "super duper!" Although self-selection in KiSS Program participation may play something of a role, this skew towards higher levels of confidence is also not surprising since all of the daily problems were a review of fundamental skills learned in or prior to the course that students had just successfully completed (Calculus 1).

Confidence Phrases and Emojis	not at all! 	not very 	meh 	somewhat 	super duper! 
Percentage	5%	8%	14%	26%	47%

Figure 4: Percentage of confidence ratings of each level

Confidence and Accuracy

After rating their confidence, students were presented with the daily multiple-choice problem. Figure 5 shows the relationship between accuracy and level of confidence. As can be seen, there was a positive relationship between accuracy and confidence. In addition, although there were very few times when a student did not respond to the problem after having rated their confidence, these were all instances in which the student had low or very low confidence in their ability to solve the problem, even just by selecting an option from five possible answers.

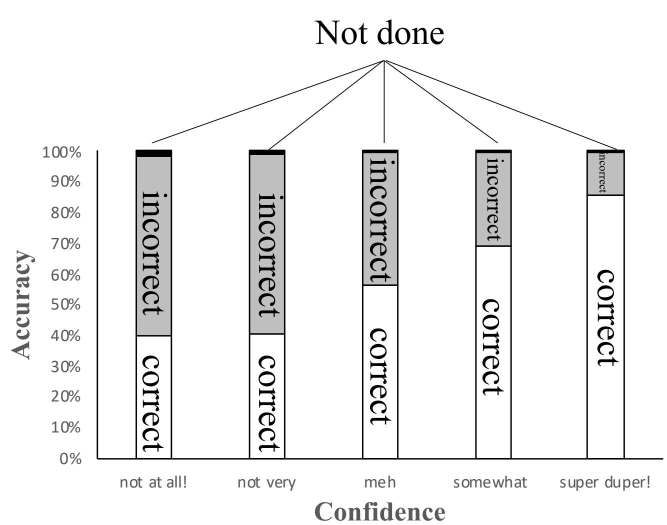

Figure 5: Accuracy of First Attempt on Daily Problem by Confidence (n=5273)

Accuracy and Path

After rating their confidence on their ability to solve the daily problem, students could take various paths through the daily review depending on their accuracy. Students who got the problem correct could choose to do a related challenge problem, view the solution to the daily problem, or simply exit for the day. Students who got the problem incorrect could choose to view a hint and retry the problem, view the solution, or exit. As can be seen in Figure 6 (left), for students who got the initial daily problem correct, confidence was related to the likelihood of engaging in a challenge problem afterwards. Students who were more confident in their ability to solve the initial problem were more likely to opt for a more challenging problem. Students who got the initial problem wrong (right), however, tended to choose to retry a problem, regardless of their confidence. Figure 6 also shows that exiting was more prevalent for students after getting the initial problem correct rather than incorrect, and that, in general, higher versus lower confidence was more characteristic of exiting the program for the day.

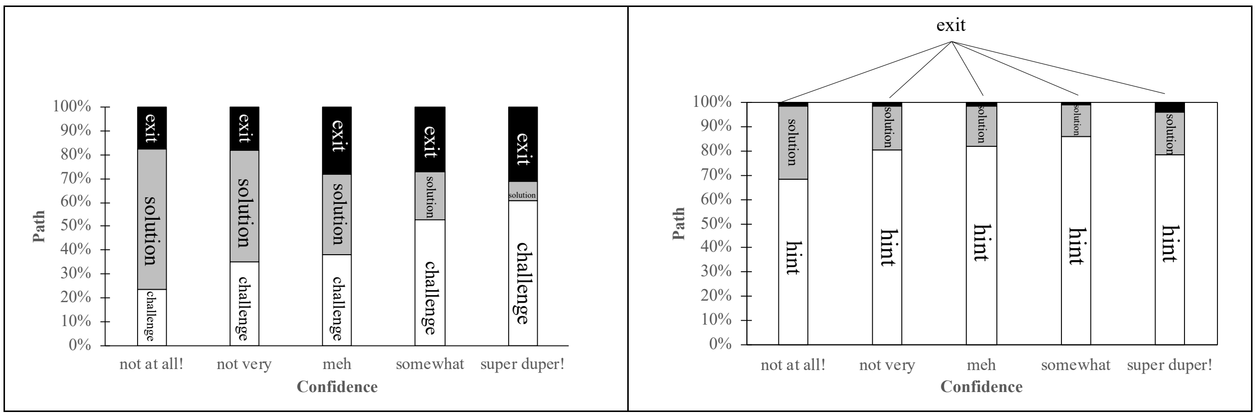

Figure 6: Path Taken by Students Who Got the Daily Problem Correct (left: n=3757) and Incorrect (right: n=1487)

Accuracy Following Challenge and Second Attempt

Depending on whether or not a student got the initial daily problem correct, they had the option to do a challenge problem or view a hint and retry the problem. Figure 7 depicts the relationship between confidence in being able to solve the initial problem and accuracy of the challenge problem (left) and accuracy of the second attempt of the initial problem (right). As seen in Figure 7 (left), confidence in being able to solve the initial problem was positively related to accuracy on the related challenge problem. However, for students who got the initial problem incorrect and then tried it a second time, the hints were helpful regardless of confidence (Figure 7, right). In addition, very few students changed their mind and exited for the day after choosing to view a challenge problem or a hint. However, these instances tended to occur more for students with low confidence who indicated that they wished to try a challenge problem after correctly solving the daily problem.

Path Following Second Attempt

Students who got the initial daily problem incorrect and chose to view a hint then had a second chance to attempt the problem. Whether or not they got the problem correct on this second attempt, students could choose to view the solution or exit for the day. As shown in Figure 8 (left), initial confidence was negatively related to viewing the solution for students who got the problem correct on their second attempt. More confidence initially, even though they got

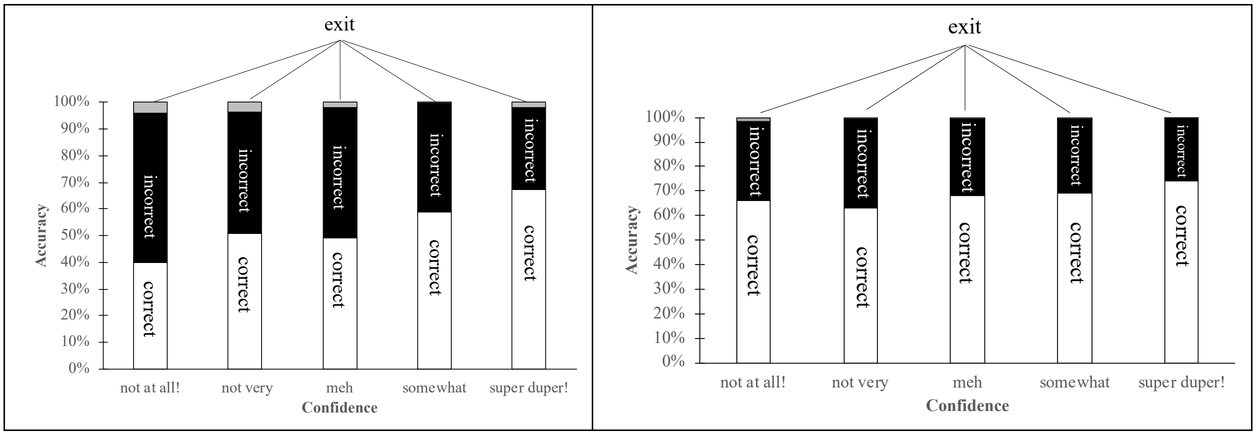

Figure 7: Accuracy on Challenge (left: n=2028) and Second Attempt of Daily Problem (right: n=1202) by Initial Confidence

the problem incorrect on the first attempt, was more associated with exiting, rather than viewing the solution, whereas less confidence initially was more associated with viewing the solution, even though the second attempt at solving the problem was successful. However, students who got the problem incorrect on their second attempt, regardless of their initial level of confidence, almost always chose to view the solution before exiting. As on the first attempt at the problem (Figure 6), exiting was much more characteristic of students after a successful versus an unsuccessful second attempt.

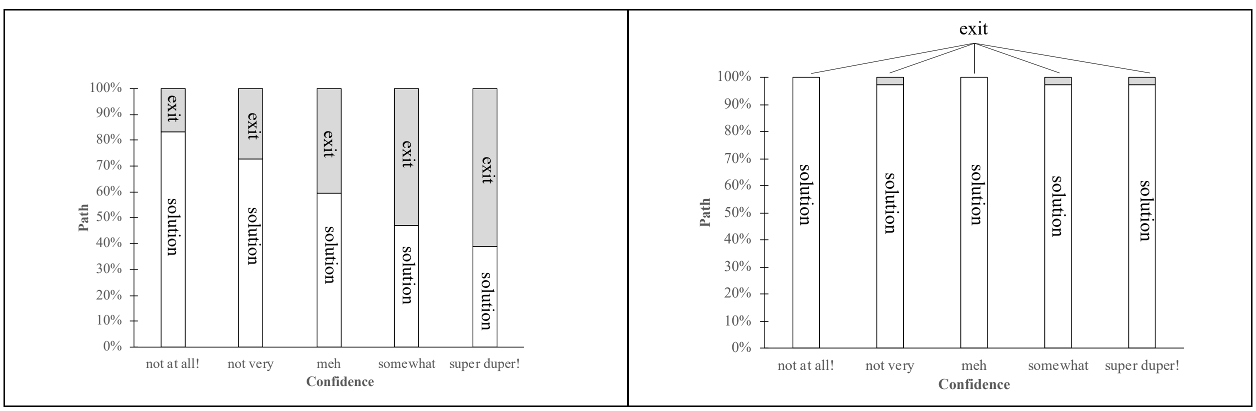

Figure 8: Path Taken by Students Who Got the Second Attempt of the Daily Problem Correct (left: n=827) and Incorrect (right: n=367) by Initial Confidence

Path Following Challenge

Students who got the initial daily problem correct could elect to do a second related challenge problem. Whether or not they got this more challenging problem correct, students could choose to view the solution or exit for the day. As seen in Figure 9 (left) initial confidence was negatively related to viewing the solution for students who got the challenge problem correct to some extent. Students who were unsure of their ability to solve the initial problem were more likely to look at the solution after getting both the initial and the challenge problem correct. This tendency was much more marked for students who got the challenge problem incorrect. Students who lacked confidence in their ability to solve the initial problem and then were unable to correctly solve the challenge problem almost always viewed the challenge problem solution. As on the first and second attempts at the problem (Figure 6 and 8), exiting was more characteristic of students after a successful versus an unsuccessful attempt.

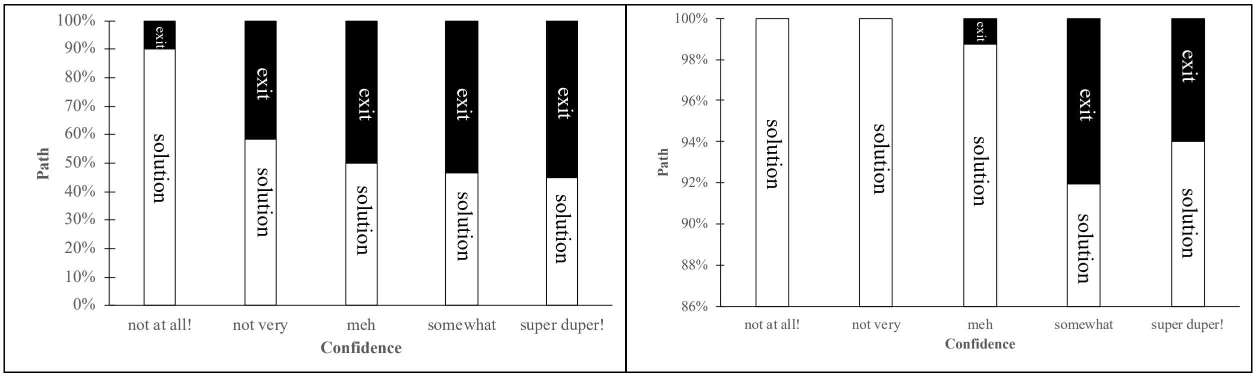

Figure 9: Path Taken by Students Who Got the Challenge Problem Correct (left: n=1243) and Incorrect (right: n=723) by Confidence on Daily Problem

Discussion

The KiSS Program illustrates the use of technology to engage students in productive struggle outside of the classroom during academic breaks. Since most students would otherwise not be testing themselves daily, the amount of voluntary participation is quite promising and could presumably be increased with more sophisticated and targeted marketing efforts. Also, the low exit rates after opening and engaging with the daily review activities indicate that the KiSS Program is successful at capturing student interest. The KiSS program therefore addresses the challenge of getting students to participate in beneficial regular retrieval practice (Kallookaran & Robra-Bissantz, 2017), even during breaks from formal instruction.

Implementing the KiSS Program and collecting judgments of learning (Rhodes, 2016) allowed us to explore metacognitive monitoring or metacomprehension (Dunlosky & Lipko, 2007) in an authentic setting. In particular, we were able to trace the relationship between confidence in the ability to solve a review problem and accuracy on that problem, as well as on a related more challenging problem. This use of technology also gave us insight into the role confidence plays on help seeking, although it did not provide a detailed account of how students used the various program resources.

Reaching students, especially those that are vulnerable, and helping them feel connected to their instructors and to their studies in a normal and predictable fashion is especially critical in light of unexpected disruptions, such as the recent pandemic, when students are overcome with feelings of alienation, uncertainty, and anxiety (Dziech, 2020). Therefore, the study of how technology can be used to deliver regular review opportunities outside of class, while simultaneously framing a positive mindset, is particularly timely and warrants attention and exploration. The tracing of the relationship between confidence, accuracy, and decision making in the KiSS Program sets the stage for future work to investigate the ways in which students use review program resources, how particular classes of problems affect student confidence, and how we can design popular and accessible review programs to build confidence and help students realize their full potential as they prepare for their future studies and careers.

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DOING MATH IN THE DIGITAL AGE: AN ANALYSIS OF ONLINE MATHEMATICS PLATFORMS

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We present a typology for characterizing online student-facing mathematics platforms that examines how they position students as learners, exemplified by 9 commonly used platforms. We identify three types of student learning experiences: instruction and practice, practice and support, and conceptual games and activities, and describe each one in terms of the relationships among instructional guidance, student agency, and the mathematical rigor of tasks. We find that within and across categories, there is substantial variation in cognitive demand and student agency, offering implications for further research, school decision makers and platform designers.

Keywords: elementary school education, instructional activities and practices, technology, curriculum.

Introduction

Increasingly, teachers and school systems in the U.S. are using a range of digital resources and tools to supplement regular mathematics instruction (Choppin, et al., 2014; Kauffman, et al., 2020). In this paper, we look closely at one type of resource that we call *online student-facing platforms* (OSFPs). These platforms represent an array of software programs, such as Dreambox, IXL, or Zearn, that have what Choppin et al. (2014) refer to as "individual learning designs." We present an analysis of 9 commonly used mathematics OSFPs and propose a typology for characterizing the nature of the learning experiences they offer elementary students.

Our interest in OSFPs has arisen out of evidence that students in the U.S. are using them in increasing numbers and as a substantial portion of their weekly mathematics learning. In our own interview study of elementary teachers' use of curriculum resources (Remillard et al., under review), 9 out of the 10 teachers in the U.S. reported assigning OSFPs at least once/week. In 3 cases, students used them 3 to 5 hours a week during dedicated periods and 5 teachers reported that OSFP use was dictated by school policy. These platforms can be understood as an often overlooked, but significant, component of mathematics learning. From this perspective, our analysis aims to consider the types of mathematics learning experiences available to students in OSFPs in U.S. elementary schools.

In order to achieve this goal, we draw on frameworks typically used to examine classroom instruction and features of digital resources. While research on digital tools often separates out components, we consider how components work together to frame comprehensive student learning experiences. Building on analyses of similar platforms in Choppin et al. (2014), Kay and Kwak (2018), and Cayton-Hodges et al. (2015), we consider how these platforms frame and structure teaching and learning interactions.

Our study contributes to what is known about OSFPs in a number of ways. First, by analyzing them using a teaching and learning framework, we suggest that students' experiences with supplemental tools matter for their mathematics learning. We do not assume that OSFPs should function in the same way or accomplish the same goals as classroom instruction, and we understand them to be supplemental in nature and useful for particular instructional goals. Nevertheless, given the increasingly extensive use of these platforms, understanding their affordances and constraints is important for teachers and researchers when considering their impact on students. Second, OSFPs are often thought of as similar in design and approach. By making visible the mathematical and pedagogical positions of some of the most commonly used OSFPs, our analysis demonstrates the diversity of these resources.

Background and Theoretical Framing

Our theoretical approach integrates frameworks that inform research and practice in mathematics teaching and learning with research on design components of digital learning tools that impact students' mathematical learning opportunities.

Summaries of research on mathematics teaching and learning speak to four critical domains of importance: a) mathematics content, b) types of learning experiences, c) learners' dispositions, and d) the role of teacher. There is general agreement that ideal *mathematics content* should be rigorous, meaning it should integrate procedural knowledge with conceptual understanding and opportunities for application of knowledge to familiar and novel contexts (NRC, 2001; Stein, et al., 1996). Recall of facts and fluency with procedures are both necessary but are only useful when connected with meaningful applications. We also know that *student learning* is increased when students are actively involved in tasks that require them to think through problems, make decisions on how to solve them, monitor their progress, and struggle with mathematical tasks (Hiebert and Grouws, 2007). These types of problem-solving experiences support students in developing *productive dispositions*, which include how students see themselves as mathematics learners in terms of identity, mindset, and agency (Boaler, 2016; Jackson, 2009; NRC, 2001). Research on the *role of the teacher* in supporting students' learning of rigorous mathematics through active engagement and fostering the development of productive dispositions is extensive. Several important themes stand out: Teachers play a critical role in scaffolding student learning, not by reducing task rigor, but by ensuring that all students can access the task (Jackson et al., 2013) and responding to students' needs during productive struggle. Some distinguish between "just-in-case" support, which provides all students with guidance prior to students engaging with problems, and "just-in-time" support which provides targeted guidance to particular students when they need it, allowing more opportunity for productive struggle (Dixon, 2020). This type of effective scaffolding is informed by knowledge of student understanding produced by ongoing, short-cycle formative assessment (Black & Wiliam, 1998; Copur-Gencturk & Rodrigues, 2020). By continuously monitoring students' understanding of a problem, teachers have the opportunity to provide scaffolds to students as necessary.

Research on the affordances of digital tools is still under development, and few studies have examined technological tools with an eye toward how they might contribute to mathematics education (Kay & Kwak, 2018). No studies focus exclusively on what we are calling OSFPs, instead classifying them as one type of broader collection of digital apps or tools. We build our analysis, in part, on a small set of studies that attend to the learning experience supported by these platforms or other digital mathematics apps, but do so in substantially different ways.

Clayton-Hodges, Feng, and Pan (2015) developed a framework for analyzing and assessing mathematics apps, which proposed four dimensions: a) quality of mathematical content (mathematical accuracy and richness), b) feedback and scaffolding, c) richness of interactions (modes of interaction and item types), and d) scoring and adaptability. To a large extent, Choppin and colleagues' (2014) analysis of digital curriculum programs hones in on the "richness of interactions" category. They focus on how learners interact with the platforms and found that they generally engage learners in one or more of three distinct types of activities: a) view video presentations, b) practice procedures that have been demonstrated, and c) manipulate representations to solve problems. Kay and Kwak (2018) offer the dimension of purpose as an additional category. Based on a review of research on mathematics apps, they found five different purposes: instructive, practice, constructive, productive, and game-based. They also offer a list of eight characteristics around which the apps they examined tended to vary, many of which overlap with or add detail to Clayton-Hodges et al's (2015) dimensions: types of learning valued, quality of the content addressed, clarity of learning goals, usability, engagement, adaptability to differing levels, mode of feedback, and opportunities for collaboration. Each of these classification systems contributed to the development of our analytical framework. Because we were interested in the learning experience offered by the OSFPs, we selected dimensions from these frameworks that most aligned with research on the nature of mathematics teaching and learning. These are described in the following section.

Design and Methods

The data for this study come from an analysis of student-facing mathematics platforms that are frequently used in the United States. We constrained our selection to platforms that students use by logging on and working individually as a supplement to primary mathematics instruction. We began with the OSFPs identified by teachers in a related study on teachers' use of digital resources (Remillard et al., under review). We then added platforms based on reports of OSFPs most commonly used by teachers in the United States (Kauffman, et al., 2020) and our own awareness of available platforms with unique features. After completing the first phase of analysis, we searched for additional OSFPs to test the viability of our emergent typology. In this paper, we report on a reduced subset of 9 platforms that exemplify the range and variation of each category.

To begin our analysis of each OSFP, we immersed ourselves in the student experience. Through completing multiple tasks and exploring the learning pathways of the platforms, we became familiar with the program organization, types of tasks, instructional supports, and responses to correct and incorrect student entries. We also read and watched instructional and promotional materials for teachers and reviewed teacher resources to understand the intended purposes of the platform features. Although the platforms included a number of teacher-facing features, we restricted our analysis to student-facing components, given our aim of understanding the student learning experience. From this phase of analysis, we wrote detailed memos that summarized the characteristics and features of each OSFP and outlined the overall nature of mathematics teaching and learning available in each platform.

Based on our initial exploration of the platforms, we developed a set of categories that roughly aligned platform features with constructs from the literature on mathematics teaching and learning and incorporated key dimensions included in analyses on platforms and apps. These are summarized in Table 1. The majority of categories were emergent, developed through successive rounds of description, comparison, and refinement. The levels of cognitive demand, introduced by Stein et al. (1996) and widely used in subsequent studies, were the only *a priori* categories applied.

Table 1: Categories and descriptions for OSFP analysis

Constructs	Platform Features	Categories
Mathematical Rigor	Cognitive demand (Stein et al., 1996)	Memorization (Mem); procedures without connections (PWOC); procedures with connections (PWC); doing mathematics (DM)
	Types of tasks	Examples: Replicating procedures, problem solving, creating models, test-like questions, conceptual activities, mathematical games
Instructional Guidance features	Instructional supports	How direct instruction is provided through videos or interactive lessons
	Responsive supports	How just-in-time supports are provided through feedback, hints, corrections, and optional instruction
	Learning pathway	How student's pathway through tasks is determined (type of adaptive tools)
Student agency/dispositions	Student decision making	How students monitor/guide their own work; availability of student choice

Many platforms we analyzed included a range of features, multiple task types, and several modes of interaction, making categorization challenging. For the purpose of this study, we focused on the most prominent features and approaches related to critical domains of teaching and learning mathematics, discussed earlier, in each OSFP. Many platforms included some timed practice of computation facts, which varied little across platforms and are not considered here. In addition, platforms used a range of gamification or engagement strategies (e.g., points, badges, characters, music, avatars, non-mathematical games), which we did not include in our analysis.

By analyzing the platforms using emergent categories and short descriptions, we were able to explore patterns in the combinations of features across different platforms. When grouped according to these patterns, we observed the potential for different learning experiences for students. We identified three primary types of *student learning experiences* offered by the platforms: Instruction and Practice, Practice and Support, and Conceptual Games and Activities. These categories overlap with and add depth to analytical frameworks from Choppin et al. (2014), Clayton-Hodges et al. (2015), and Kay and Kwak (2018).

Findings

We organize our findings around the three types of student learning experiences that emerged from our analysis. As illustrated by Table 2, the instructional guidance, student agency, and mathematical rigor of platforms intertwine to shape a distinct student experience. Within each type, we found variation across the subcomponents, which we illustrate through several exemplar platforms.

Table 2. Description of three OSFP types with exemplar platforms.

Student Learning Experience Categories	Nature of Interaction	Student Agency	Task Types with Levels of Cognitive Demand (Stein et al. 1996)
Instruction and Practice (Dreambox, iReady, Zearn)	Videos, sometimes with interactive tasks, followed by practice with identical tasks with hints or explanations	Minimal	Replicating procedures to create visual models and/or solve test-like word problems Mem, PWC
Practice and Support (IXL, Khan Academy, Prodigy, Study Island)	Practice tasks with hints or explanations; support may be corrective or conceptual	Choose type and amount of support	Test-like questions. Mem, PWOC, PWC
Conceptual Games and Activities (Beast Academy, Math Playground)	Concept-building mathematical games and activities; guided lessons or self-discovery of increasingly complex concepts.	Choose problem solving approach	Students solve game-like or progressively complex challenges Mem, PWC, DM

Instruction and Practice

Our first OSFP type, *instruction and practice,* demonstrates concepts or skills through direct or guided instruction and then has students replicate them through highly similar procedural tasks with little student agency. Within this type, we found several variations. Dreambox and iReady follow a gradual release approach to instructional guidance (I Do-We Do-You Do), where students first watch a demonstration, then participate in filling out incremental, predetermined steps, and then practice the same steps on their own. Zearn's approach is more guided, using interactive videos, during which the student is frequently asked to answer questions, interpret models, or demonstrate understanding along a conceptual pathway before they practice independently.

In terms of mathematical rigor, all 3 platforms employ visual models to connect procedures to underlying concepts (PWC tasks). iReady also includes mathematical games that allow students to apply their learning in a game context, for example by adding weights (negative) or balloons (positive) to a submarine to move it along a vertical number line and pass through a goal. Dreambox, however, uses a didactic approach of demonstrating each activity twice with a focus on completing steps. which reduces the rigor level, despite the use of interactive visual models.

We found student agency in these platforms to be minimal or superficial, as they require students to move through all lesson components in order. For example, in Zearn, video lessons start playing automatically, but students can skip through explanations of the steps, and in Dreambox students can select the topic of the next task while still proceeding down the same lesson path. Several of these platforms offer individualized learning paths based on initial diagnostics, though we found that in Dreambox students are often forced to practice many tasks at the same level despite having repeated correct answers. Zearn has no adaptivity; all students complete the same tasks in the same order.

Overall, the approach of *instruction and practice* OSFPs reflects a perspective that learning involves replicating procedures correctly in repeated practice, with iReady and Zearn at the more conceptual end of the spectrum, as students are introduced to conceptual models, and Dreambox at the more procedural end, with students using visual models in a rote manner. These OSFPs use a just-in-case model and appear to position students as passive learners, who may view mathematics as a set of steps to memorize.

Practice and Support

OSFPs that we describe as *practice and support* have students begin by directly practicing tasks and receiving instructional support only when they choose it or demonstrate need through incorrect responses. Students may access written/visual or video instruction at any time, encouraging them to select the level of support that they need. There is still substantial variation among platforms in this type, shaped by how the mathematical rigor of tasks and related instructional guidance work in tandem to modulate the level of student agency.

In *practice and support* platforms, tasks are formatted like test questions, including multiple choice, multi-select, fill-in-the-blank, drop-down, and interpreting or creating visual models. This range of task type, which includes word problems and visual models, reflects those commonly used in online assessments and textbooks in the U.S. We found that *practice and support* platforms vary in both the cognitive demand of tasks (procedural or conceptual) and the nature of instructional guidance (corrective or conceptual), often in tandem. At one end of the spectrum, Study Island provides tasks, videos, and explanations that focus on completing rote procedures without conceptual understanding (PWOC). The accompanying student feedback is corrective, providing a generic description of the solution strategy and then the correct answer. At the other end, IXL and Khan Academy use more complex tasks, based on visual models and alternative algorithms to develop conceptual understanding (PWC). Students are supported by conceptual feedback in the form of optional, stepped-out hints, explanations, and conceptually-focused video lessons. Khan Academy gives students a menu of support options and also uses a point system to encourage them to try solving first on their own. Prodigy lies in the middle, with more rigorous test-like questions, but the hints provide a small amount of supporting information or indicate the early steps of a solving strategy, and it only provides corrective feedback.

Practice and support platforms offer a higher level of student agency than *instruction and practice* platforms because they allow students to determine when and how to seek instructional guidance. Yet there is still a substantial range in student agency among *practice and support* platforms, depending on both the task complexity, the kinds of support offered, and students' level of choice in accessing the supports.

Overall, the components of *practice and support* OSFPs work together to suggest a model of learning in which students are encouraged to try solving problems independently and seek or be given help if they need it. This design positions students as active learners when practicing or figuring out a range of problem types and, in some platforms, gives them the opportunity to self-monitor their understanding. While IXL and Khan Academy offer somewhat more cognitively complex tasks and supports than Prodigy, all three use an underlying just-in-time model that begins with student practice.

Conceptual Games and Activities

OSFPs that we categorize as *conceptual games and activities* invite students to make sense of tasks and experiment with problem solving strategies. Many of these tasks offer minimal or no language introducing the problem or its objectives, inviting students to explore.

These platforms offer two types of tasks: problem progressions and logic games. Problem progressions start with a simple problem that requires only basic computation or memorization to solve and then provide increasingly more complex problems that organically lead to multi-step or algebraic thinking. For example, one Math Playground progression begins by showing two identical candies with a total cost of 6¢, which students drag to the 3¢ jar. The task quickly progresses to involve multi-step algebraic thinking to find the cost of an unknown candy when a 24¢ candy and two identical unknowns together cost 52¢, all without any instructions or demonstrations (doing mathematics; DM).

Math logic puzzles and games rely on an understanding of procedures but encourage conceptual thinking (PWC). Beast Academy and Math Playground both contain logic games where a limited set of numbers must be arranged to produce correct sums/products along horizontal, vertical, or diagonal lines, supporting flexible number sense. Additionally, some of the operational automaticity games in Math Playground support flexible number sense, for example by having students select different pairs of bubbles that have a sum of 8. (These flexible automaticity games are also available in iReady).

The instructional guidance features differ in our two exemplars. In Math Playground, feedback is minimal and corrective (e.g., a beep indicating a wrong answer), with no hints, explanations, or instruction. However, the responsive support is also unlimited; students can make endless attempts using different strategies until they find the correct solution. Beast Academy, meanwhile, provides instructional support that is more similar to a *practice and support* platform, where students may seek help at any time by clicking on a video or an illustrated lesson, and students are shown correct answers with brief explanations after two attempts.

Overall, the *conceptual games and activities* OSFPs suggest a view that students learn through solving increasingly complex tasks, using their own strategies, with minimal or no instruction. While Beast Academy provides a higher level of instructional guidance than Math Playground, the creative and rigorous tasks in both platforms position students to have agency in determining their own solution paths.

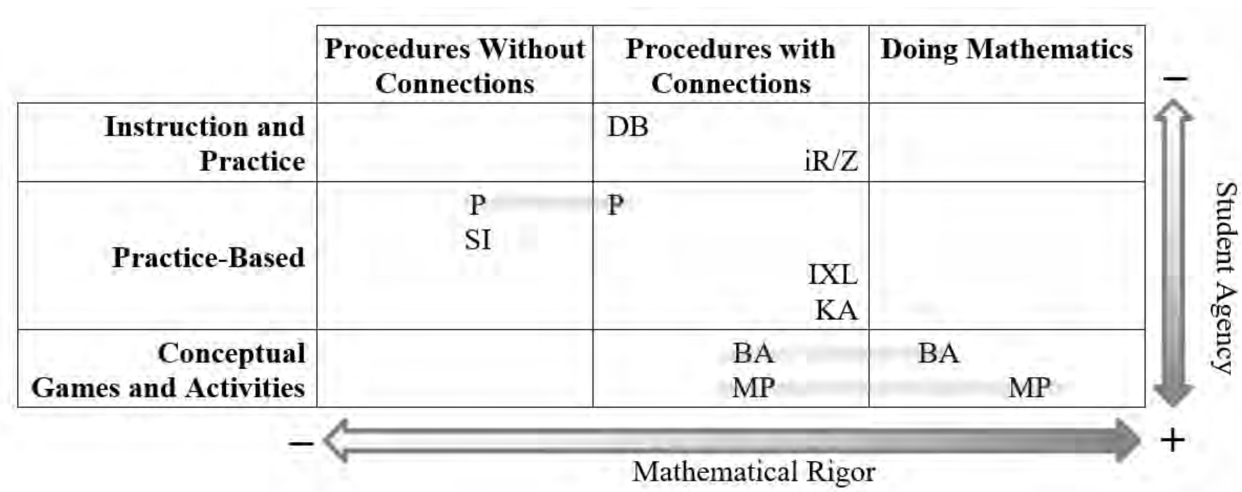
Discussion and Significance

Given the prominent use of OSFPs in U.S. classrooms, one aim of our study was to understand and make visible the types of learning experiences available to students when using these platforms. Though OSFPs are often overlooked as supplements to a core curriculum, the hours students spend with them has the potential to substantially shape their ideas about the nature of mathematics and themselves—whether they see themselves as active or passive learners, and whether mathematics involves replicating procedures, understanding them, or creatively solving problems (Boaler, 2015; Skemp, 1978).

Our typology offers a framework for analyzing OSFPs beyond the 9 that we showcase here, supporting researchers and school decision makers in understanding the affordances of these resources. While building on prior work that compared platform components and features (Cayton-Hodges, et al., 2015; Choppin, et al., 2014; Kay & Kwak, 2018), our analysis takes a more integrated approach. By putting components and analytical categories for conceptualizing mathematics teaching and learning in relation to one another, our framework offers a fuller picture of how platforms create mathematics learning experiences.

In particular, our approach illustrates the utility of our three analytical categories for examining OSFPs, instructional guidance, mathematical rigor, and student agency, while also highlighting how they work in relation to one another to shape a particular learning experience. There are substantial differences, for example, in the overall design of the three types we identified in how they position the nature of mathematics and students' roles. Moreover, we found that differences in how these components are combined leads to further variation, with the potential for substantially different students' experiences.

We have attempted to capture the way variation in mathematical rigor and student agency interact in our nine exemplar platforms in Figure 1. Moving horizontally, the platforms increase in mathematical rigor from left to right. Gray bars indicate platforms with tasks at two levels of cognitive demand. For example, Prodigy (P) contains tasks at both the PWOC and PWC levels. Student agency is placed along the vertical axis, increasing from top to bottom. Each platform is placed in relation to both continua and shown within our analytical types. In addition to showing how the platforms we analyzed differed, the figure demonstrates how the different categories influence the overall learning experience.



Key: DB = Dreambox, iR = iReady, Z = Zearn, P = Prodigy, SI = Study Island, KA = Khan Academy, BA = Beast Academy, MP = Math Playground. Gray bars indicate platforms with tasks at two levels of cognitive demand.

Figure 1: Relationship between student agency and mathematical rigor across and within each OSFP category.

Finally, we argue that the use of student agency as an analytical lens when examining digital platforms draws attention to important and unexamined aspects of OSFPs. Student agency differs qualitatively across the types of platforms and is supported by different features. In *practice and support* platforms, students are encouraged to monitor and manage their work on tasks, getting support only when they need it. In *conceptual games and activities* platforms, students can develop agency through sense making and directing their own approach to problem solving. Both types of agency can support students to develop productive dispositions and mindsets related to mathematics learning (Boaler, 2016). *Instruction and practice* platforms, on the other hand, provide limited attention to this consequential aspect of students' mathematical identity development, though Zearn suggests that this can be increased through interactive lessons where students complete some steps before they are modeled. We recommend that increasing opportunities for the development of student agency would be a fruitful path for designers of OSFPs to pursue.

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COMPLEXITY OF WORD PROBLEMS THROUGH ITS READING IN STUDENTS AND PROSPECTIVE TEACHERS

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The main goal of this study is to compare the relationships between the reading ability of prospective teachers and secondary school students measuring the average reading time per word as a proxy of the complexity of mathematical problem statements. Besides that, we study students' performance in word problems where the problem statement involves the fraction acting as an operator and the reconstruction of the whole. The results show that the proposal to express reading ability through a scale adapted to the distribution of the studied population through quartiles and interquartile range is consistent with the results obtained in the existing literature. In addition, due to the greater reading ability of prospective teachers, their success in solving problems is greater compared to secondary school students.

Keywords: reading, problem-solving, prospective teachers, students

Introduction

Reading skills are a basic tool for text comprehension and academic performance in the Primary stage (Alegría, 2006; Sweet & Snow, 2003). There are reading skills related to decoding processes, while other skills involve understanding processes themselves. As reading skills are automated, the understanding process also is developed, since cognitive resources are released that can be dedicated to understanding. However, reading does not necessarily ensure comprehension (Oakhill & Cain, 2007a, 2007b; Oakhill, Cain & Bryant, 2003). Although, there are studies, such as that of Sanz et al. (2020) who showed that the average reading time per word used when reading and solving the statement of a word problem allows predicting performance in solving the problem posed.

In this context, in the area of mathematics, there is an interaction between the processes of understanding the verbal statement and mathematical knowledge, so that reading comprehension plays an important role (Kintsch & Greeno, 1985). Different studies (De Corte and Verschaffel, 1991; Hegarty, Mayer and Monk, 1995) demonstrate the relationship between difficulty in correctly representing problem statements and mathematical performance.

But not only the student body deserves to be studied, authors such as Waller (2012) or Olfos et al. (2014) determined that teachers' knowledge is significantly associated with student performance.

Research purpose

In this context, we follow a methodology where the users' reading time is considered as a measure of the complexity of a word problem, with the particularity that given there are two actors in the teaching-learning process, we consider secondary students and prospective teachers as our study population.

Thus, the specific goals in this work are:

1. To design a proposal to compare the complexity of the statement of a word problem measured through the reading ability of prospective teachers and secondary students.
2. To verify the proposal designed to compare the complexity of word problem statements in the two populations considered.

Methodology

The process to measure the complexity of a word problem and classify the reading ability of a user

Following Sanz et al. (2020), the global complexity of the statement will be measured as the average reading time per word of the propositions that make up the statement.

Regarding the reading ability of a user when reading word problems (Eq.1), it will be calculated as the average of the time spent by the student in each proposition (t_{ijs}) for the words (w_{ij}) read.

$$rh_s = \frac{\sum_{i=1}^{k}\sum_{j=1}^{p_i} t_{ijs}}{\sum_{i=1}^{k}\sum_{j=1}^{p_i} w_{ij}}, \qquad (1)$$

where s corresponds to the student, p_i the number of propositions of each task, k is the number of tasks.

In this study, we design a proposal to be able to compare the complexity of a word problem statement in the two study populations selected characterized by integrands having different reading skills.

Following the methodology of PIRLS (2017), in which the level of school performance is expressed on a scale, in the present work, reading ability is adapted to a scale based on the distribution of the population studied. To do this, interquartile ranges are used to evaluate the dispersion of the distribution and determine four reading abilities levels or intervals (Table 1). It should be noted that values above the advanced or low level will be considered outliers and will not be analyzed.

Table 1: Reading ability levels. * o minimum ** o maximum

Reading ability level	Range
Advanced	[Q_1-1.5·R.I.C*, Q_1[
High	[Q_1, Q_2[
Intermediate	[Q_2, Q_3[
Low	[Q_3, Q_3+1.5·R.I.C**]

Technological environment

To obtain the reading time in each proposal, a technological environment named R&L will be used to design research experiments on reading comprehension in text and image-related learning tasks (Sanz et al., 2020). Experiments in R&L can include enriched texts with a list of questions and answers. A number of configuration settings are available, such as the possibility of accessing the statement at any time or only under certain conditions, the effect of alternatively hiding and showing parts of texts by clicking on them.

Covering parts of the text (including multiple choice answers) makes them remain hidden and can only be made visible by clicking on them. This allows obtaining a record of the time that the user spends reading said hidden parts.

Experiment design

We conducted a quantitative descriptive study with a sample of 113 participants (62 women) who belong to two populations of the educational system. In particular: a) 43 prospective primary school teachers (35 women), with an average age of 21.78 ± 1.44 years; and b) 43 male students and 27 female secondary students aged between 15 and 16 years.

First, we analyze the differences based on the participant's profile (Figure 1), obtaining that there are statistically significant differences at a 5% level of significance (TM = 58.520; p-value <0.0001).

In this context, Table 2 shows the levels of reading ability for each group population.

Table 2: Reading ability levels (seconds) secomdary students and prospective teachers

Reading ability level	Secondary students	Level	Prospective teachers	Level
Advanced	[1.424, 4.738[	A_4	[0.822, 2.708[	D_4
High	[4.738, 6.909[	A_3	[2.708, 3.321[	D_3
Intermediate	[6.909, 9.244 [	A_2	[3.321, 4.098 [	D_2
Low	[9.244, 14.846]	A_1	[4.098, 5.790]	D_1

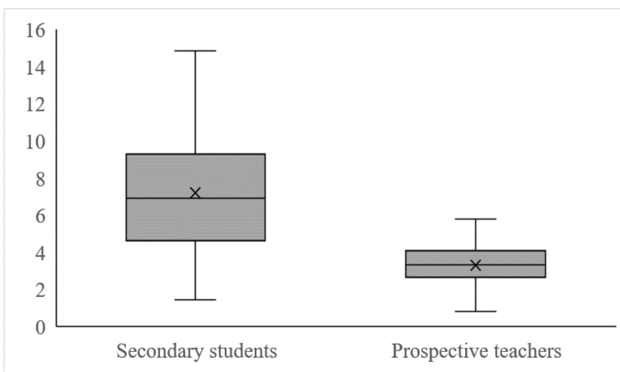


Figure 1: Reading ability classified by study population

Participants were asked to solve the following word problems:

P1. We have thirty candies. Two-thirds of them are strawberry flavored. How many strawberry candies do we have?;

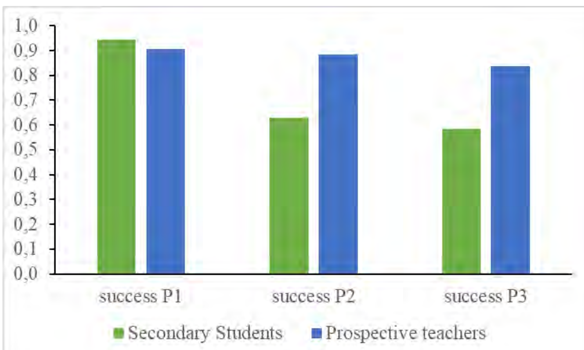
P2. I have one-half of a pizza. Two-thirds of it is pepperoni. What fraction of the pizza is pepperoni?;

P3. In a fitness club, five-sixths of the 600 members do gymnastics, two-fifths of the rest do swimming. How many members do swimming?

Results and Discussion

Results show that the distributions of secondary students and prospective teachers performance, measured through the success rate when solving problems, are the same (Figure

2a). Thus, if the complexity of a word problem were determined from the probability of success, it could be said that P (success P1)> P (success P2)> P (success P3), which would indicate that P1 would be simpler than P2 and this in turn, simpler than P3. When evaluating the average time per word used when reading the problems statements (Figure 2b), it is observed that the data distributions coincide in both populations, and, in addition, the results match with the probabilities of success (Figure 2a), but they area inversely related. That is, the higher the success rate on a problem, the less time spent on solving the problem. These results are in line with the ones of Ivars & Fernández (2015) that conclude that the success rate can measure the complexity of a problem; so, then reading time can be used as a good proxy. Likewise, given the same distributions, in secondary students as in prospective teachers, the teachers must be studied to determine the difficulties of the students (Waller, 2012; Olfos et al., 2014).

a)


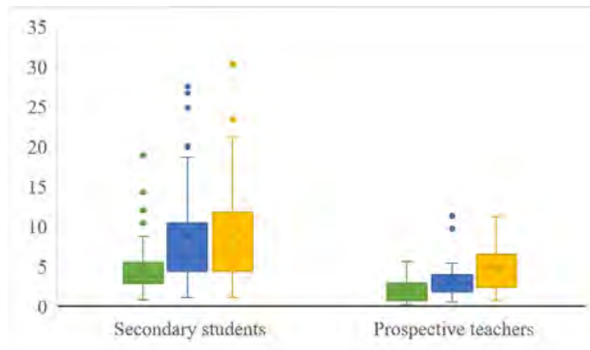
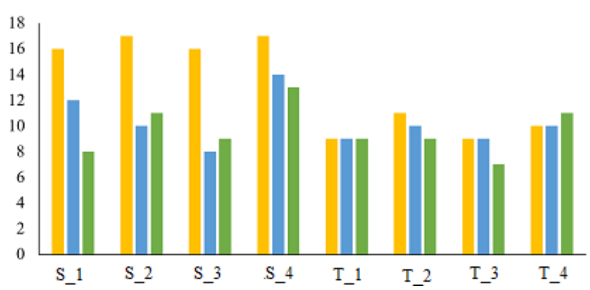
b)


Figure 2: Results of the global statement in both populations a) Success of resolution measured as the probability of success; b) Success of resolution measured as reading time per word.

On the other hand, analogous results are obtained with the same analysis but taking into account the levels of reading ability for each population considered (Figure 3). Furthermore, it can be observed that at the advanced level (A_4 and D_4) the number of students who correctly solve the problems is greater. These results indicate that reading, and in particular reading level, is related to student's performance (De Corte & Verschaffel, 1991; Hegarty, Mayer & Monk, 1995; Alegría, 2006; Sweet & Snow, 2003).

a)


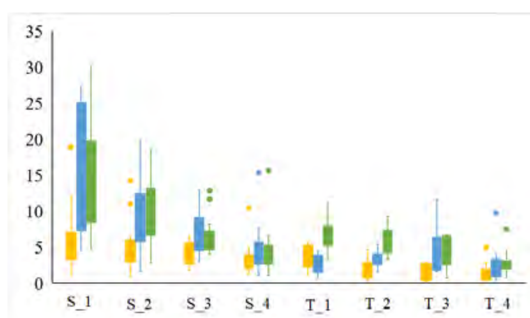
b)


Figure 3: Results of the global statement in both populations a) Number of participants who have been correct; b) Resolution success, reading time per word per population.

Conclusion

This work opens a line of research on the use of technological environments and data analytics to determine the complexities of word problems, measuring the level of understanding of each of the statements through reading time and dealing with the mathematical concepts that they make it more difficult to solve. Likewise, it allows determining the user's reading ability.

The next steps include the design of a longitudinal study by age of the students that analyzes the evolution of the concepts and the possible blockages that may occur. Future work will also help define an index that allows you to create statements with pre-set complexities by weighting the propositions in the statement, according to their level as established in Sanz et al. (2020).

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THE BESSIE COLEMAN PROJECT: BROADENING HISTORICALLY-EXCLUDED STUDENTS' PARTICIPATION IN COMPUTER SCIENCE

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This research report presents the results of The Bessie Coleman Project (BCP), which was named for the first African-American and Native woman to receive a pilot's license. Urban and rural students from historically-excluded backgrounds (i.e., Black, Latinx, Indigenous, and female) were recruited to participate in the BCP. Computer modeling, 3D printing, flight simulation, and drones were used as interventions to enhance students' computational thinking (CT) skills and STEM efficacy. Project staff and facilitators implemented the project in Pennsylvania, Colorado, and Wyoming in Year 2. This research report describes the results of the Year 2 BCP study before the COVID-19 pause, specifically as they relate to student identity and to broadening underrepresented students' participation in STEM.

Keywords: Computational Thinking, Computing and Coding, Technology

Wing (2006) describes Computational Thinking (CT) as a "problem-solving approach that draws on concepts fundamental to computer science by 'reformulating a seemingly difficult problem into the one we know how to solve, perhaps by reduction, embedding, transformation, or simulation'"(p. 33). Thus, CT is a process that includes formulating problems, algorithmic thinking, generalization, and learning transfer (Barr et al., 2011; Repenning, 2012). Yet, CT is not just for computer scientists; children are able to engage in algorithmic thinking approaches similar to those posited in mathematics to carry out the problem-solving process (Sengupta et al., 2013). This study was conducted to expose historically and contemporarily excluded children to CT through computer modeling, flight simulation, and drones in order to broaden their participation in STEM.

Theoretical Framework

The theoretical frameworks that support this study is Critical Race Theory (CRT) and Tribal Critical Race Theory (TribalCrit). CRT began as a form of legal scholarship that examined how the law intersects with race showing the limitations of meritocracy and affirmative action policies (Anderson, 2019; Bell, 1987; Crenshaw, 1988). TribalCrit, an offshoot of CRT, provides a framework to support Indigenous values, culture, and identity (Brayboy, 2005). A common thread in both theories is naming and resisting the root causes of systemic racism. The main tenet of CRT is that racism is endemic to society such that "whiteness as an explicit cultural product [takes] on a life of its own" (Apple 2003, p. 113) to normalize practices in schools and society that privilege whites and marginalize people of color. In contrast, the main tenet of TribalCrit is that colonization is endemic to society, which is evident by government policies that subjugated Indigenous peoples and attempted to assimilate them into Western culture (Brayboy, 2005, p. 429). In response, both CRT and TribalCrit draw on counternarratives to dispel myths about their culture and ways of knowing while also asserting self-determination, self-identification, sovereignty and autonomy (Brayboy, 2005; Duncan, 2005). Research takes on new meaning when examined through the lens of historically-excluded children, especially when decolonized research principles are used to empower them to tell their own stories. These children's voices are critical to conducting research that broadens their participation in STEM.

The Bessie Coleman Project: Purpose and Objectives

The Bessie Coleman Project (BCP), named for the first Black and Native American woman to earn a pilot's license, was developed to engage students of color in computer modeling, 3D printing, and coding experiences. The primary objective of the BCP was to provide meaningful technology-based learning experiences to enhance students' CT while also using culture and place to make connections to their backgrounds. A second objective was to produce counternarratives about the STEM participation of Black, Latinx, and/or Indigenous students. Teachers, facilitators, and guest speakers with STEM expertise—some of whom were from the same background as the students—were recruited to deliver the content. The teachers participated in 15-18 hours of professional development where they learned how to code in Tinkercad and Sculptris. They also learned how to use flight simulation and operate drones prior to implementing the BCP with students. The research questions that guided data collection during the second year of the three-year study were as follows:

1. In what ways did emerging technological tools (i.e., computer modeling, 3D printing, flight simulation, and/or drones) influence participating students' CT and self-efficacy in STEM?
2. What impact, if any, did the technological tools and STEM professionals have on students' STEM content learning and STEM interest?

Methods

Mixed methods were used to collect data in the BCP. Quantitative measures included a survey instrument that consisted of four subscales. Qualitative measures consisted of field notes, focus group interviews, and journal logs (Creswell, 1998). In Year 2, the Computational Thinking Self-Efficacy (CompTSE) scale (Coenraad et al., 2020) was developed, field-tested, and added as a subscale to the survey instrument. This scale provided insight about students' computational thinking abilities and is useful in discerning how exposure to the interventions influenced students' interest in STEM.

Sample

Fourteen instructors and 85 students participated in the Year 2 study. Table 1 shows student demographics in the analytic sample by type of implementation. The BCP was implemented after school in three public schools in Philadelphia, Pennsylvania, in spring 2019. In addition to the curriculum intervention, Philadelphia students took a day trip to the National Air & Space Museum in Washington, D.C., and interacted with a guest speaker from NASA Goddard who was a solar scientist. Weeklong summer camps were held in June 2019 at a Boys & Girls Club in Denver, Colorado, and in July 2019 at two school sites in Wyoming—one in Riverton and the other on the Wind River Indian Reservation (WRIR). Guest speakers in Denver included Capt. Ed Dwight, the first Black astronaut candidate. Students in Denver also had a field trip to the

Wings Over the Rockies Museum. Students in Riverton visited the Wyoming Dinosaur Center in Thermopolis where they participated in a fossil dig. Arapaho students took a day trip to Beaver Rim to learn about buffalo jumps and to fly drones.

Data Sources and Data Analyses

A survey instrument that measured student efficacy in computing, science, and technology (Ketelhut, 2010) was administered to students at the beginning and near the end of the intervention. All of the subscales on the survey instrument were reliable (i.e., Cronbach $\alpha \geq 0.80$). Qualitative data sources included transcripts of student focus group interviews and journal logs, which were analyzed for themes and patterns using the constant comparative method (Strauss & Corbin, 1990).

Results

Computer modeling, flight simulation, and drones provided students with applications that allowed them to develop new knowledge and understanding of complex systems. Pre-post surveys were analyzed using paired *t*-tests (see Table 2). Results show a significant decrease on science self-efficacy (SciSE) ($t[40]= -2.744$ [$p = 0.009$]) in Philadelphia and no change on three other subscales. There were significant increases on computing self-efficacy (CSE) ($t[16]= -3.058$ [$p = 0.008$]) with medium effect and CompTSE ($t[16]= -2.463$ [$p = 0.026$]) with small effect at the Denver Boys & Girls Club. Results also revealed significant increases on five subscales in the Wyoming summer camps. These results should be interpreted with caution given the small sample sizes in some settings and lack of control groups.

Qualitative data also reveal important findings about student learning, STEM participation, and STEM professionals. Comments made by focus group students in Philadelphia and Denver and excerpts from Arapaho students' journal logs are presented below:

> *I liked all the 3-D printer. I liked that. I liked how we could design it and make whatever we want with it and also learn how to measure it and make sure that it's like a certain size or length.* Black male, age 12 (Philadelphia)

> *I'm not trying to offend him [NASA scientist], but I feel like we should get like, umm…. He's great. I'm not complaining, but I feel like we should get like a Black woman [speaker] because…like for us…I feel like we should get a Black woman because we need that….* Black female, age 11 (Philadelphia)

> *Tinkercad was really fun, and interesting but really hard. You have to know what to do and how to do it. Say you're trying to make a circle but only have a square. You can use a hole around the square by pulling it. We thought of a company, [and] modeled something we wanted to sculpt. Then we did Tinkercad and reimagined our thing then printed our idea.* Black Female, age 13 (Denver)

> *[Black astronauts] went through tough things, and you have to be a really smart person to get there. Like Black people couldn't fly and now they can.* Black male, age 12 (Denver)

> *Today, I learned drones can follow people, and I seen a drone video with…on a skateboard. He falls, but he's okay I think. Drones can fly themselves home when they are about to die..* Arapaho male, age 14 (WRIR)

> *I had fun recording the river, then everyone seeing it. So I learned that the drones can go in a perfect circle by just pressing a button.* Arapaho female, age 15 (WRIR)

Discussion

The BCP was successful in broadening STEM participation among Black, Latinx, and Indigenous students and females. In terms of quantitative results, Philadelphia students and predominantly Black children at the Denver Boys & Girls Club made significant gains on CSE and CompTSE, and Arapaho students made significant gains on technology interest and use (TIU) and CompTSE. Qualitative data revealed students enjoyed computer modeling, 3D printing, and flying drones. One student mentioned learning about measurement and another described how to make shapes in 3D. The latter is an example of CT: "*Say you're trying to make a circle but only have a square. You can use a hole around the square by pulling it.*" Moreover, an Indigenous student used a drone to track everyday activities like skateboarding. Implicit in their excerpts is the freedom to engage in tasks that were of interest to them: "*I liked how we could design it and make whatever we want; We thought of a company, [and] modeled something we wanted to sculpt; I had fun recording the river*." Furthermore, some students expressed opinions about the guest speakers: "*[Black astronauts] went through tough things; I feel like we should get like a Black woman [speaker]….*" These excerpts reveal the importance of students' seeing STEM role models who share the same gender/racial identity (Lane & Id-Deen, 2020). Although the guest speakers could have been more diverse, they helped students to learn STEM content and also understand that resilience was needed to persevere in STEM. While the BCP focused on out-of-school contexts, online modules [www.bessieproject.com] were developed during the COVID-19 pandemic that were piloted during the school day in Tennessee and Wyoming (Leonard et al., 2022). The results of the Year 3 project are forthcoming.

Table 1: Analytic Sample by Race, Ethnicity and Gender (Year 2)

Cohort	Program Type	*N*	Gender		Race/Ethnicity†					
			Male	Female	White	Latinx	Black	Native	Asian	Other
Philadelphia	afterschool	41	54%	46%	13.5%	11.5%	69.2%	2%	2%	
Denver	summer	17	65%	35%	10%	30%	70%	10%	5%	
Riverton	summer	14	36%	64%	80%	20%		33%		
WRIR	summer	13	48%	52%		11%		100%		5%

† Percentages may exceed 100% due to student selection of multiple race/ethnicity categories.

Table 2: Student Survey Results Year 2

Cohort	Program Type	Subscale	*N*	Mean Pretest	SD	Mean Posttest	SD	t-value	P value	Cohen's *d*
Philadelphia†	afterschool	CT	41	3.77	0.62	3.61	0.68	1.927	0.061	
		CSE	41	3.70	0.59	3.81	0.59	-1.534	0.133	
		SciSE	41	5.46	0.69	5.19	0.80	2.744	**0.009**	.361
		TIU	41	3.31	0.26	3.33	0.24	-0.818	0.418	
Denver B&G	summer	CT	16	3.33	0.73	3.50	0.99	-1.034	0.317	
		CSE	17	3.32	0.67	3.68	0.60	-3.058	**0.008**	.566
		SciSE	17	4.77	0.88	4.83	0.89	-0.359	0.724	
		TIU	16	3.06	0.41	3.22	0.39	-1.772	0.097	
		CompTSE	17	3.29	0.77	3.57	0.86	-2.463	**0.026**	.343

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Wyoming	summer	CT	27	3.49	0.78	3.74	0.65	-2.513	**0.019**	.348
		CSE	27	3.26	0.63	3.60	0.63	-2.943	**0.007**	.540
		SciSE	26	4.76	1.08	5.08	1.05	-2.565	**0.017**	.300
		TIU	25	2.98	0.35	3.13	0.33	-3.082	**0.005**	.441
		CompTSE	25	3.20	0.54	3.65	0.60	-4.499	**0.000**	.788

† Pretest did not consist of CSTE items.

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THE EFFECTS OF INSTRUCTORS AND STUDENT ACTIVITY IN LEARNING FROM INSTRUCTIONAL CALCULUS VIDEOS

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We report the results of an investigation into the factors that affect students' learning from calculus instructional videos. We designed 32 sets of videos and assessed students' learning with pre- and post-video questions. We examined how students' engagement and self-identified ways of interacting with the videos connected to their learning. Our results indicate that there is a complicated relationship between the student, curriculum, instructional practices, and the video content, and that the effectiveness of instructional videos may be contextualized by both instructional practices and the extent to which the understandings supported in the videos are compatible with the meanings promoted during instruction.

Keywords: Online and Distance Education, Calculus

In recent years, "flipped" classrooms and massive open online courses have been promoted as effective ways to support students' active learning (e.g., Schroeder, McGiveny-Burelle, & Xue, 2015) and to deliver instruction remotely. Although there is increased interest in using these techniques and a growing body of research literature on student learning in flipped classrooms (e.g., Maxson & Szaniszlo, 2015), there is still minimal data to support claims of their efficacy.

With a few exceptions (e.g., Weinberg, Martin, Thomas, & Tallman, 2018; Weinberg & Thomas, 2018), there have been virtually no studies that have investigated how students utilize and learn from out-of-class video resources. Other research (e.g., Deslauriers, Schelew, & Wiemann, 2011) has largely been based on an implicit empiricist epistemology (Simon, 2013), assuming that exposure to out-of-class resources is sufficient to promote students' learning.

The dearth of empirical data on students' use of and learning from out-of-class resources suggests that it is imperative to investigate how mathematics students engage with and learn from instructional videos. In this report, we investigate the characteristics of students' and instructors' use of calculus video lessons that affect student learning outcomes.

Theoretical Framework and Research Questions

Both our instructional videos and research design were informed by Mayer's (2014) cognitive theory of multimedia learning. From this perspective, students are active participants in the process of learning from a multimedia presentation: they actively attend to, select, and organize information presented in the multimedia and integrate it into coherent mental representations. Thus, students' learning is influenced by the ways they engage in the video-watching process, their mental actions while they watch, and their prior knowledge and ways of thinking about the subject matter. Instructors can also play a role in the students' learning by supporting their development of particular knowledge structures and asking students to interact with the instructional media in particular ways.

Based on our theoretical perspective, we explored the following research questions:

1. Do differences between groups of students and different instructors influence how much students learn from watching instructional calculus videos?

2. How does student engagement with the videos affect their learning?
3. Do the instructors and the ways students report being asked to interact with the videos have an effect on student learning?

Methods

Materials

We created 56 instructional videos for 30 topics commonly taught in first-semester calculus. The videos were designed using Mayer's (2020) 12 principles of multimedia learning. We created pre- and post-video questions for each video set grounded in Tallman et al.'s (2021) theoretical principles of calculus assessment design. For each video set, we created a website that included a set of 2-4 multiple-choice pre-video questions, instructional videos, and post-video questions. The students were not informed whether their answers to the pre-video questions were correct but were informed of the correctness of their answers to the post-video questions and provided with unlimited opportunities to revise answers. The website collected information about when they paused or skipped while watching a video.

At the end of the semester, students were asked to complete a survey to report demographics such as gender, race, and major; indicate the mathematics classes they had previously completed; and report the ways their instructor asked them to interact with the videos (such as telling students which concepts they should learn from the video or giving credit for watching the videos). The overall response rate to this survey was approximately 32%.

Participants

In addition to one of the PI institutions (a large public university where all calculus instructors participated), fifteen instructors from fourteen institutions participated; these institutions ranged from regional liberal arts colleges to large public research institutions, located in eleven states and one international location. Data collection occurred during the fall 2019 and spring 2020 semesters; eight instructors participated during both semesters. Each instructor selected one or more video sets to assign and invited their students to participate in the study.

In addition to students who did not give consent, we excluded instances where less than 25% of an instructor's students completed a particular video set. We inferred that these responses were from students who were completing a set voluntarily rather than as part of an assignment, and might not be representative of their class as a whole. Overall, 1,166 students participated.

Data Analysis

We measured whether students' solutions on the pre-video questions were correct. For the multiple-choice post-video questions, we measured whether students' solutions were correct on their first attempt or, for the free-response questions, whether their solutions were correct by their second attempt. We used a modified version of normalized change (Marx & Cummings, 2007) to measure students' gains from pre- to post-video:

$$c = \begin{cases} \dfrac{post - pre}{100 - pre} & post > pre \\ drop & post = pre = 100 \\ 0 & post = pre \neq 100 \\ \dfrac{post - pre}{pre} & post < pre \end{cases}$$

We counted the number of times each student paused or skipped backward in each video; we called these instances "revisits." We computed the average rate of revisits for each student and each set of videos by dividing the total number of revisits the student made for a set of videos by the total length (in minutes) of the videos in the set.

Results

Overall Learning

Overall, the students demonstrated a mean normalized change of 7.77% (SD=59.21%). Thus, there was a considerable amount of variation in the students' learning. When we investigated learning on the separate video sets, we found that there was a significant effect of the particular video set on mean normalized change (F(29, 16818)=82.27, $p < 2 \times 10^{-16}$).

Student Characteristics and Learning

Student Engagement with the Videos. We first investigated whether the ways students interacted with the videos was associated with learning. We hypothesized that a "revisit"—an instance where a student either paused or skipped backward in the video—reflected the students' active engagement with the video content. Overall, only 19.9% of the student-video set pairs had a non-zero rate of revisits per minute, with a mean of 0.373 (SD=0.373). After excluding outliers, a simple linear regression to predict the normalized change based on the revisits per minute had non-zero slope (F(1,16086)=11.45, p=0.00717), but this was not practically significant, with b=0.031384 (t(16086)=3.384, p=0.00717). Thus, the students' engagement with the videos does not appear to predict their normalized change in a practically significant way.

Instructor Relationship with Student Learning. We investigated whether different instructors were associated with different levels of student learning. For the fall 2019 semester, a two-factor ANOVA using instructor and video set as factors within each semester showed a significant effect of instructor on normalized change (F(20, 8294)=2.351, p=0.0006) as well as a significant interaction between instructor and video set (F(344, 8294)=1.298, p=0.000225). In the spring 2020 semester, there was a significant effect of instructor on normalized change (F(14, 7817)=2.807, p=0.000337) but the interaction between instructor and video set was not statistically significant (F(300, 7817)=1.077, p=0.175556).

The Role of Curriculum. One of the participating institutions in our study included multiple sections of calculus each semester in which the content and pacing in the classes were centrally coordinated. We repeated the previous analysis at this institution and found that, in the fall 2019 semester, there was neither a significant effect of instructor on normalized change (F(8, 3181)=0.797, p=0.605) nor a significant interaction between instructor and video set (F(153, 3181)=1.046, p=0.338). Similarly, in the spring 2020 semester, there were neither a significant effect of instructor on normalized change (F(6, 2342)=2.094, p=0.051) nor a significant interaction between instructor and video set (F(148, 2342)=0.869, p=0.867). However, there was still variation between instructors. To investigate this, we transformed each instructor's mean normalized change on each video into a standardized score. Table 2 shows these scores, and demonstrates that some sets had consistently higher or lower scores across instructors, while there was considerable variation for other sets.

Table 2. Standardized scores for instructors by video set at the multiple-section institution.

	Instructor								
Video Set	A	B	C	D	E	F	G	H	I
Approximating Instantaneous Rates of Change	-0.17	-0.51	-1.03	-0.32		-0.38	-0.86	-1.02	-0.30

Product Rule	0.29	-0.41	-0.21	-0.29	-0.24	-0.41	0.44	-0.48	0.40
Quotient Rule	0.52	0.36	0.80	0.11	-0.39	0.43	0.86	-0.03	0.50

The Role of Instructor Interventions. We examined the total number of types of practices each student reported and compared the sum with their mean normalized change for each video set. We calculated a simple linear regression for the mean normalized change based on the sum of the total number of types of practices each student reported. The regression equation was not significant ($F(1, 8314)=1.535$, $p=0.2154$), with an R^2 of 0.00006431, and the result also was not practically significant, with $b=0.002708$ ($t(8314)=1.239$, $p=0.215$). However, when we repeated this analysis at the institution with multiple coordinated sections, the regression equation was moderately significant ($F(1, 1530)=5.816$, $p=0.016$) with an R^2 of 0.003136 although there was little practical significance, with $b=0.013574$ ($t(1530)=2.412$, $p=0.016$).

Discussion

The results of our study suggest that it is difficult to predict how much students learn from watching instructional videos and to discern how various ways students engage with video lessons influences their learning. In general, the students in this study demonstrated positive, yet modest learning from the videos, with a considerable amount of variation.

The fact that the students watched the videos and answered the pre- and post-video questions outside of their regular class meetings suggests that the in-class instruction should not have an effect on their learning. However, there was considerable variation from instructor to instructor and a significant interaction between the instructor and video set. These results highlight the complex relationship between how the instructor incorporates the videos into their pedagogy, the curriculum, and how effectively students use the videos to learn.

It would seem likely that the ways students interact with the videos would influence their learning. However, student engagement—measured by the rate at which they “revisited” the video—did not predict their learning.

The relative consistency of students’ performance at the institution with multiple coordinated sections of calculus suggests that curriculum—and its enactment— might play a role in what students learned from the videos. Additionally, the variability of instructor effectiveness by video set suggests that the effectiveness of mathematics video lessons is possibly contextualized by the extent to which the understandings supported in the videos are compatible with the meanings promoted during instruction and developed through various types of formative assessment. We hypothesize that aligning various forms of curriculum and assessment with the content of the videos would support students’ learning. However, even at this institution the students did not consistently achieve positive mean normalized change scores, and there was still variation in the relative effectiveness of the videos from instructor to instructor.

We conjecture that the key to effective learning from the instructional videos lies in the ways the instructors incorporate the videos into their pedagogy. Although we didn’t see a significant relationship between the number of practices students reported their instructors using, there were significant limitations to these data. In particular, the low response rate suggests the possibility of nonresponse bias, and our data don’t reveal the ways in which each instructor might have implemented the various types of practices, or how frequently or consistently—or on which video sets—the instructor implemented these practices.

Taken together, these results suggest that more research is needed to understand ways in which videos can be effectively incorporated into instruction. In particular, researchers need to create detailed descriptions of the ways instructors incorporate the videos into their classes and how their students enact these instructional practices, and investigate how this activity interacts with the content of the videos to support student learning.

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ESTIMATING CLIMATE CHANGE NUMBERS: HOW TOLERANCE FOR ERROR CAN SUPPORT SCIENCE LEARNING

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Texts presenting novel statistics can shift learners' attitudes and conceptions about controversial science topics. Research suggests that such science learning can be supported by bolstering targeted mathematical reasoning skills, though these benefits were found to be strongest among people with mid/high prior knowledge. Our project aimed to build on this research by identifying specific skills that might have contributed to such learning. We conducted ten think-aloud interviews with undergraduate and graduate students as they estimated climate change data before being shown the scientifically accepted value. Findings highlight that students with higher prior knowledge tended to have a higher tolerance for error in their calculations, a willingness to make casual "back-of-the-envelope" calculations, and often interpreted quantitative feedback in terms of its scientific meaning rather than in terms of a measure of performance.

Keywords: conceptual change; integrated STEM; numerical estimation; science education

Misconceptions about controversial science topics are widespread. For example, as of September 2020, 43% of adults in the USA incorrectly believe that human activities are not the primary cause of climate change (Marlon et al., 2020). Misconceptions about climate change are held even among individuals who believe that climate change is real (Thacker & Sinatra, 2019). Fortunately, there are several approaches that exist to shift climate change misconceptions, and one approach that shows promise makes use of pertinent statistical information.

Numerical data (e.g., statistics) found in the news can be a catalyst for changing minds about science topics. For example, prompting people to estimate just a handful of statistics about climate change and then presenting them with the actual value can shift their attitudes, beliefs, and misconceptions to be more aligned with scientists (Ranney & Clark, 2016). The evidence further suggests that the impact of such an intervention can be enhanced by bolstering targeted numerical estimation skills that support the processing and interpretation of numbers, and that such impacts are moderated by motivational and affective factors (Thacker, 2020). However, findings from this research showed that the benefits of mathematical instruction were strongest among individuals in the middle/upper range of prior climate change knowledge.

This project aimed to address this gap by examining the role of prior knowledge more closely and to identify specific skills among medium-high learners that support the learning that occurs when people engage with climate change data. Specifically, we compliment Thacker's (2020) prior quantitative research on this topic using data from ten think-aloud interviews with undergraduate and graduate students as they estimated climate change numbers before being shown the true value and identified specific skills that supported subsequent learning. First, we summarize relevant theory.

Theoretical Framework

Conceptual Change

The Plausibility Judgments for Conceptual Change model posits that novel information can incite conceptual change because it prompts learners to appraise or reappraise the plausibility of their existing beliefs (Lombardi, et al., 2016). This model predicts that when people encounter novel information such as novel climate change data, they first process the data for validity (e.g., Richter & Maier, 2017) and then make a judgment of the plausibility of the conception supported by the new information. Plausibility judgments can be either implicit or explicit, and the extent to which an individual explicitly evaluates the plausibility of a conception depends, in part, on their motivation, emotion, and views about knowledge. More explicit evaluations of plausibility lead to an increased potential for conceptual change. In sum, Lombardi et al.'s (2016) model predicts that the extent to which people engage with and learn from numerical data is influenced by their motivation, emotion, and their ability to process and interpret numbers.

Numerical Estimation

One way that learners process and interpret numbers is by estimating whether they seem reasonable (e.g., Reys & Reys, 2004). Research on *measurement estimation* concerns the explicit estimation of real-world measures (Bright, 1976; Dowker, 2005; Sowder & Wheeler, 1989) and is useful for understanding factors that help people judge whether real-world quantities are reasonable. Findings suggest that peoples' estimation accuracy and judgments of reasonableness improve when they use measurement estimation strategies, such a tolerance for error (Shimizu & Ishida, 1994) and use of the benchmark strategy—the use of given standards and facts that can be applied by the learner through mental iteration and proportional reasoning to better estimate and judge the plausibility of real-world quantities (Brown & Siegler, 2001; Joram et al., 1998). For example, a person's estimate of the number of jellybeans in a container is likely to be more accurate, and they will be a better judge of the reasonableness of other peoples' guesses if they are first told the number of jellybeans in a different container. Measurement estimation strategies may therefore support people's comprehension and evaluation of given real-world quantities.

Attitudes and Knowledge

Attitudes are another factor that may influence learning from climate change data. Attitudes can be thought of as the valenced (e.g., positive or negative) evaluation of an object, person, or event and are expressed as behaviors, affects, and beliefs (Eagly & Chaiken, 1993). According to Sinatra & Seyranian (2016), knowledge and attitudes are related and can be thought of as adhering to a 2x2 axis. Knowledge can either be consistent or inconsistent with scientifically accepted views and can be either positively or negatively valenced, yielding four categories of attitude and knowledge, each representing a different approach that a person might take to learning climate change. Briefly, these four combinations are *pro-justified* (favorable attitude and accurate knowledge), *pro-unjustified* (favorable attitude and inaccurate knowledge), *con-justified* (negative attitude and accurate knowledge), and *con-unjustified* (negative attitude and inaccurate knowledge; Sinatra & Seyranian, 2016). This study, in particular, concentrates on differences between justified and unjustified learners (i.e., higher and lower prior knowledge).

Building on this theoretical framework, we contend that numerical estimation is an essential mathematical skill that helps individuals evaluate and learn from scientific data. Further, such learning can be magnified when people hold attitudes that enable receptivity and deep engagement with new evidence. As such, the purpose of this research was to develop a learning intervention that leverages these ideas and to also explore specific pre-existing skills and knowledge that benefit learners when they engage with climate change data. Therefore, our research question is: *What prior skills and knowledge support the learning that occurs when people engage with an intervention that exposes them to novel statistics?*

Methods

The Estimation Game

To build upon an existing learning intervention (Thacker, 2020), we used a design-based research (DBR) approach to guide the development of an online estimation game. As is characteristic of design-based research, the design, implementation, and revision of the intervention occurred over several iterations (Anderson & Shattuck, 2012). Though design iteration cycles are still underway, the current product of this research is an online, open-source number estimation game with a built-in numerical estimation strategy intervention that can be easily shared with practitioners and the general public online (http://143.110.210.183/).

In the intervention, people are asked to estimate climate change numbers before being shown the true value. The estimation process is thought to elicit relevant background knowledge that is restructured when incorporating the true value (e.g., Ranney & Clark, 2016; Rinne et al., 2006). Half of these prompts also include a "hint," (or benchmark value, Brown & Siegler, 2001; Joram et al., 1998) that can be mathematically manipulated to better estimate the unknown value.

The estimation game can also be modified by the researcher to present participants with instruction on numerical estimation strategies prior to estimating values. This instruction consists of a short text that encourages participants to draw from their background knowledge and think mathematically when estimating numbers. Specifically, it emphasizes the use of benchmark values by rounding and rescaling them based on one's expectations. This is followed by a worked example and a check for understanding (see Thacker, 2020 for more detail). Half of our participants received this modification.

Participants and Procedure

We conducted ten audio and video recorded "think-aloud" interviews (Desimone & Le Floch, 2004) with graduate and undergraduate students as they interacted with the estimation game. Students attended a large Hispanic serving institution in the Southern USA and identified as Female (90%), Hispanic/Latino (50%), White (30%), Black (10%), and mixed-race (10%).

While "thinking aloud," these students (a) completed a pretest of prior knowledge (Lombardi et al., 2013) and climate change attitudes (Lombardi et al., 2012), (b) engaged with the estimation game, with half of the participants receiving the modification that included math instruction, and then (c) completed a post-test identical to the pretest.

Analysis. Survey data was used to identify individuals with high/low prior knowledge and positive/negative climate change attitudes. Recordings were transcribed and open-coded by two independent coders for varying dimensions of student thinking (Corbin & Strauss, 2004), with special emphasis on examining strategies used by students when estimating climate change numbers among individuals with high and low prior knowledge.

Findings

Survey Results

Results revealed that, at pretest, all participants believed that climate change is real. All ten participants rated the statement, "climate change exists and is caused by humans" as plausible, ranging from 6 (somewhat plausible) to 10 (highly plausible; M = 8.02 out of 10). Yet, despite these generally positive attitudes, about half of participants held misconceptions (e.g., 50% of students disagreed that the "average sea level is increasing"). Pretest knowledge scores averaged at 65% correct. Climate change knowledge improved at posttest (94% correct); no participants disagreed that the "average sea level is increasing" at posttest.

All participants held attitudes that were consistent with scientists, yet many held conceptions that were not. As such, we divided our qualitative analyses into categories of students with positive attitudes but low and high knowledge at pretest (pro-unjustified and pro-justified, respectively). Participants were coded as having low knowledge if they scored below the median (those who scored 57% correct or less) on the knowledge pretest. Based on this criteria, six participants were coded as "Pro-Justified" while four were coded as "Pro-Unjustified."

Qualitative Results: Pro-Justified vs Pro-Unjustified Learners

All students made use of background knowledge and most used mathematical operations to modify given information to better estimate unknown information. However, what distinguished those in the high prior knowledge group from those in the low prior knowledge group was their tolerance for error implicit in the calculations (see also Reys et al., 1982; Shimizu & Ishida, 1994). Another important difference was that these students interpreted quantitative feedback in terms of its scientific meaning (e.g., "Wow, that number is different than what I expected") rather than as a reflection of their performance (e.g., "Wow, I got the answer wrong").

To exemplify these codes, we present an excerpt from an interview with a female preservice teacher and undergraduate student who was identified as "pro-justified" and completed the modified version of the intervention. After completing the survey pretest, the participant was thinking aloud when she read the instructions to an item, "Of 195 countries in the world how many are committed to climate action." then noted that she was "gonna round down to 194." When prompted by the researcher to explain why, she said,

> I [rounded to] 194 because [halving] 195 will result in a point five calculation and there's not really half a country, so I just rounded down because, you know, down is less... Half of 194 is 97, but I'm going to put 42 countries because it's less than half of 195. [*She then enters 42 and clicks to show the scientifically accepted answer, revealing that 175 of 195 countries are committed to climate action.*] No! I mean, yes! But no, I mean yes! So that's more than half. That's significantly more than half. Wow, that genuinely surprises me a lot, I did not know that. I really thought that a lot of the countries were not committed to climate action, this is a good statistic. I'm happy with this. I mean, I'm sad that I'm wrong, but I'm happy that I'm wrong at the same time.

This excerpt illustrates the flexible approach to working with imperfect calculations that were characteristic of pro-justified participants. Notably, this student drew from her expectation that the world is not very supportive of climate action and performed a few casual "back of the envelope" computations using the given number (i.e., rounding to an even number to ease halving, and then going much lower again). These casual arithmetic manipulations seemed to support students in making meaning of the numbers. When shown the true value, this student noted that, though she was not particularly happy to learn that her estimate was inaccurate, the meaning of the scientifically accepted value was most salient. In contrast, students categorized as pro-unjustified were generally less willing to manipulate given numbers and attended more to the accuracy of their estimate when compared with the meaning associated with the true value.

Conclusion

We sought to explore what prior skills and knowledge support the learning that occurs from an intervention that involves numerical estimation of climate change data. We found that students with higher prior knowledge were more tolerant of error, willing to make informal calculations, and make meaning of feedback compared with individuals with lower prior knowledge. Instructors who wish to support science learning through casual estimates of real-world quantities might assure students that, when estimating, it is okay to tolerate some error in computations. They may also provide feedback that de-emphasizes performance outcomes and highlights meaning. Future research stemming from this project will more closely examine mathematical reasoning that supports the interpretation of scientific meaning and investigate how learners with negative climate change attitudes interact with the intervention.

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UNDERGRADUATE STUDENTS' INTERACTIONS WITH DYNAMIC DIAGRAMS WHILE SOLVING PROOF TASKS

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The purpose of this study was to investigate undergraduate students' interactions with diagrams while they were solving dynamic geometry-based proof tasks. Some of the tasks included a diagram while others did not. The participants in this case study included two senior college students, enrolled in an undergraduate geometry course at a public university. Differences in the way students interacted with the diagrams in each of these settings are described.

Keywords: Reasoning and Proof, Technology, Undergraduate Education, Mathematical Representations

Introduction and Related Literature

Mathematics education researchers have been investigating how individuals interact with diagrams for more than three decades (Chen & Herbst, 2013; Dimmel & Herbst, 2015; Duval, 1995; Fischbein, 1993; Gonzalez & Herbst, 2009; Herbst, 2004). Because dynamic geometry software (DGS) provides tools such as dragging and measuring we were interested in better understanding how students reason about geometry proof tasks with DGS. There is a large body of research on students' approaches to proof, with and without technology (e.g., Laborde, 2000; Sinclair & Robutti, 2012; Stylianides & Stylianides, 2009) and diagrams play an important role. Students' interactions with dynamic diagrams has been discussed in several research studies (Gonzalez & Herbst, 2009; Hollebrands, 2007; Mariotti, 2000), but there is limited research that compares how the presence of a diagram impacts students' reasoning. The purpose of our study was to investigate students' interaction with diagrams while they were solving dynamic geometry-based proof tasks in diagram-given and diagram-free settings.

Conceptual Framework

Two important theories (Duval, 1995; Fischbein, 1993) have been used to understand students' interactions with diagrams in mathematics education research. Based on Duval's (1995) theory, Herbst (2004) suggested a framework that includes four modes of students' interaction: empirical, representational, descriptive, and generative. Within the empirical mode of interaction, the student has a proximal, physical relationship with the diagram and uses measurements to make conjectures. In contrast, within the representational mode of interaction, the student maintains distal physical experiences with the diagram and student's conjectures stem from the definitions and properties (Gonzalez & Herbst, 2009; Herbst, 2004). In the descriptive mode, the student sets up a distal relationship with a diagram and makes conjectures based on visual perception through anticipating symbols (Gonzalez & Herbst, 2009). Contrary to the descriptive mode, the generative mode is often used to make hypotheses and build reasoned conjectures. In this mode, students make modifications to the original diagram to support their work. Gonzalez and Herbst (2009) proposed a fifth mode called the functional mode. They describe how students relate inputs and outputs when they use the dragging feature of dynamic geometry software. Within this mode, the combination of dragging and measuring provides students opportunities to explore relationships. Students may also check invariants when making changes to the diagram by dragging and may set up the same relation in several diagrams to compare and contrast them. In our study, we investigated undergraduate students' interaction with diagrams while they were solving dynamic geometry-based proof tasks. Some of the tasks included a diagram while others did not. Within these different settings, we examined whether students' modes of interaction with diagrams varies in diagram free and diagram given settings.

Context and Methods

The participants in this study included two senior college students, enrolled in an undergraduate geometry course at a public university. The first participant, Valerie (all names are pseudonyms), was a data science and statistics major and the second participant, Kate, was majoring in mathematics and mathematics education. Kate was more familiar with dynamic geometry programs and she was facile using the tools and features. Valerie had limited experiences working with dynamic geometry programs. Kate had taken a course on teaching mathematics with technology while Valerie had not.

To collect data, one Zoom interview was conducted with each participant using five tasks. Some of the tasks included a diagram while others did not. We asked students to work on their self-constructed diagrams while they were solving diagram free versions. These tasks are the proof tasks (see diagram given versions in Figure 1) which includes corresponding angles and a transversal (Task 1), the sum of the angles of a triangle (Task 2), the isosceles triangle (Task 3), the congruency between two triangles (Task 4) and a segment on a right triangle (Task 5).

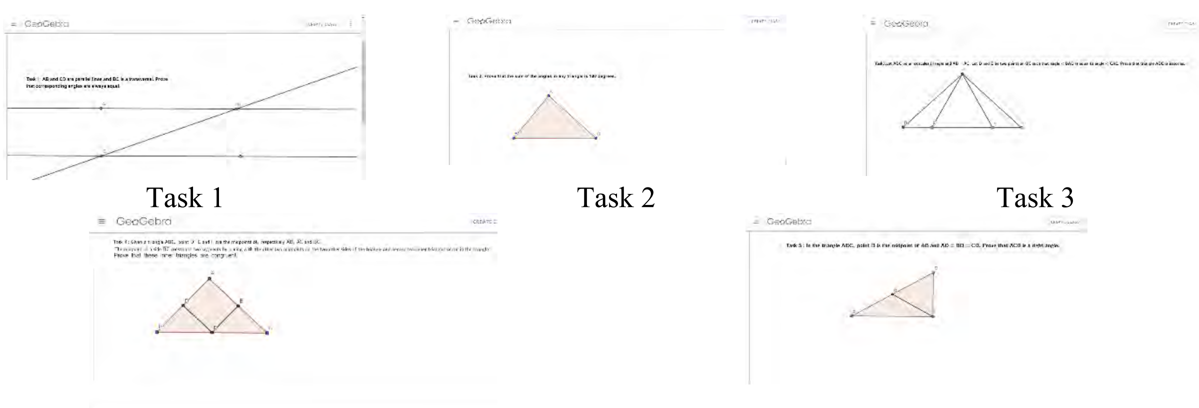


Figure 1: Screenshots of the Tasks presented in Geogebra Environment

During the interviews, we recorded participants' shared screens and voices. Aligned with the aim of this study, we presented half of the tasks with a diagram included and the others without. We analyzed the data in two cycles. In the first cycle, we focused on the students' main approach to characterize their thinking. In the second cycle, we coded participants' approach in terms of how they interacted with diagrams in each task. In order to explain the participants' approach of interaction with diagrams, the extended framework of Herbst (2004) (see updated version in Gonzales & Herbst, 2009) guided our analysis. After coding all instances, we determined how the answers varied according to the diagram given or diagram free versions of the tasks.

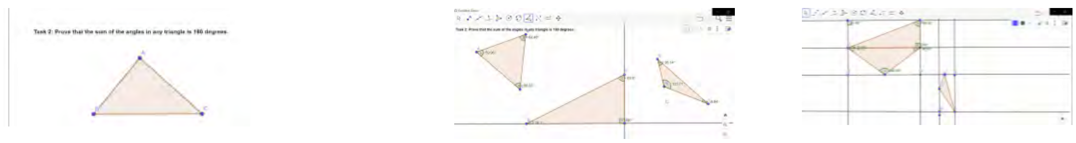
Results

The results showed that half of the instances were empirical, one instance was representational, two instances were functional and one instance was generative (Table 1). In just one instance, the student had a transitional process that is her interaction with diagrams started as descriptive in the beginning; however, it then converted to representational. When we investigated diagram given instances, most of them were empirical, only one was representational and one instance was generative. When compared to diagram given instances, the modes of interactions with diagrams in the diagram-free instances varied more. There were two functional instances and two empirical instances observed. Additionally, the transitional instance was seen while proving diagram-free tasks.

Table 1: Students' modes of interaction with diagrams in diagram given and diagram-free proof tasks.

	Diagram given	Diagram-free
Task 1	Representational (Valerie)	Functional (Kate)
Task 2	Empirical (Kate)	Empirical (Valerie)
Task 3	Empirical (Valerie)	Descriptive/Representational (Kate)
Task 4	Generative (Kate)	Empirical(Valerie)
Task 5	Empirical (Valerie)	Functional (Kate)

The most common mode of interaction with diagrams was the empirical mode. We had five instances of the empirical mode of interaction with diagrams among ten instances. Four of these five instances were Valerie's approach to the proof tasks. Whether or not the task is presented in a diagram given or diagram free setting she approaches proof tasks with empirical mode of interaction. In addition one empirical instance was shown in Kate's approach diagram given version of Task 2. In the second task, students were asked to prove that the sum of the angles in any triangle was 180 degrees. This task was presented to Kate as a diagram given and to Valerie as diagram free. In this task, the students' ways of interaction with diagrams were empirical no matter whether the diagram is given on the task or not. Kate measured the angles (a complementary angle of the angle ABC and two other interior angles) and she-used the calculator of her phone to calculate the interior angle (angle ABC). When she noticed the sum was 180, she verified her conjecture via *drag test* to convince herself or the researchers. In her diagram free setting, Valerie started to construct three different diagrams such as isosceles, scalene and right triangles (Figure 2a). She constructed a right triangle and claimed that the sum of the angles of any triangle is 180 because of the sum of the angles of the right triangle. Afterwards when the researchers asked her if her claim was valid in every case, she started to demonstrate again by constructing new three different triangles (Figure 2b). In this instance Valerie used dragging but her dominated approach was based on measurements, that is why the researchers defined her approach as empirical. While Kate's solving approach on diagram given context was familiar however Valerie's approach in the diagram-free setting was unique and not presented as the textbooks or high school geometry classes.



a. The diagram-given setting presented to Kate

b. The diagrams constructed by Valerie

Figure 2: The given diagram and the diagrams constructed by the participant

Discussion

The results of our study revealed that participants' approaches to proof in diagram given settings were generally based on drawing and measurements instead of justifying with axioms and theorems or reasoned conjectures. Based on the dominant mode of interaction with diagrams in diagram given settings, we deduced that these provided participants an environment that they were familiar from high school or undergraduate geometry classes. Even though Herbst (2004) noted that the empirical interactions with diagrams commonly occur before high school (Herbst, 2004), their approach to proof tasks may be influenced by their prior knowledge.

Based on the participants' experiences in the diagram free setting, the tasks provided participants in an open-ended context, which they may not have experienced in their courses before. Thus, before participants made conjectures on diagram free tasks, they constructed their diagrams based on the information they obtained from how they imagine the diagram implied and how they argue on this diagram. Their approach in the diagram-free instances were unique and varied unexpectedly. Surprisingly, there is one instance where the participant shifted her approach while she interacted with the diagram. Similar changes in the individuals' approach to proof in congruence based proof tasks have been observed in the study of Author et al. (in press).

Unfortunately, in some instances, diagram free tasks given to students may result in the construction of false diagrams and thus they may make conjectures based on spurious information. Also, the diagram they constructed in the diagram-free tasks may be overly constrained. When students are introduced with diagram-given tasks, their approaches are more typical of textbook approaches and they are more likely to reach the correct solution.

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A FRAMEWORK FOR ANALYSIS OF VARIANCE AND INVARIANCE IN A DYNAMIC GEOMETRY ENVIRONMENT

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Variance and Invariance are essential to the development of advanced spatial perception and understanding of geometric objects. In this paper, we present an initial framework for analysis of teachers' experiences in a dynamic geometry environment from two perspectives – continuous variation and set of examples. We synthesized relevant literature followed by an empirical study of 122 descriptions of six teachers' experiences with variance and invariance tasks. As a result, we developed an analytic framework to describe how generating examples versus experiencing continuous variation might look like in a dynamic environment. We also found cases when we struggled to characterize an experience under a specific perspective and concluded that these perspectives might not be as far apart as the literature seems to suggest. Possible implications for designing opportunities to teacher education and future research directions are discussed.

Keywords: Geometry and Spatial Thinking; Continuous Variation; Set of Examples; Technology

Purpose and Background

Variance and invariance are two powerful mathematical ideas essential to the development of advanced spatial perception and understanding of figures, shapes, and objects (Battista, 2008; Baccaglini-Frank, Mariotti, & Antonini 2009; Leung, 2008; Sinclair, Pimm, & Skelin, 2012). Such understanding is critical for the learning and teaching of mathematics at all levels (Driscoll et al., 2007; Sinclair et al., 2012). Teachers should have robust understandings of variance and invariance (Sinclair et al. 2012; Stroup 2005), but more research is necessary in this domain.

The literature suggests a distinction between two views of draggable objects in a Dynamic Geometry Environment (DGE). In the first view – a *Set of Examples (SOE)* - draggable objects can be perceived as generating numerous examples of the object (Battista 2008; Marrades & Gutierrez, 2000; Laborde 1993). In the second view - *Continuous Variation of an object (CV)* - a person can understand draggable objects as "interesting, manipulable, visual-mechanical objects that have movement constraints" (Battista, 2008, p. 349). In the current work, we set two main goals: (a) to outline and describe a set of operationalized characteristics (framework) for analyzing teachers' experiences with draggable objects based on both perspectives CV and SOE; and (b) to provide examples of how this framework was used to analyze six high school mathematics teachers' interactions with draggable objects.

Theoretical Framework

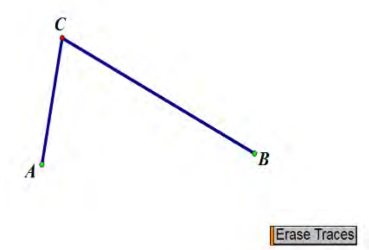
When we refer to invariance in this paper, we denote to certain geometrical properties that remain unaltered when a transformation on the object is applied (e.g., Baccaglini-Frank et al. 2009; Yerushalmy, Chazan, & Gordon 1993). Table 1 presents a summary of the differences and similarities between the two perspectives of draggable objects based on the literature we reviewed. The literature might not suggest an exhaustive list of characteristics, and it seems that some of these characteristics are described insufficiently. For instance, Battista (2008) and Laborde (1998) assert that under continuous variation the object is manipulable, but we are yet to find an explicit explanation of what 'manipulable object' means.

Table 1: Continuous variation of an object versus a set of examples

Continuous variation of an object (CV)	A set of examples perspective (SOE)
• Visual-mechanical objects (Battista, 2008) • The object has constrained movement (Battista 2008, Laborde, 1993) • Manipulable (Battista, 2008; Laborde, 1998) • It is possible to conceptualize and analyze invariant properties under continuous movement of the figure (Laborde, 1993) • Theoretical Geometry Objects that model the theoretical field and can be handle in a physical sense (Laborde, 1998) • Interesting (Battista, 2008)	• The figure is an example of something (a concept or a relationship) (Battista 2008; Marrades & Gutierrez, 2000) • Properties are derived from a set of examples (Battista 2008; Laborde, 1993) • Properties and relationships are abstracted and conceptualized from a set of examples (Battista, 2008; Laborde, 1993) • Transformation can give access to wide range of examples (Laborde, 1993; Marrades & Gutierrez, 2000)

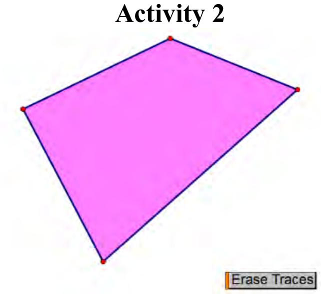
Data Sources and Methods

Activity 1



Participants could manipulate a draggable point (point C) connected to two fixed points A and B. Based on Leung's (2003) idea. Point C could be dragged to any location on the screen, and it leaves a colored trace when it is dragged. *Construction: Drawing three points, setting A and B to be fixed points, and constructing segments AC and BC.*

Activity 2



Participants were able to manipulate a dynamic polygon, by dragging one or more vertices. The vertices could be dragged to any location on the screen, and they leave a colored trace when dragged. It was possible to drag the vertices simultaneously. *Construction: Drawing four points and constructing the segments connecting adjacent points.*

Figure 1: A sample of two activities used in the study.

The Data came from a set of two 45-minutes video recorded task-based interviews with six high school mathematics teachers: Four females - Amanda, Lisa, Diana and Susan; and two males - Andy and Mark (pseudonyms). Each interview focused on a set of four activities that were designed using the Sketchpad® Explorer (Jackiw, 1991) to have teachers interact with draggable geometric objects to explore variance and invariance. Figure 1 presents two of these activities. All interviews were videotaped and transcribed verbatim. Each activity had two parts: In Part 1 (*Noticing*), participants were asked to describe what they notice when they drag one or more points in the space; and in Part II (*Maintaining*), participants were asked to think about ways to drag one or more points in a way that maintains an invariant property. All draggable aspects left a colored trace when dragged.

The analysis included two cycles of coding. In the first, we used *descriptive coding* (Miles, Huberman, & Saldaña, 2014) to look at what participants said (verbally) and did (through actions). In the second cycle, we coded characteristics related to the two perspectives as they have emerged from the literature and from our refinement process as described below. Lastly, we used a *pattern coding* method (Miles et al., 2014) to look for major themes and patterns.

Results

Operationalizing Characteristics of a Set of Examples and Continuous Variation

We found a set of six operationalized characteristics of the two perspectives. Three emerged from the literature. *Generating one or more examples* (SOE) indicates if participant generated one or more examples. *Maintaining dragging* (MD) indicates if the movement was constrained to keep an aspect as an invariance property. MD was also used to describe the movement. *Continuous movement* (CM) denotes to instances where participants moved an object continuously or explicitly described such a movement (e.g., saying "continue moving" / "if I drag this without stopping"). We added a characteristic of *Visual object* (VO) to see if descriptions might refer to objects that were not presented on the screen. Additionally, we noticed that participants, who seemed to experience continuous variation, described the object as one object that can have different formations / instantiations. Thus, we called this *One Object – Different Formations* (DF) and considered this as an indicator for CV. We also noticed that participants sometimes attempted to generalize, so we added this as *Links to Generalization* (LG) characteristic. The literature suggested some other characteristics, but we noticed that our participants used them too often and they seemed to be trivial, so we eliminated them from our scheme (e.g., describing the object as representing something – a concept or idea).

Using the Characteristics in Analyzing Teachers' Interactions with Draggable Objects

In analyzing 122 descriptions, we found 114 descriptions in which participants generated examples and/or described the object under continuous variation. We categorized them as follows: (a) cases in which one perspective was presented (108), (b) cases of ambiguity (6), and (c) cases of uncertainty (8). We distinguished cases of *ambiguity* - where we had the sense that both perspectives occurred at the same time versus *uncertainty* - where we were being careful in saying that we were not clear about how to characterize the description. To keep this paper within the word limit, we present results related only to descriptions of SOE and CV (excluding cases of ambiguity or uncertainty).

(a) (b) (c)

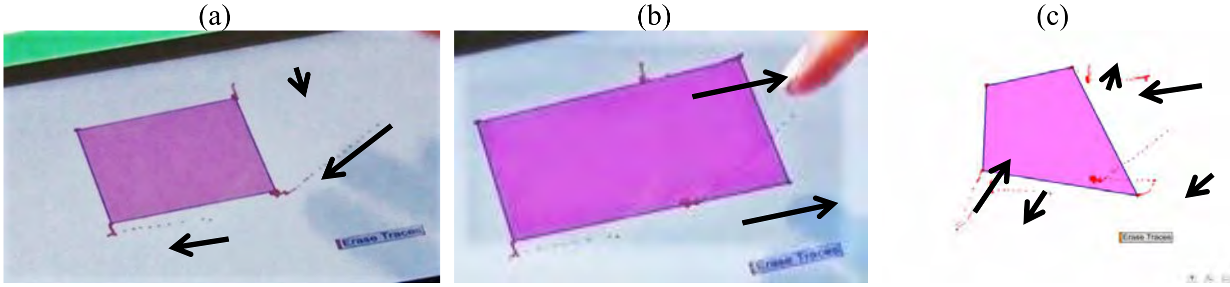
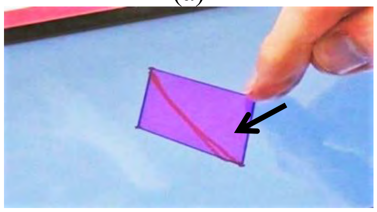


Figure 2: Diana's (a) square, (b) rectangle, and (c) kite

A set of examples. In Part 1 of Activity 2, when Diana was asked to drag one or more points and describe what she noticed, she said: "I have a quadrilateral …I could create, shapes that are quadrilaterals… I could create a square… a rectangle… I could create a kite… I could create any quadrilateral shapes that I wanted" [Figure 2]. Diana's description can be coded as generating examples of a square, a rectangle, and a kite (SOE). We did not have an indication that Diana referred to the figures as one object with different formations. Her dragging action was not maintaining dragging and the movement was not continuous. The object was visual (VO) and it seems that she tried to generalize that as long as we have a figure that is quadrilateral and it is possible to "move all four points", then it is possible to create "any quadrilateral shapes" (LG).

(a) (b)



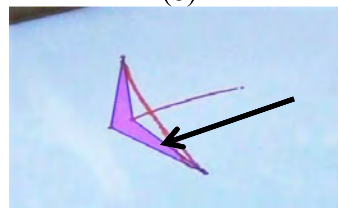


Figure 3: Mark showing, by dragging the right vertex from right to left, (a) when the polygon is still convex, (b) and when it is not convex anymore

Continuous variation. In Activity 2 Part 1, Mark shared a description suggesting an object under continuous variation to discuss when the figure is a convex polygon and when it is not:

Mark: Let me, um, draw this line very quickly [dragged the bottom vertex vertically to create a diagonal]... Right now the polygon is convex [Figure 3a]. I'm going to drag this point here in this direction [emphasized dragging the left vertex horizontally left]. If I drag it sufficiently... far enough, the polygon will no longer be convex...

Interviewer: What do you mean by far enough?

Mark: Far enough, far enough means crossing this line [pointed to the diagonal]. If this point [the right point] crosses this line [pointed to the diagonal] that I've made, it will no longer be convex. [While dragging the point horizontally left - Figure 3b] So, still convex, still convex, but then, once I cross it [stopped after passing the diagonal], no longer convex.

Mark enacted a continuous movement (CM) and constrained it horizontally as evidenced by his dragging action and his gestures (MD). His description of the polygon as being "still convex, still convex, but then… no longer convex" suggesting an object as one polygon that stays convex until it changes to have a different formation which is no longer convex (DF). Mark's description also contained a link to generalization in the sense of taking a convex polygon and maintaining it convex as long as one of its vertices is dragged horizontally but does not cross the diagonal (LG).

Discussion and Conclusions

Related to our first goal, our suggested analytic framework has implications for both research and teacher education. Future endeavor to unpack teachers' understandings of and interactions with draggable objects in the context of DGE and invariance is crucial for working toward designing learning opportunities for teacher education programs. Our framework highlights some important characteristics when interacting with draggable objects from both perspectives (SOE and CV) that are worth further consideration if one were to explore the affordances of generating examples and experiencing continuous variation in the context of DGE and invariance. For instance, we found that continuous movement can appear in descriptions related to both perspectives and it is not clear what is the role of such a movement under each perspective.

Our framework, which is drawn on the DGE literature (e.g., Battista, 2008; Laborde, 1998; Marrades & Gutierrez, 2000), looked at the perspectives of continuous variation and a set of examples as two distinct perspectives. Relevant to our second goal, we found it sometimes hard to characterize descriptions accordingly to this distinction. In every activity we coded cases of ambiguity or uncertainty, with the latest as being more dominant. This struggle suggests that it may not be a clear distinction between CV and SOE. This finding has important implications to the way the notion of draggable objects is described in the literature and how we might interpret and analyze individuals' interactions with objects in DGE. Future research should focus on examining other theoretical perspectives that might explain cases of ambiguity and uncertainty.

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FOSTERING VIRTUAL COLLABORATION IN GEOMETRY: ANALYZING TASK DESIGN AND TECHNOLOGY

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We investigated preservice mathematics teachers' engagement in virtual collaboration on geometry tasks. In online synchronous classroom environments, PSTs collectively explored four different transformations on Desmos applets and created written descriptions and exemplary figures to describe the mathematics behind each transformation. We analyzed video recordings of the online group discussion and identified emerging group actions mediated by technology, which demonstrated social and mathematical aspects of collaboration. The design features of the task (e.g., setting group norms, co-constructing deliverables) and the technological features of the virtual workspace (e.g., dynamic geometry, shared screen, shared presentation) fostered students exchanging ideas and engaging in sense-making. Limitations of the task design and the virtual environment will be discussed for future improvement.

Keywords: Technology, Online and Distance Education, Geometry and Spatial Reasoning, Preservice Teacher Education

There has been an increase in calls for effective instructional practices to facilitate student engagement in online learning environments (Carius, 2020; Iuvinale, 2020). While online instruction has become more prevalent for a variety of reasons, the COVID-19 pandemic hastened this. Given the importance of collaboration for learning, there is a need for more empirical research in mathematics education that investigates students' experience with and how to foster collaboration in virtual synchronous online environments.

This study aims to investigate how students engage in virtual collaboration for mathematics learning in an online synchronous classroom environment. We employed the design-based research methods (The Design-Based Research Collective, 2013) characterized by an iterative cycle of design, implementation, reflection, and refinement of task design to promote virtual collaboration in an online synchronous learning environment. In addition, this cyclic process involves analyzing how students socially and mathematically engage in collaborative workspace mediated by technology, which in turn, contributes to the building a theory of Virtual Collaboration for Mathematics Learning (VCML). This study serves as a pilot study of the initial task design and the first phase of the entire design-based research project with the following research question: *How can tasks be designed to foster collaboration using virtual learning tools in geometry?* We designed a Desmos task and implemented it in three mathematics and mathematics education courses for secondary mathematics pre-service teachers (PSTs). We analyzed students' online group work and identified emerging collaborative practices that demonstrated students' exchange of ideas, as constructions of shared-understanding. We discuss the features of task design that led to effective student collaboration.

Background Literature & Theoretical Perspective

Research on collaboration practices specific to geometry and technology exist (e.g., Alqahtani & Powell, 2017; Sthal, 2016). For example, Sthal (2016) explored Computer-

Supported Collaborative Learning (CSCL) with an emphasis on group-cognition (or shared-understanding) practices, contrasted from individual learning from cooperation. A longitudinal study of a small group learning online collaborative dynamic geometry identified the adoption of about 60 group practices under various categories emerged: on collaboration, technology, subject matter (mathematics), and discourse.

As a theoretical perspective, we used "mediated action" (Wertsch, 1994), a sociocultural approach that deals with the connections between human actions within context while considering the action and the setting as distinct but related. Here mediated action plays an important role in understanding sociocultural phenomena. As Wertsch explains, "Mediated action must be understood as involving an irreducible tension between the mediational means provided by the sociocultural setting, on the one hand, and the unique, contextualized use of these means in carrying out particular concrete actions, on the other" (p. 202). We use mediated action to explore students' engagement with the virtual task and how it produces opportunities for their learning. The mediating tool is the virtual task that influences how they interact with each other and with mathematics. We used mediated action to investigate the user-tool relation where users are the students and tool is the task in a virtual workspace. We used interaction between students and the virtual task as the basic unit of analysis, i.e., the actions taken by the students during the virtual task.

Methods

Setting & Participants

This research was conducted at three institutions: one Southern and two Mid-Atlantic public universities. Participants were all undergraduate students: undergraduate mathematics students enrolled in a geometry course, secondary mathematics pre-service teachers in a geometry course, and secondary mathematics pre-service teachers in a mathematics teaching methods course.

Task Design, Data Collection, & Analysis

We designed an activity that could be implemented in all three settings, with modifications as per course goals and students' needs in each course. The activity included two components: (1) a Desmos applet about geometric transformations and (2) a group product, in Google slides. The first goal of the activity was for students to learn how to engage in virtual collaboration, such as taking turns when speaking, being supportive, taking responsibility for the assigned work, etc. The second goal of the activity was different for each instructor based on their own classrooms. For example, the researcher teaching the methods course framed the activity as a mathematics teaching tool, whereas the researchers teaching the content courses used these to frame geometrical transformations. After the participants had experienced virtual collaboration they reflected on their experience and discussed its implications for teaching mathematics in their own future classrooms.

Prior to beginning the Desmos task, the students were asked to develop group norms. The Desmos task focused on the concept of transformations; students were asked to make sense of how they worked via kaleidoscopes as a context. The students first worked individually and then shared their initial thoughts with their teams. The teams were then asked to formally describe how the kaleidoscopes dynamically create transformational figures and provide pictures that best represent their description in Google slides.

Data collection included video recordings of group discussion and their submitted group documents sharing their findings. We conducted open coding (Strauss & Corbin, 1998) to identify emerging group actions: student-student verbal or non-verbal communications that may involve one or more aspects of virtual collaboration (e.g., use of technology). We then applied secondary-level open coding that fell into one of three dimensions: social (people), mathematical (content), and technological (tool). This open coding was not to isolate and distinguish episodes from others but rather to mark empirical evidence of students' virtual collaboration. The initial and secondary-level open coding were intended to highlight what made this episode interesting analytically with respect to virtual collaboration. Through this, we can identify emergent themes across those episodes for the nature of students' participation in collaboration and areas of improvement in the task design, enactment, or virtual environment.

Results

To illustrate our analysis, we describe one pivotal episode of collaboration. This episode was chosen based on the open-coding detailed above, for having all three of social, mathematical, and technological dimensions. It took place in one of the secondary geometry content courses. We draw out key features of the task that led to collaboration.

Tom, Val and Josh worked together as a group to understand how a kaleidoscope (transformations) worked. Real-time collaboration both in Desmos and shared slides were crucial: Students first individually and then together explored a Desmos applet. Then, in shared Google Slides, they had to paste a screenshot of their own drawing that best illustrated the transformation's workings and write a verbal description of the transformation.

First, the dynamic geometry environment of Desmos coupled with screen sharing allowed for real-time testing of conjectures. For example, Josh asked Tom to draw lines on the shared screen: "What happens if you draw a line...from $x = 0$ to down below?" Tom drew this with his mouse in Desmos so that the entire group could see it on his shared screen, as Josh gave more concrete instructions on how to draw what he was thinking. Upon seeing how the real-time drawing affected the mathematical space, Josh affirmed, "That's what I thought would happen." He had asked to test out a conjecture with the group and the dynamic geometry environment allowed him to visually verify that his predictions were accurate.

Second, the task component of selecting the best representation led to student justification for their choices. In deciding which drawing to select, Josh provided a rationale for why the group-generated representation was best: "I like it. I couldn't figure out if it [the transformation] was quadrants or not so this shows it much better than mine does." In needing to choose a representation, students justified their choice of one over another by referring back to the mathematics and how certain features best illustrated that.

Third, the existence of a "deliverable" led to students checking with each other and asking for reassurance. In the shared slides, Val began to type the description but seeked the group's thoughts and help by asking, "What do you guys think?" Later when Val finished typing, he again asked the group what they thought of his description. Tom provided reassurance by affirming the description, and Val then asked if Tom could pull up the Desmos app so he could see the picture. He explained a line of thinking he had considered adding to the description:

Val: I was thinking about it, maybe we should talk about how if we do cross quadrants, we'll be in a different quadrant. But it doesn't really matter because we could always just argue that that black line you drew was just on top of that quadrant and it duplicated to all the others equally.

Tom: If it's in the top right corner of quadrant one then it'll be in the top right corner of all the other three quadrants.

Val: Ok, that makes sense.

Val again was unsure, but this time voiced his thought aloud. In looking at the picture and sharing his thoughts out loud with the group, Val now felt satisfied about the description.

Discussion

Our results indicate that certain features of the task allowed the students to effectively collaborate in the virtual space: tools such as a dynamic geometry environment and screen sharing and task design features such as selecting a best representation and formally producing a group written product. For example, asking the students to present a final document as a group motivated the students to find the best possible phrasing to describe the working of the geometric transformations. This included students discussing what precise terminology to use (their use of "quadrants"). In addition, asking the students to first work individually and then share their thoughts in a group gave students time to think through the mathematics and come to the group with questions to ask and ideas to share.

When designing the task, some student experiences we had anticipated, while others emerged that were a surprise to us. We will incorporate these into future iterations of this task design. For example, we asked teams to develop group norms prior to starting the tasks; some groups did, and others did not. However, during collaboration these norms seemed to emerge organically, such as providing support when a team member was not confident about their work, allowing space for quieter group members to speak up, answering a group member's questions, etc. Not all groups developed or practiced the same norms leading to a diversity in observed collaboration.

Here technology acted as a mediator between students as they tried to make sense of the math and develop a group document together to submit. For example, the shared screen allowed students to easily look at the same mathematical environment at the same time; in person, it would be difficult for multiple students to see the same computer screen due to physical space. Students could also see what the presenter was sharing clearly at any given time, as opposed to how in real life, one may not be able to see a shared paper. Technology amplified their sharing.

Technology also served as a mediator for students' interactions with mathematics. For example, we asked students to first engage with Desmos individually and then bring their thoughts to the group for discussion. Here, technology aided their sense-making of how the kaleidoscopes worked. For example, students saw how the kaleidoscope replicated and transformed what they drew, including when they erased. This direct observation of cause and effect in real-time where students were in control aided their understanding of transformations.

While our analysis highlighted opportunities for student learning through collaboration in virtual space, there were also limitations to this approach. For example, there is vulnerability in showing individual work to a larger group and sharing it on a screen. Such limitations can be overcome by anticipating which moments of the task may feel most vulnerable and addressing them when designing the task. For example, encouraging the students to work in pairs using breakout rooms might allow them to feel comfortable when sharing their work with the larger group. In addition, students feeling comfortable sharing uncertainty with each other, asking for reassurance, and supporting each other are crucial for necessary.

As teachers we are the biggest mediators in our students' learning. Through mindful task design and implementation, limitations of a task can be overcome. For example, best practices for group norms can be modeled and encouraged. Norms to support students who are not confident in math or are worried about technology can be set. In our future work, we will develop framework(s) for how interaction occurs in virtual spaces specific to learning geometry.

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EMBEDDING EXPERT KNOWLEDGE: A CASE STUDY ON DEVELOPING AN ACCESSIBLE DIAGRAMMATIC INTERFACE

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When students with blindness and visual impairment (BVI) are confronted with inaccessible visual graphics in the geometry classroom, additional instructional supports are often provided through verbal descriptions of images, tactile and haptic representations, and/or kinetic movement. This preliminary study examined the language used by instructional experts to describe geometry images to students with and without access to a visual instructional image. Specifically, we investigated expert descriptions of geometry diagrams for 1) spatial information, 2) instructional concept information, and 3) overall description structure (e.g., length, vocabulary, image part/whole order/relationships). We found that experts used nearly twice as many words to describe diagrams in the no visual access condition. We consider the double-edged nature of this result for supporting BVI learners in classrooms and chart possibilities for future research.

Keywords: Students with Disabilities, Geometry & Spatial Reasoning, Mathematical Representations

Introduction

There are approximately 12 million people with blindness or visual impairment (BVI) in the U.S., including over 600,000 school-aged children ages 5-18 (Erikson, Lee, von Schrader, 2021). Their success in school mathematics is hampered by inadequate accessible learning technologies and curricular resources, because representations such as graphs, tables, and diagrams are generally only available in visual formats. However, the information in such visual representations can be communicated through other perhaps equally valuable sensory channels (e.g., language, sound, haptics, tactile) (Giudice, 2018; Abrahamson, Flood, Miele, & Siu, 2019). The study reported here is one segment of a larger project whose purpose is to design an interface that will allow students with visual impairments to perceive geometry diagrams through a combination of audio and tactile sensory modalities. We ask: How can geometry diagrams be apprehended through non-visual sensory modalities? As a first step toward investigating this question, we report a case study of how expert users of geometry diagrams used *natural language (NL)* (Herzog & Wazinski, 1994) to describe a core set of 2D images.

Theoretical Framework

We conceptualized geometry diagrams from a systemic functional linguistics (SFL) perspective (O'Halloran, 2005), within which diagrams are mathematical texts that use spatial (e.g., orientation, size, position) and graphical (e.g., strokes, congruence markings, labels) resources systemically to communicate mathematical concepts, properties, and relationships (Dimmel & Herbst, 2015). One technique for representing diagrams with non-visual modalities is through natural language descriptions. Natural language is a term used in psychology, linguistics, and computer science for the communication and representation of any language that has evolved naturally in humans through use and diffusion (Lyons, 1981). NL descriptions of 2D diagrams are used in studies of spatial reasoning and patterning development (Clements, 2004).

Methods

Protocol design

Instructional experts were presented a protocol that consisted of five horizontally oriented digital pictures, each representing a different geometric diagram. All the images were chosen to provide typical examples of geometry diagrams. The geometry diagram prompts are provided in Figure 1.

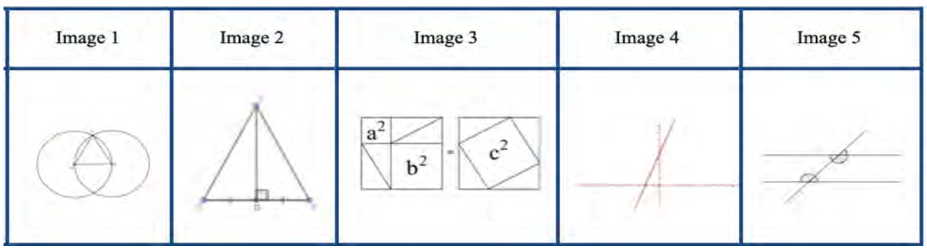


Figure 1: Diagrams used as prompts for expert NL descriptions

As a set, the diagrams were selected as realizations of a range of fundamental geometric relationships, including: *congruence* (images 1, 2, 3, and 5), *parallelism* (images 3 and 5), *perpendicularity* (images 2 and 3), *segment-angle relationships* (images 1and 2), and *incidence relationships* (each image). Images 1, 2, and 5 are examples of diagrams that might accompany typical proof problems (Herbst et al., 2009). Image 3 was selected as an example of a *proof without words,* and Image 4 was selected as a primitive example of a coordinate geometry diagram. The relationships described above are not exhaustive of the geometric concepts realized in the set of images, nor was the set of images intended to serve as a comprehensive set of primitive diagrams that would account for all possible diagrammatic variations. Rather, these images were selected because they entailed sufficient variation to provide a starting point for taking stock of how geometric concepts realized in diagrams could be described using natural language. Participants were given the following instructions and asked four questions for each of the diagrams used in the protocol:

> *Please briefly review the image and respond to the questions below for each image in the protocol. Record your responses to the questions for each image on a separate recording file.*

- How would you describe this image to a student in your class?
- How would you describe the image to a student who you are talking to over a phone or who was listening to a podcast, who cannot see the image?
- What are the most significant mathematical concepts, relationships, or features that need to be described in this image?
- How would you convey the mathematical concepts, relationships, or features to a student who you are talking to over the phone who cannot see the image or to someone listening to a podcast?

The question prompts specifically did not mention that the student who could not see the image had a vision impairment so as not to bias the expert into focusing on the student's disability, as opposed to the instructional concepts that should be explained to any student who could not directly see the image that was the focus of instruction.

Participants, data, analysis

Four experienced mathematics teachers (university or secondary;1/F, 3/M; mean age: 45.8; mean years of experience: 12.3) completed the remotely administered study. Participants recorded their responses individually and at their convenience. The experts spent approximately 30 minutes to audio record their descriptions of 5 geometry diagrams. They were permitted to ask clarifying questions via email, if necessary, but none of the experts chose to stop the recording to ask follow-up questions.

The analysis of the description data used an NLP pipeline that included the processing of raw image descriptions and an annotation process based on techniques used in Cognitive Discourse Analysis (CODA) (Tenbrink, 2015). The annotation process coded specific elements of the image description such as the whole image description, the description of a specific part of the image, as well as identifying themes and relevant features in each *segment*, which was defined as "one coherent statement about a single item/space/topic." (Suwa & Tversky, 1997; Cialone, Tenbrink, & Spiers, 2017). An annotation review was then conducted segment by segment, counting the occurrences of each linguistic feature representing a specific annotation category. The relative frequencies were calculated for each image in relation to the overall number of words produced by each expert, the total number of words produced across experts, and the total identification of instructional or spatial concepts.

Results

80 raw image descriptions (4 experts x 5 diagrams x 4 description questions) were processed and analyzed for this case study. We present here the general language patterns found across expert descriptions based on several text-analysis metrics: Total number of words (n) and average number of words(M) across all expert descriptions for each image, variety of word types (unique) or range (R) of total words used by the experts, and average sentence length (M words per sentence) based on student visual access or non-visual access to each graphic. These aggregate descriptives are reported, by diagram (columns) and condition (rows), in Figure 2.

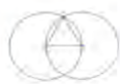
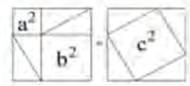
	Image 1	Image 2	Image 3	Image 4	Image 5
					
Visual Access	n= 293 words Unique =116 M= 73.25 R= 149 (18-167) M density = .73 M words per sentence = 21.4	n= 395 words Unique =138 M= 98.75 R= 101 (50-151) M density = .64 M words per sentence = 21.4	n= 1127 words Unique = 244 M= 108 R= 52 (77-129) M density = .39 M words per sentence = 23.8	n= 482 words Unique = 148 M= 120 R= 78 (87-165) M density = .53 M words per sentence = 14.27	n= 468 words Unique =155 M= 117 R= 103 (69-172) M density = .54 M words per sentence = 25.05
Non-Visual Access	n = 605 Unique = 189 M = 151.25 R= 188 (47-235) M density = .49 M words per sentence = 34.2	n = 560 Unique = 191 M = 186.66 R= 188 (63-328) M density = .55 M words per sentence = 25.8	n= 1880 words Unique = 342 M= 470 R= 511 (155-666) M density = .33 M words per sentence = 21.47	n= 412 words Unique = 170 M= 103 R= 23 (94-117) M density = 61.76 M words per sentence = 23.25	n= 1382 words Unique = 271 M= 345 R= 592(147-739) M density = 129.75 M words per sentence = 20.4

Figure 2: Summary statistics for expert description language use across the images.

Number of words in descriptions

The results reported in Figure 2 show that instructional experts used approximately two times the number of total words (n) in the non-visual access condition, including more unique words – i.e., words that were used only once in the description. This may suggest sighted instructional experts are using different description strategies (e.g., longer descriptions, a larger more diverse vocabulary set, etc.) for students without visual access to the image. This interpretation is consistent with the findings of a study using NL scene descriptions created by sighted participants for non-visual users (Doore, 2017).

Discussion

The primary findings from our analysis are that experts consistently used more words, sometimes as many as double, when describing diagrams for students without visual access to the diagram when compared to students with visual access to it. A critical question raised by these results is: *Do longer, more detailed natural language descriptions of diagrams result in more effective mental representations and conceptual comprehension for BVI students?* Future goals of this research include developing a controlled vocabulary to eventually generate automated descriptions for diagrams to be incorporated into a remote multimodal learning system that uses haptic, auditory, and NL supports for meeting the needs of BVI students in accessing diagrammatic information. The ultimate goal of the larger body of research is the development of a multimodal system that can act as an accessible personal learning environment (PLE) (Martindale & Dowdy, 2010) to support more BVI students to develop the academic skills and personal interests to enter the STEM professional pipeline as well as to increase their post-secondary attainment and employment success. This type of personal learning environment may someday provide a remotely- accessible, cloud-based system that will allow BVI students to direct their own learning.

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SUSTAINING COGNITIVE DEMAND WITH DESMOS TECHNOLOGY

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Teachers today are encouraged to incorporate digital technologies into mathematics instruction to build students' technological fluency and deepen mathematical understanding. However, the degree to which students' technology use sustains cognitive demand remains unclear. This study will contribute to the understanding of the complex relationship between technology and cognitive demand by providing a video analysis of an eighth grade mathematics teacher's task launch and the ensuing enactment by students. Our findings indicate that technology may raise cognitive demand in potentially transformative ways, although teacher actions and student familiarity with technology also play a role throughout enactment.

Keywords: Curriculum, Middle School Education, Technology

The acquisition of technological fluency, including strategic use of tools, continues to build momentum in classrooms. Unlike previous generations, there is considerable demand for teachers to instruct students with digital technologies that generate deep understanding of disciplinary concepts. Common Core Math Practice 5 states that children should select and use appropriate tools strategically, deepening their understanding of mathematical concepts (CCSSM). Yet despite general agreement toward incorporating these tools, there are few standardized practices that instruct teachers on how and when to use technologies for learning (Greenhow, Robelli, & Hughes, 2009). As a result, students may take up technology in different ways, which may impact cognitive demand. We ask: how does cognitive demand change throughout a task's enactment, and how do students use technology in relation to such changes?

Theoretical Framework

The dual foci of this study are fluctuations in cognitive demand during mathematics tasks and the use of digital technologies in relationship to demand. Stein and colleagues define a *mathematical task* as the arrangement of mathematics within the task combined with the resources provided to solve it. *Cognitive demand* refers to the cognitive processes required of students to navigate a task, identify key information, and activate prior knowledge that is useful for constructing solutions (Doyle, 1983). The current study focuses on a *high cognitive demand* task (Stein & Smith, 1998), which requires students to make conceptual connections, use multiple representations and solution strategies, and make conjectures. Cognitive demand is not a static construct; it is adapted from the task as written to the teacher's launch (Jackson et al., 2013), then negotiated by students and the teacher during the task enactment. Adaptations and negotiations have variable impacts on cognitive demand, and a key goal in this study is to examine changes in cognitive demand during the teacher's launch and student enactment.

Incorporation of digital technologies can alter mathematics tasks. This study focuses on Desmos, a technology which can automate graphing, give instantaneous feedback, and display multiple modes of representation (Sherman, 2014). The amalgamation of Desmos's features may alter the cognitive demand of the task and its outcome, which in turn may influence what math students learn and how they learn it. However, digital technology's potential to mediate task outcomes may be contingent on how teachers frame technologies to students and how students take up tools. This study proposes three classifications for examining students' technology use, adapted from Hughes's Technology as Transformation framework: students may use *technology as replacement* (using Desmos instead of a calculator), *technology as amplification* (using Desmos to do the same task faster), or *technology as potential transformation* (using Desmos to learn mathematics in new ways, potentially reshaping mathematical perspectives).

Methods

Context and Participant

This study is one component of a larger research-practice partnership (Coburn & Penuel, 2016) conducted between a large research university and Urban Unified School District (UUSD). Many students in UUSD are from non-dominant cultural and linguistic communities (UUSD, 2020). The district is committed to high-quality mathematics for all students, and developed a task-based curriculum for grades 6-8 that aligns with the Common Core State Standards for mathematics (Borko et al., 2017). To assist in implementing the new curriculum, UUSD identified and recruited teacher leaders (TLs) in each grade level to participate in professional development workshops with the intent of deeply understanding the new curriculum.

The subject of this study is Annie, a teacher, selected for her incorporation of Desmos into high demand tasks. Annie taught eighth grade mathematics at UUSD during the 2016 - 2017 school year at Pacific Middle School, and was involved in the research-practice partnership as a TL during the time of this study.

Task Selection

The Washing Machine Problem requires students to draw upon knowledge of linear equations and multiple representations to compare two scenarios of washer-dryer purchasing costs, or alternatively, using a laundromat (see Figure 1). This task meets the criteria for a *Doing Mathematics* by the Instructional Quality of Assessment (IQA) rubric (Boston & Wolf, 2006) as students solve a complex, real-world problem with potential for multiple solution strategies and representations. They must make conjectures about which washer-dryer combination is best without knowing additional details about the subject's life, justified with mathematical evidence. How students interpret assumptions impacts what washer-dryer combination they suggest, implying that a single, well-rehearsed, predictable solution is not present.

"Elena doesn't have a washing machine or dryer. She is considering buying one so that she doesn't have to go to the Laundromat. She wants to know if it makes sense financially to buy a washer and dryer. Each load of laundry at the Laundromat costs \$1.25 for the washing machine and \$1.50 for the dryer. A top-loading washer costs \$250, and each load costs \$0.26 for water and energy. A front-loading washer costs \$400, and each load costs \$0.09 for water and energy. A dryer costs \$300, and each load costs \$0.35 for energy. **What should Elena do? Give her some advice."**

Figure 1: The Washing Machine Problem (as written in UUSD's curriculum)

Data Sources and Analysis

Data for this project were obtained from video recordings of Annie's classroom while launching and enacting *The Washing Machine Problem*. Recordings contained the enactment of the task in groups of 4-5 students. All recordings were subject to additional screening for visual quality, audio quality, and completeness. Annie's launch was captured in a recording using a separate teacher camera and microphone. The research team then qualitatively analyzed the teacher launch and students' enactment to identify (1) changes in cognitive demand throughout the task, and (2) instances where Desmos was used by students to solve the task. The IQA was used to rate cognitive demand, and Hughes's framework was adapted for a student-facing stance where the researchers coded technology use as replacement, amplification, or potentially transformative. Recordings were broken into "conversation units": sequences of talk where students grappled with mathematical ideas. Throughout the analytic process, the research team used the IQA and stabilized technology codebook to rate and code independently, discuss ratings and codes during weekly meetings, and reach consensus.

Findings

Our first set of findings analyzes the teacher's launch and the resulting changes in cognitive demand from the task as written. We identify salient features that contribute to changes in cognitive demand and their hypothesized impact on students' enactment. The second set of findings focuses on three groups' enactment, changes in cognitive demand, and use of Desmos.

Teacher Launch

Annie announced students would solve a system of equations and give advice. She presented the task as a series of steps: (1) First, students would create three equations to model the two different washing machines and laundromat options, checking in with Annie once they finished. (2) Second, students would graph their equations in Desmos, framed as a step in the process of solving. (3) Finally, students would give advice for a washer-dryer combination.

We rated the cognitive demand of Annie's launch as a 2 on the IQA rubric, *Procedures without Connections.* Her launch focused on a sequence of procedures for students to follow during the task, including Desmos use framed as a procedure itself, which removed ambiguity and reduced opportunities for multiple solution strategies. Although Annie informed students to produce equation and visual representations, there was little question on how to do so. The focus of the launch seemed more suited toward completion of the task's sequence of steps than on high demand, non-algorithmic thinking.

Student Enactment: Aggregate Patterns in Cognitive Demand

The three student groups initiated enactment with low cognitive demand (level 2), then increased demand (levels 3-4) when they graphed, compared multiple representations, and gave a recommendation (see Table 1). When crafting the laundromat, top-loading, and front-loading equations, students tended toward low demand aspects of the task, and subsequentially, low demand explanations ("$1.25 is the slope, because it's 'per' load,"). Once students progressed to the graphing phase, Desmos afforded opportunities to instantaneously notice differences between equations, provide high demand explanations for those differences, and make conjectures grounded in the task's context. Students who reached cognitive demand level 4 observed that limited information about Elena's laundry habits prohibited them from making a single recommendation ("it really depends on how long she's going to use it."). In all groups, Annie monitored progress by asking open-ended questions (e.g., "what assumptions are you making?"), giving space for students to voice their thinking, and encouraging students to think beyond procedures. Despite a procedural launch, Annie appeared to have mediated cognitive demand by pressing students to consider the task's context and embedded assumptions.

Table 1: Students' Aggregate Cognitive Demand during Enactment

	Writing Three Equations		Desmos Graphing and Analysis	
Cognitive Demand	Minutes	Conversation Units	Minutes	Conversation Units
Level 1	0.0	0	0.3	4
Level 2	**15.0**	**54**	2.8	26
Level 3	13.0	37	6.7	26
Level 4	7.0	9	**15.4**	**44**

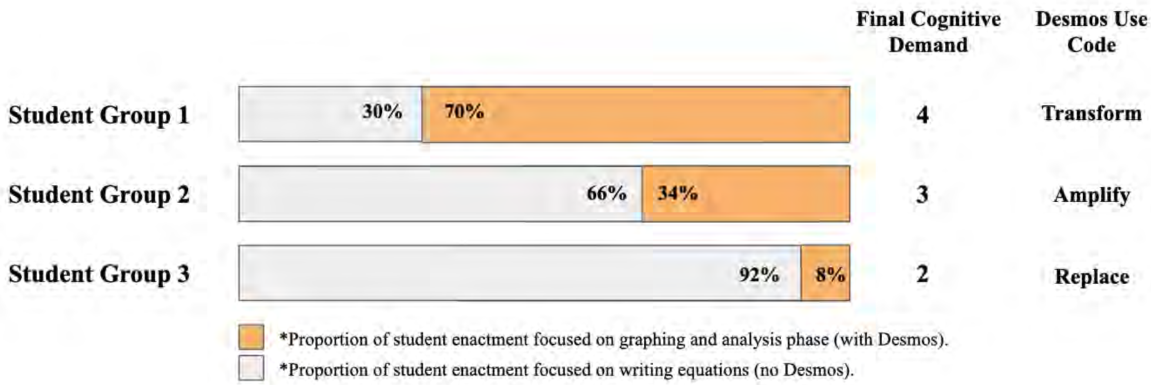

Figure 2: Cognitive Demand and Time Spent on Desmos

Student Enactment: Cognitive Demand and Desmos Use

Students who achieved higher cognitive demand used Desmos to amplify and transform the task's interpretive aspects at higher frequencies (see Figure 2). Student Group 1 graphed their linear models quickly and spent a majority of enactment discussing ambiguity and assumptions within the task. They expanded on the capacities of their laptop by searching, "*How long does a washing machine last, on average?*" into their internet browser, potentially transforming views of mathematics by triangulating multiple pieces of evidence to give a nuanced justification Student Group 2, who struggled for approximately six minutes signing into Desmos, eventually used the platform's scroll, zoom, and click features to identify points of intersection, compare rates of change, and move toward a recommendation. They made up for lost time by making use of Desmos's features, which amplified their analysis of the models. Student Group 3 spent most of enactment creating the equations, and only 8% of their time was spent constructing a graph. It took substantially longer than the other groups to input their equations, and by the time they finished, Annie called the class back to attention. We coded this group's Desmos use as replacement, as they procured a graph with a computer, but did nothing else with it thereafter.

Discussion and Conclusion

Annie's procedure-focused launch did not hamper all students' abilities to make rich mathematical meaning during enactment. Her strategic use of questioning, student voice, and press for justification from students helped two out of three groups construct equations more efficiently, which in turn, allowed them to make use of Desmos's affordances and raise demand. When Desmos was used as amplification or in potentially transformative ways, students gave more complex recommendations for which washer and dryer should be purchased, and why.

Our work suggests a synergistic relationship between the features of digital technologies, the demands of the task, and teacher actions when examining student outcomes. Desmos may have been well-suited to raise cognitive demand in this particular task because of its ability to produce multiple equations instantaneously, which shifted students' focus from graphing by hand to analyzing models and centering mathematical discussion. Simultaneously, Annie supported enactment by drawing students' attention to high demand aspects of the task during small-group interactions. Future research should further explore and build upon the relationship between cognitive demand, digital technology, and teachers' actions, taking into account different tasks, instructional tradeoffs with technology (e.g., the time teachers expend to familiarize students with new tools), and levels of teacher expertise when enacting curricula with technology.

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POSITIVE CHANGE IN MATHEMATICS TEACHER EDUCATORS' MIGRATION TO EMERGENCY REMOTE TEACHING DURING COVID-19

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Our consortium of four universities conducted survey research with mathematics teacher educators (MTEs) regarding their transition to online teaching during COVID-19. This research focuses on positive change in instruction that was identified by respondents in spring 2020. Results shared focus on general technology tools and mathematical technology tools for students' learning that were initiated by MTEs during the migration to online teaching. The data indicate MTEs altered the style of their teaching in response to the online environment, and some are likely to retain practices initiated during the emergency remote transition. The research has implications for MTEs who teach online, as well as those who are interested in incorporating more technological tools into their face-to-face instruction.

Keywords: Technology, Online and Distance Education, Instructional Activities and Practices

The unfolding response to the COVID-19 mandate to teach remotely provided a unique, one-time opportunity for groundbreaking research to study how crisis-induced changes to instruction influence faculty's teaching experience. The transition to online instruction necessitated by the COVID-19 pandemic created significant uncertainty for the nation's educators, including mathematics teacher educators (MTEs), many of whom had not previously taught online. This survey research examined MTEs' perceptions of the challenge of rapidly transitioning their teaching to emergency remote teaching, and how it impacted the quality of instruction they delivered during the COVID-19 pandemic. Data was analyzed using the technological pedagogical content knowledge ([TPACK]; Mishra & Koehler, 2006; Niess, 2005) framework to identify patterns of effective responses to the transition. These data will also facilitate the development of empirically informed policies to aid teacher education programs in navigating this current pandemic and reduce the disruption of such transitions in the future. This work will contribute fundamental knowledge on MTEs' adaptability—in the face of an urgent need—to deliver courses differently, and on improving teacher education programs by emphasizing effective ways to develop affective experiences while introducing effective technology tools for remote instruction. Furthermore, this work could help shape the design of professional development opportunities that promote adoption of research-based pedagogies and instructional technologies. The research question is: How did mathematics teacher educators change their instruction for migration to online instruction during COVID-19, and how did they perceive the value of that change?

Literature Review

Online learning is a well-discussed field in the past decade. Numerous articles and books communicate how to provide an optimal student-centered environment in synchronous online learning platforms (e.g., Flores Fahara & Lozano Castro, 2015; Goodman, 2019; Seifert, 2019). However, "there is virtually no representation of discipline specific programs for preparing college faculty to teach online" (Goodman, 2019, p. 1203). AbuZayyad-Nuseibeh (2017) examined the University of South Florida faculty's perceptions of their transition to online instruction and found that faculty were overwhelmed with the time and effort to transition, yet they designed more creative assessments and experienced increased active learning. Chiasson et al. (2015) also researched faculty's transition to online instruction using both a survey and interviews. Results indicated that faculty increased their confidence and believed they became better instructors in their face-to-face courses as a result of online teaching.

In the past year, various research studies have been conducted about how COVID-19 has changed the way we currently teach. These studies range from rethinking preservice teachers' clinical practice (Parker et al., 2020; Pike et al., 2020) to learning new ways of teaching with different digital applications, such as Twitter (Carey et al., 2020), Google forms (Khan & Jawaid, 2020), and Flipgrid (Goddard, 2020; Oliver et al., 2020). Others, such as Rapanta et al. (2020), shared ways to observe student work patterns in online learning environments.

The TPACK framework (Mishra & Koehler, 2006; Niess, 2005) guided our coding of qualitative data. TPACK extends Shulman's (1986) pedagogical content knowledge (PCK) to include the additional domain of technology, thus introducing technological content knowledge (TCK), technological pedagogical knowledge (TPK), and TPACK. This integrated knowledge of TPACK draws upon teachers' understanding of how to effectively use technology to teach mathematics.

Methods

The purpose of the survey was to gather information about MTEs' experiences with migrating their in-person classes to online classes due to COVID-19. The survey was specifically for MTEs' mathematics education or mathematics content classes that support a teacher education program for undergraduate and/or graduate students. The target population of this survey was a convenience sample of approximately 940 Association of Mathematics Teacher Educators members. The Mathematics Teacher Educators' Migration to Online Teaching in Response to COVID-19 survey was designed for this research and administered online (Nesbary, 2000). The survey consisted of 14 Likert and short-answer questions about migrating in-person classes to online classes, plus nine demographics questions. Survey topics included identification of new tools adopted for instruction; professional development participated in; benefits of online instruction over face-to-face instruction; challenges or affordances for equitable practices; as well as experiences with creating a community among students, engaging students, using general technology tools, using mathematics technology tools, and using formative and summative assessments.

To establish content validity, the survey was piloted with four MTEs and one instructional technology expert. The survey was refined and piloted with three new MTEs and one survey construction expert. Feedback from the second pilot was implemented, and the final survey was sent to AMTE members at the end of May 2020. One week after the initial email, a follow-up email was sent to ensure a high response rate.

Results

Altogether, 218 AMTE members responded to the survey. Nine of the responses were eliminated because the participants were not teaching mathematics education or mathematics content classes in spring 2020. Participants' data were collected anonymously; therefore, information is unavailable regarding which AMTE members did or did not complete the survey.

MTEs' experiences with teaching online or hybrid courses prior to COVID-19 were low. Of the 167 MTEs that responded to this question, 63 had zero experience with these modalities of teaching, while 65 had taught one to five classes, 21 had taught six to 10 classes, 7 had taught 11 to 15 classes, and 11 had taught 21 or more classes with these teaching modalities. Before spring 2020, participants were teaching 100% in-person (n = 147, 70%), no synchronous online classes (n = 196, 94%), no asynchronous classes (n = 165, 79%), and no blended classes (n = 190, 91%). Post-COVID-19, all MTEs were teaching fully online, 68% (n = 143) had some synchronous teaching, 68% (n = 143) had some asynchronous teaching, and 11% (n = 23) had some blended teaching. Therefore, spring 2020 was a shift from mostly in-person teaching—where online instruction was typically asynchronous—to all-remote instruction.

One survey question invited MTEs to share the extent of change in engaging students in learning, creating a community among students, using general technology tools, using mathematical technology tools, changing instruction to meet diverse learner needs and ensuring equitable access to content, etc. (see Table 1). MTEs used a five-point Likert scale to identify this change from *not at all* to *completely*. They also shared whether this change was positive, neutral, or negative. When combining the number of responses for *a lot* and *completely*, the opportunities for students to teach and to learn from their own teaching and the teaching of others was ranked as the most negative change. The two changes that received more positive responses than negative were general technology tools for students' learning and mathematical technology tools for students' learning.

Table 1: Extent of Change

To what extent did you change:	Not at all	Somewhat	Moderately	A lot	Completely	Blank		Positive	Neutral	Negative	Blank
the opportunities for students to teach and to learn from their own teaching and the teaching of others?	12	17	33	60	51	36		17	32	97	63
how you engaged students in learning?	4	31	38	79	22	35		20	65	64	60
the assignments/tasks/formative assessments of the course?	7	36	40	70	20	36		30	61	58	60
the use of general technology tools for students' learning?	11	28	46	67	20	37		62	75	7	65
the way you created or maintained a community among your students?	6	32	54	53	29	35		30	59	59	61
the summative assessments of students' learning?	16	37	44	46	30	36		24	81	42	62
the use of mathematical technology tools for students' learning?	31	33	39	51	18	37		64	65	14	66
your instruction to meet diverse learner needs and ensure equitable access to content in your class?	17	41	52	50	12	37		24	63	59	63
how you teach mathematics content?	8	45	57	44	17	38		21	74	54	60

To better understand the long-term implications of these changes, MTEs were asked to share some examples of how their experience with online instruction due to COVID-19 will influence their future instruction, either online or in person. Responses indicated that, while many changes were forced by the swift transition to remote teaching, participants could identify instructional changes that would persist after remote instruction ended (TPACK). Participant 143 shared,

> I also believe that I will be more intentional about structuring whole class discussion so all students have an opportunity to participate. With using collaborative interactive Google slides, we all have access to a permanent record of each person's thinking. I would love to continue this practice with my future instruction, both online and in-person instruction.

The use of specific tools was also found in the qualitative responses data. Participants described a wide range of tool use in their new teaching context. These tools included: digital manipulatives (TPACK), physical manipulatives (PCK), digital tools for instruction (TPK), and virtual meeting tools (TPK). The data indicate that tool use was a value-added activity. Participant 1 commented, "I got to learn about new ways to engage students online synchronously and some of those had benefits over my traditional teaching face-to-face."

Implications for Teacher Education Research

Our initial goal was to capture the experiences and stories of MTEs during the unprecedented pedagogical shifts that happened in spring 2020. As we analyzed the data, we wondered, "What relevance do these stories have on the future of mathematics teacher education?" It is hopeful that spring 2020 and the reality of remote-only teaching will eventually become little more than a memory. However, the authors believe that the move toward increased dependence on remote teaching strategies and tools is inevitable, and teacher education programs need to accommodate this change while considering TPACK.

It does not appear that instruction at all levels, K–12 and higher education, will go back to the pre-COVID-19-pandemic normal. Instruction is trending toward online teaching and taking advantage of digital tools—to support even in-person teaching. There is comfort in the fact that not all of this change was perceived as negative. In fact, many participants saw the shift toward new pedagogical strategies and tool uses (TPACK) as a positive that they anticipate will continue. The data in this study are revealing ways to improve and capitalize on the lessons learned from the transition to emergency remote teaching. This survey provides a more global perspective, beyond our own individual stories. A sound message from this survey data is a sense of solidarity, a focus on positive takeaways, and persevering through challenges that MTEs can learn and grow from, through their personal experiences during this historical shift to emergency remote teaching.

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HOW MIDDLE YEARS STUDENTS WITH MATHEMATICS LEARNING DISABILITIES BOOTSTRAP THEIR USE OF TECHNOLOGY

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Grounded in a learner's perspective, this case study investigates the bootstrapping resources that middle years students with mathematics learning disabilities draw on in using their personal electronic devices to support their mathematical learning. Semi-structured interviews were conducted with 14 participants in two provinces in Western Canada. Using Bereiter's categories of bootstrapping resources, the participants' reported uses of technology are discussed. Early findings suggest that participants' bootstrapping behaviors are influenced by the practices and attitudes of those around them, including teachers, family, peers, and even people they observe online.

Keywords: mathematics learning disabilities, technology, middle years, bootstrapping

Working from the learner's perspective, this paper reports the early findings of a case study exploring how middle years students with mathematics learning disabilities (MLD) learned to use, and adapt their use of, personal electronic devices to support their work in mathematics. In an earlier exploratory study (Armstrong & Gutica, 2020), post-secondary students with MLD described using technology in ways that include the following: accessing content of texts and other information sources; capturing information offered in lectures; checking accuracy of calculations; and performing algorithms so they can focus instead on mathematical concepts. Most had picked up these strategies on their own through a process of "bootstrapping." My research questions were: 1) how do middle years students with MLD use their personal electronic devices to support themselves in school mathematics and 2) how did they learn to do so? While for postsecondary students bootstrapping is often an independent, solo endeavor, my results suggest that for middle years students bootstrapping is very much grounded in their communities, influenced by the actions and attitudes of those around them.

Literature Review

Students with MLD have learning needs that may cause them to rely more heavily on digital devices for academic support than other students do. MLD has still not been adequately defined in the mathematics education literature (Lewis & Fisher, 2016) with ongoing debates about what the disability may involve, and how much the diagnosis overlaps with "math difficulties." As a number of brain processes feed into performing mathematics, including language processing, visual/spatial awareness, and working memory (Furlong et al., 2015; Willcut et al., 2013), in this paper I work from the premise that MLD can manifest themselves in a variety of ways, including difficulties with reading math texts, working with numbers, spatial reasoning, problem solving, as well as performing other mathematical tasks. It has been estimated that anywhere from 3-8% of school-age students have MLD (Brennan, 2009; Geary, 2004).

Effective strategic use of technology in teaching and learning math enables teachers and students to use digital tools in "thoughtfully designed ways" to "enhance how students and educators learn, experience, communicate and do math" (NCTM, 2015). This has the potential to let students off-load routine tasks, such as repetitive calculations, so they can focus on understanding mathematical concepts (Kaput, 1992); provide immediate feedback; offer dynamic tools that enable students to perform interactive tasks; and encourage the presentation of mathematical concepts using a variety of formats (Suh et al., 2008). Thus, technology has been recommended for teaching mathematics to students with MLD (Bouck & Flanagan, 2009).

In recent years, personal electronic technologies – electronic computer devices which are easily portable and that are used to store, display, process and transmit data (a definition adapted from Ayres et al., 2016) – have become an indispensable part of everyday life. With regular and long term access to this technology, many K-12 students have become fluent and confident in its use (Trouche & Drijvers, 2010). While some people do receive formal training in using electronic devices and applications through work or school, many of us just learn as we go, a kind of "pulling yourself up by your own bootstraps" method where you are building on your own resources to improve your situation. Bereiter (1985) suggests there are four potential resources for bootstrapping. 1) Imitation involves following a model or a mentor in learning how to use a device or software. For example, students may engage in a mathematics activity using Geogebra based on how their teacher demonstrate using the software's controls. 2) Learning support systems aim to mimic aspects of a teaching situation by scaffolding the learner's experience through offering questions with increasing levels of challenge and providing quick feedback. For instance, students may play an online math game to improve their mathematics skills. 3) Chance plus selection, or happy accidents, occur in a situation where the user performs a particular action for one reason but then discovers that it helps them do something else as well. 4) Finally, piggybacking occurs when someone uses a feature in technology for something other than what it was originally intended to do. For example, a student might use a coloured transparent plastic overlay over printed texts to improve readability and realize they can apply the same principle for screen texts by implementing a coloured background.

Bereiter suggests that people are more likely to bootstrap when they are familiar enough with a task or situation that they have mental energy left to try to improve how they are doing the task, or to explore other options or goals related to that task. In a survey of 760 middle years students, grades 6-8 (Harris Poll, 2015) 38% of middle years students considered themselves to be early adopters of technology, and 50% reported that they waited to see others try it and then tried it too. Of those whose schools either provided students with laptops/tablets to work with or had implemented a "Bring Your Own Device" (BYOD) policy, 55% considered themselves to be early adopters. This suggests that middle school students may have a comfort with technology that makes them more likely to engage in bootstrapping behaviors.

Methodology

My research is informed by the theoretical framework of enactivism as it applies to the relationship between students and technology (Li et al., 2010). Working from the premise that the systems of body, mind and environment are entangled with each other (Merleau-Ponty, 1962), enactivism suggests that learning emerges through the interactions between these systems: "cognition is effective action" (Maturana & Varela, 1987, p. 29). This points to a potential reciprocal relationship between the use of technology and cognition in the act of learning (Li et al., 2010). Enactivism has been described as an "essential tool" (Abrahamson et al., 2019) for reimagining the design of inclusive mathematics learning environments as well as theorizing how learning can occur. Rather than viewing technology as a prosthetic device for the disabled, compensating for whatever it is about their minds or bodies that is deemed to be lacking (Ayres et al., 2016), from an enactivist perspective, technology moves from being an external "add-on" to the disabled body to something that becomes embodied within the learning system in its potential to open up and extend the learner's bodily senses to new experiences and capabilities (Söffner, 2017). In this case study, I seek to investigate relationships between MLD and technology that might otherwise remain unnoticed (Yin, 2014).

To recruit participants, administrators and student services teachers in two school districts distributed information letters and consent forms to parents. All students who wished to participate in the study were interviewed by me at their schools, with 14 middle school students in total taking part (seven in grade 6, seven in grade 8; six girls and eight boys). All had been formally identified as having MLD by their student services teachers and had self-identified as being confident users of technology: they were potentially "information rich" (Patton, 1990, p. 169) with valuable insights about best technological practices for supporting their learning (Demouy et al., 2016). Semi-structured interviews included open-ended questions to elicit information about participants' technological practices and opinions (Swan et al., 2005), such as "Would you show me any technology you regularly use as part of your math schoolwork or homework? How does it help you?" and "What would you do if you needed more information about a math topic?" The interviews were 40 – 90 minutes in length and recorded by two video-cameras (offering a view of the device screen, and a view from the shoulders down of the participant to capture their gestures as they used the device). To ensure maximum participation and accessibility, I used a strengths-based approach in working with the students (Alper & Goggin, 2017). All participants were asked to bring with them the personal electronic devices that they used for their studies, so that they could demonstrate how they used them; students with learning disabilities sometimes struggle with language processing, so this enabled participants to show through doing/demonstrating/gesturing rather than having to rely on telling.

As this was an exploratory study, I used a process of constant comparative analysis (Glaser & Strauss, 1967). This analysis began with initial observations of the videos and reading of drafts of the interview transcripts by me and by two research assistants (experienced classroom teachers not associated with the schools in the study). Our individual summary notes were compared with each other to generate a further set of notes. These notes were continually refined through further reviewing of the videos and transcripts to determine emergent relationships and patterns. Two other research assistants completed an independent viewing of the transcripts and videos which offered alternate interpretations (Li, 2013). This helped to further foreground features about the participants' technological practices (Demouy et al., 2016; Kukulska-Hulme & de los Arcos, 2011) prompting a further and final round of categorization.

Findings and Discussion

As there are many learning issues that may feed into MLD, I wanted to hear from participants about any technology or strategies for using technology that they used to support themselves in learning mathematics. All the participants mentioned using at least one type of bootstrapping strategy, although some described using these strategies more than others did. Interestingly, none of the participants mentioned technology such as virtual manipulatives, online graphing calculators or dynamic geometry software.

Bootstrapping

Imitation. Participants often reported modeling their behavior on the ways the people around them used technology. For example, one participant knew she could research technologies to use because a previous teacher had done that kind of research for her. Others whose family members were active in trying out new technology reported engaging in similar activities themselves.

Modeling did not always result in more confident technological behavior, however. Although participants mentioned their teachers showing mathematics videos in class, only one participant reported looking up mathematics videos on her own to find out more information. A few participants regarded using internet resources with suspicion, voicing concerns that the mathematics online would be too confusing, too sophisticated, or it would not replicate the way their teacher wanted them to do the mathematics. However, some of these students also mentioned using videos to teach themselves other things (like new strategies for playing video games), so perhaps school mathematics was regarded as a special case. A couple of participants reported exploring technology independently, sometimes pushing against adult expectations. One student had hacked the coding of a school typing program to generate high typing speeds and had also sought answers to his mathematics questions from "experts" on Discord and Reddit.

Learning Systems. Participants reported being allowed to play mathematics games if they finished their seatwork early and felt these games could be helpful to build their mathematics skills if played regularly enough. Experiences with IXL were reported in detail by two students. One used it as his entire math program – he sat in the back of the classroom doing independent work with IXL while the rest of the class engaged in the regular lesson. He thought it was "doing something" for him although he wasn't sure what. Another student used IXL for extra practice and as a general resource, finding it useful, but sometimes was frustrated by the program because it only accepted specific answers and not mathematically equivalent ones.

Chance and Selection. A few participants mentioned "messing around" with technology in their free time, in particular finding ways to narrow their Google searches more effectively or using the Google feature "People also ask" to explore new but related topics. None mentioned any discovery that had any impact on how they used technology to learn mathematics.

Piggybacking. One participant was able to use his knowledge of Google searches as a work-around when he didn't understand the formula for surface area offered to him by IXL. Another participant had figured out how to use the colour feature in Excel to highlight data trends for a health class project and had also used coding to create a game about Pythagorean theorem for mathematics class. A few students reported using technology they had learned in other settings (often coding camps) such as Scratch, iMovie, and Mindcraft for school projects in non-math subject areas and had been teaching peers to do the same. Given that post-secondary students with MLD (Armstrong & Gutica, 2020) use online resources to supplement the content of their mathematics classes, it is likely that middle years students will realize they can do so as well.

Conclusions

Children with disabilities are "a marginalized and unheard group" and there are few studies that consider these students' views on assistive technology (Wright, Sheehy, Parsons, & Abbott, 2011, p. 5). "[A]n evidence-based understanding of students' technological experiences" (Kennedy et al., 2008) and insights is essential to improving classroom use of technology in order to develop pedagogical practices that more effectively support learning (Herrington et al., 2009; Kukulska-Hulme et al., 2011) and to improve student access and equity in mathematics. I am now investigating how the strategies of students with MLD for using technology to support themselves in mathematics emerge over time.

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PREPARING FACULTY TO TEACH WITH TECHNOLOGY: A FOCUS ON SELF-EFFICACY

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NCTM (2015), AMTE (2017), and CBMS (2012) have all pointed to the importance of preparing secondary mathematics teachers [PSMTs] to teach using mathematical action technologies (e.g., CODAP, Desmos, GeoGebra). However, a recent nationwide survey of accredited PSMT preparation programs found that while most programs do integrate technology in at least one course for PSMTs, many do not include opportunities to engage with a variety of mathematical action technologies and two programs reported that they do not include technology at all. In addition, responses to open ended questions indicated that faculty expertise is one of the reasons mathematical action technologies are not used more widely (Authors et al., 2020). To address this issue a 6 day virtual workshop on teaching undergraduate courses for PSMTs with mathematical action technologies was conducted. The workshop provided opportunities for faculty who teach mathematics, statistics, and methods courses to learn about technologies that are appropriate for use in their courses and are commonly used in secondary schools. Participants were organized into Teaching Interest Groups with opportunities to collaborate on ways in which they would incorporate what they were learning into their instruction. An overarching goal was to support the development of faculty self-efficacy for using the technologies and self-efficacy for using the technologies in their instruction.

To study faculty self-efficacy we used a survey methodology (Groves et al., 2009). Seventy participants representing 52 universities across 31 states participated in the workshop. Participants each completed a pre- and post- self-efficacy survey regarding the technologies presented in the workshop sessions. Participants rated their comfort level in using the technologies for themselves as well as teaching with the indicated technologies. The 24 question survey used a 6 point likert scale - where 1 represented “not comfortable at all” and 6 represented “extremely comfortable.” A repeated measures ANOVA was performed to assess the difference in self-efficacy as rated on the pre- and post- surveys (Norman, 2010). Preliminary findings suggest that participants began the workshop with low self-efficacy for using and teaching with the newer technologies commonly used in secondary schools today, while some reported higher levels of self-efficacy (and experience with) older technologies (e.g., TI-84 graphing calculator). A full analysis will be presented along with implications for future faculty development related to preparing to teach using mathematics and statistics technologies.

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SIMULACIONES DE REALIDAD MIXTA EN LA FORMACIÓN DE MAESTROS: ¿QUÉ CANTIDAD ES SUFICIENTE PARA QUE SEA EFECTIVO?

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El uso de simulaciones en la educación ha evolucionado de forma que ya se implementan herramientas para mejorar las habilidades de los maestros en formación (MF). Las simulaciones de realidad mixta (SRM) son tecnologías de simulación que sirven como plataforma para brindar a los MF oportunidades para desarrollar estrategias de enseñanza, en particular, aquellas relacionadas con cuestionar, evaluar y comprender el pensamiento de los estudiantes. Especialmente, para desarrollar acciones productivas en el aula de matemática [APAM] (Chapin, et al., 2009). En este estudio, mostramos los avances obtenidos en nuestra investigación en donde se examina la efectividad del uso de las SRMs para preparar maestros de primaria en formación (MPF), y como ésta sirve para desarrollar sus habilidades sobre el uso de APAMs. Así, se reponden las preguntas: ¿En qué medida la exposición de los MPFs a SRMs mejora su aprendizaje, uso, e implementación de los APAMs, en comparación con otros MPFs no expuestos a la simulaciones? (ii). ¿Puede el número de sesiones SRMs ser un factor determinante para el desarrollo del uso de APAM? Se adoptó un enfoque cualitativo (Mills & Birks, 2014) considerando un proceso de codificación deductivo (Miles et al., 2014). Participaron 40 MPF, divididos en 2 secciones, en un curso de métodos matemáticos, donde el 93% eran mujeres y el 6% hombres. Un grupo fue considerado el grupo control y el otro el experimental o de tratamiento (fue expuesto a SRMs). Todos los MPF recibieron la misma preparación en el salón de clases. El grupo experimental recibió adicionalmente tres sesiones de entre 6-10 min. de SRM. Los MPF debían realizar una entrevista clínica con un alumno de primaria utilizando ejercicios de matemática de instrucción guiada cognitivamente (Carpenter et al., 2014). De estas entrevistas se analizaron, compararon y contrastaron las transcripciones de las interacciones entre los MPF y los alumnos de primaria. Los MPF expuestos a las simulaciones mostraron implementar un 55% más de acciones productivas en el aula de matemática en comparación los MPF no expuesto a las SRM. También mostraron estar más abiertos a construir una buena relación con su estudiante de primaria, lo cual es importante cuando se busca involucrar a los estudiantes en un tema o discusiones. En una primera etapa de este estudio realizado por el Autor (2018), se siguió una metodología similar, con la diferencia de que el grupo experimental sólo fue expuesto a una sola SRM. Al comparar la primera etapa con la segunda que se presenta aquí (3 SRM) se pudo encontrar evidencia que el número de sesiones de SRM puede potencialmente influir en cómo los MPF, aprenden e implementan acciones productivas en el aula de matemáticas al realizar una entrevista clínica. Sin duda, el uso de SRM representa, como se muestra en esta propuesta, una alternativa para que durante los primeros años en los programas de formación de maestros, se puedan ofrecer experiencias que se acerquen lo más posible la práctica pedagógica en el salón de clases, sin tener que esperar hasta que los MPF se encuentres en los últimos anos donde realizan sus practicas profesionales como docentes.

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MIXED-REALITY SIMULATION IN TEACHER PREPARATION: HOW MUCH IS ENOUGH?

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The use of simulation technologies in education has been evolving in recent years and has made it possible for the implementation of technological tools to enhance pre-service teachers' skills. Mixed-reality simulations (MRSs) are a simulation technology tool that serves as a platform to provide pre-service teachers opportunities to develop high-leverage practices, in particular, those related to questioning, assessing, and eliciting students' thinking. In particular, to develop productive mathematical talk moves [PMTM] (Chapin, et al., 2009). In this study we depict results of an on-going research that examines the effectiveness of using MRSs as an alternative preparation tool to develop pre-service elementary mathematics teachers'(PEMT) ability to elicit evidence of students' mathematical knowledge and understanding through the use of productive mathematical talk moves. To this end, we answer the following question: (i) To what extent do PEMTs exposure to MRSs enhance their use of PMTMs, in comparison to other PEMTs not exposed to the simulation; (ii). Does the numbers of MRSs exposure improve PEMTs use of PMTMs to question and elicit elementary student's thoughts? In this study we took a qualitative approach (Mills & Birks, 2014) considering a deductive coding process (Miles et al., 2014). Participants were 40 elementary pre-service mathematics—divided in 2 sections—teachers taking a mathematics methods course during Fall 2019 from which 93% were females and 6% males. The elementary pre-service mathematics teachers were required as part of the course, to conduct a clinical interview with an elementary student. To prepare the elementary pre-service mathematics teachers for the assignment, 20 participants were exposed to Mixed-Reality simulations. They experienced three sessions that lasted between 8 -12 minutes each. The other half were trained in class, and both groups practiced the implantation of mathematical activities from the cognitively guided instruction framework of Carpenter et al., (2014). After the interview, the elementary pre-service mathematics provided the transcripts of their interaction with the elementary students. The transcripts were coded and analyzed following an adapted version of the framework of PMTMs. In general, the PEMT exposed to the state-of-the-art technology, have 55% more moves than the students that did not. They were also more open to build rapport with their elementary student, which is important when looking to engage students in a topic or discussions (Starcher, 2011). In a previous similar study conducted by Aguilar &

Telese (2018), they were comparing elementary pre-service mathematics teachers exposed only to a single MRSs. Although the treatment group that were prepared using MRS showed a tendency of using more moves, not all the moves were productive. In comparing with the current study (see table 3), it can be notices that the number of MRS exposure can potentially influence hw pre-service teachers use and implement productive mathematical talk moves when conducting a clinical interview. It can be inferenced that these students will also use these pedagogical strategies once they start their educational journey. It can be noticed that the use of Mixed-Reality Simulations during the teacher preparation program provide the elementary pre-service teachers an opportunity to practice teaching skills. However, we found evidence that the number of simulation exposure also play a roll when looking to develop and enhance the pre-service mathematical teacher's skills when fostering to assess, elicit, and questions their students in a formative approach.

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ELICITING KINDERGARTEN STUDENTS' MATHEMATICS WITH A CODING TOY: A PILOT STUDY ON DESIGN FEATURES

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Research is beginning to highlight the mathematics that young children demonstrate when using coding toys (e.g., Nam et al., 2019; Palmér, 2017; Shumway et al., 2021), yet little is known about how the design features of coding toys elicit children's mathematics. The purpose of this pilot study is to understand in what ways a coding toy affords mathematical ideas. The research question for this study is: *How do the design features of the Cubetto coding toy elicit kindergarten student's mathematics?*

Theoretical Framework

Research shows that mathematics-focused manipulatives (e.g., physical, virtual) afford mathematics learning (e.g., Bullock et al., 2017; Moyer-Packenham et al., 2019). Coding toys have blended characteristics of these other mathematics-focused manipulatives, so it could be hypothesized that design features of the coding toys also play an important role in how the toys support young children's mathematics. Design features in this study are the physical elements of the coding toy that offer some level of engagement to the user. Gibson's (1979) affordance theory frames our analysis of a coding toy's design features, their affordances, and the way they elicit or do not elicit kindergarten students' mathematics.

Methods

We conducted a constructivist group teaching experiment (Cobb & Steffe, 1983) with four small groups of 5-to-6-year-old students (15 total) programming the Cubetto coding toy to move to various points on a grid. Each group's 30-minute activity was video recorded (~120 minutes total) and qualitatively analyzed for ways Cubetto's design features elicited mathematics.

Results and Discussion

Literature suggests that children using digital mathematics games need to be aware of design features in order to take advantage of the potentially beneficial affordances of those features (Bullock et al., 2017; Moyer-Packenham et al., 2020). One finding of the current study was that students were unaware of the simultaneous linking features (e.g., blinking lights) because they were on a separate interface, which may have hindered students' access to the mathematics. The second important finding was that anthropomorphic design features (e.g., face on the side of the body) elicited spatial mathematical concepts. Implementing Cubetto activities, and specifically directing children's attention to the anthropomorphic features, may prove a valuable way to provide these spatial mathematical opportunities to young children. The final important finding was that the grid squares on Cubetto's mat elicited mathematical number concepts (e.g., counting on, verbal number use). This is important because it means that specific instructional strategies

Olanoff, D., Johnson, K., & Spitzer, S. (2021). *Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Philadelphia, PA.

should account for this alignment between grid squares and number concepts. Implications for instruction include prompting children to attend to Cubetto's face when learning how to use spatial rotation codes, or prompting children to attend to the grid squares when learning to count Cubetto's linear movements.

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NOTICING MATHEMATICS FROM MULTIPLE PERSPECTIVES

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A key aspect of professional noticing includes attending to students' mathematics (Jacobs et al., 2010). Initially, preservice teachers (PSTs) may attend to non-mathematics specific aspects of a classroom before attending to children's procedures and then, eventually their conceptual reasoning (Barnhart & van Es, 2015). Use of 360 videos has been observed to increase the likelihood that PSTs will attend to more mathematics-specific student actions. This is due to an increased perceptual capacity, or the capacity of a representation to convey what is perceivable in a scenario (Kosko et al., in press). A 360 camera records a classroom omnidirectionally, allowing PSTs viewing the video to look in any direction. Moreover, several 360 cameras can be used in a single room to allow the viewer to move from one point in the recorded classroom to another; defined by Zolfaghari et al., 2020 as *multi-perspective 360 video*. Although multi-perspective 360 has tremendous potential for immersion and presence (Gandolfi et al., 2021), we have not located empirical research clarifying whether or how this may affect PSTs' professional noticing. Rather, most published research focuses on the use of a single camera. Given the dearth of research, we explored PSTs' viewing of and teacher noticing related to a six-camera multi-perspective 360 video. We examined 22 early childhood PSTs' viewing of a 4th grade class using pattern blocks to find an equivalent fraction to 3/4. Towards the end of the video, one student suggested 8/12 as an equivalent fraction, but a peer claimed it was 9/12. The teacher prompts the peer to "prove it" and a brief discussion ensues before the video ends. After viewing the video, PSTs' written noticings were solicited and coded. In our initial analysis, we examined whether PSTs attended to students' fraction reasoning. Although many PSTs attended to whether 8/12 or 9/12 was the correct answer, only 7 of 22 attended to students' part-whole reasoning of the fractions. Next, we examined the variance in how frequently PSTs switched their camera perspective using the unalikeability statistic. Unalikeability (U_2) is a nonparametric measure of variance, ranging from 0 to 1, for nominal variables (Kader & Perry, 2007). Participants scores ranged from 0 to 0.80 (Median=0.47). We then compared participants' U_2 statistics for whether they attended (or not) to students mathematical reasoning in their written noticing. Findings revealed no statistically significant difference (U=38.5, p=0.316). On average, PSTs used 2-3 camera perspectives, and there was no observable benefit to using a higher number of cameras. These findings suggest that multiple perspectives may be useful for some, but not all PSTs'.

Acknowledgments

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STUDENT PERCEPTIONS FOR TECHNOLOGY LABS THAT INTEGRATE WEBWORK AND GEOGEBRA

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Keywords: Technology, WeBWorK, GeoGebra, Attitudes, Undergraduate Education

Introduction and Conceptual Perspective

Most university mathematics courses use homework to provide practice for topics and to evaluate student learning. Recently, mathematics scholars began designing cost free, innovative math resources for homework assignments that use cutting edge technology, for example WeBWorK and GeoGebra, to enhance student learning in undergraduate mathematics (May, Segal, Piercey, & Chen, 2020; The WeBWorK Project, 2021; SUMMIT-P, 2016). WeBWorK is an open-source on-line homework application supported by the Mathematical Association of America. GeoGebra is a dynamic math program that brings together geometry, algebra, spreadsheets, graphing, statistics, and calculus into one platform. Research points to positive students' perceptions for using WeBWork (Roth, Ivanchenko, & Record, 2008; Hauk & Segalla, 2013) and GeoGebra (Attard and Holmes, 2020; Radović, Radojičić, Veljković, & Marić, 2018) to complete homework and classwork. In a recent pilot study conducted by the proposal's authors at a large midwestern university, we found positive student perceptions for newly designed math co-requisite courses (three-credit course and one-credit technology lab) for business majors. For the one-credit technology lab, we also found that the interface and syntax of GeoGebra and the means for submitting technology lab assignments were not user-friendly and created unnecessary, unintended obstacles to student learning. To rectify this problem, a large project is underway to revise the curriculum for the co-requisite courses and to develop an interface that embeds GeoGebra applets into students' technology lab assignments (NewT) delivered in WeBWorK for the one-credit technology lab for business majors.

Research Question and Design and Data Collection

The research question for this proposal is the following: Is there a difference in students' perceptions for the NewT and the existing technology lab assignments (ExisT). This research project is an extension of the previously mentioned pilot study and larger project that called for the revision of curriculum and development of new technology lab assignments for the one-credit technology lab. By spring 2022, five NewT will be created to compare to the remaining ExisT. Approximately 160 students will be invited to complete a student perceptions survey and interviews. To compare student perceptions for NewT and ExisT, a paired-samples t-test will be used at alpha 0.05, where 40 students will be needed to ensure generalizable results. Five randomly selected students will be chosen to participate in a 30-minute interview comparing NewT and ExisT to gain a deeper understanding of students' perceptions. By week 6 of spring 2022, students will respond to survey questions about the five completed ExistT. During week 14, another survey will be given to compare student perceptions for the remaining NewT.

Data Analysis and Summary of Findings

Preliminary results for curriculum revisions for the co-requisite courses are positive. A student recently stated, "I am learning more from the technology lab than the three-credit course." We anticipate positive student perceptions for the NewT when compared to the ExisT.

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UNIVERSAL DESIGN OF A TIER 2 FRACTION VIDEO GAME

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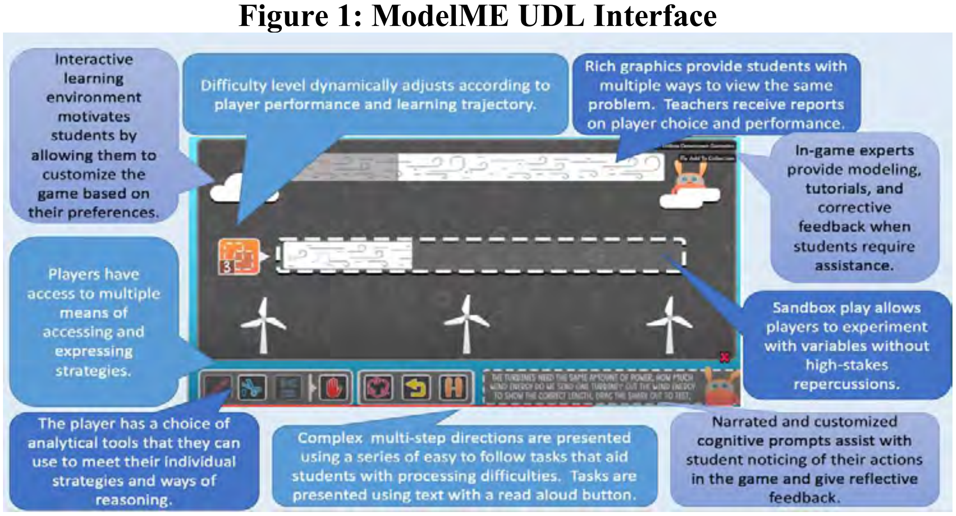
The 2017 National Assessment of Educational Progress highlights stagnant and declining performance in mathematics, foundational to STEM and ICT, in both 4th (17% proficient) and 8th (9% proficient) grades. At the same time, promoting diversity in the workforce is paramount for U.S. innovation in STEM and ICT fields. Individuals with disabilities are underutilized members of the STEM and ICT workforce. In this poster, we report on Model Mathematics Education (ModelME), a universally designed video game embedded into a student-centered Tier 2 (i.e., supplemental) mathematics curriculum.

Access, Advancement, and Power: Universal Design and Student Centered Instruction

Universal Design for Learning (UDL) addresses the need for students to access different tools to learn and express knowledge. Instruction is guided by three principles: (a) multiple means of engagement (i.e. considering how to engage students in multiple ways), (b) multiple means of representation (i.e. providing content in multiple formats), and (c) multiple means of action and expression (i.e. providing opportunities for students to demonstrate their understanding in multiple ways (see Figure 1).

Figure 1: ModelME UDL Interface



In session, we will demonstrate ModelME's use of student-centered design features and report on initial usability data with elementary school students.

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INTRODUCING FUNCTION TO MIDDLE SCHOOL STUDENTS: A NOVEL REPRESENTATION

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Introduction

The purpose of this study is to examine the effect of a specially designed applet on middle school students' ability to develop an understanding of the concept of function as a relationship between inputs and outputs with some rules about the outputs.

The *Introduction to Function* task is a series of interactive applets compiled in a GeoGebra book that consists of seven pages and an accompanying worksheet. On the first two pages are two vending machines each of which consists of four buttons (Red Cola, Diet Blue, Silver Mist, and Green Dew). When a button is clicked it produces none, one, or more than one of the four different colored cans (red, blue, silver, and green), which may or may not correspond to the color of the button pressed (see Figure 1). The students are told that the first machine on each page is an example of something called a function, and the other is not a function, with their task being to identify what is the difference between the behaviour of the machines that makes one a function and the other not.

Participants

The *Introduction to Function* applet was used in fifteen seventh grade classrooms. These classrooms were across two different states (one Northeastern state and one Southeastern state) and five different teachers for a total of 138 students who engaged with the task. These students engaged with the applet towards the end of their seventh grade year and had not yet learned about the definition of function or function notation.

Results

Participants identified Machines correctly as follows: Machines E & F: 81.3%. Machines G & H: 95.8%. Machines I & J: 86.1%. Machines K & L: 80.7%. At a first level of analysis this shows that, broadly speaking, the pairs of students were able to correctly identify which machines were functions. In a qualitative analysis of the video recordings it was seen that the main challenge to correct identification of functions is the real-world context of a vending machine and the attendant challenge of accepting an output that doesn't match the input button in colour even when it does so consistently. Also, within this study, contrary to a well-known misconception, participants may be able to recognise a constant function as a function. Finally, it should be noted that the purpose of this activity was to set the scene for a class discussion of their definitions with the goal of arriving at a shared definition. The results of the study suggest that there is a good foundation for that discussion.

Acknowledgments

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MIDDLE SCHOOL MATHEMATICS TEACHERS' COLLOQUIAL EVALUATIONS OF DIGITAL MATHEMATICS RESOURCES

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Keywords: Digital Resources, Middle School, Professional Development

This study aims to explore the benchmarks used by middle school mathematics teachers to assess the quality of digital mathematics resources for use in their classrooms. Although calls have been made to districts, teachers, and teacher educators to be cognizant of the quality of digital resources, along with the intended learning and sequencing of their use (NCTM, 2016), little attention to how stakeholders should address such issues at the level of digital resources. While many evaluative tools (such as rubrics) exist, their goals and audience are varied. In an attempt to explore what contributes to the effective evaluation of digital resources as part of an integrated mathematics classroom, the following overarching research question *What guides middle-school mathematics teachers' colloquial evaluation (CE) of digital mathematics resources?* was explored.

A total of 33 mathematics teachers, a year-long professional development (PD) participants participated in the study, consisting of 4-5 math teachers and a building's mathematics coach from the same building —a strategy to further support a collaborative experience with coaching and professional support (Darling-Hammond et al., 2017). While many other digital resources were used during the PD, teachers' CEs on the following four digital resources were analyzed: Interpreting Stories and Graphs (PBS LearningMedia, 2012), Exploring Patterns: It's a Bit Nutty (Learn Alberta, 2003), On Your Mark (Mathalicious, 2015), Algebra Tiles (NCTM, 2015). A qualitative methods approach was used to analyze teachers' evaluations of assigned digital resources. The approach stems grounded theory, "in which the researcher derives a general, abstract theory of a process, action, or interaction grounded in the views of participants" (Creswell & Creswell, 2018, p. 14).

Upon analysis of the teachers' evaluations, thirty characteristics emerged from teachers comments and were grouped into seven larger criteria. Close to half of all CE comments were related to the *digital resource features* and *learning experience* criteria. The nature of teachers' comments confirmed their perception of the digital resource in the large majority of teacher CEs. The comparison of positively and negatively viewed digital resources may also bring to light another issue in identifying the underlying reasons that some resources fared better than others. That is, the variation in the frequency of the characteristics, criteria, dimension, or other construct-related coding mechanisms could be attributed, in part, to the digital mathematics resources that were used.

The lack of an appropriate evaluation tool with respect to being teacher-friendly are essential to supporting the successful implementation of digital mathematics resources into the middle-school, mathematics classroom. While digital content evaluation is key to determining the quality of a resource and making a choice with respect to implementation, teachers often rely on peers or a quick glance at a resource. This limited judgement of the quality of the resource coupled with evaluation tools that are too robust or difficult to use and the time it costs teachers, evaluation of digital resources takes a back seat to other factors that teachers focus on in preparation for their classes.

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LEARNING MATHEMATICS WITH DYNAMIC REPRESENTATIONS: AN APPLICATION OF VARIATION THEORY

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Keywords: Mathematical Representations, Technology

A general premise of school mathematics is that through engagement with some activity, students will learn. Planning activities that are likely to lead to the intended learning outcomes is a persistent problem of practice for educators at all levels. Variation theory (Marton, 2015) can be used to analyze the design of learning activities while keeping the learning goal at the center. Studies of mathematics learning have provided compelling evidence that dynamic representations (DRs), instantiated through digital media, can support students in developing rich conceptual understanding (Roschelle, et al., 2017). On the other hand, it is clear that simply including DRs in learning activities does not always create favorable conditions for student learning. In this poster, I seek to answer the question: *What does variation theory make visible about the learning opportunities provided by dynamic representations?*

As mathematics curricula are increasingly made available and implemented through digital platforms (Remillard & Reinke, 2017), students are being provided with more opportunities to learn through the use of dynamic representational tools. In 2020, Desmos published a middle school curriculum designed to "enhance the top-rated curriculum from Illustrative Mathematics and Open Up Resources" (Desmos, n.d.). By analyzing lessons from the Desmos middle school curriculum, which was specifically designed to leverage DRas, I investigate the role of DRs in making mathematics concepts available to students.

Dynamic representations create opportunities for students to "kinesthetically and intellectually interact with the designers' construction of [mathematical entities]" (Edwards, 1998, p. 74). This interaction occurs through a series of student-initiated actions followed by "interpretable feedback" (Edwards, 1998) which is automatically generated as a result of the way the DR has been designed and programmed. Discrepancies between what an activity designer intends for students to learn and what they actually learn can occur when students do not act on the DR in ways that bring about a necessary condition.

I apply Marton's (2015) variation framework to describe the learning activities that comprise a lesson in terms of variation or invariance of critical aspects of the object of learning. Marton's theory rests on the assumption that a necessary condition of learning is the ability to discern that which is to be learned, and that the ability to discern something relies on opportunities to experience variation and invariance across all relevant aspects of the object of learning. The critical aspects of an object of learning are the aspects of the mathematical content that need to be learned in order to meet the educational objective. Variation theory centers the mathematical learning goal(s) of a lesson and provides a framework for describing the learning activity as designed and as experienced by students in commensurable terms. In this study, I combine this curricular lens with an analysis of how students must interact with the DRs in order to make the critical aspects of the object of learning visible.

This poster will illustrate the application of variation theory to the analysis of lessons within a representationally rich curriculum. A preliminary finding is that due to the individualized and exploratory nature of DRs, students' interactions with representational tools have implications for whether or not they experience variation in critical aspects of the object of learning.

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Chapter 14:

Theory & Research Methods

Olanoff, D., Johnson, K., & Spitzer, S. (2021). *Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Philadelphia, PA.

IDENTIFYING GRAPHICAL FORMS USED BY STUDENTS IN CREATING AND INTERPRETING GRAPHS

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In this paper, we describe a framework for characterizing students' graphical reasoning, focusing on providing an empirically-based list of students' graphical resources. The graphical forms framework builds on the knowledge-in-pieces perspective of cognitive structure to describe the intuitive ideas, called "graphical forms", that are activated and used to interpret and construct graphs. In this study, we expand on the current knowledge base related to the specific graphical forms used by students. Based on data involving pairs of students interpreting and constructing graphs we present a list of empirically documented graphical forms and organize them according to similarity. We end with implications regarding graphical forms' utility in understanding how students construct graphical meanings and how instructors can support students in graphical reasoning.

Keywords: High School; University Math; Cognition; STEM/STEAM; Interdisciplinary studies

Interpreting and constructing graphs that model mathematical or physical contexts is a critical competency across disciplinary fields (Driver et al., 1996; National Council of Teachers of Mathematics, 2000; National Research Council, 2012). While much previous work has examined student difficulties and non-normative reasoning related to graphing (Beichner, 1994; Glazer, 2011; Leinhardt et al., 1990; McDermott et al., 1987; Shah & Hoeffner, 2001), more work is needed that leverages students' knowledge related to creating and interpreting graphs. A new framework has recently been developed that identifies specific types of knowledge resources called *graphical forms*, that permits a finer-grained examination of how students think or reason about graphs (Rodriguez et al., 2019b). The purpose of this paper is to extend the work on graphical forms by empirically documenting and organizing a large set of graphical forms that students used to create or interpret graphs. This work permits researchers greater clarity on the cognitive work involved in constructing and interpreting graphs, and helps instructors know what types of knowledge students can develop or use for productive graphical activity.

Brief Literature Review on Student Graphical Thinking

Past research on graphical thinking has documented students' difficulties (e.g., Beichner, 1994; McDermott et al., 1987), with the consensus being that students' ability to interpret graphs depends on interaction between students' prior knowledge and the nature and content of the graphing task (Glazer, 2011; Leinhardt et al., 1990; Shah & Hoeffner, 2001). Some work has emphasized the nature of assumptions and conventions associated with graphical interpretation (Moore et al., 2019), including work that described students' use of *intuitive rules* to interpret graphs (Eshach, 2014). According to Eshach (2014), students develop a set of *intuitive rules* that share a similar ontology to diSessa's (1993) *phenomenological primitives* (discussed in more detail later) in the sense that they are constructed based on experiences. However, *intuitive rules* are more broadly useful and are not specific to explaining a physical phenomenon. This approach to considering how students interpret graphs is insightful in the way it provides explanatory power for students' reasoning that moves beyond identification of misconceptions (Beichner, 1994; Elby, 2000; McDermott et al., 1987).

Much of the literature indicates the role context plays in students' ability to extract information from graphical representations. For example, students tend to perform better when presented with decontextualized graphs in comparison to analogous graphs involving chemistry or physics content (Bollen et al., 2016; Ivanjek et al., 2016; Planinic et al., 2012, 2013; Potgieter et al., 2008). To examine context-specific graphs, recent work by the first author and colleagues has focused on students' graphical reasoning in chemistry, specifically in the context of chemical kinetics (Rodriguez et al., 2018, 2019a, 2019b, 2019c, 2019d, 2020a), which is concerned with modeling the rate of chemical reactions. A limited number of knowledge resources, called graphical forms, have been discussed in these studies, including *steepness as rate*, *straight means constant*, and *curve means change.* In some cases, graphical forms such as *steepness as rate* seem to have a particularly strong cuing priority, which, in part, could be influenced by students' tendency to inappropriately force time onto expressions and graphical representations that do not include time as a variable (Bowen et al., 1999; delMas et al., 2005; Jones, 2017; Popova & Bretz, 2018; Rodriguez et al., 2019d, 2020a, 2020b). In this paper, we build on this work by presenting several graphical forms empirically observed in students' graphical reasoning.

Theoretical Perspective: Graphical Forms

Knowledge-in-Pieces & Symbolic Forms

The construct of graphical forms is rooted in the knowledge-in-pieces (KiP) paradigm, a cognitive model that characterizes the structure of knowledge and the mechanism associated with conceptual change (diSessa, 1993). The salient feature of the KiP view is the manifold ontology of cognitive structure, in which knowledge is conceptualized as a network of fine-grained cognitive units that are activated in concert because of perceptual cuing. These cognitive units, which we call *knowledge elements* and *resources* interchangeably (see also Hammer, 2000), may reflect a variety of types of knowledge, such as ideas related to concepts, epistemology, or ontology. Building within the KiP paradigm, Sherin (2001) introduced the "symbolic forms" framework to describe mathematical resources related to symbolic equations. According to Sherin (2001), this involves associating an idea (conceptual schema) to a pattern in an equation (symbol template). Based on the introductory physics (classical mechanics) context in which the symbolic forms framework was initially developed, the symbolic forms characterized by Sherin (2001) reflected ideas associated with algebraic manipulations such as combining terms, proportional reasoning, and the role of a coefficient in scaling or tuning an expression.

Graphical Forms

The graphical forms framework reflects a natural extension of symbolic forms, providing the language to further characterize students' mathematical resources. Like symbolic forms, reasoning involving graphical forms is characterized by focusing on a structural feature and subsequently associating an idea (Rodriguez et al., 2019b). Whereas the symbolic forms framework focuses on the ideas assigned to patterns in equations, the graphical forms framework augments this work by emphasizing the ideas assigned to patterns in a *graph*. Previously, the specific feature attended to in a representation has been framed as a *registration* (Lee & Sherin, 2006; Roschelle, 1991), which in the context of graphical reasoning can vary in size—an individual may attend to and associate an idea with the entire graph or a specific region of the graph (Rodriguez et al., 2019b).

Although these resources may be activated and applied in less useful contexts, it is important to acknowledge that students have these broadly useful cognitive tools for reasoning that have the potential to guide students in the sensemaking process. Therefore, consistent with the knowledge-in-pieces perspective, research and instruction should emphasize providing insight regarding how we can support students in productively using the resources they have, rather than focusing only on cataloging misconceptions (Cooper & Stowe, 2018). Students seem to commonly draw on graphical forms such as *steepness as rate*, which can result in sophisticated conclusions regarding physical processes. In the context of interpreting graphs, this often involves initially anchoring reasoning in mathematics by drawing inferences using graphical forms and subsequently assigning discipline-specific principles to explain the observed graphical shape (Bain et al., 2019; Rodriguez et al., 2019, 2019a, 2019b, 2019c). In the case of constructing graphs, the reverse is observed in which students consider the physical scenario and subsequently utilize graphical forms as part of the drawing process to create a graphical shape that aligns with the phenomena (Rodriguez et al., 2020a).

The goal of this study is to begin to develop an empirical library of graphical forms, mirroring the current list available for symbolic forms (Rodriguez et al., 2019b); necessitating a clear definition of what constitutes a graphical form. As part of this process, we drew on extant education research related to graphical reasoning and Sherin's (2001) description of symbolic forms to consider the implications for the graphical analog. First, we draw attention to the idea that symbolic forms focused more on meaning than conventions. Second, symbolic forms emphasized the information communicated by an equation, without drawing an explicit connection to what an equation fundamentally *is* in an ontological sense. Moreover, to narrow the scope of the framework we decided to define graphical forms as assigning meaning to the curve itself, as opposed to other aspects of a graph such as the axes and graph labels (Kosslyn, 1989). In summary, our definition of a graphical form was refined to consist of a specific aspect of the graphical curve itself (e.g., a *graphical pattern*) and an intuitive *conceptual schema* associated with that aspect. Thus, our definition excludes beliefs about the nature of the graph, knowledge elements associated with the axes, or general knowledge about functions.

Methods

This paper reports on one set of outcomes from a larger study on students' graphical activity in relation to real-world contexts. In the study, twelve students across two universities at the beginning of first-semester calculus were recruited to participate in two separate interviews that occurred within a one-week timespan. One interview focused on constructing graphs that model real-world situations and the other interview focused on interpreting graphs. For space constraints, we do not present all nine tasks here, but have provided in Figure 1 one graph construction and one graph interpretation prompt that we drawn on in the Results section. The students were interviewed in pairs, and are given the pseudonyms Anna and Aria, Berto and Blaine, Cindy and Caleb, Donato and Demyan, Ellie and Eric, and Fiona and Felicity.

(A) Graph Construction

A homeowner mows the lawn once a week on Wednesday afternoon for 4 weeks in a row. Then the mower breaks and he decides not to mow the lawn for the rest of the summer. Graph the height of the grass as a function of time throughout the summer.

(B) Graph Interpretation

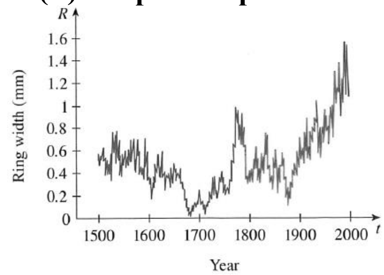


Figure 1: Prompts discussed in this paper.

Following transcription, initial data analysis involved dividing the interviews into bounded episodes based on content discussed to establish a codable unit (Campbell et al., 2013) and providing a narrative general overview of the student discussion within the episode (i.e., narrative coding) (Heisterkamp & Talanquer, 2015; Rodriguez et al., 2020b). Subsequently, we used a line-by-line analysis to analyze each statement within the episodes, focusing on the resources implied by what the student said—and did as they made the statement—also considering the context surrounding the statement, including nonverbal cues such as gestures. The process of identifying graphical resources involved a combination of deductive (previously identified graphical forms from the literature) and inductive analysis (identifying new graphical forms and other graphical resources). To refine our definition of graphical forms, we discussed together the various graphical resources we documented, which involved combining codes and creating new codes, some of which were determined to constitute graphical forms and others which were characterized more generally as "other" resources related to graphing.

Results and Discussion

We begin by providing examples of graphical forms observed in the data that have previously been identified in the extant literature. We then discuss new graphical forms identified and provide an overview list with the various graphical forms identified in the data.

Previously Identified Graphical Forms

Across the dataset various graphical forms were identified, some of which were previously discussed in the literature, such as *steepness as rate,* which involves students associating ideas about rate with the relative steepness of the graph (Rodriguez et al., 2018, 2019a, 2019b, 2019c, 2020a, 2020b). Given that this graphical form has been discussed in detail in previous work, we will not focus too much on it here, except to say that it was the one of the most frequent graphical form observed in the dataset, further building a case for its relatively high cuing priority, phenomenological basis, and its important role in graphical reasoning. For some of the previously identified graphical forms, as part of the process of developing a list, we also built on the prior descriptions, such as modifying *straight means constant* in favor of the more precise language *straight means constant rate*. This was to specify that students were focusing on *rate* as opposed to *values*. To illustrate this, one of the graph creation prompts involved a scenario related to modeling the height of grass over time (Figure 1A). When working through this prompt, Blaine and Berto initially drew the graph provided in Figure 2A, with Blaine describing the straight lines they drew as follows:

> **Blaine:** *So grass grows, um, it grows at a pretty constant rate, and you cut it every once, one or two weeks in the summer, at least where I live. Um, and then you would cut it.*

In this instance, Blaine's reasoning can be characterized as *straight means constant rate*, due to the emphasis on rate. Similar to *steepness as rate,* associations such as Blaine's above that drew a connection between straight lines and a constant rate were frequently observed in the dataset. Revisiting the distinction between specifying what is constant when describing a straight line, later in the interview Berto and Blaine drew a plateau as part of their graph (Figure 2B):

Interviewer: *... what do those horizontal flat points represent to you again?*
Blaine: *No growth.*

Here, Blaine is no longer referring to *rate* being constant, but rather the *height* of the grass being constant ("not growing"), indicated by the horizontal straight line (*horizontal as constant value*).

(A) Berto & Blaine **(B) Berto & Blaine**

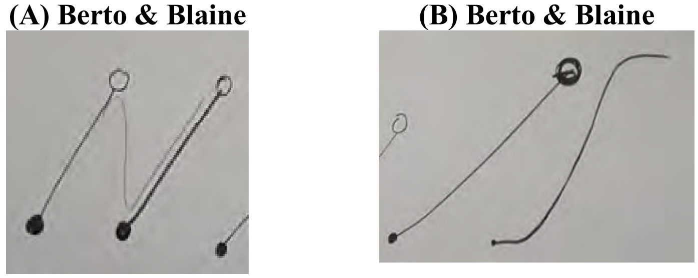


Figure 2: Two Types of "Straight": Linear-Straight as Constant Rate (A) and Horizontal-Straight as Constant Value (B).

New Graphical Forms

Although there is not space to provide student examples of all the new graphical forms identified, based on the contexts associated with the graphs provided, we discuss some of the graphical forms that emerged from the data that have not yet been discussed in the literature. For example, the nature of the grass prompt discussed previously (Figure 1A) resulted in students discussing ideas related to discontinuity, which we characterized using the graphical form *jump discontinuity means sudden*. Here, the graphical pattern of a jump/break in the graph was intuitively associated with a sudden event, such as when cutting the lawn results in a sudden decrease in height. As with other graphical forms, the name selected is intended to be descriptive for ease of communication and presentation. Another example of a new graphical form associated with the prompt in Figure 1A is *open/closed dot pair as existence*, as exemplified by Dontao and Demyan:

Demyan: *... I want to show that the height is like, this is something continual, like grass didn't like stop, uh, existing there [i.e., at the discontinuity] for like a very split microsecond while it was cut…*
Donato: *Yeah. I think that, in that case, you would do the like open circle here, closed circle there, but yeah, again, I don't think that happens-*
Demyan: *And if that is what, what, what counts, like if that makes it clear in mathematical terms that the grass is still around, it's just, you know, cut from the edge or from the bottom at three inches, then yeah, I'm down for that change.*

The graph drawn by the students did not initially have open and solid dots (only slanted lines), which bothered Demyan because it seemed to imply that the grass was no longer there because the graph was not connected (continuous). After discussing it with one another, they adopted the solid-open dot notation utilized by other students in the sample (Figure 2) to express the concept of existence at a particular point.

When analyzing student responses to the construction and interpretation prompts, we also noted graphical forms related to how features in the graph suggest realism or indicate the graph involves empirical data. For example, revisiting the grass prompt with Aria and Anna:

Interviewer: *And what does that mean that it's like a straight line segment and then a straight line segment, like as opposed to like a curve?...*
Anna: *Then we would assume it just grows at a constant rate over time, but I guess that's not true either. Cause there's a lot of factors that can affect the growing that isn't just time.*
Aria: *Yeah. Like bugs.*
Anna: *Yeah. But we're not looking at that. We're literally just looking at if grass grew in terms of time and not in terms of other things. So realistically that's probably not what it looks like. It probably is more gradual because of other factors.*

Here, the students above, as well as other students for multiple prompts, were hesitant to draw straight lines because that implies a direct linear relationship between the variables. We characterize this reasoning as *curves mean realistic*, in which students opted to draw curved lines to account for unknown factors and avoid making assumptions about the relationship between the variables. Moreover, this graphical form was complemented with *jagged means data*, which is related to *curves mean realistic* in the sense that a jagged graph with multiple sporadic increasing and decreasing regions is far from an "ideal" and "clean" linear plot. This idea was observed when students were asked to interpret the graph provided in Figure 1B:

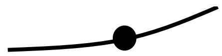
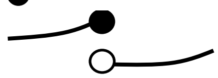
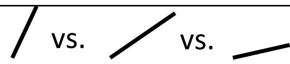
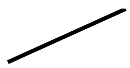
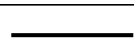
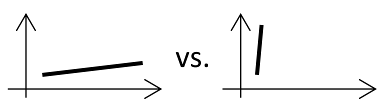
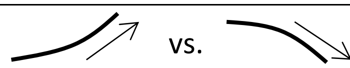
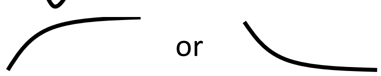
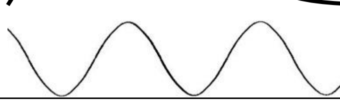
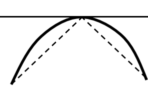
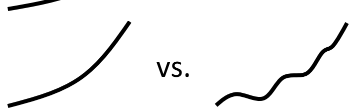
Sally: *Well, it's varying changes [Figure 1B]. It's not I guess constant in a way.*
Samuel: *Yeah, it's not like a smooth function, it's staggered in a way, I guess.*
Interviewer: *What do you mean by staggered?*
Samuel: *It was drawn like, like that [draws a graph with rigid lines]. ... I feel like it's just data plotted on the graph.*
Sally: *Yeah, and it's more abrupt, I guess.*

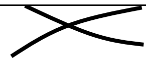
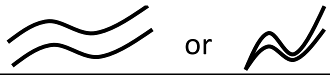
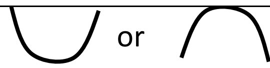
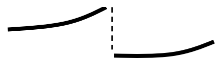
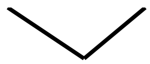
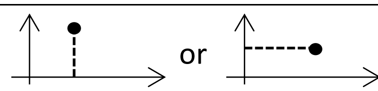
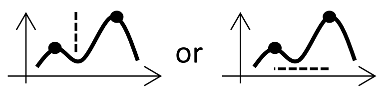
For the students, the jagged nature of the graph indicates the plot involves empirical, collected data. Combined with *curves mean realistic*, *jagged means data* reflects a productive idea for thinking about the relationship between variables and what is expected when collecting data.

List of Empirically Identified Graphical Forms

Having discussed a few graphical forms in detail in the previous section, in this section, we now present the various graphical forms observed from our students as they created or interpreted graphs (Table 1). We have organized the graphical forms into "clusters" in terms of which forms deal with similar aspects of a graph, such as points or slopes.

Table 1: List of graphical forms, organized into clusters of related graphical aspects (e.g., patterns)

Graphical form	Graphical pattern	Conceptual schema
Point Cluster		
1. Point as instance		A single point on a curve is a single instance
2. Big dot as focal point		A large dot indicates a special instance or event
3. Connecting as transition		Connecting dots transitions from one instance/event to another
4. Open/closed dot pair as existence		The closed dot defines "exists", open dot defines "nonexistence"
Slopes Cluster		
1. Steepness as rate		The steepness of the graph indicates the rate of change
2. Straight means constant rate		A straight line indicates the rate is constant
3. Curve means changing rate		A curving graph indicates the rate is changing
Cardinal Direction Cluster		
1. Horizontal as constant value		Horizontal implies a constant ("*y*") value
2. Vertical as constant value		Vertical implies a constant ("*x*") value
3. Vertical as simultaneous		Vertical means simultaneous "*y*" values at one "*x*" value
4. Running along axis		The more parallel the graph is to one axis implies more change in that axis' variable
Global Trend Cluster		
1. Shape directionality		Up-right means increasing and down-right means decreasing
2. Wavy means variation		Lots of up/down implies a lot of variation
3. Plateau as levelling off		Plateauing means variable is "levelling off" to a stable value
4. Periodic means repeated		A periodic graph means a repeating situation
Smoothness Cluster		
1. Straight lines as idealized		Straight lines give rough approximation of graph segments
2. Curved means realistic		Curved graphs are more "realistic" for real-world quantities
3. Smoothness as strength of relationship		A smoother graph implies a stronger relationship between "*x*" and "*y*" (and vice versa)
4. Jagged implies data		A jagged graph implies it depicts real-world data
Two Graphs Cluster		

1. Intersection means same		Intersection means same values at that instant
2. Transformation as same		Shifting or stretching does not change the basic *x-y* relationship
Local Feature Cluster		
1. "U" as max/min		Top of a hill or bottom of a valley is a max or min
2. Jump discontinuity means sudden		Jump discontinuity implies a very sudden change
3. Cusp as event		A cusp implies an event occurred at that instance
Position Cluster		
1. Distance as value (horizontal or vertical)		The distance from an axis indicates the other variable's value
2. Displacement as difference (horizontal or vertical)		The displacement between points indicates the difference in values

Implications and Conclusion

Building on Sherin's (2001) work related to exploring how students use knowledge resources to reason about equations, our work has detailed many resources students used to reason about graphs. It is important to note that context likely plays an important role in the activation of these graphical forms (diSessa et al., 2016; Elby, 2000; Hammer et al., 2005). Thus, if students were provided different graphical shapes or alternative coordinate systems, we would likely observe additional graphical forms. In this way we do not claim Table 1 to be an exhaustive list of all graphical forms, but we do believe it represents many important forms. Further, additional types of graphing knowledge resources exist that do not fit the strict definition of graphical forms. For example, we also saw students use knowledge about the axes or functions in creating or interpreting graphs. Beliefs about the nature of graphs also were important resources the students drew on (see Hammer et al., 2005; Hammer & Elby, 2003 for more on belief resources). However, the point of this work is to better understand the conceptual schemas coupled with specific graphical patterns (such as steepness or points) that students used in both creating and interpreting graphs. Future work will unpack the additional resources we observed and how graphical forms and these other types of resources worked in concert when students created graphs or interpreted graphs.

Our work has important theoretical and pedagogical implications. Theoretically, we have extended the initial work on graphical forms (Rodriguez et al., 2018, 2019a, 2019b, 2019c, 2020a, 2020b) to an identification of a large set of graphical forms. Such identification allows researchers to see finer-grained aspects of student reasoning when creating or interpreting graphs. It can also help researchers code for these specific knowledge resources when studying students' graphical activity, or in examining how or when specific resources might be used. Pedagogically, our work is useful for instructors in identifying knowledge they may wish to help their students develop or to draw on during in-class graphical activity. It also helps instructors gain insight into the thinking their students might be doing in-the-moment as they interpret graphs or model a situation with a graph. Lastly, our results are important in demonstrating productive knowledge resources that students have and can use to create or interpret graphs. In other words, our work helps show what students can bring to problem-solving tasks in terms of graphical reasoning, rather than focusing on what they lack.

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OPPORTUNITIES FOR MATHEMATICS ENGAGEMENT IN SECONDARY TEACHERS' PRACTICE: VALIDATING AN OBSERVATION TOOL

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The purpose of this report is to present our process and results for establishing validity and reliability of an observation tool used to investigate teaching practices that high school mathematics teachers use to engage students. We developed our tool using established practices, such as reviewing literature to develop a framework for instruction and piloting the tool to design descriptive levels for rubrics. After validating externally by consulting experts, additional rubrics regarding teaching mathematics for equity were added to the tool. We conducted a reliability study of 149 episodes of classroom instruction (equivalent to 447 10-minute segments of instruction in all), two raters per episode, to investigate the nature of coding disagreements. Most disagreements occurred due to raters noticing different evidence rather than different interpretations of rubrics, which suggested the value of two raters and resolution meetings.

Keywords: Research Methods; Instructional Activities and Practices; High School Education; Affect, Emotion, Beliefs, and Attitudes

A range of observation tools exist to support the study of mathematics instruction that supports students' learning of mathematics (e.g., Bostic et al., 2019; Boston, 2012; Hill et al., 2012; Sawada et al., 2002; Walkowiak et al., 2014). These tools enable researchers to compare teaching practices, such as features of classroom discourse, that align with frameworks for quality instruction. Although these observation tools are well-established and validated, they focus primarily on behaviors that can explain students' learning, such as mathematical task enactment; they do not investigate how mathematics teaching influences students' engagement.

It is important to identify teaching practices that can motivate and engage students in mathematics classrooms, particularly in secondary grades. It has been well documented that students' mathematics engagement decreases over time as they move through levels of education into high school (e.g., Collie et al., 2019). Students' self-efficacy, enjoyment, and sense of the utility of mathematics tends to decrease as they move from elementary school into junior high (Wigfield et al., 1991); this trend continues through high school (Chouinard & Roy, 2008). However, students' motivation and engagement is socially situated and influenced by teachers' instructional practices in the moment (Anderson et al., 2004; Shernoff et al., 2017), so it is important to investigate teaching that supports engagement. The purpose of this paper is to describe the validity and reliability of the observation tool that we developed for the SMiLES project [Secondary Mathematics in-the-moment Longitudinal Engagement Study] to investigate how secondary mathematics teaching may impact students' engagement.

Potentially Engaging Mathematics Instructional Practices

For students to learn mathematics, they must be engaged. We conceptualize engagement in mathematics classrooms as a person's cognitive, affective, behavioral, or social investment in a pedagogically relevant object, such a mathematics task or lesson, as situated in the relationship between the self, the object of engagement, and others in the environment (Middleton, Jansen, & Goldin, 2017). In a study of almost 4,000 middle school and high school students in Western Pennsylvania, higher levels of cognitive, behavioral, emotional, and social engagement predicted students' course grades in mathematics (Wang et al., 2016). According to Greene (2015), it is well-established in prior research that motivation constructs such as students' self-efficacy support students' engagement in ways that lead to learning.

Instruction is likely to support students' engagement when teachers provide students with both *social support* for working together on content and *academic support* for accessing rigorous mathematical content (Shernoff et. al., 2016). Such support can take a variety of forms. *Academic support* may include opportunities for sense-making and reasoning (Stein et al., 1996); opportunities to make conceptual connections (Hiebert & Lefevre, 1986); pressing students to explain their thinking (Engle & Conant, 2002; Kazemi & Stipek, 2001); providing students with specific and detailed feedback (Stipek et al., 1998), opportunities to solve mathematics tasks in context (Koedinger & Nathan, 2004); or some combination of these. *Social support* may include motivational discourse with a focus on learning, positive affect, and encouragement of collaboration with peers (Turner et al., 2002); positioning students as competent (Cohen & Lotan, 1995; Gresalfi et al., 2009); accountability practices in the classroom (Horn, 2017); providing opportunities for student-to-student discourse in whole class discussions (Nathan & Knuth, 2003) or small groups (Fuentes, 2018) in ways that maintain mathematical quality; attention to students' lives outside of school (Yamauchi et al., 2005); or some combination of these teaching practices. Whether these supports can foster students' mathematical engagement remains at the level of conjecture, and an observation tool could explore this conjecture.

Development Process and Use of our Observation Tool

The SMiLES project's observation tool measures the extent to which potentially engaging teaching practices are present in a lesson. The tool does not establish whether instruction was engaging for students. Student engagement in observed lessons was assessed by an in-the-moment student survey using Experience Sampling Methodology (Jansen et al., 2019; Schiefele & Csikszentmihalyi, 1995).

The final version of the tool includes fifteen rubrics to assess eight dimensions of academic support and seven dimensions of social support. Rubrics designed for academic support measured students' opportunities for sense making and reasoning, connections between representations or strategies, pressing students to explain, contexts of tasks, mathematical correctness, mathematics language use, feedback, and students' opportunities for agency and autonomy. Social support rubrics assessed whole class discourse, small group discourse, status raising and positioning students as competent, motivational discourse, enthusiasm about mathematics, attention to students' lives, and accountability and high expectations. We defined each dimension with descriptive levels. Each dimension was scored on a four-point rubric with points (0-3) assigned to index each level: absence or the opposite of ideal enactment (0), weak level of enactment (1), moderate level of enactment (2), and strong level of enactment (3). Each rubric included a definition of the teaching practice, and we defined the observable indicators for each level of enactment. We share an example observation rubric below in Figure 1.

Social Support 6: Attention to Students' Lives

This rubric captures the degree to which the teacher attempts to connect with students' lives while teaching.

Strongly Present (3)	The teacher speaks about more than one example of cultural events or outside of school events during instruction OR talks with multiple students about aspects of their lives outside of school or mathematics class in ways that are *incorporated into instruction.*
Moderately Present (2)	The teacher speaks about one example of cultural events or outside of school events during instruction OR talks with one student about aspects of their lives outside of school or mathematics class in ways that are *incorporated into instruction.*
Minimally Present (1)	The teacher mentions cultural events, outside of school events, or other information personal to any students during class, but does not incorporate it into instruction.
Not Present (0)	The teacher does not mention/discuss anything personal to students during instruction.

Figure 1: Rubric for teachers' efforts to attend to students' lives

The SMiLES project's observation tool is designed to investigate potentially engaging teaching practices during an activity within a lesson. Before each classroom observation, we asked teachers to complete an online form in which they would nominate a potentially engaging activity that would take place within the lesson. Members of our research team video-recorded the entire class period, with a particular focus on these activities, which ranged from roughly 10 to 45 minutes with a median length of 30 minutes. These teacher-selected potentially engaging instructional activities lasted between 9 minutes and 40 seconds and 45 minutes and 40 seconds, with a median length of 29 minutes and 42 seconds.

We applied the observation tool rubrics to the video recorded activities in 10-minute segments. If the last segment was under three minutes, it was not rated. Each 10-minute segment in an episode was individually rated by two coders on the research team, who then met to resolve disagreements in scoring. Coders resolved disagreements in their segment ratings for each rubric by describing the observed behavior they used as evidence when scoring and how they interpreted that behavior within the framework of a rubric. Resolved scores for each segment were assigned. Episode scores were determined by averaging the resolved segment scores for each rubric.

To calibrate rating criteria and to address potential coding drift, all observation team members met at least once per academic semester to train for rating consistency with the observation tool and to resolve any outstanding questions that had arisen during the resolution procedures. Training involved all raters coding the same episode independently and meeting to resolve disagreements as a team. Orientation to the coding concepts also included reading and discussing relevant literature as a team (e.g., Middleton, Jansen, & Goldin, 2017).

Research questions. This report consists of two studies that respectively illustrate the validity and reliability processes used for the SMiLES project's observation tool. These studies answer two research questions:

1. Validity study: To what degree did the rubrics in the observation tool align with appropriate phenomena (instructional practices that promote mathematical engagement)?

2. Reliability study: To what extent did raters in our research team reach agreement when rating observation episodes? When there was initial disagreement, what explained lack of agreement?

Method: Participants and Context

The SMiLES project team collected classroom observation data from 29 secondary mathematics teachers' lessons in two U.S. states. Sixteen of these teachers taught in a mid-Atlantic state and 13 taught in a southwestern state. Twenty-one teachers were female and eight were male. The teachers also represented a variety of racial and ethnic backgrounds, with 22 teachers identifying as white, two identifying as Black, two identifying as Latinx, and one each identifying as Asian, Black/Hispanic, and White/Asian. The teachers worked with a diverse student population. In the Mid-Atlantic, the schools' demographics ranged from 12-34% low income, 25-60% white, 27-47% Black, and 6-21% Latinx. In the Southwest, the schools' demographics ranged from 76-94% low income, 1-6% white, 1-16% Black, and 77-96% Latinx. We targeted courses at the equivalent of on-grade level mathematics for ninth and tenth grade students, which included topics-based courses in the southwestern U.S. (Algebra I, Geometry) and integrated courses in the mid-Atlantic (Integrated Math [IM] 1, IM 2, IM 3). Each class period was observed two or three times during a course. A course was either one semester (if on block scheduling, such as schools in the mid-Atlantic) or a full academic year (southwest schools). The reliability study was conducted on a subset of these data.

Validity Study

Procedures

Characterizing teaching is a qualitative practice, and we conceptualize validity as multifaceted in qualitative work. Hayashi et al., (2019) present a variety of validity frameworks for qualitative work, including the following: *Descriptive validity* concerns the ability of the report of an event to faithfully record its important features. The interdependence of observations and the descriptions of those observations must be developed from theory. *Interpretive validity* concerns the ability of the tool to help the researcher construct the meaning of the events and the behaviors of the people engaged in those events. *Theoretical validity* refers to the consistency of the analytic coding and the theoretical argument that is constructed. It is thus concerned with the truth of the concepts and classifications developed in the analysis, and the ways in which the concepts and classifications interrelate in the abstraction of the event to the (nascent or developing) theory. *Validity generalization* refers to the ability of the method to be used in other situations, times, and places. For an instrument such ours, its descriptive and interpretive frameworks and its theoretical validity should be applicable in a new context.

Our first step toward internal conceptual validity was to operationalize the construct of engaging secondary mathematics instruction grounded in a theoretical frame from research literature (theoretical validity). This framework was developed by two researchers with expertise both in mathematics teaching and learning from mathematics education and motivation and engagement from educational psychology. For rubric development, we then translated the theoretical framework into descriptive rubric levels for each teaching practice. To internally examine construct validity in the rubrics, the entire research team (composed of graduate students and faculty with expertise in mathematics education or psychology) met multiple times to discuss whether and how these levels reflected the desired teaching practices and whether the descriptions were observable and amended accordingly.

We then piloted the tool by rating publicly available video from the TIMSS video study [http://www.timssvideo.com/] (descriptive validity). This pilot study involved all members of the research team rating the same two videos using the rubrics. The team met as a whole group to compare and contrast their ratings. Disagreements were discussed and the levels of enactment for each rubric were then specified further. A rubric to describe the nature of teacher feedback was added to reflect this teaching practice as a result of piloting.

To externally examine construct validity of the observation tool rubrics and individual rubric levels, we shared the observation tool with an expert panel, the SMiLES project's advisory board, which consisted of experts in educational psychology and mathematics education (interpretive and theoretical validity). All of the researchers in the advisory board had studied mathematics or science engagement in the context of learning environments, and they had all developed methods for studying teaching practices that support students' engagement. The results of the validity study reflected the team's learning from the expert panel.

Results

We initially generated twelve dimensions or rubrics based on our review of the research literature, and the expert review panel for our validity process led to three new rubrics. At the team's first advisory board meeting, three months into the three-year project, we shared the first draft of the observation codebook. The draft reflected a review of the literature, internal construct validity meetings, and revisions to the codebook after piloting it. Advisory board members then suggested additional rubrics that supported equity in mathematics teaching and learning.

As a result of external feedback, we revised the observation tool to reflect a broader conceptualization of equity in potentially engaging mathematics teaching (interpretive validity). In our initial rubrics, we approached equity primarily as access by writing rubrics that measured opportunities for students to experience sense making and reasoning, connections, tasks in context, and other aspects of high-quality mathematics instruction. We acknowledged that access is only one dimension of equity (Gutiérrez, 2002). Some of these rubrics were more closely aligned with supporting students' identities.

We added three rubrics related to promoting equity in mathematics teaching after feedback from our advisory board, resulting in 15 rubrics in all. We added a rubric about *attention to students' lives* in mathematics teaching (see Figure 1) (Yamauchi et al., 2005). Attention to students' lives could align with identities as students could begin to see themselves reflected in mathematics. We also added an *accountability* rubric to examine whether and how teachers held students to high expectations, acknowledging that high expectations are necessary but not sufficient to achieve equity (Lubienski, 2002). One final rubric was added after the pilot year of data analysis: *student enactments of agency and autonomy.* Our initial rubrics did not appear to capture the opportunities that students had to exhibit control over their own learning (Kosko, 2016). Opportunities to enact agency are chances for students to develop productive mathematics identities (Gresalfi et al., 2009).

Reliability Study

Data Sources

To investigate whether and how our rubrics could be applied consistently across raters, we conducted a reliability study of the analysis of 149 video-recorded episodes of teacher-selected potentially engaging activities. Each of these observation episodes was rated by two analysts. We dispersed resolution assignments to ensure that duplicate pairs of raters were minimized. Across the 11 raters who were on the team at any point from 2018-2020, 43 unique pairs of raters were assigned to resolve scores. The most resolutions shared by any single pairing was 19 episodes.

The resolution process began with both raters independently scoring each 10-minute segment of the observation video across each of the 15 rubrics in the observation tool. Every score was justified by documentation of evidence from the recorded observation, including timestamps. Raters then met to discuss any discrepancies in their initial ratings and to resolve the scores for each segment. This resulted in resolved scores for each rubric by segment, as well as episodic scores which were the average of resolved segment scores for each rubric. Every observation which was coded involved this resolution process; no episode was analyzed by a single rater.

During the final round of observation resolutions, raters also identified the nature of any initial disagreement in their individual scores to better understand the reliability of the observation tool. They identified whether any initial disagreement was the result of one rater noticing additional evidence in the video (resulting in a higher or lower score) or whether the initial disagreement was the result of conceptual differences between the raters with regards to the rubrics themselves (i.e., the same evidence was given different ratings).

Analysis Procedures

Following paired ratings, the Intraclass Correlation Coefficient (ICC) for each rubric was computed to examine the reliability of initial ratings *prior* to the resolution process. An excellent ICC, or a value greater than 0.9 (Portney & Watkins, 2000), could indicate that the resolution process was unnecessary (i.e., raters almost always agreed on the rubrics in their initial ratings). In contrast, a poor ICC of less than 0.5 (Portney & Watkins, 2000) suggests value in resolving.

The ICC used to determine reliability was a 2-way random effects model. Initial segment ratings were converted to a "low" and "high" score depending on how the two initial raters scored each rubric (low and high scores would be the same value when the initial scores were the same). In total 447 segments were analyzed for the ICC.

Table 1: Reliability Statistics (Absolute Agreement)

Academic Support Instruction Rubrics	Intraclass Correlation (Average Measures)	Social Support Instruction Rubrics	Intraclass Correlation (Average Measures)
AS1: Sense-making & reasoning	.618	SS1: Whole-class discourse	.809
AS2: Connections: representations & strategies	.602	SS2: Small-group discourse	.737
AS3: Pressing students to explain	.661	SS3: Status-raising / positioning students	.675
AS4: Context of tasks	.879	SS4: Motivational discourse	.581
AS5: Mathematical correctness	.378	SS5: Enthusiasm about mathematics	.524
AS6: Mathematical language precision	.548	SS6: Attention to students' lives	.476
AS7: Feedback	.530	SS7: Accountability & high expectations	.505
AS8: Agency and autonomy	.620		

Note: N = 447

The initial rubric scores held an ICC of between .378 and .879, with an average of .610 (Table 1). Two rubrics (*mathematical correctness* and *attention to students' lives*) had poor reliability (.378 and .476, respectively). The remaining rubrics had moderate reliability of between 0.5 and 0.75 except for AS4 (*context of tasks*) and SS1 (*whole-class discourse*) which had good reliability (.879 and .809, respectively). Recall that ICCs were calculated based on initial ratings, prior to resolution meetings.

The relatively low ICC values for the majority of the scales indicated a need for some process of resolution. This led to three important questions: 1) Was a resolution meeting necessary, wherein the source of discrepancy and its nature are discussed, when the mean of the raters' initial scores could suffice?; 2) In instances when raters' initial scores differed, what was the magnitude of the discrepancy?; and 3) Regarding the nature of disagreements, did disagreements reflect attention to different evidence or disagreements about interpreting the rubric?

To address the first question, the differences between the mean of raters' initial scores and the final resolution score were analyzed for instances when initial agreement was not achieved. While individual differences would be expected here, in the aggregate such differences would balance out if the resolution meetings held no consistent sway on the resolved score--i.e., if the discrepancy were random error. The second question was answered by looking at the magnitude of any initial disagreements - describing whether these disagreements were mostly of a single rubric point or whether they represented greater disagreement among raters.

To answer the third question, raters were asked to describe the nature of any initial disagreements with observations resolved in the Spring of 2020 onward. This resulted in such data for 70 different segments, or 1,050 resolved scores spread across the 15 rubrics. For each rubric in every segment where there was a disagreement, raters identified whether this resulted from individuals observing the same evidence and rating it differently or whether different raters identified different aspects of the same phenomenon resulting in different initial scores.

Results

The most frequent outcome for resolutions was an increase of 0.5 relative to the mean of the initial scores for each rubric. The exception to this was *mathematical correctness*, which most frequently dropped 0.5 points and *accountability and high expectations* which most frequently resolved to the initial mean. The most common difference in initial ratings was 1, which held true for every rubric in the observation tool. Together these results show that, when disagreements occurred, they tended to be minor and the resolution discussions tended to result in agreement on the higher score (e.g., initial scores of 1 and 2, with a mean of 1.5, would be expected to resolve to a 2).

Within the 1,050 resolution scores analyzed to understand the nature of such disagreements, 405 initial disagreements occurred. Among these, 60 (15%) occurred when raters observed the same evidence but still initially disagreed on their rating and 346 (85%) occurred when raters observed different evidence which influenced their initial scores. The fewest disagreements arose for *whole-class discourse* (8, with 0 for same evidence and 8 for different evidence) and the most arose for *connections with representations and strategies* (40, with 4 for same evidence and 36 for different evidence).

Discussion

Our process of establishing validity suggests that the SMiLES project's observation tool has potential for measuring mathematics instructional practices that are potentially engaging for secondary students. The rubrics align with prior research about engaging mathematics instruction, as suggested from both internal and external conceptual validity investigations. This tool offers a set of rubrics that differs from existing observational tools designed to investigate high quality mathematics instruction for supporting students' learning.

Our external validity study afforded an opportunity to reflect on mathematics teaching for equity in relation to mathematics engagement. Although access to high quality mathematics instruction is important for equitable teaching and learning, it is a limited conception of equity. We revised our observation tool to capture how teaching could potentially support development of students' identities to address another dimension of equity (Gutiérrez, 2002).

Regarding reliability, the moderate to good ICCs for all but two of the rubrics showed general agreement of raters prior to resolution meetings, but not to an extent that would justify removing the resolution meetings from the observation analysis process. When disagreements did arise, they were typically minor but still afforded valuable insight when resolving scores. In some cases (~15%) the coders had conceptual differences in understanding mutually observed evidence, but in most cases (~85%) one coder had captured additional evidence which strengthened the justification of the final, resolved score.

Such results support the original intention of the resolution meetings as a way to ensure that various manifestations of these instructional supports are actually captured from the data. The data was not just "double coded" and averaged by the research team, but rather every single rubric score was discussed and agreed upon. Conceptual differences were thus addressed continually as they arose in the data, and coders had opportunities to gauge the *sum* of their evidence before committing to a rating. Since resolved ratings trended higher than the mean of the initial ratings, this could indicate that these meetings uncovered more evidence of potentially engaging mathematical instructional practices than otherwise would have been revealed.

Reliability training and double coding are valuable tools for qualitative research, but they do not transform qualitative analysis into an automated endeavor. The SMiLES project's resolution process identified one way in which the human capital of a research team can be utilized to strengthen analysis and more reliably capture relevant findings. Through this approach, disagreements are not a source of alarm but rather an opportunity to strengthen the foundation of the work itself. Initial ratings are not immutable but rather subject to interpretation and revision. Through this process, the complex nature of these engaging mathematical instructional practices – and, in turn, the work of these educators endeavoring to make them a reality – is better recognized.

The SMiLES project's observation tool demonstrates promise for investigating the presence and quality of potentially engaging instructional practices. The process we used when enacting the process of double coding provided a powerful approach for assessing instruction thoroughly. Events in a classroom are complex, and our research team found it helpful to have more than one coder noticing events that could be relevant. With this tool and this analytic process, perhaps the field can go further to understand how mathematics teaching can engage secondary students.

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A CONCEPTUAL SYNTHESIS ON APPROXIMATIONS OF PRACTICE

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Approximations of practice create valuable opportunities for preservice teachers (PSTs) by engaging them in components of teaching. By reviewing the literature, this study explored PSTs' learning through approximations and the extent approximated practices preserve the complexity – or authenticity – of teaching. A review of 25 empirical studies related to approximations of mathematics teaching indicated that mathematics teacher educators are currently exploring an expansion of opportunities through approximations wherein PSTs could experience a higher degree of authenticity. The existing conceptualization of authenticity emphasizes the complexity of practices but overlooks how approximated practices prepare PSTs for their future teaching. An alternative definition is proposed for the emerging conceptualization of authenticity to highlight how PSTs' learning through approximations prepares them for their future teaching.

Keywords: Preservice teacher education, Instructional activities and practices, Approximations of practice, Authenticity

An effective mathematics teacher education program provides preservice teachers (PSTs) with opportunities to gain practice-based experiences and develop core pedagogical practices for teaching mathematics (Association of Mathematics Teacher Educators [AMTE], 2017). PSTs often gain such practice-based experiences by engaging in *approximations of practice* (referred to as "approximations"), which represent an opportunity for PSTs to engage in aspects of practice with additional support in order to develop their professional teaching skills such as leading classroom discussions, posing purposeful questions, developing lesson plans, etc. (Grossman et al., 2009; Schutz et al., 2018). Research has indicated these approximations are often simplified versions of actual classroom teaching because they: (a) are often enacted in teacher education settings, which cannot fully reflect social and cultural aspects of school contexts: (b) often include only some components of teaching, and (c) are usually enacted with scaffolding (Janssen et al., 2015; Tyminski et al., 2014). Thus, mathematics teacher educators have cautioned that approximations do not preserve the complexity of actual practices and are not necessarily authentic (Grossman et al., 2009). These conceptualizations suggest the following three dimensions in which to measure authenticity: context/setting of practice, nature of practices (e.g., decomposed practices vs. full practices), and scaffolding in practices. These three dimensions are utilized to explore the authenticity of approximations in this paper. While the literature has shown that approximations are valuable for PSTs' learning, the extent to which those approximations prepare PSTs for addressing the improvisational and interactional nature of teaching (Hunter et al., 2015) is still underexplored. Therefore, mathematics teacher education literature was synthesized to explore how approximations of practice create opportunities for developing PSTs' teaching skills and how teacher education programs conceptualize and practice authenticity during approximations. The following research questions guided this study:

1. What teaching practices have been used in approximations-based instructional activities?
2. What influences did approximations of practice have on PSTs' learning to develop their practices?

3. To what extent were the approximated practices authentic?

Perspectives

This study is grounded on the practice-based approaches to teacher education and the concept of authenticity in approximations. In recent decades, many mathematics teacher education programs have been designing and implementing instructional activities that provide PSTs with opportunities to engage in aspects of teaching practice (Ball & Cohen, 1999; Zeichner, 2012). Such instructional activities are usually referred to as approximations of practice (Grossman et al., 2009). The actual practices are assembled during approximations either by including only some aspects of practices and/or by providing PSTs with scaffolding; and thus, approximations usually have reduced complexity of teaching (Tyminski et al., 2014). Because of the reduced complexity, educators have stated limitations of the practice-based education and approximations (Zeichner, 2012). As such, simplified practices pose a risk of creating technicians who can only apply sets of routine skills but struggle to adapt them to the school context. To address these limitations, there is an ongoing discussion around how the authenticity in approximations should be conceptualized and practiced. Teacher educators primarily have two perspectives about when and how the complexity of practices should be adjusted during approximations. From one perspective, educators argued that it can be overwhelming for PSTs to begin their teaching with complex practices, necessitating a reduction of the complexity of teaching at the beginning phases of practices (e.g., Bannister et al., 2018; Klein & Taylor, 2017). Klein and Taylor (2017) mentioned that approximations should provide PSTs with opportunities to practice in a context that is different from the natural context as it has reduced complexity. Other educators have argued that simplified practices pose the risk of creating routinely inauthentic practices, and these practices might not be transferable to school contexts (e.g., Campbell & Elliott, 2015).

This discussion about complexity suggests that authenticity is related to the complexity of teaching. Educators have discussed the dimensions of authenticity in several ways. Grossman et al. (2009) proposed the approximations that (a) have less support (scaffolding), (b) are integrated (not decomposed), and (c) have a similar setting to actual classrooms are more authentic than the approximations which involve more support from educators and involve only some components of teaching. Campbell et al. (2020) mentioned that the location of approximations (university classrooms or school classrooms), students who participated in approximations (peers or students), and teacher educator's roles (providing scaffolding or not) would determine the degree of authenticity. Tyminski et al. (2014) and Janssen et al. (2015) also defined authenticity along the continuum of three dimensions: whether or not the practice is decomposed; what the contexts of practice are; and whether or not teachers provided with scaffolding. Based on these definitions, the following dimensions of authenticity are selected: decomposition of practice, setting/context of the practice, and scaffolding.

Decomposition of Practice

Decomposition of practice indicates breaking down the practice into small components to assist PSTs to learn those small portions of practice (Grossman et al., 2009). For example, PSTs may engage in responding to students' thinking, which is only one component of the practice of teacher noticing. In the studies related to approximations, educators often engage PSTs in some components of practices. For instance, Trent (2013) listed some components of teaching that PSTs engaged in: selecting tasks, organizing students for peer or group work, learning to provide feedback to students. One way to decompose the practice is to make small components of practices in a way that PSTs would be able to master practices on those modules (Grossman et al., 2009; Janssen et al., 2015). However, educators (e.g., Campbell & Elliott, 2015; Trent, 2013) have challenged PSTs might not automatically be able to recompose those decomposed practices into complex practices. Since teaching is both improvisational and interactional that is based on students' thinking (Hunter et al., 2015), how those decomposed practices prepare PSTs for teaching is still researchable. Therefore, how the current studies related to approximations are utilizing decomposition in their studies is a focus of this investigation.

Setting or Context of the Practice

The second dimension of approximations—settings— refers to the contexts where the practice is situated. Authenticity is associated not only with the decomposition of practice but also with the setting or context of practice (Campbell et al., 2020). McDonald et al. (2013) defined the school setting where teachers and/or PSTs have opportunities to engage in the full practice as authentic settings and the one where PSTs have restrictions to conduct activities are considered as controlled or designed settings. Since the university-level course setting (i.e., methods course) is different from the actual setting, approximations might lead to the divide between theory and practice (Grossman et al, 2009). Thus, teacher educators have been trying to replicate the social and cultural complexities of actual practices in the university-level setting through approximations (Codreanu et al., 2020). There are some efforts to include responsive teaching in approximations. For example, Campbell et al. (2020) included planted students' errors and asked PSTs to respond to those errors. In this study, how the literature related to approximations have negotiated the difference between university-course settings and actual classroom settings is explored; the factors that the approximation-based literature has highlighted to replicate the social and cultural complexities of the mathematics classrooms are explored.

Scaffolding

Scaffolding during approximations refers to supports that PSTs receive to improve their teaching practices. Since teachers often enact practices independently and do not receive feedback during teaching, the extent to which PSTs receive scaffolding differentiates the approximations from the actual teaching (Grossman et al., 2009). The feedback given to PSTs could impact what they focus on during approximations. Some approximations are more loosely constrained than others. When PSTs have options to choose and approximate the whole practice, it is a loosely constrained approximation (Kavanagh et al., 2020). This study explores several forms of scaffolding and their possible influences in PSTs' learning.

Methods

A conceptual synthesis method was employed to explore the primary concepts and discussions related to approximations (Petticrew & Roberts, 2008). First, inclusion and exclusion criteria were established to identify the literature, which was the data for the study. The literature was then analyzed through top-down and bottom-up interactive modes of analysis (Chi, 1997).

Data Source: Literature Search Procedures

Twenty-five articles were identified as involving approximations of practice with PSTs. The first round of the literature search involved Key Word Processes Search (Depaepe et al., 2013). Two key phrases of, "approximations of practice" and "math," were used to search the literature in Education Resources Information Centre (ERIC) and PsycInfo, which produced a total of 32 results. This number was narrowed down to 13 by eliminating dissertations, non-peer-reviewed journals, conference proceedings, and/or book chapters. Only empirical peer-reviewed journal articles were included in the study because the aim was to understand and analyze (a) how researchers have utilized approximations in teacher education programs, (b) what specific teaching skills PSTs developed by engaging in those approximations, and (c) what challenges are associated with the development of teaching skills. The second-round search involved a Journal Search Method. Articles were selected from *Journal of Mathematics Teacher Education* (JMTE), as it was the sixth most-cited journal in the field of mathematics education in 2017 (Williams & Leatham, 2017) and because it publishes empirical research around teacher education and teacher development. All published issues from the beginning of 2017 to 2020 were accessed to search the relevant articles from the journal. Using the inclusion and exclusion criteria described above, 12 articles were selected. The 25 articles can be found in the reference section of this paper.

Analysis of the Selected Literature

The interactive nature of the top-down and bottom-up approach (Chi, 1997) was utilized to code the articles; by considering one case as each article, teaching practices used in the study, teacher learning from approximations, and dimensions of authenticity were summarized. To answer the first and second research questions related to teaching practices involved in approximations and their influences on PSTs learning, a bottom up approach was utilized; the practices involved in approximations and major findings of each study were summarized. The three already identified dimensions of authenticity (a top-down approach) were used to answer the third research question of exploring authenticity in approximations. Once codes were identified collaboratively, researchers independently coded about 30% of the initially identified articles to calibrate initial coding at the beginning, including the identification of new codes, and any discrepancies between coding (i.e., a bottom-up approach). For example, "analyzing and reflecting on teaching" was a new code identified during independent coding, which was not in the initial sets of codes and was later added to the set of codes.

Findings and Discussions

This section is organized by research questions. Findings for the first, second, and third research questions are answered for the first, second, and third research questions, respectively.

Practices Used in Approximations-Based Instructional Activities

Research indicated the following five primary teaching practices used in the literature related to approximations: curriculum enactment (e.g., Earnest & Amador, 2019; Santagata & Yeh, 2014); eliciting, interpreting, and responding to student thinking (e.g., Bailey & Taylor, 2015; Webel & Conner, 2017); teacher professional noticing (e.g., Amador et al., 2016; McDuffie et al., 2014); organizing classroom discussions (e.g., Ghousseini & Herbst, 2016; Weston et al., 2018); and analyzing and reflecting on teaching (e.g., Auslander et al., 2020; Cooper et al., 2020; Kinser-Traut & Turner, 2018). In almost half of the research reports (13 out of 25 research studies), PSTs were engaged in two or more practices. In particular, the studies that explicitly focused on the teacher noticing and classroom discussion also included the practice of eliciting, interpreting, and responding to student thinking. In the noticing-related literature, researchers broadly assessed which aspects of classroom events (i.e., teacher actions vs. student actions) PSTs attended to and how they responded to those events, including how PSTs attended to and interpreted students' mathematical thinking. For example, Schack et al. (2013) explored the extent to which PSTs noticed students' equitable practices of mathematical learning, which included students' competencies related to mathematical thinking.

The literature indicated that in the approximations related to teacher noticing, PSTs often engaged in the components of noticing, including attending to, interpreting, and responding to student thinking. Some studies that focused primarily on leading classroom discussion involved the practice of eliciting, interpreting, and responding to student thinking. Ghousseini and Herbst (2016) focused on the practice of leading classroom discussions and included interpreting student thinking. Thus, the practice of eliciting, interpreting, and responding to student thinking was the most highlighted practice in the literature related to approximations. Another highlighted practice through approximations was analyzing and reflecting on teaching. PSTs analyzed their own and their peers' teaching to develop reflective practices. Reflective practices include teachers' ability to identify and analyze aspects of classroom actions (e.g., task posing) in which instruction is successful and other areas which they need improvement (Alsawaie & Alghazo, 2010). For example, PSTs reflected on teaching by analyzing classroom practices (Kinser-Traut & Turner, 2018) and by discussing their experiences of sequencing learning activities (Cooper et al., 2020).

Some studies did not explicitly emphasize specific practices; they focused broadly on curriculum enactment without highlighting one or more practices. In these studies, PSTs planned and taught lessons (or parts of lessons) either in a virtual classroom setting (e.g., video simulations; Amador et al., 2016) or in a real classroom setting. For example, in Earnest and Amador (2019), PSTs planned a lesson and enacted the first five minutes of their lessons through the given animation, indicating this approximation did not focus on only one teaching practice.

Types of approximations used to develop practices. Researchers from the identified articles primarily used two mediums of approximations: virtual mediums and role play. Virtual mediums often included enacting lessons using simulation tools (e.g., Earnest & Amador, 2019; Weston et al., 2018) and/or interacting with virtual student characters (Bannister et al., 2018; Webel & Conner, 2017). Role play involved rehearsals of one or more components of teaching, and PSTs rehearsed teaching either with their peers or during field teaching. For instance, in Tyminski et al. (2014), PSTs practiced teacher questioning with their peers in a teacher education classroom setting, while in Santagata and Yeh (2014), PSTs rehearsed teaching in a school setting. PSTs typically used role-playing or simulated classroom scenes to communicate what they noticed or to demonstrate their anticipated student responses with virtual mediums. Estapa et al. (2018) had PSTs use animated software to represent pivotal classroom moments that they noticed from a video lesson. PSTs also engaged in eliciting, interpreting, and responding to student thinking through both role-play and simulations. For instance, in Schack et al. (2013), PSTs interpreted and discussed diagnostic interviews as well as practiced diagnostic interviews (role play) to interpret and respond to student thinking. PSTs often engaged in leading classroom discussion through role-play with their peers. In Ghousseini and Herbst (2016), PSTs chose mathematical tasks that called for reasoning and communication, led classroom discussions in a constructed classroom dialogue, and practiced leading discussions during their field teaching.

PSTs' Learning Through Approximations

The literature revealed that approximations of practice afforded PSTs opportunities for understanding and enacting aspects of practices in a scenario simulating a classroom context. For instance, Campbell and Elliott (2015) mentioned that PSTs identified actual learning goals while role-playing leading classroom discussions. Earnest and Amador (2019) discussed that their approximated practices prepared PSTs for using the curriculum to design instructions; however, those practices could not prepare PSTs for how to select specific materials in their classes. PSTs conceptualized and applied some dimensions of equitable and responsive teaching after they engaged in approximations of responsive and equitable teaching. In responsive teaching, "teachers' instructional decisions about what to pursue and how to pursue are continuously adjusted during instruction in response to children's content-specific thinking" (Jacobs &

Empson, 2016, p. 185). For instance, in Bannister et al. (2018), PSTs began to focus on students' strengths rather than their deficit perspective as they engaged in approximated practices related to deficit thinking. Initially, PSTs highlighted students' mistakes and problems and defined those mistakes as problems, while PSTs began interpreting those mistakes as learning opportunities at the end. PSTs learned to attend to students' thinking and pose tasks to respond to student thinking from approximations (Estapa et al., 2018), suggesting that approximations contributed to cultivating PSTs' ability to develop responsive teaching.

Researchers claimed that approximations related to simulations also prepare PSTs for responsive teaching. PSTs often need to predict both teachers' and students' roles when creating classroom scenes using simulations (Amador, 2017; Schack et al., 2013). For instance, in Earnest and Amador (2019), PSTs created a classroom scene wherein they selected speech bubbles by anticipating students' responses to their questions, requiring them to predict and respond to student thinking. Similarly, in Amador et al. (2016), PSTs anticipated their students' responses and planned for how they would respond to those students. The findings from these research reports indicated that these virtual tools encouraged PSTs to anticipate and analyze students' thinking and specific responses, indicating that the approximations related to virtual simulations also prepared PSTs, to some extent, for responsive teaching (e.g., de Araujo et al., 2015).

These findings indicated that PSTs' learning through approximations was reported in terms of which practices PSTs developed at the end of approximations. Indeed, there was less attention to the extent PSTs are able to transfer the practices learned from approximations to their teaching. For instance, teacher educators have been exploring how PSTs develop noticing skills by comparing what PSTs notice at the beginning and at the end of approximations. Thus, there is a less attention in the literature on how PSTs possibly apply noticing skills in their future teaching. Since approximations have often different settings than the real practices, PSTs' learning at the end of approximations does not necessarily suggest they can improvise the learning in their actual teaching.

Degree of Authenticity in Approximations of Practices

The earlier defined dimensions (decomposition, setting, and scaffolding) were used to explore authenticity in approximations.

Decomposed vs. full practices. A review of the literature suggested the following three types of decompositions in approximations: (a) focused on only some components of a practice throughout the study, (b) began with a component of practice and gradually added more components, and (c) engaged PSTs in full practices without decomposition. The first category of studies focused on only some components of practice throughout practice enactment sessions. For instance, in Bannister et al. (2018), PSTs engaged in learning to notice students' strengths rather than their deficit thinking, which is a component of teacher professional noticing. The second category of studies decomposed a practice into small components by engaging PSTs in one component of practice at a time. For instance, in Estapa et al. (2018), PSTs engaged in a core practice of teacher noticing at the beginning by having PSTs practice only one component of teacher professional noticing, namely attending to classroom events. After a time, PSTs explained what they *attended to* and how they would *respond to* those events. The third category did not decompose practices; they provided PSTs with opportunities to engage in a full practice (e.g., teacher noticing) or in planning and enacting lessons. In Earnest and Amador (2019), PSTs planned and enacted a lesson using simulation technology.

As discussed, some studies provided opportunities for PSTs to engage in recomposed practices at the end of approximations. In the process of recomposition, several components of practices are combined together; thus, recomposed practices are more complex than decomposed practices (Janssen et al., 2015). Even though PSTs engaged in both decomposed and recomposed practices in some studies, they did not have opportunities to learn the ways to recompose practices. Thus, PSTs may not have learned skills of improvising decomposed practices during their teaching. Further, the literature does not suggest which kinds of decomposition is more beneficial in developing PSTs' teaching skills. Consequently, the decomposed practices without concrete ways of recomposition might pose a risk of creating a boundary between teacher education programs and school contexts because PSTs cannot experience the social and cultural complexities of school contexts (Grossman et al., 2009; Campbell & Elliot; 2015).

Less authentic vs. more authentic setting. As defined in the literature, a factor determining the degree of authenticity is the extent to which the context of an approximation is similar to a school classroom setting. The literature suggested that PSTs engaged in the practice of teaching in three different settings: simulated environments (e.g., Amador et al., 2016; Webel & Conner, 2017), teacher education classrooms (e.g., Lampert et al., 2013; McDuffie et al., 2014), and school classrooms (e.g., Santagata & Yeh, 2014; Schack et al., 2013). PSTs often enacted lessons either in a simulated environment or in their classrooms; PSTs considered their peers and/or virtual student characters as their students, suggesting that PSTs could not experience student interactions with the social and cultural complexities of classrooms through approximations.

Researchers (e.g., Janssen et al., 2015) identified simulated environments as less authentic than PSTs' classroom contexts. They claimed that simulated environments do not necessarily preserve the complexity of teaching as PSTs have limited opportunities to develop skills to make moment-to-moment decisions and to respond to their students in real-time while creating animated classroom scenes (e.g., de Araujo et al., 2015). The studies involving role play claimed that they preserved authenticity by engaging PSTs in tasks that were similar to tasks they would do in school settings. For instance, Tyminski et al. (2014) claimed that PSTs were asked to consider and write authentic students' problem-solving strategies. However, in both simulations and in role play, PSTs could not experience ways of understanding and responding to students' cultural and social backgrounds. While the literature has indicated several approximations that provided PSTs with opportunities to practice anticipating and responding to students' thinking, the literature does not suggest how simulated virtual environments and teacher education classrooms prepare PSTs for the social and cultural complexities of school settings.

Scaffolded vs. independent enactment. The literature indicated that PSTs were scaffolded in different ways and in different times (before, after, and/or during lesson enactment). Scaffolding was provided in the form of specific frameworks/protocols, constructed dialogues, instructor-modeled activities, instructor or peer feedback, and lectures and tutorials. For instance, in McDuffie et al. (2014), PSTs were given noticing lenses (i.e., teaching, learning, task, power, and participation lens) to develop their noticing skills. In Ghousseini and Herbst (2016), teacher educators used constructed dialogues and asked PSTs to fill in the portions that were removed from those dialogues. In Bailey and Taylor (2015), teacher educators modeled problem posing in order to enhance PSTs' abilities to elicit and interpret students' thinking. In Leavy and Hourigan, PSTs were given lectures and tutorials on problem-posing skills. Overall, PSTs were scaffolded during approximations, and there were several forms of scaffolding. Since teachers are required to make most instructional decisions individually and independently, scaffolded practices are considered to be less authentic than independently enacted practices (Janssen et al., 2015).

Conclusion

As previously mentioned, eliciting, interpreting, and responding to student thinking was the most highlighted practice in approximations, suggesting that teacher educators attempted to prepare PSTs for responsive teaching through approximations. To develop this practice, PSTs were engaged with their peers, students' work samples, and planted students' errors. Review of the literature also suggested that PSTs' learning from approximations were explained in terms of what PSTs learned at the end of approximations. One example of such learning is the gain in PSTs' skills to respond to students' thinking at the end of approximations (Monson et al., 2020). Since teaching is both improvisational and interactional, which is based on students' thinking (Hunter et al., 2015), how these learned skills prepare PSTs for their actual teaching is a critical aspect of PSTs' learning through approximations. Yet, there is less attention on the literature about to what extent PSTs are able to improvise their responsive teaching skills that are learned through approximations to respond to students' thinking in real time during their teaching.

As discussed earlier, the degree of authenticity is a cause for concern in approximations, and researchers have often considered setting, scaffolding, and decomposition as three dimensions determining the degree of authenticity. Some approximations are more authentic than others, depending on what construct is aimed to be developed through approximations. In the literature, approximations involving role playing are characterized as more authentic than approximations involving simulations, which may only be valid for some specific practices. For example, role play would be more authentic than simulation if it aims is to develop PSTs' practice of 'responding to student thinking' because PSTs engage in a setting similar to the actual practice of role playing. However, approximations involving simulations would be more authentic than role playing if the focused practice is "anticipating student thinking" as PSTs anticipate students' responses while creating virtual scenes. Even though approximations are perceived to be less authentic both in terms of the nature of the activities and the setting of the practice, they seem productive for providing PSTs with opportunities to anticipate students' roles in their classes. Researchers have assumed that scaffolding reduces the authenticity of practices because PSTs do not get direct scaffolding during teaching. This study suggests that scaffolding does not always reduce authenticity; when and how PSTs receive scaffolding would determine the degree of authenticity. As such, if PSTs get feedback before or after engaging in a practice, it serves as scaffolding without reducing authenticity. After all, teachers are expected to get feedback and continue to improve their teaching throughout their teaching careers (Conference Board of the Mathematical Sciences, 2012). However, feedback provided in the middle of practice might decrease the authenticity as PSTs' decisions potentially depend on the feedback.

Based on these findings and discussion, teacher educators may not be able to fully determine the authenticity in approximations based on what PSTs learned at the end of approximations because teachers should be "seen as complex, sensible people who have reasons for the many decisions they make" (Leatham, 2006, p. 100). Even though teacher educators might perceive a practice to be transferable to school, novice teachers might not transfer because they struggle to negotiate the power dynamics within schools and teacher education programs (Trent, 2013). In fact, there is a less attention in the literature about the extent to which PSTs transfer the practices learned from approximations in their future teaching, suggesting a need for extending the research to examine the ways in which approximations can be most productive in informing PSTs' future teaching. Thus, the expansion of an alternative definition of authenticity is needed that incorporates the extent to which PSTs transfer the skills learned from approximations in their future teaching. This alternative conception of authenticity proposes that the degree of authenticity should be based on the extent to which approximations provide a way to recompose small components of a practice in order to improvise practices learned from approximations in school contexts. Collectively, this dimension of authenticity enhances PSTs' ability to enact components of practices in their teaching. With an acknowledgement that this alternative definition does not include all possible dimensions of authenticity, it brings more depth to the current dimensions of authenticity in approximations.

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REVEALING MATHEMATICAL ACTIVITY IN NON-FORMAL LEARNING SPACES

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We offer this synthesized framework as a tool to reveal mathematical activity in a non-formal makerspace. In particular, we connect research at different grain sizes to illustrate and explain how mathematics plays a crucial, if often implicit, role in this activity. We begin with describing the Approximate Number System and the Ratio-Processing System, explain how those systems connect to both embodied cognition and Thompson's (1994) conceptualization of quantities. We then examine how prediction and anticipation relate, with a particular emphasis on how social feedback guided the emergent mathematical activity. Finally, we synthesize across the two frameworks, in order to better reveal mathematical activity in low-notation environments, as the first step towards a framework for understanding mathematical learning in non-formal and low-notation contexts.

Keywords: Informal Education, Learning Theory, Technology

Identifying mathematical cognition in non-formal contexts where formal notation plays very little role can be a difficult proposition. In particular, mathematical notation-based performances are often taken as evidence of mathematical learning, and a tempting corollary is that mathematical learning is thus evidenced by mathematical notation. As a consequence, learners engaging in activities that have little or no formal notation can be seen as not engaging in mathematical learning, even when they may be experiencing a *mathematical* activity that merely lacks the explicit outward signs of such learning. In our research on mathematical play in a non-formal makerspace (Shokeen et al., 2020; Katirci et al., 2021), we have developed a new framework for identifying mathematical activity in a low-notation environment, and we share that framework here.

We build a theoretical argument that takes a multi-pronged approach: first, we develop a theoretical framework that builds from two primitive structures in the brain - the Approximate Number System and the Ratio-Processing System (Matthews et al., 2015) - tie those neural structures to Alibali and Nathan's (2012) embodied cognition view of perception and action, and interpret both of those frameworks through Thompson's (1994) conceptualization of *quantities*. Second, we describe a framework based upon prediction (Bieda & Nathan, 2009) and anticipation (Tzur, 2007) as components of mathematical learning, and tie those directly to our work on feedback and failure (Williams-Pierce, 2019). Lastly, we illustrate (or apply) the synthesis of these frameworks as a way to better identify and understand the mathematical activity that is taking place in the social context of an informal collaborative group.

Methodological Background

Our primary methodological approach for this paper is theoretical but we built our theory directly through observing and analyzing video data with the aim of examining it for evidence of mathematical play. In this section, we describe that video data. In the later sections, we provide illustrative examples of that data in order to illuminate how our comprehensive framework revealed mathematical learning and activity.

Our video data is composed of three video records of the same 20 minutes of a collaborative robotics activity with five fourth-grade students (2 M; 3F). Two of the video records were from the perspective of two students wearing GoPro cameras, while the third was a standing camera that captured the entire group's activity from a slight distance. The activity took place within the context of a physical classroom, although it was treated as a non-formal makerspace, and the participants were present voluntarily. The robotics activity had two phases: Phase 1, the group put masking tape on the floor to establish a path for a different group (who did not consent to be videotaped); and Phase 2, the group moved to the masking tape path established by the other group, and sought to measure the path and program a robot, Dash, to successfully travel it. Figure 1 illustrates the group putting down the masking tape path in Phase 1 (A), the iPad interface for programming Dash (B), and an image from the standing camera of the group measuring the path and watching Dash move in Phase 2 (C).

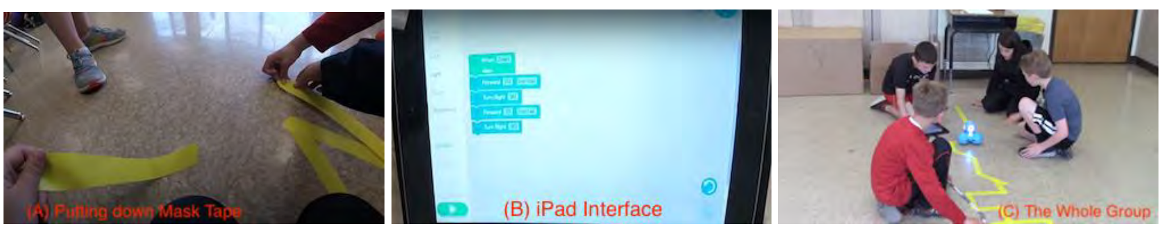

Figure 1: (A) Phase 1; (B) iPad interface; (C) Phase 2.

The research team who analyzed the data is composed of four regular members with varying areas of expertise. Two are experts in embodied cognition, in both physical and digital learning contexts; one specializes in mathematics learning in makerspaces (and originally collected the video data); one specializes in mathematical play. All four have considerable expertise with mathematics learning in both formal and informal contexts. The multidisciplinary nature of the team is how we developed our comprehensive framework over time, as our collaboration during analysis revealed both the need and the expertise for developing this framework.

Theoretical Background A: Approximate Number System to Quantities

In this section, we discuss on how perceptions, gesture, action, and the physical context relate to Thompson's (1994) conceptualization of *quantities*. We begin by describing the underlying neural systems that influence perception of magnitude (section A1); describe how perception, gesture, and action are complexly related in cognition (section A2); then describe how Thompson's quantities (1994) fit into that theoretical system (section A3).

A1: Underlying Neural Systems

Our physical bodies have perceptual systems that influence our cognition. Alibali and Nathan (2012) describe perception and simulations of perception: "When humans perceive objects, they automatically activate actions appropriate for manipulating or interacting with those objects (Ellis & Tucker, 2000; Tucker & Ellis, 1998). Thus, imagining an object can evoke simulations of perception (i.e., of the actions associated with perceiving the object) or of potential actions involved in interacting with the object" (p. 254). These perceptions and the perceptual systems that underlie them can have primitive neurological bases. For example, the Approximate Number System (ANS) ties estimation of a number of objects directly to certain animal neuron activation patterns, including humans (e.g., Dehaene, 1997; Matthews et al., 2015). A human adult, glancing at a set of three objects on a table, immediately subitizes: they know automatically and without conscious thought that there are three objects present (e.g., Miller, 1994). If that human adult is shown three objects repeatedly, the part of their brain responding to those three objects begins firing less actively as the perceiver becomes habituated to the number of objects being subitized. In such situation, if a fourth object is added, there is a small increase in relevant brain activity; whereas if three more objects are added (making six in total), a larger increase in relevant activity occurs. In other words, when the number of objects being perceived increases slightly, there is little increase in brain activity; but if the number increases considerably, so does the brain activity (e.g., Dehaene, 1997; Piazza et al., 2004).

Building upon the ANS, Matthews et al. (2015) describe the Ratio-Processing System (RPS) as a neural system in which we intuitively and immediately perceive and compare magnitudes of objects. (Although Matthews et al. (2015) describes quantities as an inherent quality of magnitude of an object or representation, we instead refer to that as *magnitude*, and reserve the term *quantity* for Thompson's (1994) definition.) With the ANS and the RPS as primitive structures that perceive and compare magnitudes, certain components of perception are built directly into our brains. Building upon those structures into more complex forms of perception (such as recognizing relevant tools in our environment, the social structures of a group, and so on), is more complex. Specifically, perception and action are reciprocal: our perception guides our action, and in turn our action reflects and guides our perception. These actions and perceptions are grounded in our physical environment, including the social, material, and structural aspects of our surroundings (Alibali & Nathan, 2012) *and* our neurological structures (Matthews et al., 2015).

A2: Perception, Action, and Gesture

Perception, whether based upon primitive numerical structures or otherwise, leads to action (such as gesture, physical movement upon the environment, or spoken language), and that action leads back into our perception. This feedback loop of perception, action, and imagining is described as mental simulation (Alibali & Nathan, 2012), and together compose the embodied nature of our cognition. This feedback loop can be evidenced through spoken or written language, physical movements that impact the physical world, or - often - can only be inferred by an outside observer through expression of gestures. These gestures are communicative acts that reveal perception and action in a variety of ways, such as through pointing (deictic) gestures that connect spoken language with objects or people in the physical environment or representational (such as iconic or metaphoric) gestures that directly reflect the state of perceptions and planned actions of the gesture. Consequently, we rely upon action and gesture as both composing and revealing perception, action, and their composite into cognition.

We now describe Thompson's (1994) conceptualization of *quantities*, then tie this conceptualization into Alibali and Nathan (2012) and Matthews et al. (2015) through illustrative examples of our data.

A3: The Role of Quantity in Perception, Action, and Gesture

Thompson (1994) specifically defines *quantity* as a conceptual entity - that is, quantity does not reside in the object, but rather in the perceiver. As noted above, our references to *magnitude* should be taken to refer to both Matthews et al.'s (2015) use of the term *quantity*, and to the *perceived* quality of an object or representation of taking up space (re: Thompson's (1994) definition).

Thompson (1994) goes on to define *quantity* as a schematic that involves "an object, a quality of the object, an appropriate unit or dimension, and a process by which to assign a numerical value to the quality" (p. 184). For example, a piece of masking tape that is "too big" (as stated by Peter; see Figure 2-A) indicates that the speaker perceives the magnitude of the tape, and compares it to some internal standard in order to determine that the piece needs to be shortened (See Figure 2-B&C). This perception and comparison of magnitudes (one physical, one imagined) occurs through Matthews et al.'s (2015) primitive structures. Then, the judgment of "too big" indicates that the speaker is perceiving the length of the masking tape as a *quantity* by Thompson's (1994) definition: the masking tape is the object; the length of the piece of masking tape is the quality they are considering; and the internal standard for magnitude is an appropriate unit or dimension. Although our participants did not have access to a measuring tape in order to assign a numerical value to the quality of length, they would have been able to do that measuring if the tool had been present (as they used such a tool in Phase 2). In other words, the speaker who says "too big" is using quantity as conceptualized by Thompson (1994), and that quantity is perceived and compared with a simulated perception (Alibali & Nathan, 2012) of appropriate unit or dimension. This perception and comparison of length is rooted in the speaker's ANS and RPS: although a lack of discrete or explicit measurement makes it difficult to determine how their ANS is contributing, the comparison of the physical length's magnitude with their imagined unit's magnitude can be directly attributed to their RPS.

Figure 2: (A) "Too big!"; (B) Shortened the piece; (C) Final decision- cut the tape

The speaker's comparison of the magnitude of the tape with their internal standard presents a communicative problem, as they must externalize their internal standard in some fashion for their group mates. One potential method of externalizing might be gesturing what "too long" is - while this does not externalize the internal standard, it indicates what magnitude the speaker is considering to be too much, which implies that the desired length of tape should be shorter. Another potential method was to engage the action of ripping the tape in half: this would serve to indicate what an appropriate length of tape would be, while requiring fellow perceivers to examine the magnitude of a resulting piece of tape in order to evaluate whether the new pieces are perhaps "too short." When a piece was too short, the choice of actions was different: they were crumpled up and thrown away, or used to extend a pre-existing length of tape already on the floor. While these actions and gestures may differ, they each indicate the same perception of magnitude, the quantification of that magnitude, and a comparison to an internal standard.

Now we shift to our second theoretical background.

Theoretical Background B: Prediction to Social Feedback

Now that we've described how perception, action, and gesture relate to magnitudes and quantities, we will focus on detailing prediction (Bieda & Nathan, 2009) and anticipation (Tzur, 2007) as components of mathematical learning, and tie them directly to our previous work on feedback and failure (Williams-Pierce, 2019). We then focus specifically on how prediction, anticipation, feedback, and failure contribute to *social feedback* with illustrations from our data. We focus on social feedback in particular due to its crucial role in the collaborative mathematical activity present in our data. We introduce Theoretical Background B with as little reference to Theoretical Background A as possible, as we plan on focusing on that final step of synthesis in our Synthesis of Theories section. As the majority of the mathematical reasoning that occurs in this activity is grounded directly in perception and quantities, and much of it also involves mental simulation (i.e., aspects of Theoretical Background B), we give less mathematical examples here that do not require attention to quantities or mental simulation.

B1: Prediction and Anticipation

Bieda and Nathan (2009) describe prediction as looking at a pattern, and predicting a later instance of that pattern, whether near or far. The vast majority of our prediction examples in the data are intertwined with students perceiving and simulating quantities, but we present two examples of prediction that rely less upon quantities, both of which revolve around the teacher-facilitator warning the teams that they were running out of time. During Phase 1, the students changed their tape-laying pattern from trying to make the path 'zig-zag' (Shokeen et al., 2020, accepted), to simply placing a single long piece of tape to complete the path across the room. In other words, they were predicting that following their zig-zag pattern would not result in completing a path across the room, so they adjusted their activity accordingly to ensure they reached across the room within the allotted amount of time. A similar moment happened towards the end of Phase 2 when the teacher-facilitator gave a 45 second warning that the activity was almost over. One of the students from the other team was overheard by the target team saying, "We are not gonna make it" and Ryan responded across the room to them as he kept measuring the tape paths: "Neither are we." In that moment, Ryan was looking ahead in time, and predicting that if they continued programming Dash as their current speed, they would not be able to get Dash to the end of the tape path. As mentioned earlier, these are not particularly rich examples, mathematically, but the students are looking at the results of their activity thus far, comparing how much time that activity took, and predicting the results of continuing with exactly the same activity in the short amount of time that is left. In the first example, they modified their activity in order to achieve their goal of getting across the room; in the second example, there was no such modification available to similarly speed up their progress.

We now shift from prediction to Tzur's (2007) description of two stages in mathematical activity: participation and anticipation. During the participatory first stage, the learner has a mathematical understanding that emerges only when prompted by the activity at hand, and cannot be independently demonstrated without the contextual cues or tools. Tzur (2007) describes the "the well-known 'oops' experience" (p. 277) in the participatory first stage, where a student does something, notices a mistake as it manifests in their activity, and goes on to adjust it in the moment. During the anticipatory second stage, however, "the learner can independently call up and utilize an anticipated activity-effect relationship proper for solving a given problem situation" (p. 278) - in other words, they are able to use their mathematical understanding without engaging in the activity first. In our activity, participation and anticipation often manifested through prediction, feedback, and failure. We give specific examples about anticipation and its relationship to prediction in Section B3, as failure and feedback (Section B2) play a critical role in identifying Tzur's (2007) stages within this activity.

B2: Feedback and Failure

Our initial goal when we began analyzing this data was to identify how zones of mathematical play that emerged in concert with mathematical video games (Williams-Pierce & Thevenow-Harrison, 2021) might manifest in this new non-formal context. We began by attending particularly to feedback and failure, as Williams-Pierce (2019) defined failure and feedback as tightly paired in digital contexts and crucial to mathematical play. In particular, it is through failing and getting feedback that players, through their own actions, engage in learning the underlying mathematical content in the game (Williams-Pierce, 2019). This type of paired feedback and failure is instantaneous and often direct, in both videogames and the current activity. For example, in Phase 2, if the path in front of Dash was measured to be 80 cm and the programmer enters that measurement into the code, but Dash goes too far and ends up off the path, Dash's location manifests failure paired with feedback. The programmer then learns from the feedback (*Dash has gone too far*) to input a smaller distance into the program. This occurrence of failure and feedback is similar to that found in videogames, but we also found that *social* feedback played a crucial role in this non-formal collaborative context: students observing Dash's failure to stop at the correct spot on the tape often amplified the feedback of Dash's location - in this case, one student said, "That is a little far away" (*too far*). As a result, the programmer reprogrammed Dash to go 70 cm instead of 80, and Dash stopped at the desired location on the tape path. In short, in this type of situation, the paired feedback and failure may be direct - as in video games - may have social components, or may be fully social.

In fact, in Phase 1 the paired feedback and failure was often fully social. For example, at one moment, Peter was holding the roll of tape, and tearing off pieces to hand out to other students, who spontaneously formed a line to wait their turn for a piece of tape. Ryan, however, tried to cut in line immediately after they had just placed a piece of tape, but Peter did not let them, forcing them to go to the back of the line to wait their turn. This is an example of social feedback and failure: Ryan was essentially informed that they were performing a social activity that was not permitted within this community, and given feedback on how to actually get their next piece of tape in an appropriate fashion. As another example of social feedback, students would often disagree on how long a piece of tape should be, or what angle it should be placed relative to the path. However, this paired feedback and failure relies heavily upon perceiving or mentally simulating quantities, so we will discuss that further in the Synthesis of Theories section.

B3: The Relationships between Prediction, Anticipation, and Social Feedback

Prediction, anticipation, and social feedback have a complex but crucial relationship. For example, when Dash went too far in the Phase 2 example above, the students had input a centimeter measurement that they anticipated and predicted would lead Dash to the correct location on the tape. Consequently, when Dash stopped at the wrong place, the students received that paired feedback and failure, and amplified that feedback and failure through talking about it (e.g., social feedback). However, often feedback and failure are not clearly evident, because if what the students predicted would happen did, they had no need to remark upon it. In situations like this, where feedback and failure are missing, and the students move on to the next step, we concluded that they were content with their previous work. We also posit that this may be an indicator that students have shifted from *participatory* to *anticipatory*, because they have learned/internalized what to do or not to do, which results in no failure and often no social feedback. For example, in Phase 2, students are using a measuring tape and a pencil to measure a part of the path, and then write their measurement of that strip of tape on the path. After measuring and writing down the measurement, they move directly on to measuring the next part of the path without commenting, because they have successfully completed a step of the measurement. Measuring by itself is an activity that can be successful or unsuccessful in itself, even before Dash enacts the measurement – but the data showed no example of the students accidentally flipping the measuring tape to the inches side, or noticing any other potential measuring issues that could happen. This illustrates the other side of the 'oops moment,' because it is a 'we measured the path appropriately and are not surprised by it' moment. Sometimes, the students are successful but remark on their success, such as when Aaron coded Dash to traverse the first three lines and the angles within them, and after Dash ended up in the correct spot, Aaron said, "That's perfect!" We consider this to be an example of social feedback paired with success, rather than failure, and an illustration of the participatory stage rather than anticipatory, because they were at least mildly surprised that it worked (e.g., they lacked confidence in their prediction), unlike when using the measuring tape.

Synthesis of Theories

This section is the culminating synthesis of the theoretical groundings introduced above. In particular, we will describe how quantities and embodied cognition relate to prediction and anticipation, through the lens of failure and feedback in this context. We give two examples - one from each Phase - detailing exactly how the students were engaged in the activity, and conclude with another example that highlights the role of social feedback in particular.

In Phase 1, as students were placing down tape, they were using their perception of length and angle to guide their placement. At one point, Ryan places down tape at what he perceives to be and says is a "ten degree angle." The teacher-facilitator notices, and says it is "too tight" for Dash to traverse. Then Ryan pulls up that tape and re-places it, using his perception of quantities to increase the angle. This is an example of using perceived acceptable quantities of angle: Ryan uses his own perception of length and angle to mark 10 degrees, which he perceives as a perfectly appropriate angle for Dash to execute; then Ryan adjusted his understanding of an appropriate (perceived) angle quantity according to external guidance by the teacher, who knows Dash's limitations. This is an example of participatory first stage, where they have an oops! moment, but the shift to anticipatory second stage occurs immediately, as we see by a complete lack of other too-tight angles in Ryan and the group's remaining activity. Similar quantity and perception-based moments occur around the length of the masking tape, such as when Aaron is placing a long piece of tape, and Peter says, "But that's too long though." Aaron immediately adjusts the tape length by ripping some of it off, so that Peter's perception of an acceptable quantity of tape is respected. All of these interactions are based directly upon perceptions of quantity - whether of length or angle - and involve nuancing each students' view of what an appropriate quantity is for the task at hand.

In Phase 2, as students code a new length of the path for Dash, they are using quantities that have already been evaluated (measured) by their teammates with the measuring tape. However, in re-coding a length that they've already tested with Dash, they are using two different evaluated quantities alongside perceptual quantities. The centimeter measurement written on the path is one evaluated quantity, while the second – how far Dash was programmed to go – is another evaluated quantity, while the comparison between the two is purely perceptual. The two different evaluated quantities are both technically centimeters, but evaluated in two different ways: the first is 'centimeters as measured by the measuring tape'; while the second is 'centimeters as enacted by Dash.' When those two evaluated quantities do not match up, the students must perceptually evaluate the difference between the two, and mentally simulate a comparison that supports them in re-programming Dash accurately. When Dash is programmed, the students are predicting that Dash needs to go the programmed distance in order to reach the correct spot; and they respond to the failure or success of that prediction accordingly, indicating their placement in the participatory or anticipatory stages.

The role of social feedback was particularly crucial, as there was a lack of mathematizing tools: each student had to use their own perception and mental simulation of quantities, as no more precise method was at hand. For example, at one point in Phase 1, the students decided that they wanted to lay the path underneath two chairs that are tucked under a table. As one student began laying the tape underneath the chairs, another student, Hannah, said something in a doubtful tone (not captured on audio), while tracing the floor under the chairs. Aaron says, "No no, that would work" and Ryan agrees, also tracing the floor under the chairs. As Hannah spoke, she was mentally simulating her perception of the size (quantities) of Dash, comparing that mental simulation with her perception of the space available underneath the chair, and visualizing a conflict between those two perception-based simulations such that Dash would run into the chair, rather than go smoothly underneath it. Aaron and Ryan, though, are either engaging in different mental simulations – one in which Dash fits under the chairs – or are merely thinking of Dash following the path (a participatory view), while Hannah was anticipating, and using that anticipation to predict that some issues would arise. Aaron and Ryan keep placing the tape, and then Peter joins to place the last piece of tape that brings the path out from under the chairs. As Peter finishes, he says, "We should move the chairs out, too, if it doesn't fit," and Ryan says, "Yeah." Then, when Phase 1 is ending, and the group is leaving their tape path for the other group to use, this group runs back to remove the chair from the path, indicating that the mental simulations of others (Peter and Hannah) have convinced the others that Dash probably will not fit - in other words, this is a moment of social feedback.

Conclusion

We offer this synthesized framework as a tool to reveal mathematical activity in a non-formal makerspace. In particular, we connected research at different grain sizes to illustrate and explain how mathematics plays a crucial, if often implicit, role in this activity. We began with describing the Approximate Number System and the Ratio-Processing System (Matthews et al., 2015), explaining how those systems connect to both embodied cognition (Alibali & Nathan, 2012) and Thompson's (1994) conceptualization of *quantities*. We then examined how prediction (Bieda & Nathan, 2009) and anticipation (Tzur, 2007) relate, with a particular emphasis on how social feedback guided the emergent mathematical activity. Finally, we synthesized across the two frameworks, in order to better illustrate the implicit mathematical activity in our data.

This theoretical framework is the first step in our efforts to better identify mathematical cognition in low-notation environments. We have connected multiple layers of research that emphasize mathematical cognition, and used it to reveal mathematical activity - and our next goal in this line of research is to connect that non-formal, low-notation mathematical activity with direct identification of the resulting learning. As yet, we do not claim a direct relationship between the revealed mathematical activity and learning, but rather focus on establishing the necessary groundwork for investigating that relationship. However, as increasing numbers of educators examine mathematical learning in informal environments such as ours, we consider the ability to identify the types of implicit, perceptual, and embodied mathematical cognition that emerge from these environments to be a necessary contribution to the field. Additionally, this identification requires using knowledge and frameworks from multiple fields examining different layers - from neurons to social interactions - in order to solidly ground each moment of mathematical activity. We offer this framework as the first step in this endeavor.

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FROM THEORY TO METHODOLOGY: GUIDANCE FOR ANALYZING STUDENTS' COVARIATIONAL REASONING

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The fields of quantitative and covariational reasoning boast a wide range of powerful theoretical tools, which are described carefully in the literature. Less frequent and explicit attention, however, has been paid to writing down detailed, practical guidance for operationalizing these theoretical constructs. Some guidance is provided by covariational reasoning frameworks, but much is left unsaid concerning the inherent complexities and ambiguities involved in analyzing students' moment-by-moment behaviors and what these behaviors convey about their covariational reasoning. In an effort to more clearly link theory to analytic methodology, we share three lessons about analyzing students' covariational reasoning to make research more accessible to newcomers and better address what is often left unsaid in the covariational reasoning literature.

Keywords: Research Methods, Cognition, Precalculus, Calculus

The fields of quantitative and covariational reasoning boast a wide range of powerful theoretical tools. These tools have grown to be complex so they can more productively model students' thinking. The theoretical relationships between these elements are often detailed in theoretical synthesis papers, such as Thompson and Carlson (2017), and in the theoretical framework sections of empirical papers. But less frequent and explicit attention has been paid to writing down detailed, practical guidance for *operationalizing* these theoretical constructs for analysis. Experienced covariational reasoning researchers undoubtedly reflect on these important analytic considerations for each study they conduct; yet, these reflections are rarely reported in the literature.

In this paper, we build on prior work in which we reflected on and critiqued our analytic techniques for studying covariational reasoning to a) improve our own methodologies and analytic techniques and b) increase the accessibility of covariational reasoning research (Drimalla et al., 2020). In line with the theme of this year's PME-NA conference, we have decided to share our own productive struggles with designing and conducting a study that assessed students' covariational reasoning (Boyce et al., 2019). To center the *productive* aspect of these struggles, we will present the content of our reflections in the form of three lessons we have learned. Each of these lessons focus explicitly on developing analytic techniques and operationalizing theoretical constructs in quantitative and covariational reasoning research.

This paper is primarily intended for newcomers to covariational reasoning hoping to learn from our productive struggles. We also invite experts to engage with our experiences and critiques so that we can work together as covariational reasoning researchers to a) make research in this area more accessible to newcomers, b) be more open, clear, and explicit when writing about and sharing our analysis techniques and methodologies, and c) address potential flaws and gaps in the literature.

Theoretical Background

Covariational Reasoning

Carlson et al. (2002), building off Saldanha and Thompson's (1998) quantitative approach to covariational reasoning, described covariational reasoning to be "the cognitive activities involved in coordinating two varying quantities while attending to the way they change in relation to each other" (p. 354). Saldanha and Thompson (1998) further described how two quantities can be thought of simultaneously using the concept of a *multiplicative object*. They wrote, "Our notion of covariation is of someone holding in mind a sustained image of two quantities' values (magnitudes) simultaneously. It entails coupling the two quantities, so that, in one's understanding, a multiplicative object is formed of the two" (p. 299). Thus, covariational reasoning entails constructing two separate varying quantities as well as a multiplicative object, an object formed by simultaneously uniting the attributes of both quantities.

Quantitative reasoning. As covariational reasoning is a form of quantitative reasoning, the construction of quantities is foundational. Thompson (1994) described a *quantity* as a conceptual entity which "is composed of an object, a quality of the object, an appropriate unit or dimension, and a process by which to assign a numerical value to the quality" (pp. 7–8). For Thompson (2011) then, "quantification is the process of conceptualizing an object and an attribute of it so that the attribute has a unit of measure, and the attribute's measure entails a proportional relationship (linear, bi-linear, multi-linear) with its unit" (p. 37). Thus, quantitative reasoning is an individual's conception of quantities and their understanding of how the quantities relate. For example, a person could conceive of the height of an airplane, the distance it has traveled, and the relationship between the two.

Nonnormative graphing schemes. Covariational relationships are often represented graphically and, subsequently, the study of covariational reasoning is further complicated by the variety of ways students understand graphs. Graphing schemes that are the norm amongst mathematics education researchers can differ from students' graphing schemes (Moore et al., 2019). These nonnormative graphing schemes mainly stem from people's different meanings for coordinate systems (Lee et al., 2019), points in the coordinate system (Tasova & Moore, 2020), and curves with respect to the coordinate system (Moore & Thompson, 2015).

To attend to how students represent quantities graphically, Joshua et al. (2015) defined a *frame of reference* as "a set of mental actions through which an individual might organize processes and products of quantitative reasoning" (p. 32). In particular,

> an individual conceives of measures as existing within a frame of reference if the act of measuring entails: 1) *committing to a unit* so that all measures are multiplicative comparisons to it, 2) *committing to a reference point* that gives meaning to a zero measure and all non-zero measures, and 3) *committing to a directionality of measure comparison* additively, multiplicatively, or both. (p. 32, emphasis added)

Students who carry out each of these mental actions when representing a quantity, have constructed not just a quantity, but a framed quantity.

Operationalizing Theoretical Constructs for Covariational Reasoning

Our research group turned to two of the most widely-cited covariational reasoning frameworks—those of Carlson et al. (2002) and Thompson and Carlson (2017)—for guidance on operationalizing the above theoretical constructs to study students' covariational reasoning.

Because this is where we suspect newcomers will begin, too, we start by attending to the combined guidance provided by these frameworks with regards to developing analytic frameworks for covariational reasoning. We then highlight the areas where the literature on these frameworks does not provide clear guidance.

Thompson and Carlson (2017) framework. Thompson and Carlson (2017) offered a framework of covariational reasoning levels based on the theoretical building blocks of quantity, multiplicative object, and variational reasoning. Each level is described based on what a person envisions while carrying out quantitative mental actions. Thompson and Carlson emphasized that when using this framework it is "essential to attend to how students are thinking that quantities' values vary and how they are uniting quantities' values when considering their meanings for covariation" (p. 443). To exemplify what it means to attend to these elements of student thinking, Thompson and Carlson shared examples of student approaches representative of each level of covariational reasoning for the classic Bottle Problem (Swan & Shell Centre, 1985) as well as a graphing task from Castillo-Garsow (2012).

The sample approaches to the Bottle Problem include one specific example of what a student at the gross coordination level might say; however, much of the discussion concerns what students at various levels might envision, focus on, or imagine—all internal processes. Less attention is paid to specific student behaviors that might be indicative of these internal processes.

For the graphing task, three kinds of student graphs are linked to specific covariational reasoning levels. Thompson and Carlson clarified that these graphical answers represent "at most" a certain level of reasoning because, for example, students "might have connected points simply because they thought that this is what one does when sketching a graph" (p. 442). In other words, they highlighted the need for researchers to carefully attend to the mental actions and images a student uses while creating a graph before attributing a corresponding level of covariational reasoning. Specific behaviors that might be indicative of these mental actions as students sketch their graph are not provided, however.

Carlson et al. (2002) framework. The levels of the Carlson et al. (2002) framework similarly highlight "images of covariation" that support particular mental actions. Unique to this framework is that each mental action is tied directly to 1–2 specific indicative behaviors. Some of the behaviors are involved in sketching a graph while others are what students say. Carlson et al. clarified that "Some students have been observed exhibiting behaviors that gave the appearance of engaging in [advanced mental actions] . . . When asked to provide a rationale for their construction, however, they indicated that they had relied on memorized facts to guide their construction" (pp. 361–362). As in Thompson and Carlson (2017)'s framework, then, each behavior corresponds to "at most" a particular level of covariational reasoning.

Critical Analysis of Both Frameworks. A primary benefit of the Thompson and Carlson (2017) framework is that it synthesizes several theoretical constructs found in the (co)variational reasoning literature. However, a newcomer using only this framework may struggle to operationalize these levels in their own research due to the lack of student behaviors tied explicitly to each level of covariational reasoning and the complexity of the theory. This reality reduces the analytic utility of this framework. On the other hand, the Carlson et al. (2002) framework provides specific graphical and verbal behaviors that correspond to each mental action/level but does not account for current theoretical advances in quantitative and covariational reasoning.

After reading the literature on both frameworks, newcomers may still wonder: What are behaviors I can observe that indicate a student has constructed and is (co)varying one or more (framed) quantities? Also, as most examples of indicative behaviors from the literature on these frameworks focus on final products (e.g., the shape of a student's graph), how should one make sense of students' in-the-moment reasoning based on their constitutive behaviors that lead to these products? Such behaviors could be graphical or verbal—as in the Carlson et al. (2002) framework—but may also be gestural and inscriptional (either graphical or non-graphical).

Three Lessons for Analyzing Covariational Reasoning

We asked similar questions while developing and refining our research group's methodologies for analyzing students' covariational reasoning. Our goal was to assess students' levels of covariational reasoning using the Thompson and Carlson (2017) framework, but this proved more difficult than we initially imagined. Although the literature details and carefully connects the theoretical aspects of quantitative and covariational reasoning, practical guidance for connecting this theory to analysis through task design and task analysis is much sparser.

Our Research Context

We learned the lessons we are about to share by reading the literature, through trial and error of analyzing undergraduate first semester calculus students' covariational reasoning for a prior study (Boyce et al., 2019), and through careful, systematic reflection both individually and as a research group (see Drimalla et al., 2020). Conversations with the last author—an expert in quantitative and covariational reasoning who was not a member of our research group—helped us frame and generalize our reflections to ensure that the lessons we learned apply beyond our experiences.

The covariational reasoning task we discuss in this paper is the Reverse Bottle Problem based on the classic Bottle Problem that has been used frequently in covariational reasoning research (Carlson et al., 2002; Paoletti & Moore, 2017; Stalvey & Vidakovic, 2015). The problem statement and the accompanying diagram we provided students is shown in Figure 1, alongside the main prompts we verbally asked participants.

To ground each lesson, we provide examples of student thinking. We focus on one student—Neal—in an attempt to accurately depict the challenges and ambiguities a researcher has to grapple with to analyze even one student's behaviors across a multi-part task.

Imagine this jug has been completely filled with water. It is then left indoors in a sunny window and left untouched until all the water has evaporated.

1) Describe how the height of the water in the jug would change as the volume of the water in the jug decreases.

2) Sketch a graph that gives the height of the liquid in the jug as a function of the volume of the water in the jug.

3) Describe the rate at which the height of the water is changing with respect to volume.

Figure 1: The Reverse Bottle Problem Task

Lesson 1: Use an Additional Analytic Frame to Track the Quantities Students Construct

Thompson's definition of quantity provides a helpful *theoretical* framing of what it means for students to construct a quantity. In practice, though, we found that an *analytic* frame for quantity is necessary to keep track of how students are conceptualizing and representing quantities on diagrams. Two primary difficulties necessitated the addition of an analytic frame for quantity.

First, we suspected that students used different words to refer to the same quantity they had constructed. However, our research team had no common method for gathering evidence that this was the case, leading to a hodge-podge of (implicit and potentially incompatible) perspectives being used, making consensus decisions difficult. A common question that arose is whether a student was simply reciting the words in the prompt or if they had truly constructed a variable quantity. We needed a way to make decisions on this front.

For example, Neal used the word “volume” six times when discussing the Bottle Problem in tandem with the initial diagram; however, he also used the phrase “amount of water” twice. Had Neal constructed two separate quantities? It was only after we used an analytic frame based on the language and gestures participants used that we noticed each gesture Neal used alongside “amount of water” was a subset of the gestures he used exclusively to refer to “volume.” This observation provided strong evidence that Neal was discussing the same quantity despite using different language. By saying “amount of water,” Neal used what we call *fresh* language—unique wording introduced by the interviewee, rather than the interview protocol or the interviewer—further cementing our conclusion that Neal was not merely repeating the word he expected the interviewer would want him to use in his response.

Second, the inclusion of volume necessitated that students construct a 3D quantity, but we initially had difficulty discerning how they represented such a quantity with fidelity when restricted to a 2D graph or diagram. Prior to the interview, Neal had encountered a problem in his class that involved a different shaped bottle being filled with water. The interviewer began by asking him to describe his recollection of that task. Neal initially described the bottle as “circular” (a 2D wording) but, within a few seconds, corrected himself and said that the bottle was “spherical” (a 3D wording). We credit this usage of fresh language as evidence of Neal’s awareness of dimensionality, which is key to understanding the quantities he constructs. Later, we again noted his awareness of dimensionality when Neal used unidimensional gestures to refer to most quantities aside from volume (e.g., “height” paired with a strictly vertical motion; “width” paired with a strictly horizontal motion). The primary gesture Neal used for volume (see Figure 2) appeared to be an attempt to represent the higher dimensionality of the volume quantity he had constructed in two or even three dimensions. This led us to conclude that he was not bound by the strictly 2D representation of the bottle when reasoning about his quantity for volume of the water in the jug.

(a) **(b)**

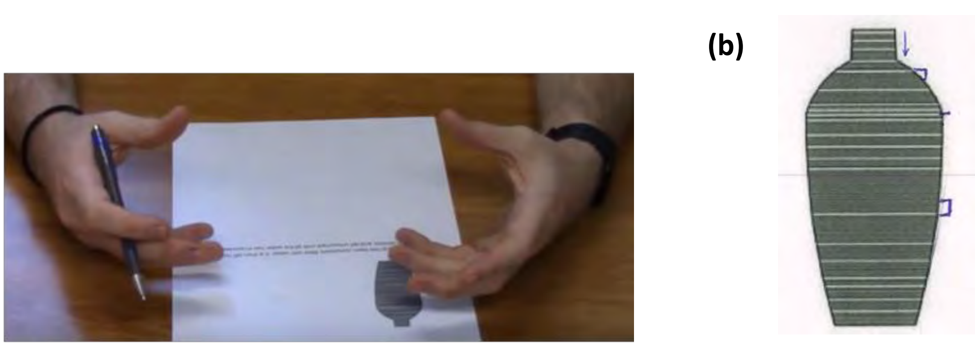

Figure 2: (a) Neal’s cupping gesture for volume (b) Neal’s bottle inscriptions

The lesson. The above two examples provide only a glimpse into the value our research

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group found in adding an analytic frame to better understand how students conceptualized and communicated about the quantities they constructed. The lesson we learned is not that we needed to use this particular analytic framework for quantity but rather that choosing *an* analytic framework is essential for discerning a) whether a student has constructed a quantity, b) how a student conceptualizes a quantity, and c) how they communicate about their constructed quantity. Because such data are necessary to understand the extent to which students have varied or covaried quantities, we believe an analytic framework is essential for covariational reasoning research.

Lesson 2: Disentangle Graphing Schemes from Covariational Reasoning Schemes

Whenever graphical contexts are used to study someone's covariational reasoning, there is a risk of either over- or under-assessing that student's level of covariational reasoning. For example, some people can sketch a correct graph using their graphing schemes without relying on the highest levels of covariational reasoning (Carlson et al., 2002, pp. 361–362). Without careful analytic tools, a researcher may overassess such a person's capability for covariational reasoning. On the other extreme, students with highly nonnormative graphing schemes may struggle to graphically exhibit their highest level of covariational reasoning (Drimalla et al., 2020), resulting in an underassessment of their capability for covariational reasoning. This issue may persist even when people are asked to explain (their) graphs. Even Thompson (2016)—a seasoned covariational reasoning researcher—was surprised by the range of non-quantitative ways of thinking elicited by a graph explanation task that was carefully designed to assess covariational reasoning (pp. 448–450). These kinds of non-quantitative responses do not indicate a person *cannot* or *does not* reason covariationally; they merely indicate no present evidence of covariational reasoning.

We ran headfirst into these methodological issues when assessing Neal's covariational reasoning level based on his graphical activity. Previously, Neal had spent just under 2.5 minutes establishing a quantitative frame (Moore & Carlson, 2012) and discussing the covariational relationship between the height and volume of water using the provided diagram (See Figure 1). On the other hand, graphing this relationship and crafting an explanation of the graph that Neal found satisfactory and consistent with his prior diagram-based explanation took approximately 20 minutes. We suspect much of this time was spent grappling with nonnormative graphing schemes and frames of reference for the quantities of height and volume, rather than exhibiting Neal's ability to reason covariationally.

Directionality of the x-axis and conventions for slope. Neal first labeled his x-axis with "volume water in jug" and his y-axis with "height of water" then added the words "empty" and "full" to fix a directionality of measure along both axes consistent with normative conventions (i.e., on the x-axis, quantities increase from left to right; on the y-axis, quantities increase from bottom to top). Next, Neal plotted the initial point of his graph at (empty, full) because "the height of the water is at the maximum . . . and we want to start at the maximum point." Moments later, though, he realized that the initial point should be (full, full) and flipped the directionality of his volume axis using labels of "full" and "0" to make it so.

Neal then sketched a curve with the correct shape. However, his justifications for this shape were based almost entirely on "the rate" without mention of any quantities. The interviewer asked Neal to provide a value for the rate of change of height with respect to volume using his graph. Neal explained that the rate of change in the beginning is -1 because "every time it moves down one it goes over one." Here, Neal used the usual convention for calculating slope without attending to the nonnormative directionality of his x-axis. He justified his claim by citing that the graph is decreasing, but then experienced a perturbation, claiming "that's weird," and noting that his x-axis decreased from left to right. But it was not until after the interviewer prompted Neal to add units on each axis that he performed a rise over run calculation and concluded that the slope was, indeed, positive, overriding the normative convention that a graph that "goes down" from left to right must have a negative slope.

The lesson. Neal was one of the only study participants who eventually overrode his normative graphing schemes to justify the graph he drew. Clearly, this is no easy feat, as it took Neal nearly 20 minutes to adapt to these nonnormative conditions and draw covariational conclusions he agreed were consistent with what he had deduced using the bottle diagram.

From this, we learned how important it is to have built-in methodology or an analytic tool to distinguish people's quantitative and covariational reasoning schemes from graphing schemes based on memorized conventions. There are a few ways this could be accomplished: 1) Establish a baseline for a participant's idiosyncratic graphing schemes earlier in the interview to contextualize their covariational reasoning schemes on tasks that likely invoke nonnormative graphing schemes. This could be done using careful task design that gradually increases in difficulty and the extent to which nonnormative graphing schemes likely need to be used. 2) Use an analytic framework for quantity in the graphical context to better foreground any quantitative reasoning that does occur relative to students' (possibly nonnormative) frames of reference and graphing schemes. This approach mirrors Lesson 1.

In our case, we went with option 2 and attended carefully to how Neal represented relevant quantities graphically (in normative and nonnormative ways) using the frames of reference analytic framework. This additional analytic framework enabled us to observe that only after Neal committed to a unit for both height and volume did he recognize that the slope of his graph was actually positive all along. Although Neal had constructed *partially* framed quantities for height and volume early on, it was not until he constructed fully framed quantities that he stopped relying on normative graphing schemes and began to reason with the quantities he had previously represented along the axes.

Lesson 3: Attend to Non-Graphical Behaviors Indicative of Covariational Reasoning

After becoming aware of the delicate issue of using graphs to assess covariational reasoning in Lesson 2 (see also Drimalla et al., 2020), we sought non-graphical behaviors that could be indicative of covariational reasoning. Primarily, we investigated participants' interactions with the bottle diagram (see Figure 1) when they were prompted to describe how the height of the water in the jug would change as the volume of the water in the jug decreases. Often, participants would instead discuss how only one of these quantities varies with time, rather than how the quantities themselves covary. To assess participants' capacity for covariational reasoning, though, we wanted to be as certain as possible that the quantities a student was envisioning as covarying were the water's height and volume. Thus, we distinguish between conceptual time, which is a constructed quantity, and experiential time, which is not. This distinction—of explicitly making a measured note of time—is relevant for understanding individuals' reasoning about multiple quantities (Thompson & Carlson, 2017) since conceptual time is sometimes included as one of the quantities to covary. After all, varying just one quantity (with respect to experiential time) is not **co**variational reasoning. And even if a participant had been covarying a quantity with *conceptual* time, we had no means to determine whether they had constructed time as a quantity. In the following section, we analyze some of Neal's tracing actions and wonder out loud whether his actions are sufficient evidence of covariational reasoning.

Tracing the side of the bottle. After Neal had established some typical gestures for both height and volume, he traced downwards along the final stretch of the right side of the bottle while simultaneously describing the relationship between the height and volume in the final portion of the bottle: "When the jug starts to get thinner the height at which—er, the *rate* at which the height is decreasing is going to raise." Next, Neal singled out a segment of the bottle in the section he had just traced along by drawing a bracket (see Figure 2b) and justified his prior answer: "Because the volume of water which is decreasing at a constant rate will take smaller—uh, sorry—*larger* and *larger* heights [traces vertically down bracket] to contain that equal amount of volume [cups hands in volume gesture, see Figure 2a]." The fluency and simultaneity of Neal's actions, gestures, and verbal explanation led us to believe that these behaviors were evidence Neal had engaged in continuous covariational reasoning. In particular, Neal's selection of an arbitrary sub-interval of bottle in which he discussed that the same volume would take up "larger and larger heights" led us to believe he had envisioned these quantities as having values which change. This seems to indicate Neal was engaging in more than gross covariational reasoning.

But was this sufficient evidence? Alternatively, perhaps Neal was merely using this tracing action as a way to track experiential time. When later drawing the shape of his graph, he often re-traced the side of the bottle, potentially envisioning what the water in the bottle looks like as time passes. Was Neal actually envisioning the values of height and volume as covarying, or just the gross variation in each quantity with respect to experiential time? As we remarked earlier, it was not until the interviewer suggested Neal add a scale to his axes (about 15 minutes into graphing) that he explicitly established units for each quantity. Had he implicitly been thinking in units as far back as when he was first tracing the bottle diagram? Possibly. But there is certainly some ambiguity present which affects the conclusions we can draw from his earlier tracing behavior.

The lesson. In this lesson, we sought to highlight the difficulty of drawing covariational conclusions from a person's observable behaviors. We believe there is great value in attending to non-graphical behaviors to assess covariational reasoning—both because it helps make the theory of covariational reasoning more accessible to newcomers but also because it expands the methodologies available to study covariational reasoning. At present, much of the literature only reports briefly or implicitly on the many ambiguities present in interpreting these behaviors. We hope this changes and encourage other covariational reasoning researchers to make this a reality.

For newcomers hoping to study covariational reasoning *now*, we envision two approaches to help resolve such ambiguities in observable behaviors. First, plan to ask targeted follow-up questions in your interview protocol to clarify whether participants are reasoning about quantities' values. In our case, we had already conducted interviews and could not do this, but in retrospect we suspect that asking Neal a simple follow-up question after he had traced the bottle may have resolved our uncertainty. Second, incorporate additional subtasks to ascertain whether participants are reasoning about quantities' values. To support this endeavor, there is research concerning task design and sequencing in covariational reasoning research (e.g., Johnson, 2015).

Conclusion

By sharing these lessons, we hope to have drawn attention to the unwritten complexities of analyzing students' moment-by-moment behaviors while studying their covariational reasoning. Namely, the importance of attending carefully to the quantities people construct (Lessons 1 & 2), the challenge of disentangling normative graphing schemes from covariational reasoning schemes (Lesson 2), and the ambiguities in interpreting non-graphical behaviors as indicative of covariational reasoning (Lesson 3). Given the long history and complex theoretical nature of quantitative/covariational reasoning research, such conversations are *essential* if we wish to keep this subfield accessible, rather than daunting, to newcomers. We hope we have contributed to this end; however, we cannot do this alone. We welcome others—experts and *especially* newcomers—to write explicitly on these topics and forge more detailed connections between theory and methodology in the covariational reasoning literature.

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INTEGRATING THREE PLANES OF TEACHER LEARNING: THE CASE OF SIDE-BY-SIDE COACHING

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Attempts to understand and design for mathematics teacher learning have frequently identified key disconnects between teachers' contexts, the professional development experience, and, ultimately, teachers' practice. In this paper, we offer a theoretical approach to understand these discontinuities drawing on Rogoff's (1995) three planes of analysis of sociocultural activity. We then apply these three planes to illustrate one example of a teacher learning environment designed for coherence: side-by-side coaching.

Keywords: Professional Development, Learning Theory

Theoretical Framework

The learning of individuals cannot be separated from those they learn with, those they learn from, or the environment in which that learning occurs (Rogoff, 1995). Learning is embedded in a complex system of interaction and influence among individuals and contexts. To understand this system, Rogoff proposed observing learning in three "inseparable, mutually constituting planes" of activity (1995, p. 139): the *apprenticeship plane, guided participation plane*, and *appropriation plane*. The apprenticeship plane encompasses community activity, including histories, social structures, and cultural organization. This involves taken-for-granted policies, practices, and structures that shape activity. The guided participation plane refers to the ways individuals involved in shared activity "communicate and coordinate efforts" (1995, p. 142). Guidance includes the influence of social interactions and the shared values of the community. The appropriation plane comprises "how individuals change through their involvement" in activity (1995, p. 142). In this plane, individual uptake and transformation of practices is central.

Each of these planes offers one way of understanding learning that occurs in activity, though none alone captures the full system. Indeed, the three planes do not just distinguish layers of activity; they enable understanding how these layers relate to and influence one another as a constantly adapting system. All three planes are needed to understand learning, though researchers can foreground one plane to study its components while still acknowledging the role of the other planes (Rogoff, 1995). While researchers in mathematics education have taken up this analytic approach to examine students' learning environments (e.g., Anghileri, 2006), few have used this approach to shed light on teacher learning or professional development (PD). In this theoretical paper, we explore how applying Rogoff's framework to teacher learning can support designing effective professional development that integrates all three planes.

The apprenticeship plane asks for analysis of the context and systems. We contend that in the case of teacher learning, there are two facets to examine: the context and systems of the teacher learning experience (i.e., the PD setting) and also the context and systems in which the teachers are expected to apply their learnings (i.e., in their classrooms). The guided participation plane of analysis entails examining the relationship and interactions among teacher learners and PD providers. Additionally, analysis of the guided participation plane must account for the relationships among teachers and their students. Finally, the appropriation plane of analysis requires an examination of individual teachers in multiple contexts. In what ways does the teacher appropriate new ideas in the PD setting? In what ways does the teacher appropriate new ideas in their classroom?

The value of applying the three planes of analysis to understand teacher learning is in highlighting areas where discontinuities across the planes might be present. For example, research on PD has shown teachers face challenges in applying ideas from PD settings to their individual instructional contexts (Borko, 2004). There also exists literature that focuses on one plane at the exclusion of some others (e.g., Givvin & Santagata, 2011), thereby potentially missing important facets of teacher learning. Designs for teacher learning that address these discontinuities need to make deliberate connections between the three planes. While this does not mean that all PD activities need to focus equally on all three, an overarching professional learning plan needs to include opportunities to account for and integrate these three planes. Doing so would support teachers to connect their activity in guided participation with the real constraints of their context and thereby support appropriation in their practice.

Integrating the Three Planes of Teacher Learning: The Case of Side-by-side Coaching

We examine one such teacher learning design, side-by-side coaching, to show how it provided opportunities to integrate the apprenticeship, guided participation, and appropriation planes. We first introduce the context of our study and then explore how side-by-side coaching connected the three planes, making them public and salient to both the teacher and coach.

Our description of side-by-side coaching is drawn from a larger study (Munson, 2018) investigating teacher learning. In our study, one coach (Munson) was working individually with three elementary teachers at a single school. The school was situated in a largely bilingual community with a high percentage of immigrant families, and with many families experiencing poverty and homelessness. All of the teachers participated in four weeks of coaching, with approximately two coached mathematics lessons each week. All of the coaching sessions were video recorded and transcribed for analysis. The focus of the coaching in this study was the practice of conferring, in which the teacher talks with small groups of students while they are solving a mathematical task (Munson, 2019).

Apprenticeship plane

Side-by-side coaching is a professional development activity nested within the practice of teacher coaching (e.g., Wei et al., 2009). During side-by-side coaching, the teacher and coach are positioned to co-participate in an instructional practice the teacher wishes to develop, each taking the lead at different moments and engaging in professional discourse about the ongoing events of the classroom and associated teacher decision making. Coaching is responsive to emergent student thinking, classroom interactions, problems of practice, and teacher questions.

Contextually, side-by-side coaching was enabled by two related factors: how the teachers were positioned with agency by their administration and by the coaching itself. This plane was made visible in the teacher-coach discourse when teachers made verbal reference to their larger contexts, bringing them into dialogue with their teaching and professional learning. The district delegated to teachers the authority to choose instructional materials, pedagogies, and the order and pacing of content. Without being bound to district-constructed approaches and timelines for teaching, the teachers had relative agency to engage in their own professional learning alongside students and make instructional decisions based on their ongoing observations of emergent student thinking. This was particularly important given the heterogeneous needs and prior experiences of students in this diverse district. Similarly, side-by-side coaching positioned the teachers to choose when and if to lead instruction and when and if to ask the coach to do so. Teachers could and did make direct and immediate requests to the coach for support, continually co-designing their professional development experience. Teacher agency played a pivotal role in the design of side-by-side coaching as an effective form of PD.

Guided participation plane

Guided participation can include deliberate instruction, incidental noticing, engagement via observation, and joint, hands-on involvement (Rogoff, 1995). All these forms of learning-in-partnership emerged in this case of side-by-side coaching. As in other forms of PD, at the beginning and end of each lesson, the coach and the teacher engaged in conversation together to discuss their goals, questions, and reflections. Beyond these pre- and post-lesson discussions, a key feature of side-by-side coaching is that teacher-coach interactions also appeared in the midst of instruction. The teacher or the coach would pause conferring with students and engage in joint sensemaking or decision-making. For example, during one conversation between a teacher and student, the coach jumped in and said, "What are you noticing? Can we pause for a second?" The teacher responded, "The explanation is based on the numbers, but not on the context." The coach and teacher then determined together the best route forward to support the student.

Side-by-side coaching included the teacher observing the coach and the coach observing the teacher. However, the plane of guided participation was most visible when the coach and teacher engaged in shared instruction and conversation with students. In these lesson segments, the teacher and coach were both vocal participants, asking and answering questions of and from students with relative frequency. The following excerpt illustrates such an interaction, which occurred as a pair of students were working to find the difference between two numbers and the teacher and coach prompted the students to contextualize the operation to support sensemaking.

> Teacher: Can you just say a quick story around these two numbers?
> Student: Matias went to the beach and he found 83 shells. Then his brother Noelle found some more. Noelle put it in Matias' bucket and now Matias has 93 shells.
> Coach: What's the question at the end of that story?
> Student: How many shells did Noelle give to Matias?

This pattern of shared leadership offered unique opportunities for the teacher and coach to learn together, build a shared understanding of student thinking and decide how to respond.

Appropriation plane

Inherent to side-by-side coaching is the goal that teachers will incorporate their learning into their practice in a meaningful way, within and beyond the context of the coaching itself; the interactions are intended to be transformative. Teachers in this study had two venues for appropriation: taking leadership of teaching during side-by-side coaching and connecting coaching to teaching outside the presence of the coach. When the coach was present, the teachers took leadership over instructional practices for bounded moments in which they could try the practices being learned. The coach was positioned side-by-side with the teacher to co-witness these instructional events, and the teacher could recruit the coach back into guided participation or debrief with the coach immediately afterward. In this way guided participation and appropriation were commingled, and the teacher had agency to move between planes.

The side-by-side coaching in this study was interlaced with days in which coaching did not occur and teachers had sole responsibility for teaching. Teachers had the opportunity to connect the interactions within guided participation with their independent teaching as they worked to appropriate the practices being coached. We saw evidence of connections between coached and independent lessons as teachers made deliberate discursive moves to weave these events together, either referencing their appropriation in a prior lesson or their plans for a future lesson when the coach would not be present. For instance, a second-grade teacher told the coach, "the other day we were doing a lesson and [the students] were doing this thing where they were counting forward and then going back, and I was so confused." In doing so, she invited the coach to make sense of student thinking with her, knowing that such strategies might appear again. Teachers also connected to future teaching, as when the coach and teacher discussed making a chart the next day with the task clearly written out after some confusion during the coached lesson; the teacher commented, "I can imagine if we're gonna start dealing with remainders, that it's only gonna happen more often," reinforcing the need for clarity. These connections between lessons with and without side-by-side coaching represented bridges between guided participation and ongoing appropriation.

Bringing the planes together

Because side-by-side coaching is embedded in the daily work of teaching, this coaching activity integrates context, guided participation, and appropriation in a single professional learning experience in a way that other professional development structures might not. While guided participation is the most prominent plane of interaction, with the coach serving as a more knowledgeable other (Wenger, 1998), that participation is fully coherent with the teacher's context for practice and provides continual opportunities for appropriation. We argue that this coherence is mutual, in that while the teacher experiences a more coherent learning environment, the coach gains access to the information necessary to provide that coherence.

Designing for Teacher Learning

Research in teacher learning and PD have long pointed out the discontinuities between the PD learning environment and teachers' practice in the classroom, often attributing these to the design of teacher learning experiences (c.f., Givvin & Santagata, 2011). We argue that one way of understanding the source of these discontinuities is the lack of attention to all three of Rogoff's planes of analysis, which can be seen, alternatively, as *planes of design*. In designing for effective teacher learning, we argue that teacher educators, professional development providers, and coaches must consider how to acknowledge, address, and create opportunities across the apprenticeship, guided participation, and appropriation planes.

Side-by-side coaching offers one structure that can do so, but such activities must be situated within a larger framework for ongoing teacher learning, one that incorporates learning opportunities in and out of the classroom, to learn and enact practices collectively and independently. Sequencing opportunities such that they integrate teachers' larger context with guided participation and opportunities for appropriation across a broad and intentional PD arc could create coherence for teacher learning, particularly in light of the field's call for ongoing and sustained PD (e.g., Darling-Hammond et al., 2017; Desimone, 2009). Side-by-side coaching is then one piece of this larger landscape and proof of concept that such integration is possible.

Those designing for teacher learning should consider questions such as: Where does (or could) the PD design attempt to bridge these planes? What professional learning activities might do so? How can these kinds of bridging activities be integrated into the arc of ongoing PD?

Analytically, we call on researchers to apply the lens of Rogoff's three planes to PD designs to augment previously established ideas about effective professional development.

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ABSTRACTS FOR ASSESSMENTS: DESCRIBING A SUMMARY STATEMENT

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Quantitative assessment development is a challenging process. The ways in which an assessment might be used, as well as how its score can be interpreted should be clear to intended users. This manuscript provides a discussion about important and useful elements that should be provided by assessment developers. In turn, this information can foster greater usability and portability of quantitative assessments, which can support scholarship focusing on a specific issue.

Keywords: Assessment

Quantitative research requires the use of measures, instruments, or assessments that allow users to draw conclusions from data they collect. When a research purpose aligns with a previously created assessment, then it seems plausible to use it. On the other hand, if there are no assessments to measure a desired construct, then an individual or team must choose whether to develop one. In either case, a statement describing how to use the assessment might be employed to make scholarly decisions. At present, there is little guidance about what goes into such a statement for potential users of quantitative assessments. The purpose of this manuscript is to present list of recommendations to include in a summary statement for an assessment and then provide an example summary statement to highlight the recommendations.

Related Literature

The *Standards for Educational and Psychological Testing* ([*Standards*] AERA et al., 2014) describe five validity sources: test content, response process, relations to other variables, internal structure, and consequences from testing/bias. Reliability is a related component of the *Standards* but is not one of the five sources. These sources describe categories in which evidence may be grouped in order to make score interpretations and effectively use a measure. The *Standards* note that it is inappropriate to use phrases such as “the validity of the test” or that a test is valid and instead, encourage a focus on validation as “...the degree to which evidence and theory support the interpretation of test scores for proposed uses of tests” (p. 11). Thus, validation is a process and the validity evidence supports or refutes the score interpretations and uses (Kane, 2006; 2016). This may seem like a small language shift but such a shift has serious implications because a valid test conveys a different idea as opposed to a valid score interpretation from a test.

Historically speaking, this shift has been slow to happen in some mathematics education communities. For example, Bostic and colleagues (2019) analyzed *Journal for Research in Mathematics Education* (1970-2017) manuscripts examining elementary students’ learning outcomes to discern (a) whether validity evidence for uses was provided and if so, (b) how was the evidence presented. One result from that analysis was that seven of 97 manuscripts (7%) that used quantitative measurement with elementary students’ outcomes described any validity evidence associated with their instruments. It was most common to describe test content evidence from an expert panel review as well as a reliability statistic. From the 1980s onward, it became increasingly common for *Journal for Research in Mathematics Education* authors to discuss an author-created measure. It is difficult to determine whether the instruments described in these published articles might be useful for a scholar's purpose because it is unclear how to administer, score, and interpret results from the measure. A framework to help measure developers describe these aspects has potential to increase assessment usability and assist scholars seeking to conduct quantitative research within mathematics education contexts. While in-depth descriptions of assessments are still warranted (e.g., validation research), succinct and explicit summaries are needed for readers to quickly and effectively discern whether to consider an assessment for a desired use. Concomitantly, peer-reviewed assessment and validity research within mathematics education contexts has increased substantially in the last five years; hence, a need to have a shared framework for communicating a summary statement related to a quantitative assessment. This manuscript responds to the question: What should a summary statement for an assessment contain?

Method

Context and Participants

This qualitative study stems from work during a NSF-sponsored conference. The conference lasted two days. Attendees were selected from an application process that brought together 41 scholars with expertise in mathematics education, mathematics, psychometrics, or applied measurement. This group included 35 terminally-degreed individuals working in industry and university settings, as well as six graduate students. A major goal of the conference was to identify a set of recommendations for the elements to include in an interpretation and use statement (aka purpose statement).

Data Collection and Analytical Process

A set of elements were initially generated by the conference leaders. These elements were based on important elements highlighted in the *Standards* and provided to conference participants. Conference attendees were asked to draft an example summary statement for an assessment around a construct of their choice using these elements as a starting point. They were asked to note elements to include and eliminate from the provided list, and to add additional elements to discuss for inclusion. This small group work time was followed by a whole group discussion on the common elements to include in the summary statement. These small and whole group recommendations were incorporated in the elements/questions list and the document was further expanded to provide a draft description of each element/question. A revised document was used by small groups of participants to draft a new summary statement and provide feedback on the elements in the revised document. A whole group discussion was held and IUS element suggestions were solicited. The small group notes, example IUSs, videorecordings from the conference, and field notes from the whole group discussion were analyzed following the conference and used to craft a set of reporting recommendations for elements of the summary statement. Four researchers (i.e., the leaders of the conference) used inductive analysis (Creswell, 2012) as a tool to develop the summary statement. The inductive analysis started with re-reading (or re-listening) to materials (e.g., written work and recorded statements from the conference). Step two was to make memos consisting of initial ideas stemming from this examination of the data. Step three was to reflect on those memos as a way to synthesize them into support (or not) for aspects of the summary statement. This is needed as evidence to ground the summary statement in validity. Step four was to search for evidence within the data sets to support components of the summary statement. Step five was to search the data for counter evidence. Impressions with a paucity of counter evidence and a large set of evidence were retained. The sixth and final step was crafting clearly elements to share broadly as a summary statement.

Findings: A summary statement

We present the recommendations for a summary statement first, then describe some of the comments surrounding its development. The ten elements were grouped to better visualize three different aspects of a quantitative measure: Construct articulation, operationalization and administration, and scores. Construct articulation provides justification for measuring the construct and clarifying its importance. Operationalizing and administering the measure is intended to give information about how the measure should be used. Scores and scoring describe aspects related to scoring and the limitations/delimitations related to the measures' scores.

Aspect	Interpretation and Use Element
Construct Articulation	#1. Why measure this construct?
	#2. Why is it important to measure this construct?
Operationalization and Administration	#3. How is the construct measured?
	#4. Who is the target population?
	#5. What is the intended context for administration?
	#6. What are associated costs with using the instrument?
Scores and Scoring	#7. How are scores determined?
	#8. What are intended interpretations for scores?
	#9. How should scores be used?
	#10. What known warnings or cautions are important to consider?

Figure 1. Summary statement to describe score interpretations and uses of a quantitative assessment

There was consensus that the summary statement should be written for an end-user to make a decision about (a) whether the score interpretation from an assessment aligns with an intended use and (b) the degree to which the assessment aligns with a desired purpose. One of the participants working in the assessment industry, Melissa, said that "You still need the details for an instrument and its uses. A summary statement is a quick read." Lucas, a university faculty, said that "This summary statement can tell you whether the instrument actually measures what it says it does. It can also show where there are gaps in the validity argument to further explore." As a result of video, audio, and written data, we reached the conclusion that the summary statement provided necessary and sufficient evidence for potential end-users.

An Instantiation of the Summary Statement

We present an example summary statement for a problem-solving measure developed by an author of this manuscript. There are numerous peer-reviewed manuscripts detailing validity evidence and arguments for this problem-solving measure (PSM); hence, it provides a brief overview for potential measure users and administrators. It should be interpreted cautiously and provide readers with an example of a potential summary statement for an actual instrument.

The PSM3 measures students' problem-solving performance within the context of the third-grade Common Core State Standards for Mathematics (CCSSO, 2011). Past research has demonstrated that problem-solving measures (a) are large-scale in nature (e.g., PISA), (b) measure problem solving but the mathematics content does not align with instructional standards, or (c) measure problem solving without drawing on mathematics content (see Bostic & Sondergeld, 2015). Thus, the PSM3 fills a need as a problem-solving measure that aligns with instructional standards used in many states within the USA. It has 15 items displayed as word problems. Each is presented as a constructed response task. Students are asked to clearly write their answer on a provided line. The target population is English-speaking, grade-level students. PSM3 administration is typically performed during instruction for and can last 120 minutes; however, most students finish within 90 minutes. There is no difference in students' outcomes due to the completing the PSM in one sitting or across multiple sittings (e.g., six, 20-minute sittings). Calculators are not allowed for administration. Those interested in using the PSMs may contact the authors for pricing. Each item is scored dichotomously, which conveys the same information as partial credit scoring (Carney et al., accepted). Respondents' scores may be calculated as percent correct. Scores may also be analyzed using Rasch to explore how students' performance compares to norms. Results from Rasch analysis should be interpreted as information about students' problem-solving performance related to CCSSM content. Such Rasch results also convey students' outcomes compared to peers and norms. PSMs are designed to complement other data about students' mathematics outcomes and be interpreted as a single touchpoint of students' outcomes. PSM data are suitable for research, evaluation, and school-based needs and as seen in this manuscript, robustly address validity *Standards* (AERA et al., 2014). Results are not intended to track students into different mathematics classes.

Discussion and Implications

This summary statement is intended to provide scholars working within mathematics education contexts a shared perspective to convey information about their quantitative assessments. It functions much like an abstract serves a manuscript or proposal – offering at-a-glance information. This summary statement also addresses the five validity sources, which may be further unpacked. For example, the ways in which scores are analyzed using Rasch analysis tells a reader that the PSM3 results are measured in logit units, which cues a reader to deciding whether that suits their needs. One implication from this research is to further engage the mathematics education scholarly community in ways that encourage sharing measures, replication studies, and offer greater access to quantitative measures. Kane (2016) and the Standards (AERA et al., 2014) have recommended that clearly identifying key information about measures has potential to improve measurement practices.

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ATTENDING TO AIMS IN ALGEBRA: THE JUGGLE STRUGGLE

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In this report, we explore the nature of aims for algebra instruction. First, we examine the major aims that have informed algebra education and curriculum reform from the 1960s into the current era. The relationships between aims are marked by compatibility as well as tension. We argue for researching and viewing aims as enacted priorities that are revealed through the everyday choices algebra educators make.

Keywords: Algebra and Algebraic Thinking, Curriculum, Research Methods, Teacher Beliefs

Algebra is a versatile subject. Scholars argue that algebra fosters generalization (Usiskin, 1995) and the recognition and use of structures (Kieran 1989). Algebra instruction can raise students' awareness of social injustice (Gutstein, 2006) and encourage autonomy (Kosko, 2016) and creativity (Chiu, 2008). However, the multiplicity of aims for algebra education can also bring a real challenge to today's algebra teachers: How should we coordinate the many aims for algebra learning? Should teachers organize their teaching with an equal emphasis on all possible aims that algebra education can carry? How do teachers make decisions on the aims that they pursue? The purpose in this paper is to discuss the need to identify and coordinate the many aims for algebra education.

Four Enduring Curriculum Aims

It is helpful to organize aims into a framework that captures the most central and enduring purposes for teaching algebra. One useful framework was developed by Kliebard (2004) while characterizing education during the early twentieth century. Kliebard proposed four major groups: humanists, developmentalists, social efficiency proponents, and social meliorists.

7. Humanists cherished western cultural heritage and the disciplinary value of classical subjects that increase students' mental power.
8. Developmentalists believed that the natural cognitive or psychological development of children should be given first priority when determining teaching content.
9. Social efficiency proponents concerned themselves with the needs of schools in a rapidly changing society, turning to the standardized techniques of industry and business.
10. Social meliorists believed education should actively foster social equity.

These four categories offer a means to explore the aims mathematics educators have emphasized throughout distinct historical periods.

During the 1960s, an array of curriculum projects known as the New Math (Phillips, 2014) took place in which mathematicians sought to ground school mathematics in the structure of the discipline. For instance, algebra during the new math movement was taught as an axiomatic system. Educators emphasized the importance of revealing the inherent and hidden structures behind algebra, such as set theory and concepts from abstract algebra (Herrera & Owens, 2001). Because this curriculum trend was led by mathematicians and emphasized the disciplinary value of mathematics, we might regard their aims during this period as in line with humanists.

By the 1980s, dissatisfaction with New Math made space for alternatives. Constructivism was one important response. Constructivists (e.g., Steffe & Kieren, 1994; Confrey, 1990; Ernest, 1994) felt that the psychological realities of young children, rather than the professional norms of mathematics as a discipline, dictated the aims of teaching and learning mathematics. Through research into student thinking and learning, algebra teachers started to recognize a significant gap between formal mathematics and students' own experiences with mathematics, and topics such as the transition from arithmetic to algebra gained attention in the field (e.g., Filloy & Rojano, 1989; Schoenfeld & Arcavi, 1988). Inspired by prominent psychologists such as Piaget and Dewey, constructivism can be characterized as a shift from the earlier humanist approach toward a developmentalist approach.

In 1983, the National Commission on Excellence in Education published a report titled *A Nation At Risk*, using an alarmist tone to bring national attention to perceived weaknesses in the American education system. The report's galvanizing influence formed the backdrop for efforts to create standards, measurement tools, and accountability policies for systemic educational improvement. These efforts targeted the efficiency of mathematics education as a system, working to cultivate mathematical knowledge as widely and effectively as possible. In the pursuit of efficiency through standardization and accountability, other educational aims were sometimes pushed aside when achievement was used as the indicator of national prosperity (Berliner, 2011). Teachers in algebra classes have felt pressured by the need for test preparation, adopting pedagogies with certain compromises and sacrifices (see Gutstein, 2006).

An ongoing movement in mathematics education that can be associated with aims of social meliorists is known as the "sociopolitical turn" (Gutierrez, 2013). In recent years, an increasing number of socially-minded mathematics educators proposed that teachers of mathematics should use their instruction to take part in solving social problems to create a more equitable society. In algebra education, scholars have promoted more culturally relevant pedagogies and equity-centered problem-solving approaches in teaching (e.g., Ligocki, 2017; Boaler & Sengupta-Irving, 2016; Gutstein, 2006).

The history of algebra education suggests that at a broad level, aims differ, aims can rise and fall, compete for attention, and overlap in complicated ways.

Aims and Priorities

The overview above suggests that there has not been a single, uniform idea about what constitutes "good algebra." Rather, educators' visions of algebra education have fluctuated throughout history in response to different but persisting educational aims. How then should educators navigate the existence of different aims?

We argue the first step in navigating aims is to think of aims as potential priorities. Priorities are objectives that require intentional effort. Consequently, the tension between aims plays out in a subtle dynamic as educators at all levels make countless choices about what should be taught and how. Therefore, the struggle between aims is not only an ideological debate but is also a practical challenge that algebra educators face every day.

To illustrate such a point in more detail, we explore one form of tension in algebra teaching between two fabricated teachers named Jack and Rose. Both teachers are preparing lessons with the main purpose of helping students to become familiar with multiplication.

Jack graduated with a master's degree in mathematics education and enjoyed reading research about students' algebraic conceptions. Therefore, in preparing the lesson, Jack decides to mimic an activity that Kaput (1999) highly praised, where the teacher helped students to informally prove the commutative property of multiplication by using arrays of sticks. Jack structures the lesson by planning to first ask students to use arrays of sticks to represent the products of different integers, such as 4×7. He expects students will likely generate at least two ways of representing the product (4 rows of 7, and 7 rows of 4). Jack will leverage those activities and invite students to think about whether different representations will have different total numbers of sticks. Then, Jack may guide students to realize that reversing the order of multiplication is exactly like rotating the number of rows with the number of columns for arrays of sticks. Since transposing rows with columns does not change the total number of sticks, changing the order of multiplication should preserve the product. In general, Jack may hope the students can both practice multiplication through this project and engage in other desired mathematical activities such as generalizing and creating mathematical representations.

Rose also graduated with a master's degree in mathematics education and enjoyed reading research about equity in mathematics classrooms. Therefore, in preparing her lesson, Rose decides to create a mini social project similar to what Gutstein (2006) has shared. Rose selects water consumption as the central issue. Rose may start to provide students a list of common water-consuming activities along with the average water use of each and ask students to first decide the gallons of water they think are needed for an average person or family per day. (For example, washing one's hands uses 2 gallons of water, so a person who washes his or her hands 4 times per day requires 8 gallons in total.) Then, Rose may provide students with information about how different nations have different average rates of water consumption per individual and ask students to calculate an average person's possible water-consuming activities depending on the country of residence. Through careful sequencing and structuring, Rose hopes that the students not only complete a list of multiplication problems but also use the results of their multiplication to have a broad understanding of the international inequality of water consumption and develop good habits of conserving water.

Jack and Rose may or may not know about any theoretical categories of aims describing their choices. Still, consciously or unconsciously, every pedagogical choice that Jack and Rose have adopted is also a choice between different educational priorities and aligns with different educational aims. Indeed, research on teacher beliefs has widely reported the following: a) teachers develop a complicated set of values and beliefs; b) those values and beliefs guide and influence their everyday teaching, planning, and assessment; c) those values and beliefs frequently do not need evidence to back them up; and d) direct training in certain pedagogical models shifts teachers' beliefs and principles (e.g., Rimm-Kaufman et al., 2006; Kagan, 1992; Richardson, 1996). Building from this literature, we argue that different educational aims act as distinct educational priorities that influence almost every instance of small or large decision making throughout the educational process. Teachers face choices between educational priorities when they are picking which task or activity to implement, researchers face choices between educational priorities when they are deciding which topic to research, and administrators face choices between educational priorities when they are judging which curriculum and policy to use.

Such a conceptualization of aims as educational priorities is consistent with our earlier discussion of the historical fluctuation of aims in the algebra curriculum. Researchers and educators who advocate a particular type of aim rarely deny the value of other possible aims. However, they do tend to make an intentional effort to prioritize their own preferred aim over others during research and curriculum reform. Thus, in a sense, aims are commensurable, as one's choice in picking a certain priority does not suggest one's denial of the value of other priorities. But aims conflict with each other as the options contributing to different aims compete for educators' intentional effort. The question we are left with is how the conceptualization of aims as educational priorities brings new insight into the work of juggling between aims.

Implications and Recommendations for Future Research

We derive several important implications from conceptualizing aims as priorities. The first implication is that aims are not cost-free and tensions between aims will inevitably persist. If competition between aims is viewed as a philosophical dispute, then aims might be reconciled at a theoretical or ideological level by weaving aims together into some grand, comprehensive quilt. However, situating the juggling of aims as an empirical reality of making choices between educational priorities suggests that tensions will persist and prioritizing aims will always have its costs.

Second, the tension and coordination between aims should be informed by research. All teachers, researchers, and policy makers are constantly picking their own priorities in their decision making and selecting their own preferred aims for their work. Consequently, there is a need to develop theoretical constructs in helping educators in all branches to conceptualize the tension and tradeoff between each aim along with an aim's relative affordances and constraints. To make aims an explicit object of research calls for expanding existing branches of research. Much of mathematics education research can be summarized as design science (see Cobb, 2007) in which teacher-researchers attempt to study and improve mathematics teaching and learning by drawing from various paradigms of scientific inquiry. When researchers conduct design science, they choose aims somewhat freely and they study the settings in which those aims can most profitably be observed and improved. Given an aim that is deemed valuable *a priori*, what are the principles by which to attain it? This research is useful, but we call for new research that adopts a different underlying premise: Given a setting with competing aims at work, what are those aims, where do they originate from, how are they prioritized and negotiated, and what are the consequences or implications of attaining or failing to attain each aim? Such research intentionally surveys and coordinates different aims by addressing the "economy" behind various priorities. For instance, not all aims are equally viable in different content areas or settings. Similarly, some aims can be satisfied with a small amount of intentional effort while others require more. Some aims have broad implications, others do not. Knowing the economy of aims helps researchers and practitioners to prioritize aims via a rigorously informed and justified process. (For interested readers, we recommend Pais (2013), Lundin (2012), and Wagner (2017) as some relevant work.)

The third implication is to respect educators holding different aims. This report does not call for a hierarchical ranking of all aims. Rather, research provides perspective to select aims more clearly. This proposal echoes Rorty's (1979/2009) idea of hermeneutic philosophy and Piaget's (2013) view that a central objective of philosophy is the "coordination between values" (p. 3). Research and scholarship about the aims of algebra education do not function as supreme guidance which teachers ought to follow, but rather as an instructive knowledge base that educators consult when selecting values, setting aims, and working to attain them (see Hiebert, 1999). We respect people’s right to pursue different aims, but just as importantly, we hope every choice can become an increasingly informed and justified choice.

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RETHINKING AGENCY IN CRITICAL MATHEMATICS EDUCATION

REDEFINIENDO AGENCIA EN LA EDUCACIÓN MATEMÁTICA CRÍTICA

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This paper focuses on critical mathematics education (CME) and its attention to children using mathematics to effect change and develop critical consciousness. Given these tenets, in CME common characterizations of agency as children's ability to act on the world according to their own choices are tautological. A poststructuralist perspective can strengthen and further specify a conceptualization of agency. Accordingly, I argue that in CME agency involves merging, reshaping, and relocating discourses available to the child. Recognizing these emergent discursive practices entails a paradigm shift from viewing children as incomplete individuals in transition to adulthood, toward recognizing the multiple ways in which children can transform their own lives and conditions. I draw on examples from a third-grade classroom to illustrate the analytical possibilities of this CME-aligned, poststructuralist conceptualization of agency.

Keywords: Elementary School Education; Equity, Inclusion, and Diversity

Research on critical mathematics education research has yielded abundant descriptions of children showing (sometimes) newfound confidence. These children take initiative to generate their problem-solving strategies, as opposed to passively following others' strategies. Such descriptions frequently serve to illustrate agency (see, for example, Turner, 2012). These portrayals of agency have served the important purpose of highlighting that children are capable of solving mathematical problems in creative ways that make sense to them. These examples, however, rarely include children who remain silent during class discussions and children who resist teacher requests to engage with mathematical tasks (Langer-Osuna, 2018). Children's initiative and acting on the world seems valued as long as it is directed toward the pedagogical game set in place by the teacher (Valero, 2005). This notion of supporting agency (simplistically defined by now as children taking initiative) while simultaneously enforcing compliance with teachers' expectations is the apparent contradiction that this paper explores. This exploration is particularly important in critical mathematics education (CME) because of its focus on liberation and inclusivity (Frankenstein, 1983; Gutstein, 2007; Skovsmose, 1994). Accordingly, in this paper I propose a conceptualization of agency that honors the commitment of CME to liberation. I argue that in CME agency involves merging, reshaping, and relocating discourses available to children. In the following section, I begin by discussing agency as constitutive of CME.

Agency in Critical Mathematics Education

Grounded on critical theory and critical pedagogy, critical mathematics education questions taken for granted practices that perpetuate the oppression of specific communities (Freire, 1970; Kincheloe, 2004). This commitment is taken into action by continuously developing pedagogies that support liberation (Kincheloe, 2004). Such pedagogies challenge problematic assumptions in the teacher-student dichotomy, according to which the adult takes on the role of knowledge-bearer and decision-maker that liberates the unknowledgeable child. Instead, in CME, students and teachers come together in a relationship of collaboration where each person brings in unique and valuable expertise and skills. Broadening the educational goal beyond developing children's mathematics conceptual understanding, CME reconfigure the student-teacher relationship, the content of instruction, and how children use what they learn (Brantlinger, 2013; Gutstein, 2007; Skovsmose, 1994). In short, students and teachers work collaboratively using mathematics to resist oppression (Gutstein, 2007).

In CME, children and teachers make joint decision about what social situations to study (Brantlinger, 2013). In these explorations, CME rejects the traditional practice of teachers explaining how to solve a problem and children reproducing the procedure presented (Lawler, 2012). Instead, students solve problems in their own creative ways, assessing their own and each other's strategies and determining courses of action to transform social circumstances. Simultaneously, children develop critical consciousness about whose ideas are valued and about their own marginalization in mathematics education (Gutiérrez, 2002; Skovsmose, 2000).

In light of these tenets, defining agency in CME as children taking initiative becomes a pleonasm, as the teacher-child hierarchy is challenged and they both decide Overlooking children who do not engage in these processes in the ways that teachers expect, or insisting that these children conform to the goals and practices the adult in the classroom established results in the agency contradiction discussed above. That is, defining agency as taking initiative and focusing on children who do what they are expected to do involves subjugating children in ways that CME attempts to interrupt. Unexamined definitions of agency can inadvertently maintain well-intentioned practices that, in the end, limit agency, as I discuss in the next section.

Traditional Definitions of Agency

Frequently, agency in CME is defined as children's acting on the world following their own choices (Martin, 2000; Turner, 2012). This definition, however, appears to be incomplete because, as de Freitas and Sinclair (2011) explain, "given the complexity of agency, all definitions will prove to be inadequate" (p. 134). Sometimes, agency is used interchangeably with initiative, choice, and confidence (Davies, 1991). Other times, definitions of agency are implicit, contributing to an "I know it when I see it" approach (see, for example, Boaler & Greeno, 2000). Efforts to elucidate what agency is have narrowed the focus to academic agency, which in CME means mathematical agency. Mathematical agency emphasizes that children can generate and assess their own problem-solving strategies (Lawler, 2012). This focus on intentional thinking, is reminiscent of Valero's (2005) myth of the schizo-mathematics-learner "with a clearly divided self: that one that has to do with mathematics, and the one that has to do with other unrelated things. Of course, those other unrelated things are secondary" (p. 5). Instead of presenting a mathematician identity as available for children, the overemphasis on mathematical agency imposes such identity. Choice—a component of agency—becomes an illusion, as lines of action are limited to those where the child pursues mathematical ideas.

These traditional definitions of agency have implications regarding what is valid evidence of agency. This evidence frequently involves observable behaviors or actions attributed to an individual (Chao & Jones, 2016; Martínez & Ramírez, 2018). Agency, however, is also portrayed as something that an individual senses and feels (Louie, 2019). In either case, agency is defined in a binary: It either is or is not present, felt, or used. Although this approach allows to operationalize analysis of agency, the contradiction mentioned earlier emerges again, as certain instances seem to simultaneously show agency and subjugation. Such is the case discussed before about making a mathematician identity available but also imposing such identity. In the next section, I discuss how a poststructuralist definition of agency can help make sense of this apparent contraction.

A Poststructuralist Definition of Agency in CME

In poststructuralism, discourses are “the complexes of signs and practices that organize social existence and social reproduction. In this view, a discourse delimits the range of possible practices under its authority and organizes how these practices are realized in time and space” (Peirce, 1989, pp. 403-404). Discourses guide what an individual and a social group see as possible. At any given moment, multiple discourses influence simultaneously our interpretations of reality (Davies, 1991; St Pierre, 2000). Although we may not be able of escaping the discourses that influence us, we may become aware of these discourses and their effect in our decisions (Davies, 1991). This awareness allows us to reconfigure and resist discourses (Davies, 1991; James, 2009). In CME, a few studies have begun to advance a notion of agency as children transforming and resisting available discourses (Louie, 2019; Stinson, 2013).

Drawing on a poststructuralist perspective, I propose agency in CME as children acting on the world by merging, reshaping, and relocating discourses available to them. Consistent with CME’s focus on critical awareness, in this conceptualization of agency children become aware of how they are constituted through dominant discourses. Moving awareness to action, children may take up, transform, or resist discourses, including discourses about what agency should look like. This kind of agency allows for children to resist aspects of CME, some teacher decisions, and the imposition of specific identities. This means rewriting master narratives that limit what children and adults see as possible.

In this conceptualization, agency is interpretative and comes into existence as felt, experienced, used in multiple concurrent ways. The adult is no longer the one who gets to decide on the presence or absence of agency. Instead, in any given situation a child can simultaneously feel a sense of agency when interpreted from one discourse and a sense of subjugation when interpreted from another. Both interpretations are legitimate and constitute reality for the child. An adult observer can interpret this situation differently, drawing on discourses available to them. Iterative interpretations from multiple discourses may create a nuanced story of agency.

Classroom Example

I illustrate this poststructuralist definition of agency in CME with an example from a third-grade Spanish-immersion classroom in the Midwest of the United States. There were 21 children in this classroom. The teacher was a Spanish-English bilingual, US-born Latina. I, a Spanish-English bilingual, Latino researcher visited this classroom regularly as part of a larger study.

This example comes from a teaching unit on what it means to take something away, addressing both the mathematical connotation (i.e., subtraction) and other social connotations (e.g., taking away people’s rights or properties). During one of the lessons, the class discussed how to solve five take away eight. While the discussion went on, a child, Anna (names are pseudonyms), went to a cabinet and took a box with linking cubes. She marched to the front of the classroom, carrying the box, and she raised her voice, demanding the class’s attention. Her classmates quieted down. Anna indicated that it was necessary to act out the situation. She grabbed five cubes and said she needed to give eight to me (the researcher in the classroom). She handed out one cube at a time and she counted from one to five, stopping when she had no cubes left. She announced that, since she did not have any more cubes, it was not possible to take away eight from five. The teacher worried that only one child would monopolize the whole class discussion, so she instructed children to think quietly about Anna’s explanation. After a minute, the teacher randomly picked a popsicle stick from labeled with one of the children's names. Reading the name, the teacher called on Joanna, who shrugged and, in a low voice, said that it was not possible to take away eight from five.

Following common definitions of agency, this interaction could be taken as evidence of agency, attributed to Ann as an individual. Ann took the initiative to use the cubes and act out the problem, without any adults or children having told her what to do. An interpretation of Joanna' agency is more nuanced. Although well-meaning, trying to distribute talking time, the teacher unintentionally subjected Joanna to comply with her expectations. Joanna was put in a position where she had to prove she had been following what Ann did and that she had something to say about it. Ann complied with these expectations, but she seemed more interested in deflecting attention. This example illustrates the contradiction between apparently supporting agency while simultaneously limiting the child's possible lines of action to those that complied with the teacher's expectations. Joanna could have decided not to answer, ignore the teacher, or express that she was not interested in the discussion. These lines of action, however, would likely have turned her into the target of an intervention for her to eventually engage with the mathematical task. In short, Joanna is encouraged to express her agency as long as it involves following adult-approved lines of action.

Discussion

Supporting agency on the condition that it be used only on a predetermined endeavor renders agency irrelevant, at best, and illusive, at worst. This is particularly important in CME because of its goal of promoting liberation and because it challenges hegemonic discursive practice. To further align definitions agency and CME tenets, I proposed a definition of agency that avoids presenting some children as agentless. This poststructuralism-informed definition argues for making room for children to merge, reshape, and relocate discourses available to the them. Instead of suppressing discursive practices such as silence, resistance, and apathy, children and teachers can work collaboratively to make sense of why these discursive practices emerge and how they should be handled. This collaboration entails a paradigm shift from viewing children as incomplete individuals transitioning to adulthood, to recognizing them as skillful and active participants who can transform their own lives and circumstances. That is, this entails an authentic and complete engagement with the CME purpose of promoting liberation.

Recognizing agency as interpretive, I provided a classroom example that shows how agency can come into existing in simultaneous and contradicting ways, depending on the discourses from which we draw. From discourses made available to a teacher, an interaction can appear as supporting agency by making room for a child to speak up in class. This interaction, however, can simultaneously be interpreted as subjugating the child to comply with the performative task of showing attention, interest, and mathematical understanding. Agency, then, appears as multifaceted and taking on multiple meanings when considered from the perspectives that specific discourses make possible.

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A TWO-LAYERED APOS ANALYSIS OF INEQUALITY NUMBER LINE GRAPHS

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Reasoning with inequalities and their solutions is important in mathematics. We provide a two-layered APOS genetic decomposition of the meaning of number line graphs as representing solution sets.

Keywords: Algebra and Algebraic Thinking, High School Education, Learning Theory

Inequalities and their solution sets are an essential aspect of various mathematics classes and topics. Number lines display solution sets of real numbers; the number line graph of a statement of the form P(x) displays all values of x that make P(x) true. We say a number c *satisfies* P(x) to mean that P(c) is true. For example, graphing $|x-9|\leq 4$ shows all the numbers between 5 and 13, since these are the numbers that satisfy $|x-9|\leq 4$ (Fig.1)

Figure 1: A number line graph of $|x-9|\leq 4$

In Figure 1, the number line graph is a visual representation of an inequality. Indeed, to graph an inequality on a number line is to graph its solution set. The limited research on solution sets demonstrates that students generally do not interpret inequality graphs as representing solution sets (Bicer et al., 2014; Frost, 2015; Mirin, 2020). We use Action-Process-Object-Schema (APOS) to describe ways that students might understand number line graphs as visual displays of solution sets. This genetic decomposition can provide a basis for an instructional intervention.

APOS is a cognitive framework that helps explain how individuals can develop action and process conceptions to ultimately encapsulate into mathematical objects. Dubinsky and McDonald (2001) explain an action as a "transformation of objects perceived by the individual as essentially external and as requiring, either explicitly or from memory, step-by-step instructions on how to perform the operation." (p. 276). A process comes from interiorizing an action or series of actions. An individual with a process conception can envision "performing the same kind of action, but no longer with the need of external stimuli" (Dubinsky & McDonald, 2001, p.276). We use the word "procedure" in its colloquial form to agnostically refer to both actions and processes. An object is the result of encapsulation of a process: "an object is constructed from a process when the individual becomes aware of the process as a totality and realizes that transformations can act on it" (p. 276). A schema involves actions, processes, and objects and how someone coordinates them. A genetic decomposition is an account of components of a schema. Specifically, it describes how someone can progress to an object conception by first developing an action then process conception.

APOS theory has been used in a variety of contexts, both finite and infinite. For a good example of a straightforward genetic decomposition, see Dubinsky and McDonald's (2001) account of left cosets -- in this case, the procedure being reified is the construction of individual cosets by performing the group operation. Dubinsky et al. (2005) use APOS to explain how individuals come to understand infinite objects using the set $\mathbf{N}=\{1,2,3\ldots\}$. An *action conception* involves enumerating members of the set by adding 1 repeatedly to result in a finite subsegment of **N**. We start with 1, we add 1 to get 2, we add 1 again to get 3, etc., for any finite number of steps. Since, as discussed earlier, an action conception involves actually performing the actions, it would be impossible to obtain the entire infinite set via an action conception. A *process conception*, however, involves interiorizing these actions without performing them and enables the individual to imagine carrying out the procedure *ad infinitum* and hence imagine infinitely many natural numbers. An individual might encapsulate such a process into an object (in this case, the infinite set **N**). The authors explain that encapsulation is what enables individuals to compare cardinalities and perform arithmetic on infinite sets. We observe that, with at least some infinite processes, there appears to implicitly be *two* action-process-object layers; the infinite one discussed (which we refer to as *global*), and a finite one corresponding to each "step" (which we refer to as *local*). Below, we elaborate on these two layers in the case of inequalities.

An Action-Process-Object View of Inequality Graphs

Unlike with the infinite set example described above, enumerating an individual member of our set (in the case of inequalities, a solution set) involves more than simply adding 1 to a number; it involves conceptualizing that number as satisfying the inequality (described in detail below). Additionally, a student must coordinate multiple (usually infinitely many) solutions and understand that these solutions simultaneously make the inequality true (to form a solution set). Accordingly, we use a two-layer approach.

The Local Layer

The *local layer* pertains to individual solutions. *action* understanding at the local layer involves being able to instantiate the inequality (substituting a value for x and computing each side), evaluate as true or false, then plot accordingly if true. For example, a student with a local action conception can plug in, say $x=8$, into $|x-9|\leq 4$, evaluate the resulting inequality ($1\leq 4$) as true, and then plot a single dot to represent $x=8$.

Interiorization enables the student to think about this procedure without carrying out the procedure themself; they can imagine the procedure as having been performed. The student has the knowledge that if they *were* to plug in 8 for x in the inequality, they would end up with a truth-value which could result in a dot on the number line. In other words, when a student is thinking of a process, they are not thinking of step-by-step computation, but instead thinking of the transformation from x-value to truth-value as a whole. Observe that at the local level, the action and process conceptions closely mirror the action and process conceptions for the concept of function as described in Breidenbach et al. (1992). However, instead of a particular value of x being mapped to a number, it is mapped to a truth-value and potentially a dot on the number line. At this stage, students are envisioning whether a particular number will be noted on their number line. A student with this conception understands that *if* they were to plug in a particular value, it would either satisfy or not satisfy the inequality (and if it did satisfy the inequality, then it is graphed).

The object conception is where our account radically diverges from Breidenbach et al.'s single-layer (1992) account of *function*. In the case of function, the object conception is an entire, possibly infinite set of ordered pairs. For our local conception, our object is more akin to a *single* pair; continuing with the example above, the relevant object is our value of x (8), paired with the corresponding truth-value (True) (i.e. the ordered pair (8, True)). The number line dot is a visual representation of this coupling. The local level is concerned only with individual solutions rather than the solution set.

At the local layer, the action, process, and object conceptions entail different conceptualizations of the dot on the number line. With a local action conception, the dot is an artifact of a mathematical procedure that the student performed step-by-step (action). With a local process conception, the dot is an artifact of an envisioned interiorized procedure (process): one that the student might not have performed themself but can envision having been performed. With a local object conception, the dot need not involve a procedure at all; instead, it represents a value of x (e.g. the number 8) statically coupled with a truth-value (True). However, some students with object conceptions might de-encapsulate to attend to the underlying procedures in some problem-solving situations.

The Global Layer

The global layer goes beyond individual values and involves how a student might come to understand the solution set in its entirety. This layer parallels the APOS analysis of the set of natural numbers described above: an action conception accounts for the one-at-a-time enumeration of members of the solution set. Instead of adding 1 like we did with the enumeration of **N**, the following procedure takes place: a value of x is substituted, the inequality is evaluated, a truth-value is noted, and a dot is possibly placed on the number line. An action conception enables this enumeration procedure to be repeated for multiple, but not infinitely many, values of x.

Like with constructing **N**, interiorizing the global procedure of repeatedly considering individual values is what produces a *global process conception*. Someone with a global process conception need not imagine enumerating every single value themself. To continue with our analogy with the construction of **N**, the process conception is what allows someone to imagine 1 being added an infinite amount of times to produce an infinite set. Similarly, the global process conception for graphing inequalities is what enables someone to imagine infinitely many values of x. Thus, a global process conception enables someone to give an infinite enumeration of values of x and the visual representation that that would produce: having a dot on every single number that satisfies $|x-9|\leq 4$ (Figure 1). Through encapsulation a student can move to an object understanding. A student with a global object understanding can produce the complete solution set to the inequality on a number line and recognize that it simultaneously represents all values that make the inequality true. It is at this level that a student can potentially operate on solution sets with Boolean operations (e.g. in absolute value problems) and recognize when operating on an inequality leaves the solution set invariant.

At the global layer, the action, process, and object conceptions entail different conceptualizations of the number line graph. As shown in Figure 2, an action conception only allows for finitely many actions and hence only finitely many members of the solution set. An action conception can result in a number line that looks indistinguishable from an infinite solution set. However, in the student's mind, they are understanding the number line in a way that is closely represented by Figure 2 below:

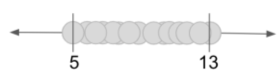

Figure 2: A global action conception of $|x-9|\leq 4$

For a student with a global action conception, their number line graph is an artifact of their finite mathematical activity of repeated work at the local level that they performed themself (plotting various values). Notice that this graph is incomplete, as the student understands it as involving only finitely many values. We can contrast this with a global process conception, in which the number line graph (a visual representation of the solution set) is an artifact of an envisioned infinite process. Like with the global process conception, the global object conception involves understanding the number line graph as complete and representing infinitely many values. However, it differs in that the number line graph need not represent any sort of procedure of instantiating individual values of x; instead, someone with such a conception understands the number line as statically showing the solution set.

Coordinating Layers

Like with other applications of APOS theory, we do not expect a student to progress linearly through each stage. Instead, a student's schema can involve a mix of various understandings at the local and global levels, within some constraints. A student with any of the local conceptions can have a global action conception, for example. How an individual element is enumerated depends on the local understanding of that student - the enumeration could be through substituting and evaluating (local action), envisioning such substituting and evaluating (local process), or simply listing a solution (local object). A student with a local and global action conception understands each individual point as an artifact of their own mathematical activity that they perform themself, and they understand the number line as a record of their repeated mathematical activity of plugging in, evaluating, and plotting individual values. A student with a global action conception and local object conception would understand each individual point on their number line graph as representing a solution to the original inequality, yet at the same time see their number line graph as an artifact of their own finite mathematical activity of enumerating individual solutions. Notice that local-object and global-action closely mirrors the "action" conceptions that Dubinsky et al. (2005) describe for enumerating an infinite set where enumeration of individual elements is straightforward and unproblematic, but the enumeration itself only allows for finitely many steps. While there is not enough space here to elaborate every way in which a student could have a y local conception with a z global conception, global and local conceptions are not entirely independent. For example, it would not make sense for a student to develop a global object conception and a local action conception. This is because it would not make sense to understand the number line graph as representing a solution set without understanding the notion of a point representing an individual solution.

Conclusions and Implications

Our two-layered genetic decomposition can provide the basis for a future learning trajectory, which is much needed in light of the literature that suggests that students do not attend to the ideas of solution, solution set, or truth-value (Bicer et al., 2014; Frost, 2015; Mirin, 2020). The number line can play an important role in this trajectory; it can act as a record-keeping device, which can help the student transition from a global action to a global process conception. This genetic decomposition suggests instructional interventions that involve substitution and evaluation of truth. For example, students often are confused by whether to place a closed or open circle at the endpoints (Bicer et al., 2014; El-Khateeb, 2016). Connecting the number line to the truth-value for that particular substitution would elucidate that if the value satisfies the inequality, a closed circle should be placed on the graph. Importantly, substitution can orient students to attending to the notion of truth-value.

In conclusion, the genetic decomposition outlined here can serve as both the basis for an instructional intervention and a framework for understanding students' development in their reasoning about the relationships between inequalities, solution sets, and number line graphs.

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STRUCTURING AND ENUMERATION: A PRELIMINARY DISCUSSION OF SPATIAL-TEMPORAL-ENACTIVE STRUCTURING

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Prior research has identified spatial structuring—the mental process of constructing an organization or form for an object or set of objects—as critical to students' development of spatial-geometric reasoning and understanding. We propose an alteration to this construct to include aspects of structuring that are especially salient in combinatorial enumeration (though also present in geometric contexts). Specifically, we replace "spatial" with "spatial-temporal-enactive" (or S) to include temporal and enactive aspects of forming and iterating spatial composites. Further, enumeration involves numerical structuring—the mental process of constructing an organization or form for a set of computations, formulas, or expressions. In meaningful, conceptualizations-based enumeration, students can link numerical structuring to S* structuring, a process we call S*-numerical linked structuring (or S*NLS).*

Keywords: Cognition; Advanced Mathematical Thinking; Geometry and Spatial Reasoning

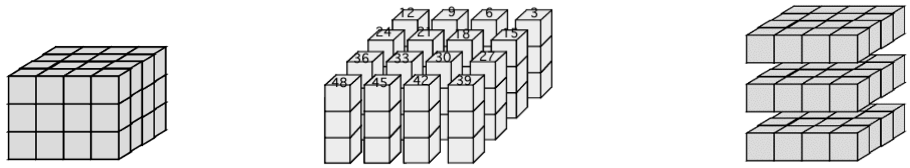
In this paper, we provide a preliminary discussion of a framework for conceptualizing students' thinking and reasoning about enumeration. Our focus is on combinatorial enumeration, though our framework can also be used to analyze student reasoning in additional, non-combinatorial enumeration contexts. We begin with an overview of the original definitions of spatial structuring (Battista & Clements, 1996) as well as numerical structuring and spatial-numerical linked structuring (Battista et al., 2018). We then introduce and discuss spatial-temporal-enactive structuring as an elaborated version of spatial structuring.

Spatial Structuring

Battista and colleagues have argued that spatial structuring is critical to students' progressions toward increasingly abstract and powerful spatial-geometric conceptualizations and forms of reasoning. Originally, Battista and Clements (1996) defined *spatial structuring* as "the mental act of constructing an organization or form for an object or set of objects" (p. 282), which includes, but is not limited to, establishing units, relationships between units, and composite units. In more recent research, Battista et al. (2018) analyzed the relation between spatial and geometric reasoning. In so doing, they identified two additional mental processes: *numerical structuring* and *spatial-numerical linked structuring* (or SNLS). Numerical structuring is "the mental act of constructing an organization or form for a set of computations" (p. 211), and SNLS is "a coordinated mental process in which numerical operations … are performed based on a linked spatial structuring … in a way that is consistent with properties of numbers and measurement" (p. 211).

As an example, consider the task of enumerating unit cubes in the $4\times4\times3$ cube array shown in Figure 1a. The SNLS in Figure 1b suggests a spatial structuring of the array into columns, with 3 cubes in each column. A student with such a spatial structuring might count by iterations of three, a numerical structuring linked directly to their spatial structuring (thus an instance of SNLS): 3, 6, 9, …, 48. Alternatively, a student might spatially structure the array into three 4×4

layers, which they might link to the numerical structuring 3×16, as shown in Figure 1c.



(a) (b) (c)

Figure 1: (a) A $4\times4\times3$ Cube Array, (b) A Column-Based SNLS, (c) A Layer-Based SNLS

Extended Version of Spatial Structuring: Spatial-Temporal-Enactive Structuring

In this paper, we introduce an elaboration of spatial structuring and SNLS, in particular to include combinatorial forms of enumeration. Rather than "spatial" structuring, we focus on an elaborated mental process of "spatial-temporal-enactive" (or S*) structuring. Structuring is a mental process of construction and is thus *active* and occurs *over time*. S* structuring highlights the temporal and enactive aspects of structuring by focusing on the organized sequences of actions and operations that students use to structure and ultimately enumerate spatial composites.

The temporal and enactive aspects of structuring became especially salient through our analyses of students' combinatorial reasoning in contexts where the spatial aspect seemed to be less important. For instance, consider the problem of enumerating the possible outcomes of a six-person race. The task is prominently situated within a temporal-enactive modality; the spatial nature of the task is implicit and is representational in nature. Nevertheless, visualization/imagery occurs in all sensory modalities (Barsalou, 2008; Moulton & Kosslyn, 2009), and even in contexts with a less prominent spatial component, students' mental representations (and mental models) are likely spatial (Winter et al., 2015).

Further, we distinguish two distinct forms of S* structuring that are especially pertinent to combinatorial forms of enumeration: inter-composite structuring and intra-composite structuring. We define *inter-composite structuring* as the temporal process by which an S* structuring is established for the *composites* within a set of spatial composites. We define *intra-composite structuring* as the temporal process by which an S* structuring is established for an *individual* composite (out of units and, potentially, sub-composites).

For example, consider the task of enumerating all possible towers 3-cubes-high, each tower containing 1 red, 1 blue, and 1 green cube. An inter-composite structuring might be: "RGB, RBG; BRG, BGR; GRB, GBR." But a different inter-composite structuring would be: "RBG, BRG, GRB, RGB, BGR, GBR." An intra-composite structuring might be: "I start with a red cube on bottom. Then I connect a blue cube on top of the red, then a green cube on top of the blue." A different intra-composite structuring could be: "I start with a blue cube in the middle. I then attach a red cube to the bottom of the blue cube, then a green cube to the top of the blue cube." Both intra-composite structurings yield the same tower (RBG), but they have important differences. A student using the first intra-composite structuring to construct towers might organize the *set* of towers based on the color of the bottom cube, but a student using the second intra-composite structuring might organize the towers based on the color of the middle cube. Thus, different intra-composite structurings can lead to different inter-composite structurings.

Additionally, we define S*-numerical linked structuring (or S*NLS) in a way that is consistent with Battista et al.'s (2018) original SNLS formulation. S*NLS is the mental process of coordinating S* structuring and numerical structuring in reasoning so that the corresponding

S* and numerical structures are conceptually linked to each other. According to Battista et al., the normal order of activation is S* structuring, *then* numerical structuring. That is, students construct a numerical structuring that is conceptually linked to an established S* structuring. However, meaningful S*NLS reasoning can also occur with the reverse order of activation, as is often the case in combinatorics. For instance, a student may be given a numerical/algebraic expression, then be asked to provide a combinatorial interpretation of the expression—that is, conceptually link the expression to S* structuring processes that describe what the expression might be used to count. Or, the student may be given more than one numerical/algebraic expression and be asked to explain why they can be used to enumerate a given set. One important example of the latter form of S*NLS is in proving that two expressions are equal by arguing (via S* structuring) that each expression counts the same finite set (Erickson & Lockwood, 2021).

We hypothesize that our framework provides an elaboration of Lockwood's (2014) set-oriented perspective and Lockwood's (2013) model of combinatorial thinking. Her model consists of three interrelated components: sets of outcomes, counting processes, and formulas/expressions. A set-oriented perspective emphasizes the role of sets of outcomes in solving counting problems. According to Lockwood (2014), "Individuals with such a perspective would recognize that different ways of structuring a set of outcomes might reflect different respective counting processes" (pp. 31-32). We believe our framework will shed light on the processes by which students structure and conceptualize sets of outcomes, and conceptually link these constructions to appropriate numerical/algebraic expressions. See (Antonides & Battista, in progress) for further discussion.

An Example of Using S* Structuring and S*NLS to Analyze Student Reasoning

We recently conducted one-on-one constructivist teaching experiments (Steffe & Thompson, 2000) with five undergraduate students, none of whom had studied combinatorics previously. Due to COVID-19, the teaching experiments were conducted remotely via Zoom. The goal of each teaching experiment was to instructionally guide each student to construct concepts and forms of reasoning about permutations, partial permutations, and combinations. In a pilot study, we used a sequence of tasks calling for the enumeration of n-cube towers, each containing n different colors of cubes, to motivate students' development of concepts and reasoning about permutations. Given these findings, we used a similar instructional sequence in this study, using squares instead of cubes (often, however, the students would refer to squares as cubes).

One of our students was AR, a third-year preservice elementary school teacher who used she/her pronouns. In her preassessment, and in her first few teaching sessions, AR exhibited a scheme in which she attempted to enumerate permutations by squaring the given number of objects being arranged. For instance, consider AR's response to the Recess Task.

Recess Task: Ms. McFrederick has 24 students in her 5th-grade class. In how many different ways could the students line up to go to recess?

AR: Um … I think you could just do, maybe like 24 times 24 to figure out how many ways they can line up. You can get the answer.

TR: Okay, 24 times 24. Can you relate this problem to a problem about towers of cubes?

AR: Yeah, so, if they were in a line, that could be like a tower. And each student will be able to be in each of the spaces from the tower, or the line I guess. So since there's 24 students, there's going to be 24 spaces for them to line up. And then each student will be able to be in one space, so since there's 24 spaces they can each be in 1 space. So you can multiply all the students by the number of spaces, just like we did.

From AR's description, we infer she conceptualized a given "line" of students as consisting of 24 spaces—a spatial structuring. Further, she reasoned that each of the 24 students could stand, or be placed, in each of those 24 spaces—a spatial-enactive structuring. Based on this S* structuring, AR reasoned multiplicatively: 24×24. However, she struggled to justify why multiplication was appropriate. We interpret this as an example of numerical structuring partially, but not fully, linked to S* structuring, as her S* structuring was (we infer) underdeveloped and consequently unable to support a viable numerical structuring.

To help AR refine and further develop her S* structuring, she was prompted to consider an analogous task with a smaller number of objects being enumerated (cf. Lockwood, 2015) and to write a list of the way in which students could line up (cf. Lockwood & Gibson, 2016).

TR: Do you want to try think about it, at least for now, with smaller numbers? Let's say Ms. McFrederick had just 4 students instead of 24.

AR: Yeah, um. So I believe you can still just do the 4 times 4, 4 students for each of the 4 spaces, maybe.

AR drew four circles arranged horizontally. She wrote "1" under the first (left-most) circle, then said "2, 3, 4." She then crossed out the "1" and rewrote it under the second circle, then wrote "2, 3, 4" under the remaining circles. At this point, AR began to realize there would be more than 16 possibilities.

AR: I think you would approach it differently than just the 4 times 4 then. Because … I think, when you arrange it—so yes, each student can be in each of the 4 spots. Like student 1 can go in all 4 of these spots, but for each time that student 1 is in a different spot, there's multiple ways for them [students 2-4] to be arranged. …

TR: It might be helpful if you just list out all the possibilities using 1, 2, 3, and 4 as your students. And try to list them in such a way that you can be sure that you listed them all.

AR: [Wrote 1234, 1243, 1324, 1342, 1432, 1423, then 2134, 2143, 2341, 2314, 2431, 2413.] So this right here is 12, altogether. And there would be 2 more groups just like this, because we would do the same thing for student 3 being at the front of the line and then student 4 being at the front of the line. So that would be 2 more groups, which would come to 12 just like this. So then we would end up with 24 ways.

We infer that asking AR to construct a list prompted her to organize—that is, establish an S* structuring for—the set of permutations of 1, 2, 3, and 4. Her inter-composite structuring was systematic, and her numerical structuring, 12 + 2(6), was linked directly to her S* structuring—an example of S*NLS.

Summary

In this paper, we have introduced and exemplified an elaborated version of spatial structuring, S* structuring, specifically so that we may accurately capture and analyze the structuring processes that underlie much of combinatorial enumeration. Naturally, due to space constraints and the preliminary nature of this report, we could not include a full discussion of S* structuring and S*NLS. However, a paper is underway (Antonides & Battista, in progress) in which we discuss these ideas in greater depth and provide multiple examples of S* structuring and S*NLS in students' combinatorial reasoning.

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MATHEMATICS TEACHER DECISION-MAKING AND THEIR SELECTION OF ONLINE CURRICULAR MATERIALS

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In this conceptual paper, I reflect on the growing influence of online lesson plan sharing websites on mathematics teachers' curricular decisions. Using the theoretical frameworks of teacher decision-making and curriculum deliberation, I explore individual and group heuristics that may impact mathematics teachers' lesson planning. I then offer a conceptual framework that details the considerations mathematics teachers engage with when choosing online curricular materials. I conclude with the implications and potential contributions to the field.

Keywords: Curriculum, Instructional Activities and Practices, Technology

Introduction

There are differing curricular policies in place for teachers in different districts and schools: several use textbooks, a number of schools and districts have adopted prescriptive curricula, some have created materials for teacher use, and others allow teachers to set up their own units and lessons (Gewertz, 2015; Pittard, 2017; Timberlake et al., 2017). As more instruction moves to online learning platforms and as teachers gain more freedom to provide curricular materials of their own in their classrooms, teachers are more often exploring online spaces for lessons and supplemental materials (Greene, 2016; Pittard, 2017; Tosh et al., 2020). The COVID-19 pandemic has additionally necessitated this approach across the nation. To date, there has been little investigation into teachers' decision-making process while selecting which materials to use from online lesson plan sharing websites and even less at the secondary mathematics level. Understanding teachers' decision-making provides a lens into how curriculum is chosen or created, implemented, and adapted according to student needs and the instructional environment.

Background

Websites such as Google, Pinterest, and Teachers Pay Teachers allow mathematics educators to share best practices, build professional learning communities across states, and support each other in developing instructional materials. However, these online spaces rarely control what lessons are posted on the sites, so there is no indication of true standards-alignment, appropriate breadth and depth of content, and accuracy of that content (Gallagher & Swalwell, 2019; Greene, 2016). Pedagogical and time considerations can also complicate mathematics teachers' decisions to download certain materials (Gallagher & Swalwell, 2019). As a result, teachers may not be utilizing standards-aligned content, or they may not know how to alter these lessons appropriately. Therefore, it is important to identify what impacts mathematics teachers' decision-making process when choosing curricular materials, so they can be supported in becoming critical consumers and making curricular decisions couched in research-based practices.

Theoretical Perspective

This work draws from the theoretical frameworks of teacher decision-making and curriculum deliberation to investigate how teachers choose and use online curricula. Stein et al. (2007) "temporal phases of curriculum use" framework provides insight into what teachers consider when using curriculum. Schwab's (1969) idea of curriculum deliberation at the individual and group levels provides a vehicle into understanding the complexity of decision-making.

Decision-Making

The idea of teacher decision-making has been extensively researched over the past 50 years or so (Holstein & Keene, 2013; Shavelson & Stern, 1981; Smith et al., 2018). Stein et al. (2007) present a framework on the temporal phases of curriculum use to highlight the phases of decision-making when teachers utilize curriculum in the classroom with the aim of student learning. Within each of these four phases, teachers engage in cognitive processes that are influenced by their beliefs, content and pedagogical knowledge, and judgement, thus altering the written curriculum even before it is enacted (Shavelson & Stern, 1981). Additionally, Shavelson and Stern (1981) assert that, during the planning phase of curriculum use, teachers balance activity flow, predictability during the lesson, choice of content, student needs, and instructional style, all while taking into account external pressures from policies, administrators, or other educators.

Mathematics Educator Decision-Making

In addition to teacher decision-making, Dingman et al. (2019) include the dimension of mathematical decisions, "defined as those decisions that influence students' opportunity to learn mathematics, and teachers' reasoning for those decisions" (p. 44). These decisions are made throughout the four phases of curriculum use and researchers have spent decades exploring what decisions teachers make during planning and enactment of lessons and why they make these decisions (Bush, 1986; Choppin, 2011; Dingman et al., 2019). Decisions surrounding curriculum materials are affected by teacher beliefs about the materials, the strategies teachers use to understand the curriculum, how they envision the materials will be enacted, and their capacity to use the materials in the classroom (Choppin, 2011).

Curriculum Deliberation

The cognitive processes of decision-making are mediated by the practice of curriculum deliberation both at the individual and group levels. First introduced by Schwab, curriculum deliberation involves contemplating the practical aspect of teaching and discussing curriculum concerns with a variety of educators to tease out multiple perspectives and approaches to curriculum and instruction, including the roles of the teacher, learner, subject matter, milieu (context), and curriculum-making (Johnston, 1993; Reid, 2010). Researchers have explored how deliberation impacts curriculum enactment and student learning. Johnston (1993) found that teachers' individual decision-making is not confined to instruction in the classroom; they also consider the curriculum through practical and theoretical lenses to connect it to instructional choices. Reid (2010) investigated curriculum deliberation in group-level interactions; his findings revealed that teachers considered the teacher, learner, subject matter, milieu, and curriculum-making process in all of their discussions.

Potential Obstacles to Curriculum Deliberation

While curriculum deliberation can be an invaluable component in curriculum decision-making, there are a variety of potential obstacles to implementing it effectively, both at the individual and group levels.

Individual level. When educators have time to engage in deliberation, they may make better decisions throughout the four phases of curriculum use. Unfortunately, teachers often do not have the luxury of time (Tichenor & Tichenor, 2019) and may rely on existing heuristics to make quicker decisions, namely availability, representativeness, and anchoring and adjustment (Tversky & Kahneman, 1974). The availability heuristic places importance on the frequency of an event (Tversky & Kahneman, 1974). In the context of education, this heuristic could draw teachers to lessons or ideas that they have often seen associated with subject matter concepts before. This could limit a teacher to only considering connections to real-world contexts for which they are familiar, such as teaching slope through the analogy of two runners in a race. The representativeness heuristic links the occurrence and characteristics of an event to another. Using this heuristic, a teacher's confirmation bias could lead to using the strategy of "drill-and-kill" because they were successful in that type of environment when they were in school. This belief could lead to a reproduction of this traditional, behaviorist approach to teaching mathematics. The anchoring and adjustment heuristic involves adjusting an initial value to get to the final answer (Parmigiani, 2012). An educator could exhibit this heuristic by making assumptions about students' prior knowledge and planning a lesson based on that assumption.

Group level. Change is constant in the education field, but rarely is it well-resourced and well-implemented (Calderhead, as cited in North et al., 2018). In particular, changes to curriculum necessitate time for deliberation and shared decision-making. Despite this, teachers have historically been shortchanged on common planning opportunities (Ross, 1993; Tichenor & Tichenor, 2019). As a result, when teachers are given any time for group deliberation, it can lead to social phenomena like "groupthink" – the desire to form a unified approach due to external pressures (Jaeger, 2020). For instance, a mathematics department could feel pressured to plan and "teach to the test" if they perceive the administration to place more importance on standardized test scores over conceptual understanding of content.

Technology as a Milieu

Digital resources have given teachers the opportunity to make lesson-planning decisions using content available on websites and by consulting individuals they may have connected with on these platforms. These options have added new aspects to individual and group curriculum deliberation and may reproduce the above-mentioned heuristics in this environment.

Individual level. At the individual level, educators may still consider all of the state, district, and school policies, as well as student constraints and needs for each lesson. However, each of the three heuristics (i.e., availability, representativeness, and anchoring and adjusting) take on new forms. For instance, teachers could rely on the availability heuristic to download a lesson that is in line with an activity they have seen multiple times, such as the "design a town" task used in geometry classrooms. Educators could use the representativeness heuristic to search for standards or objective-based lessons and download a lesson plan without confirming its alignment. Lastly, when searching for lesson plans online, a teacher could purchase or download a unit of instruction based on the content of one lesson in the unit. This "tunnel-vision" approach could result in the teacher now using low quality resources or unfamiliar content.

Group level. When the Common Core State Standards were adopted by majority of states in the United States, teachers, feeling isolated and short on time, began to utilize the internet to replace the in-person group deliberative process (Pittard, 2017). Teachers were able to use social media sites like Instagram and Facebook to connect with educators across the nation and do so on their own schedules. However, paid lesson plan websites like Teachers Pay Teachers could offer an interesting look into the "groupthink" theory, as teachers may visit these sites due to external stressors like time and availability of resources. The most noticeable features on each listing are ratings, reviews, and number of followers on each seller's account. This may lead to an overreliance on existing ratings, reviews, and seller popularity for lessons, as opposed to critical consumption of the materials presented on these sites.

Given the complex nature of curriculum use and the decision-making that may occur prior to the implementation in the classroom, I offer a conceptual framework in Figure 1 to understand the process that a teacher may engage in prior to choosing written curriculum to enact in their classroom. This conceptual framework represents the potential stages a teacher goes through prior to using a curriculum in their classroom.

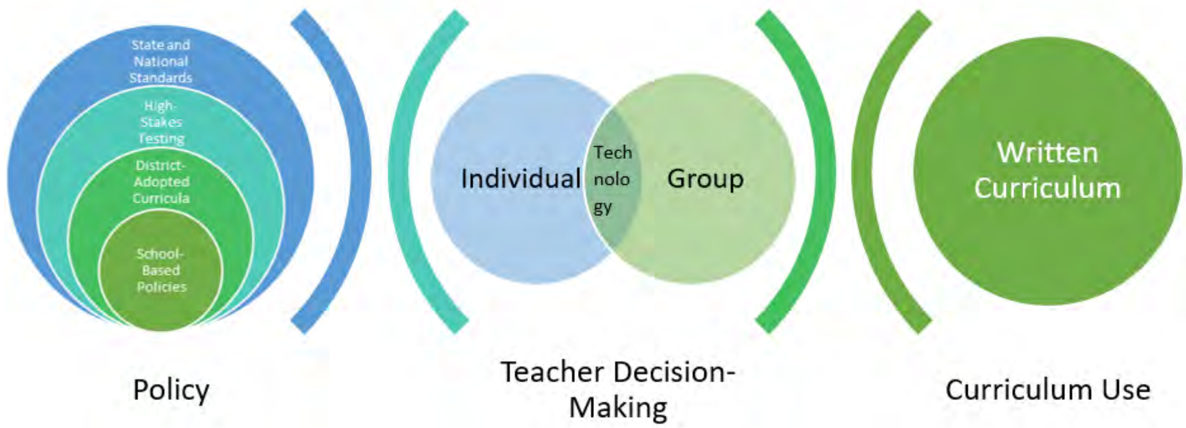

Figure 1: Online Curriculum Selection Stages

The first stage considers policy through state and national standards as the broadest level, followed by state-level high-stakes testing requirements, then district-level curricula specifications, and the innermost circle represents school-based policies. Each of these considerations has a direct or indirect effect on what curricular and instructional materials the teacher uses in their classroom. The state standards adopted and high-stakes testing make up the official curriculum that is to be given priority in instruction. Districts make curricular decisions based on the standards set – some districts provide textbooks and mandated resources, whereas others create suggested materials for use at the school level (Gewertz, 2015). Lastly, schools set the policies that most immediately impact teachers, potentially providing scripted curricula and requiring teachers to use specific teaching strategies (Timberlake et al., 2017).

Once policy restrictions for standards and curricula are established, then the teacher engages in individual decision-making on curricula and may rely on the heuristics of availability, representativeness, and anchoring and adjustment (Tversky & Kahneman, 1974). In addition, group level decision-making may be impacted by the groupthink heuristic. This framework adds the dimension of technology in teachers' deliberative process, as each of these heuristics may exist through online lesson planning decisions as well. Finally, the teacher chooses the written curriculum and begin the curriculum use phases as outlined by Stein et al. (2007).

Implications

Mathematics educators do not teach in a vacuum and their role is constantly changing in response to their environment. Teachers' beliefs, their reasoning about curriculum materials, and their capacity to enact these resources all influence the decisions they make and the opportunities students have to learn mathematics (Choppin, 2011). As teachers continue to engage with curricular materials in online spaces, their decision-making practices are potentially altered by individual and group heuristics. By identifying the heuristics that play the greatest role in these decisions, mathematics teacher educators can prepare pre-service teachers to be critical consumers of mathematics content found in online spaces.

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POINTS AND POSITIONS: AN INTERSECTION OF TWO FRAMEWORKS FOR REASONING WITH GRAPHS OF FUNCTIONS

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In this theoretical report, we examine the intersection of two previously-recognized dimensions of students' reasoning about how symbolic notations represent elements of graphs of functions. One dimension distinguishes location-thinking, where function notation refers only to a point's location on a graph, from value-thinking, where a point is treated as a multiplicative object. The other distinguishes a nominal interpretation of expressions, where expressions refer to positions in the plane, from a magnitude interpretation, where expressions measure a length. Taken together these dimensions provide four distinct ways students reason about expressions on graphs; each case reveals new meanings indicated by the interplay between the dimensions.

Keywords: Cognition, Mathematical Representations, Graphical Interpretations

The use of visualizations to illustrate concepts is central to the teaching and learning of mathematics (NCTM, 2000, 2014). However, the use of graphical representations, in particular, may pose challenges for students (Leinhardt et al., 1990). To better support student learning involving graphs, researchers have proposed theoretical frameworks to characterize various distinctions in students' graphical interpretations (e.g., Lee et al., 2019; Moore & Thompson, 2015; Paoletti et al., 2018). Yet, these frameworks do not account for how students may connect symbols to graphs and what such symbols represent, which can significantly impact students' graphical reasoning (e.g., Knuth, 2000). We present a conceptual analysis that considers two additional frameworks that account for two dimensions of students' interpretations of symbols on graphs of functions: (1) David et al.'s (2019) value-thinking and location-thinking framework which relates students' interpretation of points and (2) Parr's (in press) description of nominal and magnitude interpretations which distinguishes students' interpretations of expressions to signify positions in graphs. Together, these pairs intersect to create four distinct ways students may think about how graphs and symbolic expressions are related. By examining this intersection of frameworks, we uncover nuances in students' graphical interpretations that may afford or constrain student sense-making of graphical representations.

The Intersection of Two Theoretical Frameworks

Inherent in both theoretical frameworks are notions of notation and interpretation. We frame these concepts in the language of semiotics (Barthes, 1957). A notation such as $f(b)$ is a sign, comprised of the symbols (signifier) and that which they indicate or represent (signified). The signified can be a mental object or another mark or collection of marks (which itself could be a signifier). An interpretation is then the association between the symbol and the signified.

Value-Thinking & Location-Thinking: Two Ways Students Reason about Points

We draw on David et al.'s (2019) constructs of value-thinking and location-thinking to distinguish students' reasoning about points on curves in the Cartesian plane. A student engaged in *value-thinking* views points as representing a pair of values simultaneously, typically an input and output value of a function, consistent with the notion of a multiplicative object (Saldahna & Thompson,1998; Thompson & Carlson, 2017). When a student uses value-thinking, $f(a)$ refers to the vertical component of the associated point on the graph above the horizontal axis, which can be denoted on the vertical axis. A student using *location-thinking* refers to and focuses on the *location* of the point in the plane, rather than reasoning about it as a multiplicative object. Because of this, students engaged in location-thinking often label outputs of a function, such as $f(a)$, at points along the curve, and reason about the output as referring to the location of the point. In other words, they treat the signifier $f(a)$ as referring to a point on the graph. To a student engaging in location-thinking, this referent (i.e., the point) is a monolithic, non-decomposable entity. When value-thinking, the point is multifarious— a multiplicative object of two components, an input and output. By coordinating more meanings for points on a graph, value-thinking more readily affords further mental actions (David et al., 2019; Sencindiver, 2020). Figure 1 (left) summarizes the signification involved in value-thinking and location-thinking.

Magnitude & Nominal Interpretations: Two Ways Students Interpret Expressions in Graphs

A second framing of students' understanding of graphs offered by Parr (in press) describes how students relate expressions (e.g., $f(b)–f(a)$) with graphs. Parr (in press) describes a magnitude interpretation as treating the expression as a measure of a quantity, one that is based on particular positions represented in the plane. A magnitude interpretation of an expression often involves representing an amount of a quantity as a distance or length of a segment on a graph. In contrast, a *nominal* interpretation of expressions treats expressions as labels without quantitative significance, much like the use of labels in an anatomical diagram. A student who interprets expressions nominally may place an expression on a graph to label a particular position in the Cartesian coordinate system. Thus, a nominal interpretation may be limited to a comparison of equality between two expressions based on their spatial positions.

When students use a nominal interpretation of a symbolic expression, its referent (a position in the plane) is a monolithic entity. When they use a magnitude interpretation, this referent is multifarious— the symbolic expression signifies a position, which itself indicates a relevant endpoint for a measurement. A student using a magnitude interpretation mentally constructs a portion of a graph (e.g., a segment, an arc length) from a reference point to the relevant position and uses the expression to also signify the measurement of the length of this portion of the graph. The signification involved in a magnitude interpretation coordinates additional meanings for positions in the plane, which is why Parr (in press) found that it more readily affords further mental actions. Figure 1 (right) summarizes the signification involved in the magnitude and nominal interpretations of expressions in the plane.

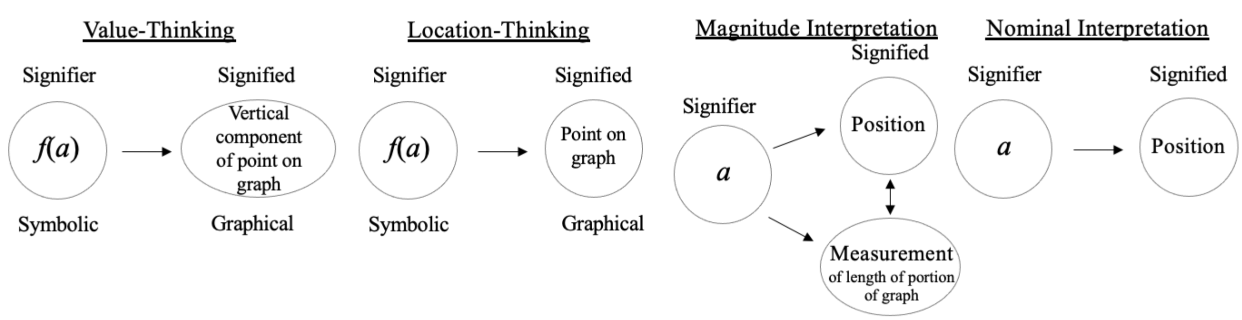

Figure 1: Signification in Value-Thinking and Location-Thinking (left) and Signification in Magnitude and Nominal Interpretations (right).

The Intersection of Interpretations of Expressions and Interpretations of Points

Each of these frameworks describes distinct aspects of students' interpretations of graphs. Table 1 shows how each of these two dimensions intersect to create four ways of thinking and uses function notation as an example in each case. To be clear, the four categories created are meant to characterize a student's thinking with a particular task or in a particular instance, rather than characterize all of the ways a student is capable of thinking. In fact, we suspect that students may demonstrate reasoning indicative of different ways of thinking within the same task. We describe each of the four cases below, focusing on Cases 2 and 3 which became more salient through the semiotic analysis.

Table 1: Four Ways of Interpreting Function Notation on Graphs

		Ways of thinking about points (David et al., 2019)	
		Value-Thinking	Location-Thinking
Ways of interpreting expressions on graphs (Parr, in press)	Magnitude Interpretation	Case 1: $f(a)$ means the measure of the vertical distance from the horizontal axis to the point	Case 2: $f(a)$ means the measure of a distance to the point along the graph from a reference point
	Nominal Interpretation	Case 3: $f(a)$ means the vertical position of a point on the graph	Case 4: $f(a)$ means the position of a point on the graph

Case 1 (Magnitude Interpretation + Value-Thinking):

For the point $(a, f(a))$, a student thinking in these ways reasons with a as the horizontal distance from the origin to the point, and f(a) as the vertical distance, indicated along axes or parallel to them. They reason about the point's position as representing a multiplicative object of two orthogonal distances. As an empirical example, see Micah's reasoning in Parr (in press).

Case 2 (Magnitude Interpretation + Location-Thinking):

A student thinking in these ways would interpret expressions as signifying lengths or distances in the graph, yet would be thinking of points as locations. After such a student identified a point in the plane, they would create labels and reason in ways that do not acknowledge the point as a multiplicative object. For example, a student may associate magnitudes with an arc length between two points, or may measure in reference to other perceivable features presented in the graph. To illustrate this case, we refer to the example of Lisa in Sencindiver (2020). Lisa recognized points in the graph as features to measure between and the curve as a place to measure along. After identifying two points on the curve of a function f, Lisa described $f(a)$ and $f(a+h)$ as arc lengths from the x-intercept to those points on the curve, and $f(a+h)– f(a)$ as the difference of the two arc lengths.

Case 3 (Nominal Interpretation + Value-Thinking):

A student using these ways of thinking interprets inputs and outputs, such as a and $f(a)$, as labels for particular positions, and as horizontal and vertical components of the ordered pair of the point $(a, f(a))$. This student may reason with a point similar to one reporting battleship coordinates, by coordinating two positions to give a third position, still forming a multiplicative object. However, this is not a multiplicative object of multiple distances— a and $f(a)$ are thought of as positions on the axes, without coordinating the distances from the positions on the axes to the origin. Likewise, a student can decouple a point into two positions by projecting vertically and horizontally to positions on the x-axis and y-axis, respectively. We refer to the example of Martha in Parr (in press). She claimed that the $f(c)$ and $f(d)$ she labeled on a monotone decreasing graph were not equal and could never be equal. She explained by saying, "because $f(c)$ and $f(d)$

are separate values… *c* and *d* are gonna be, if they're [*c* and *d*] separate values, like they're [*c* and *d*] labeled separately." Martha engaged in value-thinking and conceived of points as pairs of two positions labeled on the axis. Martha then considered *c* and *d* to be unequal ("separate values") because of how she was interpreting these expressions nominally.

Case 4 (Nominal Interpretation + Location-Thinking):

A student using these ways of thinking interprets expressions, such as $f(a)$, as labels for particular positions, and points as outputs along the curve, signified by function notation. Thus, a student may use an output label, such as $f(a)$, for a position along the curve. This position may correspond with the input *a*, but is not an indication of a measurement along an axis. In reasoning this way, the student may coordinate an input *a* with an output label $f(a)$ (thought of as a point), but in the moment, the student does not conceive of the point as a multiplicative object. For an empirical example of this type of reasoning, see Zack from David et al. (2019). As explained by David et al. (2019), Zack engaged in location-thinking, conceiving of the points he labeled on the graph solely as outputs. Further, Zack interpreted $f(a)$ and $f(b)$ nominally on a constant function, considering them as labels for positions and reasoned about these positions, rather than any measurement associated with these positions. These ways of reasoning led him to conclude that $f(a)$ and $f(b)$, which he labeled at two different points, were not equal.

Discussion

Our theoretical findings shed light on students' ways of interpreting function notation in terms of points and positions on graphs. We elaborate on the notion of "value" in value-thinking by contrasting Case 1 and Case 3. Although value-thinking as described in David et al. (2019) may be interpreted as only referring to Case 1 (value+magnitude), this work highlights the reality that students may conceptualize a point as a multiplicative object of positions on the axes, without reference to measurements as in Case 3 (value+nominal).

The coordination of these two frameworks allows us to see important parallels and interplays between them. Location-thinking's view of *point* is monolithic, while value-thinking's view is multifarious. Similarly, a nominal interpretation of *position* is monolithic, while a magnitude interpretation of it is multifarious. By overlaying these, multiple shades of meaning and signification become apparent that we might otherwise have missed. For instance, in Case 1 (value+magnitude), the meaning of point *and* position are both multifarious for the student, suggesting she has formed *a multiplicative object of coordinated distances* from the axes.

All four cases reveal affordances and constraints for the mathematical activity potentially available to students and suggest ways instructors can support this activity. For instance, Case 4 (location+nominal) may afford reasoning in geometric contexts where horizontal and vertical components are not privileged. However, a student thinking this way may have difficulty reasoning about complex statements involving graphs of functions such as the Mean Value Theorem. Likewise, aspects of reasoning with Case 2 (location+magnitude) afford conceptualizing distance along the curve as a measurable quantity which is critical for multiple topics in Calculus (e.g., arc length, line integrals of vector-valued functions), as well as reasoning about quantities within situational coordinate systems (Lee et al., 2018). However, this sort of thinking may constrain students' activity when a graph is representing information along orthogonal axes. Case 3 (value+nominal) may be sufficient for students in finding numerical values from graphs, in instances when they do not need to construct and reason about other quantities, such as $f(b) - f(a)$, within the graph.

The intersection of the two frameworks we described may help instructors account for differences in students' reasoning about points and positions on graphs. Further research in this area may include teaching experiments to study the extent to which cases of student's thinking afford or constrain their mathematical activity. Such studies may also shed light on what factors support students in transitioning from one way of thinking to another.

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A PROFESSIONAL DEVELOPMENT MODEL FOR IN-SERVICE TEACHERS BASED ON IDENTIFICATION PROCESSES AND TEACHER COLLABORATION

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Based on data and results that are part of a large longitudinal study, we built a model for the professional development of in-service middle school math teachers in Mexico. This study accounts of information on the general characteristics of the official programs for professional development implemented by the Ministry of Education in Mexico. The empirical work started by identifying teachers' personal philosophies or images of mathematics (Ernest, 2007, 2012), and a blending of theoretical constructs was utilized.

Keywords: Professional development, Teachers' beliefs, Instructional activities, and practices.

Introduction

Hiebert et al. (2003), who suggested that progress can be made by designing programs that could influence the nature and quality of this practice. In addition, Hiebert and colleagues noted that teacher training programs have an expiration period, which makes necessary to continuously review their design and implementation. And, according to Marcelo (2002), initial training provides the teacher with baggage of knowledge that must be complemented throughout teacher active professional life. All of this combined with educational contexts marked by the implementation of large-scale school mathematics curriculum reforms makes imperative to offer professional development programs for in-service teachers (Montecinos, 2003). In this regard, Mexico is particularly not an exception given a constant series of reforms to the school mathematics curriculum (Ministry of Education –SEP by Spanish sigla, 1993; 2004; 2006; 2011; 2018). The main objective in the case study that is presented here was to build a cyclic model of professional development for in-service teachers focusing on the learning of new pedagogical practices (those induced by the introduction of new contents in the curriculum reform) and based on their reflection on what constitutes their profession and practice into the classroom. Finally, it is also important to point out the need, as Cobb (2005) stablished, to approach or to establish links between the theoretical knowledge on teaching and teacher's practices into the classroom.

Characteristics of the Official Teacher Development Programs in Mexico

In the successive changes that have been implemented through the different school mathematics curriculum reforms in Mexico, teachers have been considered always as protagonists of the educational transformation (Ministry of Education –SEP by Spanish sigla, 1993; 2004; 2006; 2011). The central features of the 2011 integral curriculum reform, which concentrate the guidelines and developments of the 2006 reform, in relation to the professional development of teachers, are based on the recognition that reflection and educational practice in the school are key to strengthening the continuous training of teachers and the additional academic staff, and to promote collaborative school management processes. In particular, the different approaches of curriculum reforms have appealed to the commitment and professional development of teachers to consolidate them (SEP, 1993; 2004; 2006; 2011). In this way, teacher professional development has been considered a fundamental axis in the reform process that has been carried out in Mexican middle schools, since it has been highlighted as the possibility of generating substantive transformations in pedagogical practices.

This fact has been fully recognized by the Ministry of Public Education itself, underlining that promoting professional development is the best tool to improve teacher performance in classroom (Ortega et al., 2005). But, unfortunately, all these considerations, recognitions, calls, and underlining of promoting the professional development of mathematics teachers have mainly resulted to be only rhetorical. For example, according to Sandoval (2001), one of the characteristics of the official programs of professional development for in-service teachers in Mexico, has been the scarce teacher's participation. These programs of professional development, according to Martínez (2005), end up being framed in a course, which only acquires meaning for its recipients if it awards points for a promotion on the official teaching career. On the other hand, the additional academic figures (as supervisors) whose formal function is to guide teachers in their work, scarcely attend schools and when they do it, their work is carried out in a purely administrative format. That is the way they do not constitute an important reference in teaching practice and even less they constitute educational support within the framework of teacher professional development (Sandoval, 2001). In this context, the professional development of in-service math teachers has become a model where trainer's activities during the course focus on carefully developing their own new materials and use the organization of the new courses to disseminate reforms' proposals. It is to say that the training model of the official courses for teacher professional development is focused on the trainer and based on the implementation of homogeneous courses.

Theoretical Frame

The Concept of Document in the Documentary Genesis

Based on the documentary genesis by mathematics teachers and in their construction of collaborative design using digital resources, Gueudet and Trouche (2009) have extended the concept of instrumental genesis proposed by Verillon and Rabardel (1995) to the one of documentary genesis. In this respect, it is illustrative the following schematic representation of both concepts (see Hoyos, 2012). Therein, one can notice that the concept of document connects teacher practice with their images of mathematics (Ernest, 2012), really through managing the resources they have at hand, as mathematics curriculum prescriptions, textbooks, digital technology, etc. But "whether one wishes it or not, all mathematical pedagogy, even if scarcely coherent, rests on a philosophy of mathematics" (Thom, 1973, p.204. Cited in Ernest, 2012, p.9). In words of Gueudet & Trouche (2009), and Sabra (2010), the documentary approach provides tools for the analytical study of the processes that underlie the professional development of math teachers, both individually and collectively.

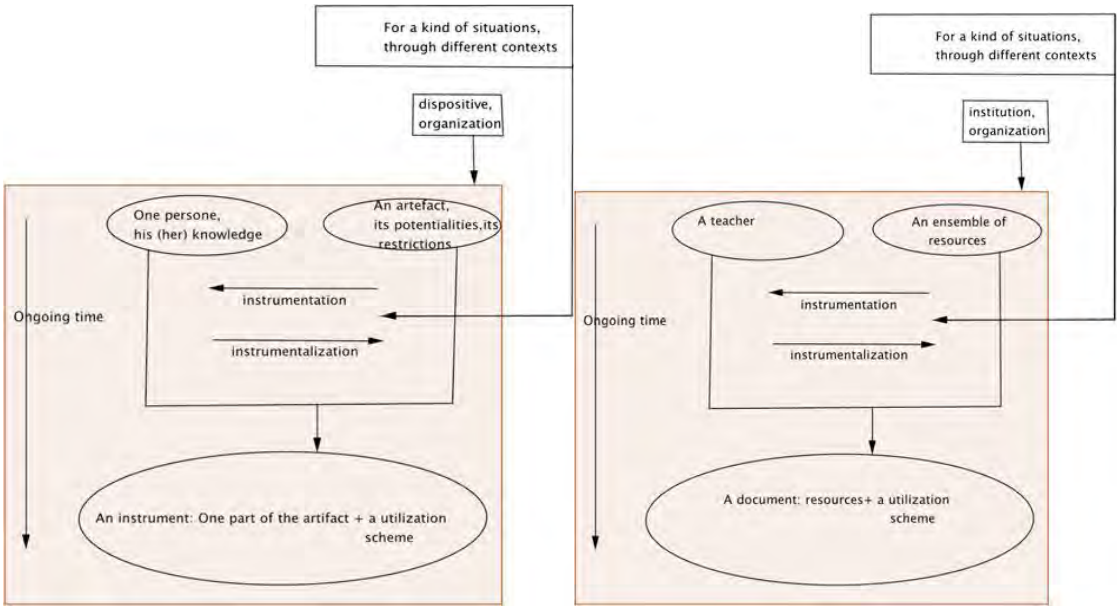


Figure 1: On the Left, Schematic Representation of Instrumental Genesis & Schematic Representation of Documentary Genesis, On the Right. (Extracted from Trouche, 2010. Cited in Hoyos, 2012).

Teacher Personal Philosophies or Images of Mathematics

Ernest (1994) argued that differences in mathematics teachers' practices cannot be explained sufficiently attending only to mathematics knowledge. Such differences may be attributable to particular belief systems about what is mathematics, and on its teaching and learning, which in particular constitutes the rudiments of certain personal philosophies or images of mathematics that teachers maintain, although often these personal philosophies are non-articulated in a coherent manner. Personal philosophies of mathematics provide a general epistemological and ethical framework, under which the conceptions about the teaching and learning of mathematics are considered, and they are subjected to limitations and opportunities of social context.

Identification Processes

The work of Cerulo (1997) provides an antithesis to traditional identity studies, and mainly the works cited here refocus scholarly attention from the individual to the collective. According to this author, many of her reviewed studies have approached identity as a source of mobilization rather than a product of it, and particularly in relation to identification processes, attention to collectives (p.394) has reenergized scholarly interest in the identification process itself. In this way, "a growing literature explores the mechanics by which collectives create distinctions, establish hierarchies, and negotiate rules of inclusion" (Cerulo 1997, p. 394).

Emergence of teacher new collaborative and pedagogical practices from teacher professional development

Although teacher collaboration wasn't considered during the activity developed by participant teachers in a specific professional development program implemented by Hoyos (2012-2016), this program was an important antecedent for the case study presented here, because it highlighted the potential of starting teacher activity during professional development from materializing the knowledge of the teacher about teaching, to move towards another level in the development of new collaborative and pedagogical practices.

Methodology for Construction and Obtention of Data

The case study we are presenting here was developed in two phases. First phase was specifically developed through enacting teacher identification processes to know of collective teacher images of mathematics and about their teaching. The second phase turned around teacher design of new lessons or activities (to be implemented in classroom) on new curricular contents included in the 2006 school mathematics curriculum reform, namely the generalization of patterns, for the learning of school algebra. The participants in this research were 21 in- service math teachers belonging to public middle schools in Mexico City. All of them were experienced teachers in math middle school. Meetings were carried out face-to-face, in an official center for teacher professional development or specific workshops, during the months of October 2007 and November 2008 (in the first phase of the study), and from January to June 2009 (during the second phase of the study). Teachers were always grouped in teams of three, or four participants, and the general objective of the meetings was to analyze the new 2006 official teaching approach for the development of the mathematical contents in the classroom.

First general task

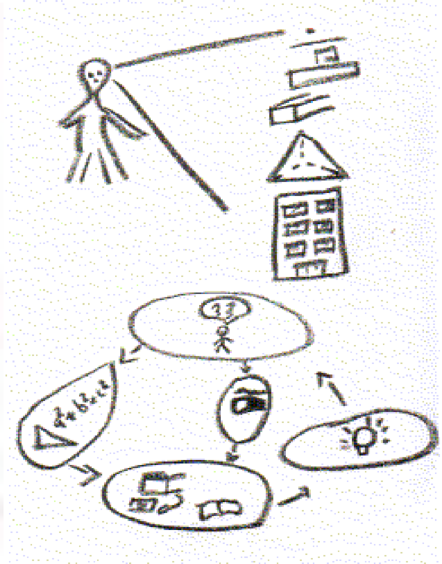
Task 1. Setup in a diagram or schematic drawing the important pedagogical elements you display to address mathematical issues with pupils.

Some of the complete teachers collaboratively produced diagrams are showed next. They were here titled as Diagram 1, and Diagram 2. Each of these diagrams were reached by teacher negotiation within the team to draw a single schema. In fact, each diagram reflected a teacher team product, because of teacher collaboration and negotiation within respective teams. Finally, is important to note that after each team had finalized their drawing, it was implemented a discussion on the meaning of their production. These discussions were managed and registered (by taking notes) by the second author of this paper (R. Garza), who really have played the role of teacher educator, or more precisely, as a teacher team tutor or person in charge of the whole implementation of teacher activities during this investigation.

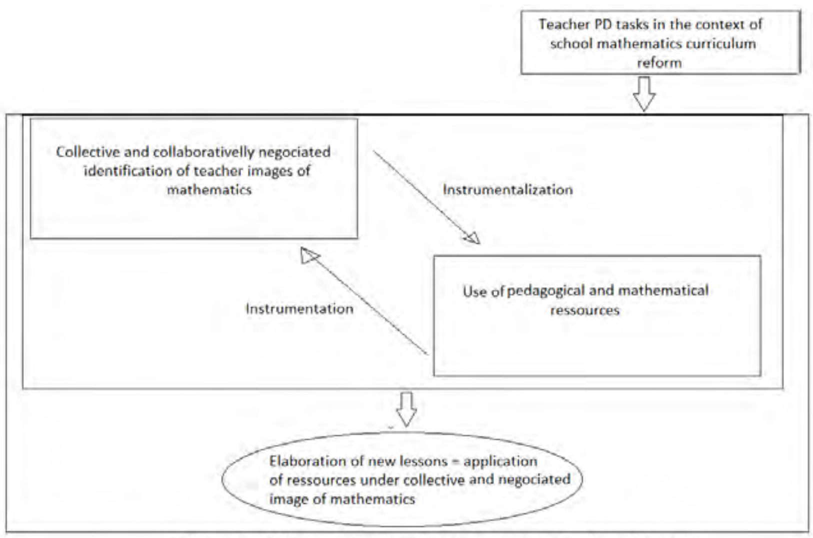
Elements of Analysis

It is important to highlight that the collective identification process accomplished here, evidenced by teacher production, integrated different levels of elaboration, abstraction, and generality, as well as different forms of representation, those sustained and negotiated by all teachers in the team. Moreover, in the second phase of this study it was important too to note how participant teacher teams used of pedagogical and mathematical resources to collectively create or design new lessons, to articulate and project their teacher identities (Cerulo 1997).

Annex 1: Teacher Productions of Diagrams 1 & 2



Annex 2: Our Cyclic Model for the Professional Development of In-Service Mathematics Teachers



Cyclic model for the professional development of in-service mathematics teacher

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THE MESSINESS OF RESEARCH: A MONTAGE

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Keywords: Research Methods, Systemic Change

Conducting research is a messy endeavor where the pursuit of new knowledge requires one to play and experiment with data and theoretical framings. This messiness includes bouncing between literature, analysis, and writing. False starts are inevitable and difficult choices are made to start over or attempt unfamiliar directions. Moreover, we have to consider how our positionality, subjectivities, and ideologies further complicate our work and the ethics of conducting research. Cai and colleagues (Cai et al., 2019a; 2019b; 2019c) attempted to clean the messiness of mathematics education research by establishing norms and needed directions for "how to conduct and report high-quality research in mathematics education" (Cai et al., 2019a, p. 114). Maxwell (2013) argued, however, that qualitative research needs to be flexible and is "inductive rather than following a strict sequence or derived from a single decision" (p. 18). Therefore, we question the desire to clean, but instead wish to explore and understand the messiness of mathematics education research.

Siy (2019) called for the mathematics education research community to do more to share both the processes *and* the products of our work by exposing the messiness of research. This includes emphasizing how our failures, reflections, and false starts have led to opportunities. In response to the call, the members of our research group have worked to reflect on our own processes as we work towards a product (i.e., composite counter-stories). We aspire to follow the work of critical race scholars (e.g., Cook, 2013; Solórzano & Yosso, 2002), specifically Pérez Huber (2009), in recognizing our position as critical race researchers and Scholars of Color in academia to "build and develop ways of doing research that counter traditional research paradigms and lead to a more complete understanding of experiences of People of Color within and beyond educational institutions" (p. 640). We believe by demonstrating the messiness of our research, we are challenging the whiteness of academia (see Brunsma et al., 2020). Mathematics education researchers have done little to investigate their own practices (Dubbs, 2021) and thereby, the whiteness of the institution (Martin, 2015).

In this poster, we provide our processes working to amplify the voices and experiences of elementary Raza students learning mathematics in predominantly white schools through composite counter-stories. We position this work within two theoretical spaces: 1) Latinx critical theory (Solórzano, 1998; Solórzano & Yosso, 2002) and 2) Borderlands theory (Anzaldúa, 1987). Our goal is to "reveal our work" (Siy, 2019, p. 17) by intentionally exposing the messiness, misfires, false starts, and challenges we faced.

Acknowledgements

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PERSPECTIVE TAKING AND THE CONSTRUCTION OF AN INTERSUBJECTIVE VIEW

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The use of video data for research is widespread (e.g., Hamel & Viau-Guay, 2019; Nassauer & Legewie, 2021), and new camera technology such as Go-Pros[TM] is becoming more prevalent (**Authors**, 2019; 2021; Burbank et al., 2018). This enables a more subjective, humanistic, collaborative, and participatory approach (Harwood & Collier, 2019; Lahlou, 2011; Pink, 2015). We argue that the use of *multiple* go-pro cameras and analysis of individual views allowed us as researchers to "see" more of an event with each new viewing of video data, affording our research team the opportunity to piece together different student perspectives to form a larger possibility space of "truth" of one event (i.e., perspective taking). We utilized third space as a framework to guide this method of collecting and analyzing data (Gutiérrez, 1999). Specifically, we conceptualized the third space as a hybrid space in which we as researchers attempted to enter the particulars of an event (e.g., human movement) through the perspectives of students wearing Go-Pro cameras, as well as through our ongoing discourse (Simpson & Feyerabend, 2021; Hulme et al., 2019). It is a space that allowed us to frame the emic (participants' view)-etic (researchers' views) as complements rather than in opposition to one another (Pink, 2005).

The video data used to highlight our approach was collected from one group of fifth-grade students (2 female and 4 male students) tasked by their teacher to first construct a masking tape-path for a robot, Dash, to traverse (Phase 1). Next, the group used the app Blockly (Phase 2) to program Dash to stay on a path created by another group of fifth-grade students. Three of the six students volunteered to wear a go-pro camera on their chest, while we also recorded the overall group interaction using one stand-alone camera. The data was analyzed using interaction analysis (Jordan & Henderson, 1995), which added an additional layer of perspective taking through the use of go-pros. In our poster, we provide examples of our approach and highlight how our view of an event moved past individual perspectives to a layered view of the collective. For instance, we initially perceived the group of students as being monopolized by one student, Olive, who positioned herself as "good at this stuff." The analysis of additional video data shifted our perspective to view the group as developing their own collaborative structure within the activity in a way that had not been visible when watching Olive's perspective.

Intersubjectivity and a story line was developed as we called into question interpretations of evidence gathered from earlier views and stitched together a collaborative view of events in which no one learner's experience was absolutely understood or could be separated from the others. Instead taken as a whole the collective event became storied by temporal events for which we had multiple perspectives, such as laying the path and controlling the iPad. Events not captured by multiple videos were conditional and created pockets of subjectivity within the dominant intersubjective narrative. Therefore, this approach, including interaction analysis, afforded us as a research team a space to question our assumptions and "truths" of events with each new viewing of video data from a student perspective. We further contend that collecting and analyzing multiple student perspectives through the use of go-pro cameras is an approach that can enhance various methodologies and theoretical perspectives.

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UNDERSTANDING STUDENT BEHAVIOUR AS EVIDENCE OF STUDENTS' CONCEPTIONS AND INSTRUCTIONAL NORMS

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Keywords: Cognition, Learning Theory, Teacher Knowledge

One of the main responsibilities of a mathematics teacher is to support their students in learning mathematics. It is widely accepted that teachers are most effective at this when they base their instructional decisions on their understandings of what their students know at any given point, which they infer from students' mathematical activities (e.g., work, utterances, gestures). To support teachers in making such inferences, researchers of students' conceptions have strived to construct viable models of the *mathematics of students* (Steffe & Thompson, 2000). For instance, some researchers have focused on students' rate of change schemes (Carlson et al, 2002) and others on students' fraction schemes (e.g., Hackenberg & Lee, 2015; Steffe & Olive, 2010). One of the main methodologies used to uncover these conceptions has been the task-based cognitive interview (e.g., Steffe & Thompson, 2000) in which students are asked to complete a series of tasks that are designed to test hypotheses that the researcher develops throughout the course of the interview and/or developed prior to the interview. For the researcher, it is often strategic to choose a novel task rather than a familiar one, as students often default to completing familiar tasks using learned techniques that they believe they are expected to use, which reveal less of the idiosyncrasies of their conceptions and the mental operations involved in their reasoning. While this is a sensible decision for the researcher, the results of such research have been critiqued as difficult to apply by a teacher who is faced with the very problem that the researcher seeks to avoid: interpreting students' understanding of mathematical concepts from their performance on familiar tasks, as most tasks that students are assigned in school are familiar (Doyle et al., 1985).

This was the problem space that drew us—a scholar of student thinking and a scholar of instructional norms (i.e., shared beliefs about what teachers or students will do, and/or ought to do, in situations of a particular type)—together. In this presentation, we invite the audience to think with us about this problem space to develop together a set of possible directions for research. To do so, we consider what teachers and researchers might gain from simultaneously thinking of students' performance on familiar tasks as both decisions to follow or deviate from norms and as evidence of their mathematical conceptions. To base this conversation in a concrete case, we consider a task that is familiar to students in high school algebra: solving a linear equation in one variable. In this case, the more general question becomes: What might teachers and/or researchers gain from considering a student's attempts to solve an equation as both a potential attempt to satisfy the teacher's expectations to solve equations by moving algebraic terms to one side of the equals sign, numeric terms to the other, simplifying both sides, then dividing by the coefficient of x (Buchbinder, Chazan, & Capozzoli, 2019), as well as their conceptions (e.g., of the equals sign, equations, or variables) and the mental operations involved in doing so (e.g., unitizing, covariational reasoning) (Thompson, 2013)? We call researchers to consider the broader question of what researchers and teachers may gain from thinking of student behaviour as influenced by both individual factors (e.g., conceptions) and social factors (e.g., norms). Lastly, we raise questions regarding methods for investigating student thinking and the types of theories that can guide and come from the use of such methods.

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THEORETICAL FRAMEWORK FOR A MATHEMATICAL OBJECT IN RELATION TO MULTIPLE COORDINATE SYSTEMS

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This study outlines a theoretical framework which describes different ways of thinking involved in mathematical activities such as graphing and symbolizing. Finding a coordinate system as a representational resource, students may engage in two forms of representational activity within a single coordinate system, which I note *"Naming"* and *"Locatin*g". The two forms of students' activity may apply to the situations where there are multiple coordinate systems, offering two perspectives to interpret a mathematical object with multiple coordinate systems, which I note *"Coordinates-open"* and *"Coordinates-closed"*.

Two forms of students' activity within a single coordinate system: *"Naming"* vs *"Locating"*

In "*Naming"*, students 'name' a mathematical object such as point, vector, line, curve located in coordinate space. Students assign numerical value(s) to the object consistently, which a coordinate system defines. The naming activity also applies to a spatial object located in a physical space. An example for the naming activity is a point being coordinatized as an ordered pair (2,-3). On the other hand, in "*Locating",* students 'locate' numerical descriptions such as n-tuples, function equations into a coordinate or physical space in a consistent way which a coordinate system defines. An example of the locating activity is to graph a function equation in a coordinate plane. A coordinate system comes with a unit measure, directionality, and reference point in either way of naming and locating activity.

Two perspectives for an object in relation to multiple coordinate systems: *"Coordinates-open"* vs *"Coordinates-closed"*

The *"Coordinates-open"* view is closely related to the *"Naming"* activity. We consider any mathematical objects as composed of points, and any coordinate system describes those points' locations. Having another coordinate system leads to "Re-Naming" activity while the object's location stays the same in a coordinate plane in this view. In other words, the object's points are "coordinatized" by the system laid atop the object. Even if the coordinate system is transformed to a new system, the objects do not move; rather, they attain new coordinates according to the new system laid atop it.

On the other hand, *"Coordinates-closed"* is closely related to the *"Locating"* activity. We consider any objects as having numerical expressions, and any coordinate system determines those expression's locations. Having another coordinate system leads to "Re-Locating" activity while the object numeric remains the same in this view. The object's shape is defined relative to the coordinate system, which means the object is transformed into a different shape by a transformation that maps points in the first coordinate system to points in the second coordinate system. Even though the transformed object looks different from how it looked in its original, one shape is a different image of the other.

Utilizing the framework described here as a lens to review undergraduate mathematics textbooks (Lay, 2012; Stewart, 2008; Strang, 2019), many textbooks seem to be approaching only from *"Coordinates-open"* perspective when introducing a new coordinate system.

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Chapter 15:

Working Groups & Research Colloquia

GROUP DISCUSSIONS DISCUSSION GROUP

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This inaugural meeting of the Group Discussions Discussion Group will convene to consider current research on group discussions in mathematics. In this series of meetings, the goal of the discussion group will be to share work (both in progress and completed), engage in collaborative analysis of data, jointly identify areas for future research, and establish connections within the PMENA community for future collaborations.

Keywords: Classroom Discourse; Equity, Inclusion, and Diversity

This new working group will focus on students' peer-peer discussions in mathematics classrooms. The primary goals of the working group are (a) to share recent research on group discussions in mathematics and (b) to create a venue in which conference participants can engage in collaborative planning for future investigations of students' group discussions of mathematics. If we are successful, then we anticipate that this working group will lead to future working groups and the creation of concrete products.

Theoretical background and Prior Research

We know that classroom discussions of mathematics can be powerful for fostering student learning (Chapin & O'Connor, 2012; O'Connor et al., 2015). Alongside whole-class discussions, small group, student-student discussions provide a format in which students can learn through participating in talk and interaction. Small group peer discussions differ from whole class mathematical discussions because students must manage the task of negotiating mathematical ideas with peers, without the direct supervision of or support from their teacher. In addition to mathematical negotiations, students also navigate and reinscribe interpersonal and broader social relations during these discussions. When peer discussions appear most productive, students may engage in sophisticated collaborative reasoning and draw upon the group as a resource for sustaining engagement (Barron, 2000; Boaler, 2008).

Prior analyses of peer discussions have drawn on a wide array of theories and frameworks such as positioning theory (e.g., Bishop, 2012; Wood, 2013), Systemic Functional Linguistics (DeJarnette & González, 2015; González & DeJarnette, 2015), and qualitative discourse analysis (Esmonde, 2009; Langer-Osuna, 2015) to unpack the work students accomplish during group discussions. Though these approaches have different theoretical heritages, they all emphasize the role of language in group interactions. When studies of group interactions in mathematics classrooms focus on learning, one critical mediator of learning is language(s) and symbolic tools used to do mathematics collaboratively.

We know from prior research that students in small groups tend to model their teacher's practices in terms of the types of questions posed to one another and the types of help given (Webb et al., 2006). We also know that students can (and do) exercise different types of authority (e.g., intellectual, social) related to status in small groups (Engle, et al., 2014; Langer-Osuna, 2016; Langer-Osuna et al., 2020). And we have models, both within and outside of mathematics education, that support students to engage more equitably and productively with one another (Cleaves, 2008; Cohen & Lotan, 1995; Featherstone et al., 2011; Jansen, 2020; Zahner, 2012) Yet, there are also several open questions for researchers and mathematics educators. For example, What are the appropriate timescales at which to analyze student discussions to answer questions related to equity and productivity in small groups? How might notions of productive group discussions vary according to context, person, or theoretical perspective?

Structure of the Working Group

The working group will be structured so that participants, both live and joining remotely, will have opportunities to engage in collaboration. Additionally, given the hybrid format of the conference we will plan each day's session so that participants can be meaningfully engaged in the discussions, even if they are unable to attend all three sessions, or if they are joining the conference remotely. The organizers plan to mix framing presentations with active group discussions. Applying the best practices from research on using group work in the classroom, we will use strategic grouping in order to bring together participants from diverse settings (geographic locations, type of institutions, research foci, etc.) for discussion. During the discussions we will provide guiding questions as well as create shared spaces (such as Jamboard) that will facilitate collaborative hybrid discussions.

Day 1: Introductions, followed by overview of the state of the field, facilitated by Jessica Pierson Bishop. This will be followed by two illustrative examples of current work (a) a presentation of Langer-Osuna's work related to positioning, identity formation, and "on task" and "off task" work in the context of group discussions (b) a presentation by DeJarnette of current work using SFL to describe how students' positioning and construction of mathematical meanings inform one another during small-group discussions. This will be followed by open discussion time with two guiding questions: (a) What theories and frameworks are you using in your work? (b) What frameworks would you like to learn more about?

Day 2: Introduce a "data dive" activity using data from two different sources. One option will be to examine patterns of small group and whole class talk by examining excerpts of group discussions in a bilingual classroom where the primary language of instruction was English. We will ask, how do group discussions play out in classrooms where students and teachers have different linguistic resources? (facilitated by Zahner). The second option will be an analysis of 6th-grade group discussions in a classroom where the teacher and students will be defining/learning group roles for the first time. Participants will be able to analyze these data using the frameworks introduced on Day 1 (but with the assumption that not all participants attended day 1). These analysis activities will be structured with guiding questions facilitated by the session organizers (45 min). At the end of day 2 we will have an open discussion: What did you learn today? What more would you like to learn? Participants to complete a short survey that will be used to create groups for the final day of the working group.

Day 3: The final day the working group will be structured as a working space for participants. We will form small groups and create break out rooms (both live and hybrid) to allow participants to share their own work, discuss current struggles in their work and set up opportunities to get future feedback on their work.

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USING SELF-BASED METHODOLOGIES TO UNPACK MATHEMATICS TEACHER EDUCATORS' WORK

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Narrative inquiry, self-study, and autoethnography (i.e., self-based methodologies) are methodologies used by mathematics teacher educators (MTEs). These methodologies have opened up the field by unpacking and unearthing MTEs' work communicating findings from their practices. Building from our previous working groups at PME-NA 2018-2020, we sustain a community where MTEs can feel supported in their study design, implementation, representation of findings, and publication using self-based methodologies. At PME-NA Philadelphia, we will continue our work at PME-NA Mexico on self-based methodologies to develop perspectives on philosophical underpinnings of self-based methodologies and addressing trustworthiness and authenticity in our reports.

Keywords: Research Methods, Sustainability, Teacher Educators

We are a group of mathematics teacher educators and researchers (MTERs) committed to creating professional development spaces for MTERs to learn and conduct studies using self-based methodologies (Suazo-Flores et al., 2018, 2019, 2020). This motivates us to propose a Working Group at PME-NA 2021, where we can connect with MTERs, collaborate, and receive support on the design and documentation of studies using self-based methodologies (Chapman et al., 2020). Self-based methodologies (Chapman et al., 2020) "privilege self in the research design, recognizing that addressing the self can contribute to our understanding of teaching and teacher education" (Hamilton et al., 2008, p. 17). These methodologies include narrative inquiry (Clandinin & Connelly, 2000), self-study (LaBoskey, 2004), and autoethnography (Ellis & Bochner, 2000). A slowly growing number of research reports using self-based methodologies have been published in mathematics education journals (Kastberg et al., 2018; Di Martino & Gregorio, 2019; Goodell, 2006; Hjalmarson, 2017; Nardi, 2016; Nicol et al., 2020; Nolan, 2018; Xenofontos, 2016) with many more in teacher education journals (Brandenburg, 2021; Brandenburg & Davidson, 2011; Hourgin & Leavy, 2021; Martinie et al., 2016; Schuck, 2009; Simpson, 2019; Stoehr, 2017). These papers include a focus on identity development and practices. For instance, Simpson (2019) described ways her development as a MTER for elementary mathematics preservice teachers from a background in secondary education paralleled that of her students. Nolan (2018) shared her experiences reconceptualizing her practices supervising preservice mathematics teachers. MTERs also have used self-based methodologies to communicate people's experiences with mathematics and call for new approaches (e.g., Nardi, 2016; Stoehr, 2017). We see MTERs' studies using self-based methodology as professional development spaces they create to learn about themselves, their practice, and contribute insights about practical knowledge within the research domain of mathematics teacher education (Chapman, 2020).

In mathematics education, calls for expanding research methodologies and methods used in published work (Cannon, 2020; Inglis & Foster, 2018), highlight the need for MTERs to gain more insight into conducting and reporting research using self-based methodologies. Addressing the current views of so-called rigor in research in mathematics education has the potential to illustrate ways the use of self-based methodologies contributes to mathematics education. In the reporting of such research, two areas of focus can help researchers communicate about their approaches: philosophical underpinnings (Ernest, 2012) and trustworthiness (Lincoln & Guba, 1985). Philosophical underpinnings of self-based methodologies illustrate how researcher's work belongs to the larger body of mathematics education research by connecting such work to the ideas about being, knowing, and feeling that have informed mathematics education. Drawing on expanded notions of trustworthiness called for by Lincoln and Guba (1985) we focus on addressing authenticity in research reports of studies using self-based methodologies (Lincoln & Grant, 2021, in press). Authenticity illustrates ways that our studies, while situated in particular contexts and not generalizable, contribute to ongoing discussions of mathematics teaching, learning, and curriculum. To support the ongoing development of research in mathematics education using self-based methodologies we endeavor to explore these factors of work in progress among the working group members using self-based methodologies. In addition, we will prepare for and organize a collection of research reports from members of the working group for submission to a special issue while also brainstorming new publication opportunities for newer members of our group.

Session Information

We have regularly met to continue creating professional development spaces where MTERs can communicate their findings and experiences using self-based methodologies (Suazo-Flores et al., 2018, 2019, 2020). MTERs are invited to join our Working Group to learn about self-based methodology studies (Chapman et al., 2020) and benefit from discussions to support the design, implementation, analysis, and representation of findings from such studies. Concerning the session activities, on Day 1, we will present literature reviews of self-based methodology studies conducted in the last five years and discuss their philosophical underpinnings. On Day 2, we will invite MTERs to present their studies using self-based methodologies to identify philosophy and Trustworthiness/Authenticity. On Day 3, we will develop action items and discuss new projects such as writing a proposal for PME-International.

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CREATING SPACE FOR PRODUCTIVE STRUGGLE TOWARD A MORE EQUITABLE FUTURE: PERSEVERING THROUGH CHALLENGES FROM WITHIN

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This working group is a consistent space for equipping, informing, and challenging mathematics education researchers to "frame equity as a continually evolving process of growth". Since its inception this working group has continued to productively orient, inspire, and organize mathematics education researchers to move toward outcomes in our field that prioritize anti-racist mathematics education as a mechanism for change. The challenge of this working group remains one of moving from collective reflections around issues of equity and diversity in mathematics education to actions that become catalysts for change. We acknowledge that this year's call for "productive struggle" is necessary, however it needs people and community to support such efforts.

Keywords: Equity, Inclusion, and Diversity, & Social Justice

"We frame equity as a continually evolving process of growth rather than as a destination that can be reached." (PMENA, 2019, p. 1)

In recent years many of the major mathematics organizations have created statements or calls to actions for their membership to address issues of equity, diversity, and social justice within mathematics education. Due to the increased racial injustices and Black murders that captured the world's attention beginning in Spring 2020, we also have a sense of urgency for "action to dismantle racism as it exists in our schools, institutions, and even our own organizations, such as PME-NA." (PME-NA, 2020). Specifically, the Access and Equity Principles states, "An excellent mathematics program requires that all students have access to a high-quality mathematics curriculum, effective teaching and learning, high expectations, and the support and resources needed to maximize their learning potential" (NCTM, 2014, p.5). Such calls to action represent both a reality to the future of PMENA membership and the critical expectations for our field in imagining the possibilities for what can be done to strengthen and nurture mathematics education as a whole. But where lies the support? This working group is designed as a space for any who attend to advance in their understanding and abilities to help shape their work toward meeting these calls for more equitable and critical outcomes in mathematics education. A space for all types of knowers to interact around issues relevant to their specific contexts so that all may advance more centrally as a doer of mathematics. To be explicit, the outcome is not to produce a product but to make the space for the building of relationships and awareness of those looking to do this work. Insofar to say, relationships are central to the formation and sustainability of research collectives that need space to productively struggle with others engaged in equity, diversity, and anti-racist lifework.

History of the Working Group

This working group was first formed in 2009. The connection to history along with the focus of the people and scholarship of this working group's longevity has allowed for the tapering of content in each iteration of the working group as it continues to center those in the margins. In 2017 explicit work was done to reset the working group to an acclimation and incubator space where "current issues affecting our field" can be discussed, organized, and introduced "into the bigger conversation about the research of mathematics and mathematics education" (PME-NA, 2019, p. 3). With the unforeseen interruption to this working group due to COVID-19 for PME-NA 42, and the protests and fires that raged due to the continued loss of Black lives in the U.S. in 2020, there remains a greater need than ever to reflect and act upon our complicity in perpetuating inequities in mathematics education.

Organization of the Working Group

Each 90-minute session will build on previous sessions, beginning with a facilitated conversation around the previously stated purpose of the working group to enact and move forward PME-NA's stated purpose to continuously "recenter education" and "recenter equity and criticality" (PME-NA, 2019). The format for the sessions will include:

- SESSION 1-Orientation & Reflection: The working group will start with a brief history of the working group and explicitly state its purpose to be a safe space for collaboration and growth in moving toward the evolving destination of equity in mathematics education (see four actions from June 2020 update to the PME-NA Equity Statement). Using collaborative structures, working group participants will share experiences, challenges, and triumphs over the past year and how they have seen equity as being a part of the work they do in mathematics education. Special attention will be made toward offering those new to the space points of entry to orient themselves to the conversations and questions from within the working group. Finally, questions of interest will be collected, organized, and posted for use during the following sessions of the working group.
- SESSION 2-Interconnectedness & Common Ground: Using participant generated questions and emergent themes, participants will be organized into small groups to engage in knowledge sharing and orientation to provide an asset-oriented perspective on all participants in the working group. Small groups will engage in structured activities to help synthesize critical questions, helpful resources, and productive collaborations for moving forward on areas of common interest.
- SESSION 3-Encouragement & Action: The final day of the working group will be a space for working group participants to continue the work of session two and more open-ended conversation and dialog on areas of interest. Participants will be encouraged to talk about next steps with the emphasis being this is only the start of the conversation, but the connections made extend beyond the limits of any conference and organization.

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MATHEMATICAL PLAY: ACROSS AGES, CONTEXT, AND CONTENT

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In the proposed working group, we will build from the foundation of the past two years' working groups as well as our members' continuing collaborations with researchers outside of this group. Specifically, we propose three days of activity, each advancing different aspects of developing the body of mathematical play research. We have planned the three following foci: adapting existing mathematical tasks and curricula to increase opportunities for play (Day 1); adapting voluntary play activity to support mathematical learning (Day 2); and collaborating with members of the EMIC research community through an intra-working-group discussion session to explore play as an embodied approach to mathematics learning (Day 3).

Keywords: Instructional activities and practices; Affect, emotion, beliefs & attitudes; Informal education

Over the past four years, members of this Mathematical Play working group have developed a community of colleagues focused on identifying and characterizing productive theoretical lenses and methodological approaches to investigate students' mathematical play, an important, yet under-investigated domain within mathematics education research (e.g., Holton et al., 2001; Wager & Parks, 2014). Central to this work has been the emergent characterization of mathematical play as (1) voluntary engagement in cycles of mathematical hypotheses with occurrences of failure (Williams-Pierce & Thevenow-Harrison, 2021), (2) often spontaneous and self-directed toward a player's emerging goals (e.g., Wager & Parks, 2014), and (3) supported or discouraged through physical or digital interactions (e.g., Sinclair & Guyevskey, 2018).

In response to shifting demands, our working group made the difficult decision to withdraw from the online conference earlier this year even though our proposal to discuss our respective progress investigating mathematical play had been accepted. Accordingly, in preparation for this year's working group proposal, the co-organizers wish to incorporate the planned presentations from the cancelled working group along with discussions surrounding new directions for mathematical play research. Throughout the past year, our organizing group situated our work based on the degree to which it might be characterized along two dimensions: pure play and structured mathematical instruction. This focus is consistent with what Wager and Parks (2014) discussed as two seemingly contrasting ideologies: increased focus on teacher-directed instruction and scholarship confirming that children learn best in play-based environments (p. 223). Wager and Parks (2014) call to identify practices that bridge the two ideologies. In direct response to these calls, this working group will build on theoretical frameworks for instruction and play by incorporating additional perspectives (e.g., Weisberg et al, 2013; Zosh et al, 2018).

Specifically, we maintain our prior goal to address the divide between play and instruction by discussing theory and collaborating around results from several projects that participating researchers might situate along the dimensions of play and instruction. These conversations will address how specific activities and instructional interventions might support shifts along those dimensions. We will focus on two shifts: firstly, how might an education researcher who has traditionally situated their work within more traditional mathematical tasks alter their existing approaches to afford greater opportunities for play? Secondly, we will address the complement: how might we support the work of educators interacting with students in play-based settings to better foster meaningful mathematical development?

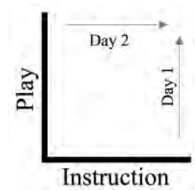

Figure 1: Graphic organization of two shifts – increasing the playfulness of high instructional tasks (Day 1) and increasing the instructional utility of high play tasks (Day 2)

Continuing the success of the prior Mathematical Play PME-NA Working Groups, we have developed the following goals for this year's working group: (1) to engage participant researchers in conceptualizing the two shifts illustrated in Figure 1; (2) to share and discuss existing projects that are making or have made these shifts, specifically identifying frameworks and perspectives to support such shifts; and (3) to summarize these conversations and promote a synergistic dialogue with the EMIC working group.

Day 1 will showcase examples from projects that originated as instructional activities but have shifted toward investigating the conceptual affordances of incorporating more playful activities for students (Ellis and Plaxco leading). The working group leaders will briefly introduce their projects and engage participants in active exploration of chosen tasks from these projects. They will then lead the group on identifying and discussing what play frameworks might be productive for supporting the transition from typical instructional tasks to play-based activities. The group will synthesize this discussion as a starting point to conceptualize how educators might incorporate playful activity within their classrooms.

On Day 2, we will take a contrasting perspective as we explore design and facilitation practices that leverage mathematical play for learning (Reimer, Molitoris-Miller, and Simpson leading). Leaders will engage group members in interactive play and board game activities with a focus on the mathematics that players draw on during their play. Whole-group discussion will focus on the pedagogical approaches and practices to support learning during mathematical play.

On Day 3 we will meet with the EMIC working group (Nathan et al., 2017) to explore areas of overlapping interest and potential convergence (Williams-Pierce leading). Members of both groups will engage in intra-working-group conversations to highlight common theoretical and methodological approaches and identify opportunities for collaborative dialogue (i.e., mathematical play as an embodied way of learning, design considerations for embodied mathematical play, etc.).

Acknowledgments

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RESEARCH COLLOQUIUM: MODELS AND MODELING PERSPECTIVES

PERSPECTIVAS (NORTEAMERICANAS) HACIA LA MODELACIÓN MATEMÁTICA

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The Models and Modeling Working Group was initiated with PME-NA itself in 1978, and it has met frequently since then. This year, we propose to carry the conversation forward in the new Research Colloquium format: our goal is to establish the foundation for articulating a coherent North-American voice in the international modeling community. In fact, we conjecture that a distinctively Pan-American perspective can be articulated, identifying the common emphases and complementary strengths among modeling researchers in North, Central, and South America. The colloquium is a testing ground for this premise, looking toward ICME and ICTMA.

Keywords: Modeling, Problem Solving, Classroom Discourse, Mathematical Representations.

History of the Models and Modeling Working Group

Over the 43 years of its existence, the Working Group has offered researchers in the Models and Modeling Perspective (MMP) a vehicle for coordinating collaborative research, for welcoming new researchers into the community, and for building mentoring relationships. Early in its history, the Group focused heavily on the design and analysis of particular, self-contained activities that enabled groups of learners to engage realistic and deep forms of modeling and that produced an *auditable trail* of thinking, making learners' thought processes visible to teachers and researchers. In this phase of the field's development, a primary focus was elaborating design principles for these *Model-Eliciting Activities,* or *MEAs* (Doerr & English, 2006; Lesh & Doerr, 2003; Lesh, Hoover, Hole, Kelly, & Post 2000; Lesh, Hoover, & Kelly, 1992; Hjalmarson & Lesh, 2007; Zawojewski, Hjalmarson, Bowman, & Lesh, 2008), and articulating images of *idea development* that they promoted (Lesh et al, 2000; Lesh & Doerr, 2003). Quickly, MEAs were recognized for their potential not only as contexts for research into idea development, but also as powerful *learning environments*. This "turn" to connect with classroom ecologies raised questions about how different student groups' work could be "processed" by a whole class, bringing out common themes and connecting them to more conventional mathematical terminology, algorithms, and procedures. The construct of a Model Development Sequence, or MDS emerged in response (Ärleback, Doerr, & O'Neil, 2013; Doerr & English, 2003; Hjalmarson, Diefes-Dux, & Moore 2008; Lesh, Cramer, Doerr, Post, & Zawojewski, 2003), as one of several candidate forms of organizing classroom modeling at a larger grain size (Brady et al, 2020). These questions begin to suggest a distinctive agenda of *research themes* on modeling and idea development, which investigate:

- interrelations among nested social levels (individual, small group, and classroom community; and then, in turn, broader levels of community beyond the classroom)

- forms of learning that can be documented at these distinct social levels,
- larger-timescale emergence of connected networks of ideas through modeling, and
- larger-timescale views of the emergence of modeling as a classroom practice.

The first two of these themes call for investigations of the role and importance of the *social* and *cultural* context of classroom modeling (Brady & Jung, 2021). The last two point to questions about *temporal* and *developmental* dimensions (Brady & Lesh, 2021).

Areas for Discussion in the Colloquium

These perspectives can offer important and complementary contributions to the current international discourse on modeling. The international community is dominated by two strong cognitivist traditions, both of which have been centered in Europe but have gained traction in Australia and Asia. A first tradition, focusing on *applications*, aims at the objective articulated in Hans Freudenthal's phrase, "to teach mathematics so as to be useful" (Freudenthal, 1968); this approach is often described under the heading "modeling and *applications*." A second tradition, focusing on *heuristics*, seeks to formulate generalizable strategies that can support problem-solvers across domains (Polya, 1945). In framing modeling and articulating research questions, these traditions tend to place both social and developmental questions in the background.

Applications-oriented perspectives focus on how previously learned mathematical concepts and skills are adapted when attempts are made to use them in "real life" situations. Such problems tend to foreground particular expert solutions that wield the given mathematical tools optimally. In contrast, the MMP tradition has focused on creating problem situations that generate the need for key mathematical constructs and press learners to construct original mathematizations that serve to interpret the situation. Here, the *diversity* in learners' ways of thinking tends to be foregrounded, both in fueling the modeling engine itself and in responding to the range of distinct mathematizations that emerge in MEA solutions.

Heuristics-oriented perspectives aim at generalizable strategies (Schoenfeld, 1992) and typologies – types of problems and learnable categories of response. Such approaches tend to consider strategies in a cognitive vein and view modeling as a procedural skill that can be directly taught (Polya, 1945). In contrast, the MMP is more amenable to a *practice-oriented* view of modeling (Boaler, 2000; Cobb et al, 2001; Kobiela & Lehrer 2015), viewing the teaching and learning of modeling as the development over time of classroom modeling communities.

Our Colloquium will engage the PME-NA community—both prior members of the Models and Modeling Working Group and new participants—in articulating a research perspective that foregrounds social and developmental perspectives that can complement these two traditions.

Organization of the Sessions

As we have done within the former Working Group format, we will attend both to the needs of "newcomers" and "old hands" to the MMP—flexibly adjusting based on the participant group. Our three principal facilitators will describe recent work that foregrounds the classroom community as a learning entity, emphasizing both social and developmental questions, as well as work that considers the classroom as embedded in larger social and cultural systems. Each session will begin with a precis of recent study, followed by structured discussion of issues highlighted by that study. These recent efforts will not be presented as "finished products;" instead, we will share the questions of *theory* and *method* that we have encountered, with the goal of opening the conversation. The broader leadership group will help to direct the conversation toward key messages we want to be articulated at ICME and ICTMA, to represent the research commitments that we are seeing in work across North, Central, and South America.

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EXPLORING PRODUCTIVE STRUGGLE IN MATHEMATICALLY-RICH CONTEXTS

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This Working Group will explore how productive struggle is normalized and engaged in mathematically-rich informal contexts. From debugging in computer programming, to "unknitting" in textile art, to the constant repair of play, and to reconfigurations in dance, we find that struggle is a central and unproblematic element of many inherently mathematical practices. However, the productivity of this struggle and its relation to mathematical thinking is rarely identified and often overlooked. The goal of this working group is to connect scholars whose work has explored contexts for productive struggle that might not have been historically labeled as mathematical, with the goal of exploring overlaps and possibilities for mathematics.

Keywords: Computing and Coding; Informal Education; Integrated STEM / STEAM

Framing and Goals of Working Group

It has been well established that students' ideas about, and relationships to, the field of mathematics develops through their experiences in the world (Bishop, 2012; Boaler & Greeno, 2000). Although there are many experiences in our everyday lives that are richly mathematical, the experiences that are most typically labeled "mathematical" happen in schools. While as a field we are constantly working on refining and reforming mathematics instruction, it continues to be the case that for many students, mathematics is experienced as a discrete body of facts to be learned and remembered; a set of calculations rather than a series of interconnected ideas (Louie, 2017). In that framing, mathematical struggle is often taken as an indication of mathematical incompetence; uncertainty, time, and errors are to be avoided at all costs, and when they do occur, they are hidden away and forgotten. Not only is this stance towards mathematics problematic with respect to what students learn about mathematics (Vogelstein, 2021), it is also damaging to the likelihood of students coming to see themselves as mathematical persons, as these narrow views of mathematical competence routinely work to convince students that their future is *not* mathematical (Ladson-Billings, 1998; Larnell, 2016).

The invitation to *struggle* with mathematics, and the idea that struggle can be *productive*, is an important reframing to the historical cultural practices of school math. And yet we know that changing existing structures is challenging, particularly as doing so requires re-imagining current work. For that reason, we have found it productive to explore mathematical possibilities outside of school math to offer a vision of how cultures of mathematics might be transformed (Brady, Gresalfi, Steinberg, & Knowe, 2020; Gresalfi & Chapman, 2017; Kafai, Franke, Ching, & Shih, 1998; Vogelstein, 2021; Vogelstein, Brady, & Hall, 2019 Wager & Parks, 2014; Weintrop et al., 2016). To that end, this working group will explore the ways that *productive struggle* is normalized and engaged in mathematically-rich non-school contexts. From debugging in computer programming, to "unknitting" and stitch picking in textile art, to the constant repair of play, and to reconfigurations in dance, we find that struggle is a central and unproblematic element of so many practices, practices that are inherently mathematical although potentially not always labeled as such.

This proposal is for a new working group—a new initiative that has not previously been a part of PME-NA. The goal of this working group is to connect scholars whose work has explored contexts for productive struggle that might not have been historically labeled as mathematical, with the goal of exploring overlaps and possibilities for mathematics. For example, the rising popularity of STEM and STEAM initiatives (Takeuchi, Sengupta, Shanahan, Adams, & Hachem, 2020), and the increased focus on the potential of interdisciplinarity, all involve central mathematical practices, and such initiatives have been of great interest to the mathematics education community. We therefore anticipate that this working group will be popular among the many scholars whose work seeks to connect mathematics with other disciplines, and therefore we anticipate that these meetings would easily lead both to future submissions and, potentially, future working group meetings.

Strategies and Activities

Because of the nature of working groups—that participation is unpredictable across three days, we plan to develop a set of activities that build on each other but that each result in a product that can become the focus of the subsequent day, thus allowing newcomers to participate without having attended previous sessions. The first day will create an opportunity for experiencing productive mathematical struggle in mathematically-relevant contexts of debugging, sewing, knitting, play, or dance. For example, if the meeting is in person, we will engage participants in a set of "debugging" tasks with a spherical robot called *Sphero*, invite them to participate in a choreography activity that includes a large mylar square, and/or offer yarn and fabric (depending on skills) to begin to create clothing or squares that cover an object. If the meeting is online, we will provide virtual opportunities to engage in mathematical struggle or share videos of students engaging in such tasks to allow participants to analyze others' debugging activities. In either case, the goal is to experience productive struggle either directly or vicariously, and to produce an account of "what happened," with respect to the struggle and its relationship to disciplinary engagement. This will result in analysis artifacts to be used later in the workshop.

The second day of the working group will invite discussion about these interdisciplinary activities by focusing on the analysis artifacts that were developed in day 1. Each group will share a brief overview of the context they engaged and their thinking about how productive struggle was invited and experienced. The goal of these discussions is to both begin to think about how we see mathematics in interdisciplinary contexts, what *counts* as mathematics in these contexts, and whether and how the struggle that is inherent in these contexts appeared to contribute to the opportunities to learn and engage.

The final day of the working group will push towards synthesis and future work, exploring how the targeted contexts and those that participants are already working to consider whether and how those contexts can be connected to student engagement, or student mathematical engagement. This collective work will focus on identifying productive areas for future research and, perhaps, identifying themes for a future working group or conference. At a minimum, a collective document will be generated to identify a set of questions or concerns that could be explored by small groups over the course of the year.

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CONCEPTIONS AND CONSEQUENCES OF WHAT WE CALL ARGUMENTATION, JUSTIFICATION, AND PROOF

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Argumentation, justification, and proof are conceptualized in many ways in extant mathematics education literature. At times, the descriptions of these objects and processes are compatible or complementary; at other times, they are inconsistent and even contradictory. Regardless of the descriptions of these processes, however, given the importance of argumentation, justification, and proof to the discipline of mathematics, as well as their valued roles as learning practices, it is critical to query the relationship between engaging students in such processes and the promotion of equitable learning spaces and outcomes. This year, working group leaders aim to facilitate discussions and collaborations among researchers to advance our collective understanding of argumentation, justification and proof through an equity and inclusion lens.

Keywords: Reasoning and Proof; Advanced Mathematical Thinking; Equity, Inclusion, and Diversity

Theoretical Background

Given the importance of argumentation, justification and proof to the discipline of mathematics, as well as their valued roles in supporting student sense making and understanding, it is critical to query the relationship between engaging students in such processes and the promotion of equitable and inclusive learning spaces and outcomes. Two constructs potentially crucial for examining the role of these processes in creating more equitable outcomes are access and agency (Gutiérrez, 2002), as these practices can provide students with access to powerful mathematics, and opportunities to come to understand themselves as knowers and doers of mathematics, as well as having their voices matter and influence the classroom. As we seek to promote strong mathematics learning, as well as rehumanize mathematics classrooms, we need to further examine and theorize the relationship between argumentation, justification and proof and equitable engagement. We assert this against a backdrop where mathematics, and its ways of knowledge development, have a history of exclusion that not only impacts participation in mathematics education (Louie, 2017) but also in the discipline more broadly. For example, proof has been positioned as a high-status process, and thus can be positioned as exclusionary (see, e.g., Knuth, 2002; Otten et al., 2020).

History of the Working Group

The *Conceptions and Consequences of What We Call Argumentation, Justification, and Proof* Working Group (AJP-WG) met for the first time during the 37th Annual Meeting of the North American Chapter of the Psychology of Mathematics Education (PME-NA) in 2015

(Cirillo et al., 2015). The group then met for three additional years in 2016, 2017, and 2018 (Staples et al., 2016; Conner et al., 2017, & Conner et al., 2018). The AJP-WG sessions were well-attended each year, and the group has been active in between meetings. Following the 2018 meeting, AJP-WG members began work on an edited book, which will be published in the coming year (Bieda et al., in press). Across the authoring teams of the AJP-WG conference papers, the book, and the white papers (i.e., Cirillo et al., 2016; Staples et al., 2017), over 50 scholars have been involved in the group's work, including many graduate students. A Research Colloquium, where we shared analyses from the forthcoming book, was facilitated at the 2020 conference (Cirillo & Bieda, 2020). We now return to the Working Group format with a focus on AJP through an equity and inclusion lens.

Plan for the Working Group

The sixth gathering of the working group will focus on the degree to which argumentation, justification, and proof (AJP) may or may not promote equity and inclusion in mathematics education. To do this, we will engage participants in vignettes focused on the interrelationship between each construct of AJP and specific aspects of equity and inclusion.

On Day 1, we welcome participants and facilitate introductions. We then focus on justification and how justification can be an *equity practice* in classrooms (in addition to a mathematical practice and a learning practice). We focus on the constructs of *access* and *agency* as we analyze classroom artifacts to consider how justification can support access and agency for students, and thus potentially more equitable outcomes. We also discuss how justification can be positioned as an *inclusive* rather than *exclusive* practice and consider *students'* choices and decision-making processes with respect to their mathematical justifications.

Day 2 will begin with a 30-minute AJP networking activity before shifting into a discussion on argumentation. Mathematical argumentation is a social activity that should be facilitated by teachers providing a context that is positive and supportive for students (Wood, 1999). Discourse in this context can be supported or hindered by either the teacher or students (Kosko, 2015; Kotsopoulos, 2008). However, not all students are provided equitable access to the social and sociomathematical norms of argumentation in mathematics lessons, and students' cultural experiences may influence their comfort level with challenging others' ideas (Civil & Hunter, 2015; Lubienski, 2000). Thus, Day 2 will focus on an interactive discussion regarding the role of relational and cultural factors in promoting (or hindering) mathematical argumentation. Vignettes of mathematical argumentation will be used to facilitate discussion regarding how attention to social and cultural contexts may lead to more equitable participation in argumentation across K-16 mathematics.

On Day 3, we will focus on proof through an equity and inclusion lens. Although some researchers have argued that proof "is not a thing separable from mathematics itself" but rather an essential component of doing mathematics (see, e.g., Schoenfeld, 1994, p. 76), other researchers (e.g., Knuth, 2002; Otten et al., 2020) have provided evidence that classroom teachers sometimes hold exclusionary beliefs about which students should be provided with opportunities to engage with proof. After we act out a vignette that explores this issue, participants will engage in small-group discussions about the consequences of such beliefs through the lens of equity and inclusion. We will then bring the small groups together in a whole-group discussion to debrief and consider strategies for dealing with these tensions. In the last 30 minutes of Day 3, we will look across the WG activities and discussions of the past three days with respect to AJP to make connections and consider next steps for the group.

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COACHING THE COACHES AND OTHER EFFORTS TO DEVELOP MATHEMATICS TEACHER EDUCATORS FOR INSERVICE TEACHERS

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This working group will engage educators who conduct professional development for, and research on, mathematics teacher educators in the context of inservice mathematics teacher education. One project that will be discussed is a professional development model designed to support mathematics coaches from rural districts. We encourage others who focus on developing teacher educators or designing professional development for teacher educators to join us. Goals for this working group are to share insights from the work of preparing teacher educators, to discuss the challenges of extrapolating understanding of teacher development to the development of teacher educators, and to develop potential collaborations for future work. We intend for this working group to continue into future PME-NA conferences as we build on this initial collaboration to influence our individual work and the field at large.

Keywords: Professional Development, Teacher Educators, Research Methods, Instructional Leadership

Working Group Rationale

There is a need to expand efforts to develop and research the development of mathematics teacher educators, particularly in inservice contexts. Relative to the literature on the professional development of teachers, there is much less literature on growing the capacity of those who design and conduct teacher professional development. Mathematics teacher educators, including coaches, need to develop capacity related to content knowledge, pedagogical knowledge, interpersonal skills, big-picture visioning and planning, and change theory to support teachers in improving instruction and student learning (Saphier & West, 2010). There are few formal training programs for mathematics teacher educators, which has led to the promotion of teachers to coaching positions based on their strengths as classroom teachers, rather than their experience or skills as a coach or trainer (Chval et.al., 2010; Hartman, 2013). Given the lack of formal training opportunities for coaches or other mathematics teacher educators providing professional development, it is reasonable to assume that most mathematics teacher educators learn practices through their experiences in the context of their work. Building on our Working Group in PME-NA42 focused on content-focused coaching specifically, we want to engage in discussions of how mathematics teacher educators develop capacity to work with inservice teachers in varying contexts. While we had some rich discussions with participants at PME-NA42, we want to engage more mathematics teacher educators who work with inservice teachers beyond those who engage in coaching. While we will initially draw from a project focused specifically on developing coaches (NSF Grant DRL-2006263), all authors have extensive experience working with mathematics teacher educators supporting inservice teachers more broadly.

Our Project

We designed, implemented, and researched an innovative fully online professional development model for mathematics coaches. We engaged coaches, as mathematics teacher educators, in a three-part professional development model that included (a) a professional development course on content-focused coaching, (b) one-on-one video-based "coaching the coach" cycles, and (c) a video coaching club (see Carson et al., 2019; Choppin et al., 2020; & Choppin et al., in press). Mentor Coaches - project personnel recruited for their expertise in content-focused coaching - supported the practicing coaches in coaching their colleagues and reflecting on their practice.

We will share key reflections and lessons learned from designing, implementing, and researching the three-part professional development model for mathematics coaches, and the challenges in doing so, in relationship to the implications for supporting mathematics teacher educators more generally. We will share perspectives from Participant Coaches, Mentor Coaches, and facilitators, as well as our main data streams. This sharing can launch discussion and engagement of session participants as they connect to their own work.

Working Group Organization and Structure

In this working group, we will draw on these experiences and those of the participants, to engage in dialogue with the goal of developing a more robust and shared understanding of how to support mathematics teacher educators, the challenges, and gaps in the field with respect to research and practice. Each session will focus on a different component related to designing and researching professional development for coaches and other mathematics teacher educators in working with inservice teachers. Session one will explore models of support and challenges; session two will explore research on supporting coaches and mathematics teacher educators who work with inservice teachers; and, session three will be dedicated to organizing future work. Each session will include opportunities for small and full group discussions, as well as for participants to share their own experiences and projects with the group in service to networking and generating new ideas and learnings related to informing future work. In addition, in session one, ideas for discussion topics will be solicited from the group in order to guide subsequent sessions related to the intention of the working group. This will allow participants to pose and discuss questions important to their work within a supportive and structured environment, and to draw on the talent that will be in the room. Each session will focus on the following themes:

- Session 1: (a) exploring models of developing coaches and other mathematics teacher educators who work with inservice teachers; and, (b) exploring challenges in implementing professional development models for coaches and mathematics teacher educators.
- Session 2: (a) what has been/is being studied related to the development of mathematics teacher educators who support inservice teachers, and what processes are there for research, and, (b) what are areas of need for further contributions to the field.

- Session 3: (a) exploring ideas and interests generated by the group and continuing conversations from the previous day, and, (b) discussing next steps and future possibilities.

To support follow-up and ongoing collaboration of participants, group notes and documents will be shared and distributed via a Google folder that will be set up for this working group. The use of Google documents will allow members to create an institutional memory of activities during the working group that we will continue to use. This shared folder will also provide a shared space for future collaborations and writing projects related to the working group members.

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STATISTICS EDUCATION: NEW CONNECTIONS AND FUTURE DIRECTIONS

EDUCACIÓN ESTADÍSTICA: NUEVAS CONEXIONES Y FUTURAS DIRECCIONES

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In continuing from past working groups on statistics education, this working group will seek to present new findings and connections, as well as work toward future directions. Each day of the working group will be based on a different theme including: international efforts, measurement, and issues of equity and social justice. Each day will start with short 3-5 minute mini-presentations of a few projects in line with the theme, followed by discussion of the theme in statistics education, and ending with a discussion of future directions related to the theme. Such structure affords easy movement of people in and out of the group based on interest and affords more focused discussion and possible future efforts.

Keywords: Data Analysis and Statistics, Measurement, Equity, Inclusion, and Diversity

Statistics education is growing by leaps and bounds. The pandemic has brought data and data-based arguments front and center in the media and people are increasingly able to openly access data for themselves (Ancker, 2020). Data science has also found a new hold in the K-12 mathematics curriculum in various locations (Bargagliotti et al., 2020) and has begun to be researched in educational settings. With the needs of the workforce and society shifting dramatically, there is significant need to think about statistics education's role and transformation in the face of a changing world. This working group is aimed at continuing discussions begun at previous meetings, and also to create space for new work and collaborations. In an effort to create an open environment that can also focus on specific elements of the field, the working group will be structured around three themes.

Themes

The working group will be structured around three main themes: international efforts, measurement, and issues of equity and social justice. Such structure will allow people to freely flow in and out of the group based on interests. The themes will also help focus discussion and create an environment for creating tangible next steps to support scholarship and collaborations.

International Efforts

Statistics education is a field that crosses both disciplinary and geopolitical boundaries (Ben-Zvi et al., 2018). Though there are contextual differences in education from country to country, there is also a lot that can be learned through cross boundary collaborations and considerations. The International Association of Statistics Education (IASE) is one avenue of sharing such work to a broad audience, however this venue is still dominated by English speaking researchers due to it being the dominant language of research. In this theme we will discuss international statistics education as well as consider ways for creating more opportunities for sharing research and collaborating across boundaries.

Measurement

Recently, there have been calls for increased focus on not only the need for developing instruments backed by strong validity arguments, but also for examination of validity evidence for existing instruments and closer attention to instruments being used for research purposes (Lavery et al., 2019). The NSF-funded Validity in Measurement in Mathematics Education Statistics Education synthesis group has spent the past two years documenting instruments used in statistics education research, as well as the types of validity evidence that accompany them. In this theme, we will discuss findings and implications, and broadly discuss issues of validity.

Issues of Equity and Social Justice

The global pandemic and recent protests for racial justice have highlighted the severity of inequities and systems of injustice that operate in society. Because of the centrality of context to statistical practice, the discipline is well positioned to interrogate issues of equity and social justice and create spaces in the often neutrally positioned mathematics curriculum (Author, 2019). In this theme, we will discuss how data investigations can be used to interrogate issues of equity and justice and how we might consider how such issues exist in our own field and scholarship.

Structure of Sessions

Each day will be focused on a different theme but will follow the same basic structure (see Table 1). The session will start with a brief ten minute introduction to the theme. Introductions will be followed by three 5-minute brief presentations of current work related to the theme. After setting the stage with introductions and brief presentations the next 35 minutes will be dedicated to talking about the theme with everyone in attendance. The final 30 minutes of each session will be focused on brainstorming next steps and future directions. The goal is for the next steps time to also involve discussion of possible scholarly products or forming collaborations to support the development of new work or projects in the field.

Table 1: Overview of Session Structure

Time	Day 1: International Connections	Day 2: Measurement	Day 3: Issues of Equity and Social Justice
Introduction to Theme	International Outlets, Conferences, and Themes	Discussion of VMED project	Statistical Investigations of Sociopolitical Issues and Areas of Concern
Brief Presentations	Brief discussion of three projects from various countries	Brief discussion of VMED findings	Brief discussion of current efforts in curriculum development and policy
Discussion	Discuss international collaborations	Discussion of ongoing measurement work and needs	Discussion of issues, outlining current problems
Future Directions	How can international collaborations be started? How can discussion across boundaries be facilitated?	What measurements do we still need? What measurements could we draw upon from other fields? How can we encourage such projects?	How do we incorporate more discussion of issues of equity and social justice in statistics education? How do we diversify the membership of the field and issues under investigation?

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COMPLEX CONNECTIONS: REIMAGINING UNITS CONSTRUCTION AND COORDINATION

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Students' construction, coordination, and abstraction of units underlie success across multiple mathematics domains. This working group aims to facilitate collaboration between researchers and educators with the particular aim of extending research on units coordination and construction across numerical contexts and constructs.

Keywords: Cognition, Learning Theory, Number Concepts and Operations

Theoretical Background, Purpose, and History

Units coordination and construction refers to the number of levels and types of units a person can construct and bring into a situation (Steffe & Olive, 2010). In Steffe's 2017 plenary for PME-NA, he substantiated particular needs for investigating how children develop mental operations when constructing and coordinating units. The working group began at PME-NA 2018, with the aim of facilitating collaboration amongst researchers and educators sharing Steffe's concerns about (a) the need for supporting units construction and coordination for all learners and (b) the need for accompanying learning trajectories (curricula) appropriate for students' current level of units across grade levels. The purpose of the working group is to provide opportunities for participants to sharing and build upon units coordination research.

Working Group Goals and Strategies, Past and Present

In the first year of the working group, goals included generation of related research topics of interest to PME-NA attendees, including the role of units coordination in early childhood education, special education, and secondary and post-secondary education, and teacher education. Products included (1) the creation of a website for organizing and collecting tasks used for assessing students' units coordination and (2) links to research papers addressing particular topics relating to units coordination: https://unitscoordination.wordpress.com/.

Since this first working group, we have hosted discussions with novice and experienced researchers who focused much of their work on (1) students' co-variational reasoning when coordinating units, (2) students' probabilistic reasoning when coordinating units, (3) students and prospective teachers' units coordinating, (4) students' algebraic reasoning when coordinating units, and (5) students' engagement with visual material (e.g. hands-on and virtual manipulatives) when constructing and coordinating units. To continue to build on the productive discussions we will focus on issues related to assessments of units coordination in different settings (within classrooms, in virtual and hybrid classroom settings, via written instruments, via clinical interviews, and via individual or paired-student teaching experiments) and continue efforts to bridge emerging research connecting units coordination across mathematical domains.

Session 1: Emerging Assessments with Units Coordination in Discrete and Continuous Numerical Contexts

GOAL: Explore and bring to focus the role of tasks and assessments when examining emerging research investigating connections between units construction and coordination across age groups. ENGAGEMENT: Prior to the working group, we will administer an entry survey of participants to determine interests and goals for collaboration. In the first day's meeting, participants will discuss what constitutes a "unit" and reflect on their own actions with units when solving tasks in discrete and continuous number contexts used to assess and support units construction and coordination (listed on the units coordination website). Participants will form at least three groups, one focused on units construction and coordination with young children, one focused on units coordination with upper elementary and middle school students, and one focused on units coordination of pre-service teachers and their mathematical knowledge for teaching. Participants will share their own tasks, assessments, and modalities used in their work. Notes will be recorded in a Google doc.

Session 2: Connections between Units Coordination and other Constructs

GOAL: Explore and bring to focus the role of tasks and assessments when examining emerging research investigating connections between units coordination and other constructs (e.g., subitizing, co-variational reasoning, quantitative reasoning, algebraic reasoning, combinatoric reasoning). ENGAGEMENT: In small groups, members will discuss how their work across constructs connects to units construction and coordination. The discussions will include both empirical and theoretical connections. Notes will be recorded in a Google doc.

Session 3: Next Steps

GOAL: Embark on planning collaborations of interest to participants. ENGAGEMENT: As a whole group, we will discuss the results from the previous sessions and share prospective projects involving units coordination. Then, form small groups for each of these goals: (1) new directions for participants' research and teaching projects, and (2) creation of content for the webpage. Administer exit survey of participants' interests and goals for collaboration.

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EMBODIED MATHEMATICAL IMAGINATION AND COGNITION (EMIC) RESEARCH COLLOQUIUM

The EMIC Working Group aims to connect, engage, and inspire colleagues in this growing community of discourse around pedagogical, technological, theoretical, and methodological developments for advancing the study of embodied cognition for mathematics education. This year, our seventh at PME-NA, we organize our interactions around the theme of "productive struggle" by promoting inclusive mathematics education research and principles through a broad range of embodied activities, practices, and emerging technologies that contribute to teaching, learning, and assessment of mathematical reasoning, and study of these phenomena.

Keywords: Cognition; Design Experiments; Equity, Inclusion & Diversity; Learning Theory; Embodiment, Gesture, Multi-modal Discourse

Pedagogical, technological, empirical, theoretical, and methodological developments in embodied cognition and gesture studies support the continuation of the regularly held Embodied Mathematical Imagination and Cognition (EMIC) Working Group for PME-NA. Members of this group have met annually at PME-NA since 2015 and we are excited to expand into a **research colloquium** this year. The central aims of EMIC are to attract, engage, and inspire colleagues to understand the emerging impact of embodied learning and invite scholars to participate in this growing community of discourse. The group seeks to advance the study of embodied cognition for mathematics education, including reasoning, instruction, assessment, technology, and learning in and outside of formal settings.

Views of learning as embodied experiences have grown from developments in philosophy, psychology, anthropology, education, and the learning sciences that frame human communication as multimodal interaction, and human thinking as multimodal simulation of sensory-motor activity (e.g., Abrahamson et al., 2020; de Freitas & Sinclair, 2014). Four ideas exemplify the plurality of ways EMIC is relevant for the study of mathematical understanding: (1) Grounding abstractions in perceptuo-motor activity as an alternative to amodal symbol systems; (2) Cognition emerges from perceptually guided action; (3) Mathematics learning is always affective, never detached from body-based feelings and interpretations; (4) Mathematical ideas are conveyed via multimodal forms of communication, e.g., gestures, drawing, and objects.

The interplay of multiple perspectives is vital for the study of embodied mathematical cognition to flourish. While there is significant convergence, there remain questions to be addressed through empirical means: (1) When and how can teachers best incorporate principles of embodied design? (2) How can theory inform good assessment practices for nonverbal math knowledge (e.g., intuition)? (3) How do embodied education practices support differentiated instruction? and (4) Does embodiment identify limits to humans' mathematical reasoning?

Past Achievements, Current Organizers, and the Future of EMIC

As Figure 1 and Table 1 shows, several activities in and beyond PME-NA have emerged to connect scholars and provide resources, such as www.embodiedmathematics.com, our web portal. Two NSF-funded workshops for K-16 researchers and instructors grew from this: "The

Future of Embodied Design for Mathematical Imagination and Cognition" (May 20-22, 2019); and "EMIC: Professional Development for Undergraduate Mathematics Instructors" (October, 2021). An edited book is planned for the "Research in Mathematics Education" Series as is a third NSF sponsored workshop. As the EMIC group matures, we are broadening the set of organizers to represent a range of institutions, perspectives, and applications. This enriches the colloquium experience and the long-term viability of the community. Mitchell Nathan (U. Wisconsin) will be this year's coordinator, with organizers: Hortensia Soto (Colorado State University), Erin Ottmar and Avery Harrison Closser (Worcester Polytechnic Institute), Janet Walkoe (University of Maryland, College Park), and Dor Abrahamson (Berkeley).

EMIC 2021: Embodiment in Mathematics for Inclusion

Our 2021 theme is **embodiment is an effective way to promote inclusive mathematics education research and practices**. This past year, members of the EMIC community conducted two online EMIC workshops to disseminate best practices to teachers and parents during the COVID pandemic. We will explore these topics with participants to explore what the research says about how embodiment facilitates grounded and embodied learning and collaboration, instruction and assessment. This includes how everyday household objects and activities, such as games, crafts, cooking, etc. can foster grounded mathematics learning. We will also identify ways that mathematics education is impaired with the loss of co-located, in-person interactions.

To demonstrate additional growth, we will join the Mathematical Play WG on Day 3, building upon the online joint-gathering for the 2020 PME-NA (held June, 2021).

On **Day 1**, we will introduce the aims of EMIC, present our past progress, and discuss the theme for PME-NA 2021 of "productive struggle." As is customary with EMIC, we will anchor this to hands-on and whole-body mathematical activities (e.g., making human-scale polyhedra). Participants will collaborate in small groups to identify how our bodies support mathematical reasoning and communication. Groups will reflect on inclusive activity designs across grade levels and math topics (based on who attends). Organizers will relate these to the 4 EMIC themes: Grounding, emergence, affect, and multimodality.

On **Day 2**, we will explore mathematical reasoning when physical and embodied resources are unavailable. Teachers may draw from their experience with distanced and remote learning in the past year. Reflections will center on accommodations to foster grounded learning, as well as the types of reasoning and assessment that are unavailable to learners and teachers.

On **Day 3**, EMIC will meet with the Mathematical Play WG to consider overlapping interests and questions. The organizers of both will also discuss aims of a joint conference proposal to NSF that is currently under review. The session will include small activity groups with Math Play participants who want to think about their work as embodied and EMIC participants who want to think of their work as math play opportunities. After, we will facilitate an open discussion that reviews Days 1 and 2, considers how our aims overlap, proposes ways to enhance future PME-NA conferences, as well as the broader ways that embodiment, imaginative thinking, and play can be used to promote inclusivity. We will conclude by discussing continued engagement and dissemination opportunities available with both communities.

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Figure 1: A small selection of embodied activities created by EMIC organizers and experienced by EMIC participants. Clockwise from top left: experiencing geometric transformations, acting out geometry conjectures, constructing icosahedra first as small, then at human scale, and enacting topological relations.

Table 1. List of events organized by the EMIC community that grew from PME-NA 2016.

2015 EMIC Working Group at **PME-NA 37**. Theme: Embodied Mathematical Imagination and Cognition. The launch of the working group and community at PME-NA.
2016 EMIC Working Group at **PME-NA 38**. Theme: Finding Common Interests for Creating an Online Community of Scholars
2017 EMIC Working Group at **PME-NA 39**. Theme: Embodiment at the Crossroads of Mathematical Collaboration.
2018 EMIC Working Group at **PME-NA 40**. Theme: Extending Theoretical Frameworks of Embodied Cognition to Assessing and Assisting Mathematics Learning.
2019 EMIC Working Group at **PME-NA 41**. Theme: Co-design of Novel Embodied Instructional Activities for Mathematics Education.
2019 EMIC Workshop with 50 international researchers, educational practitioners, and graduate students. All attendees presented. Theme: The Future of Embodied Design for Mathematical Imagination and Cognition. Funded by NSF. Hosted by University of Wisconsin-Madison.
2020 (May). EMIC virtual workshop, educators and graduate students. Theme: Instructional Gestures for Classrooms and On-Line Mathematics Learning. Designed for audience of K-16 teachers, parents and mathematics education researchers. Hosted by embodiedmathematics.com.
2020 (October). EMIC virtual panel-led. Theme: The Future of Embodied Design for Mathematics Teaching and Learning. A reading group discussion with break-out rooms with the authors of a recently published (and open access!) EMIC paper (Abrahamson et al., 2020) in *Frontiers*. Hosted by embodiedmathematics.com.
2021 (June; postponed from June, 2020). EMIC Virtual Working Group at **PME-NA 42**. Theme: How Embodied Mathematics Can Bridge Cultural Divides.
2021 (October; postponed from June, 2020). EMIC Workshop. Funded by NSF. Theme: EMIC: Professional Development for Undergraduate Mathematics Instructors. Hosted by Colorado State University.
2021 (October). Proposed EMIC Research Colloquium at **PME-NA 43**. Theme: Promoting Inclusive Mathematics Education Research and Practices.

PRESERVICE TEACHER LEARNING OF PRACTICE THROUGH SIMULATED TEACHING EXPERIENCES BEFORE, DURING, & AFTER COVID

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This working group is a continuation of a 2019 PME-NA working group focused on the challenges and opportunities of using simulations of teaching practice as an educative tool for preservice teachers focusing on simulation use in the context of the COVID-19 pandemic. Initially, we will share the takeaways from the 2019 working group. Next, we will discuss our experiences implementing simulated teaching within mathematics methods courses that experienced challenges due to COVID-19 conditions. Finally, we aim to identify the pieces of this work that are worth preserving after the pandemic.

Keywords: Simulation, Preservice Teacher Education, Online and Distance Education, Technology, Classroom Discourse

It is essential for preservice teachers (PTs) to have authentic practice-based experiences during their professional preparation (Ball & Forzani, 2009; Forzani, 2014). Among the three key components of professional education (Grossman, Hammerness, & McDonald, 2009), approximation of practice provides "opportunities to rehearse and enact discrete components of complex practice in settings of reduced complexity" (Grossman et al., 2009, p. 283). Approximations of practice provide opportunities for PTs to learn from their mistakes, experiment with various instructional approaches, and enhance their teaching knowledge and skills (Girod & Girod, 2008). Because PTs are not interacting with actual students, they can develop their professional skills in a safe environment without worrying about the possibility of negatively impacting student learning. Simulations have become a popular means of providing approximations of practice in a variety of professions, including medicine, aviation, and the military. The impact of using simulations to improve teachers' practices has been investigated in recent years (e.g., Straub et al., 2014; Howell & Mikeska, 2021). Challenges emerging from the COVID-19 pandemic of 2020, however, have brought simulations, and particularly digital simulations, into the mainstream as never before, with widespread endorsement from professional organizations such as the American Association of Colleges for Teacher Education ([AACTE], 2020). It has also, in response to difficulty in accessing field placements, led to the rapid expansion of their use in lieu of work with children, a use that was never intended in the design of most such simulations. We see this as productive struggle; while necessity in the face of unprecedented challenge may have been the primary driver of this rapid expansion in simulation use it has also allowed us to learn from this natural experiment in simulation use and re-imagine what we may want to retain from the approach as we emerge from the pandemic and re-invent our approaches to teacher preparation.

Focus of Work

This working group seeks to explore the following questions:

1. What affordances and/or challenges did you see in using, adapting, and integrating digital simulations before the pandemic?

2. How and why did the affordances and/or challenges change during the pandemic?

3. What affordances and/or challenges do you see in using, adapting, and integrating digital simulations after the pandemic?

Organization and Plan for Active Engagement

The overall goal of this continued working group is to expand the community of researchers, teacher educators, and practitioners from the initial working group to explore how simulations of practice can be optimized to provide opportunities for teacher learning during the pandemic and in a post-pandemic world. Prior to convening in Philadelphia, we will ask participants to complete a brief survey on their experiences using teaching simulations. We will use this information to frame the working group discussions.

The working group will consist of three sessions during the conference followed by virtual meetings through the following year. We organize the sessions and focus roughly along the timeline of before/during/after, focusing first on what is known about simulation design and use, next on foregrounding emergent learnings from participants' uses of simulation during the pandemic, and finishing by looking forward to where we collectively see value in the approach moving forward. In each session, participants will have opportunities to share their experiences and collaboratively design simulation tasks based on lessons learned from the community.

Session 1. A Principled Start: Pre-Pandemic Simulation Design and Best Practice

In this session, we will begin by sharing the takeaways from a 2019 PME-NA working group focused on the challenges and opportunities of using simulations in a pre-COVID-19 pandemic. The 2019 working group explored three key topics: the theories of action by which teacher learning is expected to result from engagement in simulation activities, design principles grounded in those theories of action, and how to leverage simulations to measuring the development of mathematics knowledge for teaching (MKT), teaching practice, or other valued outcomes.

Session 2. Meeting the Challenge: Affordances & Challenges in Simulation Use during COVID-19

We will then transition into our experiences using simulations during the COVID-19 pandemic and within the context of restricted access to field experiences. Specifically, the facilitators and participants will share lessons learned from redesigning methods courses around simulation-based activities, including the participation of a subset of the authors in grant activities that were conceptualized to provide exactly such opportunities (Bondurant, 2020; Lee & Freas, 2020; Schwartz, Lee, Gonzalez, & Belford, 2020).

Session 3. Crisis to Opportunity: Simulations in Post-Pandemic Teacher Preparation

Finally, we aim to discuss the components of simulation experiences that are worth preserving after the pandemic. Participants will reflect on how the simulated experiences compared to their pre-pandemic PT field experiences. They will share how they plan to incorporate simulations in their programs moving forward as well as their justifications for these decisions.

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NEW WORKING GROUP: MATHEMATICS CURRICULUM RECOMMENDATIONS FOR ELEMENTARY TEACHER PREPARATION

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This new working group seeks to produce research-based recommendations that support teacher preparation programs in providing an effective curriculum for the mathematics instruction pre-service elementary teachers receive.

Keywords: Preservice Teacher Education, Teacher Educators, Mathematical Knowledge for Teaching, Elementary School Education

Standards and policy documents from professional organizations (AMTE, 2017; CBMS, 2012), accreditation bodies (e.g., CAEP, 2018), and governmental authorities (e.g., Michigan Department of Education, 2020) set forth mathematics coursework recommendations and requirements for elementary teacher preparation programs to follow. Mathematics courses for PSTs are critical because teachers' mathematical knowledge for teaching makes a difference in the quality of instruction their students receive (Hill et al., 2008). However, there remains variation in what mathematics elementary pre-service teachers (PSTs) in the United States study and to what depth (Malzahn, 2020). Such variation also exists in Canada, which has a provincially decentralized approach to coursework standards for teacher preparation (NCEE, 2016). Although Mexico now has a more centralized approach to teacher preparation, teacher training colleges have been slow to change their practices in response to the 2013 educational reforms (Hrusa et al., 2020). Across North America, some institutions implement survey courses that attempt to teach the entire breadth of elementary mathematics, whereas programs like the Elementary Mathematics Project (EMP; Chapin et al., 2021) focus on mathematics content identified as "high-leverage" (TeachingWorks, n.d.) and intentionally omit other topics.

Organizers of this working group (Corven et al., 2019; DiNapoli et al., 2018; Hiebert et al., 2017) have presented conference sessions detailing results of research on specialized content knowledge (SCK) retained by graduates of an elementary teacher preparation program. Corven et al. (2019) found that the amount of classroom instructional time on topics addressed in teacher preparation explained over 13% of within-person variance in graduates' SCK for those topics immediately after graduation. This relationship persisted up to two years after graduation. Quantitative models suggested that (on average) about 450 minutes of high-quality instruction on one mathematical topic were needed to develop the SCK to teach it well. Following each presentation, rich discussions arose regarding how to use these results to improve elementary mathematics teacher preparation. One presenter noted that the program's faculty had made the decision to intentionally restrict the content "covered" by their elementary mathematics courses based on *their* perceptions and opinions of importance of the topics for elementary PSTs. He suggested convening a working group to synthesize a broader range of perspectives on the issue and to develop research-based recommendations that could be adopted by teacher preparation programs and adapted to local contexts. This new working group is meant to fill that purpose.

Goals of the Working Group

The goals of this working group are two-fold. First, we want to serve as a forum to share research results that could inform the design of a more unified curriculum for elementary mathematics teacher education. Second, we want to develop a document that sets out specific recommendations for such a curriculum. We believe this curriculum skeleton document (e.g., a scope and sequence of topics with a recommended range of instructional time for each topic) would serve multiple purposes. First, it would be a resource for mathematics teacher educators designing content and methods courses in their local contexts that provides more specific guidance than the AMTE (2017) and CBMS (2012) recommendations. Second, it would help mathematics teacher educators advocate to remove structural barriers to implementing these recommendations (e.g., increasing the required number of mathematics content or methods courses elementary PSTs must take), either at the institutional or the governmental level.

We expect that the work of this group will span multiple PME-NA conferences. Our focus at this conference will be on critically examining what mathematics content should be taught in courses for elementary PSTs. In other words, we want to come to a consensus on what topics should be taught (or not taught) and approximately how much time should be spent on each. Meetings at future conferences will consider recommendations for the "how" of teaching this content because what happens during the instructional time is also important (Copur-Gencturk et al., 2019; Stallings, 1980). Once the recommendations stabilize and research supporting them is provided, we would present a colloquium at a future PME-NA conference to share the curriculum skeleton document and the underlying research.

Organization and Presentation Plan

The first session will start with a discussion among all participants about their elementary teacher preparation programs, the contexts in which they exist, and challenges they are facing in ensuring elementary PSTs are prepared to teach mathematics well. This discussion will generate a list of concerns that all participants will think about to prepare for future sessions. The organizers will then give a brief overview of research on the effectiveness of elementary teacher preparation related to mathematical knowledge for teaching. We will share theoretical frameworks, including the EMP design principles (Chapin et al., 2021) and knowledge- vs. thinking-oriented approaches to teacher preparation (Li & Howe, 2021), to ground our work.

During the second session, participants will collaboratively generate content, structure, and instructional time recommendations for ideal elementary content and methods courses. At the end of this session, we will see where there is consensus and where there is disagreement. In the final session, we will share the results of our analysis of the recommendations and lead a discussion about them. Finally, participants will discuss ideas for how the previously generated list of concerns could be addressed in a recommended mathematics curriculum for elementary teacher preparation in North America (while still attending to important differences in local and national contexts), and research findings that support those recommendations or practices.

By the end of the conference, the group will have produced a working draft of a curriculum recommendations document that will be shared with all participants. We will collaborate to design and organize pilot studies that could provide evidence supporting the recommendations. Finally, we will discuss next steps for the working group (e.g., organizing a special issue of a journal or an edited book with research that highlights the importance of depth rather than breadth for elementary mathematics teacher preparation). Additionally, we intend to continue this work with broader participation from the mathematics teacher educator community by organizing a similar working group for the 2022 AMTE conference.

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CONTINUOUS IMPROVEMENT LESSON STUDY WORKING GROUP

OPTIONAL: GRUPO DE TRABAJO EN LA MEJORAMIENTO CONTINUA DEL ESTUDIO DE LECCIONES

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This working group is a new initiative aimed at providing a structure for Mathematics Teacher Educators (MTEs) to engage in Continuous Improvement Lesson Study (CILS), a process of MTE professional development that involves working collaboratively to design a lesson, improve preservice teacher (PST) learning, and MTE practice. The goals this year are to share the CILS process and experiences, and form sub-groups of MTEs with common interests to engage in the CILS process to develop an educative lesson. The eventual goal is to collaborate with sub-groups to share lessons and experiences in an edited book or a special issue of a journal.

Keywords: Teacher Educators, Preservice Teacher Education, Mathematical Knowledge for Teaching, Professional Development

The Continuous Improvement Lesson Study (CILS) Process

The CILS process integrates lesson study (Lewis & Hurd, 2011) and the Continuous Improvement process (Berk & Hiebert, 2009). Lesson study is a collaborative process of investigating instruction with the goal of improving student learning through a single lesson. This teacher-driven professional development focuses on analyzing student learning and leads to enhanced instructor knowledge (e.g., Watanabe, 2004; Demir et al., 2013). Lesson study consists of four phases: 1) study the curriculum and formulate goals, 2) plan the lesson, 3) teach the researched lesson, and 4) reflect on learning (Lewis & Hurd, 2011). While traditional lesson studies occurred with participants physically located in the same room to observe the intricacies of the research-based lesson, there have been attempts to broaden the geographic implementation of lesson studies with the use of technology (e.g., Soto et al., 2019).

Hiebert and colleagues at the University of Delaware developed a model for improving MTE lessons and instruction at their university (Berk & Hiebert, 2009). The model includes an iterative process of “planning, enactment, analysis, and revision” guided by three principles: 1) specify critical learning goals for PSTs; 2) collect and use evidence of students’ learning to drive revisions; and 3) gather and store knowledge in a final version of the lesson that becomes a shared product (p. 339). Unlike traditional lesson studies where the primary outcome of the process is student and teacher learning, the primary outcome is the educative lesson itself.

We have integrated these two methods of researching and improving instruction and student outcomes, lesson study and continuous improvement to inform an integrated process that we call Continuous Improvement Lesson Study (CILS). This new, integrated process (Figure 1) follows the lesson study process (blue) and incorporates the iterative processes of continuous improvement (yellow) that results in a final lesson product. The incorporation of continuous improvement within lesson study allows for a focus both on PST learning and on lesson revision as a lesson is taught multiple times by each participating MTE. In addition, we have included research (green) as a part of this iterative process because we found returning to literature to situate PST learning after reflecting on the teaching of the lesson was imperative for our lesson revisions (Appelgate et al., 2020).

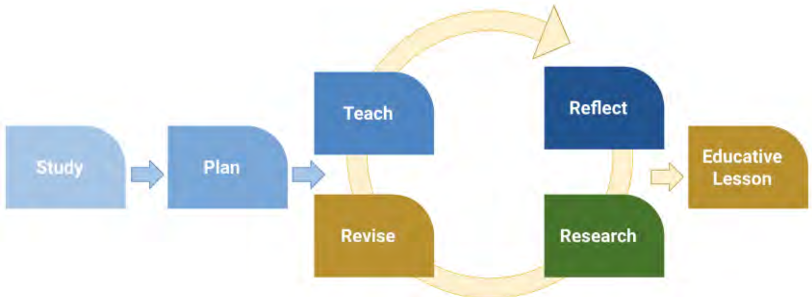

Figure 1: The Continuous Improvement Lesson Study (CILS) Process

We have found that engaging in the CILS process supports MTEs to develop an educative curriculum and collaborative teaching environment. MTEs are engaged in the act of teaching teachers mathematical knowledge for teaching (MKT) (e.g. Ball et al., 2008) thus we are working to develop our own mathematical knowledge for teaching teachers (MKTT) (Castro Superfine et al., 2020) throughout the process. Unlike with lesson study and the continuous improvement method, CILS is designed to be used by MTEs across institutions, and therefore most of the meetings and lesson outcomes are shared with each other virtually (Soto et al., 2019).

Working group Organization

This working group is a new initiative that aims at providing a structure for the participants to engage the CILS process to improve their practice. Table 1 contains the proposed structure. The plan will be to continue meeting at future PME-NAs with an eventual goal of publishing a special journal issue or an edited book.

Table 1: Overview of the Proposed Working group Session

	Activities	Take-Aways
Session One	1. Introductions and Agenda 2. Overview of CILS and goals of the WG 3. Sharing of resources; discussion of potential topics for the subgroups 4. Formation of subgroups and work time	1. Formation of subgroups 2. Resources for the next session
Session Two	1. Subgroup sharing from previous session 2. Work time- solidifying topic, research focus, and resource brainstorming 3. Brief sharing of the subgroup work	1. Topic Chosen 2. Finding resources to share
Session Three	1. Subgroup sharing from previous session 2. Work time - sharing and developing a plan for future meetings and collaboration	1. Research Focus of the subgroups 2. Future plans for

	3. Brief sharing of subgroups 4. Final reflections and plans for future working groups with eventual goal of a special issue or edited book	collaboration and sharing with ME community

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AUTHENTICITY IN MATHEMATICS EDUCATION ASSESSMENT

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We propose the establishment of a new working group that critiques a means-end view of assessment in K-12 mathematics education and recognizes the human endeavor of learning through authenticity in assessment. Through engagement in assessment tasks and discussions, we aim to develop a shared understanding of authentic assessment that leads to the creation of guiding principles and example tasks.

Keywords: Assessment

Assessment "is a process whose primary purpose is to gather data that support the teaching and learning of mathematics" (NCTM, 2014, p. 89) that requires a holistic view of students' mathematical proficiencies (Kilpatrick, Swafford, Findell, 2001) and identities as mathematics learners (Heyd-Matzuyanim & Sfard, 2012). Despite well-established perspectives, classroom assessment practices in mathematics struggle to align and evolve from an emphasis on measurement to learning-through-assessment (McMillan, 2013).

Goals and Strategies

We have two goals with related strategies for the working group:

- Develop a shared understanding of authenticity in mathematics education assessment and illustrate with specific examples; to be reached through discussion built on theoretical framings and lead to establishing collaboration for research through commitment to action;
- Develop guiding principles for what authentic assessment looks like in mathematics and types of associated tasks; to be reached through collective experience and discussion and lead to outlining a paper with guiding principles to be fleshed out and published.

Theoretical Background

We are revisioning authentic assessment as a movement away from a product-based view of assessment to an ontological view of assessment. Vu and Dall'Alba (2014) stated "authentic assessment is not an end in itself; rather, it is an opportunity for students to learn to become who they endeavour to be" (p. 779). This working group will explore authentic assessment with the framing of the opportunities that assessment provides to both students and teachers to become who they endeavour to be. Mathematics teachers in K-12 classrooms are working within constraints of assessment discourse around timing (formative/summative) and means-end (utilitarian) purpose of solely informing toward occasioning assessment to be educational. For some of these teachers, the constraints that they find themselves working within do not fit who they want to be. We offer in this working group a way for mathematics educators to reconceptualize authentic assessment as a process that both teachers and students engage in that promotes "calling things into question, challenging assumptions, and engaging in renewal" (p. 788). Additionally, DeLuca and Wickstrom (2021) envision assessment as pedagogy where assessment is integrated with learning not as an event to happen after learning.

When challenging assessment methods in mathematics education it is necessary to acknowledge mathematics as a multidimensional subject. Kilpatrick et al. (2001) indicates five strands of mathematical proficiency that should be sought after in both instruction and assessment. Understanding mathematics learning from a comprehensive and diversified perspective contests individual, discrete assessments, moving towards the idea of balanced sets of assessments. The working group will invite mathematics educators to use the framework of the five mathematical proficiencies to attend to the multifaceted growth of students' mathematical learning. In keeping with the conference theme, we will engage with the productive struggle of teachers to engage authentically with assessment and with their students.

Session 1: Authenticity of Assessment in Mathematics Education

The session's aim is to develop an initial understanding of authentic assessment in mathematics education. The first session of the working group will use the conceptualization of authentic assessment from Vu and Dall'Alba (2014) as a provocation for the participants to consider. Questions we will discuss (90 min) include: How can an assessment allow both a student and a teacher to be authentic and develop authenticity as a human being in the world? What assessment practices do we engage in that are due to the constraints of the environment we work within? Are those practices consistent with our beliefs about teaching and learning and what we value as teachers? Can we change our assessment practices to reflect our evolving beliefs and understandings? At the end of the session, we will articulate our beginning understandings of authentic assessment with these questions in mind.

Session 2: Being and Becoming Mathematically Proficient

The session's aim is to reflect on what authentic assessment in mathematics entails and on how it can support students to be and become mathematically proficient. After a brief introduction about the strands of mathematical proficiency (15 min), participants will engage in a discussion that challenges the taken-for-granted procedural approaches in mathematics assessment and considers the multidimensionality of mathematics learning (30 min). Participants will share examples of where they have seen the strands of mathematical proficiency assessed and will analytically reflect on that practice through the lens of authentic assessment (45 min). The use of the strands of mathematical proficiency to inform and improve assessments is not a widespread practice (Corrêa & Haslam, 2021) and will be further explored in the working group. The session will enable the refinement of the guidelines for authentic assessment, providing participants with deeper underpinnings to join session 3 activities.

Session 3: Enabling Teachers and Students to Grow

The session's aim is to identify a range of assessment tasks that promote growth as a moment of authenticity. We will engage participants in a geometry assessment task (15 min) that exemplifies a novel concept: generative assessment. Generative assessment (GA) includes tasks which move beyond evaluating student competency and informing teaching decisions, to sponsoring teacher- and student-growth through interactions (McFeetors & Marynowski, 2017). Participants will evaluate a GA task, share their own examples of similar tasks, and develop awareness of how assessment can also be moments of growth (30 min). These interactions will culminate in a discussion about what qualities of tasks make them GA and the range of various tasks that could entail GA as authentic assessment, and in the production of authentic assessment guidelines (45 min). Participants will reflect on their research in mathematics assessment to commit to incorporating authentic assessment in project design and/or data analysis.

Follow-up Activities

As a continuation of the activities proposed during the conference, the working group will publish a theoretical paper with guidelines for authentic assessment in mathematics education. Moreover, we intend to bring back results of incorporating authentic assessment in research to a working group in the 2022 conference.

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WORKING GROUP ON GENDER AND SEXUALITY IN MATHEMATICS EDUCATION: EMERGING CONCEPTUAL AND METHODOLOGICAL FRAMEWORKS

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The Working Group on Gender and Sexuality in Mathematics Education has convened during the three previous PME-NA conferences. These meetings have resulted in a shared foundational knowledge of the research area and have helped us to develop understandings related to how linguistic and conceptual choices in gender and sexuality research influence research methods, results, and interpretations. At the June 2021 PME-NA conference, we aimed to expand our communal knowledge to more fully utilize theories of gender and sexuality within our work in mathematics education, attending to and problematizing the concept of identity. The October 2021 working group is organized to continue these discussions while also focusing on emerging conceptual and methodological frameworks. Members of the working group will continue to develop partnerships in order to respond to theoretical and methodological dilemmas.

Keywords: Gender; LGBTQIA+; Equity, Inclusion, and Diversity

The Working Group on Gender and Sexuality in Mathematics Education has met during the past three PME-NA conferences to develop a shared knowledge base of the current research on this topic, the theories that surround this work, and the ways that language and research methodology influence the work that is being done. As a result, many research collaborations have formed through this Working Group. Furthermore, a special issue on the topic of innovations within this work will be published in early 2021 in the *Mathematics Education Research Journal* (MERJ), with many members of this Working Group serving as contributors to the special issue. As a means to continue moving forward in this area of research, we have designed the October 2021 PME-NA Working Group sessions to focus on the emerging frameworks discussed in this MERJ special issue. Although this Working Group includes the presentation of prior work, it is not a colloquium because the presentations are meant to serve as a launching point to provoke and stimulate conversations so that the Working Group members may grapple with theoretical and methodological dilemmas. The goal of this discussion-based design is to support all attendees of the working group to continue to think about their own research in innovative ways that will advance the field.

Theoretical Background

In previous research on gender in mathematics education, scholars have studied girls' and women's experiences in mathematics (Forgasz et al., 2010), distinctions between sex-based and gender-based studies in mathematics education (Leyva, 2017; Lubienski & Ganley, 2017), and sociocultural factors that affect achievement and participation (Leyva, 2017). Additionally, scholars have considered the role of identity research in mathematics education (Damarin & Erchick, 2010; Darragh, 2016) alongside the theorization of identity in gender and queer studies (Butler, 2004). In this Working Group, we continue to develop our understandings and the implications of such theoretical framings and methodologies. Our working group this year will be guided by the following guiding questions: (a) What has been and what are the current foci of gender and sexuality research in mathematics education? (b) How might emerging theory inform the methodologies of these foci? and (c) How might evolving theories and methodologies create new areas to explore?

We offer working group participants Risman's (2018) conceptualization of *gender as social structure* as a heuristic to engage with the guiding questions. Risman argued that gender operates as "a stratification system that has implications at the individual, interactional, and macro *levels* of analysis" (p. 30). At each of these three levels, there are both material and cultural *aspects* which creates six dimensions of gender and sexuality research. Although this conceptualization was developed to discuss gender, we may utilize the heuristic to also discuss sexuality. Sexuality can be made analytically distinct from gender; however, Butler (2004) argued the two are inextricably entangled. In Risman's (2018) model, it is important to recognize that "social structures not only act on people; people act on social structures" (p. 30). As such, we invite critique on these dimensions. Further, *paradigm of inquiry* (Stinson & Walshaw, 2017) will be offered as an additional resource for participants to identify and reflect on their interests and methods in relation to others, in order to discuss and advance their work.

Organization and Structure of the Working Group

The organization and structure of the working group were created to provide a common grounding of the field of gender and sexuality in mathematics education, while also maximizing participation so that members can engage in meaningful conversations that pertain to their work.

Day 1: We will begin Day 1 with a brief presentation of *gender as social structure* (Risman, 2018) and *paradigms of inquiry* (Stinson & Walshaw, 2017). Kersey (2020) and Przybyla-Kuchek (2020), two authors from the MERJ special issue will share examples of applied emerging theories and methodologies used in their work. After the presentations, an assigned discussant in the Working Group will engage the participants in conversations surrounding this work, particularly in relation to the guiding questions.

Day 2: In continuing the conversations from Day 1, we will extend the discussion of emerging theories and methodologies to consider the field's direction more broadly. Moore (2020) and Wiest (in press), two authors from the MERJ special issue will share their research. A discussant for each article will then engage the working group participants in conversations surrounding this work, particularly in relation to the guiding questions.

Day 3: We will discuss future directions, share resources, and develop plans for Working Group members to collaborate over the next year. Jennifer Hall, a co-editor of the MERJ special issue, will briefly present an updated perspective on the trends of gender and sexuality research in mathematics education, based on the experience as a co-editor of the special issue. We encourage members to continue to create strategic partnerships to pursue interests and projects to share at future PME-NA meetings. The day will be organized as a combination of whole-group discussion and small break-out groups based upon the needs of Working Group participants. We encourage working group participants to read the four noted MERJ articles prior to the working group sessions in order to get the most out of our time together.

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NEW[ISH] WORKING GROUP: TEACHING MATHEMATICS FOR JUSTICE AND LIBERATION IN THE CONTEXT OF UNIVERSITY COURSES

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Goals

There are three goals for this new[ish] working group: 1) To continue to create a community of mathematics teacher educators (MTEs) who are (or are interested in) collaboratively teaching mathematics for justice and liberation (TMJL) in their university content and/or methods classes. 2) To continue to collaboratively select/develop/modify TMJL tasks and implement those in mathematics content/methods classes. 3) To research the implementation of the TMJL tasks. At PME-NA 2020 the inaugural meeting of this working group led to the following:

Refocusing of Teaching Mathematics for Social Justice (TMfSJ) to Teaching Mathematics for Justice and Liberation (TMJL), building on the work of Paolo Freire (1996,). Mathematics for liberation focusses on justice and aims to avoid re-traumatization by exploring harm done. Contexts are examined with the goal of liberation rather than oppression, thus a focus is not on examining actions of harm but on how mathematics was used to justify the harm, as well as creating learning opportunities that center on the joy, resiliency and and ways of knowing learners bring, particularly those typically marginalized in classrooms, to promote critical literacies, agency, and action. In the 2020 working group we identified 3 main themes to examine further: (1) Understanding histories (of math, math ed, TMSJ work, etc.) in order to move to liberation, (2) Math as a tool (for liberation, curiosity, etc.), not an obstacle - what does this mean for the classroom (broadly), the system, etc. (3) Investigating data towards liberation

Examining Struggles Implementing TMJL Tasks. In our group we began examining the following issues: (1) Navigating Student or Colleague Pushback to TMJL, (2) Setting up the (TMJL) Classroom Community for Shared Authority and Participation, and (3) Balancing Content and Context Goals

Examining Contexts of TMJL Tasks. We began examining: Gentrification, Mutual Aid, Tracking, Black and Asian Solidarity.

Strategies to Reach Those Goals

Our goal for this working group is to continue to create a community of MTEs who will collaboratively develop and implement TMJL tasks in their university courses and research the implementation for (in no particular order): (a) preservice teacher (PT) learning about the mathematics, (b) PT learning about the context, (c) impacts on PTs' view of mathematics and/or teaching mathematics, and (d) the potential for TMJL in university methods or content courses to ignite a call for action.

Background

Mathematics educators face a moral and ethical imperative to support students in their struggles to make sense of and mobilize against injustices in and out of the mathematics classroom (Stinson, 2014; Turner et al., 2009). In alignment with this call to action, we framed our first working group with the Teaching Mathematics for Social Justice framework (TMfSJ). The TMfSJ framework was grounded in: (1) the commitment to connect school mathematics to issues of social (in)justice, and (2) the conviction that mathematics can be taught through the study of social justice issues in order to develop PK-12 students' mathematical literacy, critical consciousness, and positive mathematical identities (Chao & Marlowe, 2019; Esmonde, 2014; Gutstein, 2003; Raygoza, 2016; Turner et al., 2009). This second working group builds upon the TMfSJ framework by positioning school mathematics as a tool for justice *and* liberation (TMJL). The potential and power of mathematics as a tool for liberation has a storied and rich history in the mathematics education community (e.g., Jett, 2009; Martin, 2010; Moses & Cobb, 2001) and in the larger scholarly community and society (e.g., Anderson, 1970; Freire et al., 1997). Mathematics for liberation intentionally moves away from mathematics tasks that are rooted in contexts of oppression and marginalization, e.g., the causes and impacts of redlining, to mathematics tasks rooted in the stories and collective power of historically marginalized voices and communities, e.g., the need for and impact of mutual aid networks (Yeh et al., 2021).

Many PTs enter their coursework believing that mathematics is neutral or universal (Greer, Verschaffel, & Mukhopadhyay, 2007; Keitel & Vithal, 2008). Yet, given the intersectional diversity of PK-12 students and human history of oppression and colonization, it is vital for MTEs to intentionally plan and implement learning opportunities for PTs to critically unpack the pervasive and misleading belief that mathematics is somehow neutral and classroom spaces are safe spaces (Frankenstein, 1983; Gutiérrez, 2013; Yeh & Otis, 2019). One pedagogical tool is to have PTs experience TMJL tasks in their mathematics courses so that they can develop their own critical consciousness about oppression and liberation alongside their content knowledge (Ball et al., 2008; Gutiérrez, 2017). TMJL tasks also open pedagogical space for MTEs and PTs to consider the power and possibilities of implementing TMJL tasks with the PK-12 learners with whom the PTs will work in their future classrooms (Bartell, 2013; Jong & Jackson, 2016). We seek to explore PTs learning *about* TMJL as well as *through* TMfSJ in this community of MTE at PME-NA while also continuing to collaborate on challenges inherent in the work, e.g., PT resistance when integrating issues of social (in)justice into mathematics (Aguirre, 2009; Ensign, 2005; Felton-Koestler, Simic-Muller, & Menéndez, 2012; Rodríguez & Kitchen, 2004).

Participant Engagement

Session 1: 1) Organizers will present on outcomes of PME-NA 42. 2) Participants and organizers will continue to discuss the successes and struggles in implementing TML tasks. 3) We will introduce one/several context(s) to focus on for the next two sessions (continue PME-NA 42 contexts and/or add new ones) as well as an online media platform for continued participation. **Session 2:** Entry points for TMJL tasks: 1) We will discuss various entry points (focus on math and/or context) for TMJL tasks. 2) We will collaboratively engage in the use of one context in our classes and potential tasks that could go with that context. 3) Participants will share their own experiences and how they may envision using such a context in their class. **Session 3:**1) We (in small groups) will collaboratively create/adapt TMJL task(s) to participants' localized contexts to use in their teaching. Participants will leave with a more nuanced understanding of TMJL tasks/implementation. 2) We will set up structures to follow up via online media after implementations. 3) The goal will be to meet at next year's PME-NA.

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